ON SPECTRUM OF THE LINEAR WATER-WAVE PROBLEM IN CUSPIDAL DOMAINS

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Academic dissertation

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Jussi Martín
LIST OF INCLUDED ARTICLES

This thesis consists of an introductory part and the following articles:


1. Overview

Theory of linear water-waves is a linearised version of a more complicated non-linear model of water-waves; description of the non-linear model and the linearisation scheme can be found from [3] and [8]. Even though this theory is only an approximation of the more realistic one, it has been found useful in many applications.

In this thesis we study the spectrum of a boundary value problem (see (3.1)-(3.3) below) that arises after separation of variables for velocity potentials of time-harmonic linear water-waves. We cover situations where the domain has a cuspidal singularity on the boundary, which causes the continuous spectrum to appear. This is done in the articles [A] and [B]. In the article [C] we study a similar case for sloshing waves on an interface in a two-layer liquid.

One particular motivation for studying linear water-waves in cuspidal domains, is the possible existence of so called “black holes”, which are structures that absorb waves. Furthermore, the spectral problem (3.1)-(3.3) has also mathematical interest of its own, due to its unusual form, where the spectral parameter appears on the boundary condition.

Previously the problem (3.1)-(3.3) has been studied in cuspidal domains in [6], where it was shown for the first time that continuous spectrum in a bounded cuspidal domain can be non-empty. This was also done in a bounded domain where the cuspidal shape is created by a submerged object touching the surface at one point. In this later domain, it was also shown that if the sharpness exponent of the rotational cusp is 2, then the essential spectrum contains an interval $[\lambda_0, \infty)$, for some $\lambda_0 > 0$, which depends on the geometry of the domain.

The article [C], on the other hand, is the first one on the two-layer liquid case in a domain with cuspidal shape on one of the liquid domains.

Techniques that are used in studying the spectrum of the problem (3.1)-(3.3) in cuspidal domains, were first developed for the related Steklov problem in peak shaped domains. In the Steklov problem the spectral boundary condition (see (3.3) below) appears on the whole boundary. Results concerning the Steklov problem in peak shaped domains can be found from [4] and [5].

2. Linear surface waves

2.1. The setting. In order to make a distinction between vertical and horizontal coordinates, we will denote in the sequel a point $x \in \mathbb{R}^3$ as $x = (y_1, y_2, z)$.

Let $\Omega$ be a domain in the lower half-space $\mathbb{R}^3_+ = \{x \in \mathbb{R}^3 : z < 0\}$, which is bounded between two $C^2$-surfaces $\Lambda$ and $\Sigma$ in such a way that $\Lambda$ is a flat surface in the horizontal plane $\Pi = \{x \in \mathbb{R}^3 : z = 0\}$, $\Sigma$ is a surface in $\mathbb{R}^3_-$, and they share a mutual boundary, which is a closed simple $C^2$-contour $\gamma \subset \Pi$. The domain $\Omega$ describes a pond or a lake, where $\Lambda$ and $\Sigma$ represent the free surface of the water and the bottom of the pond $\Omega$.

2.2. Equations of motion. In the theory of linear surface waves the water flow is assumed to be incompressible, which leads to the equation

$$\nabla_x \cdot \mathbf{v}(x, t) = 0, \quad x \in \Omega, \ t \geq t_0$$

for the velocity field $\mathbf{v}$, with starting time $t_0$. The flow is also assumed to be irrotational, implementing that

$$\nabla_x \times \mathbf{v}(x, t) = 0, \quad x \in \Omega, \ t \geq t_0.$$
This implies the existence of velocity potential \( \phi \), for which

\[
\nabla_x \phi(x, t) = v(x, t), \quad x \in \Omega, \quad t \geq t_0.
\]

Together, (2.1) and (2.3) imply that

\[
\Delta_x \phi(x, t) = 0, \quad x \in \Omega, \quad t \geq t_0;
\]

where \( \Delta_x \) is the Laplacian. The flow does not go through the bottom, which leads to the equation

\[
\partial_n \phi(x, t) = 0, \quad x \in \Sigma, \quad t \geq t_0;
\]

where \( \partial_n \) is the derivative along the outward normal \( n \). Finally we come to the surface boundary condition

\[
\partial^2 \phi(x, t) + g \partial_n \phi(x, t) = 0, \quad x \in \Lambda, \quad t \geq t_0;
\]

where \( g \) is the acceleration caused by gravity. This equation describes the linear wave motion of the water surface.

3. Spectral problem for linear surface waves

3.1. Time-harmonic surface waves. By separation of the time and space variables, we see that the velocity potentials of time-harmonic\(^1\) linear water-waves have the form

\[
\phi(x, t) = \text{Re} \left( \Phi(x) e^{-i\omega t} \right),
\]

where \( \Phi \) is a solution of the spectral boundary value problem

\[
\begin{align*}
-\Delta_x \Phi(x) &= 0, \quad x \in \Omega, \\
\partial_n \Phi(x) &= 0, \quad x \in \Sigma, \\
\partial^2 \Phi(x) &= \lambda \Phi(x), \quad x \in \Lambda;
\end{align*}
\]

moreover, \( \lambda = \omega^2 / g \) is the spectral parameter proportional to the square of the frequency of harmonic oscillations. The condition (3.3) is called a Steklov spectral boundary condition.

In order to give exact meaning to spectral concepts, such as continuous spectrum, we shall formulate the problem (3.1)–(3.3) as a standard spectral problem for a certain operator \( K_\Omega \), defined on a function space over the domain \( \Omega \). For this purpose we shall now give a more precise definition for the domain \( \Omega \).

In the \( \delta \)-neighbourhood \( \mathcal{U}_\delta \subset \Pi \) of the contour \( \gamma \subset \Pi \) we introduce the natural system of the curvilinear coordinates \( (\nu, \tau) \), where \( \nu \) is the oriented distance to \( \gamma \), \( \nu > 0 \) inside \( \Lambda \), and \( \tau \) is the curve length on \( \gamma \). We assume that in the vicinity of \( \gamma \subset \mathbb{R}^3 \) the domain \( \Omega \) is given by the inequalities

\[
-\nu^m g(\nu, \tau) < z < 0, \quad \nu > 0,
\]

where \( m > 1 \) and \( g \) is a positive \( C^2 \)-function on \([0, \delta] \times \gamma\). In the article [A] the definition of the domain \( \Omega \) is slightly different; this definition will be described later in Section 7.

\(^1\)A wave is said to be time-harmonic if it is sinusoidal at every point \( x \), and the frequency of this sinusoidal oscillation is independent of the point \( x \).
3.2. **Operator formulation of the problem** (3.1)-(3.3). We define the Hilbert space $\mathcal{H}(\Omega; \Lambda)$, as the completion of the space $C^\infty_c(\bar{\Omega} \setminus \gamma)$ of compactly supported smooth functions with respect to the norm $\|\Phi; \mathcal{H}(\Omega; \Lambda)\| := (\Phi, \Phi)_{\Omega}^{1/2}$, where
\[
(\Phi, \Psi)_{\Omega} = (\nabla_x \Phi, \nabla_x \Psi)_{\Omega} + (\Phi, \Psi)_\Lambda
\]
and $(\cdot, \cdot)_\Omega$ and $(\cdot, \cdot)_\Lambda$ are the intrinsic inner products in the spaces $L^2(\Omega)$ and $L^2(\Lambda)$, respectively.

One obvious property of the space $\mathcal{H}(\Omega; \Lambda)$ is that the trace on $L^2(\Lambda)$, denoted by $\Psi(\cdot, 0)$, exists for all $\Psi \in \mathcal{H}(\Omega; \Lambda)$. In the sequel the trace $\Psi(\cdot, 0)$ will be often abbreviated as $\Psi$, for the sake of simplicity.

If $\lambda > 0$ and $m > 1$, then a classical solution of (3.1)-(3.3) must vanish on $\gamma$ for (3.2) and (3.3) to hold simultaneously, since the $z$-direction is parallel to the normal direction of $\Sigma$ on $\gamma$ if $m > 1$. Also, the first Green formula implies that
\[
(\nabla_x \Phi, \nabla_x \Psi)_{\Omega} = \lambda (\Phi, \Psi)_\Lambda, \quad \Psi \in C^\infty_c(\bar{\Omega} \setminus \gamma)
\]
if $\Phi$ is a classical solution of the problem (3.1)-(3.3) and $\Psi \in C^\infty_c(\bar{\Omega} \setminus \gamma)$ is arbitrary. Thus it is natural to define a weak solution of (3.1)-(3.3) to be a function $\Phi \in \mathcal{H}(\Omega; \Lambda)$ for which
\[
(\nabla_x \Phi, \nabla_x \Psi)_{\Omega} = \lambda (\Phi, \Psi)_\Lambda \quad \text{for all} \quad \Psi \in \mathcal{H}(\Omega; \Lambda).
\]
This leads us to the operator formulation of the spectral problem (3.1)-(3.3).

Namely, we define the operator $K_\Omega$ from the space $\mathcal{H}(\Omega; \Lambda)$ to itself by the relation
\[
(\Phi, \Omega)_{\Omega} = (\Phi, \Psi)_{\Lambda} \quad \text{for all} \quad \Phi, \Psi \in \mathcal{H}(\Omega; \Lambda).
\]
The operator $K_\Omega$ defined this way is clearly symmetric, bounded and therefore self-adjoint. Adding the term $(\Phi, \Psi)_\Lambda$ to both sides of the equation (3.5) and multiplying them by $\mu = (1 + \lambda)^{-1}$, we see that the problem (3.5) is equivalent to the equation
\[
K_\Omega \Phi = \mu \Phi, \quad \Phi \in \mathcal{H}(\Omega; \Lambda).
\]
The spectrum of the problem (3.1)-(3.3) and the spectrum of the operator $K_\Omega$ are now related in the following way. Let $\lambda \in \mathbb{C}$; then $\lambda$ is in the spectrum of the problem (3.1)-(3.3) if and only if $\mu = (1 + \lambda)^{-1}$ is in the spectrum of $K_\Omega$. The same relation holds also for discrete, continuous and essential spectra.

Since $m > 1$, the constant functions are contained in the space $\mathcal{H}(\Omega; \Lambda)$, even though they are not contained in the space $C^\infty_c(\Omega \setminus \gamma)$. They form the eigenspace of eigenvalue $\mu = 1$. The eigenspace of the eigenvalue $\mu = 0$, on the other hand is infinite dimensional, and consists of functions $\Phi \in \mathcal{H}(\Omega; \Lambda)$ with $\Phi(\cdot, 0) \equiv 0$.

Furthermore, for the operator norm of $K_\Omega$ we obtain
\[
\|K_\Omega\| = \sup_{\Phi \in \mathcal{S}} \langle K_\Omega \Phi, \Phi \rangle_{\Omega} = \sup_{\Phi \in \mathcal{S}} (\Phi, \Phi)_\Lambda
\]
\[
\leq \sup_{\Phi \in \mathcal{S}} (\Phi, \Phi)_\Omega = 1;
\]
where $\mathcal{S}$ denotes the unit sphere in $\mathcal{H}(\Omega; \Lambda)$. And for a constant function $\Phi \equiv \text{mes}_2(\Lambda)^{-1/2}$, where $\text{mes}_2(\Lambda)$ denotes the 2-dimensional Lebesgue-measure of $\Lambda$, we get
\[
\langle K_\Omega \Phi, \Phi \rangle_{\Omega} = (\Phi, \Phi)_\Lambda = (\Phi, \Phi)_\Omega = 1;
\]
hence the operator norm of $K_\Omega$ equals to 1.
Since the operator $K_\Omega$ is positive and $\|K_\Omega\| = 1$, its spectrum is real valued and contained in the interval $[0, 1]$; thus the spectrum of the problem (3.1)–(3.3) is contained in the interval $[0, \infty)$.

In the articles [A] and [B] the spectrum of the problem (3.1)–(3.3) is studied further, and these results are stated in Section 7.

4. Linear interface waves

4.1. The setting. In the article [C] we study linear time-harmonic waves in a two-layer liquid, which appear both on the free surface and in the interface between the two liquids. The two liquids could for example consist of warm and cold water, or sweat and salted water. These type of waves are referred to as linear interface waves in the sequel. Equations of motion for the linear interface waves are analogous to those described in Section 2, and for that reason they will be omitted in this presentation.

In this section we shall describe the geometry of the domain, and in the next one we shall formulate the spectral problem for these linear interface waves.

We assume that the liquid domain is a infinitely long cylinder with curved cross-section as in Fig. 1.1. The domain is assumed to be such that the shape of the cross-section is constant in the direction where the domain tends to infinity, and the boundary curve of the cross-section creates a cuspoidal singularity in one of the two liquid domains.

For practical purposes we now change the notations in following ways: the points in $\mathbb{R}^3$ will be denoted as $(x, y, z)$, and when considering linear interface waves, $\Omega$ will always denote the 2-dimensional cross-section of the liquid domain.

4.2. Geometry of the cross-section. We set $z$-direction to be the unbounded direction. The cross-section $\Omega$ of the cylinder is bounded from above by the line segment $S = \{(y, z) \in \mathbb{R}^2 : z = 0, |y| < L\}$, which presents the free surface. And the rest of $\partial \Omega$ is a smooth arc $B$ connecting the points $(\pm L, 0)$ inside the lower half-plane $\mathbb{R}^2$; thus the arc $B$ forms the bottom and walls of $\Omega$. The domain $\Omega$ splits into two subdomains

\[
\Omega_0 = \{(y, z) \in \Omega : z > -d\} \quad \text{and} \quad \Omega_1 = \{(y, z) \in \Omega : z < -d\}
\]

at the level $z = -d \in (0, -b_0)$, where $b_0$ is the smallest $z$-coordinate of the points of $B$. By rescaling, we reduce the half-length $L$ to 1 and thus make the coordinates $(y, z)$ dimensionless. The subdomains $\Omega_0$ and $\Omega_1$ are filled with two immiscible liquids, which have densities $\rho_0$ and $\rho_1$, respectively, with $\rho_1 > \rho_0 > 0$ due to gravity.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure1_1.png}
\caption{Two-layer-liquid with cuspoidal geometries at the interface.}
\end{figure}

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure1_2.png}
\caption{Alternative geometries.}
\end{figure}
The interface $I = \partial \Omega_0 \cap \partial \Omega_1$ between the two liquids is assumed to be a segment \{(y, z) : y \in [-L, L], z = -d\} of length $l = l_+ + l_- > 0$, however, the main results of the article [C] remain valid also in situations of Fig. 1.2 a, b.

The cuspidal geometry is related to the interface $I$ as follows: we assume that the curve $B$ is given in the vicinity of the points $P^\pm = (\pm l, d) \subset I$ by the equations
\begin{equation}
(4.2) \quad z = -d + h_{\pm}(y),
\end{equation}
where
\begin{equation}
(4.3) \quad h_{\pm}(\pm l_{\pm}) = \partial_y h_{\pm}(\pm l_{\pm}) = 0, \quad b_{\pm} := \partial^2_y h_{\pm}(\pm l_{\pm}) \neq 0.
\end{equation}
in other words, $B$ becomes tangential with the segment $I$ at its endpoints and therefore one of the subdomains $\Omega_j$ loses its Lipschitz property. On the other hand, no cuspidal geometry is supposed to appear in the vicinity of the free water surface: the curve $B$ is defined to intersect the $y$-axis in the points $(\pm L, 0)$ at the angles $\theta_{\pm} \in (0, \pi)$, see Fig. 1.1 a and b. Hence, the domains $\Omega_0$ in Fig. 1.1 a and $\Omega_1$ Fig. 1.1 b are Lipschitz while, respectively, $\Omega_1$ and $\Omega_0$ are not. We set $J = 1$ in the first case and $J = 0$ in the second one.

5. Spectral problem for linear interface waves

5.1. Time-harmonic interface waves. Similarly, to surface waves, after separation of variables we see that the velocity potentials of time-harmonic waves in the two-layer case have the form
\begin{equation}
(5.1) \quad \phi_j(y, z, t) = \text{Re}(\varphi_j(y, z)e^{-i\omega t}), \quad j = 0, 1,
\end{equation}
where $\varphi_j$ satisfy the Laplace equation in their domains of definition, that is
\begin{equation}
(5.2) \quad -q_j \Delta \varphi_j(y, z) = 0 \quad (y, z) \in \Omega_j, \quad j = 0, 1.
\end{equation}
At the free surface $S$ we have the Steklov spectral boundary condition
\begin{equation}
(5.3) \quad q_0 \partial_z \varphi_0(y, 0) = q_1 \lambda \varphi_0(y, 0), \quad |y| < L,
\end{equation}
and at the interface $I$ the spectral transmission condition
\begin{equation}
(5.4) \quad q_0 (\partial_z \varphi_0(y, -d) - \lambda \partial_z \varphi_0(y, -d)) = q_1 (\partial_z \varphi_1(y, -d) - \lambda \partial_z \varphi_1(y, -d)), \quad y \in (-l_-, l_+),
\end{equation}
with $\lambda = \omega^2 / g$, where frequency $\omega$ comes from (5.1). A physical interpretation of the conditions (5.3) and (5.4) can be found in [3] and, e.g., [2, 7].

At the surfaces $B_0 = \{(y, z) \in B : z \in (-d, 0)\}$ and $B_1 = \{(y, z) \in B : z < -d\}$ we have Neumann boundary condition
\begin{equation}
(5.5) \quad \partial_n \varphi_j(y, z) = 0, \quad (y, z) \in B_j \quad j = 0, 1.
\end{equation}
Further more, the normal velocity is assumed continuous at the interface:
\begin{equation}
(5.6) \quad \partial_z \varphi_0(y, -d) = \partial_z \varphi_1(y, -d), \quad y \in (-l_-, l_+).
\end{equation}

Next we shall formulate the problem (5.2)–(5.6) as a spectral problem for a certain operator $T$, analogously to what we did for the problem (3.1)–(3.3). Some details of this construction are similar to those seen in Section 3.1 and will be skipped.
5.2. Operator formulation of the problem (5.2)-(5.6). Let us define bilinear forms

\[
D(\varphi, \psi) = \varrho_0(\nabla \varphi_0, \nabla \psi_0)_{\Omega_0} + \varrho_1(\nabla \varphi_1, \nabla \psi_1)_{\Omega_1},
\]
\[
T(\varphi, \psi) = \varrho_0(\varphi_0, \psi_0)s + \frac{1}{\varrho_1 - \varrho_0}(\varrho_1 \varphi_1 - \varrho_0 \varphi_0, \varrho_1 \psi_1 - \varrho_0 \psi_0)t,
\]

where \((\cdot, \cdot)_X\) denotes the intrinsic inner product in the space \(L^2(X)\). And by \(\mathcal{P}\) we denote the set of corner or cuspidal points, marked by \(\bullet\) in Fig. 1.1.

Let also \(C^\infty_c(\overline{\Omega_0} \setminus \mathcal{P}) \times C^\infty_c(\overline{\Omega_1} \setminus \mathcal{P})\) denote the space of functions \(\varphi = (\varphi_0, \varphi_1)\) with components \(\varphi_j \in C^\infty_c(\overline{\Omega}_j \setminus \mathcal{P})\), which are smooth in the closure \(\overline{\Omega}_j\) but vanish in a neighbourhood of the set \(\mathcal{P}\).

We define the Hilbert space \(\mathcal{H}\), as the completion of the space \(C^\infty_c(\overline{\Omega}_0 \setminus \mathcal{P}) \times C^\infty_c(\overline{\Omega}_1 \setminus \mathcal{P})\) with respect to the norm

\[
\|\varphi; \mathcal{H}\| = (D(\varphi, \varphi) + T(\varphi, \varphi))^{1/2}.
\]

It is a Hilbert space, endowed with the inner product

\[
\langle \varphi, \psi \rangle = D(\varphi, \psi) + T(\varphi, \psi),
\]

since the bilinear form \((\cdot, \cdot)\), defined this way, is positive and non-degenerate.

Let now \(\varphi\) be a classical solution of the problem (5.2)-(5.6). By multiplying the Laplace-equation (5.2) with a test function \(\psi \in C^\infty_c(\overline{\Omega}_0 \setminus \mathcal{P}) \times C^\infty_c(\overline{\Omega}_1 \setminus \mathcal{P})\), integrating by parts, using boundary conditions (5.4), (5.6), rewriting (5.4), (5.6) as

\[
\partial_z \varphi_0 = \partial_z \varphi_1 = \frac{\lambda}{\varrho_1 - \varrho_0}(\varrho_1 \varphi_1 - \varrho_0 \varphi_0) \quad \text{on} \quad I,
\]

and summing the result we see that the problem (5.2)-(5.6) is equivalent to equation\(^2\)

\[
D(\varphi, \psi) = \lambda T(\varphi, \psi).
\]

Thus we define a weak solution of (5.2)-(5.6) to be a function \(\varphi \in \mathcal{H}\) which satisfies

\[
D(\varphi, \psi) = \lambda T(\varphi, \psi) \quad \text{for all} \quad \psi \in \mathcal{H}.
\]

This leads us to operator formulation of the problem (5.2)-(5.6). We define the operator \(\mathcal{T}\) in \(\mathcal{H}\) by formula

\[
\langle \mathcal{T} \varphi, \psi \rangle = T(\varphi, \psi) \quad \text{for all} \quad \varphi, \psi \in \mathcal{H}.
\]

It is continuous, positive and symmetric, hence self-adjoint. Moreover, its norm is equal to 1 and \(\mu = 0\) is an eigenvalue of infinite multiplicity, having the eigenspace

\[
\{ \varphi \in \mathcal{H} : \varphi = 0 \text{ on } S, \ \varrho_1 \varphi_1 = \varrho_0 \varphi_0 \text{ on } I \}.
\]

Adding the term \(T(\varphi, \psi)\) to both sides of the equation (5.11) and multiplying them by \(\mu = (1 + \lambda)^{-1}\), we see that the problem (5.11) is equivalent to the equation

\[
\mathcal{T} \varphi = \mu \varphi \quad \text{in} \quad \mathcal{H}
\]

with new spectral parameter

\[
\mu = (1 + \lambda)^{-1}.
\]

\(^2\)This variational formulation for a two-layer liquid is done according to [7].
The spectrum of the problem (5.2)–(5.6) and the spectrum of the operator $\mathcal{T}$ are now related in the following way. Let $\lambda \in \mathbb{C}$; then $\lambda$ is in the spectrum of the problem (5.2)–(5.6) if and only if $\mu = (1 + \lambda)^{-1}$ is in the spectrum of $\mathcal{T}$. The same relation holds also for discrete, continuous and essential spectra.

Similarly, to the operator $K_\Omega$, we see that the spectrum of $\mathcal{T}$ is real valued and contained in the interval $[0, 1]$; thus the spectrum of the problem (5.2)–(5.6) is contained in the interval $[0, \infty)$.

The results of article [C], concerning further properties of the spectrum of (5.2)–(5.6), are stated in Section 7.

6. On spectra

6.1. Definitions of spectra. The definitions of continuous and essential spectra differ in different books, we use those introduced in [1]. Namely, the point spectrum of an operator $A$ is denoted by $\sigma_p(A)$, the continuous spectrum is the set

$$\sigma_c(A) = \{ \lambda \in \mathbb{C} : R(A - \lambda I) \neq R(A - \lambda I) \}$$

where $R$ denotes the range of given operator, the essential spectrum is the set

$$\sigma_e(A) = \sigma_c(A) \cup \sigma^\infty_p(A)$$

where $\sigma^\infty_p(A)$ is the set of eigenvalues with infinite multiplicity, and the discrete spectrum is the set

$$\sigma_d(A) = \sigma_p(A) \setminus \sigma^\infty_p(A).$$

It should be noted, that with these definitions it is possible that $\sigma_p(A) \cap \sigma_c(A) \neq \emptyset$.

6.2. Function spaces and the spectra. Before stating the results of the articles [A], [B] and [C], we still need to clear out one ambiguity that arises from the use of complex valued function spaces in the study of spectral problems (3.1)–(3.3) and (5.2)–(5.6).

The formulas (3.6) and (5.12) define the operators $K_\Omega$ and $\mathcal{T}$, whether the function spaces $\mathcal{H}(\Omega; A)$ and $\mathcal{H}$ are defined for real or complex valued functions. In the articles [A], [B] and [C] we study the spectra in complex valued function spaces, however linear water-waves are described by real vector valued velocity fields; which would suggest that it might be more natural to define the problems (3.1)–(3.3) and (5.2)–(5.6) in real valued function spaces.

This raises a question whether the spectra of the operators $K_\Omega$ and $\mathcal{T}$ might depend on the choice of real or complex valued function spaces. It turns out that there is no dependence on this choice, since the operators are self-adjoint.

We now verify this in a more general setting. For a real Hilbert space $\mathcal{H}_R$ we may define its complexification

$$\mathcal{H}_C := \{ \Phi_1 + i\Phi_2 : \Phi_k \in \mathcal{H}_R, k = 1, 2 \},$$

equipped with the inner product

$$(\Phi_1 + i\Phi_2, \Phi'_1 + i\Phi'_2)_C := (\Phi_1, \Phi'_1)_R + i(\Phi_1, \Phi'_2)_R - i(\Phi_2, \Phi'_1)_R + (\Phi_2, \Phi'_2)_R;$$

where $(\cdot, \cdot)_R$ denotes the inner product in the space $\mathcal{H}_R$. And for an operator

$$A_R : \mathcal{H}_R \rightarrow \mathcal{H}_R$$
we define its complexification

\[ A_C : \mathcal{H}_C \to \mathcal{H}_C, \quad \Phi_1 + i\Phi_2 \mapsto A_R\Phi_1 + iA_R\Phi_2. \]

The identities we need are now stated in the following lemma.

6.1. **Lemma.** If \( A_R \) is self-adjoint, then

\[ \sigma_0(A_C) = \sigma_0(A_R) \quad \text{for} \quad \circ = c, d, e. \]

In particular \( \sigma(A_C) = \sigma(A_R) \).

**Proof.** Let \( A_R \) now be self-adjoint, then \( \sigma_p(A_R) \subseteq \sigma(A_R) \subseteq \mathbb{R} \). Thus

\[ A_C\Psi = \lambda\Psi \quad \text{for} \quad \Psi = \Phi_1 + i\Phi_2 \]

if and only if

\[ A_R\Phi_k = \lambda\Phi_k \quad \text{for} \quad k = 1, 2; \]

hence \( \sigma_p(A_C) = \sigma_p(A_R) \). It is also easily seen that \( \sigma^\infty_p(A_C) = \sigma^\infty_p(A_R) \), which implies that \( \sigma_d(A_C) = \sigma_d(A_R) \).

Since the space \( \mathcal{H}_C \) can be identified with the space \( \mathcal{H}_R \times \mathcal{H}_R \) via the mapping

\[ \Phi_1 + i\Phi_2 \mapsto (\Phi_1, \Phi_2), \]

we have that \( R(A_C - \lambda I) \neq \overline{R(A_C - \lambda I)} \) if and only if

\[
R(A_R - \lambda I) \times R(A_R - \lambda I) \neq \overline{R(A_R - \lambda I)} \times \overline{R(A_R - \lambda I)}
\]

\[
\iff \quad R(A_R - \lambda I) \neq \overline{R(A_R - \lambda I)},
\]

since \( \overline{B \times B} = B \times \overline{B} \). Thus we see that \( \sigma_c(A_C) = \sigma_c(A_R) \), and using this together with the identity \( \sigma^\infty_p(A_C) = \sigma^\infty_p(A_R) \), we also obtain \( \sigma_\epsilon(A_C) = \sigma_\epsilon(A_R) \).

It can be easily seen that for a self-adjoint \( A_R \) the unique self-adjoint extension to \( \mathcal{H}_C \) is its complexification \( A_C \). Thus the Lemma 6.1 implies that the spectra of \( K_\Omega \) are independent of the choice of real or complex valued function spaces, since it is self-adjoint in both cases. Same holds also for \( \mathcal{T} \).

7. **Summary of articles**

In the article \( [\Lambda] \) the shape of the domain \( \Omega \) is slightly different than the one defined in Section 3, the definition of \( \Omega \) in \( [\Lambda] \) is as follows.

As in the definition in Section 2, \( \Lambda \) is the free surface of the pond \( \Omega \), but now its boundary curve \( \gamma \) is assumed to be only piecewise \( C^2 \)-smooth. Furthermore, the surface \( \Sigma \) is now allowed to have walls. It is union of the bottom surface \{ \( x = (y, z) : z = -h(y_1, y_2) \} \), where the smooth function \( h > 0 \) is the depth function of the pond, and the walls, which are smooth surfaces parallel to \( z \)-axis connecting the free surface \( \Lambda \) and the bottom. As in section 2, \( \partial\Omega = \Lambda \cup \Sigma \cup \gamma \) and the domain \( \Omega \) consists of points \( (y, z) \) with

\[ -h(y) < z < 0 \]

Finally, it is assumed \( \gamma \) contains the origin \( \mathcal{O} = (0, 0, 0) \), where it is \( C^2 \)-smooth, and that the coordinates are chosen so that \( \gamma \) is tangential to the \( y_2 \) axis at \( \mathcal{O} \) and \( \Lambda \) is contained in the right half-plane \{ \( (y, 0) : y_1 > 0 \} \) of \( \Pi \).

As a consequence of the assumptions, in particular the local \( C^2 \)-smoothness of \( \gamma \) at \( \mathcal{O} \), \( \Lambda \) contains a subdomain \( \mathcal{W} \times \{0\} \) with

\[ \mathcal{W} = \{ y : \alpha y_2^2 < y_1 < \delta \}, \]
for some large enough $\alpha > 0$ and small enough $\delta > 0$.

The main result of the article [A] is stated in the following theorem.

7.2. **Theorem.** Assume that for some subdomain $\mathcal{W}$ (cf. (7.1)) and constant $b > 0$, the depth function $h(y)$ of the pond $\Omega$ equals $by_1^2$ asymptotically at $O = (0,0,0)$, more precisely,

\[
|h(y) - by_1^2| \leq c(y_1^3 + y_1y_2^2 + y_2^4) \quad \text{and} \quad |\nabla_y h(y) - \nabla_y by_1^2| = c'(y_1^2 + y_1|y_2| + y_2^2)
\]

for some constants $c, c' > 0$, for all $y \in \mathcal{W}$. Then the essential spectrum of the problem (3.1)–(3.3) contains the interval $[b/4, \infty)$.

In the article [B] we assume that the domain $\Omega$ is defined as in Section 3, and the main result of this article is the following theorem.

7.4. **Theorem.** If $m > 2$, then the continuous spectrum of the problem (3.1)–(3.3) constitutes the whole spectrum, and it is the half-line $[0, \infty)$.

In the end of the article [B] we also make a remark, which states how the results of articles [6] and [A] concerning the essential spectrum, hold even in a sharper form. Namely, it is observed that interval of the form $[c, \infty)$, which is shown to be contained in the essential spectrum, is in fact contained in the continuous spectrum.

As mentioned earlier, the article [C] deals with the spectral problem (5.2)–(5.6). The main result in the article [C] is stated as follows.

7.5. **Theorem.** Assume that the water domain has the cuspidal geometries at the interface as described above. Then the water wave problem (5.2)–(5.6) for two layer liquid has the continuous spectrum

$$\sigma_c = [\lambda_1, \infty)$$

with the cut-off value

$$\lambda_1 = \frac{1}{4}\min\{b_+, b_-\}\frac{1}{g_J}(\varrho_1 - \varrho_0)$$

where $J = 1$ and $J = 0$ in Fig. 1.1 a and b, respectively. The interval $[0, \lambda_1)$ contains the discrete spectrum, and in particular $\lambda = 0$ is an eigenvalue of multiplicity 2.
REFERENCES


