Education Quality in an Endogenous Growth Model with Congestion Costs and Public Infrastructure

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This thesis studies the implications of quality of education in the steady state of an endogenous growth model for optimal spending allocation decisions with educated labor, mandatory schooling, congestion costs and infrastructure spillovers. Education quality is characterized by the past knowledge taught to the students, and the degree of congestion in schools. Congestion costs are defined as the ratio of teachers to students in the population, and as the proportion of government spending on education to the teaching capacity given the public infrastructure. The transitional dynamics associated with an increase in the degree of congestion, and the spending share on education, are analyzed in the balanced-growth path. It is shown that a revenue-neutral spending reallocation has an ambiguous effect in the final consumption and human capital accumulation levels. A growth-maximizing share of government spending on education is required to avoid ambiguous results. This is shown to depend either on the production function parameters only or on a combination of these with education parameters and a negatively related congestion parameter, accordingly to the congestion cost considered. Implications for increasing the years of mandatory schooling are also discussed.
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List of Symbols

\( \alpha \) Output elasticity of public infrastructure and private capital ratio
\( \beta \) Output elasticity of effective educated labor
\( \gamma \) Degree of congestion of students
\( \epsilon \) Congestion parameter, knowledge acquired previously by students
\( \eta \) Elasticity of education at the balanced growth path with respect to the fraction of tax revenue spending in education
\( \theta_s \) Fraction of population in school (students)
\( \theta_r \) Fraction of population working as teachers
\( \kappa \) Educated labor elasticity of effective teaching capacity
\( \Lambda \) Admission policy variable ratio
\( \mu \) Educated labor elasticity of government spending on education
\( \pi \) Constant fraction of school equipment
\( \rho \) Discount rate
\( \varsigma \) Educated labor elasticity of the education quality
\( \tau \) Final output tax rate
\( \phi \) Congestion parameter, ratio of having a high proportion of students compared to the proportion of teachers
\( \omega \) Extent of externalities of the public infrastructure services
\( \Omega \) Balanced – growth path
\( a \) Admission policy constant ratio
\( C \) Consumption
\( E \) Stock of educated labor
\( \dot{E} \) Flow of educated labor
\( G_e \) Government spending on education
\( G_i \) Public infrastructure services

\( G_h \) Government spending on each service (education or infrastructure)

\( K \) Aggregate stock of private capital

\( \dot{K} \) Accumulation of capital

\( L \) Number of students

\( \dot{L} \) Flow of the number of students

\( N_j \) Number of workers

\( Q \) Quality factor

\( r \) Rate of return to capital

\( v_E \) Fraction of tax revenue spending in education

\( v_I \) Fraction of tax revenue spending in infrastructure

\( v_h \) Fraction of tax revenue spending on each service (education or infrastructure)

\( w_T \) Wage of the teachers

\( w_p \) Wage rate in the private sector

\( Y \) Aggregate output of the firms
1. Introduction

The impact of education on growth has been a subject of much attention in recent academic research and debates. The implications of human capital accumulation have been stressed as a determinant of growth in per capita income in several works since the seminal contribution of Lucas (1988). Many researchers have considered schooling variables such as enrolment ratios and average years of attainment as a measure in previous works. However, they only measure the quantity of schooling and not the quality. Many people might believe that one year of education in the countryside of a developing country is not the same as in the capital city of a developed country. Therefore, considering the quantity of schooling as a measure will not give an adequate comparison between countries.

Although the quality of schooling is not the same among countries and even within, it takes little analysis to see that education levels differ dramatically between developing and developed countries. The possibility to quantify education quality in several ways makes it usable for comparisons and analysis, Barro and Lee (2001) discuss the available cross-country aggregate measures of the quality of education. Such as pointed by Hanushek and Wößmann, (2007), ignoring quality differences significantly distorts the picture about the relationship between education and economic outcomes.¹

One indicator of schooling quality is the students’ international test scores. Hanushek and Kimko (2000) find that test scores are positively related to growth rates of real per capita GDP in cross-country regressions. Knighton and Bussière (2006) find that higher scores at age 15 in the PISA examination, lead to significantly higher rates of post-secondary schooling of Canadian 19 year olds.² Moreover, Hanushek and Kimko (2000) find clear evidence that international test performance relates to productivity differences.³ The combinations of these results indicate that the quality of schooling, in addition to the quantity is an important ingredient of human capital and economic performance.

¹ (Hanushek and Wößmann, 2007, 2)
² The OECD tested random samples of 15-year-old students across participating countries under the PISA program in 2000.
³ (Hanushek and Kimko, 2000, 1204)
Another indicator of quality of education is from the implications of congestion in schools. As many researchers have worked in these topics, it is of great importance to consider their results for the measurement of quality.

One of the most common congestion cost is teacher-student ratio in schools. A teacher-student ratio is expected to be negatively correlated with test scores because students can learn more rapidly by having more frequent interactions with teachers in smaller classes (Barro and Lee, 2001, 468). Angrist and Lavy (1999) also confirm class size effects by identifying significant differences in test scores between Israeli students who were subject to different class sizes by the exogenously given Maimonides’ rule.\(^4\) In spite of, Hoxby (1998), who using an instrumental variables approach based on natural population variations in Connecticut, found no significant effects of class size on performance. Nonetheless, it is commonly agreed that overcrowded classrooms affect the education in quantity and quality in schools.

Another congestion cost is infrastructure related, which has been studied in different works in the endogenous growth literature. Eicher and Turnovsky (2000) consider that the use of public capital is congested by the use of private capital. On a different approach, Albala-Bertrand and Mamatzakis (2004) found that in Chile, there is a positive effect of the public infrastructure on the private investment and therefore on growth. Brenneman and Kerf (2002) consider a positive impact of infrastructure services on educational attainment. A better transportation system, safer road network, greater access to safe water and sanitation, electricity are some of the impacts they consider can help to raise school attendance, study time and improve the learning process. Heynemen and Loxley (1983) show that school resources have much stronger effects on achievement in developing countries than in develop ones: in a sample of 29 countries.\(^5\) Hence, consider infrastructure as a congestion cost is important for the purpose of this work.

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\(^4\) According to Maimonides’ rule, a maximum of 40 students is allowed to enroll in a class. Therefore, when enrolment exceeds 40, an average class size drops sharply. Maimonides’ rule can thus cause exogenous variations in class size.

\(^5\) (Heynemen and Loxley, 1983, 1185)
These previous considerations and the importance of education quality for economic outcomes, turns the study to important policy issues. As Agénor (2009) mention, the production of human capital requires not only government spending on education services but infrastructure capital as well. Therefore, the government faces a decision problem since the determination of the optimal expenditure ratios is marked by the fiscal structure, which represents a critical issue. Is it reasonable to believe that spending a higher amount in education services will increase the economic growth? Or is it better to spend more in infrastructure? Acemoglu and Angrist (2001) mentioned that the evidence that the returns to education have implications for both economic policy and economic theory, in the private returns to education is on the order of 6-10 percent. Hence, if education represents an important component of the economic theory and likewise quality is relevant in education, it is essential to study some of its implications.

These thesis focuses on the quality of education as all of the previously mentioned studies suggest; quality may have a greater impact than the years of schooling on economic growth. The endogenous growth model used for this thesis is taken from Agénor (2012), which originally was based on the Uzawa (1968) and Lucas (1988) model. The basic structure is extended in many ways, at first by a former paper from Agénor (2008), which considers that the endowed raw labor must be educated instead of a simply stock of human capital. Based on Hanushek and Kimko (2000), the labor force quality has a consistent, stable, and strong relationship with economic growth and knowledge is thus embodied in educated workers. The main reason for this modification is that as quality is relevant for growth, the raw labor will exist even if no quality in education is present in the schooling process. Hence, this model takes into account that quality creates educated labor force, which is the main driver of the economy.

A second incorporation of the model in Agénor (2008) is a consequence of the previous consideration. As the creation of human capital requires being educated, that implies that workers must go through the education system. Therefore, the education technology

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6 (Hanushek and Wößmann, 2007, 3)
7 (Agénor, 2009, 126)
8 (Acemoglu and Angrist, 2001, 22)
9 (Hanushek and Kimko, 2000, 1203)
includes the number of students in it, contrasting with the basic model in Agénor (2012) that does not have it. Derived from this condition, it must be present an admission policy imposed by the government, which is also added to the basic model. The admission policy is considered being constant as most countries have, but further in the thesis, it is variable consequently from increasing the years of mandatory schooling. These modifications change the conditions of a stock of human capital into a flow of educated labor in the economy.

One essential difference in Agénor’s (2012) model is regarding the interpretation of the congestion costs and the presence of them. He considers both congestions cost relevant but only applying one at a time in any economy (mostly developing countries), I compare them as a close representation of developed and developing countries. While the developed countries have infrastructure services well provided, the developing countries do not and face the same conditions adding the infrastructure congestion cost to the schooling congestion. The main purpose of Agénor (2012) is to obtain the optimal spending share allocations; this thesis compares the optimal levels and the effects on the growth with each approach in addition to the former’s objective.

Finally, a new congestion parameter is introduced to the model which is not considered in any of the based references. This is defined as the past knowledge that the students have in average within a class and that teachers require to enhance the learning process. This knowledge is acquired from the previous teachers, students, exams and education quality in general, denoting students who attended an education of good quality will possess more knowledge than the ones who did not. This parameter represents a value easy to obtain which may measure quality, since it can be taken as an approximation from the international test scores. The consideration of this parameter enables to compare the quality of education between countries in the same conditions.

The remainder of the thesis is organized as follows. Chapter two presents the basic framework, in which the production, household preferences, human capital and government conditions are given in detail. The third chapter discusses the balanced-growth equilibrium and the dynamic properties of the model. Chapter four studies the dynamics of the balanced growth path by considering an increase in the schooling congestion. The following chapter examines the implications of a revenue-neutral shift
in spending in education, whereas section six provides the optimal values of a growth-maximizing spending allocation of public expenditure between education and infrastructure. Chapter seven presents a brief analysis of the implication of the new parameter added to the model. Chapter eight presents a modification to the model when the government increases the mandatory years of education and the implications in the growth path and in the government spending share allocation. The final part of the thesis summarizes the main results, discusses the implications of the analysis and offers some suggestions for further research.
2. The Basic Framework

Consider an economy populated by a single, infinitely-lived household and a continuum of identical firms, which are normalized to unity for simplicity $i \in (0,1)$. The economy includes a productive sector, preferences of the household, government that provides services such as infrastructure and education. Production sector requires the use of human and physical capital and also public infrastructure services. The services that are required by each firm consist of roads, electricity, communication services, sewers, and water systems among others. The household consumes only one single good, which can be traded or used in consumption or investment. The government is responsible for providing public services to population free of charge, such as school equipment, school supplies, sanitation services, electricity, and so on. The school equipment consists of computers, microscopes, laboratories, screen projectors, tables, chairs while the school supplies considers books, pencils, pens and teacher’s material.

The population as the firms is of a constant size and is normalized to unity for simplicity. Schooling is mandatory in the economy but just in primary and secondary school, so higher education is not included in the category. However education can be received at any moment in time at a constant fraction of $\theta_s \in (0,1)$, which represents the populations of students. The rest of the population is divided by a constant fraction of teachers $\theta_r \in (0,1)$, who are paid by the government, and laborers paid by the private sector $(1-\theta_s-\theta_r) \in (0,1)$. These two educations’ sectors have a wage arbitrage condition that determines the allocation of each one among the non-studying population. Human capital accumulation is described as the educated labor that possesses enough knowledge for basic work or study in the next level. Besides this condition the human capital accumulation depends on the proportion of students and teachers, government expenditure and a quality indicator that is responsible for the knowledge acquired by students. For simplicity students and workers are considered as full time in their own activity, so there is no possibility of being a student and a laborer (teacher or private worker) at the same time.

The economy has some congestion costs which can be understand as the limitations or negative effects in the use of a public good or service. Public infrastructure faces a sort
of congestion when one firm partially excludes another firm from the use of it making it partially rival. There is also congestion in schools, which affects negatively the quality of the educated labor created in the education system. With these conditions both public infrastructure and education are non-excludable and partially rival.

2.1 Production

The output of each representative firm, \( Y_i \), in the economy is produced with public infrastructure services, \( G_i \), and the two basic inputs, human and private physical capital, \( K_i \). The public infrastructure is affected by the congestion in proportion to the stock of private capital \( \bar{K} \). Moreover, the physical capital is referred as effective labor that can be obtained when multiplying the number of workers, \( N_i \), in each firm by the stock of educated labor, \( E \), in the economy. The production function exhibits a constant return to scale in some of the firm inputs, in this case effective labor and private capital

\[
Y_i = \left( \frac{G_i}{K} \right)^{\alpha}(N_i E)^{\beta} K_i^{1-\beta}
\]  

(1)

where \( \alpha, \beta \in (0,1) \) and \( \bar{K} = \int_0^1 K_i \, di \) represents the aggregate stock of private capital, considering that all firms are equal in the economy. Because all the firms have access to the same stock of educated labor and public infrastructure they are all affected in the same way. As mentioned before there is congestion related with the use of public infrastructure which is measured by the aggregate capital stock.

Assuming that the markets are competitive, the public infrastructure and the aggregate stock of private capital are given to all the firms, each firm has the objective of maximize its own profit. As a result of the firms maximization, they have to pay the private inputs at their marginal rate, \( w_p = \beta Y_i / N_i E \), and \( r = (1-\beta) Y_i / K_i \), where, \( w_p \) represents the wage rate in the private sector and \( r \) the rate of return to capital.

Noting that all the firms and population in the economy are equal and normalize to unity, the aggregate stock of capital can be considered as \( \int_0^1 K_i \, di = \bar{K} = K \) \( \forall i \). In a similar way the number of laborers can be aggregated into a common variable, \( N \), resulting as the remaining proportion of the population which is not studying and
neither working as teacher, in the form \( \int_{0}^{1} N_i di = N = (1 - \theta_s - \theta_t) \). Considering these previous modifications the profit maximization conditions result

\[
\begin{align*}
  w_p &= \beta \frac{Y}{(1 - \theta_s - \theta_t)E}, \\
  r &= (1 - \beta) \frac{Y}{K}.
\end{align*}
\]  

(2)

Given these conditions now the aggregated effective labor is \((1 - \theta_s - \theta_t)E\) which requires that the proportion of students and teachers do not represent the whole population. Therefore the following condition has to be established in order to avoid a corner solution with no workers in the economy:

**Assumption 1.** \( \theta_s + \theta_t < 1 \).

Finally the aggregate output will have an \( AK \) technology with respect on the proportion of workers, the congestion in the public infrastructure, \( E/K \) and private capital stock. This result if the first three elements mentioned before are constant in the following equation

\[
Y = \int_{0}^{1} Y_i di = \left( \frac{G_i}{K} \right)^{\alpha} \left[ \frac{(1 - \theta_s - \theta_t)E}{K} \right]^{\beta} K.
\]  

(3)

### 2.2 Household Preferences

The representative infinite-lived household faces a future utility maximization problem based only in the consumption \( C \) and the discount rate \( \rho > 0 \). Consumption is considered in a logarithmic form, in which case the income and substitution effects cancel out, and also that the household’s propensity to save or invest are independent of the rate of return on capital as in Agénor (2012). For this model leisure is not included so the household do not faces a negative utility between the decisions of being productive or working and spending time in leisure. Another specification is that the utility does not include any positive effect with the acquisition of education. Therefore the discounted stream of future utility is given by

\[
\max_{\tilde{c}} V = \int_{0}^{\infty} \ln C \exp(-\rho t) dt.
\]  

(4)

The household budget constraint will consist of the spending in consumption and the accumulation of capital that for simplicity does not depreciate over the time. Because as
mentioned before free education is provided by the government equal to the use of public infrastructure which has no user fees, the household benefits from an implicit rent from the public expenditure. The household budget constraint is

\[ C + \dot{K} = (1 - \tau)Y + w_t \theta_t E \]  

(5)

where \( \tau \in (0,1) \) is a final output tax rate and \( w_t \) represents the wage of the teachers which is not taxed to keep an arbitrage condition between the sectors.

To get the flow of consumption over time it is necessary to maximize the household utility function (4) subject to the budget constraint (5) taking into account the aggregate output (3) as shown in the Appendix A

\[ \frac{\dot{C}}{C} = s \left( \frac{G_i}{K} \right)^{\alpha} \left[ \frac{(1-\theta_s - \theta_t)E}{K} \right]^{\beta} - \rho \]  

(6)

where \( s = (1 - \tau)(1 - \alpha - \beta) \) and considering the salary, proportion of teachers and educated labor as given will yield the Euler equation stated above with the transversality condition \( \lim_{t \to \infty} (K/C) \exp(-\rho t) = 0 \).

2.3 Human Capital Accumulation

The human capital accumulation will be described as the flow of educated labor, \( \dot{E} \), in the economy. Raw labor grows at a constant rate but it must be educated before it can be used in the production market. As mentioned before, the model analyses the effects of the congestion costs over the growth rate in the quality variable.

The congestion cost associated with the proportion of students and teachers in a school is considered as the first approach. The second approach is defined as a particular case of the first approach. In this case the congestion cost affecting the infrastructure and materials available to teach are added to the congestion in schools parameter. Moreover, this work considers a third congestion cost which affects both of the previous, so it can be integrated in both of the previous. It can be explain as the quality of the education received by the students that then transforms into knowledge. Therefore, if the quality of the education is good enough the students will have more knowledge making it easier for the teachers to give lectures and increase the abilities of the students.
For the first approach, the production of educated labor requires the combination of a quality factor $Q$, the gross proportion of students $(1 + \theta_s)$, the effective teaching capacity $(1 + \theta_T)E$, the government expenditure in education $G_E$, and the number of students $L$.

Then the flow of educated labor in the economy will be

$$\dot{E} = Q^\zeta (1 + \theta_s) [(1 + \theta_T)E]^\kappa G_E^\mu L^{1-\kappa-\mu}$$

(7)

where $\zeta \in (0,1)$ making the quality of education having decreasing marginal returns to scale and $\kappa, \mu \in (0,1)$. Thus, the education technology in this case displays constant returns to scale with the effective teaching capacity, the government expenditure in education and the number of students. It is important to indicate that in (7) the effective teaching capacity is used instead of the number or the proportion of teachers because this can capture more the essence of the teaching knowledge. As knowledge requires good conditions, it also depends on the fact that the teachers belong to the educated labor. As Goldhaber and Brewer (1999) mentioned teachers who are certified in mathematics, and those with bachelor’s or master’s degree, are identified with high test scores, so that these variables influence student achievement.\footnote{Goldhaber and Brewer, 1999, 520}

The same way as in Agénor (2008), it is assumed that $L$ is large enough at all times to ensure that the flow of educated laborers is not greater than the number of students, resulting in the following condition

$$\dot{E} \leq L.$$  

(8)

Additionally, the quality term has an inverse relationship with the degree of congestion in the education system, affecting directly the capability of the economy to increase the flow of educated labor. In this case the quality term is measured as a rate of the gross proportion of teachers divided by the gross proportion of students, as follows:

$$Q = \frac{(1 + \phi_T)^e}{(1 + \phi_S)^\phi}$$

(9)

\footnote{The use of the gross proportion of teachers $1 + \phi_T$ and students $1 + \phi_S$ ensures that even with $\phi_T$ and $\phi_S$ close to zero, the growth will still be positive.}
where, $\varepsilon$ and $\phi > 0$ represents the congestion parameters. The first one represents the past knowledge acquired by students while the second term shows the ratio of having a high proportion of students compared to the proportion of teachers. Hence, the quality will depend on the proportion of students relative to the proportion of teachers, abating or increasing the flow of educated labor. Additionally, the gross proportion of teachers having the other congestion parameter can be explained as, if in average the students have the basic knowledge needed to be in that level, then the teacher can focus in teaching the topics established without the need of reviewing basic subjects before or delaying the class in explaining to the students who do not know it. As depreciation of knowledge is not considered and every student possesses a small amount of basic knowledge the parameter must satisfy the following condition:

$$0 < \varepsilon \leq 1$$

(10)

where having $\varepsilon = 1$ will represent that in average all the students have the basic knowledge require by the teacher to be effective. Therefore, the quality of the education will be present in all levels of the schooling making it a congestion cost additional to the first parameter. As the impact of the first congestion cost is measured by $\phi$; the only possible way that there will be a proportional congestion can be obtained if $\varepsilon = \phi = 1$. In this case the quality can be explained purely as the ratio of the proportion of students and the proportion of teachers or in other words the class size. As Tamura (2001), and so many literature considered a common indicator of efficiency in the education systems, where class sizes determinate the quality of the education.

Combining the previous definition of quality (9) with the flow of educated labor (7) yields, and after rearranging,

$$\dot{E} = (1 + \theta_T)^{\varepsilon} (1 + \theta_S)^{1-\varepsilon} \left( \frac{G_k}{L} \right)^{1-\kappa} \left( \frac{E}{L} \right) \left( \frac{G_k}{L} \right)^{\mu} L.$$

(11)

For this first estimation, the model considers as in Agénor (2008) that the admission policy for students remains equal, trying to keep the student-teacher ratio as a constant from previous periods. Hence, the number of students has to be divided by the effective teaching capacity, such as,
where $a > 0$ for obvious reasons. For this condition to be satisfied, requires that once located in the balanced growth path where the educated labor grows at a constant rate, the number of students must grow at the same level. Furthermore, to ensure this will imply that the population in the economy, considering total raw population, must grow at the same level of the educated labor in the steady state. In optimal conditions this will mean that all the population is accepted and therefore transformed into educated labor. However, the persons who are not accepted in the education system or drop it will then turn into a non-market activity which is not consider in this model.

Using the government admission policy (11), is possible to rewrite the latest flow of educated labor (12) as

$$
\frac{L}{(1+\theta_t)E} = a
$$

(12)

Thus, (13) represents the growth rate of the stock of educated labor which depends on the constant rate of admission policy, the proportion of teachers, the proportion of students and the public spending on education. As exhibit in the equation, there is still an assumption to be made to avoid getting a negative growth rate in the gross share of students. If the congestion $\phi$ is too high and the parameter $\zeta$ close to 1 or large enough also, then the net effect of the growth rate might be negative. This will mean that students are not productive in this education technology, which cannot give any logical result in the analysis. Therefore, the following assumption is established:

**Assumption 2.** $1 - \phi \zeta > 0$.

This assumption ensures that the gross proportion of students can be productive in the model. The proportion of teachers, $(1+\theta_r)$, do not require any special assumption because for all cases $1 + \alpha \zeta - \mu > 0$ making both, teachers and students, productive in the growth rate of the educated labor.
As mentioned before, the model considers another congestion cost where public infrastructure, school supplies and equipment may affect the education technology. This will also change the ability of the economy to produce educated labor. The explanation of this idea is that as far as the knowledge is obtained by the students, the requirements of better technology or infrastructure are needed. This will make the productivity of the teachers greater than without it. If the government spends enough money in providing good infrastructure and equipment then the teacher will face fewer constraints in their lectures and the quality of the education will increase. Hence, if these conditions are optimal the students must have enough knowledge and the teacher can be productive. As identified by Glewwe and Kremer (2006), the lack of adequate school equipment in schools has an important role in improving or constraining the education quality.

To capture this consideration in the model, the human capital accumulation equation must include the impact of infrastructure services, \( G_f \), in the student population, \( L \). This will represent a special case of (7) since the first approach considers this impact equal to 1. As mentioned before, these services are partially rival, so their use decreases with an increase in the flow of the number of students \( \dot{L} \)

\[
\dot{E} = Q^\gamma (1 + \theta_\gamma) G_f^\mu \left(1 + \theta_f E\right)^\kappa \frac{G_f}{L} L^{1-\mu-\kappa-\omega} 
\]

(14)

where \( \gamma \geq 0 \) measures the degree of congestion of students and \( \omega > 0 \) measures the extent of externalities of the public infrastructure services. To understand better the idea, if too many students require the computers at the same time the use of the equipment will be limited and this will reduce the public service usefulness. Another example can be referred as too many students using the internet services will slow the access of the remaining students making it more difficult to use all its advantages. Even in poor conditions if students do not have individual tables or chairs, they will distract themselves by sharing any of these. The equation (14) exhibits constant returns to scale in government spending in education, effective teaching capacity, infrastructure services and the number of students. It is possible to notice that equation (7) can be obtained if \( \omega = 0 \). This is equivalently as considering that equation (7) fits more into a developed country environment, when infrastructure services are well provided within the population. Then equation (14) represents more developing countries conditions.
For this case, the quality term also changes to capture the idea so now the school equipment represents a constant fraction $\pi \in (0,1)$ of the total spending on education which is divided by the effective teaching capacity. Moreover, the proportion of teachers faces also the constraint of the knowledge that the students have $0 < \varepsilon \leq 1$ with the same congestion cost $\phi > 0$ as before. The quality indicator will be defined as

$$Q = \frac{\pi G_E}{(1 + \theta_T)^\xi E} \cdot (15)$$

Substituting (15) in (14) and after rearranging yields,

$$\dot{E} = \pi \xi (1 + \theta_S) \left( \frac{G_E}{E} \right)^{\mu + \xi} \left(1 + \theta_T\right)^{\xi - \phi \varepsilon} \left( \frac{E}{L} \right)^{\kappa} E^{-\xi} \left( \frac{G_L}{L} \right)^{\alpha} L^{1+\xi} \cdot (16)$$

and considering (8) is possible to take into account that $\dot{E} = \dot{L}$. If this holds, then the growth of students must be equal the growth of educated labor at the steady state. Thus, combining this condition and using the same admission policy (12) into (16), after some manipulations equation (16) will be replace by

$$\frac{\dot{E}}{E} = \left( a^{1-\mu-\kappa-\alpha} \pi \xi (1 + \theta_S)(1 + \theta_T)^{1-\mu-\alpha-\phi \varepsilon} \left( \frac{G_E}{E} \right)^{\mu + \xi} \left( \frac{G_L}{E} \right)^{\alpha} E^{\xi(1-\phi-e \varepsilon \zeta)} \right) \cdot (17)$$

This equation, represents the growth rate of the stock of educated labor with the new conditions. It also needs the following assumptions to be imposed in order to avoid adverse effects:

**Assumption 3.** $1 - \mu - \omega - e \phi \zeta > 0$.

This assumption is in order to make the teachers productive, contrary as in the first approach now the proportion of teachers are facing a constraint in there growth. Hence, to avoid getting a negative effect from the productivity of the teachers **Assumption 3** has to be considered.

**Assumption 4.** $\zeta (1 - \phi) - \gamma \omega = 0$
Equivalently to (8) this assumption ensures that the stock of educated labor grows at a constant rate in the steady state. This condition will imply that \( \omega = \zeta (1 - \phi) / \gamma \), so that \( 0 < \phi < 1 \) to keep the value of \( \omega > 0 \). This shows that there is a negative relationship between these two parameters. Intuitively this condition says that the stronger of the congestion cost existing in the stock of educated labor on the production of knowledge, the less effect of the infrastructure services to generate a higher growth at the steady state, also adjusted by the congestion in the flow of the number of students. Different from the first approach, it was possible to get a proportional congestion between \( \phi \) and \( \epsilon \), in this case it can only happen if \( \phi = \epsilon \neq 1 \).

2.4 Government

The government as stated before is responsible for providing public infrastructure and education services. Additionally, the government has to pay the teacher’s salaries, \( w_r \), to the proportion of the educated population who works in the schooling system. Assuming that the government cannot issue debt and must maintain a balanced budget from collecting a proportional tax rate \( \tau \in (0,1) \) on the market output, the government budget constraint is given by\(^\text{12}\)

\[
w_r \theta_r E + G_e + G_i = \tau Y. \tag{18}
\]

The entire population can decide either to work in the private market or in the education system as teachers. Card and Krueger (1992) mentioned that a teacher’s salary and education level would be indicators of a teacher’s quality. Higher salaries attract more qualified and productive teachers who can contribute more effectively to students’ achievement. Therefore, it has to exist an arbitrage condition which allow them to work either in one of the sectors without any disadvantage or disutility associated, resulting as,

\[
w_r = (1 - \tau)w_p \tag{19}
\]

which implies that the teacher’s salaries absent of taxation must be equal to the taxed wage of the private workers. Government expenditure on infrastructure and education services represent a fraction of the tax revenue

\(^{12}\) Substituting (5) in (18) is possible to get the standard GDP identity in the form \( C + \dot{K} + G_e = Y \)
where $v_h \in (0,1)$ and $h = E, I$. As derived in Appendix A, using the government expenditure on each service (20) and combining the wage rate in private market (2) into the arbitrage condition (19) to then put both into the government budget constraint (18) will result to the equilibrium proportion of teachers in the model, as:

$$
\theta_t = \frac{\tau \beta^{-1}[1 - (v_E + v_I)]}{1 - \tau + \tau \beta^{-1}[1 - (v_E + v_I)]}(1 - \theta_s) \in (0,1)
$$

which exhibits dependency on the expenditure terms of the government as from the fraction of students. This brings a logical interpretation as considering $v_E + v_I < 1$ because otherwise the government must issue debt to pay the teacher salaries and that case is not considered in the model. Moreover it keeps considering Assumption 1 to hold $\theta_s < 1$ guaranteeing that there exist a fraction of teachers in the economy. Hence, the next proposition is established:

**Proposition 1.** A reallocation of the expenditure of the government spending between public infrastructure and education has no implications in the proportions of teachers, as long as it is a revenue neutral reallocation ($dv_E + dv_I = 0$).
3. The Balanced – Growth Path

The full derivation of the balanced-growth path (BGP) for this approach is shown in the Appendix B. Thus, the BGP can be defined as a set of conditions \( \{c,e\}_{t=0}^{\infty} \) such that the dynamic equations in (6), (B14), (13) with the transversality condition \( \lim_{t \to \infty} c^{-1} \exp(-\rho t) = 0 \) are satisfied. Also, it is required that the stock of educated labor, consumption and physical capital grow at the same constant rate as \( \Omega = \dot{c}/c = \dot{E}/E = \dot{K}/K \). Hence, the transversality condition is satisfied locally because consumption, educated labor and the stock of private capital grow at the same constant rate implying that \( c = C/K \) and \( e = E/K \). Therefore, the rate of growth of output can be shown into a system of two non-linear equations which characterize the equilibrium in the model as,

\[
\Omega = sB(v_{1}v_{\tau})^{\alpha/\alpha-1} \bar{e}^{1-\alpha} - \rho \tag{22}
\]

\[
\Omega = B^{\mu}D\left(v_{1}v_{\tau}(v_{1}v_{\tau})^{\alpha/\alpha-1}/\bar{e}^{1-\alpha}\right) \tilde{e}^{-\tilde{e}} \tag{23}
\]

where

\[
B \equiv (1-\theta_{S}-\theta_{T})^{\beta/(1-\alpha)}
\]

\[
D \equiv a^{1-\chi-\mu}(1+\theta_{T})^{1+\theta_{S}^{\omega}-\mu}(1+\theta_{S})^{1-\phi_{S}}
\]

\[
\chi \equiv (1-\alpha-\beta)\mu/(1-\alpha)
\]

and \( \tilde{e} \) represents the steady state value in that variable. As can be seen the proportion of workers among the schooling system or the private market affects the growth rate in the model as located in B and D. Moreover, the parameter, \( \epsilon \), affects also the growth rate of the model in the first approach.

For the ongoing analysis of the BGP, the system is linearized and the following chapters it is shown the dynamics after some manipulations to the model. Each of the approaches will be considered in the analysis to illustrate how each of them reacts. For this purpose a phase diagram is used where the phase line \( CC \) from equation (22) represents the combination of \( c \) and \( e \) when the consumption and the stock of physical capital ratio are constant at \( c = 0 \).
The phase line $EE$ from equation (23) represents the combination of $c$ and $e$ when the stock of educated labor and the stock of physical capital ratio are constant at $\dot{e} = 0$ for the BGP of the first approach. The saddlepath has a positive slope that is denoted by $SS$ and this phase line requires that the slope of $EE$ must be steeper than $CC$ at the point where they intersect, as in Figure 1.

The BGP for the second approach considers also equations (6), (B14), (17), and the same transversality condition as before. In order to get a steady state with a stable equilibrium it is also necessary to take into account Assumption 4. Then after some manipulations it is possible to rewrite the stock of educated labor in (17) to the following equation:

$$\frac{\dot{E}}{E} = J \left( \frac{G_E}{K} \right)^{\mu+\gamma} \left( \frac{G_I}{K} \right)^{\phi (1-\phi)} \left( 1 - \frac{1}{1-\gamma (1-\phi)} \right)$$

(24)

where

$$J = \alpha^{(1-\mu-\kappa)} \gamma^{\phi (1-\phi)} \pi^{\theta_S} (1+\theta_S) (1+\theta_F)^{(1-\mu-\kappa)\gamma (1-\phi)}$$

Fig 1. The steady-growth equilibrium with the student-teacher congestion cost
\[ \Gamma = (\mu + \xi)^r + \zeta(1 - \phi). \]

This equation boils down to a geometric transformation including new variables such as the admission policy rate value, the fraction of spending in public infrastructure and the congestion in public infrastructure, compared to the previous approach. Thus, solving for the BGP with this second approach (see the Appendix C), the steady state is given again by (22) and instead of (23),

\[
\Omega = sB(v_i^r) \left( \frac{\alpha}{1-a} \right)^{\frac{\beta}{1-a}} - \rho 
\]

\[ \Omega = \left( Mv_i^r \right) \left( \frac{\alpha}{1-a} \right)^{\frac{\beta}{1-a}} - \rho \quad (22) \]

\[
M = J^{|\mu + \xi| r + \zeta(1 - \phi)}.
\]

where

Again the allocation of workers between the sectors affects the growth rate. In this case the variable \( J \) contains the gross proportion of teachers with different conditions than before so the impact of each will vary.

The same methodology as before will be used. This implies that the phase line \( CC \) from equation (22) as previously represents the combination of \( c \) and \( e \) when the consumption and the stock of physical capital ratio are constant at \( c = 0 \). However, because the schooling quality is defined differently, the curve \( EE \) obtained from equation (25) will now represent the combination of \( c \) and \( e \) when the stock of educated labor and the stock of physical capital ratio are constant at \( e = 0 \).

In this case, equation (25) is a non-linear equation which is represented in Figure 2 as the curve \( EE \). If comparing the BGP of both approaches, it can be seen that even when the initial point of \( EE \) is the same, the steady state value is lower in Figure 2. This comes from the non-linear behavior of the curve \( EE \) and intuitively because this approach has more variables involved for the growth in the economy.
Fig 2. The steady-growth equilibrium with the impact of infrastructure.
4. Increase in Schooling Congestion

Once derived the BGP of the model with each of the channels of congestion, it is possible to make some modifications. Therefore, consider that each approach face a permanent increase in the schooling congestion, which can be acquired by an increase in parameter $\phi$ or by an increase in the share proportion of students $\theta_q$. Intuitively this can be explained as an increase in the number of students or by a reduction of the proportion of teachers compared to the students.

Consider the first approach, Figure 3, and starting from the initial BGP denoted at point $A$, if the congestion parameter $\phi$, increases then the composite variable $D$ will be reduced and hence $\psi$ (see Appendix E). This will be represented as a leftward shift of $EE$ to a new $EE'$ curve. Thus, this will also reduce the consumption and physical capital stock ratio in the steady state $\bar{c}$, denoted by the point $A'$, and then slowly go back to the point where the curves intersect again in $A'$. In this case an increase in the parameter $\phi$ does not affect the curve $CC$, since the congestion parameter is not present in the consumption function, so the movement is just represented in the curve $EE$. As a final result, there is a lower consumption-physical capital ratio and also a lower stock of educated labor-physical capital ratio than in the initial conditions.

![Figure 3. Increase in the schooling congestion by an increase in parameter $\phi$](image-url)
For the second approach as equation (25) is non-linear, the transitional dynamics will be the same but with slightly different conditions as the specifications are different. It is possible to see that even if the initial point of the curve $EE$ is the same in both cases, the first approach will have higher values of the stock of educated labor and physical capital stock ratio, $\tilde{e}$, and the consumption and physical capital stock ratio $\tilde{c}$ in the steady state. Moreover, the shift of curve $EE$ of the second approach is not as large as it is in the first approach.

The previous result comes from the presence of the congestion parameter in equation (25) which affects almost all other variables in it. Hence, having an increase in congestion parameter, $\phi$, derives a positive shift in some variables given the conditions of it, while in most of the other variables it has a negative effect. The total effect is a leftward movement of the curve $EE$, which reduce the consumption and physical capital stock ratio, $\tilde{c}$, and the stock of educated labor and physical capital stock ratio, $\tilde{e}$ in a smaller quantity than in the first approach result. Figure 4 shows the behavior of the model with this condition.

In case where the increase of congestion is caused by a higher proportion of students $\Theta_s$, both curves will move to lower values but in different magnitudes than in the previous
case. Because $CC$ contains $B$ which includes the proportion of students, $\theta_S$, the impact of an increase will make the curve move downwards to the level of $CC'$. While $EE$ contains $D$ and $B$, both of them containing $\theta_S$, the first variable will increase while the second will be reduced but given diminishing returns then the reduction effect hampers the new equilibrium point at $A'$. Contrasting the previous analysis, this modification will shift curve $EE$ to $EE'$ less than before as a shared participation of variable $\theta_S$ in equation (23).

In the case of the second approach the leftward shift of curve $EE$ to $EE'$ is increased by the diminishing returns of the composite variable $M$ included in equation (25). First a reduction in the consumption and physical capital stock ratio, $\tilde{c}$, from point $A$ to $A'$ which keeps the same flow of educated labor initially. Later the saddlepath moves the stock of educated labor and physical capital stock ratio, $\tilde{\ell}$, to a new value. Thus, the total effect will set the new equilibrium point at $A'$ to lower values of the consumption and physical capital stock ratio, $\tilde{c}'$, and the stock of educated labor and physical capital stock ratio, $\tilde{\ell}'$, as illustrated in Figure 6.

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13 See Appendix E
Fig 6. Increase in the schooling congestion by an increase in variable $\theta_s$. 
5. Reallocation of Spending to Education

As in the previous analysis, in this case the phase diagrams have the same notation. The BGP system will be evaluated when there is a reallocation of the spending and this gives more weight to the spending on education. As the reallocation is revenue-neutral the total government expenditure remains the same, just the parameters of their own share change their value.

Consider that the first approach faces this reallocation of spending to education, on one hand this will make the curve $CC$ to move downwards to $CC'$. On the other hand the curve $EE$ will have an ambiguous shift. It can shifts to the right or to the left. It depends now in how productive the infrastructure services are in the production of goods, but also how much they influence the generation of educated labor in the economy. This ambiguity leads to a general optimal spending allocation which will be covered in the next chapter with more detail.

The first case (right shift) is presented when the increase of the educated workers are used in the private market leading to a higher production. This results because with a higher rate of educated labor able to work in the production of goods, gives a larger productivity even with less infrastructure that may still lead to an increase in the stock of educated labor and physical capital stock ratio to $\bar{e}'$ and also to a bigger consumption and physical capital stock ratio to $\bar{c}'$, as illustrated in Figure 7.

Fig 7. Budget-neutral increase in the share of public spending on education
The second case (left shift) occurs when the supply of infrastructure is reduced affecting the educated labor also. As mentioned before infrastructure services affect the generation of educated labor and it tends to lower it by diminishing the supply of these services. Therefore, changes in production affect both of the variables of consumption and the educated labor, so the steady state will depend on the technology parameters. So in the case of a larger $\alpha$, it is expected that the educated labor and physical capital stock ratio $\bar{e}$ and the growth rate will decrease as shown in point $A''$.

Regarding about the second approach case, the effects on the steady-state are the same as explained before, just adapted to the properties it has. Nonetheless, the impact on a reallocation share of spending will be larger in this case than in the first one at both scenarios (positive and negative). The fact that the BGP now includes the direct parameter $\omega$ related to infrastructure services in the education technology, which for simplicity is modified into other parameters, gives a straight reaction in the model.

Fig 8. Budget-neutral increase in the share of public spending on education

It is not entirely surprising that in all the modifications and even in the initial BGP, the second approach has lower values. The explanation can be given because the model is more affected to the share of spending and the congestion parameters affect in a greater amount the model.
6. Growth-Maximizing Spending Allocation

For the purpose at hand, one of the important concepts to illustrate is the effect of schooling quality. Therefore, showing the optimal spending allocation between education and infrastructure services within the presence of the congestions costs represents a key issue in this work. Given the conditions of the model, it is only possible to examine the growth maximizing spending allocation solution using Proposition 1 \( (d v_E + d v_I = 0) \). This means that under a revenue-neutral reallocation of government spending the model will keep a constant proportion of teachers. This represents a convenient assumption because with that it is possible to get a feasible analytical solution. Moreover, considering that government spending must hold the following inequality \( v_E + v_I < 1 \), the growth maximizing share of spending on education can be obtained for both approaches (Appendix D).

Recalling that both congestions costs approaches have equation (22) in their BGP, it is necessary to obtain the impact of an increase in \( v_E \) on the growth rate. This result will lead to

\[
\eta_{\bar{v}/v_E} = \frac{\alpha v_E}{\beta v_I},
\]

which represents the elasticity of the educated labor and private capital ratio at the steady state with respect to the fraction of tax revenue spending in education. It is possible to interpret this elasticity as it is the proportion of the parameter \( \alpha \) and \( \beta \) multiply by the proportion of the tax revenue spending in education and infrastructure.

Regarding the first approach conditions, equation (23) is used as it represents the second equation in the BGP system. Hereby, the previous considerations from Proposition 1 become useful as the composite variables \( B \) and \( D \) can be considered as given (Appendix E). It can be established that the growth-maximizing share of spending on education is,

\[
v_E \bigg|_{v_E + v_I - \omega} \approx \frac{\beta}{\alpha + \beta}.
\]
From equation (27) is viable to interpret that although the degree of congestion in schooling affects the economic growth in the long term; it is not included in the optimal spending allocation between education and infrastructure services. The properties of this explanation can be summarized in the following proposition:

**Proposition 2.** The growth-maximizing spending allocation in education is increasing in $\beta$ and decreasing in $\alpha$, however, it is not affected by the degree of congestion in schooling nor any other parameter described in the schooling technology.

Consider now the second approach conditions, with equation (25) in addition with the results previously taken, it can be established that the growth-maximizing share of spending on education is given now by

$$
\nu_{E}^{2} = \frac{\gamma\beta}{(\alpha\gamma + \beta)(\mu + \zeta)(\gamma + \zeta(1 - \phi))} \in (0,1].
$$

(28)

The previous equation, considers as before that the proportion of teachers does not change and under revenue neutral reallocation equation (27) is a specific case of (28). This particular case is when the congestion parameters, $\gamma = \phi = 1$ and the spending component $\mu + \zeta = 1$. The properties of equation (28) can be summarized as follows:

**Proposition 3.** The growth-maximizing spending allocation in education with the congestions costs affecting the infrastructure and teaching capacity, depends positively on the proportion of schooling technology with respect to parameter, $\mu$, which is the component of spending and negatively on the parameter, $\phi$.

This can be explained as a situation when the degree of the congestion parameters in schooling is high, the optimal share of spending in education services will be reduced. The reason for this is that the negative effects of the large congestion on the flow of educated labor that will affect the final outcome can be compensated by increasing the spending share on infrastructure services.
This kind of reallocation of spending will benefit first the production in the economy and then the education technology to diminish the congestion costs. For example, if government increase the number of schools or classrooms, then the congestion parameter, $\phi$, will be reduce on the optimal spending allocation on education. However, these considerations will require some modifications such as increasing the number of teachers, so that the student-teacher ratio remains constant. Some of these considerations will be discussed later.
7. Impact of Parameter $\varepsilon$

In sight of this work it is important to evaluate the model and the impact of the new additional congestion parameter, $\varepsilon$. Recalling that this parameter is a measure of the knowledge that the students have in average from the previous years with respect to the required by teachers. This parameter is added to the Agénor’s (2012) model by affecting the gross proportion of teachers as an exponent. In the former paper this parameter is not present but it can be considered to have a value of 1 keeping constant returns of the teaching skills. With that consideration this parameter must have a positive value because no depreciation of knowledge is considered in the model. This assumption comes from the fact that even in cases where students cannot fail any school year as in Finland (Sahlberg 2007, 155) or even if they fail they will have a minimum amount of knowledge in the average from the spillovers or help from other students, which will be higher than nothing denoted as $0 < \varepsilon$.

Other consideration that must be clarified in the explanation of parameter $\varepsilon$, as depreciation of knowledge is not consider in the model, and knowledge itself does not possess a limit, a value where $\varepsilon > 1$ is also feasible. In spite of this possibility, considering such case will bring an exponential growth to the model. The fact that this value is greater than unity will describe increasing returns to scale from the knowledge of the teaching capacity, which in fact will be counterfactual at some point and lead to an unstable model.

Empirically, this will represent a special case where the students possess more knowledge than the expected from teachers bringing the possibility of getting extra knowledge in that schooling period. Moreover, besides the good effects that this might bring, it implies two fails in the education system. It represents a fail in the current education programs as the level of required knowledge is exceeded which means is not optimal. It also represents a fail in the future education system given that at a certain point all additional knowledge will be increasing the required knowledge for the following years, creating a lower value of $\varepsilon$ for the upcoming generations. Therefore, this possibility is not considered in this paper given that the model is based in a homogenized population of students and no special education is considered.
For the present work the values of the parameter are continuous, included from $0 < \varepsilon \leq 1$. The situation when $\varepsilon = 1$ will represent the condition when students have enough knowledge so the teaching capacities of the instructors will not be affected. Nonetheless, it is difficult to reach this level because it requires two conditions. The first one is that the collective knowledge of the students must be similar, meaning a small difference in the knowledge acquired previously between them. The second is that the lack of abilities and education from the students below the optimal level must be compensated by a higher knowledge from other students above this level. This effect can bring good effects as spillovers of knowledge from some students to the others reducing the gap between the less educated and the good educated students. However, this condition might also bring bad effects as free riding behavior increasing the gap between these two groups of students. For the purpose of this work analyzing the value of each group type is not in the scope so just denoting these effects will give an idea of the potential outcomes.

As parameter $\varepsilon$, affects only the teaching capacities, it does not have any effect on the BGP equation (22) which explains the consumption behavior in the economy. It only affects the equation regarding of the stock of educated labor. Consider the case when $\varepsilon = 1$ into the BGP equation (23) of the first approach, so that it becomes

$$\Omega = (1 + \theta_s)^{1+\mu} B^\mu D^\mu \left( v_\theta \tau(v, \tau) - \frac{a}{\varepsilon} \right)^{-\varepsilon}$$

where

$$D = a^{1-\mu}(1+\theta_s)^{1-\mu}$$

and represents the new BGP equation of the system given the optimal level of knowledge from students. From (29) is possible to verify that given the optimal conditions for teachers the parameters involved in their teaching capabilities are the marginal returns to quality, $V$, and the elasticity characterizing the government spending in education $\mu$. In this case the congestion parameter will enhance the teaching conditions but the government spending in education elasticity will be negatively related to it. As long as the marginal returns to quality are equal or greater than the
elasticity of government spending in education, $\zeta \geq \mu$, the teaching capacities will have constant or increasing returns given this conditions.

Consider now the second approach case with parameter $\varepsilon = 1$ into the BGP equation (25), yields

$$
\Omega = \left(1 + \theta_x\right)^{(1-\mu)\gamma-(1-(1-\gamma)\phi)\zeta} M_1 \left\{ V_E^T \right\}^{\mu+\varepsilon} \left( V_i^T \right)^{-\varepsilon} \left( \frac{a\gamma(1+\varepsilon)\phi}{1-\alpha} \right) \left( \frac{-G}{(1-\alpha)\gamma} \right) \left( \frac{1}{1+\varepsilon(1-\phi)} \right)
$$

(30)

where

$$M_1 = J_1 B^{(\mu+\varepsilon)\gamma-(1-\phi)\zeta(1-\phi)}$$

$$J_1 = a^{(1-\mu-\kappa)\gamma-(1-\phi)\zeta} \pi \zeta (1 + \theta_y)$$

which represents the new BGP equation of the system given the optimal level of knowledge from students. From equation (30) gross proportion of teachers are negatively affected by the elasticity of government spending in education $\mu$ with the congestion parameter $\gamma$, and the combination of the marginal returns to quality $\zeta$ with the congestion parameters $\phi$ and $\gamma$. Given that this model consider all the congestion costs at the same time as the ratio of student-teacher with the infrastructure services available, it is logical to see that all of them are involved in the BGP equation for educated labor. To ensure that the teachers remain productive in this particular case it is necessary the following assumption:

**Assumption 5.** $(1 - \mu)\gamma > (1 - (1 - \gamma)\phi)\zeta$.

The lower limit $\varepsilon \approx 0$ represents the situation when students in average do not have any of the required knowledge. As this condition is very difficult to hold given the admission policy restriction the results do not give a clear result in this environment. Moreover, modifying the admission policy will bring a logical interpretation to these results as explained in the next chapter. Hence verifying the effect on the BGP when $\varepsilon \approx 0$ is important to evaluate the impact of this parameter in the growth of this model.
Consider the first approach where the proportion of students and teachers in a school define the congestion cost and using equation (23) with the previous consideration, gives

$$\Omega = (1 + \theta_T)^{1-\mu} B^\alpha D_i \left( v_E \tau (v_j \tau)^{1-\alpha} \right)^{\frac{\mu}{\epsilon - \chi}}$$

(31)

where the new BGP has a different value affecting the gross proportion of teachers. From the previous equation (31) is possible to see that now the teachers are negatively related by the elasticity characterizing the government spending in education $\mu$. Compare to (29) where the marginal returns to quality was positively related in this case; as the required knowledge is not as needed, the quality has no impact.

To evaluate the modifications of the BGP in the second approach, using equation (25) with the same value of $\epsilon \approx 0$, so that

$$\Omega = \left(1 + \theta_T\right)^{1-\mu} \gamma_{(1-\phi)} M_i \left(v_E \tau (v_j \tau)^{1-\alpha} \right)^{\frac{\alpha\gamma (\mu + \tilde{\chi}) + \tilde{\chi} (1-\phi)}{(1-\alpha)\gamma}} \hat{e}^{\tilde{\gamma} (1-\phi) \gamma (1-\phi)} \right)^{\frac{1}{1+\tilde{\gamma}(1-\phi)}}$$

(32)

becomes now in the new BGP equation. It can be seen from the previous equation that if the required knowledge of the students is not as required, the gross proportion of teachers will be related to the elasticity of government spending in education, $\mu$, affected by the congestion parameter, $\gamma$, and the marginal returns to quality, $\zeta$, by the impact of the congestion effect, $\phi$.

Using Assumption 4 is possible to see that $(1 - \mu)\gamma - \zeta (1 - \phi) = 1 - \mu - \omega$, which can be explain as the infrastructure externalities and elasticity of the government spending in education are the only parameters affecting the productivity of the teachers in this case. With these considerations the marginal returns to quality does not affect the effectiveness of the teachers implying the same condition as with the first approach.

This comes from the fact that given the average knowledge of the students is non existing the teacher, has to teach from the basics. The quality parameter is excluded since teaching will start creating the value of the quality parameter $\zeta$. Lower conditions
will require more effort to the teachers regardless of how good or bad their abilities are. At this case there is no possibility of having increasing returns to quality since the knowledge of the students does not allow that.

These final results contrast with the basic model from Agénor (2008). With the results obtained with the parameter $\varepsilon$ inclusion and the flow of educated labor condition instead of stock of human capital, the model keeps the parameters $\mu$ and $\omega$ in the BGP equation for each approach.

Hence, combining the two limits of the parameter $\varepsilon$, is possible to conclude that the lower the average knowledge acquired from previous years the more affected the teaching capacities will be by the elasticity of government spending in education $\mu$ and the extend of externalities from infrastructure $\omega$. Additionally, the higher the value of the parameter $\varepsilon$ the more importance of the marginal returns to quality from parameter $\zeta$. 
8. Increasing Years of Mandatory Schooling

So far the model considered in this work is based on the assumption of an admission policy such that it keeps the student-teacher ratio constant. This condition, in most of the cases, is appropriate for developing than industrial countries. Industrial economies consider more an admission policy based on the education spending per student as Agénor and Neanidis (2006) use. However, if this model is based on the characteristics of developing countries it would be important to consider some modifications regarding the admission policy.

On one hand if more students are able to study and transform into educated labor, the benefits will be higher than with a constant number of students. On the other hand modifying the admission policy allowing more students to get into a higher education level is feasible but will affect the quality of the education itself. Reducing the requirements in the admission test will implicitly describe that the education system is facing a problem, as no students have the necessary abilities or the lack of interest in certain topics.

Education is a public service provided by government but it is not equal between countries. Besides the quality, infrastructure and materials, also the years of mandatory years of school vary among each country. While most of the developed countries have a bigger number of mandatory years of schooling than developing countries, increasing the years of mandatory schooling will not give the same results. As for instance in Eckstein and Zilcha (1994) a small level of compulsory education gives a higher level of output, capital and human capital in an economy. Moreover, compulsory education also represents a public policy that enhances the growth and makes the distribution of income more equal among the individuals. Thus, the population in an educated economy is better off in the long run if they have a high level of education, obtained mostly from public education. It is natural to believe that a productive development strategy would be to raise the schooling levels of population.14

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14 Hanushek E. A. and Wößmann, L (2007, 3)
Taking into account the previous statements, increasing the years of mandatory education is supposed to bring additional growth to a country. Adding some extra years to the education system brings some changes to the growth rate stability. Consider the case when the government increases the years of mandatory school. Given that the government has a limited number of facilities and infrastructure, this modification will imply that the admission policy has to change. In other words, there was a filter of students trying to keep a student-teacher ratio constant, but now this will have to be modified to accept the gross number of students in the next years of education.

The admission policy besides keeping a student-teacher ratio constant also keeps an optimal level of knowledge among the students, accepting the ones more competent. Thus, an additional number of students given the gross proportion of teachers will change the student-teacher ratio and rise the value of the congestion parameter $\phi$ and $\gamma$. This will reduce the average knowledge of the students’ parameter, $E$, and affect usage of the limited amount of infrastructure provided by the government. Keeping in mind that the education is provided by government the same as infrastructure, adding new classrooms to the education system is not as immediate as changing a policy so the previous effects will take some time to disappear.

In the foregoing analysis, it was assumed that the admission policy was constant as in equation (12). After the modifications of the admission policy, equation (12) will no longer have a constant value $a$, therefore becoming:

$$L = \frac{L}{(1 + \theta)E} = \Lambda$$

(33)

where $\Lambda > 0$ and will no longer be constant as long as the infrastructure facilities are optimal for the gross population of students at the extra years of education. This new equation will have implications likewise in the quality of education. The previous constant student-teacher ratio $a$ from (12) will be taken as a threshold to determine if the quality of the schooling technology is ‘high’ or ‘low’, such as

$$Q_i = \begin{cases} 
Q_H = Q^{S_H}, & \text{if } \Lambda \leq a \\
Q_L = Q^{S_L}, & \text{if } \Lambda > a
\end{cases}$$

(34)
where \( Q \) again is defined as either (9) or (15) with the modification that the marginal return to quality will show if it has a ‘high’ or ‘low’ value compared to the threshold value \( a \).

The model with this specification can generate multiple BGPs, depending on the value of the new admission’s policy of the student-teacher ratio \( \Lambda \) and the type of quality definition (first or second approach). Deriving a full characterization of these new outcomes is not considered in this word, however they will give similar results as the already given with some differences. Furthermore, the two general cases of \( \zeta_H \) or \( \zeta_L \) will have different growth-maximizing share allocations between the education and infrastructure services. In the case where \( \zeta_H \) the optimal share \( v^*_E \) will be higher than in the \( \zeta_L \) situation. Nevertheless, in the \( \zeta_L \)-regime the optimal \( v^*_E \) will be lower as the optimal \( v^*_I \) will increase its value. These results can be summarized in the following proposition:

**Proposition 4.** Increasing the years of mandatory schooling raises the government spending share on public infrastructure at the steady state, in order to reduce the congestion in schools.

As the model is based on educated labor instead of raw labor, if the years of education are not enough for achieving the level of new educated labor, the growth will decline. If the new years of mandatory school increase the number of students given the previous conditions, the quality of the education will be reduced as the congestion parameters increase their values. The years needed to reach the level of educated labor will be higher and the final output will be reduced. Therefore, the government will need to reduce the student-teacher ratio by building new classrooms and consequently hiring more teachers. If none of the previous measures are done, a huge difference in the knowledge between students might happen. Allowing all type of students into a new education level will induce that in some cases students with not even the minimum requirements in the same class as the highly skilled students or even a full class of the same type. Hence this will represent the model behaving as in equations (31) and (32).
9. Concluding Remarks

This thesis studied the implications of quality as a congestion cost in the education system for an optimal level creating educated labor that enhance the economic growth. The analysis in Chapter 3 was based on an endogenous growth model considering free access to public services as education, which is mandatory until certain level and infrastructure spillovers. These public services are provided by the government which is financed by a tax rate on final output. The importance of including infrastructure services in the model is because students need access to roads, electricity to study (at night), to use a computer, adequate sanitation and school facilities among many things which the government must provide. The inputs of the production function include private capital and labor, which instead of being raw must be educated to become productive. The quality of education was inversely related to the degree of congestion in schools, which was denoted in two ways. The first approach considers the proportion of students and teachers in the population and the second approach as the ratio of the government spending on education to the effective teaching capacity. An own feature added to the model was the inclusion of parameter $\varepsilon$. It measures the knowledge that the students have for the following education level, which affects the teaching capacities if it is not optimal.

The BGP of the model was derived with the conditions of the saddlepath stability were examined in Chapter 4 and Chapter 5. It was shown that the equilibrium was locally determined. The models presented a linear behavior and a concave curve shape.

The transitional dynamics associated with an increase in the degree of congestion was analyzed with the two approaches in Chapter 6. It was shown that given the increase of the congestion parameter the final output will be reduce in both approaches. The same negative effect was obtained by increasing the degree of congestion by augmenting the proportion of students but with higher impact in both models.

The transitional dynamics associated with a revenue-neutral shift in the composition of public spending from infrastructure services to education were analyzed in Chapter 7. It was shown that in general that shift will have an ambiguous effect on the growth rate and the steady state values of consumption and the supply of educated labor, which are
measure by the proportion of private capital. Such ambiguity is determined by the impact of the infrastructure services in the production function. If the impact of is less than the effective labor then there will exhibit a higher value of growth. In the other case, the shift will reduce the growth even with more educated labor in the economy as infrastructure services have a more important role in the production function.

Chapter 8 analyzes the growth-maximizing share allocation in each approach. With the first approach when the student-teacher ratio define the quality, the growth-maximizing share of government spending on education services was shown to depend only in parameters which characterize the production function and no education parameters. In the second approach the growth-maximizing share was shown to depend negatively on the congestion parameter and the parameters characterizing the education technology.

Chapter 9 analyzes the impact of the quality education parameter $\varepsilon$, showing that as the value of this parameter decreases the teaching capacities will be affected in a larger proportion by the government spending in education and the extent of the infrastructure spillovers. Contrasting this case, if the value of parameter $\varepsilon$ increases, the importance of the marginal return to quality parameter $\zeta$ rises.

The final chapter presents the scenario when a modification in the admission policy is made by lowering because of an increase in the mandatory years of school. No characterization was developed but the analysis of the possible outcomes was taken into account. This change had shown that the modification of the admission policy by increasing the years of mandatory school raises the growth-maximizing share on public infrastructure to compensate the reduction in the final output.

Although the model developed used different considerations than the ones initially used in Agénor (2008) and (2012) it can be extended in a variety of directions. One extension would be to analyze the impacts of the growth in the economy given a more detailed description of students. Hence, for this case differentiating the highly skilled from the less skilled students and the impact they have in the quality of the education system. A second possibility of extending the model would be to consider private education, in addition to public education. This would bring a different analysis since the education system will be one whole but for the students schooling will be presented as a decision
making between substitutes. Additional assumptions must be included in the model such as subsidies to private schools to make it feasible to all the population. These considerations would let the analysis of the growth and distributional effects of each system as in Glomm and Ravikummar (2003), in the presence of trade-offs between public education and infrastructure spending. A third extension of the model would be including the condition where the parameter $\epsilon$ may have values higher than unity. This would be reasonable if special education in any private or public way is available in the model. The impact of this special group might be focus on particular sectors of the growth and this could make possible the consideration of highly educated labor among the rest of the population, preventing a brain drain in the economy.
References


Appendix A

Profit Maximization to get (2)

\[ \Pi_i = \left( \frac{G_i}{K} \right) ^\alpha (N_i, E)^\beta K_i^{1-\beta} - (N_i, E)w_p - K_i \]

\[ \frac{\partial \Pi_i}{\partial (N_i, E)} = \beta \left( \frac{G_i}{K} \right) ^\alpha (N_i, E)^{\beta-1} K_i^{1-\beta} - w_p = 0 \]

\[ w_p = \beta \frac{Y_i}{N_i, E} \]

\[ \frac{\partial \Pi_i}{\partial (K_i)} = (1-\beta) \left( \frac{G_i}{K} \right) ^\alpha (N_i, E)^\beta \frac{K_i^{1-\beta}}{K_i} - r \]

\[ r = \frac{(1-\beta)Y_i}{K_i} \]

\[ \int_0^1 N_i \, di = N = (1-\theta_s - \theta_r) \]

\[ \int_0^1 K_i \, di = K = K \]

Using the previous definitions, yields

\[ w_p = \beta \frac{Y}{(1-\theta_s - \theta_r)E} \]

\[ r = (1-\beta) \frac{Y}{K} \]

Household Utility Maximization to get (6)

\[ \max_c V = \int_0^\infty \ln C \exp(-\rho t) \, dt \]

\[ \rho > 0 \]

\[ C + \dot{K} = (1-\tau)Y + w_r \theta_r E \]

\[ \frac{\dot{C}}{C} = s \left( \frac{G_i}{K} \right) ^\alpha \left[ \frac{(1-\theta_s - \theta_r)E^{\gamma\beta}}{K} \right] - \rho \]

\[ s = (1-\tau)(1-\alpha - \beta) \]

\[ \lim_{t \to \infty} (K/C) \exp(-\rho t) = 0 \]
Using the previous equations, the current-value Hamiltonian for this problem will be:

\[ H = U(C) + \lambda \overset{\cdot}{K} \]

where \( \lambda \) is a co-state variable corresponding to the state variable \( K \). It can be interpreted as the shadow value of private capital. It evolves according to:

\[ H = \ln C + \lambda \left[ (1 - \tau)\left( \frac{G_i}{K} \right)^{\alpha} \left( \frac{1 - \theta_s - \theta_f}{K} \right)^{\beta} K + \theta_f E - C \right] \]

The maximization of the Hamiltonian with respect to \( C \), as in the first-order condition \( \frac{\partial H}{\partial C} = 0 \) and the co-state condition \( \frac{\partial \Lambda}{\partial K} = \rho \lambda - \overset{\cdot}{\lambda} \), optimality conditions for this problem can be written as:

\[
\frac{\partial H}{\partial C} = \frac{1}{C} - \lambda = 0
\]

\[ C^{-1} = \lambda \]

\[ C = \lambda^{-1} \]

\[ \log C = -\log \lambda \]

\[ 1/C = \lambda \]

\[ \lambda = \rho \lambda - \frac{\partial \Lambda}{\partial K} \quad (A1) \]

\[
\overset{\cdot}{\lambda} = \rho \lambda - \left[ (1 - \tau)\left( \frac{G_i}{K} \right)^{\alpha} \left( \frac{1 - \theta_s - \theta_f}{K} \right)^{\beta} K \right] \lambda \\
- \alpha (1 - \tau)\left( \frac{G_i}{K} \right)^{\alpha} \left( \frac{1}{K} \right)^{\beta} \left( \frac{1 - \theta_s - \theta_f}{K} \right)^{\beta} K \lambda \\
- \beta (1 - \tau)\left( \frac{G_i}{K} \right)^{\alpha} \left( \frac{1 - \theta_s - \theta_f}{K} \right)^{\beta} \left( \frac{1}{K} \right)^{\beta} K \lambda \\
= \rho \lambda - \left[ (1 - \tau)(1 - \alpha - \beta)\left( \frac{G_i}{K} \right)^{\alpha} \left( \frac{1 - \theta_s - \theta_f}{K} \right)^{\beta} \right] \lambda \\
\]

\[
\overset{\cdot}{\lambda} = \left\{ \rho - \left( 1 - \tau \right)(1 - \alpha - \beta)\left( \frac{G_i}{K} \right)^{\alpha} \left( \frac{1 - \theta_s - \theta_f}{K} \right)^{\beta} \right\} \lambda \\
\]

\[
\overset{\cdot}{\lambda} = \left\{ \rho - \left( \frac{G_i}{K} \right)^{\alpha} \left( \frac{1 - \theta_s - \theta_f}{K} \right)^{\beta} \right\} \lambda \\
\]

(A2)
Together with the budget constraint, the transversality condition, and combining (A1) and (A2) will give,

\[
\frac{\dot{C}}{C} = d \log C dt = - \log \lambda = - \frac{\dot{\lambda}}{\lambda} = - \left\{ \rho - \left[ s \left( \frac{G}{K} \right)^\alpha \left[ \frac{(1-\theta_S-\theta_T)E}{K} \right]^{\beta} \right] \right\} \lambda
\]

which rearranging a bit will result as (6) in the text,

\[
\frac{\dot{C}}{C} = s \left( \frac{G}{K} \right)^\alpha \left[ \frac{(1-\theta_S-\theta_T)E}{K} \right]^{\beta} - \rho.
\]

(A3)

Estimating the equilibrium proportion of teachers (21)

Combining the wage rate in private market \( w_p = \beta \frac{Y}{(1-\theta_S-\theta_T)E} \) into the arbitrage condition \( w_r = (1-\tau)w_p \) to then put both into the government budget constraint \( w_r, \theta_r, E + G_x + G_y = \tau Y \) considering the government expenditure on each service \( G_h = v_h \tau Y \), will result to the equilibrium proportion of teachers as:

\[
(1-\tau) \frac{\beta Y}{(1-\theta_S-\theta_T)E} \theta_r E + V_r \tau Y + V_r Y = \tau Y
\]

\[
(1-\tau)\beta Y \theta_r E + V_r \tau Y ((1-\theta_S-\theta_T)E) + V_r \tau Y ((1-\theta_S-\theta_T)E) = \tau Y ((1-\theta_S-\theta_T)E)
\]

\[
\beta \theta_r - \tau \beta \theta_r - V_r \tau \theta_r - V_r \tau \theta_r + \tau \theta_r = - V_r \tau (1-\theta_S) - V_r \tau (1-\theta_S) + \tau (1-\theta_S)
\]

\[
(\beta - \tau \beta - V_r \tau - V_r \tau + \tau) \theta_r = (\tau V_r \tau - V_r \tau)(1-\theta_S)
\]

\[
(\beta - \tau \beta - V_r \tau - V_r \tau + \tau) \theta_r = (1-V_r - V_r)(1-\theta_S)
\]

Separating \( \theta_r \) to one side of the result we can get the proportion of teachers.

\[
\theta_r = \frac{(1-V_r - V_r)(1-\theta_S)}{\beta - V_r \tau - V_r \tau + \tau} \left( \frac{1}{\beta} \right)
\]

(A4)
Appendix B

This Appendix derives the dynamic form of the model and with it the solution of the steady state. This part only considers the congestion cost associated with the proportion of students and teachers in school (first approach).

**Dynamic form**

Considering that $G_j = v_j \tau y$ from (20) and using (3), yields

$$ G_j = v_j \tau \left( \frac{G_j}{K} \right)^{\alpha} \left[ \frac{(1-\theta_S - \theta_T)E^\beta}{K} \right] K. \quad (B1) $$

Substituting (21) into (B1) and after rearranging

$$ G_j = v_j \tau \left( \frac{G_j}{K} \right)^{\alpha} \left[ \frac{(1-\theta_S - \theta_T)E^\beta}{K} \right] K $$

$$ \frac{G_j}{K} = v_j \tau \left( \frac{G_j}{K} \right)^{\alpha} \left[ \frac{(1-\theta_S)(1-\tau + \tau \beta E^\beta)}{K} \right] $$

$$ \frac{G_j}{K} = v_j \tau \left( \frac{G_j}{K} \right)^{\alpha} \left[ \frac{(1-\theta_S)(1-\tau + \tau \beta E^\beta)}{K} \right] $$

or equivalently

$$ \left( \frac{G_j}{K} \right)^{1-a} = v_j \tau A^\beta \left( \frac{E}{K} \right)^\beta $$

$$ \frac{G_j}{K} = (v_j \tau)^{1-a} A^\beta e^{1-a} \quad (B2) $$

where $e = E / K$ and

$$ A \equiv \frac{(1-\theta_S)(1-\tau)}{1-\tau + \tau \beta E^\beta} < 1 \quad (B3) $$

which shows that, with $v_T + v_j \rightarrow 1, A \rightarrow (1-\theta_S)(1-\tau)$. 

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Putting (21) into (3)

\[
Y = \left( \frac{G_l}{K} \right)^\alpha \left[ \frac{1 - \theta_s - \tau \beta^{-1} [1 - (v_E + v_i)] (1 - \theta_s)}{1 - \tau + \tau \beta^{-1} [1 - (v_E + v_i)]} \right]^\beta \]

and considering (B3) and rearranging, yields

\[
Y = A^\beta \left( \frac{G_l}{K} \right)^\alpha e^{\beta K}. \quad \text{(B4)}
\]

From (18) and the fact that \( G_h = v_h \tau Y \), \( h = E, I \)

\[
w_r \theta_r E + v_E \tau Y + v_i \tau Y = \tau Y
\]

which can be rewritten as,

\[
w_r \theta_r E = [1 - (V_E + V_i)] \tau Y. \quad \text{(B5)}
\]

Substituting (B5) in (5) yields

\[
C + \dot{K} = ((1 - \tau) + [1 - (V_E + V_i)] \tau) Y. \quad \text{(B6)}
\]

Separating \( \dot{K} \) and putting (B4) in (B6), gives

\[
\dot{K} = [1 - \tau (V_E + V_i)] Y - C
\]

\[
\dot{K} = [1 - \tau (V_E + V_i)] A^\beta \left( \frac{G_l}{K} \right)^\alpha e^{\beta K} - C. \quad \text{(B7)}
\]

Combining (B2) with (B7) as,

\[
\frac{\dot{K}}{K} = [1 - \tau (V_E + V_i)] A^\beta \left( \frac{G_l}{K} \right)^\alpha e^{\beta K} - \frac{C}{K}
\]

\[
\frac{\dot{K}}{K} = [1 - \tau (V_E + V_i)] A^\beta A^{\frac{\beta}{\gamma - \alpha}} (v_i \tau) \frac{1}{\gamma - \alpha} e^{\frac{\beta}{\gamma - \alpha} - \frac{\alpha}{\gamma - \alpha} v_i \tau e^{\frac{\beta}{\gamma - \alpha}}} - \frac{C}{K}
\]

\[
\frac{\dot{K}}{K} = [1 - \tau (V_E + V_i)] A^\beta A^{\frac{\gamma \beta}{\gamma - \alpha}} (v_i \tau) \frac{1}{\gamma - \alpha} e^{\frac{\beta}{\gamma - \alpha}} e^{\frac{\alpha v_i \tau}{\gamma - \alpha}} - \frac{C}{K}
\]
\[ \frac{\dot{K}}{K} = \left[1 - \tau (V_e + V_f)\right] A^{\frac{\beta}{1-a}} (v_f \tau)^{\frac{\alpha}{1-a}} e^{\frac{\beta}{1-a}} - \frac{C}{K}. \]  

(B8)

Setting \( \varphi = [1 - \tau (V_e + V_f)] \in (0, 1) \), \( B \equiv A^{\frac{\beta}{1-a}} \equiv (1 - \theta_s - \theta_f)^{\frac{\beta}{1-a}} \), and \( c = \frac{C}{K} \), (B8) can be written as

\[ \frac{\dot{K}}{K} = \varphi B (v_f \tau)^{\frac{\alpha}{1-a}} e^{\frac{\beta}{1-a}} - c. \]  

(B9)

Substituting (A2) in (6) and also using (12) is possible to get

\[ \frac{\dot{C}}{C} = s \left( A^{\frac{\beta}{1-a}} (v_f \tau)^{\frac{1}{1-a}} e^{\frac{\beta}{1-a}} \right)^{\frac{\alpha}{1-a}} \left[ \frac{(1 - \theta_s - \theta_f) E}{K} \right]^{\beta} - \rho \]

\[ \frac{\dot{C}}{C} = s A^{\frac{\alpha \beta}{1-a}} (v_f \tau)^{\frac{\alpha}{1-a}} e^{\frac{\beta}{1-a}} \left[ \frac{(1 - \theta_s - \theta_f) E}{K} \right]^{\beta} - \rho \]

Recalling that \( B \equiv A^{\frac{\beta}{1-a}} \) is possible to write the previous result as,

\[ \frac{\dot{C}}{C} = s B (v_f \tau)^{\frac{\alpha}{1-a}} e^{\frac{\beta}{1-a}} - \rho \]  

(B10)

which is (22) in the text.

Similarly, noting that \( G_e = v_f \tau Y \) and using (3) with (21)

\[ G_e = v_f \tau \left( \frac{G_l}{K} \right)^{\alpha} \left[ \frac{(1 - \theta_s - \theta_f) E}{K} \right]^{\beta} K \]
\[ G_E = v_e \tau \left( \frac{G_i}{K} \right)^\alpha \left[ \left( 1 - \theta_S - \tau \beta \left[ 1 - (V_e + V_j) \right] (1 - \theta_S) \right) K \right]^\beta \]

considering (B3) and arranging the previous equation can be written as

\[ \frac{G_E}{K} = v_e \tau \left( \frac{G_i}{K} \right)^\alpha A^\beta e^\beta. \]  

(B11)

Using (B2) in (B11) yields

\[
\frac{G_E}{K} = v_e \tau A^{\frac{\beta}{1-\alpha}} \left( v_j \tau \right)^{\frac{1}{1-\alpha}} e^{\frac{\beta}{1-\alpha}} A^\beta e^\beta
\]

\[
\frac{G_E}{K} = v_e \tau A^{\frac{\beta}{1-\alpha}} (v_j \tau)^{\frac{1}{1-\alpha}} e^{\frac{\beta}{1-\alpha}} A^\beta e^\beta
\]

\[
\frac{G_E}{K} = v_e \tau A^{\frac{\beta}{1-\alpha}} (v_j \tau)^{\frac{1}{1-\alpha}} e^{\frac{\beta}{1-\alpha}} A^\beta e^\beta
\]

(B12)

Equivalently, substituting (B12) in (17) gives

\[ \frac{\dot{E}}{E} = a^{1-x-\mu}(1+\theta_T^{1+\theta_\lambda-\mu})(1+\theta_S^{1+\theta_\lambda}) \left( \frac{G_E}{K} \right)^\mu \left( \frac{K}{E} \right)^\mu \]

so it is possible to write equation (17) as

\[ \frac{\dot{E}}{E} = D \left( \frac{G_E}{K} \right)^\mu e^{-\mu} \]  

(B13)

where \( D = a^{1-x-\mu}(1+\theta_T^{1+\theta_\lambda-\mu})(1+\theta_S^{1+\theta_\lambda}) \). Which substituting (B7) in (B13) yields

\[ \frac{\dot{E}}{E} = v_e \tau A \left( v_j \tau \right)^{\alpha} e^{\frac{\beta}{1-\alpha}} e^{-\mu} \]

\[ \frac{\dot{E}}{E} = D \left( \frac{G_E}{K} \right)^\mu \left( \frac{K}{E} \right)^\mu \]

\[ \frac{\dot{E}}{E} = D \left( v_e \tau A \left( v_j \tau \right)^{\alpha} e^{\frac{\beta}{1-\alpha}} \right)^\mu e^{-\mu} \]
so that,

\[
\frac{\dot{E}}{E} = B^\mu D \left( \frac{v_\epsilon \tau (v_i \tau)^{\alpha}}{1 - \alpha} \right)^\mu e^{-\chi}
\]  

(B14)

where \( \chi = (1 - \alpha - \beta) \mu/(1 - \alpha) > 0 \), giving the same result as equation (24) from the text.

Combining equation (B9) and (B10) gives

\[
\frac{\dot{C}}{C} = \frac{K}{K} = s B(v_i \tau)^{\alpha} \frac{\beta}{1 - \alpha} e^{\frac{\beta}{1 - \alpha}} - \rho - \left( \phi B(v_i \tau)^{\alpha} \frac{\beta}{1 - \alpha} e^{\frac{\beta}{1 - \alpha}} - c \right)
\]

\[
\frac{\dot{C}}{C} = \frac{\dot{K}}{K} = \frac{c}{c} = (s - \phi) B(v_i \tau)^{\alpha} \frac{\beta}{1 - \alpha} e^{\frac{\beta}{1 - \alpha}} - \rho + c
\]

\[
\frac{\dot{C}}{C} = \frac{\dot{K}}{K} = \frac{c}{c} = (s - \phi) B(v_i \tau)^{\alpha} \frac{\beta}{1 - \alpha} e^{\frac{\beta}{1 - \alpha}} - \rho + c.
\]  

(B15)

From the definitions of \( s \) and \( \phi \), it can be established that

\[ s - \phi = (1 - \tau)(1 - \alpha - \beta) - \left[ 1 - \tau (V_\epsilon + V_i) \right] \]

that is

\[ s - \phi = -(\alpha + \beta) + \tau(\alpha + \beta) - \tau + \tau (V_\epsilon + V_i) \]

or

\[ s - \phi = -(1 - \tau)(\alpha + \beta) - \tau + \tau (V_\epsilon + V_i) \]  

(B16)

Hence \( s - \phi < 0 \) as \( \tau (V_\epsilon + V_i) < (1 - \tau)(\alpha + \beta) + \tau \) because \( V_\epsilon + V_i \leq 1 \) in all the cases.

Similarly combining (B9) and B(14) yields

\[
\frac{\dot{E}}{E} = \frac{\dot{K}}{K} = B^\mu D \left( \frac{v_\epsilon \tau (v_i \tau)^{\alpha}}{1 - \alpha} \right)^\mu e^{-\chi} - \left( \phi B(v_i \tau)^{\alpha} \frac{\beta}{1 - \alpha} e^{\frac{\beta}{1 - \alpha}} - c \right)
\]

\[
\frac{\dot{E}}{E} = \frac{\dot{K}}{K} = \frac{\dot{e}}{e} = \psi e^{-\chi} - \phi B(v_i \tau)^{\alpha} \frac{\beta}{1 - \alpha} e^{\frac{\beta}{1 - \alpha}} + c
\]
\[
\frac{\dot{e}}{e} = \psi e^{-\chi} - \phi B(v, \tau) \frac{a}{1-a} e^{1-a} + c
\]  
(B17)

where
\[
\psi = B^\mu D \left( v_\tau (v, \tau) \right)^{\alpha \beta}.
\]  
(B18)

**Steady State**

For the expressions of the steady state given in the text as (22) and (23) concerning the first approach and correspond in this Appendix as (B10) and (B14). To determine the values of the steady state in the system of \(c\) and \(e\), it is necessary to set \(c = e = 0\) in (B15) and (B17).

First from (B15)
\[
0 = (s - \phi)B(v, \tau)^{\alpha \beta} e^{1-a} \rho + \tilde{c}
\]
\[
\tilde{c} = \rho - (s - \phi)B(v, \tau)^{\alpha \beta} e^{1-a} > \rho\]  
(B19)

Equivalently for (B17)
\[
0 = \psi e^{-\chi} - \phi B(v, \tau)^{\alpha \beta} e^{1-a} + \tilde{c}
\]
\[
\tilde{c} = \phi B(v, \tau)^{\alpha \beta} e^{1-a} - \psi e^{-\chi}\]  
(B20)

Combining (B19) and (B20) through eliminating \(\tilde{c}\) as
\[
\rho - (s - \phi)B(v, \tau)^{\alpha \beta} e^{1-a} = \phi B(v, \tau)^{\alpha \beta} e^{1-a} - \psi e^{-\chi}
\]

is possible to get an expression in the form
\[
F(\tilde{c}) = 0 \iff \rho - sB(v, \tau)^{\alpha \beta} e^{1-a} + \psi e^{-\chi} = 0
\]  
(B21)

To investigate the dynamics by linearizing the system (B15) and (B17) in the vicinity of the steady state gives,
\[
\begin{bmatrix}
\dot{\tilde{c}} \\
\dot{\tilde{e}}
\end{bmatrix} =
\begin{bmatrix}
a_{11} & a_{12} \\
a_{21} & a_{22}
\end{bmatrix}
\begin{bmatrix}
c - \tilde{c} \\
e - \tilde{e}
\end{bmatrix}
\]  
(B22)
where the \( a_j \) are given by

\[
    a_{11} = \tilde{c}, \quad a_{21} = \tilde{c},
\]

\[
    a_{12} = \frac{\tilde{c}\beta}{1-\alpha} B(s-\varphi) v^T \frac{\alpha}{1-\alpha} \tilde{\varphi}^{-1} < 0,
\]

\[
    a_{22} = -\chi \psi \tilde{\varphi}^{-1} - \frac{\beta}{1-\alpha} \varphi B(v^T \frac{\alpha}{1-\alpha} \tilde{\varphi}^{-1} < 0
\]

where \( \tilde{c} \) and \( \tilde{c} \) denote the stationary values of \( e \) and \( c \). Concurrently in (B22) \( e \) is predetermined at any given moment in time, whereas \( c \) is a jump variable. For the saddlepath stability is required one unstable (positive) root, which must come from the determinant of the Jacobian matrix of partial derivatives of the dynamic system (B22).

To ensure this condition holds the determinant must be negative, that is, \( \Delta = a_{11}a_{22} - a_{21}a_{12} < 0 \). Hence using the previous definition yields

\[
    \Delta = -\tilde{c} \chi \psi \tilde{\varphi}^{-1} - \frac{\tilde{c}\beta}{1-\alpha} B(s-\varphi) v^T \frac{\alpha}{1-\alpha} \tilde{\varphi}^{-1} < 0
\]

or equivalently \( -a_{12}/a_{11} < -a_{22}/a_{21} \), that is, the phase line \( EE \) has to be steeper than the phase line \( CC \). It can be seen that the slopes of the phase lines are positive and are given by

\[
    \left. \frac{dc}{dc} \right|_{CC} = -\frac{a_{12}}{a_{11}} = -\frac{\beta}{1-\alpha} B(s-\varphi) v^T \frac{\alpha}{1-\alpha} \tilde{\varphi}^{-1} > 0
\]

\[
    \left. \frac{dc}{dc} \right|_{EE} = -\frac{a_{22}}{a_{21}} = \chi \psi \tilde{\varphi}^{-1} + \frac{\beta}{1-\alpha} \varphi B(v^T \frac{\alpha}{1-\alpha} \tilde{\varphi}^{-1} > 0.
\]

With the above definitions it is possible to verify that the equilibrium is unique in the system. Taking into account that from (B21) and the derivative of it gives \( F_\varphi < 0 \), then it cannot cross the horizontal axis from below because \( \rho > 0 \) for the case when \( F(0) > 0 \). Considering that \( F(\tilde{c}) \) is a continuous monotonically decreasing function of \( \tilde{c} \), only one unique positive value of \( \tilde{c} \) can satisfy \( F(\tilde{c}) = 0 \).
Among the previous linearization, the stable manifold is given by

\[ e = \bar{e} + (e - \bar{e}) \exp(v \tau), \quad c = \bar{c} + \omega_1 (e - \bar{e}) \]

where

\[ \omega_1 = \frac{v - a_{22}}{\bar{e}} = -\frac{a_{12}}{\bar{c} - v} > 0 \]

represents the slope of the saddlepath, denoted as mentioned before as SS and \( v \) denotes the negative root of (26). Noting that \( a_{12} < 0 \) and \( \bar{c} - v > 0 \) the value of \( \omega_1 \) will be positive. To verify if SS is flatter than CC, it is enough to note that \( \omega_1 < -\frac{a_{12}}{\bar{c}} \), as \( a_{12} v > 0 \).
Appendix C

This Appendix derives the dynamic form of the model and with it the solution of the steady state and the stability properties. This part only considers the congestion cost associated with the infrastructure and materials available to teach and enhance the abilities of teaching (second approach).

Dynamic form

As for this approach, the model changes only in the flow of educated labor conditions, which enable the analysis to keep the same equations from Appendix B from (B1) to (B12). Consider now that schooling quality equation is given by (15) and the flow of educated labor is (14). By substituting (B15) into (B14) gives

\[
\dot{E} = \left[ \frac{\pi G_E}{(1+\theta_3)^\gamma E^\rho} \right]^\kappa (1+\theta_3) G_E^\mu (1+\theta_T) E^\kappa \left( \frac{G_I}{L} \right)^\alpha L^{-\mu-\kappa-\alpha}
\]

and after rearranging the terms, yields

\[
\dot{E} = \pi^\gamma (1+\theta_3) \left( \frac{G_E}{L} \right)^\mu (1+\theta_T)^{\kappa-\phi_3} \left( \frac{E}{L} \right)^\kappa \left( \frac{G_I}{E} \right)^\alpha L^{\gamma+\zeta}.
\]  

(C1)

Given that from (12), \( \dot{E} = \dot{L} \), and substituting in (C1)

\[
\dot{E} = \pi^\gamma (1+\theta_3) \left( \frac{G_E}{a(1+\theta_T)E} \right)^\mu (1+\theta_T)^{\kappa-\phi_3} \left( \frac{E}{a(1+\theta_T)E} \right)^\kappa \left( \frac{G_I}{E} \right)^\alpha \left( a(1+\theta_T)E \right)^{\gamma+\zeta}
\]

followed by some modifications gives

\[
\begin{align*}
\dot{E}^{1+\gamma_0} & = a^{1-\mu-\kappa-\alpha} \pi^\gamma (1+\theta_3) \left( \frac{G_E}{E} \right)^\mu (1+\theta_T)^{1-\mu-\alpha-\phi_3} E^{1+\zeta-\phi_3} \left( \frac{G_I}{E} \right)^\alpha \left( \frac{E}{E} \right)^{\gamma_0} \\
\left( \frac{\dot{E}}{E} \right)^{1+\gamma_0} & = a^{1-\mu-\kappa-\alpha} \pi^\gamma (1+\theta_3) \left( \frac{G_E}{E} \right)^\mu (1+\theta_T)^{1-\mu-\alpha-\phi_3} E^{\zeta-\phi_3-\gamma_0} \left( \frac{G_I}{E} \right)^\alpha
\end{align*}
\]
which is the same as (17) in the text.

Again using (B12) into (C2)

\[
\frac{\dot{E}}{E} = \left( a^{1-\mu-\kappa-\omega} \pi^\varphi (1 + \theta_s)(1 + \theta_T) \right)^{\mu+\varphi} \left( G_E \frac{G_E}{E} \right) \left( K \frac{K}{E} \right) \varphi^\omega \left( K \frac{K}{E} \right)^\omega \left( 1 + \gamma + \frac{\pi^\varphi}{\gamma} \right) \]

\[
\frac{\dot{E}}{E} = \left( a^{1-\mu-\kappa-\omega} \pi^\varphi (1 + \theta_s)(1 + \theta_T) \right)^{\mu+\varphi} \left( G_E \frac{G_E}{E} \right) \left( K \frac{K}{E} \right) \varphi^\omega \left( K \frac{K}{E} \right)^\omega \left( 1 + \gamma + \frac{\pi^\varphi}{\gamma} \right) \]

Consider the restriction from Assumption 4 \( \varphi (1 - \phi) / \gamma = \omega \), this equation is

\[
\frac{\dot{E}}{E} = \left( a^{1-\mu-\kappa-\omega} \pi^\varphi (1 + \theta_s)(1 + \theta_T) \right)^{\mu+\varphi} \left( G_E \frac{G_E}{E} \right) \left( K \frac{K}{E} \right) \varphi^\omega \left( K \frac{K}{E} \right)^\omega \left( 1 + \gamma + \frac{\pi^\varphi}{\gamma} \right) \]

that is (24) in the text,

\[
\frac{\dot{E}}{E} = J \left( G_E \frac{G_E}{K} \right)^{\mu+\varphi} \left( G_I \frac{G_I}{K} \right) \varphi^\omega \left( K \frac{K}{E} \right)^\omega \left( 1 + \gamma + \frac{\pi^\varphi}{\gamma} \right) \]

with

\[
\frac{\dot{E}}{E} = J \left( G_E \frac{G_E}{K} \right)^{\mu+\varphi} \left( G_I \frac{G_I}{K} \right) \varphi^\omega \left( K \frac{K}{E} \right)^\omega \left( 1 + \gamma + \frac{\pi^\varphi}{\gamma} \right) \]

(33)
and

$$\Gamma = (\mu + \varsigma)\gamma + \varsigma(1 - \phi). \quad \text{(C5)}$$

Using (C5) and putting (B12) and (B2) in it, gives

$$\frac{\dot{E}}{E} = \left( J \left( v_e \alpha B(v_e \tau) \frac{\alpha}{1 - \alpha} \frac{\beta}{e^{1 - \alpha}} \right)^{\mu + \varsigma} \left( A^{1 - \alpha} (v_e \tau)^{1 - \alpha} e^{1 - \alpha} \right) \right)^{\frac{1}{\gamma + \varsigma(1 - \phi)}}$$

$$\frac{\dot{E}}{E} = \left( J \left( v_e \alpha B(v_e \tau) \frac{\alpha}{1 - \alpha} \frac{\beta}{e^{1 - \alpha}} \right)^{\mu + \varsigma} \left( A^{1 - \alpha} (v_e \tau)^{1 - \alpha} e^{1 - \alpha} \right) \right)^{\frac{1}{\gamma + \varsigma(1 - \phi)}}$$

$$\frac{\dot{E}}{E} = \left( J \left( v_e \alpha B(v_e \tau) \frac{\alpha}{1 - \alpha} \frac{\beta}{e^{1 - \alpha}} \right)^{\mu + \varsigma} \left( A^{1 - \alpha} (v_e \tau)^{1 - \alpha} e^{1 - \alpha} \right) \right)^{\frac{1}{\gamma + \varsigma(1 - \phi)}}$$

which can be rewritten as (25) in the text as

$$\frac{\dot{E}}{E} = \left( M(v_e \alpha B(v_e \tau) \frac{\alpha}{1 - \alpha} \frac{\beta}{e^{1 - \alpha}} \right)^{\mu + \varsigma} \left( A^{1 - \alpha} (v_e \tau)^{1 - \alpha} e^{1 - \alpha} \right)^{\frac{1}{\gamma + \varsigma(1 - \phi)}}$$

\[ \text{(C6)} \]

where

$$M = JB^{\mu + \varsigma(1 - \phi)}. \quad \text{(C7)}$$

Combining (B9) and (C6) yields

$$\frac{\dot{E}}{E} = \left( M(v_e \alpha B(v_e \tau) \frac{\alpha}{1 - \alpha} \frac{\beta}{e^{1 - \alpha}} \right)^{\mu + \varsigma(1 - \phi)} \left( A^{1 - \alpha} (v_e \tau)^{1 - \alpha} e^{1 - \alpha} \right)^{\frac{1}{\gamma + \varsigma(1 - \phi)}}$$

$$\frac{\dot{E}}{E} = \left( M(v_e \alpha B(v_e \tau) \frac{\alpha}{1 - \alpha} \frac{\beta}{e^{1 - \alpha}} \right)^{\mu + \varsigma(1 - \phi)} \left( A^{1 - \alpha} (v_e \tau)^{1 - \alpha} e^{1 - \alpha} \right)^{\frac{1}{\gamma + \varsigma(1 - \phi)}}$$

\[ \text{(C8)} \]

\[ \text{(C9)} \]
\[ \sigma = \frac{1}{1 + \zeta (1 - \phi)} \tag{C10} \]

**Steady State**

For the expressions of the steady state given in the text as (22) and (25) concerning the second approach and correspond in this Appendix as (B10) and (C6). To determine the values of the steady state in the system of \( c \) and \( e \), it is necessary to set \( c = e = 0 \) in (B15) and (C8).

First from (B15)

\[ 0 = (s - \varphi)B(\nu_1) \frac{a}{1-a} \frac{\beta}{e^{1-a}} - \rho + \tilde{c} \]

\[ \tilde{c} = \rho - (s - \varphi)B(\nu_1) \frac{a}{1-a} \frac{\beta}{e^{1-a}} > 0 \tag{C11} \]

equivalently for (C8)

\[ 0 = \theta^\sigma \tilde{c} \frac{\sigma(\gamma - \alpha y - \beta)}{(1-a)\gamma} - \varphi B(v, \tau) \frac{a}{1-a} \frac{\beta}{e^{1-a}} + \tilde{c} \]

\[ \tilde{c} = \varphi B(\nu_1) \frac{a}{1-a} \frac{\beta}{e^{1-a}} \theta^\sigma \tilde{e} - \theta^\sigma \tilde{e} \frac{\sigma(\gamma - \alpha y - \beta)}{(1-a)\gamma} \tag{C12} \]

Combining (B19) and (B20) through eliminating \( \tilde{c} \) as

\[ \rho - (s - \varphi)B(\nu_1) \frac{a}{1-a} \frac{\beta}{e^{1-a}} = \varphi B(\nu_1) \frac{a}{1-a} \frac{\beta}{e^{1-a}} - \theta^\sigma \tilde{e} \frac{\sigma(\gamma - \alpha y - \beta)}{(1-a)\gamma} \]

is possible to get an expression in the form

\[ F(\tilde{e}) = 0 \iff \rho - sB(\nu_1) \frac{a}{1-a} \frac{\beta}{e^{1-a}} + \theta^\sigma \tilde{e} \frac{\sigma(\gamma - \alpha y - \beta)}{(1-a)\gamma} = 0 \tag{C13} \]

To investigate the dynamics in the vicinity of the steady state, the system (B15) and (C8) can be linearized to give

\[ \begin{bmatrix} \dot{c} \\ \dot{e} \end{bmatrix} = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \begin{bmatrix} c - \tilde{c} \\ e - \tilde{e} \end{bmatrix} \tag{C14} \]
where again the $b_{ij}$ values are given by

\[
\begin{align*}
    b_{11} &= \tilde{c} , \\
    b_{21} &= \tilde{c} \\
    b_{12} &= \frac{\tilde{c}\beta}{1-\alpha} B(s-\varphi) v \varphi B(v) \left( \frac{\alpha}{1-\alpha} \right)^{\frac{\beta}{1-\alpha}} < 0
\end{align*}
\]

and

\[
b_{22} = -\frac{\Gamma\sigma(\gamma-\alpha\gamma-\beta)}{(1-\alpha)\gamma} \tilde{c}^\sigma \epsilon \left( \frac{\Gamma\sigma(\gamma-\alpha\gamma-\beta)}{(1-\alpha)\gamma} - \frac{\tilde{c}\beta}{1-\alpha} B(v) \varphi B(v) \left( \frac{\alpha}{1-\alpha} \right)^{\frac{\beta}{1-\alpha}} < 0.
\]

Same as before one unstable positive root is required entailing a negative determinant from the Jacobian matrix of the dynamic system (C14) as $\Delta = b_{11}b_{22} - b_{12}b_{21} < 0$. Thus, this will yield

\[
\Delta = -\frac{\Gamma\sigma(\gamma-\alpha\gamma-\beta)}{(1-\alpha)\gamma} \tilde{c}^\sigma \epsilon \left( \frac{\Gamma\sigma(\gamma-\alpha\gamma-\beta)}{(1-\alpha)\gamma} - \frac{\tilde{c}\beta}{1-\alpha} B(v) \varphi B(v) \left( \frac{\alpha}{1-\alpha} \right)^{\frac{\beta}{1-\alpha}} < 0.
\]

Therefore the slope of the phase line $CC$ will remain positive as before while the conditions of the new curve $EE$ will show a concave shape given that

\[
\left| \frac{d\tilde{c}}{d\tilde{e}} \right|_{EE} = -\frac{a_{22}}{a_{21}} = \frac{\Gamma\sigma(\gamma-\alpha\gamma-\beta)}{(1-\alpha)\gamma} \tilde{c}^\sigma \epsilon \left( \frac{\Gamma\sigma(\gamma-\alpha\gamma-\beta)}{(1-\alpha)\gamma} - \frac{\tilde{c}\beta}{1-\alpha} B(v) \varphi B(v) \left( \frac{\alpha}{1-\alpha} \right)^{\frac{\beta}{1-\alpha}} > 0,
\]

\[
\left| \frac{d^2\tilde{c}}{d\tilde{e}^2} \right|_{EE} < 0.
\]

As before the stable manifold is given by

\[
e = \tilde{c} + (e - \tilde{c}) \exp(v\tau), \quad c = \tilde{c} + \omega_2 (e - \tilde{c})
\]

where in this case

\[
\omega_2 = \frac{v - b_{22}}{\tilde{c}} = -\frac{b_{12}}{\tilde{c} - v} > 0
\]

represents the slope of the saddlepath $SS$ and $V$ denotes the negative root of (C14).
Appendix D

This Appendix derives the value of the growth-maximizing spending share on education for each approach. Consider the first approach, where the school quality is given by equation (9). Using the BGP equation (22) together with Proposition 1 as \( dv_e = -dv_i \) the effect of an increase in the government’s spending share on education, \( v_E \) on the growth rate will be given by

\[
\frac{d\Omega}{dv_E} = -\alpha \left( \begin{array}{c} sB(\tau) \frac{\alpha}{\alpha + \beta} - sB(v_i, \tau) \frac{\alpha \beta}{1 - \alpha} \end{array} \right) = 0
\]

or rewriting it as,

\[
\frac{d\Omega}{dv_E} = \left( \Omega + \rho \right) \left[ -\frac{\alpha}{1 - \alpha} v_i^{-1} + \frac{\beta}{1 - \alpha} \tilde{e}^{-1} \left( \frac{\partial \tilde{e}}{\partial v_E} \right) \right] = 0. \tag{D1}
\]

Which then multiplied by \( v_E \) yields

\[
\left[ -\frac{\alpha}{1 - \alpha} v_i^{-1} + \frac{\beta}{1 - \alpha} \tilde{e}^{-1} \left( \frac{\partial \tilde{e}}{\partial v_E} \right) \right] v_E = 0
\]

hence, the growth-maximizing share has to satisfy

\[
-\frac{\alpha}{1 - \alpha} \left( \frac{v_E}{v_i} \right) + \frac{\beta}{1 - \alpha} \left( \frac{v_E}{\tilde{e}} \right) \left( \frac{\partial \tilde{e}}{\partial v_E} \right) = 0. \tag{D2}
\]

From (D2) it can be taken into account that \( v_i^{-1} \) is such that the elasticity of \( \tilde{e} \) with respect to \( v_E \) is equal to

\[
\left( \frac{v_E}{\tilde{e}} \right) \left( \frac{\partial \tilde{e}}{\partial v_E} \right) = \alpha \left( \frac{v_E}{v_i} \right) \left( \frac{1 - \alpha}{\beta} \right)
\]

which is referred in the text as (26) as the following equation

\[
\left( \frac{v_E}{\tilde{e}} \right) \left( \frac{\partial \tilde{e}}{\partial v_E} \right) = \eta \left( \frac{v_E}{v_i} \right) = \alpha \left( \frac{v_E}{v_i} \right) \tag{D3}
\]
Similarly, using the BGP equation (23) together with Proposition 1 as 
\[ (d_{v_{E}} = -d_{v_{I}}) \] the effect of an increase in the government’s spending share on education, \( v_{E} \) on the growth rate yields,

\[
\frac{d\Omega}{dv_{E}} = \mu B^\mu D \left( \tau(v_{I}) \right)^{\alpha-1} v_{E}^{\alpha-1} \bar{e}^{-x} - \frac{\alpha \mu}{1-\alpha} B^{\mu} D \left( v_{E}^{\alpha-1} \bar{e}^{-x} \right) \left( v_{I} \right)^{\alpha-1} \bar{e}^{-x} \\
- \chi B^\mu D \left( v_{E}^{\alpha-1} \bar{e}^{-x} \right) \left( \frac{\partial \bar{e}}{\partial v_{E}} \right) = 0
\]

or equivalently as

\[
\frac{d\Omega}{dv_{E}} = B^\mu D \left( v_{E}^{\alpha-1} \bar{e}^{-x} \right) \left( \mu v_{E}^{\alpha-1} - \frac{\alpha \mu}{1-\alpha} (v_{I})^{-1} - \chi \bar{e}^{-1} \left( \frac{\partial \bar{e}}{\partial v_{E}} \right) \right) = 0
\]

Same as before (D4) multiplied by \( v_{E} \) will give

\[
\left[ \mu v_{E}^{\alpha-1} - \frac{\alpha \mu}{1-\alpha} (v_{I})^{-1} - \chi \bar{e}^{-1} \left( \frac{\partial \bar{e}}{\partial v_{E}} \right) \right] = 0
\]

thus, the growth-maximizing spending share must also satisfy

\[
\mu - \frac{\alpha \mu}{1-\alpha} v_{E} - \chi \frac{v_{E}}{\bar{e}} \left( \frac{\partial \bar{e}}{\partial v_{E}} \right) = 0,
\]

\[
\mu - \frac{\alpha \mu}{1-\alpha} v_{E} - \chi \frac{v_{E}}{\bar{e}} \left( \frac{\partial \bar{e}}{\partial v_{E}} \right) = 0
\]

\[
\frac{v_{E}}{v_{I}} = \frac{\mu - \chi \eta_{\bar{e}/v_{E}}}{1-\alpha} \frac{1-\alpha}{\alpha \mu}
\]

\[
\frac{v_{E}}{v_{I}} = \frac{1-\alpha}{\alpha} - \frac{\chi}{\mu} \frac{1-\alpha}{\alpha \eta_{\bar{e}/v_{E}}}
\]

(D5)

Using the definition of \( \chi \equiv (1 - \alpha - \beta) / (1 - \alpha) \) as \( \chi / \mu = (1 - \alpha - \beta) / (1 - \alpha) \) and (D3), equation (D5) becomes
\[
\frac{v_E}{v_I} = \frac{1 - \alpha}{\alpha} - \frac{1 - \alpha - \beta}{1 - \alpha} \left( \frac{\alpha v_E}{\beta v_I} \right)
\]

\[
\frac{v_E}{v_I} = \frac{1 - \alpha}{\alpha} - \frac{1 - \alpha - \beta v_E}{\beta v_I}
\]

\[
\frac{v_E}{v_I} = \frac{1 - \alpha}{\beta} - \frac{1 - \alpha}{\alpha}
\]

\[
\frac{v_E}{v_I} = \frac{\beta}{\alpha},
\]

(D6)

noting that \( v_E + v_I \rightarrow \Gamma \); and then substituting in the (D6) it is straightforward to see that

\[
v_E \equiv \frac{\beta}{\alpha + \beta},
\]

(D7)

is the same as equation (27) in the text.

Consider now the second approach, where the school quality is given by equation (15). Using the BGP equation (25) together with \( Proposition \ 1 \) as \( (dv_E = -dv_I) \) the effect of an increase in the government’s spending share on education, \( v_E \) on the growth rate will be given by

\[
\frac{d\Omega}{dv_E} = \frac{\Omega}{1 + \zeta(1 - \phi)} \left[ \mu + \zeta(v_E)^{-1} - \frac{\alpha\gamma(\mu + \zeta) + \zeta(1 - \phi)}{(1 - \alpha)\gamma} (v_I)^{-1} \right] = 0
\]

(D8)

which then multiplied by \( v_E \) yields

\[
\left[ \mu + \zeta(v_E)^{-1} - \frac{\alpha\gamma(\mu + \zeta) + \zeta(1 - \phi)}{(1 - \alpha)\gamma} (v_I)^{-1} - \frac{\Gamma(\gamma - \alpha\gamma - \beta)}{(1 - \alpha)\gamma} \left( \frac{\partial\bar{e}}{\partial v_E} \right) \right] = \frac{1}{v_E}
\]

thus, the growth-maximizing spending share must also satisfy

\[
\mu + \zeta - \frac{\alpha\gamma(\mu + \zeta) + \zeta(1 - \phi) v_E}{(1 - \alpha)\gamma} \frac{v_I}{v_E} - \frac{\Gamma(\gamma - \alpha\gamma - \beta)}{(1 - \alpha)\gamma} \frac{v_E}{\bar{e}} \left( \frac{\partial\bar{e}}{\partial v_E} \right) = 0
\]

that if (D3) is plugged in, gives

\[
\mu + \zeta((1 - \alpha)\gamma) = \alpha\gamma(\mu + \zeta) + \zeta(1 - \phi) \frac{v_E}{v_I} + \Gamma(\gamma - \alpha\gamma - \beta) \left( \frac{\alpha v_E}{\beta v_I} \right).
\]

(D9)
Using the definition of \((C5), \Gamma = (\mu + \zeta)\gamma + \zeta(1 - \phi)\), equation (D9) becomes

\[
(\mu + \zeta)(1 - \alpha)\gamma = \left[ \gamma(\mu + \zeta)\alpha\beta + (\gamma - \alpha\gamma - \beta)\alpha + \zeta(1 - \phi)\beta + (\gamma - \alpha\gamma - \beta)\alpha \right] \frac{v_E}{v_i},
\]

from which separating \(\frac{v_E}{v_i}\) gives

\[
\frac{v_E}{v_i} = \frac{(\mu + \zeta)(1 - \alpha)\beta\gamma}{(\mu + \zeta)(\gamma - \alpha\gamma)\alpha\gamma + \zeta(1 - \phi)\beta + (\gamma - \alpha\gamma - \beta)\alpha}.
\]

Again noting that \(v_E + v_i \rightarrow 1\); and then substituting in the (D10) with some

manipulations as shown yields

\[
\begin{align*}
\frac{v_E}{v_i} &\approx \frac{(\mu + \zeta)(1 - \alpha)\beta\gamma}{(\mu + \zeta)(\gamma - \alpha\gamma)\alpha\gamma + (1 - \phi)(\gamma - \alpha\gamma - \beta)\alpha + (\mu + \zeta)(1 - \alpha)\beta\gamma} \\
&\approx \frac{(\mu + \zeta)(1 - \alpha)\beta\gamma}{(\mu + \zeta)(1 - \alpha)\beta\gamma + \zeta(1 - \phi)(\gamma - \alpha\gamma + \beta)} \\
&\approx \frac{(\mu + \zeta)(1 - \alpha)\beta\gamma}{(1 - \alpha)(\gamma + \beta)(\mu + \zeta)\gamma + \zeta(1 - \phi)} \\
&\approx \frac{(\mu + \zeta)\beta\gamma}{(\gamma + \beta)(\mu + \zeta)\gamma + \zeta(1 - \phi)},
\end{align*}
\]

This is equation (28) in the text.
### Appendix E

#### Composite variables

<table>
<thead>
<tr>
<th>Equation</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>[ \mathcal{G} = M \left( v_E \right)^{\frac{\alpha}{\mu + \zeta} (v_f \tau)^{\frac{\alpha}{1 - \alpha}} } ]</td>
<td>Composite variable ( \mathcal{G} )</td>
</tr>
<tr>
<td>[ \Gamma = (\mu + \zeta) \gamma + \zeta (1 - \phi) ]</td>
<td>Parameter ( \Gamma )</td>
</tr>
<tr>
<td>[ \sigma = \frac{1}{1 + \zeta (1 - \phi)} ]</td>
<td>Parameter ( \sigma )</td>
</tr>
<tr>
<td>[ \varphi = [1 - \tau (V_E + V_f)] ]</td>
<td>Parameter ( \varphi )</td>
</tr>
<tr>
<td>[ \chi = (1 - \alpha - \beta) \mu / (1 - \alpha) ]</td>
<td>Parameter ( \chi )</td>
</tr>
<tr>
<td>[ \psi = B^\mu D \left( v_E \tau (v_f \tau)^{\frac{\alpha}{1 - \alpha}} \right)^\mu ]</td>
<td>Parameter ( \psi )</td>
</tr>
<tr>
<td>[ A \equiv \frac{(1 - \theta_f)(1 - \tau)}{1 - \tau + \tau \beta \left(1 - (V_E + V_f) \right)} ]</td>
<td>Parameter ( A )</td>
</tr>
<tr>
<td>[ B \equiv A^{\frac{\beta}{1 - \alpha}} \equiv (1 - \theta_S - \theta_f)^{\frac{\beta}{1 - \alpha}} ]</td>
<td>Parameter ( B )</td>
</tr>
<tr>
<td>[ D = a^{1 - \kappa - \mu} (1 + \theta_f)^{1 + \xi - \mu} (1 + \theta_S)^{1 - \phi \xi} ]</td>
<td>Parameter ( D )</td>
</tr>
<tr>
<td>[ D_1 = a^{1 - \kappa - \mu} (1 + \theta_S)^{1 - \phi \xi} ]</td>
<td>Parameter ( D_1 )</td>
</tr>
<tr>
<td>[ J \equiv a^{(1 - \mu - \kappa) \gamma - \zeta (1 - \phi) \pi \xi} (1 + \theta_f) (1 + \theta_S) (1 - \mu - \xi \kappa) \gamma - \zeta (1 - \phi) ]</td>
<td>Parameter ( J )</td>
</tr>
<tr>
<td>[ J_1 \equiv a^{(1 - \mu - \kappa) \gamma - \zeta (1 - \phi) \pi \xi} (1 + \theta_S) ]</td>
<td>Parameter ( J_1 )</td>
</tr>
<tr>
<td>[ M \equiv J B^{(\mu + \zeta) \gamma + \zeta (1 - \phi)} ]</td>
<td>Parameter ( M )</td>
</tr>
<tr>
<td>[ M_1 \equiv J_1 B^{(\mu + \zeta) \gamma + \zeta (1 - \phi)} ]</td>
<td>Parameter ( M_1 )</td>
</tr>
<tr>
<td>[ s \equiv (1 - \tau)(1 - \alpha - \beta) ]</td>
<td>Parameter ( s )</td>
</tr>
</tbody>
</table>