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FUNCTIONS OF BOUNDED VARIATION AND THE AM -MODULUS IN $\mathbb{R}^n$

VENDULA HONZLOVÁ EXNEROVÁ, JAN MALÝ, AND OLLI MARTIO

ABSTRACT. Moduli of path families are widely used to study Sobolev functions. Similarly, the recently introduced approximation (AM -) modulus is helpful in the theory of functions of bounded variation (BV) in $\mathbb{R}^n$ [14]. We continue this direction of research. Let Γ_E be the family of all paths which meet $E \subset \mathbb{R}^n$. We introduce the outer measure $E \mapsto AM(\Gamma_E)$ and compare it with other $(n-1)$ -dimensional measures. In particular, we show that $AM(\Gamma_E) = 2\mathcal{H}^{n-1}(E)$ whenever E lies on a countably $(n-1)$ -rectifiable set. Further, we study functions which have bounded variation on AM -a.e. path and we relate these functions to the classical BV functions which have only bounded essential variation on AM -a.e. path. We also characterize sets E of finite perimeter in terms of the AM -modulus of two path families naturally associated with E .

1. INTRODUCTION

The approximation modulus, abbreviated as AM -modulus, was introduced in [14] to study functions of bounded variation (BV) in metric measure spaces, see also [11] and [15]. The AM -modulus offers a counterpart to the Fuglede theorem [10] which states that functions in the Sobolev space $W^{1,p}(\mathbb{R}^n)$, $p \geq 1$, are absolutely continuous on every path in $\mathbb{R}^n$ except of a family whose M_p -modulus is zero and in [14] it was shown that BV functions in metric measure spaces have bounded essential variation on AM almost every path. In Section 4 we show that BV functions in $\mathbb{R}^n$ have bounded variation in the classical sense, and not only bounded essential variation, on AM almost every path in $\mathbb{R}^n$. This leads to a new characterization of BV functions in $\mathbb{R}^n$.

Let Γ_E denote the family of all paths which meet a set $E \subset \mathbb{R}^n$. It turns out that the set function $E \mapsto AM(\Gamma_E)$ is a non-trivial measure in $\mathbb{R}^n$ and we compare this to the $(n-1)$ -dimensional Hausdorff measure $\mathcal{H}^{n-1}(E)$ and pay special attention to the case when E lies on a “regular” set $A \subset \mathbb{R}^n$, for instance on a countably $(n-1)$ -rectifiable set A in Section 5.

In the last section, Section 6, we give two characterizations for sets E of finite perimeter in terms of the AM -modulus. The perimeter of E

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coincides with the AM -modulus of the family of curves connecting the measure theoretic interior to the measure theoretic exterior of E and with the AM -modulus of the family of curves $\gamma : [a, b] \rightarrow \mathbb{R}^n$ which meet the measure theoretic boundary of E at some point $\gamma(t)$, $t \in (a, b)$.

Note that there is a parallel approach to measuring curves [4], which is related to modulus [2] and can be also used to characterize BV spaces [1].

Our notation is standard and more specialized symbols and concepts are explained in due course.

2. PRELIMINARIES AND AM -MODULUS

Let $A \subset \mathbb{R}$ and $f : A \rightarrow [-\infty, \infty]$. The *total variation* of f on A is

$$V(f, A) = \sup \left\{ \sum_{j=1}^m |f(x_j) - f(x_{j-1})| : \right. \\ \left. x_0 < x_1 < x_2 < \dots < x_m, \quad x_j \in A \right\}$$

where $|f(x_j) - f(x_{j-1})| = \infty$ if $|f(x_j) - f(x_{j-1})|$ is undefined, i.e. of the form $|\infty - \infty|$. If A is (Lebesgue) measurable and f is measurable in A , then the *essential variation* of f on A is

$$\tilde{V}(f, A) = \inf \{ V(u, A) : u = f \text{ a.e. in } A \};$$

obviously $\tilde{V}(f, A) \leq V(f, A)$ and $\tilde{V}(f, [a, b]) = \tilde{V}(f, (a, b))$. If $f \in L^1_{\text{loc}}((a, b))$, then by [3, Theorem 3.27], see also [8, 5.10.1 Theorem 1],

$$(1) \quad \tilde{V}(f, (a, b)) = \sup \left\{ \int_A f \varphi' dt : \varphi \in C_0^1((a, b)), |\varphi| \leq 1 \right\}.$$

For the properties of $V(f, A)$ and $\tilde{V}(f, A)$ see e.g. [3, Section 3.2], [8, Section 5.10.1] and [14].

A continuous mapping $\gamma : [a, b] \rightarrow \mathbb{R}^n$ is called a *curve*. We say that a curve γ is a *path* if it has a finite and non-zero total length; in this case we parametrize γ by its arclength. We consider paths with various lengths but (with some abuse of notation) do not mark the dependence of the total length ℓ on γ . The *locus* of γ is defined as $\gamma([0, \ell])$ and denoted by $\langle \gamma \rangle$.

If γ is a path and $f : \langle \gamma \rangle \rightarrow [-\infty, \infty]$, we write $V(f, \gamma) = V(f \circ \gamma, [0, \ell])$ and $\tilde{V}(f, \gamma) = \tilde{V}(f \circ \gamma, [0, \ell])$ provided that $f \circ \gamma$ is measurable on $[0, \ell]$.

We refer to [14] and [11] for the properties of the AM -modulus and to [5] and [10] for those of the M_p -modulus. For completeness we recall the definition for the AM -modulus. Let Γ be a family of paths in $\mathbb{R}^n$. A sequence (ρ_i) of Borel functions on $\mathbb{R}^n$ with values in $[0, \infty]$ is said

to be *AM-admissible* (or simply *admissible*) for Γ if for every $\gamma \in \Gamma$

$$\liminf_i \int_{\gamma} \rho_i ds \geq 1, \quad .$$

and we define the *AM-modulus* of Γ as

$$AM(\Gamma) := \inf \left\{ \liminf_i \int_{\mathbb{R}^n} \rho_i dx : (\rho_i)_i \text{ is admissible for } \Gamma \right\}.$$

The *AM-modulus*, as the M_p -modulus, is monotone and countably subadditive and we say that a property holds for *AM*-a.e. curve if the property fails on the family Γ such that $AM(\Gamma) = 0$.

With an arbitrary set $E \subset \mathbb{R}^n$ we associate the family Γ_E of all paths that meet E . Further, let Γ_E^c be the family of all paths γ in $\mathbb{R}^n$ that *cross* E , i.e. there is an *interior* point $t \in (0, \ell)$ such that $\gamma(t) \in E$. Although Γ_E looks more natural, in the subsequent sections it becomes evident that $AM(\Gamma_E^c)$ is better related to the $(n-1)$ -dimensional Hausdorff measure in $\mathbb{R}^n$ than $AM(\Gamma_E)$.

The following lemma clarifies the roles of Γ_E and Γ_E^c .

Lemma 1. $AM(\Gamma_E) = 2 AM(\Gamma_E^c)$.

Proof. Let (ρ_i) be an *AM*-admissible sequence for Γ_E and $\gamma \in \Gamma_E^c$. Since $\gamma(t_0) \in E$ for some $t_0 \in (0, \ell)$, the paths $\gamma_1 = \gamma|_{[0, t_0]}$ and $\gamma_2|_{[t_0, \ell]}$ belong to Γ_E and hence

$$\begin{aligned} \liminf_i \int_{\gamma} \rho_i ds &= \liminf_i \left(\int_{\gamma_1} \rho_i ds + \int_{\gamma_2} \rho_i ds \right) \\ &\geq \liminf_i \int_{\gamma_1} \rho_i ds + \liminf_i \int_{\gamma_2} \rho_i ds \geq 2. \end{aligned}$$

Thus the sequence $(\rho_i/2)$ is *AM*-admissible for Γ_E^c and we obtain $AM(\Gamma_E^c) \leq AM(\Gamma_E)/2$.

For the reverse inequality let (ρ_i^c) be an *AM*-admissible sequence for Γ_E^c and let $\gamma \in \Gamma_E \setminus \Gamma_E^c$. We can assume that $\gamma(\ell) \in E$. Let $\tilde{\gamma}$ be the path $\tilde{\gamma}(t) = \gamma(t)$ for $t \in [0, \ell]$ and $\tilde{\gamma}(t) = \gamma(2\ell - t)$ for $t \in (\ell, 2\ell]$. Then $\tilde{\gamma} \in \Gamma_E^c$ and

$$1 \leq \liminf_i \int_{\tilde{\gamma}} \rho_i^c ds = 2 \liminf_i \int_{\gamma} \rho_i^c ds.$$

Thus $(2\rho_i^c)$ is an *AM*-admissible sequence for Γ_E and, consequently, $AM(\Gamma_E) \leq 2 AM(\Gamma_E^c)$. $\square$

The following auxiliary lemma is often useful.

Lemma 2. *If $U \subset \mathbb{R}^n$ is open and $E \subset U$, then $AM(\Gamma_E) = AM(\Gamma_E(U))$ where $\Gamma_E(U) = \{\gamma \in \Gamma_E : \langle \gamma \rangle \subset U\}$.*

Proof. If $\gamma \in \Gamma_E \setminus \Gamma_E(U)$, then γ has a subpath $\tilde{\gamma} \in \Gamma_E(U)$ and thus $AM(\Gamma_E(U)) \geq AM(\Gamma_E)$. On the other hand, $\Gamma_E(U) \subset \Gamma_E$ and thus $AM(\Gamma_E(U)) \leq AM(\Gamma_E)$. $\square$

For $E \subset \mathbb{R}^n$ we set $\phi(E) = AM(\Gamma_E^c) = \frac{1}{2}AM(\Gamma_E)$ and call ϕ the *AM-modulus measure*. The following theorem justifies the name.

Theorem 1. *ϕ is a metric outer measure on $\mathbb{R}^n$. Therefore, all Borel sets are ϕ -measurable.*

Proof. Obviously $\phi(\emptyset) = 0$ and ϕ is monotone. For the countable subadditivity let $E = \cup_j E_j$. We may assume that $\phi(E) < \infty$. Choose $\varepsilon > 0$. For each j find an *AM*-admissible sequence $(\rho_i^j)_i$ for $\Gamma_{E_j}^c$ such that

$$\int_{\mathbb{R}^n} \rho_i^j d\mu \leq AM(\Gamma_{E_j}^c) + 2^{-j}\varepsilon$$

for each j and i . This is possible by passing to subsequences. Now

$$\rho_i = \sum_j \rho_i^j$$

is *AM*-admissible for Γ_E^c and we obtain

$$\begin{aligned} \phi(E) &= AM(\Gamma_E^c) \leq \liminf_i \int_{\mathbb{R}^n} \rho_i dx \\ &\leq \liminf_i \left(\sum_j AM(\Gamma_{E_j}^c) + 2^{-j}\varepsilon \right) \leq \sum_j AM(\Gamma_{E_j}^c) + \varepsilon \end{aligned}$$

and letting $\varepsilon \rightarrow 0$ we obtain

$$\phi(E) = AM(\Gamma_E^c) \leq \sum_j \phi(E_j)$$

as required. We have shown that ϕ is an outer measure in $\mathbb{R}^n$.

It remains to show that ϕ is a metric outer measure, i.e. if sets $E_1, E_2 \subset \mathbb{R}^n$ are such that $d(E_1, E_2) > \delta$ for some $\delta > 0$, then

$$(2) \quad \phi(E_1) + \phi(E_2) = \phi(E_1 \cup E_2).$$

Now there are disjoint open sets U_i , $i = 1, 2$, with $E_i \subset U_i$ and then (2) follows from Lemma 2. Therefore ϕ is a metric outer measure and, consequently, all Borel sets are ϕ -measurable. $\square$

For the set theoretic terminology below we refer to [9, 2.2.10] and [13].

Theorem 2. *Let $E \subset \mathbb{R}^n$ be an arbitrary set. Then there exists a ϕ -measurable (co-Suslin) set $F \subset \mathbb{R}^n$ such that $E \subset F$ and $\phi(F) = \phi(E)$. The *AM*-modulus measure ϕ also has this property.*

Proof. In this proof we use the version $\phi(E) = \frac{1}{2}AM(\Gamma_E)$ of the definition of ϕ . Since closed sets are ϕ -measurable, each co-Suslin set is ϕ -measurable, [9, Theorem 2.2.12]. We follow the proof of Proposition 1.5 in [6]. Let $\mathcal{L}$ be the space of all curves $\gamma: [0, 1] \rightarrow \mathbb{R}^n$ with Lipschitz constant ≤ 1 equipped with the distance $d(\gamma_1, \gamma_2) = \sup_{t \in [0, 1]} |\gamma_1(t) - \gamma_2(t)|$.

Then $\mathcal{L}$ is a complete metric space. If (ρ_i) is an admissible sequence for Γ_E , consider the mapping $\Phi_i : \mathcal{L} \rightarrow [0, \infty]$,

$$\Phi_i(\gamma) = \int_{\gamma} \rho_i ds, \quad \gamma \in \mathcal{L}.$$

Then Φ_i are Borel measurable, see [12] (see also [6]), and thus

$$\mathcal{L}_1 = \{\gamma \in \mathcal{L} : \gamma \text{ is nonconstant, } \liminf_i \Phi_i(\gamma) < 1\}$$

is also a Borel set in $\mathcal{L}$. Now, the evaluation mapping $\kappa : \gamma \mapsto \gamma(0)$ is continuous and thus $G = \kappa(\mathcal{L}_1)$ is a Suslin set. Set $F = \mathbb{R}^n \setminus G$. If $x \in G \cap E$, then there exists $\gamma \in \mathcal{L}_1$ such that $\gamma(0) = x$. Thus $\gamma \in \Gamma_E$, which is a contradiction, as $\mathcal{L}_1 \cap \Gamma_E = \emptyset$. Hence $E \subset F$. The sequence (ρ_i) is admissible for Γ_F and thus

$$\phi(F) \leq \frac{1}{2} \liminf_i \int_{\mathbb{R}^n} \rho_i dx.$$

Taking infimum over all admissible sequences we obtain $\phi(F) \leq \phi(E)$, whereas the converse inequality is obvious because $E \subset F$. $\square$

Remark 1. Lemmata 1 and 2 and Theorems 1 and 2 hold in metric measure spaces with similar proofs.

We need estimates between $\phi(E) = AM(\Gamma_E^c)$ and the $(n-1)$ -dimensional Hausdorff measure

$$\mathcal{H}^{n-1}(E) = \sup_{\delta > 0} \mathcal{H}_{\delta}^{n-1}(E),$$

where

$$\mathcal{H}_{\delta}^{n-1}(E) = \inf \left\{ 2^{1-n} \alpha_{n-1} \sum_{i=1}^{\infty} (\text{diam } E_i)^{n-1} : E \subset \bigcup_{i=1}^{\infty} E_i, \text{diam } E_i < \delta \right\}$$

are the Hausdorff δ -contents and $\alpha_{n-1} = \mathcal{L}^{n-1}(B^{n-1}(0, 1))$.

Let $G(n, n-1)$ be the Grassmannian manifold of $(n-1)$ -dimensional linear subspaces of $\mathbb{R}^n$. With each $V \in G(n, n-1)$ we associate the orthogonal projection $\Pi_V : \mathbb{R}^n \rightarrow V$. For $E \subset \mathbb{R}^n$ we define

$$\mathcal{G}^{n-1}(E) = \sup_{\delta > 0} \mathcal{G}_{\delta}^{n-1}(E),$$

where

$$\begin{aligned} \mathcal{G}_{\delta}^{n-1}(E) = \inf \left\{ \sum_{i=1}^{\infty} \sup \{ \mathcal{H}^{n-1}(\Pi_V(E_i)) : V \in G(n, n-1) \} : \right. \\ \left. E_i \text{ Borel, } E \subset \bigcup_{i=1}^{\infty} E_i, \text{diam } E_i < \delta \right\}. \end{aligned}$$

This is the $(n-1)$ -dimensional *Gross measure*, see [9, 2.10.4].

We let $\mathbb{H}_n$ denote the coordinate plane $\{x \in \mathbb{R}^n : x_n = 0\}$ and identify $\mathbb{R}^{n-1} = \mathbb{H}_n$.

Inequality (3) below (with a different constant) was considered in metric measure spaces in [14, Theorem 3.17].

Theorem 3. *If $E \subset \mathbb{R}^n$, $n > 1$, then*

$$(3) \quad \phi(E) \leq c_n \mathcal{H}^{n-1}(E),$$

where

$$c_n = \frac{n^n \alpha_n}{2(n-1)^{n-1} \alpha_{n-1}}.$$

Proof. We may assume that $\mathcal{H}^{n-1}(E) < \infty$. For $j = 1, 2, \dots$ choose a covering E_i^j , $i = 1, 2, \dots$, of E such that $d_i^j := \text{diam } E_i^j \leq 2^{-j}$ and

$$2^{1-n} \alpha_{n-1} \sum_{i=1}^{\infty} (d_i^j)^{n-1} \leq \mathcal{H}_{2^{-j}}^{n-1}(E) + 2^{-j} \leq \mathcal{H}^{n-1}(E) + 2^{-j}.$$

Set

$$\rho_j(x) = \sum_i \frac{1}{2r_i^j} \chi_{E_i^j + B_i^j}(x)$$

where $B_i^j = B(0, r_i^j)$ and

$$r_i^j = \frac{1}{2(n-1)} d_i^j.$$

Then ρ_j are Borel functions and the sequence (ρ_j) is AM -admissible for Γ_E^c . Indeed, if $\gamma \in \Gamma_E^c$, then there is $t_0 \in (0, \ell)$ such that $\gamma(t_0) \in E$. The path $\gamma|_{[0, t_0]}$ is non-constant and hence there is j_0 such that for $j \geq j_0$, $\gamma|_{[0, t_0]}$ meets both E_i^j and the complement of $E_i^j + B_i^j$ for some $i = i(j)$. Now the path $\gamma|_{[0, t_0]}$ travels in $E_i^j + B_i^j$ at least distance r_i^j . The same applies to the path $\gamma|_{[t_0, \ell]}$, possibly with a different j_0 , and hence γ travels in $E_i^j + B_i^j$ at least distance $2r_i^j$. Thus

$$\liminf_j \int_{\gamma} \rho_j ds \geq 1$$

as required. Using the isodiametric inequality we estimate

$$\begin{aligned} AM(\Gamma_E^c) &\leq \liminf_j \int_{\mathbb{R}^n} \rho_j dx \leq 2^{-n} \alpha_n \liminf_j \sum_i \frac{1}{2r_i^j} \text{diam}(E_i^j + B_i^j)^n \\ &= 2^{-n} \alpha_n \liminf_j \sum_i \frac{1}{2r_i^j} (d_i^j + 2r_i^j)^n \\ &= 2^{-n} \left(\frac{n}{n-1} \right)^n \alpha_n \liminf_j \sum_i \frac{n-1}{d_i^j} (d_i^j)^n \\ &= \frac{n^n \alpha_n}{2(n-1)^{n-1} \alpha_{n-1}} \liminf_j 2^{1-n} \alpha_{n-1} \sum_i (d_i^j)^{n-1} \\ &\leq \frac{n^n \alpha_n}{2(n-1)^{n-1} \alpha_{n-1}} \liminf_j (\mathcal{H}^{n-1}(E) + 2^{-j}) \end{aligned}$$

and (3) follows. $\square$

Theorem 4. For every $V \in G(n, n-1)$ and $E \subset \mathbb{R}^n$

$$(4) \quad \mathcal{H}^{n-1}(\Pi_V(E)) \leq \phi(E)$$

and if E is a Borel set, then

$$(5) \quad \mathcal{G}^{n-1}(E) \leq \phi(E).$$

Proof. We first prove (4). Let $V \in G(n, n-1)$. We may assume that $V = \mathbb{H}_n = \mathbb{R}^{n-1}$. Let (ρ_i) be an AM -admissible sequence for Γ_E^c . Since each ρ_i is a Borel function, the set

$$A = \left\{ y \in \mathbb{R}^{n-1} : \liminf_i \int_{-\infty}^{\infty} \rho_i(y, t) dt \geq 1 \right\}$$

is a measurable set in $\mathbb{R}^{n-1}$. For each $x' \in \Pi_V(E)$ choose $x \in \Pi_V^{-1}(x') \cap E$ and define

$$\gamma_{x'}(t) = x + (t-1)e_n, \quad t \in [0, 2].$$

Now $\gamma_{x'}$ belongs to Γ_E^c . By the Fubini theorem and the Fatou lemma

$$\begin{aligned} \liminf_i \int_{\mathbb{R}^n} \rho_i dx &\geq \liminf_i \int_A \left(\int_{-\infty}^{\infty} \rho_i((y, t)) dt \right) dy \\ &\geq \int_A \left(\liminf_i \int_{-\infty}^{\infty} \rho_i((y, t)) dt \right) dy \geq \mathcal{H}^{n-1}(A) \geq \mathcal{H}^{n-1}(\Pi_V(E)) \end{aligned}$$

and since this holds for every AM -admissible sequence, the estimate (4) follows.

If E is a Borel set and $\delta > 0$, we write E as a pairwise disjoint union of Borel sets E_i with $\text{diam } E_i < \delta$. Using countable subadditivity of $\mathcal{G}_\delta^{n-1}$, countable additivity of ϕ on Borel sets and (4), we obtain

$$\mathcal{G}_\delta^{n-1}(E) \leq \sum_i \mathcal{G}_\delta^{n-1}(E_i) \leq \sum_i \phi(E_i) \leq \phi(E).$$

Passing to the supremum over δ we obtain (5). $\square$

Remark 2. It is easy to see that $\mathcal{H}^{n-1} = \phi$ for $n = 1$.

Remark 3. Note that for the estimate (4) we do not need to suppose that E is a Borel set. We do not know if (5) holds for all sets $E \subset \mathbb{R}^n$.

3. BVC FUNCTIONS IN $\mathbb{R}^n$

Let Ω be an open set in $\mathbb{R}^n$ and $u: \Omega \rightarrow [-\infty, \infty]$. We say that u is a *BVC function*, $u \in \mathcal{BV}\mathcal{C}(\Omega)$, if there exists a sequence (g_i) of non-negative Borel functions in Ω such that

$$(6) \quad |u(\gamma(\ell)) - u(\gamma(0))| \leq \liminf_i \int_\gamma g_i ds$$

for each path γ in Ω and

$$(7) \quad \liminf_i \int_\Omega g_i dx < \infty.$$

Equation (6) is understood in the sense that if the left hand side is undefined, i.e. of the form $|\infty - \infty|$, then its value is ∞ . The sequence (g_i) is called a *BV upper bound* for u in Ω .

Note that we assume no a priori regularity, besides (6), for $u \in \mathcal{BVC}(\Omega)$. However, in Theorem 5 below we show that $u \in L^1_{\text{loc}}(\Omega)$.

If $u \in \mathcal{BVC}(\Omega)$, then [14, Theorem 4.1] shows that for AM -a.e. path in Ω and for every *BV upper bound* (g_i)

$$(8) \quad V(u, \gamma) \leq \liminf_i \int_{\gamma} g_i ds < \infty.$$

The condition (6) can be relaxed and we say that a sequence (g_i) of non-negative Borel functions in Ω is a *weak BV upper bound* for u in Ω if (7) is satisfied and (6) holds for AM -a.e. path in Ω . The next lemma shows that weak upper bounds can be used instead of ordinary *BV upper bounds*.

Lemma 3. *If (g_i) is a weak BV upper bound for u and $\varepsilon > 0$, then there exists a BV upper bound $(\tilde{g}_i)$ for u and*

$$\liminf_i \int_{\Omega} \tilde{g}_i dx \leq \liminf_i \int_{\Omega} g_i dx + \varepsilon.$$

Proof. Let Γ_0 be the family of all paths in Ω such that

$$V(u, \gamma) > \liminf_i \int_{\gamma} g_i ds.$$

Since $AM(\Gamma_0) = 0$, for each $\varepsilon > 0$ there exists a sequence (v_i) of non-negative Borel functions such that

$$\liminf_i \int_{\gamma} v_i ds = \infty$$

for every $\gamma \in \Gamma_0$ and

$$\int_{\Omega} v_i dx < \varepsilon$$

for every i , see [11, Theorem 7 and Remark 6]. Let $\tilde{g}_i = g_i + v_i$. Now $(\tilde{g}_i)$ satisfies 6 and

$$\liminf_i \int_{\Omega} \tilde{g}_i dx = \liminf_i \int_{\Omega} (g_i + v_i) dx \leq \liminf_i \int_{\Omega} g_i dx + \varepsilon$$

and so $(\tilde{g}_i)$ is the required *BV upper bound* for u . $\square$

Lemma 3 shows that (8) holds for each weak upper bound for $u \in \mathcal{BVC}(\Omega)$ and consequently u needs to be defined only in $\Omega \setminus E$ where E satisfies $AM(\Gamma_E) = 0$ and so the values of u on E are immaterial. For example, precisely represented functions u in the Sobolev class $W^{1,p}(\Omega)$, $p \geq 1$, are *BVC* functions since they are absolutely continuous on M_p -a.e. path and hence on AM -a.e. path in Ω . The constant sequence $(|\nabla u|)$ is a weak *BV upper bound* for u .

For an open set $U \subset \Omega$ we define

$$|D_{BVC}u|(U) = \inf \left\{ \liminf_j \int_{\Omega} g_j dx : \right. \\ \left. (g_j) \text{ is a } BV \text{ upper bound for } u \text{ in } U \right\}.$$

By Lemma 3 weak BV upper bounds can be used as well.

Next we show that a BVC function u in Ω belongs to $L^1_{\text{loc}}(\Omega)$. Since a BVC function has bounded variation on a.e. line parallel to coordinate axis one might think that this property already implies measurability. However, W. Sierpiński [17] constructed a non-measurable set $A \subset \mathbb{R}^2$ such every line (not only parallel to coordinate axis) meets A at most in two points. The function χ_A has thus bounded variation on every line segment but it is not measurable. Hence the role of the weak BV upper bound (g_j) is essential.

We employ a weak type estimate for a maximal function associated with a weak BV upper bound (g_j) of a BVC function u in Ω . We set $g_j = 0$ in $\mathbb{R}^n \setminus \Omega$. To simplify the notation, we use the symbol $\mathbf{g}$ for the sequence (g_j) and introduce the maximal function

$$M\mathbf{g}(x) = \liminf_j M g_j(x)$$

where $M g_j$ denotes the ordinary Hardy–Littlewood maximal function of g_j .

Lemma 4. *Let $\mathbf{g} = (g_j)$ be a sequence of nonnegative measurable functions on $\mathbb{R}^n$ and $t > 0$. Then*

$$\mathcal{L}^n(\{x \in \mathbb{R}^n : M\mathbf{g}(x) > t\}) \leq \frac{C_n}{t} \liminf_j \int_{\mathbb{R}^n} g_j(z) dz,$$

where C_n depends only on n . Consequently, $M\mathbf{g}(x) < \infty$ for a.e. $x \in \mathbb{R}^n$.

Proof. Set

$$I = \liminf_j \int_{\mathbb{R}^n} g_j(z) dz.$$

Choose $\varepsilon > 0$. We may assume that

$$\sup_j \int_{\mathbb{R}^n} g_j(z) dz \leq I + \varepsilon.$$

Fix $t > 0$ and let

$$E_k = \{x \in \mathbb{R}^n : \inf_{j \geq k} M g_j(x) > t\}.$$

Then $E_1 \subset E_2 \subset \dots$ and

$$E := \{x \in \mathbb{R}^n : M\mathbf{g}(x) > t\} = \bigcup_k E_k.$$

The sets E_k are measurable. By the Hardy-Littlewood maximal theorem,

$$\mathcal{L}^n(E_k) \leq \mathcal{L}^n(\{x \in \mathbb{R}^n : Mg_k(x) > t\}) \leq \frac{C}{t} \int_{\mathbb{R}^n} g_k(z) dz \leq \frac{C(I + \varepsilon)}{t}.$$

Letting $E_k \nearrow E$ and $\varepsilon \searrow 0$ we conclude

$$\mathcal{L}^n(E) \leq \frac{CI}{t}.$$

□

Theorem 5. $\mathcal{BVC}(\Omega) \subset L^1_{\text{loc}}(\Omega)$.

Proof. We first show that $u \in \mathcal{BVC}(\Omega)$ is measurable. It suffices to show that if u is a BVC function in $B'_0 = B(x_0, 2r_0) \subset \Omega$, then u is measurable in $B_0 = B(x_0, r_0/2)$. Let (g_j) be a BV upper bound for u in Ω and set $g_j = 0$ in $\mathbb{R}^n \setminus \Omega$.

If $r \in (0, 3r_0/2)$, $x \in B_0$ and $y \in B(x, r)$, then

$$\begin{aligned} |u(y) - u(x)| &\leq \liminf_j \int_{\gamma_{x,y}} g_j ds = \liminf_j \int_0^1 g_j(x + t(y-x)) |y-x| dt \\ (9) \quad &\leq \liminf_j \int_0^1 r g_j(x + t(y-x)) dt =: v^x(y) \end{aligned}$$

where $\gamma_{x,y}(t) = x + t(y-x)$, $t \in [0, 1]$, is the line segment from x to y . For a fixed x the function v^x is measurable since the functions g_j are Borel functions. Now, the Fubini theorem and the Fatou lemma yield

$$\begin{aligned} (10) \quad \int_{B(x,r)} v^x(y) dy &= \int_{B(x,r)} \left(\liminf_j \int_0^1 r g_j(x + t(y-x)) dt \right) dy \\ &\leq \liminf_j \int_{B(x,r)} \left(\int_0^1 r g_j(x + t(y-x)) dt \right) dy \\ &= \liminf_j \int_0^1 r t^{-n} \left(\int_{B(x,tr)} g_j(z) dz \right) dt \\ &\leq \alpha_n r^{n+1} \liminf_j \int_0^1 M g_j(x) dt = \alpha_n r^{n+1} M \mathbf{g}(x), \end{aligned}$$

where the change of variable from y to $z = x + t(y-x)$ is also used.

To complete the proof let $x_1, x_2 \in B_0$ with $r = |x_2 - x_1| > 0$. Write $B = B(a, r/2)$, where $a = \frac{1}{2}(x_1 + x_2)$. Then $B \subset B(x_1, r) \cap B(x_2, r) \subset B'_0$ and for $y \in B$ we have

$$|u(x_2) - u(x_1)| \leq |u(x_2) - u(y)| + |u(x_1) - u(y)| \leq v^{x_2}(y) + v^{x_1}(y)$$

and thus integrating the inequality

$$|u(x_2) - u(x_1)| \leq v^{x_2}(y) + v^{x_1}(y)$$

over $y \in B$ we obtain from (10)

$$\alpha_n(r/2)^n |u(x_2) - u(x_1)| \leq \alpha_n r^{n+1} (M\mathbf{g}(x_1) + M\mathbf{g}(x_2)).$$

This shows that

$$|u(x_2) - u(x_1)| \leq 2^n |x_2 - x_1| (M\mathbf{g}(x_1) + M\mathbf{g}(x_2)).$$

Let $A_i = \{x \in B_0 : M\mathbf{g}(x) \leq i\}$, $i = 1, 2, \dots$. Then $\mathcal{L}^n(B_0 \setminus \cup_i A_i) = 0$ as $\mathcal{L}^n(\{x \in B_0 : M\mathbf{g}(x) = \infty\}) = 0$ by Lemma 4. In the set A_i the function u is Lipschitz and hence measurable and from this it follows that u is measurable in B_0 .

Now it easily follows that $u \in L^1_{\text{loc}}(\Omega)$. First note that $|u(x)| < \infty$ for a.e. $x \in \Omega$ by the Fubini theorem, as the family Γ of line segments parallel to a coordinate axis on which u fails to be of finite variation has AM -modulus zero by Theorem 3. Then we find $x \in B_0$ such that $|u(x)| < \infty$ and $M\mathbf{g}(x) < \infty$. An integration of (9) over $y \in B_0$ and (10) show that $u \in L^1(B_0)$. $\square$

Remark 4. In fact, we can obtain better integrability $\mathcal{BVC}(\Omega) \subset L^{n/(n-1)}_{\text{loc}}(\Omega)$. This can be obtained following the standard argument by Gagliardo and Nirenberg, or as a consequence of the inclusion $\mathcal{BVC}(\Omega) \subset BV_{\text{loc}}(\Omega)$ presented later (Theorem 7).

4. BVC VERSUS BV FUNCTIONS

The M_1 -modulus cooperates well with the Sobolev spaces $W^{1,1}$, but not so well with the BV spaces. In this respect the AM -modulus is a more efficient tool. Next we show that a precise representative of a BV function in Ω is a BVC function.

We recall some concepts from the theory of BV functions in $\mathbb{R}^n$, see [3] and [8].

A function $u \in L^1_{\text{loc}}(\Omega)$ has *bounded variation* (BV) in Ω if its distributional gradient Du can be represented as a finite vector-valued signed Radon measure Du in Ω , i.e.

$$\int_{\Omega} u \operatorname{div} \varphi \, dx = - \int_{\Omega} \varphi \cdot Du$$

for every $\varphi \in C^1_0(\Omega, \mathbb{R}^n)$. Equivalently there are signed Radon measures $D_i u$, $i = 1, \dots, n$, such that

$$\int_{\Omega} u \partial_i \varphi \, dx = - \int_{\Omega} \varphi D_i u$$

for every $\varphi \in C^1_0(\Omega)$. We let $|Du|$ and $|D_i u|$ denote the total variation of the measures Du and $D_i u$, $i = 1, \dots, n$, respectively. By the Riesz representation theorem, see e.g. [8], we have

$$(11) \quad |Du|(\Omega) = \sup \left\{ \int_{\Omega} u \operatorname{div} \varphi \, dx : \varphi \in C^1_0(\Omega, \mathbb{R}^n), |\varphi| \leq 1 \right\}$$

and for each $i = 1, \dots, n$

$$(12) \quad |D_i u|(\Omega) = \sup \left\{ \int_{\Omega} u \partial_i \varphi \, dx : \varphi \in C_0^1(\Omega), |\varphi| \leq 1 \right\}.$$

A function $u \in C^1(\Omega)$ with $\int_{\Omega} |\nabla u| \, dx < \infty$ has bounded variation and

$$|Du|(\Omega) = \int_{\Omega} |\nabla u| \, dx, \quad |D_i u|(\Omega) = \int_{\Omega} |\partial_i u| \, dx.$$

We denote the space of all BV functions in Ω by $\mathcal{BV}(\Omega)$. This is a Dirichlet type space; we leave the symbol $BV(\Omega)$ for the usual BV -space, namely

$$BV(\Omega) = \mathcal{BV}(\Omega) \cap L^1(\Omega).$$

The space $BV(\Omega)$ is equipped with the BV norm

$$\|u\|_{BV} = |Du|(\Omega) + \|u\|_1.$$

Similarly, we define

$$BVC(\Omega) = \mathcal{BVC}(\Omega) \cap L^1(\Omega)$$

and consider the BVC norm

$$\|u\|_{BVC} = |Du|_{BVC}(\Omega) + \|u\|_1.$$

Of course, the BV norm does not satisfy the axioms of norm on the family of BV functions. For a precise manipulation with the BV space in the framework on normed linear spaces we identify functions which coincide a.e. and mark the underlying linear space

$$\mathbf{BV}(\Omega) := \{[u]_{\mathcal{L}^n} : u \in BV(\Omega)\},$$

where

$$[u]_{\mathcal{L}^n} = \{v \in BV(\Omega) : v = u \text{ a.e.}\}$$

Concerning the BVC norm, we use a “finer” equivalence relation, so that the underlying linear space is

$$\mathbf{BVC}(\Omega) := \{[u]_{\phi} : u \in BVC(\Omega)\},$$

where

$$[u]_{\phi} = \{v \in BVC(\Omega) : v = u \text{ } \phi\text{-a.e.}\}$$

This formalism helps us to explain the difference between the BV and BVC spaces in Remark 5 below.

Let us make the convention that we will still say e.g. “ v is a representative of $u \in BV(\Omega)$ ” instead of more precise, but clumsy “ v is a representative of $[u]_{\mathcal{L}^n} \in \mathbf{BV}(\Omega)$ ”.

Let $u \in L_{\text{loc}}^1(\Omega)$. A point $z \in \Omega$ is said to be a *Lebesgue point* for u if

$$\lim_{r \rightarrow 0+} \int_{B(z,r)} |u(x) - u(z)| \, dx = 0.$$

Given $z \in \mathbb{R}^n$, $r > 0$ and $\mathbf{n} \in \partial B(0, 1)$, we write

$$B_{\mathbf{n}}(z, r) = \{x \in B(z, r) : (x - z) \cdot \mathbf{n} > 0\}$$

and say that $z \in \Omega$ is an L^1 -approximate jump point for u if there exist $a, b \in \mathbb{R}$ and $\mathbf{n} \in \partial B(0, 1)$ such that $a \neq b$ and

$$\lim_{r \rightarrow 0+} \int_{B_{\mathbf{n}}(z, r)} |u(x) - b| dx = \lim_{r \rightarrow 0+} \int_{B_{-\mathbf{n}}(z, r)} |u(x) - a| dx = 0.$$

Define the *normal vector* of u at z by $\mathbf{n}_u(z) = (b - a)\mathbf{n}$ where $\mathbf{n}$ is given by the preceding formula; the vector $\mathbf{n}_u(z)$ is unique at L^1 -approximate jump points.

We define the L^1 -approximate discontinuity set S_u as a set of non-Lebesgue points for u and the L^1 -approximate jump set of J_u as the set of all L^1 -approximate jump points for u ; now $J_u \subset S_u$.

We call $\bar{u}: \Omega \rightarrow \mathbb{R}$ a *precise representative* of u if

$$\bar{u}(z) = \lim_{r \rightarrow 0} \int_{B(z, r)} u(x) dx$$

whenever for z this limit exists and is finite. At a jump point z of u we set $\bar{u}(z) = (a + b)/2$. Now we have $\bar{u} = u$ a.e. in Ω .

If u is a BV function in Ω and $\bar{u}$ the precise representative of u , then by [3, Theorem 3.78] we have $\mathcal{H}^{n-1}(S_u \setminus J_u) = 0$ and hence by Theorem 3, $AM(\Gamma_{S_u \setminus J_u}) = 0$ and thus $\bar{u}$ is well defined on AM -a.e. path in Ω . The set $S_u \setminus J_u$ contains all points at which the value of a precise representative remains undetermined. The values of $\bar{u}$ at these points x are immaterial, even can be undefined.

Theorem 6. *Let u be a precise representative of a function from $\mathcal{BV}(\Omega)$. Then $u \in \mathcal{BVC}(\Omega)$ and u has bounded variation on AM -a.e. path in Ω . Moreover, there is a sequence of functions $v_j \in C^1(\Omega)$ such that $v_j \rightarrow u$ in $L^1_{\text{loc}}(\Omega)$ and $|\nabla v_j|$ is a weak BV upper bound for u with*

$$(13) \quad \begin{aligned} |Du|(\Omega) &= \lim_{j \rightarrow \infty} \int_{\Omega} |\nabla v_j| dx, \\ |D_i u|(\Omega) &= \lim_{j \rightarrow \infty} \int_{\Omega} |\partial_i v_j| dx, \quad i = 1, \dots, n. \end{aligned}$$

In particular,

$$|D_{BVC} u|(\Omega) \leq |Du|(\Omega).$$

Proof. As in the proof of [3, Theorem 3.9] or of [8, 5.2.2 Theorem 2] there is a sequence (v_j) of functions $v_j \in C^1(U)$ such that $v_j \rightarrow u$ in $L^1_{\text{loc}}(\Omega)$ and (13) holds. The functions v_j are constructed by the standard convolution process with a radial mollifier η , see [8, 4.2.1.], with the aid of a partition of unity which helps to define the approximating function near the boundary. Now at each Lebesgue point $x \in \Omega$ of u , i.e. $x \notin S_u$, $v_j(x) \rightarrow u(x)$, see [3, Proposition 3.64] or [8, 4.2.1 Theorem 1], and also at each jump point x of u , $v_j(x) \rightarrow u(x) = (a + b)/2$, see [3, Proposition 3.69] or [8, 5.9 Corollary 1].

Let $E = S_u \setminus J_u$. Now $AM(\Gamma_E) = 0$ and for each path γ in Ω but not in Γ_E , $v_i \circ \gamma(t) \rightarrow u \circ \gamma(t)$ for every $t \in [0, \ell]$ and hence by the lower semicontinuity of the total variation

$$V(u, \gamma) \leq \liminf_j V(v_j, \gamma) \leq \liminf_j \int_\gamma |\nabla v_j| ds.$$

Thus the sequence $(|\nabla v_j|)$ is a weak BV upper bound for u in Ω and so $u \in \mathcal{BV}\mathcal{C}(\Omega)$. $\square$

A converse for Theorem 6 was established in [15, Theorem 5.4]. We will improve the bound obtained there.

Lemma 5. *Let $u \in \mathcal{BV}\mathcal{C}(\Omega)$. Let (g_j) a weak BV upper bound for u in Ω . Then $u \in \mathcal{BV}(\Omega)$ and*

$$(14) \quad |Du(U)| \leq \liminf_j \int_U g_j dx$$

for each convex open set $U \subset \Omega$ with compact closure in Ω .

Proof. We first fix a convex open set $U \subset \Omega$ with compact closure in Ω and show that $u \in \mathcal{BV}(U)$ and, for $i = 1, \dots, n$,

$$(15) \quad \begin{aligned} |D_i u|(U) &= \sup \left\{ \int_U u \partial_i \varphi dx : \varphi \in C_0^1(U), |\varphi| \leq 1 \right\} \\ &\leq \liminf_j \int_U g_j dx < \infty. \end{aligned}$$

We may assume $i = n$ which simplifies notation. Since U is convex and has compact closure in Ω , for a fixed $y \in \mathbb{R}^{n-1}$, the set $\{t \in \mathbb{R} : (y, t) \in \overline{U}\}$ is a closed interval L_y (possibly empty) and from (8) it follows that

$$(16) \quad V(u(y, \cdot), L_y) \leq \liminf_j \int_{L_y} g_j(y, t) dt$$

for a.e. $y \in \mathbb{R}^{n-1}$. By Theorem 5, $u \in L^1(U)$. Let $\varphi \in C_0^1(U)$, $|\varphi| \leq 1$. Applying the Fubini theorem and the Fatou lemma we obtain from (1) and (16)

$$\begin{aligned} \int_U u \partial_n \varphi dx &= \int_{\mathbb{R}^{n-1}} \left(\int_{L_y} u(y, t) \partial_n \varphi(y, t) dt \right) dy \\ &\leq \int_{\mathbb{R}^{n-1}} \tilde{V}(u(y, \cdot), L_y) dy \leq \int_{\mathbb{R}^{n-1}} V(u(y, \cdot), L_y) dy \\ &\leq \liminf_j \int_U g_j dx, \end{aligned}$$

which verifies (15). To prove (14), pick a unit vector $\mathbf{n}$ such that $|Du(U)| = Du(U) \cdot \mathbf{n}$. We may assume that $\mathbf{n} = \mathbf{e}_n$; then

$$|Du(U)| = D_n u(U) \leq \liminf_j \int_U g_j dx$$

from (15).

The estimate

$$|D_i u|(U) \leq \liminf_j \int_U g_j dx, \quad i = 1, \dots, n$$

from (15) implies

$$(17) \quad |Du|(B) \leq n \liminf_j \int_B g_j dx$$

for each ball B with closure in Ω . Using a finite covering with bounded multiplicity of Ω by such balls and an associated partition of unity we obtain

$$(18) \quad |Du|(\Omega) \leq C \liminf_j \int_{\Omega} g_j dx$$

where C depends only on n . Hence $u \in \mathcal{BV}(\Omega)$. $\square$

If $u \in \mathcal{BV}(\Omega)$ is a precisely represented BV function, Theorem 6 shows that $|D_{BV} u|(U) \leq |Du|(U)$ for open sets $U \subset \Omega$. In the next theorem we present the converse sharp inequality.

Theorem 7. *If $u \in \mathcal{BVC}(\Omega)$, then $u \in \mathcal{BV}(\Omega)$ and*

$$|Du|(\Omega) \leq |D_{BV} u|(\Omega).$$

Proof. Let (g_j) be a BV upper bound for u in Ω . By Lemma 5, u is a BV function in Ω . By the Besicovitch differentiation theorem [3, Theorem 2.22], there exists a set $N \subset \Omega$ such that $\Omega \setminus N$ is contained in the support of $|Du|$,

$$(19) \quad |Du|(\Omega \cap N) = 0$$

and for each $y \in \Omega \setminus N$ there exists

$$(20) \quad \lim_{r \rightarrow 0} \frac{Du(B(y, r))}{|Du|(B(y, r))} \in \partial B(0, 1).$$

Given $\varepsilon > 0$, with each $y \in \Omega$ we associate the family $R(y)$ of all radii $r > 0$ with the properties $\overline{B}(y, r) \subset \Omega$,

$$(21) \quad |Du|(B(y, r)) \leq (1 + \varepsilon) |Du(B(y, r))|$$

and

$$(22) \quad |Du|(\partial B(y, r)) = 0.$$

This defines a fine centered covering of $\Omega \setminus N$. By the Vitali-Besicovitch covering theorem [3, Theorem 2.19], there exists a countable (possibly finite) pairwise disjoint collection $\{B(y_i, r_i)\}$ of balls such that $y_i \in \Omega \setminus N$, $r_i \in R(y_i)$ and

$$|Du|((\Omega \setminus N) \setminus \bigcup_i \overline{B}(y_i, r_i)) = 0.$$

However, since $|Du|(N) = 0$ and in view of (22), we can improve to

$$|Du|(\Omega \setminus \bigcup_i B(y_i, r_i)) = 0.$$

Now Lemma 5 and (21) yield

$$\begin{aligned} (23) \quad |Du|(\Omega) &\leq \sum_i |Du|(B(y_i, r_i)) \leq (1 + \varepsilon) \sum_i |Du|(B(y_i, r_i)) \\ &\leq (1 + \varepsilon) \sum_i \liminf_j \int_{B(y_i, r_i)} g_j dx \leq (1 + \varepsilon) \liminf_j \int_{\Omega} g_j dx. \end{aligned}$$

Since (g_j) was arbitrary, letting $\varepsilon \rightarrow 0$ we obtain the desired inequality. $\square$

The above results lead to the following characterization of BV functions.

Theorem 8. *Let u be a function on Ω . Then u is a BV function in Ω if and only if there exists a BVC function v on Ω such that $u = v$ a.e.*

Proof. This is immediate from Theorem 6 and Theorem 7. $\square$

Remark 5. We have proved inclusions

$$\{u \in BV(\Omega) : u \text{ is precisely represented}\} \subset BVC(\Omega) \subset BV(\Omega).$$

These inclusions are both strict: The characteristic function of a singleton is in $BVC(\Omega)$ but it is not precisely represented. The characteristic function of the set of all $x \in \Omega$ which have at least one rational coordinate is in $BV(\Omega)$ but not in $BVC(\Omega)$. Let us associate a precise representative $\bar{u}$ with any $u \in BV(\Omega)$. Consider the induced mappings

$$\begin{aligned} \epsilon : [u]_{\mathcal{L}^n} &\mapsto [\bar{u}]_{\phi} : \mathbf{BV}(\Omega) \rightarrow \mathbf{BVC}(\Omega), \\ \varkappa : [u]_{\phi} &\mapsto [u]_{\mathcal{L}^n} : \mathbf{BVC}(\Omega) \rightarrow \mathbf{BV}(\Omega). \end{aligned}$$

If v_1, v_2 are precise representatives of $u \in BV(\Omega)$, then $v_1 = v_2$ $\mathcal{H}^{n-1}$ -a.e. and the more $v_1 = v_2$ ϕ -a.e., so that ϵ is well defined. Both ϵ and $\varkappa$ are linear. We will show that ϵ is injective, but not surjective and $\varkappa$ is surjective, but not injective. Therefore we cannot identify the spaces $\mathbf{BVC}(\Omega)$ and $\mathbf{BV}(\Omega)$. Indeed, the injectivity of ϵ is obvious. For any $\mathbf{u} \in \mathbf{BV}(\Omega)$ we have $\mathbf{u} = \varkappa(\epsilon(\mathbf{u}))$, so that $\varkappa$ is surjective. Now, consider the characteristic function u of the singleton $\{0\}$ in $(-1, 1)$. Then $\phi(\{0\}) = 1$, hence $[u]_{\phi} \neq [0]_{\phi}$, but

$$\varkappa([u]_{\phi}) = [0]_{\mathcal{L}^n} = \varkappa([0]_{\phi}).$$

Therefore $\varkappa$ is not injective. The same u disproves surjectivity of ϵ , as $[u]_{\phi}$ does not belong to its range. Note that $|D_{BVC}u|((-1, 1)) = 2$.

The representatives u of $u \in BV(\Omega)$ can be classified as non- BVC representatives, “bad” BVC representatives and “good” BVC representatives. Here, we mean that u is a good representative if $[u]_{\phi} =$

$\epsilon([u]_{\mathcal{L}^n})$. If u is a good representative, then $|Du|(\Omega) = |D_{BVC}u|(\Omega)$ (but not conversely, consider the characteristic function of $[0, 1]$ in $(-1, 1)$).

5. AM -MODULUS AND $(n-1)$ -RECTIFIABLE SETS

A set $A \subset \mathbb{R}^n$ is called *countably $\mathcal{H}^{n-1}$ -rectifiable*, or (following [16]) simply *$(n-1)$ -rectifiable*, if there are Lipschitz maps $h_i : \mathbb{R}^{n-1} \rightarrow \mathbb{R}^n$, $i = 1, 2, \dots$, such that $\mathcal{H}^{n-1}(A \setminus \cup_{i=1}^{\infty} h_i(\mathbb{R}^{n-1})) = 0$.

For the theory of rectifiable sets see [9], [16] and [3].

We first state the lower estimate as a separate lemma.

Lemma 6. *Suppose that $A \subset \mathbb{R}^n$ is $(n-1)$ -rectifiable. If $E \subset A$, then*

$$(24) \quad \mathcal{H}^{n-1}(E) \leq AM(\Gamma_E^c)$$

Proof. Assume first that E is $\mathcal{H}^{n-1}$ -measurable. The result of [3, Proposition 2.66] states that (due to $(n-1)$ -rectifiability)

$$\mathcal{H}^{n-1}(E) = \sup \left\{ \sum_{i=1}^{\infty} \mathcal{H}^{n-1}(\Pi_{V_i}(K_i)) : V_i \in G(n, n-1), \right. \\ \left. K_i \subset E \text{ compact and pairwise disjoint} \right\}.$$

If K_i and V_i are as above, by (4) we have

$$\sum_{i=1}^{\infty} \mathcal{H}^{n-1}(\Pi_{V_i}(K_i)) \leq \sum_{i=1}^{\infty} \phi(K_i) \leq \phi(E).$$

Hence $\mathcal{H}^{n-1}(E) \leq \phi(E)$ and in the general case we use Theorem 2 to find a ϕ -measurable set $F \supset E$ such that $\phi(F) = \phi(E)$ and obtain

$$\mathcal{H}^{n-1}(E) \leq \mathcal{H}^{n-1}(F) \leq \phi(F) = \phi(E).$$

□

Comparing the previous lemma with Theorem 3 we already know that $\mathcal{H}^{n-1}$ and ϕ are comparable on $(n-1)$ -rectifiable sets. However, we can profit from rectifiability also in the sharp constant of the upper estimate and prove that $\mathcal{H}^{n-1}$ and ϕ in fact agree on $(n-1)$ -rectifiable sets.

We start with a simple estimate of Minkowski content. If $E \subset \mathbb{R}^n$ is an arbitrary set, we define

$$E_r = \{x \in \mathbb{R}^n : \text{dist}(x, E) < r\}$$

and

$$\underline{\mathcal{M}}^{n-1}(E) = \liminf_{r \rightarrow 0+} \frac{\mathcal{L}^n(E_r)}{2r}.$$

This is the *lower Minkowski content* of E .

Lemma 7. $\phi(E) \leq \underline{\mathcal{M}}^{n-1}(E)$ for every set $E \subset \mathbb{R}^n$

Proof. We find a sequence (r_i) , $r_i \searrow 0$, such that

$$\frac{\mathcal{L}^n(E_{r_i})}{2r_i} \rightarrow \underline{\mathcal{M}}^{n-1}(E)$$

and set $\rho_i = \chi_{E_{r_i}}/(2r_i)$. Let $\gamma \in \Gamma_E^c$. Then there exists $i_0 \in \mathbb{N}$ and $t \in (0, \ell)$ such that $\gamma(t) \in E$ and $[t - r_i, t + r_i] \subset [0, \ell]$ for each $i \geq i_0$. Since (by our convention) γ is parametrized by its arclength, we have $\gamma([t - r_i, t + r_i]) \subset E_{r_i}$ and thus

$$\int_{\gamma} \rho_i \geq 1, \quad i \geq i_0.$$

It follows that (ρ_i) is admissible for Γ_E^c and thus

$$\phi(E) \leq \liminf_j \int_{\mathbb{R}^n} \rho_i dx = \liminf_j \frac{\mathcal{L}^n(E_{r_i})}{2r_i} = \underline{\mathcal{M}}^{n-1}(E).$$

□

Theorem 9. *If $A \subset \mathbb{R}^n$ is $(n-1)$ -rectifiable and $E \subset A$, then*

$$\mathcal{H}^{n-1}(E) = AM(\Gamma_E^c).$$

Proof. By Theorem 3 it suffices to show

$$(25) \quad AM(\Gamma_E^c) \leq \mathcal{H}^{n-1}(E)$$

and we may assume that $\mathcal{H}^{n-1}(E) < \infty$. Also assume first that E is $\mathcal{H}^{n-1}$ -measurable. Then as a straightforward consequence of the definition, there exist compact sets $K_j \subset \mathbb{R}^{n-1}$ and Lipschitz maps $f_j: K_j \rightarrow \mathbb{R}^n$ such that $f_j(K_j)$ are pairwise disjoint subsets of E and

$$\mathcal{H}^{n-1}\left(E \setminus \bigcup_j f_j(K_j)\right) = 0.$$

Theorem [9, 3.2.39] states that $\underline{\mathcal{M}}^{n-1}(F) = \mathcal{H}^{n-1}(F)$ for each Lipschitz image of a compact subset of $\mathbb{R}^{n-1}$. Hence by Lemma 7,

$$\begin{aligned} \phi(E) &= \sum_j \phi(f_j(K_j)) \leq \sum_j \underline{\mathcal{M}}^{n-1}(f_j(K_j)) \\ &= \sum_j \mathcal{H}^{n-1}(f_j(K_j)) \leq \mathcal{H}^{n-1}(E). \end{aligned}$$

Here we have used that $\phi(N) = 0$ for each $N \subset \mathbb{R}^n$ with $\mathcal{H}^{n-1}(N) = 0$, this follows from Theorem 3.

Now we may remove the additional assumption of $\mathcal{H}^{n-1}$ -measurability. Indeed, since $\mathcal{H}^{n-1}$ is a Borel regular measure, for an arbitrary set $E \subset A$ there exists a Borel set $F \supset E$ such that $\mathcal{H}^{n-1}(F) = \mathcal{H}^{n-1}(E)$ and we can use the estimate above for $F \cap A$. □

6. PERIMETER AND AM -MODULUS

We first recall the basic concepts on sets of finite perimeter, see [3] and [8].

Let $E \subset \mathbb{R}^n$ be a Lebesgue measurable set. The *measure theoretic boundary* $\partial_* E$ of E consists of points $x \in \mathbb{R}^n$ such that $\Theta^n(x, E) > 0$ and $\Theta^n(x, \mathbb{R}^n \setminus E) > 0$ where

$$\Theta^n(x, A) = \limsup_{r \rightarrow 0} \frac{\mathcal{L}^n(B(x, r) \cap A)}{\mathcal{L}^n(B(x, r))}$$

is the upper Lebesgue density of A at x . The *measure theoretic interior* $\text{int}_* E$ of E and the *measure theoretic exterior* $\text{ext}_* E$ are the sets of points $x \in \mathbb{R}^n$ where $\Theta^n(x, \mathbb{R}^n \setminus E) = 0$ and $\Theta^n(x, E) = 0$, respectively. The sets $\partial_* E$, $\text{int}_* E$ and $\text{ext}_* E$ are Borel sets.

A set $E \subset \mathbb{R}^n$ has *finite perimeter* if $\chi_E \in BV(\mathbb{R}^n)$, i.e. E has finite Lebesgue measure and $P(E) = |D\chi_E|(\mathbb{R}^n) < \infty$.

It is also possible to consider sets E which have finite perimeter in an open set $\Omega \subset \mathbb{R}^n$, i.e. χ_E is a BV function in Ω , but, for simplicity, we only consider $\Omega = \mathbb{R}^n$.

If E has finite perimeter we let $\partial^* E$ denote the *reduced boundary* of E , which consist of all points x from the support of $|D\chi_E|$ for which there exists a unit vector $\mathbf{n}(x)$ such that

$$\lim_{r \rightarrow 0} \frac{D\chi_E(B(x, r))}{|D\chi_E(B(x, r))|} = \mathbf{n}(x),$$

see [3, Definition 3.54] [8, 5.7.1]. Then we have

$$(26) \quad |D\chi_E|(A) = \mathcal{H}^{n-1}(\partial^* E \cap A) = \mathcal{H}^{n-1}(\partial_* E \cap A)$$

for each Borel set $A \subset \mathbb{R}^n$, see [3, Theorem 3.59 and Theorem 3.61] or [8, 5.7.3 and 5.8]. From Theorems 6 and 7 we obtain

Corollary 1. *If E has finite perimeter, then the precise representative $\bar{\chi}_E$ of χ_E belongs to $BVC(\mathbb{R}^n)$ and, conversely, if $\chi_E \in BVC(\mathbb{R}^n)$, then E has finite perimeter.*

Next we use the results from the previous sections to express $|D\chi_E|(A)$ for sets $A \subset \mathbb{R}^n$ in terms of the AM -modulus.

Theorem 10. *Suppose that E has finite perimeter and $A \subset \mathbb{R}^n$ a Borel set. Then*

$$(27) \quad |D\chi_E|(A) = AM(\Gamma_{A \cap \partial_* E}^c)$$

and, in particular, $P(E) = AM(\Gamma_{\partial_ E}^c)$.*

Proof. By the structure theorem for sets of finite perimeter, see [3, Theorem 3.59] or [8, 5.7.3 Theorem 2], the reduced boundary $\partial^* E$ is $(n-1)$ -rectifiable and $|D\chi_E|(A) = \mathcal{H}^{n-1}(A \cap \partial^* E)$. Since $\partial^* E \subset \partial_* E$ and $\mathcal{H}^{n-1}(\partial_* E \setminus \partial^* E) = 0$, by [3, Theorem 3.61] or [8, 5.8 Lemma 1],

$|D\chi_E|(A) = \mathcal{H}^{n-1}(A \cap \partial_* E)$. Now the proof follows from Theorem 9. $\square$

Next we characterize the sets E of finite perimeter in terms of the path families $\Gamma_{\partial_* E}^c$ and $\Gamma_{\text{cross}}(E)$, where the latter family consists of all paths in $\mathbb{R}^n$ connecting $\text{int}_* E$ to $\text{ext}_* E$, i.e. each path in $\Gamma_{\text{cross}}(E)$ meets both $\text{int}_* E$ and $\text{ext}_* E$.

Let $E \subset \mathbb{R}^n$ be a set of finite measure. We start by showing that the condition $AM(\Gamma_{\text{cross}}(E)) < \infty$ implies that E has finite perimeter. Then we proceed to prove a sharp result in Theorem 11.

Lemma 8. *If $AM(\Gamma_{\text{cross}}(E)) < \infty$, then E has finite perimeter. Moreover, if (ρ_j) is an admissible sequence for $\Gamma_{\text{cross}}(E)$, then for each ball $B(z, r)$ and every $i \in \{1, \dots, n\}$*

$$(28) \quad |D_i \chi_E|(B(z, r)) \leq \liminf_j \int_{B(z, r)} \rho_j \, dx$$

and

$$(29) \quad |D\chi_E(B(z, r))| \leq \liminf_j \int_{B(z, r)} \rho_j \, dx.$$

Proof. Assume that χ_E is precisely represented. Fix a ball $B = B(z, r)$ and $i \in \{1, \dots, n\}$ and let Π_i denote the orthogonal projection of $\mathbb{R}^n$ onto the i^{th} -coordinate plane. Given $y \in \Pi_i(B)$, we let $\gamma_y: [0, \ell] \rightarrow \overline{B}$ denote the path which parametrizes the line segment L_y parallel to the x_i -axis, joins the opposite faces of ∂B and whose continuation passes through y . Since the set $\partial_* E$ has Lebesgue measure zero, the Fubini theorem yields that for a.e. $y \in \Pi_i(B)$, the set $L_y \cap \partial_* E$ has one-dimensional measure zero. Therefore, for any $\lambda < \tilde{V}(\chi_E, \gamma_y)$ there exists a partition $0 < t_0 < t_1 < \dots < t_m < \ell$ such that $m \geq \lambda$, all the points $\gamma_y(t_\alpha)$ belong to $\text{int}_* E \cup \text{ext}_* E$, and

$$|\chi_E(\gamma_y(t_\alpha)) - \chi_E(\gamma_y(t_{\alpha-1}))| \geq 1, \quad \alpha = 1, \dots, m.$$

Set

$$\gamma_{y, \alpha}(t) = \gamma_y(t - t_{\alpha-1}), \quad t \in [0, t_\alpha - t_{\alpha-1}], \quad \alpha = 1, \dots, m.$$

Then each $\gamma_{y, \alpha}$ belongs to $\Gamma_{\text{cross}}(E)$ and thus

$$\begin{aligned} 1 &\leq \liminf_j \int_{\gamma_{y, \alpha}} \rho_j \, ds, \quad \alpha = 1, \dots, m, \\ \lambda &\leq \sum_{\alpha=1}^m \liminf_j \int_{\gamma_{y, \alpha}} \rho_j \, ds \leq \liminf_j \int_{\gamma_y} \rho_j \, ds. \end{aligned}$$

Then letting $\lambda \rightarrow \tilde{V}(\chi_E, \gamma_y)$ we obtain

$$\tilde{V}(\chi_E, \gamma_y) \leq \liminf_j \int_{\gamma_y} \rho_j \, ds$$

and integrating with respect to y and using the Fatou lemma and the Fubini theorem we conclude that

$$\int_{\Pi_i(B)} \tilde{V}(\chi_E, \gamma_y) dy \leq \int_{\Pi_i(B)} \left(\liminf_j \int_{\gamma_y} \rho_j ds \right) dy \leq \liminf_j \int_B \rho_j dx.$$

Now from (12), (1) and the Fubini theorem we obtain (28). Note that $y \mapsto \tilde{V}(\chi_E, \gamma_y)$ is $\mathcal{L}^{n-1}$ -measurable, see [8, 5.10.2 Lemma 1].

Since (28) is true in every ball, we have

$$|D_i \chi_E|(\mathbb{R}^n) \leq \liminf_j \int_{\mathbb{R}^n} \rho_j dx$$

for every $i \in \{1, \dots, n\}$ and this implies that $P(E) = |D\chi_E|(\mathbb{R}^n) < \infty$.

To prove (29) note that there exists a unit vector $\mathbf{n}$ such that

$$|D\chi_E(B)| = D\chi_E(B) \cdot \mathbf{n}.$$

We may assume that $\mathbf{n} = \mathbf{e}_n$ and then $D\chi_E(B) \cdot \mathbf{n} = D_n \chi_E(B)$. Since $|D_n \chi_E(B)| \leq |D_n \chi_E|(B)$ the result follows from (29). $\square$

Theorem 11. *If $E \subset \mathbb{R}^n$ is a set of finite (Lebesgue) measure, then*

$$(30) \quad AM(\Gamma_{\partial_* E}^c) = P(E) = AM(\Gamma_{\text{cross}}(E)).$$

Proof. If $P(E) < \infty$, then Theorem 10 gives $P(E) = AM(\Gamma_{\partial_* E}^c)$. Suppose that $AM(\Gamma_{\partial_* E}^c) < \infty$. By Theorem 4, $\mathcal{G}_{n-1}(\partial_* E) < \infty$ and since the $(n-1)$ -dimensional integral geometric measure (with exponent 1) $\mathcal{I}_1^{n-1}$ satisfies $\mathcal{I}_1^{n-1} \leq \mathcal{G}_{n-1}$, see [9, 2.10.5–6], $\mathcal{I}_1^{n-1}(\partial_* E) < \infty$. Now, by [9, 4.5.11] it follows that E has finite perimeter (for a more elementary proof of this deep result see [7, Theorem 4.9]). Then Theorem 10 gives $AM(\Gamma_{\partial_* E}^c) = P(E)$.

It remains to consider the second equality in (30). We first show that

$$(31) \quad P(E) \leq AM(\Gamma_{\text{cross}}(E)).$$

We may assume that $AM(\Gamma_{\text{cross}}(E)) < \infty$. Then Lemma 8 yields that $\chi_E \in BV(\mathbb{R}^n)$ and we replace χ_E by its precise representative. Let (ρ_j) be an admissible sequence for $\Gamma_{\text{cross}}(E)$. Now we can proceed as in the proof of Theorem 7. Given $\varepsilon > 0$, using the Vitali-Besicovitch covering theorem [3, Theorem 2.19], the definition of reduced boundary, (26) and (29) we find a countable (possibly finite) pairwise disjoint collection $\{B(y_i, r_i)\}$ of balls such that

$$|D\chi_E|(\partial^* E \setminus \bigcup_i B(y_i, r_i)) = 0,$$

$$|D\chi_E|(B(y_i, r_i)) \leq (1 + \varepsilon) |D\chi_E(B(y_i, r_i))|,$$

and

$$|D\chi_E(B(y_i, r_i))| \leq \liminf_j \int_{B(y_i, r_i)} \rho_j dx.$$

Summing over i we obtain

$$P(E) \leq (1 + \varepsilon) \liminf_j \int_{\mathbb{R}^n} \rho_j dx$$

and since this holds for every $\varepsilon > 0$ and every admissible sequence (ρ_j) , (31) follows.

To prove the reverse inequality to (31) we may assume that $P(E) < \infty$. Now we can use the fact that χ_E is a BV function. By the approximation theorem for BV functions ([3, Theorem 3.9] or [8, 5.2.2 Theorem 2]) a sequence of BV functions $f_i \in C^1(\mathbb{R}^n, \mathbb{R})$ such that

$$\int_{\mathbb{R}^n} |\nabla f_i| dx \rightarrow P(E)$$

and $f_i(x) \rightarrow \chi_E(x)$ at every density point x of E and at every density point x of $\mathbb{R}^n \setminus E$ as $i \rightarrow \infty$. Now the sequence $(|\nabla f_i|)$ is an AM -admissible sequence for $\Gamma_{\text{cross}}(E)$ because for $\gamma \in \Gamma_{\text{cross}}(E)$ we find t_1 and t_2 such that $\gamma(t_1) \in \text{ext}_* E$ and $\gamma(t_2) \in \text{int}_* E$ and thus

$$1 = \lim_i |f_i(\gamma(t_2)) - f_i(\gamma(t_1))| \leq \liminf_i \int_{\gamma} |\nabla f_i| ds$$

and we obtain

$$AM(\Gamma_{\text{cross}}(E)) \leq \liminf_i \int_{\mathbb{R}^n} |\nabla f_i| dx = P(E).$$

□

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