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EXTREME APPRENTICESHIP METHOD IN TEACHING UNIVERSITY-LEVEL MATHEMATICS

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At the Department of Mathematics and Statistics in the University of Helsinki the courses Linear Algebra and Matrices I and II have traditionally been taught through lectures and take-home tasks. Although the pass rates have been satisfactory, the students' level of understanding of the topics has not, which has been apparent in subsequent courses. The approach chosen to tackle this problem was to use the Extreme Apprenticeship method developed for teaching computer programming and implement it into teaching university-level mathematics. The courses under study in fall 2011 were among the largest courses at the department with hundreds of participants (N=361 and N=248). The results show that the students were satisfied with the new teaching method, and the passing rate did not drop even though the workload was significantly increased and the requirement level was raised.

Extreme Apprenticeship method, Linear Algebra, Scaffolding, Tertiary Education

INTRODUCTION

Students aiming for a bachelor's degree in mathematics begin their studies with the courses *Linear Algebra and Matrices I*[3] and *Linear Algebra and Matrices II*[4]. Even though the majority of the students pass these courses, the lecturers of subsequent courses have noticed that many students lacked understanding of the concepts they had studied. The most recent development in tackling this issue has been to introduce the Extreme Apprenticeship method (XA) in fall 2011 to these courses with 361 and 248 participating students.

XA was first introduced to teach introductory computer programming courses at the University of Helsinki by Vihavainen et al. (2011a). The method emphasizes the individual work carried out by students and the continuous support given by instructors. The results have

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[3] Topics include: Systems of linear equations, matrices and matrix operations, inverse matrix, Gaussian elimination, vector space, subspace, linear independence, basis of a vector space.

[4] Topics include: Dot product, inner product, orthogonality, orthogonal projection, orthogonal complement, Gram-Schmidt process, linear mapping, kernel and image, isomorphism, eigenvalue, diagonalization, determinant.

been good with increasing course completion rates and the method has proven to be scalable for large university courses (Vihavainen et al., 2011a,b; Kurhila and Vihavainen, 2011).

This paper discusses how XA can be implemented to teaching mathematics in a university setting. We show that using XA has a positive impact on the students' engagement in learning and that the students are aware of their skills and have a positive experience of the method.

We begin by describing the theoretical background behind the apprenticeship-based teaching used in the study and then give a description of the practical arrangements that took place. We continue by presenting data from the courses and conclude with discussion of the implications of the research and further development of the method.

THEORETHICAL BACKGROUND

Cognitive Apprenticeship

Collins et al. (1989, 1991) suggested an apprenticeship-based method for teaching reading, writing and mathematics. The method called Cognitive Apprenticeship (CA) stresses modeling of tasks, coaching the students through selected sub-goals and instruction and careful observation of the students' progress (Collins and Greeno, 2011). Collins et al. (1989, 1991) use the findings of Lave (later published by Lave and Wenger (1991)) who had studied West African tailors learning their craft by observing and practicing in tailor shops under the instruction of master tailors. Schoenfeld (1992) summarizes Lave's observations as follows:

> Being a tailor is more than having a set of tailoring skills. It includes a way of thinking, a way of seeing, and having a set of values and perspectives. In Tailors' Alley, learning the curriculum of tailoring and learning to be a tailor are inseparable: the learning takes place in the context of doing real tailors' work, in the community of tailors. Apprentices are surrounded by journeymen and master tailors, from whom they learn their skills – and among whom they live, picking up their values and perspectives as well. These values and perspectives are not part of the formal curriculum of tailoring, but they are a central defining feature of the environment, and of what the apprentices learn. The apprentice tailors are apprenticing themselves into a community, and when they have succeeded in doing so, they have adopted a point of view as well as a set of skills – both of which define them as tailors. (p. 85-86)

Teaching through CA proceeds as follows: first the master, teaching the apprentice, models the task at hand through his own example. After the modeling phase the apprentice practices accomplishing the task while the master coaches him by scaffolding. Scaffolding means the temporary support given by a master so that the student is able to learn (Bruner, 1985). The scaffolding is conducted by selecting necessary sub-goals and by giving necessary instruction. The challenge of CA is to create meaningful tasks for the students that are both systematic and diverse and to make the students reflect on what they are doing (Collins et al., 1991).

In the beginning more scaffolding is needed. As the apprentice becomes more skilled, the amount of scaffolding decreases. Scaffolding should be temporary, as the idea is not to make the apprentice dependent of the master. It is also important that the apprentices are scaffolded by many masters and do not expect to be taught by one single authority (Collins et al., 1989). The teaching is seen as an interplay between the apprentice and the master, where the apprentice learns from the master and the master learns from the apprentice.

The final stage is when the master fades away and the learner is able to explore on his own (Collins et al., 1991). At this stage the student already performs as an expert and uses exploration and learning strategies he has acquired during the process.

Extreme Apprenticeship method

Building on Cognitive Apprenticeship, Vihavainen et al. (2011a,b); Kurhila and Vihavainen (2011) have implemented apprenticeship-based learning in teaching computer programming for undergraduates. Their method, called Extreme Apprenticeship method (XA)[5], focuses on the individual effort carried out by students (Vihavainen et al., 2011a).

The method centers on tasks being completed under the constant supervision of instructors. The aim is to raise the amount of actively conducted individual effort of students, while minimizing their passive activities such as listening to lectures. In computer science, as in mathematics, a typical lecture course is based on lectures and take-home tasks. In XA the tasks become the main teaching material and conducting the tasks the main method of learning. The lectures, if any, are based on and complement the tasks.

XA places a lot of emphasis on how the tasks are constructed, conducted and evaluated (Vihavainen et al., 2011a). Students begin to solve tasks from day one of the course, and the tasks are done in a suitable space where it is easy for the students and instructors to interact. This enables instructors to help and monitor the student's progress.

In XA tasks have to be correctly answered. In computer programming this means that the tasks have to be correctly solved and also follow good programming practices (Vihavainen et al., 2011a), for example Clean Code (Martin, 2009). In mathematics this translates to the principle that solutions should be finished with clear notation and structure.

The instructors, in the role of masters, accept or reject the solutions of the students. In case of a rejection, the instructors help the student to correct the solution of the task. Rejection should not be seen as a failure but as an opportunity for the student to learn and correct possible misunderstandings. Therefore, students are allowed to submit solutions many times, until they come up with an acceptable solution. Moments of success and raising confidence level of student should be enforced from day one as it will enhance the student's performance and self-esteem in their studies (Boud et al., 1985).

It is important that the instructors continuously monitor how the students perform (Vihavainen et al., 2011a). If the instructors notice that many students have problems with a

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[5] The label "Extreme" comes from the software engineering industry, where Extreme Programming (Beck, 1999) is a method where good programming practices are taken to the extreme. For example, code review is considered a good programming practice and Extreme Programming takes this to the extreme by enforcing continuous code review in the form of pair programming. XA tries to do the same for educational best practices, for example as monitoring student progress is seen as a good teaching practice, XA enforces continuous monitoring of students progress (Vihavainen et al., 2011a).

certain topic, it is addressed during upcoming lectures and tasks. These are not planned long beforehand, but change during the progress of the course. The monitoring requires dedication from the teaching staff and ample resources to be allocated to instruction (Vihavainen et al., 2011b).

APPLYING THE METHOD

In applying XA special emphasis was placed on how the scaffolding should be performed. According to Kirschner et al. (2006) students learn more deeply when guided. Especially for novices and intermediate students it is important to give strong instructional guidance. The scaffolding provided for the students' learning was two-fold: a lot of structured tasks and plenty of individual support from the teaching assistants (TAs). During the course, scaffolding was gradually decreased.

Weekly tasks

The weekly tasks of the courses were not compulsory but the students were told at the beginning of the courses to solve them if they expected to pass the course. As further incentive, the students who had correctly solved tasks were awarded extra points in the exam[6] of the course.

As the tasks are the central learning material of the course, they have to serve many purposes. They have to (1) introduce students to new concepts and notation. They have to (2) repeat previously learned concepts and (3) create relations between concepts. The tasks should not only train (4) the procedural and notational skills of students but also (5) challenge them to reflect on and articulate what they have learned.

The students were required to give well-formulated solutions with correct explanations and arguments to the tasks. Solving them was not enough. Special emphasis was placed on requiring good structure of the solutions that clearly showed what the student had intended. For the majority of students this was their first mathematics course at the university level, and therefore they found it difficult to express themselves mathematically. Many of the students also held the view that at the university level, well-formulated mathematical argumentation is not required as all involved parties are "mathematically talented".

Practicing the use of mathematical language in a highly guided environment proved to be beneficial. After a couple of weeks when the students had understood what was required of them, the structure of the solutions started to improve.

The number of tasks per week was between 19 and 30 and question types varied a lot. Problem sheets were structured so that tasks, which in the take-home format would have been one large task, were split into several smaller ones. For example, the following task includes many steps, and is demanding for the students to solve:

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[6] If a student solved 90% of the tasks correctly, 6 points were earned, if 75% then 5 points, if 60% then 4 points, if 45% then 3, if 30% then 2 and if 15% then 1 point. The exam had four questions, worth twelve points each, amounting to a total of 48 points.

Let $A = \begin{bmatrix} 1 & 0 & 2 \\ 2 & 1 & 1 \\ 1 & 2 & -4 \end{bmatrix}$. Determine whether the mapping $L_A: \boldsymbol{R}^3 \to \boldsymbol{R}^3, L_A(x) = Ax$ is injective, surjective or bijective.

In XA, the same task was presented in four parts:

1. Determine the kernel $\mathrm{Ker}\, L_A$ of the mapping.
2. What is the dimension of the kernel?
3. What is the dimension of the image $\mathrm{Im}\, L_A$?
4. Is the linear mapping L_A injective, surjective or bijective?

Now the set of tasks guides the students through the solving process. This is important especially when the topic is introduced for the first time. If the students encounter difficulties, they are able to ask detailed questions instead of just saying “I don’t understand a thing.” Since it is easy to pinpoint where the difficulties arose, it makes is possible for the instructor to give targeted guidance.

Many tasks were reflective, and the student had to articulate the solutions of previous tasks. For example, in the following exercise the students first prove a result, and then explain it in their own words:

1. Suppose that V is an inner product space and $v_1, v_2, \dots, v_n \in \boldsymbol{V}$. Consider the subspace $W = \mathrm{span}(v_1, v_2, \dots, v_n)$ generated by these vectors. Let $v \in V$ and $< v, v_i >= 0$ for each $i \in \{1,2,\dots,k\}$. Show that $< v, v_i >= 0$ for every w$\in W$.
2. Explain in your own words, without using any mathematical symbols, the result you obtained in the previous task.

Checking the students’ solutions

The students had to submit their solutions to tasks with a weekly deadline starting from the first week of the courses. The TAs checked if the solutions were correct and written with accurate mathematical notation and language. The students were then given an opportunity to amend their incorrect answers and resubmit them within one week. This meant that students were working on two sets of tasks per week, the new tasks for the week and the incorrect solutions from the previous week.

Between deadlines the students had the opportunity to utilize the special classroom set up for the courses and get instruction from the TAs. Both the instruction and the checking of the solutions proved to be a vital part of the bi-directional feedback as it gave the teaching staff an overall view of which tasks were difficult and what problems the students were facing. This also gave the students feedback on what they had learned and whether they had misunderstood concepts.

The teaching staff checked 39,399 solutions, which took approximately 800 hours in total. As the checking process was time-consuming, the students’ answers could not always be checked thoroughly and they were told not to expect an exam-like evaluation process. The TAs used common guidelines in checking the solutions, but occasionally similar answers were rated differently.

The process of coping with all the data was copied from the Department of Computer Science (Kurhila and Vihavainen, 2011). The data was entered into a collaborative spreadsheet, which provided instant overall figures for the teaching staff of the course to be used for the planning of the courses.

The role of the teaching assistants

Typically in a lecture course, TAs are responsible for guiding the demo sessions of the take-home tasks. In XA the role of the TAs is to help the learner solve the tasks by himself. Therefore the recruitment of the TAs was changed to incorporate an interview of each applicant and the applicants were scored according to their social skills and their perception of how the learning process can be enhanced through instruction. Additionally to this their study performance in mathematics was taken into consideration. Of the 34 applicants 16 TAs were recruited.

The criteria for salary payment were also changed for the TAs. Normally TAs are paid according to the demo sessions they guide. On top of that they are expected to perform additional obligations such as attending TA meetings and creating model solutions for tasks. In the new system the hourly wage was reduced, but the TAs were paid for all the work they performed including meetings and writing model solutions. This made it easier to allocate extra working hours. The TAs worked approximately eleven hours per week during the courses. Of these hours five were allocated to guidance, four for correcting students' solutions and two hours for the weekly team meeting.

Prior to the courses the TAs were given training in the procedures of the courses and in the role they should play as instructors in XA. The TAs were asked (1) not to give direct answers to the tasks, but to push the students towards the right solutions, (2) not to give students information they could easily find on their own in the course material, for example definitions, but to tell them where to find it, and (3) to encourage students and give positive feedback when they succeeded, especially if they had struggled with finding the right solution.

The TAs, the lecturer and the administration personnel had a weekly meeting. At the beginning the meetings lasted for two hours, but in order to make them more focused and to save resources they were reduced to one hour. The TAs were expected to solve the tasks prior to the meeting although they were given model solutions. In the meetings the tasks were discussed, as well as the guidelines on how to instruct and correct them.

The XA Classroom

The space chosen as the XA classroom, fitting 40 persons, was located in the heart of the department close to the social room of the students. With one wall covered with glass windows it was easy to spot people working inside.

As the instruction is not given in front of the class, but amongst the students, the classroom had to be refurnished. The desks that were previously placed into rows were replaced with bigger ones, around which the students could sit in groups. In order to turn the desks into whiteboards they were covered with acrylic glass. Also, extra blackboards were placed at the back of the room. Writing and drawing was allowed on every suitable surface in the room (including the windows). This enabled the students and the assistants to sketch and brainstorm together. In addition to discussing mathematics the students and the TAs were able to visualize their thoughts.

The idea of the XA is to offer students individual instruction that is easily available. It has been noted that students are reluctant to ask for help even if it is offered. A number of measures were taken in order to make the students feel that they were expected to ask for instruction. The instructors wore orange vests with the print "Instructor". Also, their names and photographs were posted on the wall. Students entering the classroom should not need to think twice whether they were allowed to ask for instruction. A typical scene from the XA classroom can be seen in Figure 1.

Figure 1. The instructor (in an orange vest) discusses the solution with a student in the XA classroom. [Published with permission from participants.]

RESULTS

Although the requirement level of the courses *Linear Algebra and Matrices I* (LMI) and *II* (LMII) was raised, the passing rates did not change significantly, as seen in Table 1 and Table 2. Noteworthy is the fact that although the students were asked to produce high-quality solutions with proper mathematical notation, they solved significantly more tasks than in previous years.

After the exam a post-course online survey took place. The number of students giving feedback was 206 in LMI and 138 LMII. The survey used a forced choice likert scale with the options "Strongly agree", "Agree", "Disagree" and "Strongly disagree". The student feedback was overwhelmingly positive as 83 % of the students in LMI and 88 % in LMII agreed or strongly agreed that they would like to attend courses taught with XA in the future.

(See Figure 2.) The amount of students who agreed or strongly agreed with the claim "I know which topics of the course I master" was as high as 97 % in LMI and 96 % in LMII. (See Figure 2.)

Table 1: Mathematics major students who participated in LMI and passed LM II

	Participated in LM I	Passed LM II	%
2011	163	117	72
2010	99	64	65
2009	97	71	73

Table 2: Comparison of the number of students who participated in LMI and LMII.

		Number of Students		Took part in the exam		Passed the course	
		LMI	LMII	LMI	LMII	LMI	LMII
2011	N	361	248	322	221	287	192
	%	100	100	89	89	80	77
2010	N	308	203	280	185	254	149
	%	100	100	91	91	82	73
2009	N	284	207	262	189	226	161
	%	100	100	92	91	80	78

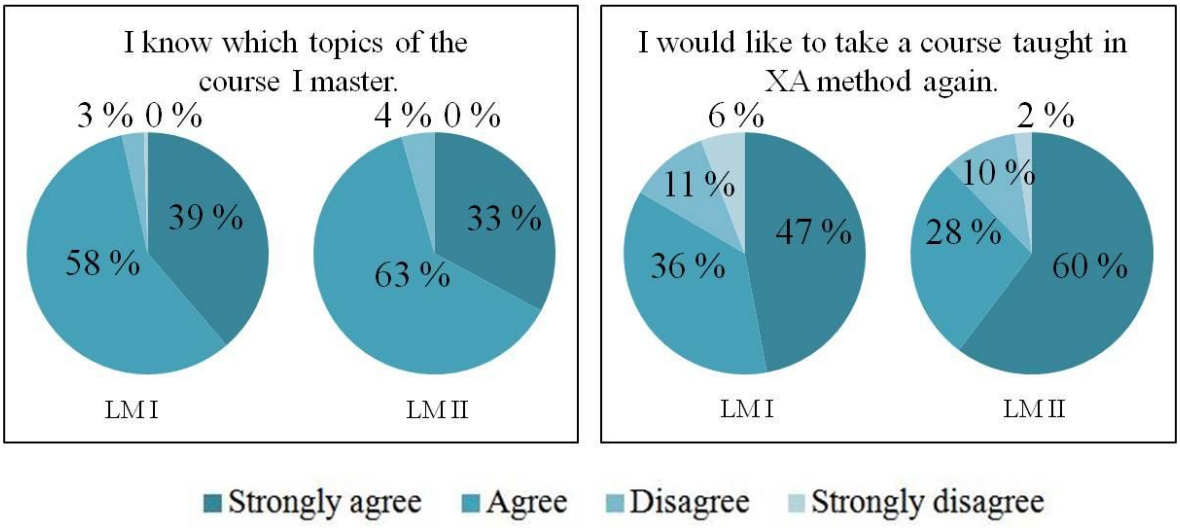

Figure 2. Results of post-course online survey.

CONCLUSION

We have shown that XA can be scaled up to mathematics courses with several hundred students. The objective of implementing XA was to improve students' learning. As was mentioned before, it was not the passing rates that were a problem, but the fact that the students did not seem to learn the content of the course very deeply. As a comment in the post-course survey describes, XA has raised the confidence level of students to pursue their studies and made them realize that they can perform well on their own:

> Questions that required some form of proof strengthened my ability to argue mathematically. [...] In my humble opinion, it [XA] developed a lot of confidence to face the numerical and theoretical exercises and solving them by myself. This method has taught me to work and read more independently than the conventional methods of learning.

Future research will show if the students indeed perform better in their consequent studies, and hence the next step is to study the long-term effects of the teaching method. This can be done by comparing students' achievement in more advanced courses between those who have studied with XA and those who have not.

As observed, the students thought they were aware how well they mastered the course topics. In the future, research will be carried out on how the awareness correlates with the actual mastery and what the underlying causes of this awareness are.

A notable observation is that even though the demand level was raised and the amount of individual work hugely increased, the students appreciated the teaching method. The high number of students who would like to take more courses taught with XA backs this observation and is summarized in a comment in the post-course online survey:

> The best thing about the course was that I learned things the hard way. At first the demanded accuracy felt unreasonable but in the end I can say that it helped me to understand things more clearly and do them correctly.

The amount of checked tasks and the working hours spent on checking were huge. These figures pose a challenge for the scalability (i.e. cost effectiveness) of XA. In computer programming the checking can be automated but in mathematics evaluating students' work is more complicated. One answer could be not to check all the solutions. In this case, the checking has to be planned carefully to ensure that the students attempt solving all the tasks regardless whether they are checked or not. The development and results of different strategies in dealing with the checking are issues for further research.

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