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An alternative capacity in metric measure spaces

Olli Martio

Dedicated to Vladimir Gutlyanskii on his 80th birthday

Abstract. A new condenser capacity $\text{Cap}_p^M(E, G)$ is introduced as an alternative to the classical Dirichlet capacity in a metric measure space X . For $p > 1$, it coincides with the M_p -modulus of the curve family $\Gamma(E, G)$ joining ∂G to an arbitrary set $E \subset G$ and, for $p = 1$, it lies between $AM_1(\Gamma(E, G))$ and $M_1(\Gamma(E, G))$. Moreover, the $\text{Cap}_p^M(E, G)$ -capacity has good measure theoretic regularity properties with respect to the set E . The $\text{Cap}_p^M(E, G)$ -capacity uses Lipschitz functions and their upper gradients. The doubling property of the measure μ and Poincaré inequalities in X are not needed.

Keywords. Capacity, M - and AM -moduli.

1. Introduction

The Dirichlet p -capacity $\text{cap}_p(E, G)$ of a condenser (E, G) , developed by G. Choquet, is the most commonly used capacity in analysis. The modulus of a curve family offers an alternative approach to capacity. In a metric measure space X , curve families play a more central role than in $\mathbb{R}^n$ since the Fubini theorem is not available in X . For example, in X , the modulus method is used to construct the so-called Newtonian spaces, which have many properties common to the first order Sobolev spaces in $\mathbb{R}^n$. The constructions require that the metric space X should be a so-called good metric space, i.e. the measure μ in X be doubling and X support the Poincaré inequality in addition to various topological properties, see [2], [7] and [13].

The purpose of this paper is to introduce an alternative capacity, the Cap_p^M -capacity, which is directly connected to the M_p -modulus for $p > 1$ and uses neither the doubling property nor the Poincaré inequalities. For $p = 1$, the Cap_p^M -capacity lies between the AM_1 - and M_1 -moduli.

The Cap_p^M -capacity, $p > 1$, offers a more straightforward approach to the classical Dirichlet capacity cap_p , and their equivalence is considered in Section 4.

2. M_p - and AM_p -moduli

Let (X, d) be a metric space equipped by a Borel regular measure μ which is finite on compact sets. We also assume that X is proper, i.e. bounded closed sets are compact. From this, it follows that X is complete.

A continuous mapping $\gamma: [a, b] \rightarrow X$ is called a *curve*. We say that a curve γ is a *path* if it has a finite and non-zero total length; in this case, we parametrize γ by its arclength. The *locus* of γ is defined as $\gamma([0, \ell])$ and denoted by $\langle \gamma \rangle$, and the length of γ by $\ell(\gamma)$.

Let Γ be a family of paths in X . A non-negative Borel function ρ is *M-admissible*, or simply admissible, for Γ if

$$\int_{\gamma} \rho ds \geq 1$$

for every $\gamma \in \Gamma$. For $p \geq 1$, the M_p -modulus of Γ is defined as

$$M_p(\Gamma) = \inf \int_X \rho^p d\mu,$$

where the infimum is taken over all admissible functions ρ .

A sequence of non-negative Borel functions $\rho_i, i = 1, 2, \dots$, is *AM-admissible*, or simply admissible, for Γ if

$$\liminf_{i \rightarrow \infty} \int_{\gamma} \rho_i ds \geq 1 \quad (2.1)$$

for every $\gamma \in \Gamma$. The *approximation modulus* Γ is defined as

$$AM_p(\Gamma) = \inf_{(\rho_i)} \left\{ \liminf_{i \rightarrow \infty} \int_X \rho_i^p d\mu \right\}, \quad (2.2)$$

where the infimum is taken over all *AM*-admissible sequences (ρ_i) for Γ .

Since the space X is proper, instead of admissible Borel functions, it is possible to use lower semi-continuous non-negative functions as admissible for the M_p - and AM_p -moduli (see, e.g., [5, Proposition 7.14]).

For the following lemma, we refer to [8], [12] and [9] for the properties of the AM_p -modulus and to [2], [6] and [1] for those of the M_p -modulus, $p \geq 1$.

Lemma 2.1. *The AM_p - and M_p -moduli are outer measures in the set of all paths in X , i.e.*

- (a) $AM_p(\emptyset) = 0$,
 - (b) $\Gamma_1 \subset \Gamma_2 \implies AM_p(\Gamma_1) \leq AM_p(\Gamma_2)$,
 - (c) $\Gamma = \bigcup_{j=1}^{\infty} \Gamma_j \implies AM_p(\Gamma) \leq \sum_{j=1}^{\infty} AM_p(\Gamma_j)$,
 - (d) $AM_1(\Gamma) \leq M_1(\Gamma)$ and $AM_p(\Gamma) = M_p(\Gamma)$, $p > 1$, for every path family Γ .
- The properties (a)–(c) also hold for the M_p -modulus.*

We employ the following notation for path families associated with an arbitrary set $E \subset X$ and an open bounded set $G \supset E$:

$$\Gamma(E, G) = \Gamma(E) \cap \Gamma(X \setminus G) \text{ and } \Gamma(E) = \{\gamma : \gamma \text{ meets } E\}.$$

3. Cap_p^M -capacity

In this section, we assume that X is a proper metric space with a Borel regular measure μ and introduce a new capacity for the condenser (E, G) , where E is an arbitrary subset of a bounded open set G in X . Since a metric space usually has plenty of Lipschitz functions but need not contain many curves, such a capacity is not possible without an assumption that guarantees plenitude of curves, and we use the quasiconvexity property of X , i.e. there is $c < \infty$ such that for all $x, y \in X$, $x \neq y$, there exists a path γ joining x to y whose length satisfies $\ell(\gamma) \leq c d(x, y)$. Note that we do not need the quasiconvexity property for G but for X .

A complete doubling p -Poincaré space X is quasiconvex, see [2, Chapter 4], but the converse is not true as simple examples show.

We mostly work with Lipschitz functions in X . For such a function u , a non-negative Borel function g is an upper gradient of u in X if for every path γ in X ,

$$|u(\gamma(\ell)) - u(\gamma(0))| \leq \int_{\gamma} g ds$$

(see [2, Chapters 1–2] for the properties of functions and their upper gradients). The lower pointwise dilatation

$$|\nabla u(x)| = \liminf_{r \rightarrow 0} \sup_{y \in B(x,r)} \frac{|u(y) - u(x)|}{r}$$

is an upper gradient of u , see [2, Proposition 1.14]. In $\mathbb{R}^n$, $|\nabla u(x)|$ is a unique minimal upper gradient for a Lipschitz function u , see [2, Examples A1].

Let G be a fixed bounded open set in X , and E an arbitrary subset of G . An increasing sequence (u_i) of non-negative Lipschitz functions in X is called *admissible*, $(u_i) \in \text{Ad}(E, G)$, for the condenser (E, G) if $u_i = 0$ in $X \setminus G$, and

$$\liminf_{i \rightarrow \infty} u_i(x) \geq 1$$

for $x \in E$. For $p \geq 1$, we define

$$\text{Cap}_p^M(E, G) = \inf \left\{ \liminf_{i \rightarrow \infty} \int_G g_i^p d\mu : (u_i) \in \text{Ad}(E, G) \text{ and } g_i \text{ is an upper gradient of } u_i \right\}.$$

It is obvious that the Cap_p^M -capacity is monotone, i.e.

$$E_1 \subset E_2 \subset G \implies \text{Cap}_p^M(E_1, G) \leq \text{Cap}_p^M(E_2, G). \quad (3.1)$$

In the rest of this section, we assume that X is proper and quasiconvex, and $G \subset X$ is a bounded open set.

Theorem 3.1. *If E is an arbitrary subset of G , then, for $p \geq 1$,*

$$AM_p(\Gamma(E, G)) \leq \text{Cap}_p^M(E, G) \leq M_p(\Gamma(E; G)). \quad (3.2)$$

For $p > 1$, the above inequalities are equalities and, in particular,

$$\text{Cap}_p^M(E, G) = M_p(\Gamma(E; G)). \quad (3.3)$$

Proof. The first inequality $\leq$ in (3.2) is classical. For completeness, we recall the proof. Let (u_i) be an $\text{Ad}(E, G)$ -admissible sequence and g_i an upper gradient of u_i . Now (g_i) is an AM -admissible sequence for $\Gamma(E, G)$ because for each path $\gamma \in \Gamma(E, G)$ with $\gamma(\ell) \in E$,

$$\liminf_{i \rightarrow \infty} \int_\gamma g_i ds \geq \liminf_{i \rightarrow \infty} u_i(\gamma(\ell)) \geq 1.$$

Hence,

$$AM_p(\Gamma(E, G)) \leq \liminf_{i \rightarrow \infty} \int_G g_i^p d\mu$$

and since this holds for all sequences $(u_i) \in \text{Ad}(E, G)$ and all upper gradients g_i of u_i , the left-hand side of (3.2) follows.

For the second inequality in (3.2), we use a modification of the method in [2, Lemmata 5.25 and 5.26]. Let $\tilde{\rho}$ be a lower semicontinuous M -admissible function for $\Gamma(E, G)$. We may assume that $\tilde{\rho} = 0$ in $X \setminus G$ and

$$\int_G \tilde{\rho}^p d\mu < \infty.$$

Let $\tau > 0$ and set $\rho = \tilde{\rho} + \tau$ in X . Now ρ is lower semicontinuous in X and, since X is proper, there is an increasing sequence of continuous functions $\rho_i : X \rightarrow [0, \infty)$ such that

$$\lim_{i \rightarrow \infty} \rho_i(x) = \rho(x)$$

for every $x \in X$. We may assume that $\rho_i \geq \tau$ in X .

For each i , define

$$u_i(x) = \inf \left\{ \int_{\gamma} \rho_i ds : \gamma \text{ joins } X \setminus G \text{ to } x \right\}$$

for $x \in G$ and $u_i(x) = 0$ for $x \in X \setminus G$. Note that each path γ that meets $X \setminus G$ and $x \in G$ has a subpath meeting $X \setminus G$ at $\gamma(0)$ only. Hence, in the definition of $u_i(x)$, $x \in G$, we can consider only the paths γ that lie in G except at $\gamma(0)$. The sequence (u_i) is increasing and we show that each u_i is an $C_i c$ -Lipschitz function, where $C_i = \sup\{\rho_i(x) : x \in X\}$ and c is the quasigeodesic constant of X . Consider first the case where $x, y \in G$. By symmetry, we may assume $u_i(y) \geq u_i(x)$. Let $\varepsilon > 0$ and choose a path γ_{xy} joining x to y with $\ell(\gamma_{xy}) \leq c d(x, y)$. By the definition of $u_i(x)$, there is a path γ_x from $X \setminus G$ to x such that

$$u_i(x) > \int_{\gamma_x} \rho_i ds - \varepsilon.$$

Joining the paths γ_x and γ_{xy} together, we obtain a path γ from $X \setminus G$ to y . Now

$$u_i(y) - u_i(x) \leq \int_{\gamma} \rho_i ds - \int_{\gamma_x} \rho_i ds + \varepsilon \leq \int_{\gamma_{xy}} \rho_i ds + \varepsilon \leq C_i c d(x, y) + \varepsilon$$

and, letting $\varepsilon \rightarrow 0$, we obtain the required Lipschitz bound for u_i . If $x \in X \setminus G$ and $y \in G$, then $u_i(x) = 0$, and choosing γ_{xy} as before, we have

$$u_i(y) - u_i(x) = u_i(y) \leq \int_{\gamma_{xy}} \rho_i ds \leq C_i c d(x, y).$$

For $y, x \in X \setminus G$, the inequality is trivial.

The function ρ_i is an upper gradient of u_i . If γ is a path joining y and x which lie in G , then, by symmetry, we can assume that $u_i(y) \geq u_i(x)$, and for $\varepsilon > 0$ we can choose a path γ_x joining $X \setminus G$ to x such that

$$u_i(x) > \int_{\gamma_x} \rho_i ds - \varepsilon;$$

and joining the paths γ_x and γ together, we obtain the path $\tilde{\gamma}$ joining $X \setminus G$ to y . Thus

$$u_i(y) - u_i(x) \leq \int_{\tilde{\gamma}} \rho_i ds - \int_{\gamma_x} \rho_i ds + \varepsilon = \int_{\gamma} \rho_i ds + \varepsilon$$

and letting $\varepsilon \rightarrow 0$, we obtain the required inequality. If $y \in X \setminus G$, $x \in G$, and γ is a path joining y to x , then

$$u_i(y) - u_i(x) = u_i(y) \leq \int_{\gamma} \rho_i ds.$$

The case $x, y \in X \setminus G$ is again trivial.

Next let $\varepsilon > 0$ and

$$E_{\varepsilon} = \{x \in E : \lim_{i \rightarrow \infty} u_i(x) < 1 - \varepsilon\}.$$

Fix $x \in E_\varepsilon$. Then there is a sequence of paths γ_i from $X \setminus G$ to x such that for each i

$$1 - \varepsilon > u_i(x) \geq \int_{\gamma_i} \rho_i ds - 2^{-i} \varepsilon. \quad (3.4)$$

Now for $j \geq i$

$$u_i(x) \leq u_j(x) \leq \int_{\gamma_j} \rho_j ds.$$

Let

$$L = \liminf_{i \rightarrow \infty} \ell(\gamma_i)$$

and reparameterize the paths γ_i as $\tilde{\gamma}_i(t) = \gamma_i(t\ell(\gamma_i))$, $t \in [0, 1]$. Since we may assume that each γ_i meets $X \setminus G$ at $\gamma_i(0)$ only,

$$\ell(\gamma_i) \leq \int_{\gamma_i} \frac{\rho_i}{\tau} ds \leq \frac{1 - \varepsilon}{\tau} < \frac{1}{\tau}$$

and so the curves $\tilde{\gamma}_i$ are $\ell(\gamma_i)$ -Lipschitz and uniformly $1/\tau$ -Lipschitz and thus an equicontinuous family of mappings from $[0, 1]$ to the compact space $\overline{G}$. By the Ascoli theorem there is a subsequence of $(\tilde{\gamma}_i)$, denoted again by $(\tilde{\gamma}_i)$, which converges uniformly to a $1/\tau$ -Lipschitz curve $\tilde{\gamma} : [0, 1] \rightarrow \overline{G}$. Clearly, $\tilde{\gamma}(0) \in X \setminus G$ and $\tilde{\gamma}(1) = x$, and by the continuity of ρ_j in $\overline{G}$,

$$\lim_{i \rightarrow \infty} \rho_j(\tilde{\gamma}_i(t)) = \rho_j(\tilde{\gamma}(t))$$

for each j and $t \in [0, 1]$.

Next let γ be the reparametrization of $\tilde{\gamma}$ by arch length and note that

$$\ell(\gamma) \leq \liminf_{i \rightarrow \infty} \ell(\gamma_i) = L.$$

Denote the length function of $\tilde{\gamma}_i$ as

$$s_i(t) = \ell(\tilde{\gamma}_i|[0, t]), \quad t \in [0, 1]$$

and the length function of $\tilde{\gamma}$ as

$$s(t) = \ell(\tilde{\gamma}|[0, t]), \quad t \in [0, 1].$$

Now $s'_i(t) = \ell(\gamma_i)$ for $t \in (0, 1)$, and at the point $t_0 \in (0, 1)$ of the differentiability of s , we have for $t_0 < t_1 \leq 1$

$$\begin{aligned} \frac{s(t_1) - s(t_0)}{t_1 - t_0} &= \frac{\ell(\tilde{\gamma}|[t_0, t_1])}{t_1 - t_0} \leq \liminf_{i \rightarrow \infty} \frac{\ell(\tilde{\gamma}_i|[t_0, t_1])}{t_1 - t_0} \\ &= \frac{1}{t_1 - t_0} \liminf_{i \rightarrow \infty} \int_{t_0}^{t_1} s'_i(t) dt = \liminf_{i \rightarrow \infty} \frac{\ell(\gamma_i)(t_1 - t_0)}{t_1 - t_0} = L \end{aligned}$$

and hence $s'(t) \leq L$ for a.e. $t \in [0, 1]$. For $j \geq i$, we have

$$\int_{\tilde{\gamma}_j} \rho_j ds \geq \int_{\tilde{\gamma}_j} \rho_i ds = \int_0^1 \rho_i(\tilde{\gamma}_j(t)) s'_j(t) dt = \ell(\gamma_j) \int_0^1 \rho_i(\tilde{\gamma}_j(t)) dt. \quad (3.5)$$

Note that the function s is absolutely continuous because $\tilde{\gamma}$ is a Lipschitz curve. Now (3.4), (3.5), and the continuity of ρ_i yield for each i

$$1 - \varepsilon \geq \liminf_{j \rightarrow \infty} \int_{\tilde{\gamma}_j} \rho_j ds \geq L \int_0^1 \rho_i(\tilde{\gamma}(t)) dt \geq \int_0^1 \rho_i(\tilde{\gamma}(t)) s'(t) dt = \int_\gamma \rho_i ds.$$

This leads to a contradiction since, by the Lebesgue increasing convergence theorem,

$$\lim_{i \rightarrow \infty} \int_{\gamma} \rho_i ds = \int_{\gamma} \rho ds \geq \int_{\gamma} \tilde{\rho} ds \geq 1$$

for every path $\gamma \in \Gamma(E, G)$ and thus

$$\lim_{i \rightarrow \infty} u_i(x) \geq 1 - \varepsilon$$

for each $x \in E$.

Now $(u_i/(1 - \varepsilon))$ is an admissible sequence for $\text{Cap}_p^M(E, G)$ and, by the Lebesgue bounded convergence theorem,

$$\text{Cap}_p^M(E, G) \leq \liminf_{i \rightarrow \infty} \int_G \left(\frac{\rho_i}{1 - \varepsilon} \right)^p d\mu = \int_G \left(\frac{\tilde{\rho} + \tau}{1 - \varepsilon} \right)^p d\mu$$

and, since $\tilde{\rho}$ is an arbitrary M -admissible function for $\Gamma(E, G)$, letting $\varepsilon \rightarrow 0$ and $\tau \rightarrow 0$, we complete the proof for the right inequality of (3.2).

Because $AM_p(\Gamma(E, G)) = M_p(\Gamma(E, G))$ for $p > 1$, the equality case in (3.2) is clear. $\square$

Lemma 3.1. *If $E \subset G$ is an arbitrary set and $p \geq 1$, then there is a Borel set $E' \supset E$ such that*

$$\text{Cap}_p^M(E', G) = \text{Cap}_p^M(E, G).$$

Proof. If $\text{Cap}_p^M(E, G) = \infty$, we can choose $E' = G$. Then the monotonicity of Cap_p^M implies

$$\text{Cap}_p^M(G, G) \geq \text{Cap}_p^M(E, G) = \infty.$$

Suppose that $\text{Cap}_p^M(E, G) < \infty$ and for each $j \in \mathbb{N}$ choose an $Ad(E, G)$ admissible sequence (u_i^j) such that

$$\text{Cap}_p^M(E, G) \geq \liminf_{i \rightarrow \infty} \int_G (g_i^j)^p d\mu - 1/j \quad (3.6)$$

where g_i^j is an upper gradient of u_i^j .

Now the set

$$F^j = \{x \in G : \liminf_{i \rightarrow \infty} u_i^j(x) \geq 1\}$$

is a Borel set and $F^j \supset E$. The set $E' = \cap_j F^j$ is a Borel set and contains E so that $\text{Cap}_p(E', G) \geq \text{Cap}_p(E, G)$. Using (3.6), we obtain the converse inequality

$$\begin{aligned} \text{Cap}_p(E', G) &\leq \liminf_{j \rightarrow \infty} \liminf_{i \rightarrow \infty} \int_G (g_i^j)^p d\mu \\ &\leq \liminf_{j \rightarrow \infty} (\text{Cap}_p^M(E, G) + 1/j) = \text{Cap}_p^M(E, G). \end{aligned}$$

$\square$

From Theorem 3.1 and Lemma 3.1, we obtain:

Corollary 3.1. *If $p > 1$, then for each set $E \subset G$, there is a Borel set $E' \supset E$ such that $M_p(\Gamma(E, G)) = M_p(\Gamma(E', G))$.*

Since $\text{Cap}_p^M(E, G) = M_p(\Gamma(E, G))$ for $p > 1$, the $\text{Cap}_p^M(E, G)$ -capacity for all sets $E \subset G$ inherits all the properties of the M_p -modulus in a quasiconvex space X . For the properties of the Choquet capacity, see Section 4.

Theorem 3.2. *The Cap_p^M -capacity, $p > 1$, has the following properties:*

(a) Cap_p^M is subadditive. i.e. if $E_i \subset G$, $i = 1, 2, \dots$, then

$$\text{Cap}_p^M\left(\bigcup_i E_i, G\right) \leq \sum_i \text{Cap}_p^M(E_i, G).$$

(b) If $K_1 \supset K_2 \supset \dots$ are compact sets in G , then

$$\lim_{i \rightarrow \infty} \text{Cap}_p^M(K_i, G) = \text{Cap}_p^M\left(\bigcap_i K_i, G\right).$$

(c) Cap_p^M is a Choquet capacity, i.e. for a Suslin set $E \subset G$,

$$\text{Cap}_p^M(E, G) = \sup \{ \text{Cap}_p^M(K, G) : K \subset E \text{ compact} \}.$$

Proof. The subadditivity of the M_p -modulus is well known, see Lemma 2.1, and hence Theorem 3.1 implies (a). For (b), let $K = \bigcap_i K_i$ and note that by the monotonicity

$$\lim_{i \rightarrow \infty} \text{Cap}_p^M(K_i, G) \geq \text{Cap}_p^M(K, G).$$

For the reverse inequality, let $\varepsilon > 0$ and choose a sequence $(u_i) \in \text{Ad}(K, G)$ such that

$$\text{Cap}_p^M(K, G) \geq \liminf_{i \rightarrow \infty} \int_G g_i^p d\mu - \varepsilon. \quad (3.7)$$

The function $u = \lim_i u_i$ is lower semicontinuous in G as a limit of an increasing sequence of continuous functions u_i . Thus the set $U = \{x \in G : u(x) > 1 - \varepsilon\}$ is open and contains K . Now there is i_0 such that $K_i \subset U$ for $i \geq i_0$ and thus $(u_i/(1 - \varepsilon)) \in \text{Ad}(K_i, G]$ for $i \geq i_0$. By (3.7),

$$\lim_{i \rightarrow \infty} \text{Cap}_p^M(K_i, G) \leq \liminf_{i \rightarrow \infty} \int_G \left(\frac{g_i}{1 - \varepsilon}\right)^p d\mu \leq \frac{\text{Cap}_p^M(K, G) + \varepsilon}{(1 - \varepsilon)^p}$$

and letting $\varepsilon \rightarrow 0$, we obtain (b).

The map $E \mapsto \text{Cap}_p^M(E, G)$ is monotone and satisfies (a) and (b). Hence, by the Choquet capacity theorem, see [4], it satisfies (c). $\square$

4. Dirichlet capacity

In this section, we compare the Cap_p^M -capacity to the classical Dirichlet capacity cap_p and we first recall its definition and basic properties due to G. Choquet. Originally, this capacity used C_0^∞ -functions in $\mathbb{R}^n$ and their gradients, but the upper gradients for Lipschitz functions work as well in a metric measure space X , see [2, Section 6.3].

We again assume that X is a proper quasigeodesic space and $G \subset X$ is a fixed bounded open set.

Let K be a compact subset of G , and $\text{Ad}_G(K, G)$ the family of all Lipschitz functions such that $u \geq 1$ in K and $u = 0$ in $X \setminus G$. Define

$$\text{cap}_p(K, G) = \inf \left\{ \int_G g^p d\mu : u \in \text{Ad}_G(K, G), g \text{ an upper gradient of } u \right\},$$

Obviously, the infimum does not change if restricted to test functions satisfying $0 \leq u \leq 1$. The condition that a test function $u \in \text{Ad}_G(K, G)$ satisfies $u = 0$ in $X \setminus G$ can be replaced, due to the continuity of u , by the requirement that u has a compact support in G .

If $U \subset G$ is open, then we set

$$\text{cap}_p(U, G) = \sup\{\text{cap}_p(K, G) : K \subset U \text{ compact}\}$$

and, for an arbitrary set $E \subset G$,

$$\text{cap}_p(E, G) = \inf\{\text{cap}_p(U, G) : U \text{ open}, E \subset U \subset G\}.$$

Now there are two definitions for $\text{cap}_p(E, G)$ when E is compact, but since the competitors are continuous, the both definitions give the same value.

The cap_p -capacity, $p \geq 1$, has the following properties:

- (i) monotonicity: $E_1 \subset E_2 \implies \text{cap}_p(E_1, G) \leq \text{cap}_p(E_2, G)$;
- (ii) subadditivity: $E_1 \subset E_2 \subset \dots \implies \lim_i \text{cap}_p(E_i, G) = \text{cap}_p(\bigcup_i E_i)$;
- (iii) $K_1 \supset K_2 \supset \dots$ compact $\implies \lim_i \text{cap}_p(K_i, G) = \text{cap}_p(\bigcap_i K_i, G)$.

By the Choquet capacibility theorem, for all Suslin sets $E \subset G$,

$$\text{cap}_p(E, G) = \sup\{\text{cap}_p(K, G) : K \subset E \text{ compact}\}.$$

For the Choquet theory, see [4] and [3], and for the Dirichlet capacity in X , see [2, Section 6.3].

We frequently use the following lemma:

Lemma 4.1. *For $p \geq 1$ and $K \subset G$ compact, the equality*

$$\text{cap}_p(K, G) = M_p(\Gamma(K, G)) = AM_p(\Gamma(K, G)) = \text{Cap}_p^M(K, G) \quad (4.1)$$

holds.

Proof. For $p = 1$, by [11, Lemma 3.3],

$$\text{cap}_1(K, G) = M_1(\Gamma(K, G)) = AM_1(\Gamma(K, G))$$

and thus, by Theorem 3.1, this equality holds in (4.1). For $p > 1$, (4.1) is well known, see e.g., [2, Chapter 5] and [1]. Note that for $p > 1$, $M_p(\Gamma) = AM_p(\Gamma)$ for every path family Γ in X , see [11, Lemma 3.3]. $\square$

Lemma 4.2. *If $E \subset G$ is an arbitrary set and $p \geq 1$, then*

$$\text{Cap}_p^M(E, G) \geq \text{cap}_p(E, G). \quad (4.2)$$

Proof. Choose a Borel set $E' \supset E$ such that $\text{Cap}_p^M(E, G) = \text{Cap}_p^M(E', G)$. Then, by Lemma 4.1 and the Choquet capacibility theorem,

$$\begin{aligned} \text{Cap}_p^M(E, G) &= \text{Cap}_p^M(E', G) \geq \sup_{\{K \subset E' \text{ compact}\}} \text{Cap}_p^M(K, G) \\ &= \sup_{\{K \subset E' \text{ compact}\}} \text{cap}_p(K, G) = \text{cap}_p(E', G) \geq \text{cap}_p(E, G). \end{aligned}$$

$\square$

Lemma 4.3. *Suppose that E is an arbitrary subset of G . For $p > 1$,*

$$AM_p(\Gamma(E, G)) = M_p(\Gamma(E, G)) \leq \text{cap}_p(E, G), \quad (4.3)$$

and for $p = 1$,

$$AM_1(\Gamma(E, G)) \leq \text{cap}_1(E, G). \quad (4.4)$$

Proof. The first equality in (4.3) is due to the fact that $AM_p = M_p$ for $p > 1$. For the inequality in (4.3), let $U \subset G$ be an open set with $U \supset E$ and choose compact sets $K_1 \subset K_2 \subset \dots \subset U$ such that $\cup_i K_i = U$. Now $\Gamma(K_i, G) \subset \Gamma(K_{i+1}, G)$ and

$$\bigcup_i \Gamma(K_i, G) = \Gamma(U, G),$$

and, since $p > 1$, we have $M_p(\Gamma(U, G)) = \lim_{i \rightarrow \infty} M_p(\Gamma(K_i, G))$, see [1]. Then, from (4.1), it follows that

$$M_p(\Gamma(U, G)) = \lim_{i \rightarrow \infty} M_p(\Gamma(K_i, G)) = \lim_{i \rightarrow \infty} \text{cap}_p(K_i, G) = \text{cap}_p(U, G).$$

Since this holds for all open sets $U \supset E$,

$$\begin{aligned} M_p(\Gamma(E, G)) &\leq \inf_{G \supset U \supset E} M_p(\Gamma(U, G)) \\ &= \inf_{G \supset U \supset E} \text{cap}_p(\Gamma(U, G)) = \text{cap}_p(E, G). \end{aligned}$$

For (4.4), we can proceed as above but now, by [8, Lemma 3.11], we have

$$\text{cap}_1(U, G) = \lim_{i \rightarrow \infty} \text{cap}_1(K_i, G) = \lim_{i \rightarrow \infty} M_1(\Gamma(K_i, G)) \geq AM_1(\Gamma(U, G))$$

and hence, for every open set $U \supset E$,

$$\text{cap}_1(U, G) \geq AM_1(\Gamma(U, G)) \geq AM_1(\Gamma(E, G)).$$

□

The following summarizes the situation with Suslin sets for $p = 1$.

Lemma 4.4. *If $E \subset G$ is a Suslin set, then*

$$\text{cap}_1(E, G) = AM_1(\Gamma(E, G)) \leq \text{Cap}_1^M(E, G) \leq M_1(\Gamma(E, G)). \quad (4.5)$$

Proof. To prove the first equality in (4.5), it suffices to show, by (4.4), that

$$\text{cap}_1(E, G) \leq AM_1(\Gamma(E, G)). \quad (4.6)$$

Since E is a Suslin set, the Choquet capacity theorem yields

$$\text{cap}_1(E, G) = \sup \{ \text{cap}_1(K, G) : K \subset E \text{ compact} \},$$

and for each compact set $K \subset E$,

$$\text{cap}_1(K, G) = AM_1(\Gamma(K, G)) \leq AM_1(\Gamma(E, G))$$

and (4.6) follows. The rest follows from Theorem 3.1 and Lemma 4.2. □

The following theorem summarizes the situation for $p > 1$ and $p = 1$.

Theorem 4.1. *If X is a proper quasigeodesic metric space, $G \subset X$ a bounded open set, and $E \subset G$ an arbitrary set, then, for $p > 1$,*

$$AM_p(\Gamma(E, G)) = \text{cap}_p(E, G) = \text{Cap}_p^M(E, G) = M_p(\Gamma(E, G)), \quad (4.7)$$

and for $p = 1$,

$$AM_1(\Gamma(E, G)) \leq \text{cap}_1(E, G) \leq \text{Cap}_1^M(E, G) \leq M_1(\Gamma(E, G)). \quad (4.8)$$

Proof. Since $AM_p(\Gamma(E, G)) = M_p(\Gamma(E, G))$, the proof for (4.7) in the case $p > 1$ follows from Theorem 3.1, as well as Lemmata 4.2 and 4.3. The inequalities in (4.8) follow from Theorem 3.1, (4.4), and (4.2). $\square$

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