FLOQUET BUNDLES FOR TRIDIAGONAL COMPETITIVE-COOPERATIVE SYSTEMS AND THE DYNAMICS OF TIME-RECURRENT SYSTEMS

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Abstract. We consider a general time-dependent linear competitive-cooperative tridiagonal system of differential equations in the framework of skew-product flows and obtain canonical Floquet invariant bundles which are exponentially separated. Such Floquet bundles naturally reduce to the standard Floquet space when the system is assumed to be time-periodic. We apply the Floquet theory so obtained to study the dynamics on the hyperbolic omega-limit sets for the nonlinear competitive-cooperative tridiagonal systems in time-recurrent structures including almost periodicity and almost automorphy.

Key words. Floquet bundles, exponential separation, tridiagonal competitive-cooperative systems, skew-product flows, hyperbolicity

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1. Introduction. In this paper we study the dynamical properties of systems of differential equations with a tridiagonal structure (such terminology is borrowed from [15, 33]), that is, systems of the form

\[
\begin{align*}
\dot{x}_1 &= f_1(t, x_1, x_2), \\
\dot{x}_i &= f_i(t, x_{i-1}, x_i, x_{i+1}), & 2 \leq i \leq n-1, \\
\dot{x}_n &= f_n(t, x_{n-1}, x_n).
\end{align*}
\]

(1.1)

We assume that the nonlinearity \( f = (f_1, f_2, \ldots, f_n) \) is defined on \( \mathbb{R} \times \mathbb{R}^n \) and that it is \( C^1 \)-admissible, by which we mean that \( f \) together with its first derivatives with respect to \( x = (x_1, x_2, \ldots, x_n) \) are bounded and uniformly continuous on \( \mathbb{R} \times K \) for any compact set \( K \subset \mathbb{R}^n \).

Equations of the form (1.1) arise, for instance, in modeling ecosystems of \( n \) species \( x_1, x_2, \ldots, x_n \) with a certain hierarchical structure. In this hierarchy, \( x_1 \) interacts only with \( x_2, x_n \) only with \( x_{n-1} \), and for \( i = 2, \ldots, n-1 \), species \( x_i \) interacts with \( x_{i-1} \) and \( x_{i+1} \). Such a hierarchy may occur in an ocean water column or on a steep mountainside or on island groups, where each species dominates a zone (depth, altitude, or island) but is obliged to interact with other species in the (narrow) overlap of their zones of dominance.

Our key assumption about the tridiagonal system (1.1) is that the variable \( x_{i+1} \) affects \( \dot{x}_i \) and \( x_i \) affects \( \dot{x}_{i+1} \) monotonically in the same fashion. More precisely, there are \( \varepsilon_0 > 0 \) and \( \delta_i \in \{-1, +1\} \) such that
(F) \[ \delta_i \frac{\partial f_i}{\partial x_{i+1}}(t, x) \geq \varepsilon_0, \quad \delta_i \frac{\partial f_{i+1}}{\partial x_i}(t, x) \geq \varepsilon_0, \quad 1 \leq i \leq n - 1, \]

for all \((t, x) \in \mathbb{R} \times \mathbb{R}^n\). If \(\delta_i = -1\) for all \(i\), then (1.1) is called strongly competitive. If \(\delta_i = 1\) for all \(i\), then (1.1) is called strongly cooperative. In this paper we do not consider predator-prey interactions. For a treatment of tridiagonal predator-prey systems we refer to [1].

Following Smith [34], we introduce new variables \(\hat{x}_i = \mu_i x_i, \mu_i \in \{+1, -1\}, 1 \leq i \leq n, \) with \(\mu_1 = 1, \mu_i = \delta_{i-1} \mu_{i-1}\). With these variables the system (1.1) transforms into a new system of the same type with

\[ \hat{\delta}_i = \mu_i \mu_{i+1} \delta_i = \mu_i^2 \delta_i^2 = 1 \]

in place of \(\delta_i\). Therefore we can always assume, without loss of generality, that the tridiagonal system (1.1) is in fact strongly cooperative, which means that

\[ \frac{\partial f_i}{\partial x_{i+1}}(t, x) \geq \varepsilon_0, \quad \frac{\partial f_{i+1}}{\partial x_i}(t, x) \geq \varepsilon_0, \quad 1 \leq i \leq n - 1, \quad (t, x) \in \mathbb{R} \times \mathbb{R}^n. \]

In particular, if system (1.1) is linear, we write it in the form

\[ \begin{align*}
\dot{x}_1 &= a_{11}(t)x_1 + a_{12}(t)x_2, \\
\dot{x}_i &= a_{i,i-1}(t)x_{i-1} + a_{i,i}(t)x_i + a_{i,i+1}(t)x_{i+1}, \quad 2 \leq i \leq n - 1, \\
\dot{x}_n &= a_{n,n-1}(t)x_{n-1} + a_{n,n}(t)x_n,
\end{align*} \]

where \(a_{i,i+1}(t) \geq \varepsilon_0, a_{i+1,i}(t) \geq \varepsilon_0\), for all \(t \in \mathbb{R}\) and \(1 \leq i \leq n - 1\).

In the case when the linear system (1.3) is time-periodic in \(t\), Smith [34] studied the Floquet theory by using an integer-valued Lyapunov function \(\sigma\), first defined by Smillie [33] (see also similar forms by Mallet-Paret and Smith [17], Fusco and Oliva [8, 9], and Mallet-Paret and Sell [16]), and related the values of \(\sigma\) to the Floquet multipliers of the linear periodic system. This function \(\sigma\) is not defined everywhere but only on an open and dense subset \(\Lambda\) of \(\mathbb{R}^n\) on which it is also continuous (see section 2). However, \(\sigma(x(t))\) is well defined for all except a finite set of points \(t\) along a nontrivial solution \(x(t)\) of the linear system (1.3). It is locally constant near points where it is defined and strictly decreasing as \(t\) increases through points where it is not defined. As a consequence, \(\sigma\) can be seen as a discrete analogue of the zero-crossing number of Matano [18] (discovered originally by Nickel [22]) for scalar reaction-diffusion equations. By utilizing the zero-crossing number, Chow, Lu, and Mallet-Paret [5] have established a Floquet theory for linear periodic scalar parabolic equations.

In the first part of the present paper, we will develop a Floquet theory for the general linear time-dependent system (1.3), and we express this theory in the language of invariant vector bundles (see, e.g., [2, Chapter I]) and the so-called exponential separation (see, e.g., [20, 23, 24] and references therein). Our approach is motivated by the work of Chow, Lu, and Mallet-Paret [5, 6] for time-dependent scalar parabolic equations and extends earlier work on linear autonomous tridiagonal equations in [33], linear time-periodic equations in [34], and linear asymptotically autonomous equations in [8].

With each \(m\) with \(0 \leq m \leq n - 1\), we associate a nontrivial solution \(x_m(t)\) of (1.3) (unique up to constant multiple) such that \(x_m \in \Lambda\) and \(\sigma(x_m(t)) = m\) for all \(t \in \mathbb{R}\). These solutions are then treated as a basis to decouple (1.3) into a system of one-dimensional ordinary differential equations. Moreover, if one writes
for the one-dimensional span of \( x_m(0) \), we show that \( W_m \) varies continuously with respect to the coefficients of (1.3) in a certain appropriate topology, and hence \( W_m(\cdot) \) forms a one-dimensional vector bundle (called the Floquet bundle of (1.3)) over a certain product space. In addition, the exponential separation property holds between the different time-dependent Floquet bundles, and hence we obtain a more delicate decomposition of invariant bundles than those induced by Sacker and Sell [25, 26] for linear skew-product flows.

The Floquet bundles obtained here are analogous to the ones obtained in [6] for time-dependent scalar parabolic equations. However, as the function \( \sigma \) is defined and continuous only on \( \Lambda \) and not on the whole \( \mathbb{R}^n \setminus \{0\} \) (while the zero-crossing number can be defined on the whole phase space \( X \) except for \( \{0\} \)), it is technically more difficult to construct the Floquet bundles. In particular, we have extra difficulties when dealing with the critical phase points at which the integer-valued Lyapunov function drops to a lower value. For the zero-crossing number, the phase space \( X \) is a Sobolev space which can be embedded into a space of smooth functions. Every critical phase point \( u \in X \), at which the zero-crossing number drops, can be treated as a smooth function possessing a multiple zero. Hence it was possible in [6] to employ a standard characterization, viz., \( u(\xi) = u_x(\xi) = 0 \) for some \( \xi \), to analyze the critical situation [6, Corollary 4.8 and Theorem 5.1]. In our case, however, the critical points do not belong to \( \Lambda \) and there is no obvious useful characterization of critical points. Therefore we have to take another, novel approach.

It is well known that the linear theory of invariant bundles plays a crucial role in the study of qualitative properties of nonlinear differential equations. In the second part of this paper, we investigate the nonlinear tridiagonal system (1.1) under assumption (1.2) via the Floquet theory developed in the first part. To be more specific, we embed (1.1) into the skew-product flow \( \Pi : \mathbb{R} \times \mathbb{R}^n \times H(f) \to \mathbb{R}^n \times H(f) \),

\[
\Pi(t, x_0, g) \mapsto (x(t, x_0, g), g \cdot t),
\]

where \( x(t, x_0, g) \) is the solution of

\[
\begin{align*}
\dot{x}_1 &= g_1(t, x_1, x_2), \\
\dot{x}_i &= g_i(t, x_{i-1}, x_i, x_{i+1}), & 2 \leq i \leq n - 1, \\
\dot{x}_n &= g_n(t, x_{n-1}, x_n),
\end{align*}
\]

with \( x(0, x_0, g) = x_0 \in \mathbb{R}^n \), and \( g = (g_1, \ldots, g_n) \in H(f), \, (g \cdot t)(\cdot, \cdot) = g(t + \cdot, \cdot) \).

Here \( H(f) \) is the hull of \( f \), that is, the closure of the set \( \{f \cdot \tau : \tau \in \mathbb{R} \} \) in the compact open topology (see [28]). Clearly, \( x(t, x_0, g) \) has the cocycle property, that is, \( x(t + s, x, g) = x(s, x(t, x, g), g \cdot t) \), for all \( s \in \mathbb{R} \) and \( g \in H(f) \).

Since \( f \) is \( C^1 \)-admissible, the Ascoli–Arzela theorem implies that the time-translation flow \( g \cdot t \) on \( H(f) \) is compact. We further assume that \( f \) is time-recurrent or, in other words, that the flow on \( H(f) \) is minimal. This means that \( H(f) \) is a minimal set of the flow, that is, it is the only nonempty compact subset of itself that is invariant under the flow \( g \cdot t \). This is true, for instance, when \( f \) is a uniformly almost periodic or, more generally, a uniformly almost automorphic function (see Definition 4.2).

In the case where \( f \) is independent of \( t \) or, equivalently, if \( H(f) = \{f\} \), Smillie [33] showed that all bounded trajectories of system (1.1) converge to equilibria. Transversality of the stable and unstable manifolds of hyperbolic equilibria was later established by Fusco and Oliva [8]. Smith [34] studied the system (1.1)
under the assumption that $f$ is time-periodic with period $T > 0$. (In this case, $H(f)$ is homeomorphic to the circle $S^1$.) He proved that every bounded solution is asymptotic to a $T$-periodic solution. For time-recurrent systems $H(f)$ is minimal and Wang [35] has shown that every minimal invariant set $K \subset \mathbb{R}^n \times H(f)$ of $\Pi$ is an almost 1-cover of $H(f)$ (which means that $\text{card}(p^{-1}(g_0) \cap K) = 1$ for some $g_0 \in H(f)$, where $p : \mathbb{R}^n \times H(f) \to H(f); (x, g) \mapsto g$ is the natural flow homomorphism), and every $\omega$-limit set $\omega(x, g)$ of $\Pi$ contains at most two minimal sets. Moreover, it was also shown in [35] that if the $\omega$-limit set is distal or uniformly stable, then it is a 1-cover of $H(f)$ (that is, $\text{card}(p^{-1}(g_0) \cap \omega(x, g)) = 1$ for every $g_0 \in H(f)$).

Inspired by the papers [30, 31] by Shen and Yi, we utilize the Floquet theory obtained in this paper to improve the above-mentioned results on the lifting property of the $\omega$-limit sets in the case when the $\omega$-limit sets are hyperbolic (see Definition 4.6 and Theorem 4.7). More precisely, we show that any hyperbolic $\omega$-limit set is a 1-cover of $H(f)$. A direct consequence of this result is that if the system (1.1) is almost periodic (almost automorphic), then any solution in a hyperbolic $\omega$-limit set is almost periodic (almost automorphic), and that the frequency module (see Definition 4.2) of such a solution is contained in that of $f$. In particular, when $f$ is quasi-periodic in time (in which case $H(f)$ is homeomorphic to the $k$-torus $T^k$), $\omega(x_0, g_0)$ is homeomorphic to the $k$-torus $T^k$. Therefore, our results here are a natural generalization of the results of Smillie [33] and Smith [34] to time-recurrent systems.

When $n = 2$ the system (1.1) reduces to a two-dimensional competitive (or cooperative) system. In the case of $T$-periodic two-dimensional competitive systems, Hale and Somolinos [11] have shown that all bounded solutions are asymptotic to $T$-periodic solutions. (See also [21] for Lotka–Volterra systems.) For the case of an almost periodic two-dimensional competitive system, Hetzer and Shen [13, Theorem A] have proved that any minimal set is an almost 1-cover of $H(f)$. Our main result, Theorem 4.7, implies that any hyperbolic omega-limit set is a 1-cover of the base flow, which improves all the results mentioned above for two-dimensional competitive systems. Moreover, Theorem 4.7 also extends all the results for the two-dimensional case mentioned above to higher dimensions ($n \geq 3$). We refer to [14] and [29] for other related extensions of the two-dimensional case.

The paper is organized as follows. The Floquet solutions and spaces of system (1.3) are constructed in section 2 by taking certain limits of periodic linear tridiagonal systems. Moreover, we also relate the values of $\sigma$ to the Floquet solutions and decouple (1.3) into a system of one-dimensional ODEs. In section 3, we define the Floquet bundles and prove the exponential separation between these invariant bundles in terms of the skew-product flow. Finally, we focus on nonautonomous nonlinear cooperative-competitive tridiagonal systems in section 4 and study the lifting properties of hyperbolic omega-limit sets using the Floquet theory obtained in the previous sections.

2. Floquet solutions and spaces. In this section, we focus on the linear tridiagonal system (1.3) with all the coefficient functions being bounded and uniformly continuous on $\mathbb{R}$. We further assume that there is an $\varepsilon_0 > 0$ such that

$$a_{i,i+1}(t) \geq \varepsilon_0, \quad a_{i+1,i}(t) \geq \varepsilon_0$$

for all $t \in \mathbb{R}$ and $1 \leq i \leq n - 1$, that is, the corresponding tridiagonal matrix $A(t) = (a_{ij}(t))_{n \times n}$ is assumed to be strongly positive.

We will construct Floquet solutions and spaces for the general time-dependent linear system (1.3). Following [33, 34], we define a continuous map...
\[ \sigma: \Lambda \to \{0, 1, 2, \ldots, n - 1\} \]
on \Lambda = \{x \in \mathbb{R}^n : x_1 \neq 0, x_n \neq 0 \text{ and if } x_i = 0 \text{ for some } i, 2 \leq i \leq n - 1, \text{ then } x_{i-1}x_{i+1} < 0 \} by
\[ \sigma(x) = \#\{i | x_i = 0 \text{ or } x_ix_{i+1} < 0\}. \]

Here \# denotes the cardinality of the set. Note that \( \Lambda \) is open and dense in \( \mathbb{R}^n \) and that \( \Lambda \) is the maximal domain on which \( \sigma \) is continuous.

**Lemma 2.1.** Let \( x(t) \) be a nontrivial solution of system (1.3). Then the following hold:

(i) \( x(t) \in \Lambda \) except possibly for isolated values of \( t \).
(ii) \( \sigma(x(t)) \) is nonincreasing as \( t \) increases with \( x(t) \in \Lambda \). Moreover, if \( x(s) \notin \Lambda \) for some \( s \in \mathbb{R} \), then \( \sigma(x(s_+)) < \sigma(x(s_-)) \).
(iii) There exists a \( t_0 > 0 \) such that \( x(t) \in \Lambda \) and \( \sigma(x(t)) \) is a constant for \( t \in [t_0, \infty) \) and for \( t \in (-\infty, -t_0] \).

**Proof.** See [34, Proposition 1.2] for the proof of (i) and (ii). It follows from (i) and (ii) that \( \sigma(x(t)) \) can drop to a lower value only finitely many times, which implies (iii).

When \( A(t) \) is periodic in time, we have the following result concerning the Floquet multipliers of (1.3) and the corresponding eigenspaces.

**Lemma 2.2.** If \( A(t) \) is periodic in time with period \( T > 0 \), then

(i) the system (1.3) has \( n \) distinct positive Floquet multipliers \( \alpha_0, \alpha_1, \ldots, \alpha_{n-1} \), satisfying \( \alpha_0 > \alpha_1 > \cdots > \alpha_{n-1} > 0 \);
(ii) if \( E_{\alpha_m} \) is the one-dimensional eigenspace associated with \( \alpha_m \), then \( E_{\alpha_m} \setminus \{0\} \subset \Lambda \) and
\[ \sigma(E_{\alpha_m} \setminus \{0\}) = m, \quad 0 \leq m \leq n - 1; \]
(iii) for a given \( m \) with \( 0 \leq m \leq n - 1 \), there exists a solution \( x_m(t) \) such that
\[ \sigma(x_m(t)) = m \quad \text{for } t \in \mathbb{R}. \]

**Proof.** For (i) and (ii), see [34, Theorem 1.3]. We prove only (iii). Fix \( m \), \( 0 \leq m \leq n - 1 \). Note that \( \alpha_m \) is a positive number. By the standard Floquet theory, there exists a nontrivial solution of (1.3)
\[ x_m(t) = e^{\mu_m t}p_m(t), \]
where \( \mu_m = \log \alpha_m \) and \( p_m(t) \) is a \( T \)-periodic function with \( p_m(0) \in E_{\alpha_m} \setminus \{0\} \). Since \( p_m(kT) = p_m(0) \in E_{\alpha_m} \setminus \{0\} \), (ii) implies that \( p_m(kT) \in \Lambda \) and \( \sigma(p_m(kT)) = m \) for all \( k \in \mathbb{Z} \). Combining this with Lemma 2.1(iii), we readily get \( x_m(t) \in \Lambda \) and \( \sigma(x_m(t)) = m \) for all \( |t| \) sufficiently large. Then Lemma 2.1(ii) implies that \( x_m(t) \in \Lambda \) and \( \sigma(x_m(t)) = m \) for all \( t \in \mathbb{R} \), which completes the proof.

The following proposition shows that the conclusion of Lemma 2.2(iii) holds also for the general time-dependent system (1.3) without the periodicity assumption on \( A(t) \).

**Proposition 2.3.** For each \( m \) with \( 0 \leq m \leq n - 1 \), there exists a solution \( x_m(t) \) of (1.3) satisfying
\[ \sigma(x_m(t)) = m \quad \text{for all } t \in \mathbb{R}. \]

**Proof.** We begin by constructing a sequence of continuous matrix-valued functions \( \{A_k(t)\}_{k=1}^{\infty} \), defined by
We prove only that to show that the corresponding solution for all \( t \) on \([-k-1, k+1]\). The matrices \( A_k(t) \) are then extended to \(2(k+1)\)-periodic functions on \( \mathbb{R} \). It is easily seen that the sequence \( \{A_k(t)\}_{k=1}^{\infty} \) is uniformly bounded and converges to \( A(t) \) uniformly on compact subsets of \( \mathbb{R} \).

For each \( k \geq 1 \), consider the \(2(k+1)\)-periodic equation

\[
\dot{x} = A_k(t)x.
\]

Note that \( A_k(t) \) is a strongly positive tridiagonal matrix. Therefore Lemma 2.2(iii) implies that for each \( m \) with \( 0 \leq m \leq n - 1 \), there exists a solution \( x_m^{(k)} \) of (2.2) on \( \mathbb{R} \) such that

\[
\sigma(x_m^{(k)}(t)) = m \quad \text{for all } t \in \mathbb{R}.
\]

We normalize these solutions by the initial condition \(|x_m^{(k)}(0)| = 1\).

Fix \( m \), \( 0 \leq m \leq n - 1 \), and consider the sequence \( \{x_m^{(k)}(0)\}_{k=1}^{\infty} \). There exists a subsequence \( \{k'\} \) such that \( x_m^{(k')}(0) \to y_m \) with \(|y_m| = 1\), as \( k' \to \infty \). Recall that \( \{A_k(t)\} \) is uniformly bounded and tends to \( A(t) \) uniformly on compact intervals. By a standard result in the theory of ordinary differential equations [10, Lemma 3.1, Chapter I], the corresponding solution \( x_m(t) \) of (1.3) with initial values \( x_m(0) = y_m \) is the limit of \( x_m^{(k')}(t) \) uniformly on compact intervals as \( k' \to \infty \).

We claim that \( x_m(t) \in \Lambda \) and that \( \sigma(x_m(t)) = m \) for all \( t \in \mathbb{R} \). Indeed, by Lemma 2.1(iii), one can find a \( t_0 > 0 \) such that \( x_m(t) \in \Lambda \) and \( \sigma(x_m(t)) = N_1 \) (resp., \( N_2 \)) for all \( t \geq t_0 \) (resp., \( t \leq -t_0 \)). On the other hand, using the openness of \( \Lambda \) and \( \lim_{k \to \infty} x_m^{(k)}(t_0) = x_m(t_0) \), we obtain that \( \sigma(x_m^{(k)}(t_0)) = N_1 \) and \( \sigma(x_m^{(k)}(-t_0)) = N_2 \) for all \( k \) sufficiently large. It then follows from Lemma 2.2(iii) that \( N_1 = N_2 = m \) for all \( t \in \mathbb{R} \), which implies that \( x_m(t) \in \Lambda \) for all \( t \in \mathbb{R} \). This completes the proof.

For integers \( 0 \leq m \leq l \leq n - 1 \), we define the set

\[
W_{m,l}(A) = \{ x_0 \in \mathbb{R}^n \setminus \{0\} : \text{the solution } x(t) \text{ of (1.3) with } x(0) = x_0 \text{ satisfies } m \leq \sigma(x(t)) \leq l \text{ whenever } x(t) \in \Lambda \} \cup \{0\}.
\]

Proposition 2.3 implies that \( W_{m,l}(A) \) contains more points than just the origin. But we can say considerably more, as the following proposition shows.

**Proposition 2.4.** The set \( W_{m,l}(A) \) is a linear subspace of \( \mathbb{R}^n \) and

\[
\dim(W_{m,l}(A)) = l - m + 1.
\]

**Proof.** Take \( x_0, y_0 \in W_{m,l}(A) \setminus \{0\} \). Let \( x(t) \) and \( y(t) \) be the solutions of (1.3) with \( x(0) = x_0 \), \( y(0) = y_0 \). Since \( \sigma(\alpha x) = \sigma(x) \) for all \( x \in \Lambda \) and all \( \alpha \neq 0 \), it suffices to show that

\[
m \leq \sigma(x(t) + y(t)) \leq l \quad \text{whenever } x(t) + y(t) \in \Lambda.
\]

We prove only that \( l \) is the upper bound of \( \sigma(x(t) + y(t)) \), as the proof that \( m \) is the lower bound is analogous.

Suppose on the contrary that there exists a \( t \in \mathbb{R} \) such that

\[
x(t) + y(t) \in \Lambda \quad \text{and} \quad \sigma(x(t) + y(t)) > l.
\]
By (ii)–(iii) of Lemma 2.1, this implies that there exists a $t_0 \in \mathbb{R}$ such that

$$
(2.3) \quad x(t) + y(t) \in \Lambda \quad \text{and} \quad \sigma(x(t) + y(t)) > l \quad \text{for all} \quad t \leq t_0.
$$

Since $\Lambda$ is an open set, there exists $\delta_0 \in (0, 1)$ such that for all $\delta \in [0, \delta_0]$, $x(t_0) + (1 - \delta)y(t_0) \in \Lambda$ and $\sigma(x(t_0) + (1 - \delta)y(t_0)) = \sigma(x(t_0) + y(t_0)) > l$. Since $x(t) + (1 - \delta)y(t)$ is a solution of (1.3), Lemma 2.1(ii)–(iii) implies that for each $\delta \in [0, \delta_0]$, there exists $a \in (0, 1 - \delta)$ such that

$$
(2.4) \quad \sigma(x(t) + (1 - \delta)y(t)) > l \quad \text{for all} \quad t \leq t_0.
$$

Clearly, $z$ is a contradiction. Thus we have proved that

$$
\lim_{k \to \infty} \left( x(t_k) + \delta_1(t_k) y(t_k) \right) \notin \Lambda \quad \text{and} \quad \lim_{k \to \infty} \left( x(t_k) + y(t_k) \right) \notin \Lambda.
$$

Now we define the following pair of sequences:

$$
\tilde{z}_k = \frac{x(t_k) + \delta_1(t_k) y(t_k)}{|x(t_k)| + |y(t_k)|} \quad \text{and} \quad \tilde{w}_k = \frac{\delta_2(t_k) x(t_k) + y(t_k)}{|x(t_k)| + |y(t_k)|}.
$$

Clearly, $\tilde{z}_k, \tilde{w}_k \notin \Lambda$ for all $k \geq 1$. By taking a subsequence of $\{t_k\}$ if necessary, we may also assume that $\frac{x(t_k)}{|x(t_k)| + |y(t_k)|} \to \tilde{z}_s, \frac{y(t_k)}{|x(t_k)| + |y(t_k)|} \to \tilde{w}_s, \delta_1(t_k) \to \delta_1s \in [0, 1 - \delta], \text{and} \delta_2(t_k) \to \delta_2s \in [0, 1]$ as $n \to \infty$. Then one obtains that $\tilde{z}_k \to \tilde{z}_s, \tilde{w}_k \to \tilde{w}_s =: w_s \notin \Lambda$ as $k \to \infty$, because $\Lambda$ is an open set. Moreover, the vector $(\tilde{z}_s, \tilde{w}_s) \neq (0, 0)$ since $0 \leq \delta_1s \leq 1 - \delta$ and $0 \leq \delta_2s \leq 1$. (Otherwise, it follows that $(\tilde{z}_s, \tilde{w}_s) = (0, 0)$, which yields that $1 = \frac{|x(t_k)|}{|x(t_k)| + |y(t_k)|} + \frac{|y(t_k)|}{|x(t_k)| + |y(t_k)|} \to |\tilde{z}_s| + |\tilde{w}_s| = 0$, a contradiction.)

Without loss of generality, we now assume that $z_k \neq 0$. For each $k \geq 1$, let

$$
z_{t_k}(t) = \frac{x(t + t_k) + \delta_1s \cdot y(t + t_k)}{|x(t_k)| + |y(t_k)|}, \quad t \in \mathbb{R}.
$$

Clearly, $z_{t_k}(t)$ is a nontrivial solution of the equation

$$
\dot{x} = A(t)x := A(t + t_k)x
$$

with initial value $z_{t_k}(0) = \tilde{z}_k$. Recall that $A(t)$ is bounded and uniformly continuous on $\mathbb{R}$. Therefore one can find a subsequence, still denoted by $\{t_k\}$, such that $A_{t_k}(t)$ converges to $A_{t_0}(t)$ uniformly on any compact interval as $k \to \infty$. Because $A(t)$ is strongly positive, it is easy to see that $A_{t_k}(t)$ is strongly positive as well.

Let $z_{t_k}(t)$ be the solution of $\dot{x} = A_{t_0}(t)x$ with initial value $z_{t_k}(0) = z_{t_0} \neq 0$. Recall that $\tilde{z}_k \to z_s$ as $k \to \infty$. It then follows from this (the proof is postponed to Lemma 2.5 below) that $z_s(t) \in \Lambda$ and $\sigma(z_s(t)) = \sigma(t) = \text{const}$ for all $s \in \mathbb{R}$. In particular, $z_s = z_{t_0}(0) \in \Lambda$, a contradiction. Thus we have proved that $W_{m,l}(A)$ is a linear space.

To prove the assertion about the dimension of $W_{m,l}(A)$ we first note that $W_{k,k}(A) \subseteq W_{m,l}(A)$ whenever $m \leq k \leq l$ and that the solutions $x_k$, obtained in Proposition 2.3, are linearly independent. This gives us the chains...
The proof of Proposition 2.4 reveals that the strict positivity of the limiting matrix
\[ W_{i,i}(A) \subset W_{i-1,i}(A) \subset \cdots \subset W_{0,i}(A), \]
where all the inclusions are proper. As a consequence, we have the following inequalities:
\[ \dim W_{0,0}(A) < \dim W_{0,1}(A) < \cdots < \dim W_{0,n-1}(A) \]
and
\[ \dim W_{i,i}(A) < \dim W_{i-1,i}(A) < \cdots < \dim W_{0,i}(A). \]

Because obviously \( \dim W_{0,n-1}(A) = n \), \( 2.5 \) yields \( \dim W_{0,l}(A) = l+1 \). Inserting
this into \( 2.6 \) and using the obvious fact that \( \dim W_{i,i}(A) = 1 \), we arrive at the
general case \( \dim(W_{m,i}(A)) = l - m + 1 \). This completes the proof. □

As in the proof of the preceding proposition, we denote in the following the \( \tau \)-shift
of \( A \) by \( A_{\tau} \), that is, for \( \tau \in \mathbb{R} \) we define
\[ A_{\tau}(t) = A(t + \tau), \quad t \in \mathbb{R}. \]

**Lemma 2.5.** Let \( A(t) \) be strongly positive and let \( x(t) \) be a nontrivial solution
of \( 1.3 \). If there exists a sequence \( t_k \to \infty \) (or \( t_k \to -\infty \)) such that \( x(t_k) \to x_* \neq 0 \) and
\( A_{t_k} \) converges to \( A_* \) uniformly on compact intervals of \( \mathbb{R} \), then the solution \( x_*(t) \)
of \( \dot{x} = A_*(t)x \), with initial value \( x_*(0) = x_* \), satisfies
\[ x_*(t) \in \Lambda \quad \text{and} \quad \sigma(x_*(t)) = \text{const} \]
for all \( t \in \mathbb{R} \).

**Proof.** We first note that for each \( t_k \), the function \( t \mapsto x(t + t_k) \) is a nontrivial
solution of \( \dot{x} = A_{t_k}(t)x \) on \( \mathbb{R} \). It follows from [10, Lemma 3.1, Chapter I] that \( x(t + t_k) \)
tends to \( x_* \) uniformly on compact intervals. So for any \( s \in \mathbb{R} \) with \( x_*(s) \in \Lambda \), the
continuity of \( \sigma \) implies that
\[ \sigma(x_*(s)) = \lim_{k \to \infty} \sigma(x(s + t_k)) = N_1, \]
where the last equality follows from Lemma 2.1(iii). On the other hand, it follows from
the strong positivity of \( A(t) \) that \( A_*(t) \) is strongly positive, too. As a consequence,
the conclusions of Lemma 2.1 hold for the solution \( x_*(t) \) of the equation \( \dot{x} = A_*(t)x \).
Based on this, \( 2.7 \) yields that \( x_*(t) \in \Lambda \) for any \( t \in \mathbb{R} \), and hence \( \sigma(x_*(t)) = \text{const} \)
for all \( t \in \mathbb{R} \). □

**Remark 2.6.** We point out that Lemma 2.1 holds under a weaker assumption
on \( A(t) \), called **strict positivity**, that is, \( a_{i,i+1}(t) > 0 \), \( a_{i+1,i}(t) > 0 \) for all \( t \in \mathbb{R} \)
and all \( 1 \leq i \leq n - 1 \) (see [34, Proposition 1.2]). Moreover, one can also prove that
Proposition 2.3 holds if \( A(t) \) is only strictly positive. However, we cannot prove that
\( W_{m,i}(A) \) is a linear space under this weaker assumption. A careful examination of
the proof of Proposition 2.4 reveals that the strict positivity of \( A(t) \) does not guarantee
that the limiting matrix \( A_* \) is strictly positive. This is the reason for assuming
strong positivity of \( A(t) \). With this assumption the conclusions of Lemma 2.1 hold
for the solution \( z_*(t) \) (or \( x_*(t) \) in Lemma 2.5) of the equation \( \dot{x} = A_*(t)x \), and this
was a crucial point in the proof.
Remark 2.7. By Proposition 2.4, it is clear that
\[ W_{m,l}(A) = \bigoplus_{k=m}^{l} W_{k,k}(A). \]
Moreover, for each \( m \) with \( 0 \leq m \leq n-1 \), the solution \( x_m(t) \) obtained in Proposition 2.3 is unique up to a constant multiple. As a consequence, we can normalize \( x_m(t) \) so that \( |x_m(0)| = 1 \) and the first coordinate of \( x_m(0) \) is positive. Hereafter we always use these normalized solutions.

We call the spaces \( \{W_{m,l}(A)\}_{0 \leq m \leq l \leq n-1} \) and the normalized solutions \( \{x_m(t)\}_{0 \leq m \leq n-1} \) Floquet spaces and Floquet solutions of (1.3), respectively.

3. Floquet bundles and exponential separation. Let \((Y,d)\) be a compact metric space with a flow \( y \cdot t \) and let \( B \) be a continuous \((n \times n)\)-matrix-valued function on \( Y \). We assume that \( B(y) \) is a strongly cooperative tridiagonal matrix for each \( y \in Y \).

In this section we apply the results obtained in the previous section to study the following \( n \)-dimensional system of differential equations with a parameter \( y \in Y \):
\[ \dot{x}(t) = B(y \cdot t)x(t). \]

We denote by \( \Phi(t,y) \) the principal fundamental matrix of (3.1).

From Proposition 2.4 and Remark 2.7, we obtain the Floquet spaces \( W_{m,l}(y) \) and the solutions \( x_m(t,y) \) associated with each \( y \in Y \), where \( 0 \leq m \leq l \leq n-1 \).

For brevity, we hereafter use the abbreviated notation \( W_m(y) \) and \( x_m(0,y) \) instead of \( W_{m,m}(y) \) and \( x_m(0,y) \). It follows from Remark 2.7 that \( W_m(y) = \text{span}\{x_m(y)\} \) and \( W_{m,l}(y) = \bigoplus_{k=m}^{l} W_{k,k}(y) \). Furthermore, we have the following result.

**Proposition 3.1.** Let \( 0 \leq m \leq l \leq n-1 \) and \( y \in Y \).

(i) \( \Phi(t,y)W_{m,l}(y) = W_{m,l}(y \cdot t) \) for all \( t \in \mathbb{R} \);

(ii) \( W_{m,l}(y) \) varies continuously with \( y \in Y \) as a subspace of \( \mathbb{R}^n \) in the sense that the basis vectors \( \{x_k(y)\}_{m \leq k \leq l} \) are continuous functions of \( y \).

**Proof.** (i) It suffices to show that for each \( m \) with \( 0 \leq m \leq n-1 \), one has \( \Phi(t,y)W_m(y) = W_m(y \cdot t) \) for all \( t \in \mathbb{R} \). Since \( W_m(y) = \text{span}\{x_m(y)\} \), we only need to prove that \( x_m(t,y) \in W_m(y \cdot t) \). To see this, fix \( t \in \mathbb{R} \) and note that both \( x_m(t+s,y) \) and \( x_m(s,y \cdot t) \), as functions of \( s \), are nontrivial solutions of \( \dot{x} = B(y \cdot t)x \). Recall that \( \sigma(x_m(s,y \cdot t)) = m = \sigma(x_m(t+s,y)) \) for all \( t \in \mathbb{R} \). It follows from Remark 2.7 that there exists a real number \( C \neq 0 \) such that
\[ x_m(t+s,y) = Cx_m(s,y \cdot t) \quad \text{for all } s \in \mathbb{R}. \]

By letting \( s = 0 \), we get \( x_m(t,y) = Cx_m(0,y \cdot t) \in W_m(y \cdot t) \).

(ii) We only need to prove that \( x_m(y) \) is a continuous function of \( y \in Y \) for each \( m \) with \( 0 \leq m \leq n-1 \). Fix \( m \) with \( 0 \leq m \leq n-1 \) and let \( y_k \to y \) in \( Y \). Then \( B(y_k \cdot t) \) converges to \( B(y \cdot t) \) uniformly for \( t \) on compact intervals. Given any subsequence \( k' \) satisfying \( x_m(y_{k'}) \to x_*(as k' \to \infty) \), it follows [10, Lemma 3.1, Chapter I] that \( x_m(t,y_{k'}) \) converges to \( x(t,y) \) uniformly for \( t \) on any compact intervals, where \( x(t,y) \) is the solution of
\[ \dot{x} = B(y \cdot t)x \]
with initial condition \( x(0,y) = x_* \). Here \( |x_*| = 1 \) and its first coordinate \( (x_*)_1 \geq 0 \) (because \( x_m(y_{k'})_1 > 0 \) by the normalization convention of Remark 2.7). Moreover,
since $\sigma(x_m(t, y_k)) = m$ for all $t \in \mathbb{R}$, one can apply Lemma 2.1 to $x(t, y)$ and obtain that $\sigma(x(t, y)) = m$ for all $t \in \mathbb{R}$. Then the uniqueness of the solutions $x_m$ (see Remark 2.7) implies that $x(t, y) = x_m(t, y)$ for all $t \in \mathbb{R}$. In particular, $x_m(y_k) \to x_* = x(0, y) = x_m(y)$ as $k' \to \infty$. Because this holds for every subsequence $\{k'\}$, one has $x_m(y_k) \to x_m(y)$ as $k \to \infty$. This completes the proof.

Remark 3.2. As a matter of fact, the constant $C$ in (3.2) equals $|x_m(t, y)|$. To see this, notice that $x_m(t+s, y) \in \Lambda$, $x_m(s, y \cdot t) \in \Lambda$ for all $s, t \in \mathbb{R}$, and hence it follows from the normalization convention $(x_m(0, y))_1 > 0$ and Lemma 2.1 that $(x_m(t+s, y))_1 > 0, (x_m(s, y \cdot t))_1 > 0$ for all $s \in \mathbb{R}$. So one has $C > 0$. But $C = |x_m(t, y)|$, because $|x_m(0, y \cdot t)| = 1$. As a consequence,

\begin{equation}
(3.3) \quad x_m(s, y \cdot t) = \frac{x_m(t+s, y)}{x_m(t, y)} \quad \text{for all } s \in \mathbb{R}.
\end{equation}

Next we define the linear skew-product flow $\pi : \mathbb{R} \times \mathbb{R}^n \times Y \to \mathbb{R}^n \times Y$, associated with system (3.1) by

\begin{equation}
(3.4) \quad \pi(t, x, y) = (\Phi(t, y)x, y \cdot t).
\end{equation}

For $0 \leq m \leq l \leq n - 1$, we define the Floquet bundles $W_{m,l}(Y)$ by

$W_{m,l}(Y) = \bigcup_{y \in Y} W_{m,l}(y) \times \{y\}$.

It follows from Proposition 3.1 that the bundles $W_{m,l}(Y)$ are $\pi$-invariant and that $m \leq \sigma(x) \leq l$ whenever $x \in W_{m,l}(y) \cap \Lambda$ and $y \in Y$. Moreover, it is easy to see that $\mathbb{R}^n \times Y = W_{0,k}(Y) \oplus W_{k+1,n-1}(Y)$ for $0 \leq k \leq n - 2$.

Hereafter, we call $W$ a subbundle of the vector bundle $\mathbb{R}^n \times H(f)$ on $H(f)$ if $W$ is a collection of linear subspaces $W(y)$ of the fibers $\mathbb{R}^n \times \{y\}$ of $\mathbb{R}^n \times H(f)$ at $y \in H(f)$ that make up a vector bundle in their own right. A subbundle $W$ of $\mathbb{R}^n \times H(f)$ is called invariant if $\Pi(W) \subset W$ for all $t \in \mathbb{R}$. In the following, we will give the definition of exponential separation (see [20, 23, 24] and the references therein) between invariant subbundles of $\mathbb{R}^n \times Y$.

**Definition 3.3.** The ordered pair $(X_1, X_2)$ of complementary invariant subbundles of $\mathbb{R}^n \times Y$ is said to be exponentially separated for $\pi$ if there exist positive numbers $K$ and $\nu$ such that

\begin{equation}
(3.5) \quad \frac{|\Phi(t, y)x_2|}{|\Phi(t, y)x_1|} \leq Ke^{-\nu t}, \quad t \geq 0,
\end{equation}

for all $y \in Y$ and $x_1 \in X_1(y)$, $x_2 \in X_2(y)$ with $|x_1| = |x_2| = 1$.

We now present the first main theorem of this section.

**Theorem 3.4.** For all $m$ with $0 \leq m \leq n - 2$, the pair $(W_{0,m}(Y), W_{m+1,n-1}(Y))$ of invariant subbundles is exponentially separated for $\pi$.

Before we prove this theorem, we need some basic concepts and definitions.

Let $(X, X)$ be a given pair of complementary invariant subbundles of $\mathbb{R}^n \times Y$. The projections of $\mathbb{R}^n$ on $X(y)$ along $X(y)$ and on $X(y)$ along $X(y)$ are denoted by $\Pi(y)$ and $\Pi(y)$, respectively. In the case of the pair $(X, X) = (W_{0,m}(Y), W_{m+1,n-1}(Y))$, we write $\Pi_m(y)$ for $\Pi(y)$ and $\Pi_m(y)$ for $\Pi(y)$, $(0 \leq m \leq n - 2)$.

We define an equivalence relation on $\mathbb{R}^n \setminus \{0\}$ by declaring $x_1 \sim x_2$ if and only if $x_1 = \alpha x_2$ for some $\alpha \in \mathbb{R}$ with $\alpha \neq 0$. The equivalence class of $x$ is denoted by $[x]$. 


Then the linear skew-product flow $\pi$ on $\mathbb{R}^n \times Y$ induces in a natural way a projective flow $\mathbb{P}\pi : \mathbb{P} \times RP^{n-1} \times Y \to RP^{n-1} \times Y$ by

$$(t, [x], y) \mapsto ([\Phi(t, y)x], y \cdot t),$$

where $RP^{n-1}$ is the real $(n-1)$-dimensional projective space (see, e.g., [27]).

Let $M \subset RP^{n-1} \times Y$ be a closed invariant subset of $\mathbb{P}\pi$. $M$ is called a uniform attractor if it has a neighborhood $U_0$ such that for every neighborhood $V$ of $M$, there is a $T > 0$ such that $\mathbb{P}\pi(t, U_0) \subset V$ for all $t > T$. If this is the case, we say that the neighborhood $U_0$ is attracted by $M$.

Let $W$ be a subbundle of $\mathbb{R}^n \times Y$. We write $\mathbb{P}W$ for the projective subbundle associated with $W$. Moreover, the cone of angle $h > 0$ about $W$ is the set

$$K(W, h) = \{ (x, y) \in \mathbb{R}^n \times Y : |\Pi(y)x| \leq h|\Pi(y)x| \}.$$

If we put $\mathbb{P}K(\mathbb{P}W, h) = \{ ([x], y) \in RP^{n-1} \times Y : (x, y) \in K(W, h), |x| \neq 0 \}$, then $\mathbb{P}K(\mathbb{P}W, h) : h > 0$ is a base of the neighborhoods of $\mathbb{P}W$ in $RP^{n-1} \times Y$ (see [3]).

**Lemma 3.5.** Let $\pi : \mathbb{P} \times RP^{n-1} \times Y \to \mathbb{R}^n \times Y$ be the skew-product flow defined by (3.4). The ordered pair $(X, X)$ of complementary invariant subbundles of $\mathbb{R}^n \times Y$ is exponentially separated if and only if $\mathbb{P}X$ is a uniform attractor for the flow $\mathbb{P}\pi$ on $RP^{n-1} \times Y$.

**Proof.** See Lemma 3 in [3].

The following lemma shows the uniqueness of exponential separation.

**Lemma 3.6.** Assume that the ordered pairs $(X_1, \bar{X}_1)$ and $(X_2, \bar{X}_2)$ of complementary invariant subbundles of $\mathbb{R}^n \times Y$ are exponentially separated and that $\dim(\bar{X}_1) = \dim(\bar{X}_2)$. Then

$$X_1 = X_2 \quad \text{and} \quad \bar{X}_1 = \bar{X}_2.$$

**Proof.** See [19, Lemma A.4].

**Proof of Theorem 3.4**. Take any $([x_0], y_0) \in \mathbb{P}W_{0,m}$. It follows from Lemma 2.1 that there is a $\tau > 0$ such that $\Phi(\tau, y_0)x_0 \in \Lambda$ and $\sigma(\Phi(\tau, y_0)x_0) \leq m$. So, one can find a neighborhood $V$ of $(x_0, y_0)$, with its closure $\overline{V} \subset (\mathbb{R}^n \setminus \{0\}) \times Y$, such that $\Phi(\tau, y)x \in \Lambda$ and $\sigma(\Phi(\tau, y)x) \leq m$ for all $(x, y) \in V$. Moreover, one has

$$\lim_{t \to \infty} \sigma(\Phi(t, y)z) \leq m \quad \text{for all } z \in [x] \text{ and } ([x], y) \in \mathbb{P}V.$$

By the compactness of $\mathbb{P}W_{0,m}$ there is a finite collection of neighborhoods $\{\mathbb{P}V_i\}_{i=1}^k$ covering $\mathbb{P}W_{0,m}$ in $RP^{n-1} \times Y$. We denote their union by $V = \bigcup_{i=1}^k \mathbb{P}V_i$. Then

$$\lim_{t \to \infty} \sigma(\Phi(t, y)z) \leq m \quad \text{for all } z \in [x] \text{ and } ([x], y) \in V.$$

Since $\{\mathbb{P}K(\mathbb{P}W_{0,m}, h) : h > 0\}$ is a basis of the neighborhoods of $\mathbb{P}W_{0,m}$, one can choose some small $h_0 > 0$ such that

$$\mathbb{P}K(\mathbb{P}W_{0,m}, h_0) \subset V,$$

and hence (3.6) is satisfied for all $z \in [x]$ and $([x], y) \in \mathbb{P}K(\mathbb{P}W_{0,m}, h_0)$.

We claim that $\mathbb{P}K(\mathbb{P}W_{0,m}, h_0)$ is a neighborhood attracted by $\mathbb{P}W_{0,m}$. In fact, by [3, Lemma 1], one only needs to show that given any $([x], y) \in \mathbb{P}K(\mathbb{P}W_{0,m}, h_0)$ and any $\varepsilon > 0$, there is a $T = T([x], y, \varepsilon) > 0$ such that

$$\Phi(t, y)x, y \cdot t) \in K(W_{0,m}, \varepsilon) \quad \text{for all } t > T.$$
To this end, suppose that there are some \((x_0, y_0) \in \mathbb{P}K(\mathbb{P}W_{0,m}, h_0), \varepsilon_0 > 0\) and \(t_k \to \infty\), satisfying

\[
|\Pi_m(y_0 \cdot t_k)\Phi(t_k, y_0)x_0| \geq \varepsilon_0|\Pi_m(y_0 \cdot t_k)\Phi(t_k, y_0)x_0|.
\]

Without loss of generality, we may assume that \(y_0 \cdot t_k \to y_\ast\) and \(\Phi(t_k, y_0)x_0 \to x_\ast \neq 0\).

As in the proof of Lemma 2.5, we obtain that \(\sigma(\Phi(t, y_\ast)x_\ast) \equiv m_\ast\) for all \(t \in \mathbb{R}\), which implies that \(\sigma(x_\ast) = m_\ast\). Noting that \(\sigma(x_\ast) = \lim_{k \to \infty} \sigma(\Phi(t_k, y_0)x_0)\), (3.6) implies that \(m_\ast \leq m\), and hence \(\Pi_m(y_\ast)x_\ast = 0\).

On the other hand, by letting \(t_k \to \infty\) in (3.7), we get

\[
|\Pi_m(y_\ast)x_\ast| \geq |\Pi_m(y_\ast)x_\ast|.
\]

It follows that \(\Pi_m(y_\ast)x_\ast = \tilde{\Pi}_m(y_\ast)x_\ast = 0\), which means that \(x_\ast = 0\), a contradiction.

Thus we have completed the proof of the claim. Consequently, \(\mathbb{P}W_{0,m}\) is a uniform attractor of the flow \(\mathbb{P}_\pi\). Lemma 3.5 now implies that the \((W_{0,m}(Y), W_{m+1,n-1}(Y))\) is exponentially separated. \(\blacksquare\)

Now we return to the parameterized linear equation (3.1). Fix \(y \in Y\) and consider the vectors \(x_0(y), \ldots, x_{n-1}(y)\) in Proposition 3.1. It is not difficult to see that their corresponding solutions \(x_0(t, y), \ldots, x_{n-1}(t, y)\) are linearly independent. Thus for any solution \(x(t, y)\) of (3.1), there exist constants \(c_0, \ldots, c_{n-1}\) such that

\[
x(t, y) = c_0x_0(t, y) + \cdots + c_{n-1}x_{n-1}(t, y) \quad \text{for all } t \in \mathbb{R}.
\]

On the other hand, note that \(\mathbb{R}^n = \oplus_{m=0}^{n-1} W_m(y \cdot t)\). Therefore there are also functions \(c_0(t), \ldots, c_{n-1}(t)\) such that

\[
x(t, y) = c_0(t)x_0(y \cdot t) + \cdots + c_{n-1}(t)x_{n-1}(y \cdot t).
\]

It then follows from (3.3) in Remark 3.2 that

\[
c_m(t) = \hat{c}_m|x_m(t, y)| \quad \text{for } m = 0, \ldots, n-1.
\]

A direct calculation yields

\[
\frac{d}{dt}|x_m(t, y)| = \frac{x_m^T(t, y)B(y \cdot t)x_m(t, y)}{|x_m(t, y)|},
\]

and hence by (3.8),

\[
\hat{c}_m(t) = \lambda_m(y \cdot t)c_m(t),
\]

where,

\[
\lambda_m(y \cdot t) = \frac{x_m^T(t, y)B(y \cdot t)x_m(t, y)}{|x_m(t, y)|^2}.
\]

Clearly, \(\lambda_m(y)\) is continuous on \(Y\) for each \(0 \leq m \leq n-1\). As a consequence, we have decoupled (3.1) into a system of one-dimensional equations (3.9). Moreover, from Theorem 3.4, we get the following estimate of the growth rate of the solutions of these linear equations.
COROLLARY 3.7. Consider the linear skew-product flow $\pi$ defined by (3.4). There exist constants $\beta \geq 0$ and $\gamma > 0$ such that

$$\int_s^t (\lambda_{m+1}(y \cdot \tau) - \lambda_m(y \cdot \tau)) d\tau \leq -\gamma(t-s) + \beta$$

for all $s \leq t$ and $m = 0, \ldots, n-2$.

Proof. This follows directly from (3.5), (3.8), and (3.9).

Next we investigate the relation between the Floquet bundles and the Sacker–Sell spectral bundles of (3.1). As before, let $\pi$ be the linear skew-product flow defined by (3.4). Let $\lambda \in \mathbb{R}$ and define $\pi_{\lambda} : \mathbb{R} \times \mathbb{R}^n \times Y \to \mathbb{R}^n \times Y$ by

$$\pi_{\lambda}(t, x, y) = (\Phi_{\lambda}(t, x, y), t),$$

where $\Phi_{\lambda}(t, y) = e^{-\lambda t}\Phi(t, y)$. It is easy to verify that $\pi_{\lambda}$ is also a linear skew-product flow on $\mathbb{R}^n \times Y$. We say $\pi_{\lambda}$ admits an exponential dichotomy over $Y$ if there exist $K > 0$, $\alpha > 0$ and continuous projections $Q(y) : \mathbb{R}^n \to \mathbb{R}^n$ such that for all $y \in Y$, $\Phi_{\lambda}(t, y)|_{R(Y)} : R(Q(y)) \to R(Q(y - t))$ is an isomorphism satisfying $\Phi_{\lambda}(t, y)Q(y) = Q(y - t)\Phi_{\lambda}(t, y)$, $t \in \mathbb{R}$; moreover,

$$|\Phi_{\lambda}(t, y)(1 - Q(y))| \leq Ke^{-\alpha t}, \quad t \geq 0,$$

$$|\Phi_{\lambda}(t, y)Q(y)| \leq Ke^{\alpha t}, \quad t \leq 0.$$

Here $R(Q(y))$ is the range of $Q(y)$. We call

$$\Sigma(Y) = \{\lambda \in \mathbb{R} : \pi_{\lambda} \text{ has no exponential dichotomy over } Y\}$$

the Sacker–Sell spectrum of $\pi$ (or of (3.1)) on $Y$. If $Y$ is connected, then the Sacker–Sell spectrum $\Sigma(Y)$ is of the form $\Sigma(Y) = \bigcup_{l=0}^{l-1} [a_m, b_m]$, where the intervals $[a_m, b_m]$ are ordered from right to left, that is, $a_{l-1} \leq b_{l-1} < a_{l-2} \leq b_{l-2} < \cdots < a_0 \leq b_0$ (see [25, 26]). We hereafter denote by $V_m$ the associated spectral bundle corresponding to the spectrum interval $[a_m, b_m]$ for $m = 0, \ldots, l-1$, that is,

$$V_m(Y) = \{(x, y) \in \mathbb{R}^n \times Y : |\Phi(t, y)x| = o(e^{a_m t}) \text{ as } t \to -\infty, \quad |\Phi(t, y)x| = o(e^{b_m t}) \text{ as } t \to \infty\},$$

where $a_m^-, b_m^+$ are any numbers such that $a_m^- < a_m \leq b_m < b_m^+$. With this notation we present a more delicate decomposition of $V(t)$.

COROLLARY 3.8. For $0 \leq m \leq l-1$ one has

$$V_m(Y) = W_{N+1}(Y) \oplus \cdots \oplus W_{N+M}(Y),$$

where $N = \dim(V_0(Y) \oplus \cdots \oplus V_{m-1}(Y)) - 1$, $M = \dim V_m(Y)$.

Proof. Fix $0 \leq m \leq l-1$. Note that the ordered spectral bundle pair $(V_0(Y) \oplus \cdots \oplus V_{m-1}(Y), V_m(Y) \oplus \cdots \oplus V_{l-1}(Y))$ is exponentially separated over $Y$ with $\dim(V_0(Y) \oplus \cdots \oplus V_{m-1}(Y)) = N + 1$. On the other hand, by Theorem 3.4 the ordered pair $(W_{0,N}(Y), W_{N+1,n-1}(Y))$ is also exponentially separated. By uniqueness (see Lemma 3.6), we obtain that $V_0(Y) \oplus \cdots \oplus V_{m-1}(Y) = W_{0,N}(Y)$ and $V_m(Y) \oplus \cdots \oplus V_{l-1}(Y) = W_{N+1,n-1}(Y)$. Similarly, one can also show that $V_0(Y) \oplus \cdots \oplus V_m(Y) = W_{0+N+M}(Y)$ and $V_{m+1}(Y) \oplus \cdots \oplus V_{l-1}(Y) = W_{N+M+1,n-1}(Y)$. Consequently, $V_m(Y) = (V_0(Y) \oplus \cdots \oplus V_{m}(Y)) \cap (V_{m+1}(Y) \oplus \cdots \oplus V_{l-1}(Y)) = W_{0,N+M}(Y) \cap W_{N+1,n-1}(Y) = W_{N+1}(Y) \oplus \cdots \oplus W_{N+M}(Y)$.
If there exists some $m$ with $1 \leq m \leq l - 1$ such that $b_m < 0 < a_{m-1}$, then $\pi$ itself admits an exponential dichotomy over $Y$. Let $V^s(y) = V_0(y) \oplus \cdots \oplus V_m(y)$, $V^u(y) = V_{m+1}(y) \oplus \cdots \oplus V_l(y)$. $V^u(y)$ and $V^s(y)$ are called the unstable space and stable space of (3.1) at $y \in Y$, respectively. By Corollary 3.8 and Proposition 3.1(ii), $V^s(y), V^u(y)$ are continuous in $y \in Y$, and moreover

\begin{equation}
\begin{align*}
\sigma(x) &\leq \dim V^u(y) - 1 \quad \text{for } x \in V^u(y) \cap \Lambda, \\
\sigma(x) &\geq \dim V^s(y) \quad \text{for } x \in V^s(y) \cap \Lambda
\end{align*}
\end{equation}

for each $y \in Y$.

We close this section by proving the following lemma, which will be useful in the next section.

**Lemma 3.9.** Let $y_1, y_2 \in Y$. If the distance $d(y_1, y_2)$ is sufficiently small, then $V^s(y_1) \oplus V^u(y_2) = \mathbb{R}^n$.

**Proof.** We first show that $V^s(y_1) \cap V^u(y_2) = \{0\}$ for $y_1, y_2 \in Y$ with $d(y_1, y_2)$ sufficiently small. If this is not the case, there are two sequences $\{y^k_1\}, \{y^k_2\} \subset Y$, with $d(y^k_1, y^k_2) \to 0$ as $k \to \infty$, such that $V^s(y^k_1) \cap V^u(y^k_2) \neq \{0\}$ for any $k$. For each $k$, choose some unit vector $w_k \in V^s(y^k_1) \cap V^u(y^k_2)$. Without loss of generality, one may assume that $y^k_1 \to y_*$ ($i = 1, 2$) and $w_k \to w$ as $k \to \infty$. The continuity of $V^u, V^s(y)$ in $y$ then implies that the unit vector $w \in V^s(y_*) \cap V^u(y_*)$, contradicting $V^s(y_*) \cap V^u(y_*) = \emptyset$. Thus, we have proved the assertion.

We complete the proof by noting that $V^s(y_2) \oplus V^u(y_2) = \mathbb{R}^n$ and $\dim V^u(y_2) = \dim V^u(y_1)$, and hence $V^s(y_1) \oplus V^u(y_2) = \mathbb{R}^n$. $\square$

**4. Nonlinear time-recurrent systems.** In this last section of the paper, we apply the Floquet theory developed in the previous sections to investigate the lifting property of $\omega$-limit sets of the nonlinear tridiagonal system (1.1).

As we mentioned in the introduction, system (1.1) can be embedded into a skew-product flow $\Pi : \mathbb{R} \times \mathbb{R}^n \times H(f) \to \mathbb{R}^n \times H(f)$,

\begin{equation}
\Pi(t, x_0, g) \mapsto (x(t, x_0, g), g \cdot t),
\end{equation}

where $x(t, x_0, g)$ is the solution of

\begin{equation}
\begin{align*}
\dot{x}_1 &= g_1(t, x_1, x_2), \\
\dot{x}_i &= g_i(t, x_{i-1}, x_i, x_{i+1}), & 2 \leq i \leq n - 1, \\
\dot{x}_n &= g_n(t, x_{n-1}, x_n)
\end{align*}
\end{equation}

with $x(0, x_0, g) = x_0 \in \mathbb{R}^n$ and $g = (g_1, \ldots, g_n) \in H(f)$. Recall that we assume that $f$ is $C^1$-admissible and satisfies the condition (1.2). It follows that these two conditions are also satisfied for each $g \in H(f)$ (see, e.g., [32, Theorem 1.3.1 and Remark 3.4.4] for details). For convenience we state these conditions explicitly: For every $g \in H(f)$,

\begin{enumerate}
\item[(G1)] $g$ is $C^1$-admissible;
\item[(G2)] $\frac{\partial g_i}{\partial x_{i+1}} \geq \varepsilon_0$, $\frac{\partial g_{i+1}}{\partial x_i} \geq \varepsilon_0$, $1 \leq i \leq n - 1$, $(t, x) \in \mathbb{R} \times \mathbb{R}^n$.
\end{enumerate}

**Remark 4.1.** For any $g \in H(f)$, let $x(t, x_1, g)$ and $x(t, x_2, g)$ be two distinct solutions of (4.2) on $\mathbb{R}$. Then the difference $x(t) = x(t, x_1, g) - x(t, x_2, g)$ satisfies the linear equation

\[ \dot{x}(t) = A(t)x(t) \]
in which the elements \( a_{ij}(t) \) of the matrix-valued function \( A(t) \) are given by

\[
a_{ij}(t) = \int_0^1 \frac{\partial g_i}{\partial x_j}(t, (1 - \tau)x(t, x_1, g) + \tau x(t, x_2, g))d\tau.
\]

As a consequence, the conclusions of Lemma 2.1 hold for \( x(t) \).

For the rest of this section, we assume that \( f \) is time-recurrent, that is, the time translation flow defined by \( (g, t) \mapsto g \cdot t \) for \( g \in H(f) \) and \( t \in \mathbb{R} \) is minimal. This is the case, for instance, when \( f \) is a uniformly almost periodic or, more generally, a uniformly almost automorphic function.

We start by defining these concepts and giving some of their properties.

**Definition 4.2.**

1. A function \( f \in C(\mathbb{R}, \mathbb{R}^n) \) is almost periodic if for every \( \varepsilon > 0 \), the set \( T(\varepsilon) := \{ \tau \in \mathbb{R} : |f(t + \tau) - f(t)| < \varepsilon \) for all \( t \in \mathbb{R} \} \) is relatively dense in \( \mathbb{R} \). \( f \) is almost automorphic if for every \( \{t_k\} \subset \mathbb{R} \) there is a subsequence \( \{t_{k_l}\} \) and a function \( g : \mathbb{R} \to \mathbb{R}^n \) such that \( f(t + t_{k_l}) \to g(t) \) and \( g(t - t_{k_l}) \to f(t) \) pointwise.

2. A function \( f \in C(\mathbb{R} \times D, \mathbb{R}^n)(D \subset \mathbb{R}^m) \) is uniformly almost periodic (uniformly almost automorphic) in \( t \) if \( f \) is admissible and almost periodic (almost automorphic) in \( t \in D \).

3. Let \( f \in C(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}^n) \) be uniformly almost periodic (uniformly almost automorphic), and let

\[
f(t, x) \sim \sum_{\lambda \in \Lambda} a_{\lambda}(x)e^{i\lambda t}
\]

be the Fourier series of \( F \). (See [7, 32] for the definition and existence of the Fourier series.) Then the set

\[
S(f) = \{ \lambda \in \mathbb{R} : a_{\lambda}(x) \neq 0 \}
\]

is called the Fourier spectrum of \( f \) associated with the Fourier series (4.3).

The smallest additive subgroup of \( \mathbb{R} \) containing \( S(f) \) is called the frequency module of \( f \) and is denoted by \( M(f) \).

**Proposition 4.3.**

(a) Let \( f \in C(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}^n) \) be uniformly almost periodic (uniformly almost automorphic). Then \( M(f) \) is a countable subset of \( \mathbb{R} \).

(b) Let \( f, g \in C(\mathbb{R} \times \mathbb{R}^m, \mathbb{R}^n) \) be two uniformly almost automorphic functions. Then \( M(g) \subset M(f) \) if and only if for every sequence \( \{\alpha_k\} \subset \mathbb{R} \),

\[
\lim_{k \to \infty} f(t + \alpha_k, x) = f(t, x) \quad \text{uniformly for } t \text{ and } x \text{ in bounded sets, }
\]

implies

\[
\lim_{k \to \infty} g(t + \alpha_k, x) = g(t, x) \quad \text{uniformly for } t \text{ and } x \text{ in bounded sets.}
\]

**Proof.** See [7, 32].

A subset \( K \subset \mathbb{R}^n \times H(f) \) is invariant if \( \Pi(t, K) = K \) for every \( t \in \mathbb{R} \). A subset \( K \subset \mathbb{R}^n \times H(f) \) is called minimal if it is compact and invariant and the only nonempty compact invariant subset of it is itself. The natural flow homomorphism \( p : \mathbb{R}^n \times H(f) \to H(f) \) is the mapping \( (x_0, g) \mapsto g \).

**Definition 4.4.** An invariant compact set \( K \subset \mathbb{R}^n \times H(f) \) is called an almost 1-cover (resp., 1-cover) of \( H(f) \), if \( p^{-1}(g) \cap K \) is a singleton for at least one \( g \in H(f) \) (resp., for every \( g \in H(f) \)).

The following lemma, which is adopted from [35], describes the structure of minimal sets and \( \omega \)-limit sets of (4.1) in terms of their lifting property.
Lemma 4.5.
(i) If $E \subset \mathbb{R}^n \times H(f)$ is a minimal set of (4.1), then $E$ is an almost 1-cover of $H(f)$.

(ii) Let $E_1, E_2$ be two minimal sets of (4.1). Then for any $(x_i, g_i) \in E_i, i = 1, 2$, one has $\sigma(x(t, x_1, g) - x(t, x_2, g)) = \text{const}$ for all $t \in \mathbb{R}$.

(iii) Every $\omega$-limit set of (4.1) contains at most two minimal sets.

Proof. See [35, Lemma 4.2, Theorems 3.6 and 4.4].

Motivated by the work of Shen and Yi [30, 31], we will now utilize the Floquet theory developed in section 3 to strengthen the above result for $\omega$-limit sets that are hyperbolic (see Definition 4.6 below).

Let $Y \subset \mathbb{R}^n \times H(f)$ be a compact invariant set of (4.1). For each $y = (x_0, g_0) \in Y$, consider the linearized equation of (4.2) along the orbit $y \cdot t := \Pi(t, x_0, g)$,

$$
\dot{z} = B(y \cdot t)z, \quad t \in \mathbb{R}, \quad z \in \mathbb{R}^n,
$$

where $B(y \cdot t) = \frac{\partial}{\partial x} g(t, x(t, x_0, g))$ is a strongly positive tridiagonal matrix-valued function.

Definition 4.6. A compact invariant set $Y \subset \mathbb{R}^n \times H(f)$ of the skew-product flow (4.1) is called hyperbolic if the linearized equation (4.4) admits an exponential dichotomy over $Y$ and the corresponding projections $Q(y)$ satisfy $R(Q(y)) = \{0\}$ for all $y \in Y$.

Now we are ready to state our main result as follows.

Theorem 4.7. Let $\omega(x_0, g_0) \subset \mathbb{R}^n \times H(f)$ be the $\omega$-limit set of $(x_0, g_0) \in \mathbb{R}^n \times H(f)$ for (4.1). If $\omega(x_0, g_0)$ is hyperbolic, then $\omega(x_0, g_0)$ is a 1-cover of $H(f)$.

Remark 4.8. (i) If $f$ in (1.1) is uniformly almost periodic (uniformly almost automorphic), then Theorem 4.7 implies that for any $(x_*, g_*) \in \omega(x_0, g_0)$, $x(t, x_*, g_*)$ is an almost periodic (almost automorphic) solution of (4.2); moreover, the solution $x(t, x_*, g_*)$ is harmonic (that is, the frequency module $M(x(t, x_*, g_*)) \subset M(f)$; see Definition 4.2(3) and Proposition 4.3). In particular, when $f$ is quasi-periodic in time $t$ ($H(f)$ is homeomorphic to the $k$-torus $T^k$), one has that $\omega(x_0, g_0)$ is homeomorphic to the $k$-torus $T^k$. As a consequence, Theorem 4.7 generalizes the results of Smillie [33] and Smith [34] for the autonomous and time-periodic cases to time-recurrent systems.

(ii) When $n = 2$, system (1.1) reduces to a two-dimensional competitive or cooperative system. Accordingly, Theorem 4.7 generalizes the results of de Mottoni and Schiaffino [21] and Hale and Somolinos [11], who proved that all bounded solutions of two-dimensional $T$-periodic competitive systems are asymptotic to $T$-periodic solutions. Theorem 4.7 even improves [13, Theorem A] by Hetzer and Shen, who investigated the dynamics of two-dimensional competitive almost periodic systems. In a certain sense, Theorem 4.7 also extends all the results for the two-dimensional case mentioned above to higher dimensions ($n \geq 3$).

In order to prove our main result, we first proceed with the characterization of the integer-valued function $\sigma$ on the local invariant manifolds of hyperbolic invariant sets. Our approach is motivated by [30, 31]. However, as mentioned in the introduction, we still need our technical Lemma 2.5 to overcome the difficulties stemming from the fact that there is no obvious characterization of the critical phase points which are not in $\Lambda$. This differs from the zero-crossing number, for which there is a standard characterization of critical phase points which can be directly used to analyze the critical situation (see [30, Theorem 4.8]).
Let $Y \subset \mathbb{R}^n \times H(f)$ be a hyperbolic compact invariant set of (4.1). For any $y = (x_0, g) \in Y$, let $z = x - x(t, x_0, g)$. Then $z$ satisfies the nonlinear equation
\begin{equation}
\dot{z} = B(y \cdot t)z + G(z, y \cdot t),
\end{equation}
where $B(y \cdot t)$ is as in (4.4) and $G(z, y \cdot t) = O(|z|^2)$. Noticing that system (4.4) admits an exponential dichotomy over $Y$, one sees using standard invariant manifold theory (see [4, 12, 30, 36]) that system (4.5) possesses for each $y \in Y$ a local stable manifold $W^s(y)$ and a local unstable manifold $W^u(y)$, and one can find constants $\alpha, C > 0$ such that for any $y \in Y$ and $x_s \in W^s(y)$, $x_u \in W^u(y)$,
\begin{align}
|\Psi_t(x_s, y)| &\leq Ce^{-(\alpha/2)t}|x_s| \quad \text{for } t \geq 0, \\
|\Psi_t(x_u, y)| &\leq Ce^{(\alpha/2)t}|x_u| \quad \text{for } t \leq 0,
\end{align}
where $\Psi_t(\cdot, y)$ is the solution operator of (4.5). Moreover, they are overflowing invariant in the sense that
\begin{align}
\Psi_t(W^s(y), y) &\subseteq W^s(y \cdot t) \quad \text{for } t \gg 1, \\
\Psi_t(W^u(y), y) &\subseteq W^u(y \cdot t) \quad \text{for } t \ll -1.
\end{align}

Now for each $y = (x_0, g) \in Y$, we define
\begin{align*}
M^s(y) &\triangleq \{x \in \mathbb{R}^n | x - x_0 \in W^s(y)\}, \\
M^u(y) &\triangleq \{x \in \mathbb{R}^n | x - x_0 \in W^u(y)\}.
\end{align*}
Then $M^s(y)$ and $M^u(y)$ are overflowing invariant to (4.2), that is,
\begin{align*}
x(t, M^s(y), g) &\subseteq M^s(y \cdot t) \quad \text{for } t \gg 1, \\
x(t, M^u(y), g) &\subseteq M^u(y \cdot t) \quad \text{for } t \ll -1.
\end{align*}

We also note that if $Y$ is connected, then $\dim M^u(y) = \dim V^u(y)$ and $\dim M^s(y) = \dim V^s(y)$ are positive integers independent of $y \in Y$; here $V^u(y)$ and $V^s(y)$ are the unstable and stable subspaces of (4.4), respectively, as defined at the end of the previous section.

In the proof of the next lemma we shall compress the notation by writing $M^{s,u}, V^{s,u}, W^{s,u}$, etc., when a statement holds for both the stable and unstable cases. Formulas containing double superscripts should be read as holding separately for the former and latter superscripts.

**Lemma 4.9.** Let $Y \subset \mathbb{R}^n \times H(f)$ be a connected compact hyperbolic invariant set of (4.1). Then, for $(x_1, g), (x_2, g) \in Y$ with $|x_1 - x_2|$ being sufficiently small, one has $M^s(x_1, g) \cap M^u(x_2, g) \neq \emptyset$.

**Proof.** By the standard invariant manifold theory (see [4, 12, 30, 36]), there are $\delta > 0, M > 0$ and bounded continuous functions $h^s : \bigcup_{y \in Y} V^s(y) \times \{y\} \to \bigcup_{y \in Y} V^u(y)$ and $h^u : \bigcup_{y \in Y} V^u(y) \times \{y\} \to \bigcup_{y \in Y} V^s(y)$ with $h^{s,u}(\cdot, y) : V^{s,u}(y) \to V^{s,u}(y)$ being $C^1$ for each fixed $y \in Y$, as well as $h^{s,u}(z, y) = o(|z|), |(\partial h^{s,u}/\partial z)(z, y)| \leq M$ for all $y \in Y, z \in V^{s,u}(y)$ such that
\begin{equation}
\begin{align*}
M^s(y) &= x_0 + W^s(y) = x_0 + \{x^s + h^s(x^s, y) | x^s \in V^s(y) \cap B_\delta\}, \\
M^u(y) &= x_0 + W^u(y) = x_0 + \{x^u + h^u(x^u, y) | x^u \in V^u(y) \cap B_\delta\},
\end{align*}
\end{equation}
where $B_\delta = \{v \in \mathbb{R}^n : |v| < \delta\}$. Moreover, $W^{s,u}(y)$ are diffeomorphic to $V^{s,u}(y) \cap B_\delta$, and $V^{s,u}(y)$ are tangent to $V^{s,u}(y)$ at 0 $\in \mathbb{R}^n$ for each $y \in Y$.  

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Let $\Phi$ be the principal solution matrix of (4.4) and define a family of continuous mappings $(\tilde{x}, t, x) \mapsto x_1^{*} - x_0^{*} + h(x_1^{*}, (x_2^{*}, z))$ where $|z| < \delta$. Noticing that $k(x, x) (0, 0) = 0$ and $\left| \frac{\partial h(x_1^{*}, (x_2^{*}, z))}{\partial (x_1^{*}, (x_2^{*}, z))} \right|$ are small for all $x \in V^{s}(x_1^{*}) \cap B_{\delta}$. It is not difficult to see from the implicit function theorem that there are smaller $\delta_2, \delta_3 \in (0, \delta_1)$ such that if $|z| < \delta_2$, then the equation $k(x_1^{*}, (x_2^{*}, z)) = z$ has a unique solution $(x_1^{*}, x_2^{*}) \in V^{s}(x_1^{*}) \cap B_{\delta_2} \cap V^{u}(x_2^{*}) \cap B_{\delta_3}$ for any $z \in \mathbb{R} ^n$ with $|z| < \delta_3$. In particular, if $|z_1 - z_2| < \min \{\delta_2, \delta_3\}$, then by letting $z = z_2 - z_1$, one obtains a unique $x_1^{*} \in V^{s}(x_1^{*}) \cap B_{\delta_2}$ and a unique $x_2^{*} \in V^{u}(x_2^{*}) \cap B_{\delta_3}$ such that $k(x_1^{*}, (x_2^{*}, z)) = x_2^{*} - x_1^{*} - 1$, that is, $x_1^{*} + x_2^{*} + h(x_1^{*}, (x_2^{*}, z)) = x_2^{*} - x_1^{*} - 1$, and $x_1^{*} + x_2^{*} - h(x_1^{*}, (x_2^{*}, z)) = x_2^{*} - x_1^{*}$. Therefore, by the representation (4.8) of $M^{s,u}(y)$ we get $M^{s}(x_1^{*}) \cap M^{u}(x_2^{*}) \neq \emptyset$ whenever $|z_1 - z_2| < \delta$ sufficiently small. This completes the proof.

**Theorem 4.10.** Let $Y \subset \mathbb{R} ^n \times H(f)$ be a connected compact hyperbolic invariant set of (4.1). For every $y = (x_0, g) \in Y$, $x^{*} \in M^{s}(y) \setminus \{x_0\}$, and $x^{n} \in M^{u}(y) \setminus \{x_0\}$, one has

\begin{align}
\sigma(x^{*} - x_0) &= \geq \dim M^{u}(Y) \quad \text{if } x^{*} - x_0 \in \Lambda, \\
\sigma(x^{n} - x_0) &= \leq \dim M^{u}(Y) - 1 \quad \text{if } x^{n} - x_0 \in \Lambda.
\end{align}

**Proof.** We prove only (4.9) as the proof of (4.10) is analogous. For any $y = (x_0, g)$ and $x^{n} \in M^{s}(y) \setminus \{x_0\}$, let $x(t) = x(t, x^{n}, g) - x(t, x_0, g)$. Since $x(0) = x^{n} - x_0 \in W^{u}(y)$, we have $x(t) \in W^{u}(y \cdot t)$ for $t \ll -1$ by (4.7), and hence

$$x(t) = x^{n}(x(t), y \cdot t) + h(x^{n}(x(t), y \cdot t), y \cdot t)$$

for $t \ll -1$, where $x^{n}(x(t), y \cdot t) \in V^{u}(y \cdot t)$ and $h^{n}$ is the $C^{1}$-function defined in the proof of Lemma 4.9. Choose any sequence $t_k \to -\infty$ such that $y \cdot t_k \to y^{*} = (x^{*}, g^{*})$. Because $W^{u}(y \cdot t)$ is contracting in reverse time, it follows that

$$\lim_{k \to \infty} \frac{x(t_k)}{|x(t_k)|} = \lim_{k \to \infty} \frac{x^{n}(x(t_k), y \cdot t_k)}{|x^{n}(x(t_k), y \cdot t_k)|} = x^{*} \in V^{u}(y^{*}).$$

Let $\Phi$ be the principal solution matrix of (4.4), (that is, (4.4) with $y$ replaced by $y^{*}$). Then $\Phi(t, y^{*}) x^{*}$ is a nontrivial solution of (4.4) and $x^{*}$ is the $C^{1}$-function defined in the proof of Lemma 4.9. Moreover, for each $t \in \mathbb{R}$, one has $x(t + t_k)/|x(t + t_k)| = \Phi(t, y^{*}) x^{*} \Phi(t, y^{*})^{-1} x^{*}$ as $k \to \infty$. By Lemma 2.1, there exists a $t_0 < 0$ such that $x(t_0, y^{*}) x^{*} \in \Lambda$. Consequently, $x(t_0 + t_k) \in \Lambda$ and $\sigma(x(t_0 + t_k)) = \sigma(\Phi(t_0, y^{*}) x^{*})$ for all sufficiently large $t$. Note also that $\Phi(t_0, y^{*}) x^{*} \in V^{u}(y^{*} \cdot t_0)$. Then by (3.11), one obtains that $\sigma(x(t_0 + t_k)) \leq \dim M^{u}(Y) - 1$ for all sufficiently large $t$.

By Remark 4.1, the conclusions of Lemma 2.1 hold for $x(t) = x(t, x^{n}, g) - x(t, x_0, g)$ and hence we have $\sigma(x^{*} - x_0) \leq \sigma(x(0)) \leq \sigma(x(t_0 + t_k)) \leq \dim M^{u}(Y) - 1$ whenever $x^{*} - x_0 \in \Lambda$. This completes the proof.

**Definition 4.11.** Let $Y \subset \mathbb{R} ^n \times H(f)$ be a compact invariant set of (4.1). A pair $((x_1, g), (x_2, g)) \in Y \times Y$ is said to be one-sided fiber distal if

$$\inf_{t \in \mathbb{R}^{+}} |x(t, x_1, g) - x(t, x_2, g)| > 0 \text{ or } \inf_{t \in \mathbb{R}^{-}} |x(t, x_1, g) - x(t, x_2, g)| > 0.$$

**Lemma 4.12.** Let $Y \subset \mathbb{R} ^n \times H(f)$ be a connected and compact hyperbolic invariant set of (4.1). Then all pairs $((x_1, g), (x_2, g)) \in Y \times Y$ with $x_1 \neq x_2$ are one-sided fiber distal.

**Proof.** In order to reach a contradiction, suppose that there is a pair

$$(x_1, g_0), (x_2, g_0) \in Y \times Y$$

such that

$$\inf_{t \in \mathbb{R}^{+}} |x(t, x_1, g_0) - x(t, x_2, g_0)| = 0 \text{ and } \inf_{t \in \mathbb{R}^{-}} |x(t, x_1, g_0) - x(t, x_2, g_0)| = 0.$$
By Remark 4.1, there exists some $t_0 \in \mathbb{R}$ such that $x(t_0, x_1, g_0) - x(t_0, x_2, g_0) \in \Lambda$. Therefore one can choose $\varepsilon_0 > 0$ so small that $x(t_0, x_1, g_0) - x(t_0, x_2, g_0) + x \in \Lambda$ and

\begin{equation}
(4.11) \quad \sigma(x(t_0, x_1, g_0) - x(t_0, x_2, g_0) + x) = \sigma(x(t_0, x_1, g_0) - x(t_0, x_2, g_0))
\end{equation}

whenever $x \in \mathbb{R}^n$ with $|x| < \varepsilon_0$. Choose two sequences $t_k \to \infty, s_k \to -\infty (k \to \infty)$ such that

$$|x(t_k, x_1, g_0) - x(t_k, x_2, g_0)| \to 0$$

and

$$|x(s_k, x_1, g_0) - x(s_k, x_2, g_0)| \to 0$$

as $k \to \infty$. Lemma 4.9 implies that for each $k$ sufficiently large, one can find

$$x^k_+ \in M^s(x(t_k, x_1, g_0), g_0 \cdot t_k) \cap M^u(x(t_k, x_2, g_0), g_0 \cdot t_k)$$

and

$$x^k_- \in M^s(x(s_k, x_1, g_0), g_0 \cdot s_k) \cap M^u(x(s_k, x_2, g_0), g_0 \cdot s_k).$$

Using the fact of $x^k_+ \in M^u(x(t_k, x_2, g_0), g_0 \cdot t_k), x^k_- \in M^u(x(s_k, x_1, g_0), g_0 \cdot s_k)$ and (4.6), we readily get

$$|x(s, x^k_+, g_0 \cdot t_k) - x(s, x(t_k, x_2, g_0), g_0 \cdot t_k)| \leq C e^{(\alpha/2)|s|} |x^k_+ - x(t_k, x_2, g_0)|,$$

$$|x(t, x^k_-, g_0 \cdot s_k) - x(t, x(s_k, x_1, g_0), g_0 \cdot s_k)| \leq C e^{-(\alpha/2)|s|} |x^k_- - x(s_k, x_1, g_0)|$$

for any $s \leq 0, t \geq 0$, and $k$ sufficiently large. In particular, we choose $s = t_0 - t_k < 0$ and $t = t_0 - s_k > 0$. Then one can find some $k_0$ sufficiently large such that

$$|x(t_0 - t_{k_0}, x^k_+, g_0 \cdot t_{k_0}) - x(t_0, x_2, g_0)| < \varepsilon_0$$

and

$$|x(t_0 - s_{k_0}, x^k_-, g_0 \cdot s_{k_0}) - x(t_0, x_1, g_0)| < \varepsilon_0.$$
where the last inequality follows from the fact $x^k \in M^u(x(s_k, x_2, g_0), g_0 \cdot s_k)$. Thus we have obtained a contradiction and completed the proof.

**Proposition 4.13.** Let $E \subset \mathbb{R}^n \times H(f)$ be a hyperbolic minimal set of (4.1). Then $E$ is a 1-cover of $H(f)$.

**Proof.** Suppose that $E$ is only an almost 1-cover and not a 1-cover of $H(f)$. It follows from the minimality of $H(f)$ that there is no one-sided fiber distal pair on $E$, which contradicts Lemma 4.12.

**Proof of Theorem 4.7.** According to Proposition 4.13, it suffices to show that $\omega(x_0, g_0)$ is minimal. Suppose on the contrary that $\omega(x_0, g_0)$ is not minimal. Then Lemma 4.5(iii) implies that $\omega(x_0, g_0)$ can be written as $\omega(x_0, g_0) = E_1 \cup E_2 \cup E_1$, where $E_i (i = 1, 2)$ are minimal sets (and hence, they are 1-covers of $H(f)$ by Proposition 4.13). Since $\omega(x, g)$ is connected, $E_1 \neq \emptyset$. Moreover, for any $(x, g) \in E_1$ one has

$$(4.12) \quad \omega(x, g) \cap (E_1 \cup E_2) \neq \emptyset \quad \text{and} \quad \alpha(x, g) \cap (E_1 \cup E_2) \neq \emptyset.$$

If $E_1 = E_2$, then we pick an $(x_1, g) \in E_1$ and let $(x_1, g) = E_1 \cap p^{-1}(g)$. By (4.12), the pair $((x_1, g), (x_1, g))$ is not a one-sided fiber pair, which contradicts Lemma 4.12.

If $E_1 \neq E_2$, then we pick an $(x_2, g) \in E_2$ and let $(x_i, g) \in E_i \cap p^{-1}(g)$ for $i = 1, 2$. By the same reason as above, we may assume without loss of generality that $\omega(x_1, g) \cap E_1 \neq \emptyset$ and $\alpha(x_2, g) \cap E_2 \neq \emptyset$. Since $E_i (i = 1, 2)$ are 1-covers of $H(f)$, it is easily seen that $|x(t, x_1, g) - x(t, x_1, g)| \rightarrow 0$ as $t \rightarrow \infty$, and $|x(t, x_2, g) - x(t, x_2, g)| \rightarrow 0$ as $t \rightarrow -\infty$. As a consequence, $x(t, x_1, g) \in M^u(\Pi(t, x_1, g))$ for all $t$ sufficiently positive and $x(t, x_2, g) \in M^u(\Pi(t, x_2, g))$ for all $t$ sufficiently negative. It then follows from Theorem 4.10 and Remark 4.1 that for any $t \in \mathbb{R}$,

$$(4.13) \quad \sigma(x(t, x_1, g) - x(t, x_1, g)) \geq N$$

whenever $x(t, x_1, g) - x(t, x_1, g) \in \Lambda$ and

$$(4.14) \quad \sigma(x(t, x_1, g) - x(t, x_2, g)) \leq N - 1$$

whenever $x(t, x_2, g) - x(t, x_2, g) \in \Lambda$. Here $N = \dim M^u(x_1, g) = M^u(x_2, g)$.

As pointed out in Remark 4.1, $z(t) := x(t, x_2, g) - x(t, x_1, g)$ is a nontrivial solution of the linear equation $\dot{z} = B(t)z$ with $B(t) = \int_0^1 \frac{\partial}{\partial x}g(t, 1 - \tau)x(t, x_1, g) + \tau x(t, x_2, g))d\tau$, which is a strongly positive tridiagonal matrix. Moreover, $B(t)$ is bounded and uniformly continuous on $\mathbb{R}$, because $g$ is $C^1$-admissible and $(x_i, g) \in E_i$ with $E_i$ being a 1-cover of $H(f)$ for $i = 1, 2$.

We now claim that there exist numbers $T > 0$, $\delta > 0$ such that$z(t) \in \Lambda := \{x \in \Lambda : \text{dist}(x, \Lambda^c) > \delta\}$

for all $|t| > T$. ($\Lambda^c$ is the complement of $\Lambda$ in $\mathbb{R}^n$.) If this were not the case, there would exist a sequence $t_n \rightarrow \infty$ (or $t_n \rightarrow -\infty$) such that

$\text{dist}(z(t_n), \Lambda^c) \rightarrow 0 \quad \text{as} \quad n \rightarrow \infty.$

Note that $x_i \in E_i$, $i = 1, 2$. One can choose a subsequence (still denoted by $t_n$) such that $x(t_n, x_i, g) \rightarrow w_i \in E_i$. Let $z_i = w_1 - w_2 \neq 0$. Then $z(t_n) \rightarrow z_i$ as $n \rightarrow \infty$. Moreover, $z_i \notin \Lambda$. By the boundedness and uniform continuity of $B(t)$, we may assume without loss of generality that $B(t + t_n)$ converges to $B_i(t)$ uniformly on compact intervals of $\mathbb{R}$. Then it follows from Lemma 2.5 that the solution $z_i(t)$ of
\[ \dot{z} = B_s(t)z \] with initial value \( z_*(0) = z_* \), satisfies \( z_*(t) \in \Lambda \) for all \( t \in \mathbb{R} \), contradicting the fact that \( z_* \notin \Lambda \). This proves the claim. As a consequence,

\begin{equation}
(4.15) \quad z(t) + y \in \Lambda
\end{equation}

whenever \( |y| < \frac{\delta}{2} \) and \( |t| > T \).

Recalling that \( |x(t, x_1, g) - x(t, x_2, g)| \to 0 \) as \( t \to \infty \), one obtains from (4.13) and (4.15) that

\[
\sigma(z(t)) = \sigma(x(t, x_2, g) - x(t, x_1, g) + x(t, x_1, g) - x(t, x_2, g))
\]

\[
= \sigma(x(t, x_2, g) - x(t, x_1, g))
\]

\[
\leq N - 1
\]

for all \( t \) sufficiently positive. On the other hand, by (4.14) and (4.15) one can similarly get that

\[
\sigma(z(t)) = \sigma(x(t, x_2, g) - x(t, x_1, g) + x(t, x_1, g) - x(t, x_2, g))
\]

\[
= \sigma(x(t, x_2, g) - x(t, x_1, g))
\]

\[
\geq N
\]

for all \( t \) sufficiently negative. This contradicts Lemma 4.5(ii) and completes the proof. \( \square \)

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