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ESSAYS ON HIGHER MOMENTS IN MACROECONOMICS AND FINANCE

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1.1 Why study higher order moments?

Risk aversion in economics is very loosely speaking a tendency of a person to prefer situations where outcomes are more certain to those where they are less certain - even if the more certain outcomes are less valuable in expectation. A risk averse person might prefer a certain payout of 80 units of currency to a lottery where he has a high chance to receive 100 units and a small chance to receive nothing.

In macroeconomics and finance this idea manifests through the preferences of the people inhabiting the models, particularly their preferences regarding the higher moments of distributions: variance and covariance, kurtosis and skewness. These quantities typically characterize the riskiness of the lotteries agents face both in practical situations and applied research in these fields. Variance and kurtosis describe the variability of outcomes: how likely are events that are far from the “usual” outcome, the mean? Covariance relates the outcomes of seemingly separate random events with each other, while skewness describes possible asymmetry in the probability of outcomes that are equidistant from the mean.

Of particular importance is the possibility of temporal variation in these quantities. This thesis studies these concepts. The primary focus is on the dynamics and influences of the second moments. These affect the decision making both in models and in reality by causing variation in risk and they thus change the behavior of risk averse individuals. Temporal variation, especially when the
variation itself is random or difficult to understand, puts the decision makers in a situation where even the variability of the lotteries they face in the future is uncertain.

An important related question is that of distributional form. In reality many macroeconomic and financial time series quite clearly do not follow the Gaussian distribution, but have instead for example fat tails. That is, they have skewness and kurtosis higher (or smaller) than the normal distribution. An important question to study - so far often ignored in the literature - is how this affects the behavior of individuals and firms compared to the case where, for convenience, researchers assume structures that are unable to reproduce this fact.

In finance one of the many ways uncertainty and risk affect investors is through asset prices, and in this context the relationships between second moments and investor behavior is, at least at the level of widely-accepted theory, quite explicit. Standard theory on the prices of financial assets postulates that under certain assumptions returns and thus prices can be determined through a linear relationship between the risk and return of that asset, and that this risk can be ultimately determined by the various second moments of the return in question in relation to some sources of risk.

In many models of macroeconomic decision making households and firms face quite similar problems as in financial applications of economic theory, although the implications of risk are usually not presented as explicitly as in the finance literature. The concepts are however fundamentally the same: uncertainty regarding future well-being affects the behavior of individuals today. In fact, what is sometimes called the fundamental pricing relationship in finance is conceptually exactly the same thing as what is called the Euler equation in the macroeconomics literature.

The thesis is divided into 3 chapters beyond this introductory chapter, each of which considers these issues from different perspectives. Chapter 2 is a study on the effects of macroeconomic uncertainty, modeled as time-varying variance, on the effectiveness of public spending in stimulating the economy. Chapter 3 studies whether macroeconomic models are able and which particular model features are needed in order to generate simulations where the moments of simulated variables match those of real data. Chapter 4 finishes with the viewpoint of finance, more specifically asset pricing, and studies the pricing of macroeconomic risk in the stock markets through time varying covariances.

The rest of this chapter is divided along these same lines: in Section 1.2 a brief
look is given at the methods and models applied in the thesis and in Section 1.3 we summarize the rest of the chapters.

1.2 Methods for modeling and analyzing higher order variation

For the sake of this thesis I categorize the tools commonly applied to analyze these questions broadly into two categories: empirical and theoretical\(^1\). Especially the literature on empirical analysis of higher order moments is enormous. This brief review is therefore not even intended to be exhaustive - I focus on those parts of the field that are directly relevant to the thesis.

This section is further divided into two subsections, the first of which looks at the issues from a macroeconomic point of view, while the second considers the empirical literature on conditional variances, originating in the time series literature.

1.2.1 Theoretical macroeconomics and higher moments

The majority of modern macroeconomic research involving theory is conducted using the framework of Dynamic Stochastic General Equilibrium (DSGE) models. These models are Dynamic in the sense that there are multiple (usually an infinite amount of) distinct periods of time, Stochastic due to the fact that they contain exogenous random components (such as time-varying aggregate productivity of firms) and they are called General Equilibrium models because all model quantities except for the Stochastic components are endogenously determined within the model.\(^2\) In this section we present a general problem - why are previous methods inadequate? - and then show in subsections two ways of incorporating modern features and particularly the effects of time-varying higher moments into DSGE models.

Before the 1980s a significant part, perhaps even the majority of macroeconomic research was based on the Keynesian IS-LM\(^3\) model of Hicks (1937), although during the post-war decades various alternative schools of thought existed. The

\(^{1}\)This distinction also matches that between the essays.

\(^{2}\)In contrast to partial equilibrium models, which would have quantities, such as prices, that are determined outside the model.

\(^{3}\)Investment-Saving-Liquidity preference-Money.
IS-LM model is also a general equilibrium model: it models two markets simultaneously, with everything determined endogenously. In one investment and savings are in equilibrium, yielding the IS curve, while in the other agents trade "liquidity" in the money market and determine the interest rate. Famous applications include the Mundell-Fleming model of short run dynamics in exchange rates and many others.

The late 1970s saw however a rise in criticism, fueled by the empirical failure of these models: the stagflation of the early 1970s was something they failed to predict or even explain, which opened a path for new avenues of inquiry. Lucas (1976) presents what is now known as the Lucas Critique, an attack on the fundamentals of the IS-LM structure. While the IS-LM model family is easy to criticize particularly because of its ad hoc nature, the Lucas Critique focuses on the empirical validation and application of models based on the then dominant paradigm. What Lucas found particularly lacking was the inability to focus on truly structural issues: especially the empirical models were unable to take into account that policy changes would have effects on what the models estimated as the "rules" of the economy.

This line of thought lead first into the development of what are now called Real Business Cycle (RBC) models, where the behavior of say, investment and labor supply are built using building blocks that are now called microfoundations. That is, the model is assumed to be populated by a set of agents, who operate under the traditional microeconomic assumptions of rationality, optimizing decisions over production and consumption. Models found in papers such as Kydland and Prescott (1982) also endowed the agents with what often seems a superhuman capability: rational expectations, a complete understanding of the structure of the economy they live in. These models succeeded in building an internally coherent structure for analyzing (some) macroeconomic phenomena, but what they failed at is to account for phenomena related to monetary issues.

Thus, after a new wave of criticism from sources such as Gali (1999), the framework was further extended. These New Keynesian DSGE models took their basic structure from the RBC literature, but also plugged in new features that could generate model outcomes resembling those observed in reality. Of particular importance was the addition of price rigidity, which made nominal prices meaningful. While the conceptual basis of what is called Calvo pricing (Calvo (1983)) can also be seen as a rather ad hoc fix, with it researchers are able to analyze, for example, the effects of monetary policy on the real economy, applied
with great success in Smets and Wouters (2007), among others.

In principle these models are highly nonlinear, as most model features are often built using quite complex functional forms. For example a common way of modeling the behavior of the households is to assume what is called the habit formation utility function of Constantinides (1990) which has, depending on the application, a form resembling

\[ U(C_t, C_{t-1}) = \frac{(C_t - hC_{t-1})^{1-\gamma}}{1-\gamma} \]  

(1.2.1)

where \( C_t \) is the consumption of a household at time \( t \) and \( 0 < h < 1 \) and \( \gamma > 0 \) are parameters. The idea behind this form is to make the agents risk averse (determined by the size of \( \gamma \)) and have their well-being tied to the past: loosely speaking, the psychology literature has found that people experience significant loss of well-being (determined by the size of \( h \)) when they experience a drop in consumption compared to the recent past.

Given a set of other equations describing the rest of the economy, such as those determining production and interest rates, the model equilibrium is given by a set of optimality conditions to the choice rules of the entities in the model together with any constraints. As the functional forms of this latter set of equations - often called equilibrium conditions - are also highly nonlinear not to mention dynamic, it is almost always the case that no analytical solution exists.

The most common workaround for this problem is linearization. That is, the equilibrium conditions are approximated with first order Taylor polynomials, either in logarithms or levels around a point of interest, which is usually the non-stochastic steady state\(^4\) of the model. In almost all cases this set of equations can then be solved in the sense that the equations can be manipulated into a form where all model quantities can be expressed using affine transformations of the same quantities in the previous model period, using for example the method of Blanchard and Kahn (1980) or Klein (2000).

Unfortunately this “workaround” also comes at a cost, which can be quite high depending on the model. For example, what was in Section 1.1 described as risk aversion is in the models a consequence of assuming nonlinear utility functions. The utility function (1.2.1) also contains a feature commonly called Constant Relative Risk Aversion (CRRA) due to the \( \gamma \) in the exponent and the denom-

\(^4\)A point where the model is “at rest”: all model quantities stay fixed at this point in consecutive periods.
inator; when the corresponding equilibrium conditions are linearized, this risk aversion is lost. Because of linearization the model becomes thus completely insensitive to variation in the higher moments of any model variables, a topic studied in the first two essays of this thesis.

The central question studied in the first essay is the effect of economic uncertainty - variation in higher moments - on the effectiveness of public spending in stimulating the economy. This is a longstanding debate in macroeconomics. First discussed in Kahn (1931), the fiscal multiplier is a direct measure of the ratio of a change in output to the change in government spending that causes it. However, due to technical difficulties in measuring its exact size, no strong consensus view has yet arisen within economics, with most empirical estimates ranging from 0.5 to 2. See Ramey (2011) for an overview of recent results.

Briefly put, differing views persist despite the use of a variety of methods such as structural vector autoregressions (Blanchard and Perotti (2002) and Fisher and Peters (2010)) or theory based DSGE models (Cogan et al. (2010)) or data sets such as annual defense spending (Barro and Redlick (2011)).

The contribution of the first essay on the debate regarding the multiplier also concerns a rather extreme form of nonlinearity, the zero lower bound (ZLB) on the nominal interest rate - a situation where the nominal interest rate cannot go below a certain threshold. A consensus of sorts has emerged recently over the point that when the economy is at the ZLB, the multiplier is likely to be larger. The lower bound is commonly modeled as

\[ R_t = \max\{1, \phi_t\} \]  

(1.2.2)

where \( R_t \) is the interest rate at time \( t \) and \( \phi_t \) some choice rule the central bank of the model uses to determine a desired interest rate. But even if the central bank wishes to set the interest rate below unity the bound prevents this, causing a strong kink to appear in the functional form. As the validity of linearized approximations of any order rests on the assumption of continuous functional forms, it is obvious that such a model feature cannot be accommodated using these traditional methods.

Furthermore, it is rather common for macroeconomic data to exhibit some form of time-variation in their higher moments, not to mention non-normality. Linearized models are in principle unable to reproduce the time-variation, but also

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5 This might be due, for example, to the fact that households refuse to deposit money into banks if the nominal rate is not large enough.
the non-normality, unless the stochastic components of the model are assumed to be non-normal. Unfortunately, the vast majority of the DSGE literature is also inadequate in this sense, and only in fairly recent advances have these features been modeled at all. The second essay considers this vital technical question: the ability of theoretical models to generate non-Gaussian realizations when simulated. This, however requires a method that is able to account for the higher moments; the method applied in this thesis is called the Generalized Stochastic Simulation Algorithm, and was first suggested by Judd et al. (2011).

1.2.2 Empirical models for time-varying moments

Most modern empirical study of time-varying higher moments originates with Engle (1982), which studies temporal variation in the variance of inflation in the United Kingdom. This seminal contribution is the origin of models of Autoregressive Conditional Heteroskedasticity (ARCH), a family of models that assume that variance today is a weighted sum of the variances in the past. A vast industry of extensions and applications was spawned. Starting with Generalized ARCH (GARCH) of Bollerslev (1986), consecutive authors considered new, more complex structures for the process describing the dynamics of the variance process.

These models become empirical when taken to data. That is, they become descriptive of reality when their parameters are estimated based on observation. In the simplest cases, such as the ARCH itself, this can be accomplished with the method of ordinary least squares. When the models are more complex, numerical procedures such as the Berndt-Hall-Hall-Hausman algorithm have to be applied to solve the problem of maximizing the likelihood function associated with the statistical model.

The model of conditional variance presented in Bollerslev (1986) is simply (in the first order case)

\[ h_t = \alpha + \beta \varepsilon_{t-1}^2 + \gamma h_{t-1} \]

(1.2.3)

where \( h_t \) is the conditional variance at time \( t \) and \( \varepsilon_t = \sqrt{h_t} z_t \) with \( z_t \) a draw from the standard normal distribution. \( \alpha, \beta \) and \( \gamma \) are parameters with \( \alpha > 0, \beta \geq 0, \gamma \geq 0 \) and \( \beta + \gamma < 1 \). Thus in the first order model, usually called GARCH(1,1), current conditional variance is the weighted sum of the previous period conditional variance and a zero mean “shock” drawn from the normal
distribution with variance $h_{t-1}$.

This framework has been subsequently extended in a multitude of ways to better accommodate the features of a particular data set. For example the GJR-GARCH of Glosten et al. (1993) was inspired by the observation that observed stock returns often exhibit asymmetry, as the distributions are often skewed. In this model a positive innovation $\varepsilon_t$ could have an effect different from that of a negative one:

$$h_t = \alpha + \beta \varepsilon_{t-1}^2 + \gamma h_{t-1} + \delta I_{t-1} \varepsilon_{t-1}^2$$

(1.2.4)

where $I_t$ is the indicator function, equal to unity when $\varepsilon_t > 0$ and 0 otherwise and $\delta$ is a parameter. See Teräsvirta (2009) for an overview of univariate GARCH models.

While the original application of ARCH models by Engle (1982) was on inflation, they have been mostly used in finance, where the applications have been varied, ranging from portfolio allocation to risk management. One of the most studied questions and a topic of the third essay in this thesis is the risk-return relationship briefly mentioned in Section 1.1. Standard theory postulates that the return $r_{i,t}$ of an asset $i$ is linearly related to the return $r_{m,t}$ on the market portfolio - a portfolio that holds small pieces of all the assets in the market, with the share of each asset in the portfolio determined by its covariances with all the other assets. Thus the theory claims that

$$r_{i,t} = \beta_i r_{m,t} + C$$

(1.2.5)

where $\beta_i$ is a parameter, measuring the sensitivity of the asset $i$ to variation in the market return and $C$ is a term containing other factors that may affect the returns. Seminal contributions studying (1.2.5) include Fama and MacBeth (1973), Fama and French (1993), and Lettau and Ludvigson (2001), among many others.

The relationship in (1.2.5) can also be expressed as

$$r_{i,t} = \gamma \sigma_{im,t} + C$$

(1.2.6)

where $\sigma_{im,t}$ is covariance between asset $i$ and the market portfolio. When asset $i$ is in fact the market portfolio itself, the relationship becomes

$$r_{m,t} = \gamma \sigma_{m,t}^2 + C$$

(1.2.7)
which is an equation that has been widely studied with the use of what are often called GARCH-in-mean models. This strand of literature includes the aforementioned Glosten et al. (1993), but also Engle et al. (1987), the origin of the ARCH-in-mean framework.

More recent extensions to the GARCH family of models have taken these ideas even further. Multivariate GARCH (MGARCH) models study the time-varying covariances between two or more time series using a wide variety of modeling approaches to deal with practical and conceptual difficulties inherent in these complex structures (see Silvennoinen and Terasvirta (2009) for an overview). A particularly convenient form applied in this thesis is the Dynamic Conditional Correlation (DCC) model introduced in Engle (2002), which is a parsimonious method applied in Chapter 4 to obtain estimates of the time-varying covariances between stock returns and a variety macroeconomic variables. With this tool, the researcher can estimate risk-return relationships of the form

\[ r_{i,t} = \gamma \sigma_{im,t} + \sum_{j=1}^{n} \lambda_j \sigma_{ij,t} \]  

(1.2.8)

where \( \lambda_j \) is the risk premium paid for sensitivity to some risk factor \( j \).

A key paper in the literature on the pricing of macroeconomic variables is Chen et al. (1986). Using U.S. data, they consider a broad set of risk factors, including industrial production, inflation, oil price and many more. They find that for example inflation, the spread between high and low grade bonds and industrial production are all paid a risk premium, while both aggregate consumption and the return on the market portfolio are not. Similarly, Santos and Veronesi (2006) find that risk associated with labor income is paid a premium. On the other hand these models have been criticized for using weak data: in comparison to financial time series, they are known to be subject to poor measurement and frequent revisions. Partly due to this reason Savov (2011) shows that when consumption is measured with a proxy known to be accurate and informative - garbage output - it appears to be priced in the market. An example of applying the DCC model to asset pricing can be found in Bali and Engle (2010), where it is shown that the return on the market portfolio, the default and term spreads are priced, as is the HML factor\(^6\), while for example SMB factor\(^7\) is not.

\(^6\) Return on the High-Minus-Low portfolio, i.e. the return on a portfolio where the average return on low book value firms is subtracted from that of high book value firms.

\(^7\) Return on the Small-Minus-Big portfolio, i.e. the return on a portfolio where the average return on big firms is subtracted from that of small firms.
For the most part MGARCH models have been applied to financial data, where they have been extremely successful both in academia and the financial industry, but there are also applications that are more closely associated with macroeconomic questions. As was noted in section 1.1, agents in macroeconomic models are risk averse - in times of higher risk they act more cautiously. An implication is that higher uncertainty regarding the future should change behavior and be in principle observable in the data. As the variance is (one) measure of risk and uncertainty, a high conditional variance should imply high risk.

Such a relationship is considered for example in Grier and Perry (1998), who study the Granger causality between inflation and inflation uncertainty, in Fountas et al. (2006) who use a bivariate model to study uncertainty in inflation and output and Kohonen (2013), who looks at the effects of joint uncertainty in stock markets and industrial production. Predicting financial volatility with macroeconomic variables has been done in several papers, including Schwert (1989), who studies the temporal variation in U.S. stock return volatility; the conclusion is that there is weak evidence for predictability of stock and bond returns with macroeconomic volatility, and stronger evidence for predictability of macroeconomic volatility with financial volatility. Similarly, Engle et al. (2013) apply a variant of the MGARCH family based on mixed data sampling and find that especially at long but also at short horizons inflation and industrial production are useful for predicting stock volatility.

1.3 Summary of the essays

1.3.1 Government Spending in a Volatile Economy at the Zero Lower Bound

In this essay I study the effects of exogenous uncertainty in future public spending and aggregate productivity on the effectiveness of government spending, i.e. the size of the fiscal multiplier, when the economy has hit the zero lower bound. As a novel feature, the essay includes stochastic volatility as a modeling device for uncertainty. The model is solved using a global solution method, based on approximating policy functions with polynomials, that is able to accommodate both key model features. The model calibration is standard relative to the literature, with the exception that the variances of shock processes have set at 0.005 - this is because the goal of the calibration is that the model hits the ZLB often
enough, approximately in 6% of periods. Why does the ZLB have this effect? Woodford (2011) presents a compelling argument for why this is the case, and similar points can also be found in Christiano et al. (2011), among others. In a nutshell, this phenomenon is most likely due to the intertemporal preferences of the households. In a “normal” situation an increase in public spending would affect the rate of return the households get on their savings. At the ZLB this is not possible, and any household that wishes to consume more will find it preferable to work more instead of adjusting their savings. This, in turn, will result in an increase in output and hence a multiplicative effect on the spending increase. In these situations previous literature, such as Fernandez-Villaverde et al. (2013), has found the multiplier to be roughly 1.5.

At the bound exogenous uncertainty is found to have an effect of its own for example on output and inflation. The effects of the two types are quite distinct: an increase in uncertainty in public spending will cause output to drop and the interest rate to rise (vice versa for a decrease in uncertainty), while a decrease in uncertainty regarding productivity will push output up. Furthermore, the various structural shocks are shown to have distinct effects on endogenous model quantities when they push the economy to the bound: shocks to the bond return and household discount rate push output and inflation down, while shocks to productivity and the nominal interest rate push output up.

The fiscal multiplier is computed to be 0.6 when the economy is not at the bound. For the reasons described above, at the bound the multiplier depends on three factors: the shock that drives the economy to the bound, the size of the spending increase and the degree of uncertainty. When the economy is at the bound due to a shock to the discount rate as often analyzed in the literature and uncertainty is mild, the multiplier is roughly 1.5, though with high spending volatility and a small spending increase it can be as high as 3. In contrast when the driving force is a shock to productivity, the multiplier peaks at 0.6 and bottoms at -1.6. Most importantly, there are clear patterns in how volatility affects the multiplier at the bound: in the typical case, high volatility in spending drives the multiplier up by affecting household expectations regarding the length of the period the economy spends at the bound, as was pointed out by Christiano et al. (2011) and Erceg and Linde (2014). Similarly, in the typical case low volatility in productivity pushes the multiplier up.
1.3.2 Can DSGEs generate non-Gaussian realizations?

It is quite typical that macroeconomic data follows some distribution other than the Gaussian. For example, the distributions of the U.S. consumer price index and the Federal Funds Rate are quite strongly skewed, while the distribution of personal consumption expenditure exhibits significant excess kurtosis in comparison to the normal distribution. A crucial question regarding the applicability of DSGE models is whether they produce simulated data that matches that observed in reality. Traditional solution methods based on linearization have however failed at this task, as was shown by Ascari et al. (2014). The contribution is to show that with sufficient nonlinearity and a proper calibration even relatively ordinary models can attain the goal.

The essay considers two different models: a simple model for analyzing real business cycles under the assumption that households have a habit formation utility function (see eq. (1.2.1)), and a New Keynesian model of prices and interest rates where the zero lower bound is assumed to be binding. The models are solved with a global solution method that yields a solution that takes into account the nonlinearity in utility and the lower bound. For both models two cases are considered: exogenous shocks that follow the normal distribution and shocks that follow Student’s t-distribution.

It is shown that with suitable choices for model parameters both models are able to generate non-Gaussian data. In the first case the parameter governing the strength of habit is crucial: with the habit parameter $\gamma$ set low at 0.1, realizations produced by the model line up with the distribution of the exogenous shocks. However, when the habit parameter is set high, at 0.7, the realizations become non-Gaussian, even if the exogenous shocks themselves are Gaussian. Similar results are obtained when the model with binding ZLB is studied; here the crucial parameters to calibrate correctly are the variances of the exogenous shocks. More specifically, it is shown that if the variances are too low (0.001) the model rarely hits the lower bound, but when they are sufficiently large (0.004), the bound is hit often. As a consequence the distributions of several endogenous variables become quite strongly skewed even with Gaussian shocks, and with t-distributed shocks even more so.

Both models are able to some extent match the moments of actual U.S. data: the RBC model matches the second moments rather well, while the NK-DSGE is able to generate significant skewness. The main finding is that DSGEs are indeed able to at least some extent able to “replicate” reality - the crucial question
appears to be in the adequate modeling of nonlinearity, which most literature so far has failed to do.

1.3.3 Dynamic Economic Forces and the Stock Market

The third essay considers the pricing of macroeconomic risk in stock markets. Since at least Merton (1973) it has been widely acknowledged that there are solid theoretical reasons for why agents in asset markets would care about variation in the macroeconomy: changes in e.g. unemployment are possibly risks that cannot be hedged away through portfolio allocation and hence assets that are sensitive to these risks should be paid a commensurate premium.

The focus is on studying whether risk regarding four macroeconomic variables - measured by monthly U.S. industrial production, unemployment, inflation (from 1948 to 2012) and a somewhat novel factor, the Case-Shiller house price index (from 1987 to 2012) - is paid such a premium. Is the risk associated with variation in industrial production hedgeable or something affecting the average investor? As the theory of Merton (1973) implies that the premia are independent of the asset, we study this question using a broad set of test assets, comprising of the monthly returns of three sorted U.S. stock portfolios.

The econometric method applied is the DCC of Engle (2002) with a twist: we assume that the covariances appear directly in the mean equations, and estimate the whole model in a single step, unlike most of the previous literature. The coefficients on the covariance terms can then be directly interpreted as risk premia. We find that the return on the market portfolio, constructed as the value-weighted return on all listed U.S. stocks, is a source of priced risk, but that variation in the house price index is not. Consistent with previous studies such as Chen et al. (1986), we also find that variation in inflation and unemployment earn a negative premium that is statistically highly significant, while variation in industrial production earns a positive premium. However, uncertainty remains regarding the size of these premia, as the estimates for the different portfolios are not consistent across the test assets, contrary to what the theory of Merton implies.
Chapter 2

Government Spending in a Volatile Economy at the Zero Lower Bound

We study the effects of exogenous uncertainty in government expenditure and aggregate productivity on the effectiveness of fiscal policy using a nonlinear solution to a model where the zero lower bound on the nominal interest rate binds occasionally. The model is solved using a stabilized stochastic simulation algorithm. We find that stochastic volatility has both an effect on the properties of the economy at the lower bound and on the effectiveness of public spending: reductions in uncertainty regarding future productivity and spending can both amplify and dampen the multiplier. The size of the multiplier depends strongly on the size of the expenditure increase and the shock that brings the economy to the bound: discount shocks lead to effective fiscal policy, while productivity shocks lead to low multipliers.
2.1 Introduction

As a consequence of the still ongoing financial crisis, in recent years both the United States and the Euro area have faced an ordeal Japan endured in the late 1990s: a lengthy period where the nominal interest rate is extremely close to zero. While the exact consequences of the economy hitting the zero lower bound (ZLB) are not entirely clear, conventional monetary policy obviously becomes more difficult to implement, yet the effectiveness of an increase in government spending in stimulating the economy remains a point of contention.

In recent research several different views have emerged on this point. Fernandez-Villaverde et al. (2013) find that the multiplier on government expenditure can be rather high when at the bound, roughly three times as large as in normal times. Braun et al. (2013) on the other hand point out that at least under Rotemberg pricing several empirically plausible calibrations exist where the multiplier is about one or less. The results of Christiano et al. (2011) are that the multiplier, which they claim is greater than one, is positively related to the loss of output caused by the ZLB episode and the expected length of the ZLB episode. The latter point is also emphasized by Erceg and Linde (2014) together with Johannsen (2014). Finally, Aruoba and Schorfheide (2013) find that depending on the monetary policy rule, the multiplier can be less than one but also significantly larger than one.

Our results indicate that depending on why the economy is at the bound the multiplier can be rather large, but also very small, and that exogenous uncertainty can play a significant role in both directions. A likely reason for these results is that both the size of the spending increase and associated uncertainty have an effect on the expected duration of the ZLB spell. As the expectation of the length of the spell depends on the spending increase - or more specifically on expectations regarding the increase - uncertainty regarding future spending also affects the households expectation regarding the length of the spell.

We reach these conclusions by adding stochastic volatility to a relatively standard New Keynesian Dynamic Stochastic General Equilibrium (NK-DSGE) model. In particular, we model the lower bound in the interest rate as a strong kink in the Taylor rule - a model feature which has been found to have an amplifying effect on the fiscal multiplier\(^1\). To fully retain the nonlinearity, we solve the model using a slightly modified version of the Generalized Stochastic

\(^1\)See, for example, Eggertson and Woodford (2003).
Simulation Algorithm (GSSA) of Judd et al. (2011). This method leaves the equation for the interest rate intact, but approximates certain decision rules with high-order polynomials of the state variables. The particular shocks we assume subject to stochastic volatility are aggregate productivity and government expenditure. This enables the study of the effects of exogenous uncertainty on the effectiveness of policy through temporal variation in the variances of these shock processes.

The GSSA employed in solving the model is best described as an extension of the Parametrized Expectations Algorithm (PEA) proposed by den Haan and Marcet (1990). The key difference is in the application of stabilization procedures and deterministic integration to deal with numerical problems that often manifest when a stochastic simulation algorithm such as the PEA is used to solve more complex models. Section 2.3 and Appendix B present details on the method.

The novel feature - at least in this strand of literature - of stochastic volatility is added to the model for two reasons. Primarily, it enables the study of the effect of variation in uncertainty itself on the effectiveness of policy. Second, as argued in Justiniano and Primiceri (2008), Fernandez-Villaverde and Rubio-Ramirez (2010) and Fernandez-Villaverde et al. (2011a), based on observed data it seems extremely likely that most macroeconomic time series feature temporal variation in volatility. On the other hand this feature is somewhat ad hoc, as this variation in uncertainty is still exogenous. In addition, it comes at a computational cost, as the state space of the model grows significantly. However, it also has quite notable effects, especially on the size of the multiplier.

Why would exogenous uncertainty matter? Previous research on the effects of uncertainty at the ZLB has pointed out several mechanisms, e.g. the inability of the central bank to smooth out fluctuations and the desire by households to adjust their consumption, savings and leisure. For example Basu and Bundick (2015) point out that at the bound uncertainty may cause the households to engage in what they call “precautionary labor supply”. They, together with Nakata (2013) and Johanssen (2014), find that uncertainty can have significantly larger effects on the economy when at the ZLB than otherwise. Other related contributions studying the effects of uncertainty include Fernandez-Villaverde et al. (2011b), who look directly at the general economic effects of spending volatility shocks, but not when the economy is at the ZLB and using a perturbation solu-

2See Plante et al. (2014) for a study on the effects of endogenous uncertainty on GDP growth at the ZLB.
tion and Bloom et al. (2012), which contains an example of analyzing the effects of uncertainty on effectiveness of policy. They however consider the effectiveness of subsidizing labor.

Previous literature on the nonlinear effects of the ZLB in general includes Gust et al. (2012), Fernandez-Villaverde et al. (2013) and Aruoba and Schorfheide (2013) who all apply a nonlinear global solution similar to ours to a model where the ZLB is imposed; Gust et al. also estimate their version of the model. A fully nonlinear solution employing a method extending the one applied in this paper can be found in Judd et al. (2012), but their emphasis is on the solution method itself.

In the case of the ZLB the nonlinearity is extremely strong, and traditional solution methods based on loglinearization are in principle unable to deal with the kink. Workarounds have been occasionally implemented: Eggertson and Woodford (2003) loglinearize all equations but the ZLB constraint. Christiano et al. (2011) loglinearize their model, but assume that the interest rate follows a two-state Markov chain. However, Fernandez-Villaverde et al. (2013) show that the decision rule obtained by applying a method that produces a nonlinear solution yields outcomes that differ significantly from those obtained from a loglinearized solution, where the lower bound is imposed ex post. The farther the state variables are from the steady state of the model, the greater is the difference between the outcomes of the solution methods. In addition, Braun et al. (2013) point out that especially when the model is loglinearized, some calibrations of the model parameters may result in a large fiscal multiplier while other, equally plausible ones yield small multipliers.\footnote{The calibration matters of course also when other solution methods are applied.}

Empirical estimates of the size of the fiscal multiplier when not at the bound are mixed. See Ramey (2011) for an overview. Briefly put, a variety of empirical methods and data sets yield significantly different results. For example, Blanchard and Perotti (2002) use a structural vector autoregression (SVAR) to identify tax shocks and find that the multiplier is 0.9 to 1.29, depending on assumptions. Cogan et al. (2010) apply the model of Smets and Wouters (2007) and conclude that the multiplier is significantly less than one. Using the excess returns to defense contractor stocks as news in a SVAR, Fisher and Peters (2010) estimate the multiplier to be 1.5. Finally, Barro and Redlick (2011) estimate that for defense spending the multiplier is 0.4-0.5. However, a recent paper by Ramey and Zubairy (2014) fails to find evidence for higher multipliers during
times of slack using a state-contingent estimation procedure.

As a preview of our findings, we point out that the size of the multiplier at the ZLB is in our framework dependent three factors: the size of the stimulus, the reason for being at the bound and exogenous uncertainty. For small increases in spending, the multiplier can be quite large, especially after a discount shock. Conversely, when a productivity shock causes the event, fiscal policy is ineffective regardless of the size of the spending increase. For spending shocks of 5 standard deviations or more the multiplier is usually less than one. Changes in exogenous uncertainty regarding public spending or aggregate productivity can both dampen and amplify the effectiveness of stimulus. For example, when at the bound due to a bond return shock, high volatility in spending can push the multiplier from 0.9 to 1.2 and after a productivity shock high volatility in productivity pushes the multiplier from 0.5 to -0.5. Finally, when not at the bound the multiplier is roughly one half.

In our model the primary exogenous drivers of ZLB events are the monetary policy shock and the bond premium shock, though large negative shocks to the discount rate and large positive shocks to productivity can also send the economy to the bound. These four shocks have quite different effects on the economy and therefore also on the multiplier. A positive productivity shock will both drive the economy to the bound and increase output, but after a positive bond return shock the economy will hit the bound while output drops.

A likely explanation for the decline in the multiplier as the size of the spending increase goes up is that it affects the expectations of the households regarding the length of the ZLB episode. As was pointed out by Christiano et al. (2011), the multiplier is strongly dependent on how severe the ZLB problem is. Similarly, Erceg and Linde (2014) endogenize the duration of the episode (a feature shared by the model of this paper) and find that the multiplier drops substantially as the government spending increases. It seems more than plausible that the patterns in the multiplier schedule that follow volatility shocks are also a consequence of changes in expectations regarding the duration of the ZLB spell.

The rest of the paper is organized as follows. Section 2.2 presents the theoretical model. The solution method is briefly described in Section 2.3 (Appendix B contains details on the implementation). Section 2.4 contains our analysis regarding stochastic volatility and ZLB events, while the main results regarding the fiscal multiplier can be found in Section 2.5. Section 2.6 concludes.
2.2 The model

The model we study is a standard New Keynesian DSGE model. We assume a continuum of identical households of unit mass who consume, save and supply labor. Production is handled by two sectors, one of which consists of monopolistic competitors who supply intermediate goods to a final good producer and use labor supplied by the households as the sole factor of production. Adjusting the price of the intermediate goods is constrained by a Calvo process. The economy is closed by a public sector, consisting of a central bank following a Taylor rule - with a binding lower bound - and a government which taxes, borrows and spends.

Variants of this basic framework have been applied in several papers. Judd et al. (2012) use the model as an expositional tool to demonstrate a new solution method. Gust et al. (2012) estimate their model under the assumption of Rotembergian price adjustment to study the Great Moderation. Gavin et al. (2013) consider the effect of the form of the Taylor rule on the economy at the ZLB. Fernandez-Villaverde et al. (2013) use it to study the general properties of the ZLB and look briefly at the fiscal multiplier. Our main contribution comes from extending the shock structure of the model to include stochastic volatility.

We assume six structural shocks: to the household’s time and labor preferences, aggregate productivity, the Taylor rule, the return on the government bond, and public expenditure. Additionally, we assume that the shock processes to both aggregate productivity and public expenditure are subject to stochastic volatility. That is, we assume that the variances of these shock processes are time-varying themselves. While the case where all six processes have time-varying variances would be interesting to study, we focus on these two for the sake of computational cost. Alternative formulations of time-variation such as Markov switching would also be applicable in principle, but numerical issues arose when applying the GSSA to solve a model with this type of shocks.

Sections 2.1-2.3 present the different components of the economy. The full derivation of the equilibrium conditions of the model is relegated to Appendix A, but they are briefly presented in Section 2.4.

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4Solving the model becomes significantly more difficult as the dimension of the state space grows.
2.2.1 Households

The households sell labor in a competitive labor market to the firms, buy Dixit-Stiglitz-aggregated goods from them and have the opportunity to invest in government bonds that are subject to a shock and pay out a return determined by the central bank.

The representative household faces the problem

$$\max_{C_t, B_t, L_t} \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t u_t \left[ \log C_t - l_t \frac{L_t^{1+\vartheta}}{1+\vartheta} \right]$$

subject to

$$P_tC_t + \frac{B_t}{b_t} \leq R_{t-1}B_{t-1} + W_tL_t + T_t + \Pi_t$$

where $C_t$ is consumption in period $t$, $B_t$ bond purchases, $L_t$ labor supply, $R_t$ the nominal interest rate, $P_t$ the price level, $W_t$ the wage rate, $T_t$ net lump sum taxes and $\Pi_t$ the profits of the firms in the economy. $u_t$, $l_t$ and $b_t$ are shocks hitting the households discount rate, labor preference and bond returns, respectively. They are assumed to follow $h_t = h_t^{\rho_h} \exp(\sigma_h \varepsilon_{h,t})$, with $\varepsilon_{h,t} \sim N(0,1)$ and $h \in \{u, l, b\}$. $\beta$, $\gamma$, and $\vartheta$ are parameters describing the households’ time preference, risk aversion and disutility of supplying labor.

2.2.2 Producers

Production is handled by two sectors: intermediate goods producers, who use labor to produce differentiated goods which they then sell to the final-goods producer, who sells output to the households.

The intermediate goods producer $i$ uses the technology $Y_{it} = a_t L_{it}$, where $a_t$ is an aggregate productivity shock following $a_t = a_{t-1}^{\rho_a} \exp(\sigma_{a,t} \varepsilon_{a,t})$, $\log \sigma_{a,t} = (1 - \varrho_a) \log \sigma_a + \varrho_a \log \sigma_{a,t-1} + \epsilon_{a,t}$, $\varepsilon_{a,t} \sim N(0,1)$ and $\epsilon_{a,t} \sim N(0,1)$. They are also subsidized by the government by the sum $\nu > 0$. Thus they face the problem

$$\min_{L_{it}} (1 - \nu) W_t L_{it}$$

under the constraint

$$Y_{it} = a_t L_{it}$$

Furthermore, the producers are subject to Calvo-style price setting: a fraction $1 - \vartheta$ is able to set prices so that their current price $P_{it}$ is equal to the optimal
price $\bar{P}_t$, while a fraction $\theta$ is not and is thus forced to keep the same price as in the previous period. The corresponding price-setting problem is

$$\max_{\bar{P}_t} \sum_{j=0}^{\infty} \beta^j \theta^j E_t \left\{ \Lambda_{t+j} Y_{t,t+j} \left[ \bar{P}_t - P_{t+j} MC_{t+j} \right] \right\}$$

subject to

$$Y_{it} = Y_t \left( \frac{P_{it}}{\bar{P}_t} \right)^{-\varepsilon}$$

where $Y_t$ is the amount of final good produced, $\Lambda_t$ is the Lagrange multiplier on the household’s budget constraint and $MC_t$ is the marginal cost of output. $\varepsilon \geq 1$ is a parameter governing the aggregation process used by the final-goods producer, also describing the degree of substitutability between the inputs.

The price setting process implies that in any given period not all firms will have the same price for the intermediate good they sell. Thus there is a degree of (inverse) price dispersion in the economy, described by

$$\Delta_t = \left[ (1 - \theta) \left[ 1 - \theta \pi_{t-1}^{\varepsilon} \right] 1 - \theta \pi_{t-1}^{\varepsilon} \right]^{-1}$$

where $\pi_t = \frac{P_t}{P_{t-1}}$ is inflation. This process, in turn, is governed by

$$\pi_t = \left[ 1 - (1 - \theta) \left( \frac{S_t}{\bar{P}_t} \right)^{1-\varepsilon} \right]^{\frac{1}{1-\varepsilon}}$$

where

$$S_t = \frac{u_t l_t}{a_t} L_t^\beta Y_t + \beta \theta E_t \left[ \pi_{t+1}^{\varepsilon} S_{t+1} \right]$$

and

$$F_t = u_t C^{-1}_t Y_t + \beta \theta E_t \left[ \pi_{t+1}^{\varepsilon-1} F_{t+1} \right]$$

as shown in Appendix A.

The final-goods producer uses the technology

$$Y_t = \left( \int_0^1 Y_{it}^{\varepsilon+1} \, di \right)^{\frac{1}{\varepsilon+1}}$$

to produce a consumption good it sells to the households: it buys $Y_{it}$ at price
\( P_t \) from the intermediate firms and sells the amount \( Y_t \) at price \( P_t \) to the households. The problem it faces is

\[
\max_{Y_t} P_t Y_t - \int_0^1 P_d Y_d \, di \tag{2.2.12}
\]

subject to the constraint imposed by the production function.

### 2.2.3 Public sector

The central bank follows a Taylor rule when setting the nominal interest rate \( R_t \). Furthermore, we assume that the zero lower bound is actually binding. That is,

\[
R_t = \max \left\{ 1, r_t (R^*)^{1-\rho} (R_{t-1})^\rho \left[ \left( \frac{\pi_t}{\pi_s} \right)^{\phi_\pi} \left( \frac{Y_t}{Y^*} \right)^{\phi_Y} \right]^{1-\rho} \right\} \tag{2.2.13}
\]

where \( \pi_s \) is the inflation target of the central bank. \( Y^* \) and \( R^* \) are the steady states of output and the nominal rate. \( \rho, \phi_\pi \) and \( \phi_Y \) are parameters. \( r_t \) is a shock, which follows \( r_t = r_t^{\rho_\sigma} \exp(\sigma_r \epsilon_r,t) \) and \( \epsilon_r,t \sim N(0, 1) \).

The government spends a share of output, which is financed by a combination of lump-sum taxes \( T_t \) and borrowing \( B_t \). However, in equilibrium the bonds are always in zero net supply. The government budget constraint is then of the form

\[
T_t + \frac{B_t}{R_t} = P_t \frac{G Y_t}{g_t} + B_{t-1} + \nu W_t L_t \tag{2.2.14}
\]

where \( G \) is the share of government spending in steady state and \( \nu \) is the subsidy paid to the intermediate good firms. \( g_t \) is an exogenous shock to government expenditure, following \( g_t = g_{t-1}^{\rho_g} \exp(\sigma_g, t \epsilon_g, t) \), with \( \log \sigma_g,t = (1 - \varrho_g) \log \sigma_g + \varrho_g \log \sigma_{g,t-1} + \epsilon_{g,t} \), \( \epsilon_{g,t} \sim N(0, 1) \) and \( \epsilon_{g,t} \sim N(0, 1) \).

### 2.2.4 Equilibrium conditions

The equilibrium conditions are briefly summarized below. See Appendix A for the full derivation. The equilibrium of the model is given by the sequence \( \{Y_t, C_t, L_t, S_t, F_t, \Delta_t, \pi_t, R_t, a_t, b_t, g_t, l_t, r_t, u_t, \sigma_{a,t}, \sigma_{g,t}, \sigma_{r,t}, \sigma_{\epsilon} \}_{t=0}^\infty \) which is determined by the equations for the shock processes

\[
a_t = a_{t-1}^{\rho_a} \exp(\sigma_{a,t} \epsilon_{a,t}) \tag{2.2.15}
\]
\[ b_t = b_{t-1}^\rho \exp(\sigma_b \varepsilon_{b,t}) \quad (2.2.16) \]
\[ g_t = g_{t-1}^\rho \exp(\sigma_g \varepsilon_{g,t}) \quad (2.2.17) \]
\[ l_t = l_{t-1}^\rho \exp(\sigma_l \varepsilon_{l,t}) \quad (2.2.18) \]
\[ r_t = r_{t-1}^\rho \exp(\sigma_r \varepsilon_{r,t}) \quad (2.2.19) \]
\[ u_t = u_{t-1}^\rho \exp(\sigma_u \varepsilon_{u,t}) \quad (2.2.20) \]

\[ \log \sigma_{a,t} = (1 - \varrho_a) \log \sigma_a + \varrho_a \log \sigma_{a,t-1} + \varsigma_a \varepsilon_{a,t} \quad (2.2.21) \]
\[ \log \sigma_{g,t} = (1 - \varrho_g) \log \sigma_g + \varrho_g \log \sigma_{g,t-1} + \varsigma_g \varepsilon_{g,t} \quad (2.2.22) \]

and those for the endogenous model variables:

- Output, labor and consumption

\[ Y_t = a_t L_t \Delta_t \quad (2.2.23) \]
\[ C_t = \left( 1 - \frac{G}{g_t} \right) Y_t \quad (2.2.24) \]
\[ C_t^{-1} = \beta \frac{b_t}{u_t} R_t E_t \left[ \frac{u_t+1 C_t^{-1}}{\pi_t+1} \right] \quad (2.2.25) \]

- Inflation and price dispersion

\[ \pi_t = \left[ \frac{1 - (1 - \theta) \left( \frac{\pi^t}{\pi_r} \right)^{1-\varepsilon}}{\theta} \right]^{-\frac{1}{1-\varepsilon}} \quad (2.2.26) \]
\[ \Delta_t = \left[ (1 - \theta) \left( \frac{1 - \theta \pi_t^{-1}}{1 - \theta} \right)^{\frac{1}{1-\varepsilon}} + \theta \left( \frac{\pi_t^*}{\Delta_{t-1}} \right)^{-1} \right] \quad (2.2.27) \]
\[ S_t = \frac{u_t l_t}{a_t} L_t^b Y_t + \beta \theta E_t \left[ \pi_{t+1} S_{t+1} \right] \quad (2.2.28) \]
\[ F_t = u_t C_t^{-1} Y_t + \beta \theta E_t \left[ \pi_{t+1}^{-1} F_{t+1} \right] \quad (2.2.29) \]

- Nominal interest rate

\[ R_t = \max \left\{ 1, r_t (R^*)^{1-\rho} (R_{t-1})^\rho \left[ \frac{\pi_t}{\pi^*} \right]^{\phi_*} \left( \frac{Y_t}{Y^*} \right)^{\phi_Y} \right\}^{-\frac{1}{1-\rho}} \quad (2.2.30) \]
2.2.5 Calibration

The model calibration we choose is standard, with the caveat that it was chosen with the goal of having the model hit the ZLB approximately in 6% of simulated periods, which is the target also adopted by Fernandez-Villaverde et al. (2013), which they claim to be the Chung et al. (2012) estimate of the frequency of hitting the bound in the U.S. based on a sample starting in 1968. It should be kept in mind that the estimate depends on methodology and sample period. In particular, including the Great Recession has a significant effect. The length of a period is assumed to be a quarter.

We start with the shock processes, which are central to the objective of the calibration. The autoregressive coefficients $\rho_a$, $\rho_b$, $\rho_g$, $\rho_l$ and $\rho_u$ are all set to 0.75, except for $\rho_r$, which we set at 0, following Fernandez-Villaverde et al. (2013). Second, we assume that the “unconditional” standard deviation of the shocks, $\sigma_h$, is 0.005, which we find is optimal for hitting the objective of the calibration for the frequency of hitting the bound: with lower standard deviations the model does not hit the bound often enough to reach the objective, while with higher variances numerical problems appear when the GSSA is applied. A likely explanation for these problems was suggested in Mendes (2011), and corroborated by Basu and Bundick (2015), who report similar problems: if the central bank follows a Taylor rule, high enough shock variances may lead to nonexistence of equilibria.\(^6\)

We set $\varrho_a = \varrho_g = 0.9$ and $\varsigma_a = \varsigma_g = 0.1$, following Justiniano and Primiceri (2008) and Fernandez-Villaverde and Rubio-Ramirez (2010). These are also loosely in line with the estimates seen in Fernandez-Villaverde et al. (2011b) for the parameters of the spending volatility process.

Our calibration of the structural parameters is also based on Fernandez-Villaverde et al. (2013). Households’ subjective time preference is set at 0.994, while the elasticity of labor supply is set as $\vartheta = 1$. The Calvo parameter $\theta$ is set at 0.75, implying prices persist on average for 4 periods. The elasticity of substitution between intermediate goods is set at $\varepsilon = 6$, so that in the steady state the mark-up is 20 percent.

Finally, the parameters of the Taylor rule are set as $\phi_x = 2$, $\phi_Y = 0.15$, and

\(^5\)Numerical problems were encountered when $\rho_b$ was set above this bound.

\(^6\)The intuition is that it is possible that the level of volatility is high enough that the expected nominal rate implied by the Taylor rule never equals the expected nominal rate implied by the Fisher condition.
\( \rho = 0.8 \) - the central bank responds somewhat more strongly to inflation than in some other calibrations. This is due to the calibration target: when \( \phi_n \) is set lower, numerical problems appear, unless the autoregressive coefficients are set lower. Finally, the inflation target of the central bank, \( \pi^* \), is set as 1.004 to ensure that the target of the calibration is met.

### 2.3 Solution method

In this section we briefly describe the method we use to solve and estimate the model. To reiterate, we apply an adjusted version the Generalized Stochastic Simulation Algorithm proposed by Judd et al. (2011) to find a nonlinear solution to the model. A more detailed description including details on our implementation of the method can be found in Appendix B. Briefly put, the GSSA is a solution method based on finding parameters for polynomials approximating model choice rules through stochastic simulation.

The vector of state variables for this model is \((\Delta_{t-1}, R_{t-1}, a_t, b_t, g_t, l_t, r_t, u_t, \sigma_{a,t}, \sigma_{g,t})\) and the choice rules we choose to approximate are \(S_t, F_t\) and \(MU_t \equiv C_t^{-1}\), which are approximated with the functions \(\psi_S, \psi_F\) and \(\psi_{MU}\) - third order polynomials in the state variables. As the name of the GSSA indicates, we proceed by simulating the model. In the first step, we use a guess for the parameters of the three approximating polynomials, compute their values at time \(t = 1\) given a realization of the shock processes and then, using the equilibrium conditions given in Section 2.4, compute the values of the rest of the model variables. This procedure is repeated until some time point \(T\). The algorithm then proceeds by computing values for the expectations appearing in the equilibrium conditions by applying a deterministic integration method on the synthetic data generated by the simulation.

Finally, the outcomes of the simulation with approximated expectations are regressed applying regression augmented with a singular value decomposition on the corresponding state variables to obtain three parameter vectors. The algorithm then returns to the first step of the simulation, with the polynomials now parametrized using the three vectors obtained at the end of the previous iteration. This iterative process is then repeated until convergence.

As has been noted in the literature (see, for example, Benhabib et al. (2001), Mendes (2011), or Aruoba and Schorfheide (2013)) it is typical that models with a lower bound on the nominal rate may have two equilibria: in one the nominal
rate is above unity and inflation is on target, and in the other the economy is in perpetual deflation and the nominal rate is stuck at one. Given that the nominal rate is a state variable in our model, the numerical method we apply implicitly selects the non-deflationary equilibrium: the solution is computed by inverting matrices of state variables, which becomes impossible if one of them is constant for all $t$ of the simulation.

2.4 Stochastic volatility and the zero lower bound

This section studies the effects of exogenous variables on the economy and the zero lower bound. We start with a section on the effects of stochastic volatility in isolation, Section 2.4.2 studies the distributions of ZLB events and Section 2.4.3 their causes - relevant for the policymaker who needs to understand properties of the economy at the bound but also for the rest of this paper, as an assessment on which shocks to focus on later is needed.

2.4.1 Effects of stochastic volatility on endogenous variables

This section considers the effects of stochastic volatility itself on endogenous model variables. How do shocks to the uncertainty of productivity and public spending affect the economy?

We start by analyzing the impulse responses of output, the nominal rate, inflation and labor in response to shocks to the variability of government spending and aggregate productivity when the economy otherwise starts at the steady state. As the model is highly nonlinear, the effect of a small increase in volatility (a small positive shock to the volatility process) will not be the same as that of a multiple of a large increase, not to mention a volatility decrease (a negative shock to the volatility process) - therefore we compute the impulse responses for several different levels, going from -5 to 5 standard deviations from the steady state. The results for the case where the economy is not at the bound are presented in Figure 2.4.1.

The effects of volatility increases (and decreases) depend on which of the two processes is hit. An increase in the uncertainty regarding spending (Figure 2.4.1, left panel) unambiguously cools down the private economy: output, the interest rate and inflation all drop. As a consequence the nominal interest
rate drops even further in the two periods after the shock, as the effects of the cooldown transmit through the Taylor rule. The nonlinearity of the model becomes apparent by comparing the effects of ±1 standard deviation shocks to those of the larger shocks: the responses are indeed more than five times larger. There is also asymmetry in the response of the economy to spending volatility: large positive spending volatility shocks have a greater effect than large negative ones. Similarly, the asymmetry between shocks to spending and productivity become clear when comparing the left and right panels of Figure 2.4.1, as the asymmetry between positive and negative volatility in productivity has an even mismatch than that observed for spending volatility. Overall, these results match those seen in for example Fernandez-Villaverde et al. (2011b), with the exception of inflation, although the effects in that paper are much stronger.

Figure 2.4.2 plots similar impulse responses in the case where the economy has been put to the bound by a shock to bond returns sufficiently large to keep it there for 4 quarters. In this case, too the asymmetry between spending and productivity volatility is clear, with shocks to productivity having a much larger effect in general. As has been the case in the literature (e.g. Nakata (2013), Johannsen (2014) and Basu and Bundick (2015)), the volatility effects are orders of magnitude stronger at the bound. Our results on the size of the effects are also broadly in line with previous studies - for example Basu and Bundick (2015) find that a positive one standard deviation shock to the volatility of the discount rate has a -0.17% effect on output under a standard Taylor rule, while we observe that a similar shock to the volatility of the spending process affects output by -0.16%.

The hump-shaped pattern seen in some of the subfigures of Figure 2.4.1 is also amplified when the economy is at the bound, with the response of for example output to shocks in either direction typically peaking in the first period following the shock. However, there is also a important change: when the economy is at the bound, positive volatility shocks will in fact have a positive effect on the economy. This puzzling phenomenon will later play an important role in determining the effects of volatility on the multiplier.
Figure 2.4.1: Impulse responses to -5, -1, 0, 1 and 5 standard deviation shocks to volatility of government spending and productivity, relative to the nonstochastic steady state, when the economy is not at the bound. Left row: volatility in spending. Right row: volatility in productivity.
Figure 2.4.2: Impulse responses to -5, -1, 0, 1 and 5 standard deviation shocks to volatility of government spending and productivity, relative to the nonstochastic steady state, after the economy has hit the ZLB due to a shock to bond returns. Left row: volatility in spending. Right row: volatility in productivity.
2.4.2 Distributions of ZLB events

We estimate the unconditional probability of the economy hitting the zero lower bound by simulating the model for 5000000 periods. We start the simulation from the steady state of the model, and then compute the probability of hitting the bound as the frequency of periods where \( R_t = 1 \), i.e.

\[
P(R_t = 1) \approx \frac{1}{T} \sum_{t=1}^{T} I(R_t = 1)
\] (2.4.1)

where \( I \) is the indicator function, equal to one when the argument is true. This computation yields 5.6% - that is, the model is at the bound slightly more often than was our target for calibration.

How does this probability depend on the history of states - or in other words, what are the conditional probabilities of hitting the bound? Using the same simulation as before, we start with the obvious case: the situation where the interest rate was at the bound in the previous period. We compute the approximation

\[
P(R_t = 1 | R_{t-1} = 1) \approx \frac{1}{T} \sum_{t=1}^{T} I(R_t = 1 \land R_{t-1} = 1)
\] (2.4.2)

which yields 0.24 - the probability of being at the bound again is roughly 4 times as high as the unconditional probability. Due to the heavily nonlinear nature of the model the probability depends on past states.

As further evidence of the time-varying nature of this phenomenon, Table 2.4.1 reports the expected number of additional periods at the bound when the economy has been at the bound for \( n \) periods and its variance, i.e.

\[
E\left(\left(\sum_{j=t}^{T} I(R_j = 1) - 1\right) | R_{t-1} = 1, R_{t-2} = 1, ..., R_{t-n} = 1\right)
\] (2.4.3)

and

\[
Var\left(\left(\sum_{j=t}^{T} I(R_j = 1) - 1\right) | R_{t-1} = 1, R_{t-2} = 1, ..., R_{t-n} = 1\right)
\] (2.4.4)

As the economy spends more time at the bound, the expected number of additional periods at the bound increases, doubling from \( n = 1 \) to \( n = 7 \). Similarly,\footnote{The nonmonotonicity is likely due to simulation related uncertainty.}
Table 2.4.1: Conditional expectation and variance of additional periods at the ZLB when the economy has been at the bound for \( n \) periods

<table>
<thead>
<tr>
<th>( n )</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>Expect.</td>
<td>0.51</td>
<td>0.57</td>
<td>0.62</td>
<td>0.66</td>
<td>0.68</td>
<td>0.82</td>
<td>1.00</td>
<td>1.04</td>
<td>1.18</td>
<td></td>
</tr>
<tr>
<td>Variance</td>
<td>1.86</td>
<td>2.11</td>
<td>2.23</td>
<td>2.44</td>
<td>2.76</td>
<td>2.92</td>
<td>2.85</td>
<td>2.69</td>
<td>2.06</td>
<td></td>
</tr>
</tbody>
</table>

the variance of additional periods increases: the future becomes more and more uncertain as more time is spent at the ZLB.

We then turn to the effects of variation in exogenous uncertainty: how do these quantities depend on the shocks to the variances of aggregate productivity and government spending? We study cases where \( \sigma_{g,t} \) and \( \sigma_{a,t} \) are either high or low, defined as \( \sigma_{j,t} > 0.008 \) and \( \sigma_{j,t} < 0.003 \), respectively, with \( j \in \{g,a\} \). Thus the probabilities we compute are

\[
P(R_t = 1 | \sigma_{j,t} > 0.008) \approx \frac{1}{T} \sum_{t=1}^{T} I(R_t = 1 \land \sigma_{j,t} > 0.008) \tag{2.4.5}
\]

for the high state, with the obvious analogue for \( \sigma_{j,t} < 0.003 \) in the low state. We find that the change in probability indeed depends on the source of uncertainty: if government spending is expected to be more certain in the future, the nominal rate is less likely, with probability 5%, to hit the ZLB today than when there is less certainty about future government spending - then the probability is 6.2%. Uncertainty regarding productivity appears to have no effect on the probability of hitting the bound: in the high case the probability is 5.4%, and in the low case 5.6%.

We finish this section with an analysis on the conditional expectations and variances of additional periods at the bound in states of high and low volatility. Table 2.4.2 reports the results of computing approximations for

\[
E \left( \left( \sum_{j=t}^{T} I(R_j = 1) - 1 \right) | R_{t-1} = 1, R_{t-2} = 1, ..., R_{t-n} = 1, \sigma_{j,t} > 0.008 \right) \tag{2.4.6}
\]

and

\[
Var \left( \left( \sum_{j=t}^{T} I(R_j = 1) - 1 \right) | R_{t-1} = 1, R_{t-2} = 1, ..., R_{t-n} = 1, \sigma_{j,t} > 0.008 \right) \tag{2.4.7}
\]
Table 2.4.2: Conditional expectation and variance of additional periods at the ZLB when the economy has been at the bound for \( n \) periods when volatility of spending \((\sigma_{g,t})\) or productivity \((\sigma_{a,t})\) is either high or low

<table>
<thead>
<tr>
<th>( n )</th>
<th>( \sigma_{g,t} )</th>
<th>( \sigma_{a,t} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Expect., high vol.</td>
<td>0.55 0.62 0.6 0.63 0.89</td>
<td>0.53 0.58 0.69 0.66 0.86</td>
</tr>
<tr>
<td>Variance, high vol.</td>
<td>1.89 2.08 2.02 2.26 2.47</td>
<td>1.96 2.05 2.43 2.2 2.21</td>
</tr>
<tr>
<td>Expect., low vol.</td>
<td>0.47 0.52 0.53 0.68 0.63</td>
<td>0.56 0.59 0.57 0.41 0.59</td>
</tr>
<tr>
<td>Variance, low vol.</td>
<td>1.77 1.84 2.15 2.25 2.61</td>
<td>1.93 1.98 1.91 1.56 1.67</td>
</tr>
</tbody>
</table>

for \( j \in \{g,a\} \) and the corresponding quantities for states of low volatility.

As can be seen by comparing Table 2.4.2 to Table 2.4.1, the changes in volatilities make a difference, albeit a small one. Low volatility decreases the expected number of extra periods at the ZLB and the corresponding variance - the latter effect is particularly strong for a decrease in the volatility of aggregate productivity. However, high volatility in either process appears to have no noticeable effect on either the expectation of additional periods or their variance. The spell-shortening effect of low volatility is, on the other hand, a key part in explaining the size of the multiplier in certain situations: both Christiano et al. (2011) and Erceg and Linde (2014) found that the expected duration of the ZLB spell has an noticeable effect on the effectiveness of fiscal policy.

2.4.3 Which shocks drive the economy to the bound?

To answer this question we first analyze the probability of hitting the bound when one of the eight exogenous processes in the model is hit by a positive or negative shock that is either one, two or five standard deviations away from the deterministic steady state. Table 2.4.3 presents these results; the middle row reports the case of no innovation for reference. The difference compared to the unconditional probability of hitting the bound computed in Section 2.4.2 (5.6%) is due to the fact that we report the probability of hitting the bound at least once within the ten periods following the innovation. These results were constructed by simulating the model 10 000 times for 11 periods, starting all state variables except the one with the innovation from the steady state. The probability is then computed as an average over the simulations, i.e. we compute

\[
P (\bigcup_{i=1}^{10} R_i = 1 | h_1 = S) \approx \frac{1}{T} \sum_{t=1}^{T} I (\bigvee_{i=1}^{10} (R_i = 1) \land h_1 = S) \tag{2.4.8}
\]
Table 2.4.3: Effect of shock innovations on probability of entering the ZLB within the next 10 periods.

<table>
<thead>
<tr>
<th>Size of innov.</th>
<th>(a_t)</th>
<th>(b_t)</th>
<th>(u_t)</th>
<th>(l_t)</th>
<th>(r_t)</th>
<th>(g_t)</th>
<th>(\sigma_{a,t})</th>
<th>(\sigma_{g,t})</th>
</tr>
</thead>
<tbody>
<tr>
<td>2 s.d.</td>
<td>0.31</td>
<td>0.43</td>
<td>0.23</td>
<td>0.24</td>
<td>0.22</td>
<td>0.27</td>
<td>0.26</td>
<td>0.26</td>
</tr>
<tr>
<td>1 s.d.</td>
<td>0.28</td>
<td>0.34</td>
<td>0.24</td>
<td>0.25</td>
<td>0.23</td>
<td>0.27</td>
<td>0.27</td>
<td>0.26</td>
</tr>
<tr>
<td>0 s.d.</td>
<td>0.26</td>
<td>0.26</td>
<td>0.26</td>
<td>0.26</td>
<td>0.26</td>
<td>0.26</td>
<td>0.26</td>
<td>0.26</td>
</tr>
<tr>
<td>-1 s.d.</td>
<td>0.24</td>
<td>0.2</td>
<td>0.28</td>
<td>0.27</td>
<td>0.28</td>
<td>0.26</td>
<td>0.25</td>
<td>0.26</td>
</tr>
<tr>
<td>-2 s.d.</td>
<td>0.22</td>
<td>0.15</td>
<td>0.3</td>
<td>0.28</td>
<td>0.36</td>
<td>0.25</td>
<td>0.26</td>
<td>0.26</td>
</tr>
</tbody>
</table>

Shock notation: \(a_t\) is productivity, \(b_t\) bond return, \(u_t\) discount rate, \(l_t\) labor supply, \(r_t\) interest rate, \(g_t\) government spending, \(\sigma_{a,t}\) vol. of productivity, \(\sigma_{g,t}\) vol. of spending.

where \(h \in \{a, b, g, l, r, u, \sigma_g, \sigma_a\}\), \(S = \pm k\sigma_h\) and \(k \in \{0, 1, 2\}\).

The first thing to note is that the primary driver of the economy to the ZLB is the shock to bond returns \((b_t)\): with relatively modest two standard deviation shocks the probability of hitting the bound increases 65%. Positive shocks to aggregate productivity \((a_t)\) and negative shocks to the discount rate \((u_t)\) and monetary policy \((r_t)\) also have noticeable effects, with two standard deviation shocks increasing the probability by roughly a fifth. Shocks to the labor supply \((l_t)\) have a lesser effect: a negative shock increases the probability slightly. Innovations to government expenditure \((g_t)\) seem to have no effect at all, as is the case with innovations to the volatility processes \(\sigma_{a,t}\) and \(\sigma_{g,t}\).

In comparison to Fernandez-Villaverde et al. (2013), most of these results are in concordance. However, our finding of the effect of the discount rate shock points to the completely different direction, as they found it a significant positive determinant of ZLB episodes. A likely explanation to this drastic discrepancy is the presence of the bond return shock in our framework. In this model a large, current shock to the bond premium has a similar effect on the behavior of the household through the Euler equation as a large future shock to the subjective discount rate has in the model of Fernandez-Villaverde et al. (2013). Furthermore, in our model the discount rate shock appears both as a current period and next period term in the Euler equation and it seems that the effect of the current period realization is stronger.

The question of which shocks lead to the bound can also be looked at from a different perspective, which also tells us how persistently the economy stays at the bound. Figure 2.4.3 graphs the expected value of \(R_t, R_{t+1}\) and \(R_{t+2}\) (on the vertical axis) when the economy has been hit by shocks of varying magnitude.
(on the horizontal axis) at time $t$. From this point of view the answer looks mostly similar to what was seen in Table 2.4.3. The shocks that can push the economy to the bound (easily) are the bond return and monetary policy shocks - the expected value of $R_t$ following a 5 standard deviation shock to the bond premium or a -5 standard deviation shock to monetary policy is very close to unity. Shocks to the discount rate and productivity can also push the expectation close to unity, but only if they are very large, roughly ten standard deviations in magnitude.

The major difference that can be seen is the importance of the labor shock: when the shock is negative enough, the expected value of the nominal rate is quite close to unity. The volatility shocks have practically no effect on the expected value. For example, spending volatility shocks of -20 and 20 standard deviations imply practically identical expected values of the nominal interest rate. As is to be expected from the quite persistent calibration of the shock processes, after hitting the bound the nominal rate stays very close to what it was at time $t$ in the following two periods. The key exception is the case of monetary policy shocks: the nominal rate starts recovering immediately at time $t+1$ - but this shock was assumed to have no persistence at all.

We conclude this section by presenting histograms of the realizations of the exogenous shocks at the ZLB and comparing them to the unconditional distributions from which they were drawn - this allows us to test which shocks are important in a third way. The histograms plot realized values of the shocks during periods when the economy is at the bound, while the dashed lines plot the corresponding theoretical densities. From Figure 2.4.4 we can see that when the economy is at the bound, the histograms of the realized volatility shocks match the corresponding theoretical distributions extremely closely. Similarly the realized distributions of the labor shock and government spending shock seem unmoved from what was expected, though the distribution of the labor shocks has rather fat tails.

At the ZLB the realizations of the other four exogenous processes do not, however, line up with the theoretical distributions from which they were drawn. Confirming the findings deduced from Figure 2.4.3, the conditional mean of the monetary policy shock is significantly below its predicted value. As was

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8 As before, we compute these by simulating the model for 10 000 periods and taking the average over the simulations.

9 The figures were computed by simulating the model for 5000000 periods, using the data from section 2.4.2.
Figure 2.4.3: Expected values of $R_t$, $R_{t+1}$, and $R_{t+2}$ when the economy is hit by a shock at time $t$. Horizontal axis: deviation from steady state, in standard deviations. Vertical axis: expected value of interest rate.
inferred from Table 2.4.3, the conditional mean of the bond return shock is positive when at the bound, as is that of the aggregate productivity shock. Finally, the discount rate shock is typically negative when the economy hits the bound.

Based on the findings seen in Table 2.4.3 and Figures 2.4.3 and 2.4.4, we later analyze ZLB events that are caused by 10 standard deviation shocks to productivity, five standard deviation shocks to the bond return, minus five standard deviation shocks to the interest rate and -10 standard deviation shocks to the discount rate. While significant portions of previous literature have focused only on a single shock (commonly to the discount rate), we will later show that the dynamics of the economy at the bound depend on which shock caused the event and thus significantly affect not only the size and time path of the multiplier, but also the effects of volatility.

2.5 Effectiveness of government spending

In this section we turn to the central question of the paper: effectiveness of increasing government spending in order to stimulate the economy when conventional monetary policy is unavailable as a tool.

At first, we assume the following basic framework, from which we deviate later to study more specific questions.

1. At time $t = 0$ the economy is at the steady state. At $t = 1$ the economy is hit by a shock to government spending and a shock that drives the economy to the ZLB, large enough that the economy is expected to hit the lower bound immediately and stay there for several periods.\(^{10}\)

2. From period $t = 2$ onwards the model is simulated normally for 9 additional periods a total of 10000 times.

3. Following Fernandez-Villaverde et al. (2013), the multiplier for each period is computed as $\frac{Y_t - Y_t^*}{G_1 - G_1^*}$ as an average over the simulations, where $G_1$ is government spending in period 1 with the spending increase, computed as $\bar{G}Y_t$ and $Y_t^*$ and $G_t^*$ are the means of output and spending at time $t$ from simulations where there is no spending increase.

\(^{10}\)Based on the findings of the previous Section, the shocks are set as 5, -5, 10 and -10 standard deviations for bond returns, monetary policy, productivity and the discount rate, respectively. These were found large enough to ensure that the economy stays at the bound for roughly 3 periods on average.
Figure 2.4.4: Distributions of exogenous shocks, unconditional and conditional on being at the ZLB. Dashed line: unconditional, theoretical density function of the shock process. Histogram: realized shocks, conditional on being at the ZLB.
It is important to note that it is not at all inconsequential which shock drives the economy to the ZLB. For example, the effect of a productivity shock large enough to put the economy to the bound on output is significantly positive but negative on inflation. A shock to the bond premium or to the discount rate, on the other hand, has a negative effect on both output and inflation. Finally, a monetary policy shock will drive the economy to the bound but also boost output and inflation. These effects are documented in Figure 2.5.1, which presents the time paths of output and inflation in these four different situations, relative to the steady state, both with and without a spending shock of 5 standard deviations. This corresponds to a 2.5% increase in spending relative to its steady state.

Furthermore, as can be seen from Figure 2.5.1, the effects of the spending increase on prices and production also depend on the cause of the ZLB episode. If the ZLB was entered due to a shock to the discount rate (third row), the spending increase is deflationary. If the economy is at the bound due to a shock to bond return (second row), the increase in spending is inflationary. Thus in some cases there is a tradeoff between inflation and (extra) output caused by increased public spending. Partly due to these effects we will throughout the rest of the paper study the fiscal multiplier in four different cases: as a consequence of either increased productivity, bond returns, a jump in the discount factor or a negative monetary policy shock.

We then turn to the primary questions of this paper. How does exogenous uncertainty affect the fiscal multiplier? To reiterate, previous literature has found evidence both for and against effective fiscal policy while at the bound. As an example of quantitative results, Fernandez-Villaverde et al. (2013) found the multiplier, when the ZLB event is caused by a discount rate shock, to be roughly 1.5.

Figure 2.5.2 depicts the multiplier when the economy is not at the bound. It plots the on-impact multipliers for shocks to the spending process of 1 to 20 standard deviations (plotted on the vertical axis) when the economy is also hit at time $t = 1$ by volatility shocks ranging from -10 to 10 standard deviations. The upper figure contains the case of spending volatility, while the lower has the case of productivity volatility. The spending shocks correspond to increases in spending, relative to the steady state, of 0.5% to 10%. The presentation is slightly nonstandard, as the size of the multiplier on the two dimensional plane is plotted using colour: the brighter the tone, the higher the multiplier.
Figure 2.5.1: Average time paths of output (left column) and inflation (right column) relative to steady state after entering the ZLB due to a shock to aggregate productivity (top row), bond return (second row), discount rate (third row), interest rate (bottom row) with and without a 5 standard deviation increase in government spending.
It can be seen that when the economy is not at the bound, fiscal policy is almost always either ineffective with a multiplier of roughly 0.6, with most areas being relatively flat colour. In extreme cases a spending increase is harmful\textsuperscript{11}, such as when volatility in productivity is extremely high and the spending increase is very large - in this case the colour in the plot is practically black. The rare exception of a large multiplier is when the spending increase is very small and the volatility of productivity is extremely low - in these cases the multiplier is significantly larger than one and the plot practically white. In addition, when the economy is not at the bound the multiplier is quite insensitive to changes in volatility.

Figures 2.5.3 and 2.5.4 present our main results - plots of multipliers in various volatility states when the economy is at the zero lower bound. In Figure 2.5.3 we have the case of volatility in spending, while Figure 2.5.4 plots the case of volatility in productivity. At the bound the multiplier varies a great deal, depending on the shock, volatility and size of spending increase: when hit by a productivity shock (subfigure a. in both figures) the multiplier rarely goes significantly above 0.5. Conversely, when the economy is hit by a shock to the discount rate (subfigure c.), the multiplier can be rather large, even well above three, as was also shown to be the case by Christiano et al. (2011). Multipliers when hit by a monetary policy shock or a bond return shock lie in between, but are typically greater than 1 for spending shocks below 5 standard deviations.

Why is government spending so effective after discount rate shocks and so ineffective after productivity shocks? The answer is in the intertemporal consumption patterns of the household. A significant increase in productivity means that the household is less likely to be willing to increase hours worked after the stimulus even further - a necessary condition for the multiplier to be high. And on the converse, when hit by a negative discount rate shock, future consumption becomes relatively less valuable, inciting the household to consume and work more today.

Comparing Figures 2.5.3 and 2.5.4 we can also see that the effect of uncertainty on the multiplier, just as the multiplier itself, depends a great deal on the cause of the ZLB event. For example, when at the bound after a discount rate shock, changes in volatility seem to have very little effect. That is, when the discount rate is low, households' sensitivity to uncertainty appears to go down. A contrary result can be seen in the case of a shock to the discount factor

\textsuperscript{11}In the sense that output actually drops when spending goes up.
Figure 2.5.2: On-impact multipliers of spending shocks from 1 to 20 standard deviations (vertical axis) when the economy is not at the zero lower bound. Upper figure: -10 to 10 standard deviation variation (horizontal axis) in volatility of spending. Lower figure: -10 to 10 standard deviation variation (horizontal axis) in volatility of productivity. Note: colour scale is not same in the two figures.
(fourth row), which causes households to be significantly more responsive to changes in volatility: in this case the multiplier varies a great deal depending on the volatility state.

Furthermore, the source of uncertainty itself affects the results. The multiplier is higher when the volatility of spending is high (Figure 2.5.3), and lower when the volatility of productivity is high (Figure 2.5.4), with the exception of the case of a shock to monetary policy. A likely explanation can again be found in the risk averseness of the households: high volatility in productivity means that future consumption is less certain, causing the households to be more sensitive to spending increases. Similarly, an increase in spending combined with low uncertainty regarding future spending implies that the probability of high taxes and less consumption in the future is high, which makes the households less willing to consume today. The other crucial factor determining the effect of volatility is the expected duration of the ZLB, as shown by e.g. Erceg and Linde (2014) - and as was seen previously, higher volatility corresponds to longer expected ZLB episodes.

The difference between the two types of uncertainty is likely due to the risk aversion of the households. When uncertainty regarding future expenditure is low, households are sure that a current spending increase is more likely to persist, and they adjust their consumption pattern to match this belief. And when uncertainty in productivity is low, a spending increase will elicit the risk averse household, now sure of the level of consumption in the future, to increase its spending.

These results regarding uncertainty have implications for policy. It is typical in the literature that the size of the spending shock is often interpreted as some sort of a policy variable of the government when exercises of the type seen here are conducted. If one is willing to extend this interpretation to uncertainty regarding spending - that the government is capable of manipulating this exogenous variable, too - then a wise policymaker will attempt to ensure that the household believes that future spending is uncertain. Or in other words, the government should tell the public that the spending increases it has just enacted are not necessarily going to persist for long.

In Figure 2.5.5 we plot the time paths of the multiplier of a 5 standard deviation spending increase in when the economy is hit by volatility shocks of ±5 standard deviations. For comparison we also present the effect for a similar expenditure increase when not at the bound. As before, the on-impact multiplier when not at
Figure 2.5.3: On-impact multipliers of spending shocks from 1 to 20 standard deviations (vertical axis) when the economy is at the bound due to a shock (a) productivity, (b) bond return, (c) discount rate, (d) monetary policy. Horizontal axis: -10 to 10 standard deviation variation in volatility of spending. Note: colour scale is not same in the four figures.
Figure 2.5.4: On-impact multipliers of spending shocks from 1 to 20 standard deviations (vertical axis) when the economy is at the bound due to a shock (a) productivity, (b) bond return, (c) discount rate, (d) monetary policy. Horizontal axis: -10 to 10 standard deviation variation in volatility of productivity. Note: colour scale is not same in the four figures.
the bound is in absence of volatility shocks approximately 0.6, roughly matching the 0.5 reported by Fernandez-Villaverde et al. (2013). While volatility does affect the on-impact multiplier, the time paths after impact remain essentially the same when the interest rate is high.

However, when not at the bound volatility changes can affect the temporal dynamics of the multiplier. As was seen in Figures 2.5.3 and 2.5.4, high volatility in spending typically amplifies the multiplier, while low volatility in productivity has the same effect. In the case where the multiplier is largest (discount rate shock) the effects of volatility are relatively the smallest. For the most part the multiplier does not persist above unity for long - for example when the economy has been hit by a discount rate shock (fourth row) the multiplier is large for just 2 periods. That is, significant stimulating effects are not very long in duration.

We finish our study of the effectiveness of government spending by a brief analysis on its effects on some of the endogenous variables. Figure 2.5.6 presents the time paths of key endogenous variables relative to the nonstochastic steady state, following an increase in government spending of 5 standard deviations in states of normal, high and low volatility - defined as ±5 standard deviation shocks to volatility - of public expenditure. Furthermore, the plots are presented as differences from the case where there is no spending increase.

As a reminder, in Figure 2.5.1 it was shown that the actual increase in output on impact of the spending increase depends heavily on which shock pushes the economy to the bound. For example, after a discount rate shock, output will drop significantly.

The economy will enter a deflationary period (relative to the case of no intervention) after being pushed to the bound by either a productivity or a discount rate shock, while the intervention is inflationary following a bond return shock. Finally, a drop in the interest rate, when caused by a monetary policy shock, will unsurprisingly cause inflation to go up, but only very slightly compared to the bond shock case. The path of the interest rate itself following a 5 standard deviation spending increase also depends on the shock, with stimulus pushing it down after a productivity or discount shock and up after a bond shock. After a monetary policy shock there is a relatively modest hump shaped response.

The relatively minor effects of volatility are also visible in Figure 2.5.6. Typically in low volatility states the effect of stimulus on output is slightly higher, and the deflationary effect weaker than otherwise. This effect is particularly pronounced.
Figure 2.5.5: Time path of multiplier (vertical axis) for 5 standard deviation increase in spending when the standard deviation of government spending (left panel) or aggregate productivity (right panel) varies from high to low and the economy is above the ZLB (top row) or at the ZLB due to a shock to aggregate productivity (second row), bond returns (third row), discount rate (fourth row) or interest rate (bottom row).
after a discount rate shock.

2.6 Conclusions

The zero lower bound poses a significant problem for governments: the effectiveness of conventional monetary policy in stimulating the real economy is reduced to practically zero. A sometimes politically infeasible alternative is to apply traditional Keynesian fiscal stimulus - increase government spending - in order to incentivize the private sector to increase its activity. In this paper we have studied the effectiveness of such policy at the ZLB when the future state of the economy is uncertain.

The primary new result is clear: the multiplier is strongly dependent on the state of the volatility in the economy and on which shock has brought the economy to the ZLB. We find that the size of the multiplier is dependent on uncertainty regarding both future spending and productivity, and in a rather complex fashion. For example, low uncertainty regarding future productivity amplifies the effects of spending increases, while low uncertainty regarding future spending often dampens the effects of stimulus. Especially the effects of spending uncertainty are likely to be due to the effects of uncertainty on the expected duration of the ZLB episode, as shown by Erceg and Linde (2014).

If the economy is at the bound due to a shock to the households’ discount rate, the multiplier is typically quite large - as was found by both Fernandez-Villaverde et al. (2013) and Christiano et al. (2011). When the economy is at the bound due to a productivity shock the multiplier can even be negative. Thus it would seem that question of how large the multiplier is cannot be reduced to a single number, as was also emphasized by Corsetti et al. (2012): the multiplier is determined by the environment. When not at the bound the multiplier is below one, regardless of exogenous uncertainty.

It should also be kept in mind that depending on the cause of the ZLB episode the spending increase can also have other effects on the economy. For example if the episode is due to an increase in bond return, an increase in government spending will both boost output and inflation - but if the episode is caused by a productivity increase, a spending increase will be deflationary. Thus there are significant tradeoffs in certain situations, and appropriate policy requires an analysis on the causes of the ZLB event.
Figure 2.5.6: Time paths of endogenous variables relative to the nonstochastic steady state after a ZLB event with 5 standard deviation government stimulus in states of normal, high and low spending volatility. Rows: 1. productivity shock, 2. bond shock, 3. discount rate shock, 4. interest rate shock. Columns: Level of output (left), inflation (middle), interest rate (right).
Several caveats - that might also be fruitful topics for further research - regarding the results are important to keep in mind. A key assumption made in the model is that the financing of the expenditure increases has no effect on the economy beyond household consumption - which is unlikely to hold in reality. A more realistic model would assume that the spending increases are financed through more realistic means, such as distortionary taxes on labor income, or possibly even that the expenditure increases are financed with a deficit under conditions where government debt is not irrelevant. Furthermore, the model abstracts away several features that have been found rather important in the literature, such as habit formation and capital, both of which are likely to affect the behavior of the household significantly.
Appendix A. Derivation of model equilibrium conditions and steady state

Households

The first-order conditions to the households’ problem with respect to $C_t$, $B_t$ and $L_t$ are

$$\Lambda_t = \frac{u_t}{C_t P_t}$$

$$P_t \Lambda_t = \beta b_t R_t E_t [P_{t+1} \Lambda_{t+1}]$$

and

$$\Lambda_t W_t = u_t l_t L_t^a$$

Combining the first-order conditions for consumption and labor implies

$$u_t l_t L_t^a = P_t \Lambda_t W_t$$

In addition, in equilibrium the households trade bonds amongst themselves so that market clearing implies $B_t = 0$.

Firms

Final good producers

The first order condition results in

$$Y_{it} = Y_t \left( \frac{P_{it}}{P_t} \right)^{-\varepsilon}$$

and by plugging this into the production function and rearranging we obtain

$$P_t = \left( \int_0^1 P_{it}^{1-\varepsilon} \, di \right)^{\frac{1}{1-\varepsilon}}$$

Intermediate good producers

The first order condition with respect to $L_t$ is

$$\Theta_t = \frac{(1-\nu) W_t}{a_t}$$

where $\Theta_t$ is the Lagrange multiplier associated with the relevant constraint.
As marginal cost is the derivative of the cost function with respect to output we have \( MC_t = \Theta_t \), and thus real marginal cost is

\[
mc_t = \frac{(1 - \nu) W_t}{P_t}
\]

The first order condition for the price-setting problem is

\[
E_t \sum_{t=0}^{\infty} (\beta \theta)^j \Lambda_{t+j} Y_{t+j} P_{t+j}^f \left[ \frac{\hat{P}_t}{P_{t+j}} - \frac{\varepsilon}{\varepsilon - 1} mc_{t+j} \right] = 0
\]

Let \( \chi_{j,t} \equiv \begin{cases} 
1 & j = 0 \\
\frac{1}{\pi_{t+j-1} \cdots \pi_t} & j \geq 1
\end{cases} \), where \( \pi_t = \frac{P_t}{P_{t-1}} \). Then

\[
E_t \sum_{t=0}^{\infty} (\beta \theta)^j P_{t+j} \Lambda_{t+j} Y_{t+j} \chi_{j,t}^{-\varepsilon} \left[ \hat{P}_t \chi_{j,t} - \frac{\varepsilon}{\varepsilon - 1} mc_{t+j} \right] = 0
\]

where \( \hat{P}_t = \frac{P_t}{P_{t-1}} \). This ratio can be expressed as

\[
\frac{E_t \sum_{t=0}^{\infty} (\beta \theta)^j P_{t+j} \Lambda_{t+j} Y_{t+j} \chi_{j,t}^{-\varepsilon} \left[ \chi_{j,t} - \frac{\varepsilon}{\varepsilon - 1} mc_{t+j} \right]}{E_t \sum_{t=0}^{\infty} (\beta \theta)^j P_{t+j} \Lambda_{t+j} Y_{t+j} \chi_{j,t}^{1-\varepsilon}} \equiv \frac{S_t}{F_t}
\]

Recursive relationships for \( S_t \) and \( F_t \) can be expressed as

\[
S_t = E_t \sum_{t=0}^{\infty} (\beta \theta)^j P_{t+j} \Lambda_{t+j} Y_{t+j} \chi_{j,t}^{-\varepsilon} \left[ \frac{1}{\pi_{t+j}} - \frac{\varepsilon}{\varepsilon - 1} mc_{t+j} \right] \quad \iff \quad S_t = \frac{\varepsilon}{\varepsilon - 1} mc_t P_t \Lambda_t Y_t + E_t \left[ \left( \frac{1}{\pi_{t+1}} \right)^{-\varepsilon} \right] E_{t+1} \left[ \sum_{t=0}^{\infty} (\beta \theta)^j P_{t+j+1} \Lambda_{t+j+1} Y_{t+j+1} \chi_{j,t+1}^{-\varepsilon} \left[ \frac{1}{\pi_{t+j+1}} - \frac{\varepsilon}{\varepsilon - 1} mc_{t+j+1} \right] \right]
\]

\[
\iff \quad S_t = \frac{\varepsilon}{\varepsilon - 1} mc_t P_t \Lambda_t Y_t + \beta \theta E_t \left[ (\pi_{t+1})^\varepsilon S_{t+1} \right]
\]

Plugging in the definition of marginal cost yields

\[
S_t = \frac{\varepsilon}{\varepsilon - 1} \frac{(1 - \nu) W_t}{P_t} P_t \Lambda_t Y_t + \beta \theta E_t \left[ (\pi_{t+1})^\varepsilon S_{t+1} \right]
\]

and since

\[
\frac{\varepsilon}{\varepsilon - 1} (1 - \nu) = 1
\]
we have, after plugging in the expression for $\Lambda_t$,

$$S_t = \frac{u_t}{a_t} L_t^\theta Y_t + \beta E_t \left[ \pi_{t+1}^\varepsilon S_{t+1} \right]$$

and for $F_t$ using a similar method we have

$$F_t = u_t C_t^{-1} Y_t + \beta E_t \left[ \pi_{t+1}^\varepsilon F_{t+1} \right]$$

**Aggregate price level**

(2.6.1) can be rearranged as

$$P_t = \left[ \int_{\text{reopt}} P^1_{it} \epsilon_{di} + \int_{\text{non-reopt}} P^1_{it} \epsilon_{di} \right]^{\frac{1}{1-\epsilon}}$$

Let $\omega_{t-1,t}(j)$ denote the share of non-reoptimizing firms at $t$ that had the price $P(j)$ at time $t-1$ and let $\omega_{t-1}(j)$ be the share of firms at time $t$ that had the price $P(j)$ in period $t-1$. Then $\omega_{t-1,t}(j) = \theta \omega_{t-1}(j)$ and furthermore

$$\int_{\text{non-reopt}} P^1_{it} \epsilon_{di} = \int_0^1 \theta \int_0^1 P(j)^{1-\epsilon} \omega_{t-1,t}(j) dj = \theta \int_0^1 P(j)^{1-\epsilon} \omega_{t-1}(j) dj = \theta P_{t-1}^{1-\epsilon}$$

implying that

$$P_t = \left[ (1-\theta) \tilde{P}_t^{1-\epsilon} + \theta P_{t-1}^{1-\epsilon} \right]^{\frac{1}{1-\epsilon}}$$

which can be rearranged as

$$1 = \left[ (1-\theta) \tilde{P}_t^{1-\epsilon} + \theta \left( \frac{1}{\pi_t} \right)^{1-\epsilon} \right]^{\frac{1}{1-\epsilon}}$$

so that

$$\tilde{P}_t = \left[ \frac{1 - \theta \left( \frac{1}{\pi_t} \right)^{1-\epsilon}}{1 - \theta} \right]^{\frac{1}{1-\epsilon}} = \frac{S_t}{F_t}$$

**Aggregate output**

Let $Y_t = a_t L_t = \int Y_{it} di$ and $L_t = \int_0^1 L_{it} di$. As $Y_{it} = Y_i \left( \frac{P_{it}}{P_t} \right)$ we have

$$\tilde{Y}_t = Y_t \tilde{P}_t^{-\epsilon} \int_0^1 P_{it}^{-\epsilon} di$$
Let \((\bar{P}_t)^{-\varepsilon} \equiv \int_0^1 P_{it}^{-\varepsilon} \, di\)

implying

\[ Y_t = Y_t \left( \frac{P_t}{\bar{P}_t} \right)^{\varepsilon} = a_t L_t \Delta_t \]

with \(\Delta_t \equiv \left( \frac{P_t}{\bar{P}_t} \right)^{\varepsilon}\).

**Dynamics of price dispersion**

The dynamics of \(\bar{P}_t\) are described by

\[ \bar{P}_t = \left[ (1 - \theta) \bar{P}_{t-1}^{1-\varepsilon} + \theta \bar{P}_t^{1-\varepsilon} \right]^{-\frac{1}{\varepsilon}} \]

so that

\[ \Delta_t = \left( \left[ (1 - \theta) \bar{P}_{t-1}^{1-\varepsilon} + \theta \bar{P}_t^{1-\varepsilon} \right]^{-\frac{1}{\varepsilon}} \right)^{\varepsilon} \]

and

\[ \Delta_t^{1/\varepsilon} = \left( (1 - \theta) \left( \frac{\bar{P}_t}{\bar{P}_t} \right)^{1-\varepsilon} + \theta \left( \frac{\bar{P}_t}{\bar{P}_t} \right)^{1-\varepsilon} \right)^{-\frac{1}{\varepsilon}} \]

implying

\[ \Delta_t = \left( (1 - \theta) \bar{P}_t^{1-\varepsilon} + \theta \left( \frac{\bar{P}_t}{\bar{P}_t} \right)^{1-\varepsilon} \right)^{-1} \]

and by plugging in \(\bar{P}_t = \left[ \frac{1 - \theta \pi_t^{-1}}{1 - \theta} \right]^{\frac{1}{1-\varepsilon}}\) we obtain

\[ \Delta_t = \left[ (1 - \theta) \left[ \frac{1 - \theta \pi_t^{-1}}{1 - \theta} \right]^{\frac{1}{1-\varepsilon}} + \theta \left( \frac{\pi_t^{} \pi_t}{\Delta_{t-1}} \right) \right]^{-1} \]

**The steady state**

The steady state is defined by the equations

\[ F_{ss} = \frac{1 - \bar{G}}{1 - \theta \pi_{ss}^{-1}} \]

and

\[ S_{ss} = F_{ss} \frac{1 - \theta \pi_{ss}^{-1}}{1 - \theta} \left( \frac{1}{1-\varepsilon} \right) \]

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\[
\Delta_{ss} = (1 - \theta \pi_{ss}) \left[ (1 - \theta) \left[ \frac{1 - \theta \pi_{ss}^{\epsilon - 1}}{1 - \theta} \right]^{\frac{\pi_{ss}}{\pi_{ss} - 1}} \right]^{-1}
\]

\[
L_{ss} = S_{ss} \left( \frac{1 - \beta \theta \pi_{ss}^{\epsilon}}{\Delta_{ss}} \right)^{\frac{1}{\theta + \gamma}}
\]

\[
Y_{ss} = L_{ss} \Delta_{ss}
\]

\[
C_{ss} = (1 - \bar{G}) Y_{ss}
\]

\[
R_{ss} = \frac{\pi_{ss}}{\beta}
\]
Appendix B. Model solution procedure

Generalized Stochastic Simulation Algorithm

The idea of the GSSA is to approximate the decision rules in dynamic economic models with polynomials of past state variables. That is, the GSSA approximation is

\[ x_t \approx c_0 + c_1 a_{1t-1} + c_2 a_{2t-1} + \ldots + c_{n+1} a_{1t-1}^2 + c_{n+2} a_{2t-1}^2 + \ldots + c_k a_{1t-1} a_{2t-1} + \ldots = \psi(a_t; c) \]

where \( x_t \) is a endogenous model variable and \( a_t \) is a vector containing all state variables in the model. \( c \) is a vector of parameters containing \( c_j, j \in \{1, \ldots, k\} \), which is “estimated” using the algorithm. If there are multiple choice rules in the model that have to be approximated then each of these will have its own polynomial in the state variables. \( c \) is found with an iterative procedure of linear regression, where the “true” model outcome of \( x_t \) (generated with the simulation) is regressed on the polynomial used to approximate this outcome. Thus the task is to find a fixed point \( c^* \) for which the outcomes prove the approximations correct. The GSSA enhances the properties of this procedure in comparison to previously used stochastic simulation algorithms by using two augmentations: a stabilization mechanism and deterministic integration.

The general outline of the GSSA procedure is as follows:

1. Initialization. Choose initial guesses \( c^1 \) for the parameter vectors, choose initial values for the state variables and draw sequences of innovations for shock processes of length \( T \).
2. Simulate the model for \( T \) periods. At GSSA iteration \( p \), compute a sequence of model variables \( \{V_{t}\}_{t=1}^{T} \) using the model equations augmented with the approximating polynomials parametrized by \( c^p \).
3. Calculate the values of conditional expectations using a deterministic integration method. In particular, if \( y_t \) is a vector approximating the conditional expectations in the model, its \( i \)th element is computed as

\[ y_{it} = \sum_{j=1}^{J} \omega_j f(V_t, V_{t+1,j}, \varepsilon_{t,j}) \]

where \( J \) is the number of integration nodes used, \( \omega_j \) are integration weights and \( \varepsilon_{t,j} \) a vector of shock processes evaluated using the integration nodes as innovations. For example, if a shock process follows \( \varepsilon_t = e^{u_t} (\varepsilon_{t-1})^\rho \) then \( \varepsilon_{t,j} = \ldots \)
\[ e^{u_j (\varepsilon_{t-1})^\rho} \] where \( u_j \) is the value of the integration node. Finally, \( V_{t+1,j} \) is a vector of model variables computed at \( \varepsilon_{t,j} \).

4. Regression. Find \( \hat{c} \) that minimizes the errors \( \epsilon_t \) in \( \hat{x}_t = \psi (a_t; c^p) + \epsilon_t \) where \( \hat{x}_t \) is analogous to \( x_t \) but with any conditional expectations approximated by \( y_t \).

5. Check for convergence: if \( \frac{1}{T} \sum_{t=1}^{T} \left| \frac{x_t^b - x_{t-1}^b}{x_t^b} \right| < \tau \), stop. Else compute new parameters as \( c^{p+1} = (1 - \xi) c^p + \xi \hat{c} \), where \( |\xi| < 1 \) is the so called damping parameter. Go to step 2.

In previous applications the fixed point \( c^* \) is found using either standard Ordinary Least Squares if the approximating function is a standard polynomial or using Nonlinear Least Squares if the approximating function is a nonlinear transformation of the polynomial, e.g. in the case of Parametrized Expectations Algorithm of den Haan and Marcet (1990). However, with more complex models or with high-order approximations the polynomials are highly correlated, causing the regression procedure to fail to converge. In particular, Judd et al. (2011) point out that multicollinearity and poor scaling of the regressors can cause the matrix \( X'X \) of simulated data to be ill-conditioned: the ratio of the largest and the smallest eigenvalues of \( X'X \) is large and thus the matrix is close to being singular. In this case the regression coefficients \( c \) behave poorly during the fixed point iteration, which can lead to divergence. Solutions this problem are discussed below.

**Stabilization method**

As pointed out by Judd et al. (2011), the parameter estimates obtained using iterated simulations with OLS are highly sensitive to changes in the simulated data between iterations. That is, as consecutive iterations produce simulated data sets that differ from each other to a significant degree, consecutive OLS estimates are also far away from each other. Furthermore, the regressors are likely to be highly collinear as some of them are polynomial transformations of each other. For example the terms \( z^2_t \) and \( z^4_t \) can be strongly correlated. Finally, the procedure is vulnerable to issues of scale: if some variables are significantly larger than others, OLS will treat the small ones as if they were zeroes.

Judd et al. (2011) suggest several remedies. First, the issues of scale can be addressed with a simple normalization. More specifically, both the independent variable \( Y \) and dependent variables \( X_i \) in the regression applied in step 4 of the
GSSA procedure are standardized to have zero mean and unit variance, and the regression is estimated without an intercept term. The coefficients corresponding to the unnormalized variables are recovered as \( \hat{c}_i = \frac{\sigma_Y}{\sigma_{X_i}} \hat{c}_i^+ \) where \( \hat{c}_i^+ \) is a coefficient from the normalized regression and \( \sigma_Y \) and \( \sigma_{X_i} \) are the standard deviations of the unnormalized variables. Finally, the intercept term is computed as \( \hat{c}_0 = \bar{Y} - \sum_{i=1}^n \hat{c}_i^+ \bar{X}_i \), where \( \bar{Y} \) and \( \bar{X}_i \) are sample means.

The problem of instability in the regression coefficients can be solved in multiple ways. Judd et al. (2011) suggest methods based on Singular Value Decomposition (SVD), different types of regularization methods and methods that otherwise avoid directly computing the matrix \((X'X)^{-1}\) - our choice is to apply SVD.

**Deterministic integration**

The approximation of conditional expectations has been handled using Monte Carlo integration in the previous literature. Judd et al. (2011) note that it suffers to a significant degree from sampling error caused by the simulation, causing a significant loss of accuracy compared to deterministic integration. They suggest instead replacing it with a deterministic integration method, either Gauss-Hermite quadrature or a monomial method.

We apply monomial integration to compute the values of conditional expectations in the models, particularly the monomial rule M1 of Judd et al. (2011). Rule M1 sets the number of nodes used in the integration to \( 2N \), where \( N \) is the number of shock processes in the model. The values of the nodes are set by considering consecutive deviations of each random variable from its expected value, holding the other random variables fixed to their expected values, while the integration weights are set as \( \omega_j = 1/N \).

**Implementation details**

The convergence parameter is set at \( \tau = 10^{-4-m}\xi \), where \( m \) is the order of the approximating polynomial. This takes into account the accuracy gained by increasing \( m \) and the convergence speed that is implied by the damping parameter \( \xi \), which we set at 0.01. This value seems to be a reasonable compromise between guaranteeing convergence (higher values usually diverge) and the number of iterations needed (lower values increase computation time significantly). We set the order of the approximating polynomials to 3. As shown in
Fernandez-Villaverde and Rubio-Ramirez (2010), this is the smallest dimension for polynomials that are able to account for variation in volatility. The length of the simulation is set at 5200, and the first 200 are discarded as burn-in when computing the regression coefficients to avoid any possible problems caused by the choice of initial values.

Adequate initial values for the polynomial approximating the choice rules were found using the method of Gomme and Klein (2011), with the adjustment of setting the coefficients involving the stochastic volatility terms to zero. Initial values for the state variables in the simulations are always set at the steady state of the model.

Finally, convergence problems were encountered when the ZLB binds too often. To avoid this problem, the model was first solved for a set of parameter values where the ZLB does not bind at all, after which the parameter values were incrementally adjusted until the desired calibration was reached.
Chapter 3

Can DSGEs generate non-Gaussian realizations?

We study the cross-sectional properties of data generated by simulating two different nonlinear Dynamic Stochastic General Equilibrium (DSGE) models. A commonly observed property of real-world macroeconomic and financial data is non-Gaussianity - a fact that especially linearized DSGE models have not been able to replicate. Two models are investigated: a standard RBC model with habit formation and a New Keynesian DSGE model where the zero lower bound is binding. The models are solved using a stochastic simulation algorithm, which is able to handle significant nonlinearity. We find that the nonlinearities do affect the distributions of endogenous variables. In particular, we show that with strong habit formation the former model exhibits non-Gaussian consumption, while in the second sufficiently large shocks result in non-Gaussianity. Both models are able to replicate some features of U.S. data.
3.1 Introduction

The assumption of the normal distribution is pervasive in most fields of macroeconomics. For example, the widely used New Keynesian model of Smets and Wouters (2007) assumes 7 shock processes, some with rather complex intertemporal dynamics, but with innovations that are Gaussian. Similarly, Justiniano et al. (2010) study the effects of investment shock volatility on business cycles, yet assume that these and other shocks are normally distributed.

However, the observed distributions of macroeconomic variables - also those used to estimate the Smets and Wouters (2007) model - are quite clearly not Gaussian. GDP growth in the United States, for example, has a kurtosis of approximately 4.55, much higher than that implied by the normal distribution. Similarly, inflation has extremely fat tails, yet models for the most part fail to account for this fact by assuming Gaussian shocks together with solution methods based on linearization. Moments and the results of the Jarque-Bera normality tests on U.S. post-war macroeconomic data\(^1\) are presented in Table 3.1.1. The null hypothesis of normally distributed data is solidly rejected for each of the six variables. Similar evidence is presented in for example Fagiolo et al. (2008), who show that for the majority of OECD countries the distribution of the growth rate of output has much fatter tails than the normal distribution.

In this paper we study the ability of Dynamic Stochastic General Equilibrium (DSGE) models to generate non-Gaussian realizations using two models. The

\(^1\)Data are obtained from the FRED database of the St. Louis Federal Reserve.

Table 3.1.1: Sample moments and p-values of Jarque-Bera (JB) tests of growth rates of quarterly U.S. macroeconomic variables, 1954:II-2013:IV.

<table>
<thead>
<tr>
<th>Series</th>
<th>Std. dev.</th>
<th>Skewness</th>
<th>Kurtosis</th>
<th>JB test, p-value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Y</td>
<td>0.009</td>
<td>-0.42</td>
<td>4.55</td>
<td>&lt;0.001</td>
</tr>
<tr>
<td>C</td>
<td>0.008</td>
<td>-0.31</td>
<td>5.68</td>
<td>&lt;0.001</td>
</tr>
<tr>
<td>P</td>
<td>0.006</td>
<td>1.34</td>
<td>4.57</td>
<td>&lt;0.001</td>
</tr>
<tr>
<td>R</td>
<td>0.035</td>
<td>0.96</td>
<td>4.29</td>
<td>&lt;0.001</td>
</tr>
<tr>
<td>L</td>
<td>0.24</td>
<td>-0.24</td>
<td>1.65</td>
<td>0.002</td>
</tr>
<tr>
<td>I</td>
<td>0.023</td>
<td>-0.77</td>
<td>5.64</td>
<td>&lt;0.001</td>
</tr>
</tbody>
</table>

Legend: (Y) GDP, (C) personal consumption expenditure, (P) consumer price index, (R) effective Federal Funds rate (level), (L) hours worked (deviation from mean), (I) fixed private investment
former is a textbook Real Business Cycle (RBC) model with habit formation in consumption and a productivity shock, chosen to demonstrate a minimum requirement for non-Gaussian simulation output. The latter is a New Keynesian (NK) model, where the firms are able to adjust prices only occasionally à la Calvo, chosen to demonstrate how simple but strong nonlinearities can result in extreme deviations from Gaussianity. In addition, strong nonlinearity is imposed on the model by assuming that the zero lower bound (ZLB) on the nominal interest rate is binding, and that there are shocks to the policy rule of the central bank, productivity and the time preference of the households. The models are simulated under either the standard assumption of Gaussian innovations or assuming t-distributed innovations. Previous examples of the latter can be found in Chib and Ramamurthy (2014) and Cúrdia et al. (2014), both of which study the Bayesian estimation of DSGEs with t-distributed shocks.

Our solution method is based on approximating the choice rules in the models using the Generalized Stochastic Simulation Algorithm (GSSA) of Judd et al. (2011). The method constructs nth order polynomial approximations to decision rules while leaving the rest of the model as is and thus allows for considerable nonlinearity compared to most previous tools, which is vital especially when the nonlinearity is as severe as the ZLB. The method is inspired by the previous literature on least-squares approximation such as the Parametrized Expectations Algorithm of den Haan and Marcet (1990). The key advances made by Judd et al. (2011) are in the use of a stabilization method to improve the convergence of the algorithm when applied to more complex models and the use of more accurate deterministic integration methods instead of Monte Carlo integration.

Similar work has previously been conducted by Ascari et al. (2014). They study the simulation output of a RBC model and the model of Smets and Wouters (2007) in second order approximations when Gaussian and Laplacian innovations are assumed. They find that the realizations of the RBC model always follow the assumed shock distribution, while in the case of the Smets and Wouters model the distributions are nearly Gaussian with normal shocks, and somewhat less leptokurtic than the Laplace distribution when Laplacian innovations are assumed. Thus under these assumptions neither model is able to produce realizations that match observed data. Neither model under any of the assumed shock distributions is able to generate skewed simulated data.

Other related recent contributions include Andreasen (2012) and Andreasen (2013). The former considers a relatively standard New Keynesian DSGE model,
but with Epstein-Zin-Weil preferences and stochastic volatility - it is shown how strong stochastic volatility can result in excess kurtosis - which however closely matches that of the shock process. The latter looks at a similar model with external habit formation within Epstein-Zin-Weil preferences, capital adjustment costs and several constraints to the production function. These strong nonlinearities are found to lead to significant skewness and excess kurtosis, even under Gaussian shocks with constant variance. However, for both models the match between real and simulated data depends strongly on the assumed shock specification. Both models are solved using a third order perturbation method.

As a preview of our findings, note that our results are not completely in conflict with those of Ascari et al. (2014). With strong habit formation, the RBC model produces consumption and investment that deviate from the normal distribution. This is in stark contrast with the result concerning RBC models obtained in Ascari et al. (2014), but with weak habit this effect is quite small. As is the case with Andreasen’s (2013) results, the output of the New Keynesian model, on the other hand, is less dependent on the shock distributions than in the model of Smets and Wouters (2007) solved using perturbations. This indicates that the ZLB brings about strong enough nonlinearity, that even with low shock variance the realizations become skewed and fat tailed and hence non-Gaussian.

In addition, we find that both models are able to match some of the key moments of real U.S. data: the RBC model matches the second moments rather well, while the New Keynesian model is able to generate significant skewness, although in general in the wrong direction compared to the data.

The rest of the paper is divided into four sections. Section 3.2 discusses the theoretical models, the shock processes and the calibration of model parameters. Section 3.3 describes the solution method we apply and its application to the models, while Section 3.4 presents the results of the simulation exercise. Section 3.5 concludes.

3.2 Models, shocks and calibration

In this section we describe the DSGE models used to generate the realizations. To reiterate, we consider two different models: a Real Business Cycle model with habit formation and a New Keynesian model with a binding lower bound on the nominal interest rate. They are covered in detail in order in the next two subsections. The calibration of the model parameters, the shock processes
and the distributions of the innovations are briefly discussed in Section 3.2.3.

3.2.1 RBC model with habit formation

This section presents our first model, the textbook Real Business Cycle model with habit formation in consumption.

We assume a continuum of identical households of unit mass. They buy goods $C_t$ from the firms and have the opportunity to invest in capital stock $K_t$, which they rent to the firms in exchange for $R^k_t$ in a competitive market.

The representative household faces the problem

$$
\max_{C_t, K_{t+1}} E_0 \sum_{t=0}^{\infty} \beta^t \left[ \log \left( C_t - h C^a_{t-1} \right) \right]
$$

subject to

$$
C_t + K_{t+1} \leq R^k_t K_t + (1 - \delta) K_t
$$

where $\beta$, $h$ and $\delta$ are parameters. Finally, $C^a_{t-1}$ denotes previous period aggregate consumption that the household takes as given.

Production in the economy is conducted by a continuum of competitive firms using capital to produce output $Y_t$ using the production function

$$
Y_t = \varepsilon^a_t K^n_t
$$

where $\varepsilon^a_t$ is a productivity shock and $\eta$ is a parameter.

The equilibrium is characterized by the equations

$$
1 = \beta E_t \left[ \frac{(C_t - h C^a_{t-1})}{(C_{t+1} - h C^a_{t})} \left( R^k_{t+1} + (1 - \delta) \right) \right]
$$

and

$$
R^k_t = \varepsilon_t^a \eta K^{n-1}_t
$$

together with the budget constraint. In addition, in equilibrium individual and aggregate consumption coincide, so that $C_t = C^a_t$.

Finally, for the purposes of the simulation we also compute investment $I_t$ from the model as $I_t = Y_t - C_t$. 

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3.2.2 New Keynesian model with Calvo pricing

This section presents a New Keynesian DSGE model where we assume that the zero lower bound on the interest rate is binding. See Fernandez-Villaverde et al. (2013) for a discussion on the general properties of a very similar model.

It deviates from the RBC model in Section 3.2.1 by assuming that instead of investing in capital goods, the households manage their savings by investing in government bonds $B_t$ which pay out a nominal rate $R_t$ set by the central bank. The households are assumed to solve the problem

$$\max_{C_t,B_t,L_t} E_0 \sum_{t=0}^{\infty} \beta^t \varepsilon_t^u \left[ \log C_t - \frac{L_t^{1+\theta}}{1 + \vartheta} \right]$$

subject to the budget constraint

$$P_tC_t + B_t \leq R_{t-1}B_{t-1} + W_tL_t + \Pi_t$$

where $L_t$ is labor supplied by the household, $W_t$ is the nominal wage, $P_t$ is the price of the consumption good, $\Pi_t$ are the profits of the firms in the economy, and $\varepsilon_t^u$ is a discount rate shock, while $\gamma$ and $\vartheta$ are parameters. The households’ first-order conditions can be arranged as

$$\varepsilon^u_t C_t^{-1} = \beta R_t E_t \left[ \frac{\varepsilon^u_{t+1} C_{t+1}^{-1}}{\pi_{t+1}} \right]$$

and

$$L_t^\theta = C_t^{-1}W_t$$

Furthermore, we assume that the production side of the economy comprises of two sectors: intermediate-good producing firms engaged in monopolistic competition and able to adjust the price of the good they produce only intermittently, and a competitive final-good producing sector. Labor supplied by the households is the only input used by the primary producers, so that the production function of firm $i$ is

$$Y_{it} = \varepsilon^u_t L_{it}$$

The producers are subject to Calvo-style price setting: a fraction $1 - \theta$ is able to set prices optimally so that $P_{it} = \hat{P}_t$, while the rest are forced to keep the same price as in the previous period. The corresponding price-setting problem
is then
\[
\max_{\hat{P}_t} \sum_{j=0}^{\infty} \beta^j \theta^j E_t \left\{ \Lambda_{t+j} \left[ \hat{P}_t Y_{i,t+j} - P_{t+j} MC_{t+j} Y_{i,t+j} \right] \right\}
\]  (3.2.11)

subject to
\[
Y_{it} = Y_t \left( \frac{P_{it}}{\hat{P}_t} \right)^{-\kappa}
\]  (3.2.12)

where \( \Lambda_t \) is the Lagrange multiplier on the households budget constraint and \( MC_t \) is the real marginal cost of output. \( \kappa \geq 1 \) is a parameter governing the aggregation process used by the final-good producer. The target price \( \hat{P}_t \) relative to the aggregate price level \( P_t \) is characterized by
\[
\hat{P}_t = \frac{E_t \sum_{t=0}^{\infty} (\beta \theta)^j P_{t+j} \Lambda_{t+j} Y_{i,t+j} \chi_{j,t}^{-\kappa} MC_{t+j}}{E_t \sum_{t=0}^{\infty} (\beta \theta)^j P_{t+j} \Lambda_{t+j} Y_{i,t+j} \chi_{j,t}^{-1-\kappa}} \equiv S_t/F_t
\]  (3.2.13)

The recursive dynamics of \( S_t \) and \( F_t \) are
\[
S_t = \frac{\varepsilon_t}{\varepsilon_t} L_t \theta Y_t + \beta \theta E_t \left[ \pi_t^{\kappa+1} S_{t+1} \right]
\]  (3.2.14)
\[
F_t = \frac{\varepsilon_t}{\varepsilon_t} C_t^{-1} Y_t + \beta \theta E_t \left[ \pi_t^{\kappa-1} F_{t+1} \right]
\]  (3.2.15)

The aggregate price level is given by
\[
P_t = \left[ (1 - \theta) \hat{P}_t^{1-\kappa} + \theta P_{t-1}^{1-\kappa} \right]^{\frac{1}{1-\kappa}}
\]  (3.2.16)

which can be rearranged as
\[
\frac{\hat{P}_t}{P_t} = \left[ \frac{1 - \theta \pi_t^{-1}}{1 - \theta} \right]^{\frac{1}{1-\kappa}}
\]  (3.2.17)

implying
\[
\pi_t = \left[ \frac{1 - (1 - \theta) \left( \frac{\hat{P}_t}{P_t} \right)^{1-\kappa}}{\theta} \right]^{\frac{1}{1-\kappa}}
\]  (3.2.18)

where \( \pi_t = \frac{P_t}{P_{t-1}} \).
The rms of the competitive final good sector use the technology
\[ Y_t = \left( \int_0^1 Y_{it}^{\kappa - 1} \, di \right)^{\frac{\kappa}{\kappa - 1}} \]  
(3.2.19)
to pack the intermediate goods into the consumption good sold to the households. Solving their profit maximization problem yields (3.2.12).

Aggregate output is then given by
\[ Y_t = \varepsilon_t^a L_t \Delta_t \]  
(3.2.20)
where \( \Delta_t \) is a measure of inverse price dispersion across firms. Its dynamics are described by
\[ \Delta_t = \left( (1 - \theta) \left[ \frac{1 - \theta \pi_t^{k - 1}}{1 - \theta} \right]^{\frac{\kappa}{\kappa - 1}} + \theta \left( \frac{\pi_t}{\Delta_{t-1}} \right) \right)^{-1} \]  
(3.2.21)
The central bank is assumed to follow a Taylor rule so that the nominal rate is set as
\[ R_t = \max \left\{ 1, \varepsilon_t^r (R_{t-1})^\rho \left[ \left( \frac{\pi_t}{\bar{\pi}} \right)^{\phi_\pi} \left( \frac{Y_t}{Y} \right)^{\phi_Y} \right]^{1-\rho} \right\} \]  
(3.2.22)
where \( \varepsilon_t^r \) is a shock to monetary policy, \( \bar{\pi} \) is the inflation target of the central bank and \( \rho, \phi_\pi \) and \( \phi_Y \) are parameters. Finally, \( Y \) is the steady state of output.

The equilibrium of this model is characterized by the dynamics of the frictionless output, the Taylor rule given above and equations (3.2.8), (3.2.14), (3.2.15), (3.2.18), (3.2.20), (3.2.21) and \( Y_t = C_t \).

### 3.2.3 Shocks and calibration

In this section we describe the shock processes and the calibration of the parameters employed in simulating the model.

The shock processes in the models include the productivity shock \( \varepsilon_t^a \) in both models and the monetary policy shock \( \varepsilon_t^r \) and the discount rate shock \( \varepsilon_t^u \) in the New Keynesian model. We assume that the processes follow
\[ \varepsilon_t^j = (\varepsilon_{t-1}^j)^{\rho_j} \exp \left( \nu_t^j \right) \]  
with \( j \in \{a, r, u\} \), \( 0 < \rho_j < 1 \) and that \( \nu_t^j \) is an identically and independently distributed random variable drawn from a continuous distribution. We consider two different families of distributions: the normal distribution with zero mean.
and standard deviation \( \sigma_j = \sigma = 0.001 \) and the Student’s t-distribution with 5 degrees of freedom, which exhibits significant excess kurtosis in comparison to the normal distribution. All draws from the t-distribution are multiplied by \( \sigma \) in order to scale its variance to a level similar to that of the normal distribution we apply, as without the scaling the shocks become too large. As mentioned previously, the normality assumption is standard in the literature, while the t-distribution is applied because it is capable of producing kurtosis comparable to that observed in real data. We set \( \sigma \) somewhat smaller than most estimates for standard deviations of shock processes seen in the literature, but this value is chosen in order to be able to better demonstrate the importance of the calibration by comparing it to the case where the standard deviations are set at 0.004.

We next turn to the calibration of the structural parameters of the models. Table 3.2.1 presents the chosen values. Any parameters that are common to the two models are calibrated the same. Overall, the calibration is rather standard relative to the literature. See, for example, Smets and Wouters (2007) for estimates that are rather close to our calibrated values. However, it is important to note that the results are sensitive to the chosen values. As we will show later by adjusting the calibration, in the RBC model the habit parameter \( h \) has an effect on the distribution of consumption and investment, and in the ZLB model the standard deviations of the shock processes affect not only the standard deviations of the endogenous variables, but their third and fourth moments too.

<table>
<thead>
<tr>
<th></th>
<th>RBC</th>
<th>NK</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \beta )</td>
<td>0.99</td>
<td>0.99</td>
</tr>
<tr>
<td>( h )</td>
<td>0.7</td>
<td>2</td>
</tr>
<tr>
<td>( \delta )</td>
<td>0.02</td>
<td>0.8</td>
</tr>
<tr>
<td>( \eta )</td>
<td>0.33</td>
<td>4.5</td>
</tr>
<tr>
<td>( \rho )</td>
<td>0.8</td>
<td></td>
</tr>
<tr>
<td>( \phi_\pi )</td>
<td>2</td>
<td></td>
</tr>
<tr>
<td>( \phi_Y )</td>
<td>0.15</td>
<td></td>
</tr>
<tr>
<td>( \bar{\pi} )</td>
<td>1.005</td>
<td></td>
</tr>
<tr>
<td>( \rho_a )</td>
<td>0.95</td>
<td>0.75</td>
</tr>
<tr>
<td>( \rho_i )</td>
<td>0.75</td>
<td></td>
</tr>
<tr>
<td>( \rho_r )</td>
<td>0.75</td>
<td></td>
</tr>
</tbody>
</table>
3.3 Solution method

In this section we briefly describe the methods used to solve the models for simulation. We apply the Generalized Stochastic Simulation Algorithm (GSSA) of Judd et al. (2011) for which we obtain initial values with the method of Gomme and Klein (2011). A more thorough description including details on our implementation of the method can be found in the Appendix. Briefly put, the GSSA is a solution method based on finding parameters for polynomials approximating model policy functions through stochastic simulation.

In the RBC model, the state variables are $C_{t-1}$, $K_{t-1}$ and $\varepsilon^a_t$. We approximate the policy function of capital

$$K_t = K (C_{t-1}, K_{t-1}, \varepsilon^a_t) \approx \psi_K (C_{t-1}, K_{t-1}, \varepsilon^a_t; c_K)$$  (3.3.1)

with the function $\psi_K$, a second order polynomial in the state variables.

In the New Keynesian model, the vector of state variables is $(\Delta_{t-1}, R_{t-1}, \varepsilon^a_t, \varepsilon^r_t, \varepsilon^a_t)$ and the choice rules we choose to approximate are those of $S_t$, $F_t$ and $MU_t \equiv C_t^{-1}$, which are approximated with the second order polynomials $\psi_S$, $\psi_F$ and $\psi_{MU}$.

As the name of the GSSA indicates, we proceed by simulating the model. In the first step, we guess for the values of the parameters of the approximating polynomials, compute the values of these polynomials at time $t = 1$, given a realization from the shock processes, and, using the equilibrium conditions given in Section 3.2, compute the values of the rest of the variables. This procedure is repeated until some point of time $T$. The algorithm then proceeds by computing the values of the expectations appearing in the equilibrium conditions by applying a deterministic integration method on “data” generated by the simulation.

Finally, the outcomes of the simulation with approximated expectations are regressed (applying ridge regression) on the corresponding state variables to obtain three parameter vectors. The algorithm then returns to the first step of the simulation, with the polynomials now parametrized using the three vectors obtained at the end of the previous iteration. This iterative process is then repeated until convergence.
3.4 Simulation results

This section presents the results of our simulation exercise, for which 1000 realizations of 5000 observations were generated for each shock distribution. We start with a subsection on the RBC model.

RBC model with habit

Table 3.4.1 presents the averages of the second, third and fourth moments of output, consumption and investment over the simulated realizations as well as the share of realizations where the Jarque-Bera normality test rejects at the 5% level of significance. Two different values of the habit parameter $h$ are considered, with the left panel presenting the case $h = 0.7$ and the right panel the alternative case of $h = 0.1$. The case of Gaussian shocks can be found in the top three rows, while the bottom three rows show these statistics for t-distributed innovations.

The first thing to note is that strength of habit formation has an effect, albeit not very large, on the distribution of endogenous variables in this model. With normally distributed shocks, changing the habit parameter from 0.7 to 0.1 results in noticeable changes in the shares of rejected normality tests. In particular, the Jarque-Bera test rejects the null hypothesis of normally distributed consumption at the 5% significance level in roughly one out of six cases when habit is strong, and only 5.7% of the time with weak habit - slightly more often than is expected based on the assumed critical value. Thus it seems that in the weak habit case the model behaves almost as the standard RBC model, which according to Ascari et al. (2014) produces realizations that follow closely the distribution of the productivity shock.

Further evidence for this last point can be seen by looking at the moments of the simulated variables when the shocks are t-distributed. Most noticeable is the fact that in this situation the Jarque-Bera test rejects practically all the time. In particular, the kurtosis of the endogenous variables seems to depend on the size of the habit parameter: in the weak case the moments of the simulation are much closer to those of the shocks. For instance, the average kurtosis of investment is 5.6, when $h = 0.1$, but only 3.9, when $h = 0.7$, and the kurtosis of the shocks following the t-distribution with 5 degrees of freedom is 9. Hence, the model seems to moderate the effect of extreme shocks more in the strong habit case.

How do the model moments match those observed in actual U.S. data? With
Table 3.4.1: Means of moments and Jarque-Bera (JB) tests of simulated variables, RBC model.

\( h = 0.7 \)

<table>
<thead>
<tr>
<th>Series</th>
<th>Std. dev.</th>
<th>Skewness</th>
<th>Kurtosis</th>
<th>JB</th>
<th>Std. dev.</th>
<th>Skewness</th>
<th>Kurtosis</th>
<th>JB</th>
</tr>
</thead>
<tbody>
<tr>
<td>Panel A. Gaussian shocks</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Y</td>
<td>0.001</td>
<td>-0.002</td>
<td>3.001</td>
<td>0.058</td>
<td>0.001</td>
<td>0.003</td>
<td>2.99</td>
<td>0.048</td>
</tr>
<tr>
<td>C</td>
<td>0.005</td>
<td>0.001</td>
<td>3.002</td>
<td>0.16</td>
<td>0.007</td>
<td>-0.01</td>
<td>2.99</td>
<td>0.057</td>
</tr>
<tr>
<td>I</td>
<td>0.017</td>
<td>0.0001</td>
<td>2.99</td>
<td>0.2</td>
<td>0.022</td>
<td>0.012</td>
<td>3.01</td>
<td>0.055</td>
</tr>
<tr>
<td>Panel B. t-distributed shocks</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Y</td>
<td>0.001</td>
<td>-0.019</td>
<td>8.83</td>
<td>1</td>
<td>0.001</td>
<td>-0.01</td>
<td>7.53</td>
<td>1</td>
</tr>
<tr>
<td>C</td>
<td>0.007</td>
<td>-0.003</td>
<td>4.13</td>
<td>0.96</td>
<td>0.009</td>
<td>-0.008</td>
<td>5.52</td>
<td>1</td>
</tr>
<tr>
<td>I</td>
<td>0.023</td>
<td>0.0001</td>
<td>3.92</td>
<td>0.94</td>
<td>0.029</td>
<td>0.01</td>
<td>5.6</td>
<td>1</td>
</tr>
</tbody>
</table>

Legend: (Y) output, (C) consumption, (I) investment.
Top panel: Gaussian shocks. Bottom panel: t-distributed shocks.
Table 3.4.2: Share of null hypotheses rejected by the Kolmogorov-Smirnov test at the 5% level, RBC model

<table>
<thead>
<tr>
<th></th>
<th>Normal</th>
<th>Student's t</th>
<th>Normal</th>
<th>Student's t</th>
</tr>
</thead>
<tbody>
<tr>
<td>$h = 0.7$</td>
<td>0.01</td>
<td>0.001</td>
<td>0.016</td>
<td></td>
</tr>
<tr>
<td>$h = 0.1$</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>

strong habit and Gaussian shocks the standard deviations of simulated consumption and investment somewhat match those of observations (for example, the simulated standard deviation of investment is between 0.02 and 0.03, while that in the data is 0.023), and with t-distributed shocks the kurtoses of the realizations resemble those in the data, too. However, the model is not able to generate significant skewness with any strength of habit-distribution combination, while in the data all the corresponding variables are noticeably skewed. An interesting detail is that with weak habit consumption and investment are for some reason more skewed than with strong habit.

Two-sample Kolmogorov-Smirnov tests were conducted to study whether the simulated growth rates and the innovations are draws from the same distribution. Table 3.4.2 presents the shares of realizations where the null hypothesis is rejected at the 5% level. Overall, it is clear that whichever calibration or shock distribution is applied, the distributions of consumption and investment do not match those of the shocks at all, while that of output appears to follow the shock distribution very closely. However, it should be kept in mind that the Kolmogorov-Smirnov test only tests for whether the samples are from the same distribution with the same parameters, and based on this evidence it can not be ruled out that simulated consumption and investment are normally distributed but with different moments than those of the innovations.

New Keynesian model

Table 3.4.3 presents the averages of the second, third and fourth moments of the simulated growth rate of consumption, inflation, interest rate and labor. The case of Gaussian shocks can be found in the top four rows, while the bottom rows show these statistics for t-distributed innovations. The left panel presents moments of realizations where the standard deviations of the shocks are low, $\sigma = 0.001$, and in the right panel we have the case where the standard deviations...
are high, i.e. $\sigma = 0.004$. While this value is rather high, on the upper end of the range seen in empirical estimates, it serves to demonstrate our point.

The first thing to note is that already in the case of normally distributed shocks with low variance, all endogenous variables but consumption have noticeably skewed distributions and interest rate and labor also exhibit significant excess kurtosis. When either the standard deviation or kurtosis of the shocks is increased, the distributions of the endogenous variables move even further away from the Gaussian distribution. This is due to the fact that with shocks that have fat tails or high variance, the frequency of the economy hitting the lower bound on the interest rate considerably increases.

The most important evidence against the variables inheriting the Gaussian distribution assumed for the shocks is thus in the top rows of the right panel of Table 3.4.3, i.e. the case of normal shocks with high variance. Here we can see that even with Gaussian shocks, the model is able to produce realizations that without doubt follow a non-Gaussian distribution. As far as consumption is concerned, the evidence is not so strong, but even in this case the Jarque-Bera test rejects for 14% of the simulated series, which is not an insignificant amount. Overall the evidence seems clear: with reasonable shock distributions and parametrizations, a model where the zero lower bound is binding, is able to produce realizations that do not follow the normal distribution. With fat tailed shocks, the evidence is even stronger. Furthermore, the distributions of those variables that are non-Gaussian even with mild shocks become quite extreme with t-shocks with high variance; for example the kurtosis of the interest rate is in this case 47, which is somewhat larger than 3, to put it mildly.

Two-sample Kolmogorov-Smirnov tests were conducted also for these simulations. The results are unambiguous: the null hypothesis is always rejected at the 5% level of significance, when the distribution of any of the simulated variables is compared to any shock distribution. As pointed out above, this does not necessarily imply non-Gaussianity of the endogenous variables.

Unlike the RBC model discussed above, in comparison to actual U.S. data, this model is able to generate quite significant skewness - but in many cases too much, especially when the shocks are t-distributed. In addition it has the wrong sign in comparison to data. Also, this model is able to generate standard deviations and kurtosis that loosely match those seen in reality: even with

\footnote{Numerical problems were encountered in the t-distributed case when going above this bound.}
Table 3.4.3: Means of moments and Jarque-Bera (JB) tests of simulated variables, NK model.

<table>
<thead>
<tr>
<th>Series</th>
<th>Std. dev.</th>
<th>Skewness</th>
<th>Kurtosis</th>
<th>JB test</th>
<th>Std. dev.</th>
<th>Skewness</th>
<th>Kurtosis</th>
<th>JB test</th>
</tr>
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<tr>
<td>Panel A. Gaussian shocks</td>
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<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>C</td>
<td>0.002</td>
<td>0.081</td>
<td>3.03</td>
<td>0.06</td>
<td>0.01</td>
<td>0.33</td>
<td>3.2</td>
<td>0.14</td>
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<tr>
<td>P</td>
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<td>-0.13</td>
<td>2.99</td>
<td>0.86</td>
<td>0.012</td>
<td>-0.48</td>
<td>3.02</td>
<td>1</td>
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<tr>
<td>R</td>
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<td>-0.51</td>
<td>3.55</td>
<td>1</td>
<td>0.002</td>
<td>-1.35</td>
<td>5.63</td>
<td>1</td>
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<tr>
<td>L</td>
<td>0.004</td>
<td>0.52</td>
<td>3.46</td>
<td>1</td>
<td>0.024</td>
<td>1.8</td>
<td>8.03</td>
<td>1</td>
</tr>
<tr>
<td>Panel B. t-distributed shocks</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>C</td>
<td>0.002</td>
<td>0.092</td>
<td>7.09</td>
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<td>0.4</td>
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<tr>
<td>P</td>
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<td>1</td>
<td>0.016</td>
<td>-1</td>
<td>8.12</td>
<td>1</td>
</tr>
<tr>
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<tr>
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<td>0.041</td>
<td>2.4</td>
<td>11.8</td>
<td>1</td>
</tr>
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</table>

Legend: (C) consumption, (P) inflation, (R) interest rate, (L) labor.
Top panel: Gaussian shocks. Bottom panel: t-distributed shocks.
Gaussian shocks, the distribution of the simulated interest rate, for example, exhibits noticeable excess kurtosis, as does the Fed Funds rate.

3.5 Conclusions

In this paper we have studied whether two macroeconomic models are able to produce realizations that have distributions resembling those of observed macroeconomic time series. The conclusion from observed U.S. data is clear: macroeconomic variables are not Gaussian. They have tails that are significantly thicker and their distributions are often skewed. The conclusion of our simulation exercises is that neither of the theoretical models is fully able to replicate these facts but that the nonlinearities do affect the distributions of the endogenous variables.

In the RBC model we found that habit formation does matter: with no habit, the Ascari et al. (2014) result is that endogenous variables follow the shock distributions. With strong, empirically plausible habit the distribution of simulated consumption deviates from the normal distribution. On the other hand, we also found that habit formation has a dampening effect on the distributions of consumption and investment. This is in contrast to the results of Ascari et al. (2014), who found that their model, which did not have habit formation, is essentially a neutral filter on the innovations, with simulations following the innovations extremely closely. In our case it appears that strong habit formation has a dampening effect on the shocks, as the realizations with t-distributed shocks are less extreme as the t-distribution itself. In addition, the standard deviations and kurtosis of the realizations somewhat match those of observations, but the model is not able to match real-world skewness.

Our findings from a New Keynesian model, with a strictly binding zero lower bound on the nominal interest rate, indicate that the model is able to produce non-Gaussian data, even with fairly mild shocks. For example, the distribution of inflation is found to be noticeably skewed with low variance Gaussian shocks. Drawing the shocks from fat tailed or high variance distributions significantly amplifies this phenomenon: it ensures that the model economy hits the lower bound more often, further skewing the distributions.

Finally, recall that Ascari et al. (2014) also studied a New Keynesian model that has several different types of real and nominal rigidities and found that the model is unable to produce non-Gaussian realizations. It seems likely that the
nonlinearities in the model they consider are not strong enough. Using a similar solution method the strongly nonlinear model investigated in Andreasen (2013) generates endogenously realizations that are clearly non-Gaussian. Crucially these realizations are also significantly skewed.

We conclude that some DSGEs are indeed able to produce non-Gaussian data. What is needed to attain this goal are model features that are able to generate sufficient endogenous nonlinearity, shocks that ensure that the nonlinearity matters and a solution method that respects this nonlinearity.
Appendix. The GSSA algorithm

The idea of the GSSA is to approximate the decision rules in dynamic economic models with polynomials of past state variables. That is, the GSSA approximation is

\[ x_t \approx c_0 + c_1 a_{1t-1} + c_2 a_{2t-1} + \ldots + c_{n+1} a_{1t-1}^2 + c_{n+2} a_{2t-1}^2 + \ldots + c_{k} a_{1t-1} a_{2t-1} + \ldots = \psi (a_t; c) \]

where \( x_t \) is an endogenous model variable and \( a_t \) is a vector containing all state variables in the model. \( c \) is a vector of parameters containing \( c_j, j \in \{1, \ldots, k\} \), which is “estimated” using the algorithm. If there are multiple choice rules in the model that have to be approximated then each of these will have its own polynomial in the state variables. \( c \) is found with an iterative procedure of linear regression, where the “true” model outcome of \( x_t \) (generated with the simulation) is regressed on the polynomial used to approximate this outcome. Thus the task is to find a fixed point \( c^* \) for which the outcomes prove the approximations correct. The GSSA enhances the properties of this procedure in comparison to previously used stochastic simulation algorithms by using two augmentations: a stabilization mechanism and deterministic integration.

The general outline of the GSSA procedure is as follows:

1. Initialization. Choose initial guesses \( c^1 \) for the parameter vectors, choose initial values for the state variables and draw sequences of innovations for shock processes of length \( T \).

2. Simulate the model for \( T \) periods. At GSSA iteration \( p \), compute a sequence of model variables \( \{V_t\}_{t=1}^T \) using the model equations augmented with the approximating polynomials parametrized by \( c^p \).

3. Calculate the values of conditional expectations using a deterministic integration method. In particular, if \( y_t \) is a vector approximating the conditional expectations in the model, its \( i \)-th element is computed as

\[ y_{it} = \sum_{j=1}^{J} \omega_j f (V_t, V_{t+1,j}, \varepsilon_{t,j}) \]

where \( J \) is the number of integration nodes used, \( \omega_j \) are integration weights and \( \varepsilon_{t,j} \) a vector of shock processes evaluated using the integration nodes as innovations. For example, if a shock process follows \( \varepsilon_t = e^{u_t} (\varepsilon_{t-1})^\rho \) then \( \varepsilon_{t,j} = e^{u_j} (\varepsilon_{t-1})^\rho \) where \( u_j \) is the value of the integration node. Finally, \( V_{t+1,j} \) is a
vector of model variables computed at \( \varepsilon_{t,j} \).

4. Regression. Find \( \hat{c} \) that minimizes the errors \( \epsilon_t \) in \( \hat{x}_t = \psi(a_t; c^p) + \epsilon_t \) where \( \hat{x}_t \) is analogous to \( x_t \) but with any conditional expectations approximated by \( \hat{y}_t \).

5. Check for convergence: if \( \frac{1}{T} \sum_{t=1}^{T} \left| \frac{x_t - x_{t-1}}{x_t} \right| < \tau \), stop. Else compute new parameters as \( c^{p+1} = (1 - \xi) c^p + \xi \hat{c} \), where \( |\xi| < 1 \) is the so called damping parameter. Go to step 2.

In previous applications the fixed point \( c^* \) is found using either standard Ordinary Least Squares if the approximating function is a standard polynomial or using Nonlinear Least Squares if the approximating function is a nonlinear transformation of the polynomial, e.g. in the case of Parametrized Expectations Algorithm of den Haan and Marcet (1990). However, with more complex models or with high-order approximations the polynomials are highly correlated, causing the regression procedure to fail to converge. In particular, Judd et al. (2011) point out that multicollinearity and poor scaling of the regressors can cause the matrix \( X'X \) of simulated data to be ill-conditioned: the ratio of the largest and the smallest eigenvalues of \( X'X \) is large and thus the matrix is close to being singular. In this case the regression coefficients \( c \) behave poorly during the fixed point iteration, which can lead to divergence. Solutions this problem are discussed below.

**Stabilization method**

As pointed out by Judd et al. (2011), the parameter estimates obtained using iterated simulations with OLS are highly sensitive to changes in the simulated data between iterations. That is, as consecutive iterations produce simulated data sets that differ from each other to a significant degree, consecutive OLS estimates are also far away from each other. Furthermore, the regressors are likely to be highly collinear as some of them are polynomial transformations of each other. For example the terms \( z_t^2 \) and \( z_t^4 \) can be strongly correlated. Finally, the procedure is vulnerable to issues of scale: if some variables are significantly larger than others, OLS will treat the small ones as if they were zeroes.

Judd et al. (2011) suggest several remedies. First, the issues of scale can be addressed with a simple normalization. More specifically, both the independent variable \( Y \) and dependent variables \( X_i \) in the regression applied in step 4 of the GSSA procedure are standardized to have zero mean and unit variance, and the
regression is estimated without an intercept term. The coefficients corresponding to the unnormalized variables are recovered as 
\[
\hat{c}_i = \frac{\sigma_Y}{\sigma_{X_i}} \hat{c}_i^+ \quad \text{where} \quad \hat{c}_i^+ \text{ is a coefficient from the normalized regression and} \quad \sigma_Y \quad \text{and} \quad \sigma_{X_i} \quad \text{are the standard deviations of the unnormalized variables. Finally, the intercept term is computed as}
\]
\[
\hat{c}_0 = \bar{Y} - \sum_{i=1}^{n} \hat{c}_i^+ \bar{X}_i \quad \text{where} \quad \bar{Y} \quad \text{and} \quad \bar{X}_i \quad \text{are sample means.}
\]

The problem of instability in the regression coefficients can be solved in multiple ways. Judd et al. (2011) suggest methods based on Singular Value Decomposition (SVD), different types of regularization methods and methods that otherwise avoid directly computing the matrix \((X'X)^{-1}\). We choose to apply a SVD on the regression.

**Deterministic integration**

The approximation of conditional expectations has been handled using Monte Carlo integration in the previous literature. Judd et al. (2011) note that it suffers to a significant degree from sampling error caused by the simulation, causing a significant loss of accuracy compared to deterministic integration. They suggest instead replacing it with a deterministic integration method, either Gauss-Hermite quadrature or a monomial method.

We apply the methods of Gauss-Hermite quadrature for the RBC model and monomial integration for the New Keynesian model to compute the values of conditional expectations in the models, particularly the monomial rule M1 of Judd et al. (2011). Rule M1 sets the number of nodes used in the integration to \(2N\), where \(N\) is the number of shock processes in the model. The values of the nodes are set by considering consecutive deviations of each random variable from its expected value, holding the other random variables fixed to their expected values, while the integration weights are set as \(\omega_j = 1/N\).

**Implementation details**

The convergence parameter is set at \(\omega = 10^{-4-m}\xi\), where \(m\) is the order of the approximating polynomial. This takes into account the accuracy gained by increasing \(m\) and the convergence speed that is implied by the damping parameter \(\xi\), which we set at 0.01 in all cases. This value seems to be a reasonable compromise between guaranteeing convergence (higher values usually diverge) and the number of iterations needed (lower values increase computation significantly). In all cases we set the orders of the approximating polynomials to 2. The length of the simulation is set at 5200. The first 200 are discarded when
computing the regression coefficients as burn-in.

As the speed of computing Gauss-Hermite quadrature is heavily dependent on the chosen number of nodes and shock processes in the model, we only apply it to the RBC model with the number of nodes set at 10. For the New Keynesian model, we apply the monomial rule.

Initial values for the parameters of the polynomial approximating the choice rule in the RBC model are selected as $c_K^0 = (0, 0.1, 0.9, 0.05K_{ss}, 0, \ldots, 0)$ where $K_{ss}$ is the steady state of $K_t$. Initial values for the NK model were found by applying the method of Gomme and Klein (2011) to obtain a second order perturbation solution. Finally, initial values for the state variables in the simulations are always set at the steady state of the model.
Chapter 4

Dynamic Economic Forces and the Stock Market

Are innovations in macroeconomic variables risks that are rewarded in the stock market? We estimate a conditional form of the Intertemporal Capital Asset Pricing Model of Merton (1973) with the multivariate GARCH model of Engle (2002) which allows for the modeling of conditional, time-varying covariances with just two additional parameters. As test assets we use a triplet (size-book-to-market, size-momentum, size-long run reversal) of sorted portfolios. Market risk is found clearly positively priced in the market, while the pricing of the macroeconomic variables is slightly ambiguous. The risk premia of unemployment and inflation are significant, while that of a house price index is not, but the coefficients are not consistent across test assets, contradicting the theory of Merton.
4.1 Introduction

The central implication of the Intertemporal Capital Asset Pricing Model (ICAPM) of Merton (1973) is that in a dynamic, continuous time setting variation in the investment opportunity set is the primary reason for variation in asset prices. That is, current asset prices are not only explained by the return on the market portfolio, but also by a set of state variables that describe the changes in the investment opportunity set. The conclusion is that adequate models of asset returns cannot be reduced to the univariate Capital Asset Pricing Model (CAPM) where only the return on the market portfolio is needed to describe asset returns. Thus there is a continuous time relationship between the excess return of asset $i$ and its covariance with the market ($\sigma_{im}$) and with the state variables ($\sigma_{iX}$) of the form

$$ Er_i - r_f = \gamma \sigma_{im} + \lambda \sigma_{iX} \quad (4.1.1) $$

Both $\gamma$ and $\lambda$ have a theoretical interpretation: as pointed out by Merton (1980), $\gamma$ is the coefficient of relative risk aversion of the representative agent and $\lambda$ is the vector of prices of risk associated with the state variables.

We test three key implications of the ICAPM. First, both $\gamma$ and $\lambda$ are uniquely determined by the derivatives of the utility function - they should be the same for each test asset, regardless of $\sigma_{im}$ or $\sigma_{iX}$. Second, $\gamma$ should be positive, as it is the relative risk aversion of the representative investor. The final implication we look at is that there should be no excess return that is not captured by one of the factors.

The classical intuition behind the result in Merton (1973) is that extending the time horizon of the representative agent beyond a single future period - contrary to the CAPM, a one period model - results in the risk-averse agent seeking to hedge his or her portfolio against unfavorable changes in state variables that will affect the returns on the portfolio in the future. Thus high covariances of returns (high risk) with variables that have a positive effect on returns should be rewarded on the market in the form of high excess returns. An alternative way of obtaining a similar result is to look at the agent’s marginal utility directly - any unhedgeable variation in state variables affecting marginal utility should be a priced source of risk. For example, in the presence of wage income risk that cannot be insured against, the level (or possibly growth rate) of unemployment should be priced. Similarly, if household utility is nonseparable in consumption...
and housing, house prices should affect asset prices.

We apply a slightly novel approach to pricing asset returns: we construct estimates of covariances of returns and candidate macroeconomic state variables at each point of time using the Dynamic Conditional Correlation (DCC) model of Engle (2002) and use these estimates directly in a cross-sectional regression. DCC is a parsimonious method of modeling the time-varying variances and covariances of multiple time series - the parsimony results from assuming that the time-variation in the correlations (and thus also in the covariances) depends on just two additional parameters. A detailed description of the model and its estimation is presented in Section 4.3.

Within the broader class of multivariate GARCH models\(^1\) the DCC is best characterized by simplicity in specification. In comparison, in the most general of multivariate GARCH models - the unrestricted VEC GARCH of Bollerslev et al. (1988) - in an \(N\) dimensional model a total of \(N(N + 1)/2 + 2(N(N + 1)/2)^2\) parameters have to be estimated. In addition to being simple to specify, the DCC model is usually rather easy to estimate. This is due to the fact that when the conditional mean does not depend on the conditional variances or covariances the estimation procedure can be split into two parts - one containing parameters related only to the conditional variances in the model and the other parameters related only to the covariances. This leads to a conceptually rather simple two step maximum likelihood procedure. In practice numerical methods, also applied in two steps, are required due to the extreme nonlinearity of the model.

A significant difference between this paper and most of the previous literature applying the DCC model, especially Bali and Engle (2010), is in the estimation procedure. While the structure of the DCC model in principle allows for a computationally rather simple procedure, in our case the conditional means are directly dependent on the conditional covariances. Thus the whole model should be estimated in a single step. While this issue can be circumvented\(^2\), our approach is motivated by the goal of full efficiency, which is attained by means of a computationally rather demanding single step procedure. These differences are described in detail in Section 4.3.

Compared to some of the previous related contributions, we consider a more

\(^1\)For an overview see Silvennoinen and Teräsvirta (2009).

\(^2\)For example Bali and Engle (2010) work around this by using a three-step procedure where the time-varying covariances between assets are estimated first using daily data and then the effect of the covariances on the means is estimated separately.
diverse set of test assets to investigate whether $\gamma$ and $\lambda$ are the same for them. Our test assets include portfolios sorted on size and book-to-market equity ratio (SB), size and momentum (SM) and size and long-run reversal (SR) constructed from monthly returns on U.S. stocks from 1948 to 2012. As a robustness check we include the three risk factors\(^3\) of Fama and French (1993) (FF3) within the state variables.

The macroeconomic factors we consider are monthly U.S. industrial production, unemployment and inflation over the same period and the Case-Shiller house price index from 1987 to 2012, each of which could be argued to be something that the average investor cares about. With the exception of the Case-Shiller index they have also been considered previously in closely related literature.

Why would the index be relevant? Housing is a major component of household wealth, and thus house prices affect the consumption and saving choices made by investors, especially through the wealth effect and by acting as collateral. For example Lustig and Van Nieuwerburgh (2005) and Piazzesi et al. (2007) find that the ratio of housing wealth to human wealth and the housing share of private expenditure in the U.S. are able to forecast excess stock returns and explain a significant amount of cross-sectional variation in U.S. stock prices.

As a preview of the findings of this paper, we point out that the market return is indeed a priced source of risk and earns a positive premium. The house price index we consider is not priced in the market. While the rest of the macro variables are also sources of priced risk, there is considerable uncertainty regarding size of the premia they earn, as the estimates are not consistent across portfolios, in violation of the theory of Merton (1973). Confirming results in Chen et al. (1986) and Santos and Veronesi (2006), the premia of unemployment and inflation are negative, and consistently so across the portfolios.

The rest of the paper is divided into four sections. The next section begins the paper with a literature review. Section 4.3 describes the theory behind the estimation. Section 4.4 presents the data and discusses the estimation results. Section 4.5 concludes.

\(^3\)Return on the market portfolio, the difference between the “small” and the “big” portfolios and the difference between the “high” and the “low” portfolios. See Kenneth French’s data library at http://mba.tuck.dartmouth.edu/pages/faculty/ken.french/data_library.html for details.
4.2 Literature review

An early study on effects of macroeconomic factors on the cross-section of stock returns can be found in Chen et al. (1986), who estimate the prices of risk for innovations in several macroeconomic (and financial) variables using U.S. data. They find that for example the long-short interest rate spread, inflation and industrial production are priced sources of risk, while oil price risk, aggregate consumption and return on the market portfolio are not.

Lettau and Ludvigson (2001) study predicting U.S. stock returns using the aggregate consumption-wealth ratio, discovering that the ratio is a good forecaster of future returns. Parker and Julliard (2005) examine the pricing of what they term “ultimate consumption risk” - variation in the cumulative consumption growth over several quarters - and show that it is able to explain a significant portion of cross-sectional variation in stock returns. Santos and Veronesi (2006) focus on the predictability of asset returns using labor income - in a theoretical model they point out how fluctuations in wages induce variation in the risk premia of stocks, and show how the labor income to consumption ratio is able to price a set of sorted portfolios. Piazzesi et al. (2007) show how the expenditure share of housing should earn a positive, yet volatile premium, and conclude that it is indeed useful in predicting excess U.S. stock returns. In quite recent work Bali et al. (2014) consider the effect U.S. macroeconomic risk (in inflation, unemployment, GDP growth and others) on the returns of hedge funds using a multivariate GARCH model to estimate the time-varying covariances of macroeconomic variables and hedge fund returns in a VAR setting. The finding is that this risk is a strong determinant of dispersion in hedge fund returns.

Studies on the pricing of macroeconomic volatility or news on the business cycle include examples such as Kim and Lee (2008) - they find a positive relationship between the business cycle and excess returns and a negative one between return volatility and the business cycle - and Jones et al. (1998), who discover that U.S. treasury bond’s excess return and volatility do react to news announcements as predicted by theory. The joint effects of consumption volatility and volatility in the stock market are studied by Duffee (2005) by estimating the price of the conditional covariance of stock returns and consumption growth using a GARCH-in-mean model with DCC covariances. The central finding is that relationship between expected excess stock returns and the conditional covariance is negative.
There is also a large related literature involving the application of univariate GARCH-in-mean models to price assets. The results obtained in these studies are in general mixed. An early example is Engle et al. (1987), who find that the risk premium on the yield of several different bonds is positive and statistically significant. Later studies, which focus on the pricing of U.S. stock returns with more complex GARCH models allowing for example volatility to have asymmetric effects, include for example Glosten et al. (1993) and Whitelaw (1994) (with a yield spread as an additional factor). They find a negative or weak relationship between the volatility and return of the market portfolio, while others, such as French et al. (1987) and Scruggs (1998) (who also uses U.S. long term bond returns as an additional factor), find a positive relationship.

The DCC model has also been previously applied to price assets with a volatility-in-mean specification. Bali and Engle (2010) and Bali et al. (2013) study the pricing of various macroeconomic and financial factors in this framework. In particular, Bali and Engle (2010) study the pricing of financial factors such as the term spread, corporate bond rates and 3-month treasury bill rate in an ICAPM framework, finding a significant risk-return relationship for market return and a significant price of risk for the term spread and the default spread. Bali et al. (2013) on the other hand study the conditional CAPM - a version of the CAPM where the coefficient on the market return is allowed to vary over time - and find that DCC based estimates for the conditional betas - parameters quantifying the relationship between the return on an asset and the market portfolio - are highly significant, unlike in many previous studies. However, they also find evidence that the conditional betas are not able to capture all the risk in asset returns, as other factors such as past returns and idiosyncratic volatility are still able to predict future returns, pointing to the need for additional risk factors.

4.3 Statistical model and estimation

We model joint dynamics of the asset returns and macroeconomic variables as the following $N \times 1$ vector process

$$Y_t = E_{t-1}(Y_t) + \varepsilon_t = E_{t-1}(Y_t) + H_t^{1/2} \eta_t$$

(4.3.1)

where $\varepsilon_t$ is a $N \times 1$ vector, $H_t$ is the $N \times N$ conditional covariance matrix of $Y_t$ and $\eta_t$ is a IID random vector with $E\eta_t = 0$ and $E\eta_t\eta_t' = I_n$. In many
applications of multivariate GARCH models it is typical that $E_{t-1}(Y_t) = 0$. However, in this paper we consider more general specifications. Furthermore, we decompose $Y_t = (r_t, S_t)$, where $r_t$ is a vector of asset returns and $S_t$ is a vector of non-return state variables. Some components of $r_t$ may also be used to price the returns.

As seen in equation (4.1.1), the ICAPM implies that asset returns can be characterized as linear functions of instantaneous covariances between the assets and state variables that affect future investment opportunities. From here on we assume that this continuous time relationship can be approximated in discrete time by

$$r_{i,t} = \alpha_i + \gamma \sigma_{im,t} + \sum_{j=1}^{n} \lambda_j \sigma_{ij,t} + \varepsilon_{it}$$ (4.3.2)

where $\varepsilon_{it}$ is an identically and independently distributed error term. We estimate two types of models for the return processes - with and without the intercept term $\alpha_i$, which should be interpreted as a pricing error. In both types of models $\sigma_{im,t}$ and $\sigma_{ij,t}$ are the covariances of return $i$ with the return on the market portfolio and state variable $j$, respectively. $\gamma$ is the coefficient of relative risk aversion and $\lambda_j$ is the risk premium on state variable $j$.

The question of whether or not to include the intercept $\alpha_i$ in the model is not trivial. As pointed out by Lanne and Saikkonen (2006), including them when it is unnecessary (i.e. the “true” value is zero) may cause the power of the Wald test for the hypothesis $\gamma = 0$ to drop significantly, although this result was only shown in the univariate case. On the other hand, including an intercept offers a convenient way of testing the validity of the model: if the $\alpha_i$ are found to be nonzero, the model misprices the assets. Therefore we run regressions of both types, which allows us to test the model in two different ways: by testing the coefficients on the covariances, to see if a factor is priced or not, and the intercept, to see if there is mispricing.

The mean processes of the $S_t$ vector are specified to follow appropriate ARMA $(p, q)$ processes, and we use the Bayesian Information Criterion (BIC) to find $p$ and $q$. Table 4.3.1 presents the results of this search over all $p$ and $q$ such that $p+q \leq 8$.

As discussed in Section 4.1, in the case of DCC of Engle (2002), the covariance matrix $H_t$ is decomposed as $D_t R_t D_t$, where $R_t$ is the correlation matrix of $Y_t$, $D_t = \text{diag}(h_{1t}^{1/2}, \ldots, h_{Nt}^{1/2})$ and $h_{it}$ is the $i$th diagonal element of $H_t$. The DCC model imposes no general restrictions on the form of the individual $h_{it}$. However,
Table 4.3.1: Optimal lag lengths for non-financial state variables

<table>
<thead>
<tr>
<th></th>
<th>p</th>
<th>q</th>
</tr>
</thead>
<tbody>
<tr>
<td>Industrial production growth</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>Unemployment 1st difference</td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>Case-Shiller index 2nd difference</td>
<td>0</td>
<td>2</td>
</tr>
<tr>
<td>Inflation</td>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>

in this paper we assume that they follow GARCH(1, 1) processes individually:

\[ h_{it} = \omega + a_i \varepsilon_{it-1}^2 + b_i h_{it-1} \forall i \] (4.3.3)

where \( \omega > 0, a_i \geq 0, b_i \geq 0 \) and \( a_i + b_i < 1 \).

Furthermore, the correlation matrix \( R_t \) is a rescaling of

\[ Q_t = (1 - d_1 - d_2)\rho + d_1 z_{t-1} z_{t-1}' + d_2 Q_{t-1} \] (4.3.4)

with \( d_1 > 0, d_2 \geq 0, d_1 + d_2 < 1 \) and \( \rho \) is the unconditional correlation matrix of the standardized errors \( z_t = \frac{\varepsilon_t}{\sqrt{h_t}} \). To ensure that \( R_t \) is indeed a valid correlation matrix with all elements less than or equal to unity in absolute value, \( Q_t \) is rescaled as \( R_t = \{Q_t\}^{-1/2} Q_t \{Q_t\}^{-1/2} \) where \( Q_t \) is a matrix with the diagonal elements of \( Q_t \) on its diagonal and zeroes elsewhere. Hence, the individual conditional correlations are computed as \( R_{ij,t} = \frac{q_{ij,t}}{\sqrt{q_{ii,t} q_{jj,t}}} \), where \( q_{ij,t} \) is an element of \( Q_t \). The parameters \( d_1 \) and \( d_2 \) are in fact the only parameters that need to be estimated in addition to those in (4.3.1) and (4.3.3).

Assuming normality, Engle (2002) derives the log likelihood function of this model to be of the form

\[ l(\theta, \phi | y) = -\frac{1}{2} \sum_{t=1}^{T} \left( n \log(2\pi) + \log |H_t| + \varepsilon_t' H_t^{-1} \varepsilon_t \right) \]

\[ = -\frac{1}{2} \sum_{t=1}^{T} \left( n \log(2\pi) + 2 \log |D_t| + \varepsilon_t' D_t^{-1} D_t^{-1} \varepsilon_t - z_t' z_t + \log |R_t| + z_t' R_t^{-1} z_t \right) \]

(4.3.5)

where \( \phi \) is a vector containing all parameters involved in the correlation process.
and $\theta$ is a vector containing the rest. Equation (4.3.5) can be rearranged as

$$l(\theta, \phi| y) = -\frac{1}{2} \sum_{t=1}^{T} \left( n \log(2\pi) + 2 \log|D_t| + \varepsilon_t^t D_t^{-2} \varepsilon_t \right) - \frac{1}{2} \sum_{t=1}^{T} \left( \log|R_t| + z'_t R_t^{-1} z_t - z'_t z_t \right)$$

$$= l_{\text{vol}}(\theta| y) + l_{\text{cor}}(\theta, \phi| y) \quad (4.3.6)$$

assuming that $E_{t-1}(y_t)$ does not depend on $\phi$. This leads to the two-step estimation procedure suggested in Engle (2002): first maximize $l_{\text{vol}}(\theta| y)$ to obtain $\hat{\theta}$ and then maximize $l_{\text{cor}}(\hat{\theta}, \phi| y)$ with respect to $\phi$.

The approach followed by Bali and Engle (2010) is somewhat different. Similar to our case, they assume returns follow (4.3.2), implying that $E_{t-1}(Y_t)$ depends on $\phi$. They however proceed in a three step method: they first estimate a univariate GARCH model for each series they use with "any autoregressive elements taken out" from the mean equations, then using standardized returns from the previous step, bivariate DCC models (and thus conditional covariance processes) for each pair of variables. Finally, the expected return equations are estimated as a panel, with the previously estimated covariances used as regressors in a seemingly unrelated regression.

In contrast, the approach applied in this study is to estimate the parameters by maximizing the log likelihood function (4.3.5) in a single step. As pointed out by Engle and Sheppard (2001), the two-step procedure often applied is actually not fully efficient. However, in their Lemma 1, they also show that the single step procedure is efficient. This comes at a significant computational cost - with a single, simple model with six test portfolios and a single state variable taking several hours to estimate\(^5\) - but also with the benefit of attaining full efficiency. Therefore we use this method in order to be able to make accurate statistical inference.

As shown by Bollerslev and Wooldridge (1992) the (quasi) maximum likelihood estimator remains consistent, whether or not the assumed normality holds. However, the commonly used estimator for the covariance matrix of maximum likelihood estimator - the inverse of the Hessian of $l(\cdot)$ - is no longer appropriate under nonnormality. Instead we use the robust covariance matrix estimator of

\(^4\)With exact details of this and the next step left for the reader.
\(^5\)On a standard desktop PC using R.
Bollerslev and Wooldridge. Assuming a sample of length $T$, it is computed as

$$\hat{V}_T = \hat{C}_T^{-1} \hat{S}_T \hat{C}_T^{-1} / T \quad (4.3.7)$$

where $\hat{C}_T = \frac{1}{T} \sum_{t=1}^{T} c_t$ and $\hat{S}_T = \frac{1}{T} \sum_{t=1}^{T} s_t' s_t$, with $s_t$ the gradient and $c_t$ the Hessian of $l(\cdot)$ at time $t$ evaluated at $\hat{\theta}$, the (quasi) maximum likelihood estimate. Note that under normality both $\hat{C}_T^{-1} / T$ and $\hat{S}_T^{-1} / T$ are appropriate as estimators of the covariance matrix.

Hypothesis testing in this case is thus based on a slightly adjusted version of the Wald statistic: for $H_0 : \tau(\theta_0) = 0$, $\tau : \theta \to \mathbb{R}^K$, with $\tau'$ the gradient matrix of $\tau$, the test statistic

$$W_T = T \tau(\hat{\theta}_T)' \left[ \tau'(\hat{\theta}_T) \hat{C}_T^{-1} \hat{S}_T \hat{C}_T^{-1} \tau'(\hat{\theta}_T) \right] ^{-1} \tau(\hat{\theta}_T) \quad (4.3.8)$$

is asymptotically $\chi^2_K$-distributed.

### 4.4 Data and estimation results

In this chapter we present our data set and the results of the estimation exercise: in Section 4.4.1 we describe the test assets and the state variables. Section 4.4.2 is divided into subsections by model specification - we start with the simple univariate models, then cover in sequence the cases where both the market return and some other state variable appear in the pricing equation, followed by a section analyzing the situation where the model includes the market return with industrial production, unemployment and inflation. For robustness we study the FF3 specification and a specification where the univariate volatility processes allow for asymmetry.

#### 4.4.1 Data

Our data set consists of two blocks: asset return data from CRSP database, which was obtained from Kenneth French’s data library\(^6\), and U.S. macroeconomic data from the FRED database of the Federal Reserve Bank of St. Louis and Standard & Poor’s.

\(^6\)http://mba.tuck.dartmouth.edu/pages/faculty/ken.french/data_library.html
4.4.1.1 Test assets

We use portfolios sorted on size and the book-to-market ratio (size-B/M, SB), portfolios sorted on size and momentum (size-momentum, SM) and portfolios sorted on size and long-run returns (size-reversal, SR). The portfolios have been constructed using all stocks listed on the New York Stock Exchange, AMEX and NASDAQ. The asset return data spans the period January 1948 - December 2012, and hence comprises a total of 780 monthly observations. The sorts are constructed by first dividing the stocks into two groups based on size and then into three groups based on the other factor, with 40% of the individual stocks in the middle portfolio and 30% in the low and high portfolios. Momentum returns are defined as the returns on the stock in the 12 months prior to portfolio formation while the long-run return is defined as the return on the stock in the 60 months preceding portfolio formation, but not including the returns in the prior 12 months. More details on the exact calculation can be found on Kenneth French’s website.

Each of our sorted portfolios focuses on a different asset pricing anomaly, although our goal is not to solve them. The SR portfolios demonstrate the finding of “long-run reversal” by DeBondt and Thaler (1985): typically stocks that performed poorly in the past 3 - 5 years outperformed in the last year the stocks that performed well in the past. Jegadeesh and Titman (1993) pointed out the existence of the momentum effect in stock returns - portfolios sorted on the returns of the past 3 - 12 months often generate abnormal returns, as past winners continue to perform well. The momentum factor - which is used to construct the portfolios - is based on this fact. Finally, the SB portfolios are an early and extremely influential example of a set of test assets where the CAPM does extremely poorly, as shown by Fama and French (1993). They exhibit the value effect, where stocks with high ratios of book equity to market equity tend to have higher returns than predicted by CAPM.

Our test assets are also well known for being rather difficult to price. Especially the simple CAPM model performs quite poorly on all of them. In each case this is due to an underlying “mispricing” by the market, often described in the literature as an anomaly. Especially in the case of high momentum returns and long-term reversal, a behavioral explanation is likely to be at the heart of the matter. On the other hand, as noted by Liew and Vassalou (2000), the size, momentum and value effects are likely to be related to future economic growth. Aretz et al. (2010) obtain similar results for a broader set of macroeconomic
variables, including inflation, the term structure and exchange rates.

4.4.1.2 Candidate state variables

Our macroeconomic data comprises monthly U.S. industrial production, unemployment and inflation from January 1948 onwards to December 2012 (from FRED) and the national\textsuperscript{7} United States Case-Shiller home price index (from Standard & Poor's), which is available at the monthly level from January 1987 to December 2012, for 310 observations.

Each of these variables could be argued to be something that the average investor cares about - it is more than plausible that they affect marginal utility growth either directly as arguments of the utility function or indirectly by affecting the investment opportunity set and thus future consumption. Unemployment and the house price index affect both consumer behavior and either income or wealth directly and thus have a direct effect on the stochastic discount factor: house prices have an effect on household consumption through the wealth effect\textsuperscript{8}, as discussed in Piazzesi et al. (2007), among others. Finally, industrial production growth is a reasonable proxy for growth in consumption.

On the other hand, none of the factors seems a particularly strong candidate for explaining the dynamics observed in our test assets. Each of the three asset pricing anomalies present is likely caused by an phenomenon not directly related to variation in our factors. This does not a priori invalidate the candidate factors, as they may still be priced sources of risk. If the marginal utility of the average investor depends on the state variables, he or she will be willing to pay (or will be paid) a premium for holding an asset sensitive to variation in them, even if the sensitivity is not direct. This however imposes an additional hurdle for the model.

Where necessary, we use an appropriate transformation on our state variables to ensure stationarity. In the typical case either the first difference or the growth rate is stationary, as shown by the Augmented Dickey-Fuller test at the 5% level. The Case-Shiller index is an unusual special case, as the unit root in the first difference cannot be rejected by the Augmented Dickey-Fuller test at this level. In this case the second difference is stationary.

\textsuperscript{7}Constructed using home price data from the Boston, Chicago, Denver, Las Vegas, Los Angeles, Miami, New York, San Diego, San Francisco and Washington D.C. metro areas.

\textsuperscript{8}For example, in the U.S. consumption in the 1990s and early 2000s was often financed by mortgage based leverage that depended on perceived increases in house values.
Finally, our set of explanatory variables includes the three well-known factors first used in Fama and French (1993). Besides the excess return on the market portfolio - defined as the value-weighted return on all listed stocks minus the 1-month Treasury bill rate - we use the so-called High-Minus-Low (HML) and the Small-Minus-Big (SMB) factors. They are constructed using the size-B/M portfolios: SMB is computed as the average return on the three small portfolios minus the average return on the three big portfolios, while HML is the average return on the two high book-to-market ratio portfolios minus the average return on the two low book-to-market ratio portfolios. More details on their construction can be found in Kenneth French's data library.

4.4.2 Estimation results

4.4.2.1 Single risk factor

Tables 4.4.1 and 4.4.2 present the estimated risk aversion and risk premium coefficients for the SB, SM and SR portfolios in models with a single state variable - either the excess return on the market portfolio, the growth rate of industrial production, the first difference of unemployment, or the second difference of the Case-Shiller home price index or inflation. Table 4.4.1 contains the estimates without an intercept term, while Table 4.4.2 contains those with it. Table 4.4.3 contains the averages across test portfolios of the absolute values of the intercept terms, which we use for an informal comparison of model performance.

The estimates of $\gamma$ are roughly of the same magnitude in all three cases, as implied by the theory. But interpreting the estimates as actual coefficients of relative risk aversion is difficult, since in previous work it has been estimated significantly greater than unity. For example, based on U.S. data, Mehra and Prescott (1985) find that for the commonly assumed Constant Relative Risk Aversion utility function $u(c) = \frac{c^{1-\gamma}}{1-\gamma}$, where $c$ is consumption, the coefficient plausibly ranges between 2 and 10. Constantinides (1990) suggests it might be greater than 8 when the agents are assumed to have habit formation utility, and in this case the coefficient would also be time-varying due to the properties of the habit formation utility function. In a setup very similar to this paper, Bali and Engle (2010) find estimates that range from 2 to 4. The results then lead us to interpret the estimated $\gamma$ as simply risk premia on variation in the return on the market portfolio.

Continuing with Table 4.4.1, the estimates for $\lambda$ are often several magnitudes
greater than the risk aversion estimates, with (absolute) values ranging from 0.005 to 2.78. The upper end of this range seems somewhat similar to those found by Bali and Engle (2010), and most of these estimates are also statistically significant. The exception is the house price index, indicating that the index is not a priced source of risk. Furthermore, there is significant variation in the estimates. For example, the estimated risk premia for industrial production growth range from 0.005 to 0.31, implying that the theory of Merton (1973) is violated: the coefficients are not constant across assets.

Turning to Table 4.4.2, including the intercept term has practically no effect on the estimates for $\gamma$, while the estimates for risk premia are in many cases quite different from those seen in Table 4.4.3. For example, two of the signs of the risk premia have switched signs, indicating that despite the seemingly small standard errors, the coefficients are possibly spurious. Overall, the standard errors cannot be said to be noticeably larger, which, given the result of Lanne and Saikkonen (2006), leads us to conclude that the intercept terms should indeed be included in the model. As the standard errors were so small to begin with, many of the estimates remain highly significant.

Based on Tables 4.4.1 and 4.4.2, we conclude that inflation and unemployment are indeed risks priced in the market and that the risk premia are negative - that is, assets that are sensitive to variation in unemployment or inflation are paid less than those that are not. The intuition behind the result is that these assets are, in part, insurance against these risks: households pay a premium to hold them in order to hedge themselves. These results match those found in Chen et al. (1986), who also find a negative premium on inflation, and Santos and Veronesi (2006), who find a negative premium on labor income risk. Furthermore, the evidence in our estimates for risk in industrial production earning a premium is weak, but in this case, too, the sign of the estimate matches that of Chen et al. (1986), who find that it is not priced.

More curious is the fact that the Case-Shiller house price index was so clearly rejected, especially since in previous research such as Lustig and Van Nieuwerburgh (2005) and Piazzesi et al. (2007) risk related to the share of housing to private expenditure and housing wealth relative to human wealth was found both a useful predictor and to earn a positive premium in a calibrated model. One possible explanation is that the finding is simply related to data - if one had measures of these ratios, different results would emerge. In addition, it is indeed plausible that the true risk in housing that households care about is its
Table 4.4.1: Risk aversion and premia estimates, single risk factor, no intercepts

<table>
<thead>
<tr>
<th>Risk factor</th>
<th>SB</th>
<th>SM</th>
<th>SR</th>
</tr>
</thead>
<tbody>
<tr>
<td>Market return</td>
<td>0.035</td>
<td>0.041</td>
<td>0.046</td>
</tr>
<tr>
<td></td>
<td>(0.001)</td>
<td>(0.001)</td>
<td>(0.001)</td>
</tr>
<tr>
<td>Ind. production growth</td>
<td>0.059</td>
<td>0.31</td>
<td>0.005</td>
</tr>
<tr>
<td></td>
<td>(0.002)</td>
<td>(0.07)</td>
<td>(0.01)</td>
</tr>
<tr>
<td>Unemployment</td>
<td>-0.87</td>
<td>-2.78</td>
<td>-0.46</td>
</tr>
<tr>
<td></td>
<td>(0.13)</td>
<td>(0.18)</td>
<td>(0.08)</td>
</tr>
<tr>
<td>Case-Shiller index</td>
<td>0.25</td>
<td>0.81</td>
<td>0.43</td>
</tr>
<tr>
<td></td>
<td>(0.29)</td>
<td>(0.42)</td>
<td>(0.77)</td>
</tr>
<tr>
<td>Inflation</td>
<td>-0.38</td>
<td>-0.33</td>
<td>-1.85</td>
</tr>
<tr>
<td></td>
<td>(0.02)</td>
<td>(0.11)</td>
<td>(0.28)</td>
</tr>
</tbody>
</table>

Note: standard errors in parenthesis

ratio to consumption and other wealth, not the absolute level of housing wealth they have. However, for now we have to conclude that risk related to variation in house prices is not priced in the market.

Table 4.4.3 presents averages of absolute values of $\alpha$ - our chosen measure of mispricing - across portfolios. It can be seen that the return on the market portfolio does a fairly good job - at least in comparison to the macro variables - in pricing all three sets of test assets. The macroeconomic variables yield rather large estimates of mispricing. They also do not price the assets equally well - industrial production growth performs particularly poorly, with average estimates of mispricing between 0.4 and 0.9. Unemployment yields similar weak results, while inflation does a somewhat better job. The Case-Shiller index, however, seems to perform the best of our four candidates, yielding $\alpha$ estimates that are quite often much smaller. It should be noted though that this superior performance may be due to the fact that any estimates using the index are computed using a different, shorter sample of the data - a sample which may be much easier to price.

4.4.2.2 Bivariate models

To study the robustness of the risk premia estimated in the previous section, we augment our model specification with an additional macroeconomic state variable. As the Case-Shiller house price index was found not to be priced in the market in the previous section, we omit any further results involving it. The models presented in this and latter sections were estimated also using the
Table 4.4.2: Risk aversion and premia estimates, single risk factor

<table>
<thead>
<tr>
<th>Risk factor</th>
<th>SB</th>
<th>SM</th>
<th>SR</th>
</tr>
</thead>
<tbody>
<tr>
<td>Market return</td>
<td>0.029</td>
<td>0.044</td>
<td>0.047</td>
</tr>
<tr>
<td>(0.004)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Ind. production growth</td>
<td>0.1</td>
<td>-0.001</td>
<td>-0.11</td>
</tr>
<tr>
<td>(0.017)</td>
<td>(0.08)</td>
<td>(0.07)</td>
<td></td>
</tr>
<tr>
<td>Unemployment</td>
<td>-0.7</td>
<td>-0.95</td>
<td>-0.26</td>
</tr>
<tr>
<td>(0.17)</td>
<td>(0.066)</td>
<td>(0.19)</td>
<td></td>
</tr>
<tr>
<td>Case-Shiller index</td>
<td>-0.91</td>
<td>1.22</td>
<td>0.41</td>
</tr>
<tr>
<td>(0.83)</td>
<td>(0.59)</td>
<td>(0.93)</td>
<td></td>
</tr>
<tr>
<td>Inflation</td>
<td>-0.69</td>
<td>-1.31</td>
<td>-2.56</td>
</tr>
<tr>
<td>(0.12)</td>
<td>(0.44)</td>
<td>(0.4)</td>
<td></td>
</tr>
</tbody>
</table>

Note: standard errors in parentheses

Table 4.4.3: Averages of absolute values of $\alpha$, single risk factor

<table>
<thead>
<tr>
<th></th>
<th>SB</th>
<th>SM</th>
<th>SR</th>
</tr>
</thead>
<tbody>
<tr>
<td>Market return</td>
<td>0.23</td>
<td>0.27</td>
<td>0.08</td>
</tr>
<tr>
<td>Ind. production growth</td>
<td>0.87</td>
<td>0.42</td>
<td>0.9</td>
</tr>
<tr>
<td>Unemployment</td>
<td>0.77</td>
<td>0.84</td>
<td>0.74</td>
</tr>
<tr>
<td>Case-Shiller index</td>
<td>0.56</td>
<td>0.61</td>
<td>0.26</td>
</tr>
<tr>
<td>Inflation</td>
<td>0.79</td>
<td>0.34</td>
<td>0.6</td>
</tr>
</tbody>
</table>

Case-Shiller index as an explanatory variable, but it was found not to be priced also in these results.

Table 4.4.4 presents estimates of the $\gamma$ and $\lambda$ coefficients for models that contain two risk factors - the return on the market portfolio and one of either industrial production growth, unemployment or inflation. First, the estimates of $\gamma$ remain roughly the same as in section 4.4.2.1 - all of the $\gamma$ estimates in Table 4.4.4 are between 0.03 and 0.055. Adding any of the other state variables into the model has practically no effect on estimated risk aversion. Second, the risk premia estimates for the macroeconomic variables are, for the most part, roughly similar to those in section 4.4.2.1 and statistically highly significant. Inexplicable exceptions are the $\lambda$ coefficients when pricing the size-reversal portfolios with unemployment or inflation as risk factors, as the signs of coefficients have flipped from negative to positive.

4.4.2.3 4-factor models

In the previous two sections we found several state variables that seem to be priced sources of risk. On the other hand, we also noticed that none of these
Table 4.4.4: Risk aversion coefficients and risk premia, two risk factors, no intercepts

<table>
<thead>
<tr>
<th>Additional risk factor</th>
<th>SB</th>
<th>SM</th>
<th>SR</th>
</tr>
</thead>
<tbody>
<tr>
<td>Ind. production growth</td>
<td>0.042</td>
<td>0.081</td>
<td>0.051</td>
</tr>
<tr>
<td></td>
<td>(0.001)</td>
<td>(0.007)</td>
<td>(0.001)</td>
</tr>
<tr>
<td>Unemployment</td>
<td>0.042</td>
<td>-0.35</td>
<td>0.046</td>
</tr>
<tr>
<td></td>
<td>(0.001)</td>
<td>(0.03)</td>
<td>(0.001)</td>
</tr>
<tr>
<td>Inflation</td>
<td>0.032</td>
<td>-0.27</td>
<td>0.045</td>
</tr>
<tr>
<td></td>
<td>(0.001)</td>
<td>(0.13)</td>
<td>(0.002)</td>
</tr>
</tbody>
</table>

Note: standard errors in parenthesis

variables were alone able to capture all the excess returns in the test portfolios we studied, as the estimated pricing errors (the $\alpha$ terms) were found to be significantly different from zero. In this section we focus on this question. Are the state variables able to describe additional priced risk compared to the univariate specifications?

In Tables 4.4.5 and 4.4.6 we have the outcomes of estimating specifications which include four different candidate state variables: excess return on the market portfolio, industrial production growth, the first difference of unemployment and inflation.

As seen in Table 4.4.5, the estimates of the risk aversion coefficient $\gamma$ match those seen in sections 4.4.2.1 and 4.4.2.2 quite closely, with values ranging between 0.035 and 0.049 - fairly close to the previously seen estimates of roughly 0.04. Furthermore, the estimates of the risk aversion coefficient remain highly significant. In addition, estimates of the risk premia for macro variables are also for the most part highly significant and also close to the estimates seen previously.

Table 4.4.6 presents estimates and standard errors of the mispricing $\alpha$s for the individual test portfolios. First, the $\alpha$s are clearly decreasing in size, increasing in the book-to-market ratio and momentum and decreasing in reversal, implying that the factors we study are indeed unable to solve the previously mentioned asset pricing anomalies\(^\text{9}\). Second, in nine cases out of 18 the estimated pricing error is significant at the 5% or smaller level, indicating that at least this set of state variables is not sufficient to price these assets.

The average $\alpha$ terms - numbers comparable to those seen in Table 4.4.3 - are 0.26,

\(^{9}\)If they were, there would not be a pattern in the pricing errors.
Table 4.4.5: Risk aversion coefficients and risk premia, four risk factors

<table>
<thead>
<tr>
<th></th>
<th>$\gamma$</th>
<th>$\lambda_{IND}$</th>
<th>$\lambda_{UNEMP}$</th>
<th>$\lambda_{INFL}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Size-B/M</td>
<td>0.035</td>
<td>0.08</td>
<td>-0.25</td>
<td>-0.37</td>
</tr>
<tr>
<td></td>
<td>(0.0001)</td>
<td>(0.011)</td>
<td>(0.12)</td>
<td>(0.07)</td>
</tr>
<tr>
<td>Size-momentum</td>
<td>0.048</td>
<td>-0.008</td>
<td>-0.86</td>
<td>-0.66</td>
</tr>
<tr>
<td></td>
<td>(0.0001)</td>
<td>(0.06)</td>
<td>(0.07)</td>
<td>(0.04)</td>
</tr>
<tr>
<td>Size-reversal</td>
<td>0.049</td>
<td>-0.18</td>
<td>-0.041</td>
<td>-0.92</td>
</tr>
<tr>
<td></td>
<td>(0.004)</td>
<td>(0.05)</td>
<td>(0.32)</td>
<td>(0.005)</td>
</tr>
</tbody>
</table>

Note: standard errors in parenthesis

Table 4.4.6: Intercept estimates, four risk factors

<table>
<thead>
<tr>
<th></th>
<th>Size-B/M</th>
<th>Size-momentum</th>
<th>Size-reversal</th>
</tr>
</thead>
<tbody>
<tr>
<td>Low</td>
<td>Mid</td>
<td>High</td>
<td>Low</td>
</tr>
<tr>
<td>Small</td>
<td>-0.028</td>
<td>0.35</td>
<td>0.44</td>
</tr>
<tr>
<td></td>
<td>(0.02)</td>
<td>(0.05)</td>
<td>(0.05)</td>
</tr>
<tr>
<td>Big</td>
<td>0.14</td>
<td>0.25</td>
<td>0.34</td>
</tr>
<tr>
<td></td>
<td>(0.005)</td>
<td>(0.14)</td>
<td>(0.08)</td>
</tr>
</tbody>
</table>

Note: standard errors in parenthesis

0.28, and 0.15 for the size-B/M, size-momentum and size-reversal portfolios, respectively. Contrasting these with the results (0.23, 0.27 and 0.08 for the model with only the market return as a risk factor) seen in Table 4.4.3 yields the conclusion that the macro variables are not able to help in driving down the pricing errors: the 4-factor model actually yields slightly higher errors.

In comparison to some of the previous research, such as Lustig and Van Nieuwerburgh (2005), Parker and Julliard (2005) and Santos and Veronesi (2006), who are able to price the portfolios sorted on size and the book-to-market ratio quite well, our factors perform poorly. This result is somewhat surprising, but most likely simply a data related issue: it seems probable that the factors they use contain information about these returns that our factors do not.

4.4.2.4 Fama-French factors

The estimation results for models that include an intercept term and the excess return on the market portfolio and the HML and SMB factors of Fama and French (1993) are presented in Tables 4.4.7 and 4.4.8, with 4.4.7 presenting the estimated risk-related coefficients and 4.4.8 the corresponding pricing errors. In the case of $\gamma$ the results are roughly in line with those found in the previous sections, with highly significant estimates that are (mostly) slightly less than
Table 4.4.7: Risk aversion coefficients and risk premia, Fama-French factors

<table>
<thead>
<tr>
<th></th>
<th>$\gamma$</th>
<th>$\lambda_{HML}$</th>
<th>$\lambda_{SMB}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Size-B/M</td>
<td>0.034</td>
<td>-0.013</td>
<td>0.09</td>
</tr>
<tr>
<td></td>
<td>(0.001)</td>
<td>(0.003)</td>
<td>(0.001)</td>
</tr>
<tr>
<td>Size-momentum</td>
<td>0.038</td>
<td>0.027</td>
<td>0.066</td>
</tr>
<tr>
<td></td>
<td>(0.01)</td>
<td>(0.0138)</td>
<td>(0.01)</td>
</tr>
<tr>
<td>Size-reversal</td>
<td>0.09</td>
<td>-0.007</td>
<td>0.21</td>
</tr>
<tr>
<td></td>
<td>(0.001)</td>
<td>(0.01)</td>
<td>(0.01)</td>
</tr>
</tbody>
</table>

Note: standard errors in parenthesis

Table 4.4.8: Intercept estimates, Fama-French factors

<table>
<thead>
<tr>
<th></th>
<th>Size-B/M</th>
<th>Size-momentum</th>
<th>Size-reversal</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Low</td>
<td>Mid</td>
<td>High</td>
</tr>
<tr>
<td>Small</td>
<td>-0.066</td>
<td>-0.21</td>
<td>-0.59</td>
</tr>
<tr>
<td></td>
<td>(0.01)</td>
<td>(0.04)</td>
<td>(0.005)</td>
</tr>
<tr>
<td>Big</td>
<td>0.1</td>
<td>0.017</td>
<td>-0.49</td>
</tr>
<tr>
<td></td>
<td>(0.02)</td>
<td>(0.006)</td>
<td>(0.01)</td>
</tr>
</tbody>
</table>

Note: standard errors in parenthesis

The validity of the ICAPM story - that additional factors are needed to properly price assets - is again confirmed by the estimated risk premia for the HML and SMB factors. They are often significant, and in the case of SMB, always the same sign. An unexpected result is obtained when estimating $\lambda_{HML}$: it is negative in two cases out of three and significant only in one case. In comparison, Fama and French (1993) find it to be significantly positive.

Turning to the intercept terms of Table 4.4.8, it can be seen that adding the HML and SMB factors does not improve the situation with respect to mispricing - in fact the opposite seems to be true. Not only are all of the estimates statistically significant, in many cases they are greater in absolute value than those seen in Tables 4.4.1 and 4.4.6 - the two additional risk factors result in quite large $\alpha$s. The size-reversal portfolios seem to be particularly strongly affected by this effect. Furthermore, the averages of the $\alpha$s are 0.15, 0.51 and 0.78 for size-B/M, size-momentum and size-reversal portfolios, respectively, again indicating weak performance in comparison to the univariate market return specification. The 3-factor model seems to be able to price the size-B/M portfolios somewhat better than the market return by itself, but it fails on the two other sets.
Table 4.4.9: Risk aversion and premia estimates, single risk factor, no intercepts, GJR-GARCH

<table>
<thead>
<tr>
<th></th>
<th>SB</th>
<th>SM</th>
<th>SR</th>
</tr>
</thead>
<tbody>
<tr>
<td>Market return</td>
<td>0.051</td>
<td>0.041</td>
<td>0.062</td>
</tr>
<tr>
<td></td>
<td>(0.002)</td>
<td>(0.001)</td>
<td>(0.002)</td>
</tr>
<tr>
<td>Ind. production growth</td>
<td>0.09</td>
<td>0.18</td>
<td>-0.09</td>
</tr>
<tr>
<td></td>
<td>(0.006)</td>
<td>(0.07)</td>
<td>(0.08)</td>
</tr>
<tr>
<td>Unemployment</td>
<td>-1.16</td>
<td>-2.75</td>
<td>-0.42</td>
</tr>
<tr>
<td></td>
<td>(0.35)</td>
<td>(0.44)</td>
<td>(0.15)</td>
</tr>
<tr>
<td>Inflation</td>
<td>-1.38</td>
<td>-0.81</td>
<td>-3.03</td>
</tr>
<tr>
<td></td>
<td>(0.027)</td>
<td>(0.13)</td>
<td>(0.39)</td>
</tr>
</tbody>
</table>

Note: standard errors in parenthesis

4.4.2.5 GJR-GARCH robustness checks

The fact that asset returns are quite often asymmetrically distributed is a widely documented phenomenon. GARCH models have been extended in various ways to accommodate this issue. One of them is the GJR-GARCH of Glosten et al. (1993), which allows for an asymmetric response in the volatility dynamics - a positive innovation may have a different impact than a negative one. More specifically, in the GJR-GARCH(1,1) model (4.3.3) is replaced with

\[ h_{it} = \omega_i + a_i \varepsilon_{it-1}^2 + b_i h_{it-1} + c_i I_{it-1} \varepsilon_{it-1}^2 \]  \hspace{1cm} (4.4.1)

where \( I_{it} = 1 \) if \( \varepsilon_{it} > 0 \) and 0 otherwise. Table 4.4.9 presents estimation results corresponding to Table 4.4.1, but under the assumption of the GJR model.

Overall the volatility specification seems to have very little effect on the estimates, as the risk aversion coefficients \( \gamma \) are again rather close to 0.05, while the the risk premia coefficients \( \lambda \) line up very close to those seen in sections 4.4.2.1 with consistently negative estimates for unemployment and inflation and mostly positive estimates for industrial production. Extremely similar results were obtained when replicating the other specifications under the GJR assumption.

4.5 Conclusions

In this paper we have studied the pricing of a set of test assets famous for being difficult for standard asset pricing models to deal with - portfolios sorted first on size and then on either the book-to-market equity ratio, momentum or long-
term reversal. Each of the three sorting variables is known to be a source of an asset pricing anomaly, known to generate portfolios that are hard to price with the standard CAPM (and in some cases even with more refined models). The ICAPM theory implies that these returns should be priced by the market based on their covariance with the market portfolio and some unspecified state variables affecting the wealth (or the marginal utility) of the investors.

The pricing of the test assets was studied and evaluated using two criteria: whether or not the price of risk associated with the candidate state variables and the pricing errors implied by the model were significantly different from zero. The first criterion tells whether or not some state variable is of interest to the average (or aggregate) investor, while the second is a check on the adequacy of the model. Due to concerns regarding statistical inference on these parameters, these tests were done using two different regressions - with and without intercept terms in the specification. This, however, was found not to play a significant role.

The excess return on the market portfolio and the returns on the small-minus-big portfolio were all found to be priced risks, as in many (but not all) previous studies, with estimates of $\gamma$, the coefficient of relative risk aversion ranging from 0.03 to 0.06, appearing to be quite constant regardless of model specification or test asset. Evidence for the pricing of the return on the high-minus-low portfolio was much weaker. In addition we found ambiguous evidence regarding the risk premia paid on variation on the HML and SMB factors. Some of the estimates had negative signs, while others were positive. As the estimated coefficients are supposed to be measures of the same “deep” parameters uniquely determined by the utility function of the representative investor and independent of the return that is studied, it appears that the theory of Merton (1973) is violated.

Similar troubles arose when estimating the risk premia of the macroeconomic state variables we studied - industrial production growth, the first difference of unemployment, inflation and the second difference of the Case-Shiller house price index. In model specifications where one of these four was the only risk factor, we found that while unemployment and inflation are clearly priced in the market, there is considerable uncertainty regarding the sizes of the risk premia. However, the findings of Chen et al. (1986) and Santos and Veronesi (2006) regarding the sign of these premia were confirmed. But overall the hypothesis of risk premia that are constant across assets is rejected by the data, implying that the theory of Merton (1973) is violated also in this case.
When model performance was evaluated using the intercept terms of the regression models - the mispricing of the test assets - best performance (smallest absolute intercepts) was delivered by the simple univariate specification using market return as the sole risk factor. As this specification outperformed the specifications with macroeconomic variables as additional risk factors, we conclude that these four macroeconomic factors do not help in pricing assets, even if some of them are priced by the market. This result is somewhat in contrast to the findings of Parker and Julliard (2005) and Santos and Veronesi (2006), who find that “ultimate” consumption and labor income are able to explain cross-sectional variation in stock prices.
Bibliography


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