Essays on Market Dynamics and Frictions

by

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M.Soc.Sc.

Dissertation

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To Pilvi, Mikko, and Iiris
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Chapter 1

Introduction
This thesis is titled "Essays on Market Dynamics and Frictions". It contributes to the dynamic price and search theory. As for the title, a multitude of alternative permutations of its key concepts – "market", "dynamics", and "frictions" – like "Studies on Dynamics and Frictions in a Market" were also considered. While the final wording is rather arbitrary and mainly aesthetic, each of these three concepts features as an important ingredient in the three main thesis Chapters 2–4.

1. **Market**: In each core chapter, there is a market with several sellers and buyers. No one seller has monopoly power nor one buyer has monopsony power. Instead, all the players have to be considerate of competition or the strategic situation in the market.

2. **Dynamics**: In each core chapter, payoff relevant information keeps arriving over time and either the sellers or the buyers have to make a dynamic decision of whether to buy from this seller or from the next seller or in which store to continue their shopping.

3. **Frictions**: In each core chapter, there are some sort of frictions. Either there is asymmetric quality information between the traders and they have to wait to have another trading opportunity or the buyers have to search more to come by additional price information.

Chapter 2 deals with the classic problem of trading under asymmetric information about the quality of the seller’s product. Chapters 3 and 4 are interlinked, though both stand-alone, analyzing retailing strategies that lock in buyers by creating in-store frictions. We provide a more detailed summary of each of these three "papers" at the end of this introductory chapter.

Next, to put our contributions properly into context and to offer a broad motivation for our work, we give a selective overview of the corner stones in the development of dynamic price and search theory with incomplete information. The exposition is necessarily quite condensed. For additional information, the reader is advised to consult the numerous references provided.
1.1 Simple price search

To gain an understanding on how the performance of a market is affected by frictions, we consider first a textbook market with several similar goods sold by many sellers.

The basic model can be developed step by step with increasing complexity and richness: starting from markets with no frictions, which serve as a benchmark, and proceeding thereafter from search under an exogenous price distribution to search under an endogenous price distribution (as in Chapters 2–4), from exogenous to endogenous search costs (like in Chapter 4) and, finally, extending the basic model for asymmetric information (like in Chapter 2).

We open this discussion by considering the Bertrand equilibrium and the Walrasian equilibrium, which arise in markets without any frictions. They lead us to touch on two counterintuitive results, the Bertrand (1887) paradox and the Diamond (1971) paradox, which arise in markets with homogenous buyers and homogenous sellers – the former in a setup without frictions, the latter in a setup with some positive frictions. We then add some heterogeneity.

1.1.1 "Law of one price"

One of the foremost observations spurring the development of price and search theory is the failure of the "law of one price". It has been repeatedly documented that similar goods are traded for different prices in almost every market conceivable (see Baye et al. (2006) and the numerous references therein). In the literature, this finding is typically regarded as telltale evidence of there being some sort of frictions in the buying process – travel costs, information processing costs etc. – which would have to be sufficiently significant to prohibit the consumers from exploiting the opportunity to arbitrage. Otherwise, it is very hard to reconcile, why a consumer would purchase for a higher price if there is also a lower price available for exactly the same good.

To understand the implications, it might be helpful to contrast this evidence of price dispersion with theoretical work. Consider the basic setup of Bertrand price competition, which features no frictions: Sellers have similar goods for sale and choose their prices. Buyers select from whom to purchase. Price information is available to the buyers with no cost. In this frictionless case where the price is the only competition instrument, the sellers are engaged in so harsh a price war that it completely eats up their price markups. In other words, in the symmetric case without cost advantage, in the unique equilibrium of this game, the sellers charge a price equaling their marginal cost and earn zero profits.\footnote{The theoretical underlying reason for this is that, when the buyers see every price, there is a discontinuity in a seller’s profit function with respect to the prices in the market such that, if the two lowest prices are the same, the market is divided equally between the two sellers but, if one of the two lowest prices is reduced anywhere below the other, the deviator captures the whole market. As a result, a seller}
This result is also called the Bertrand paradox.

At this point we would like to call some attention to the fact that the Bertrand equilibrium is much reminiscent of the Walrasian equilibrium, where the basic exercise it to find a price that clears the market by equating supply with demand. The Bertrand equilibrium is a standard approach to oligopolistic markets, the Walrasian equilibrium is one of the most important classic market models. Yet, for simplest, symmetric cases at least, the price is equal to the marginal cost in both.\footnote{Note that the Bertrand equilibrium arises under imperfect competition (price setting, two sellers) but the Walrasian equilibrium under perfect competition (price taking, a large number of sellers). The coincidence of equilibria is noteworthy, suggesting that in order to reap the benefits of competition, two sellers can be as good as a large number of sellers.}

In conclusion, for all the empirical evidence of price dispersion, these benchmark models are unfortunately not able to generate it between similar sellers. Something appears to be missing. As mentioned, we need some frictions to reconcile the coexistence of multiple distinct prices for the same good. It does not make sense to buy for a higher price if a lower price is at hand. To come up with a way to take this into account in the basic setup, it is therefore imperative that some buyers fail to find some prices. Interestingly, this is not all however. If we only add a simple search cost to the basic setup, we essentially just switch from one uniform price outcome, the Bertrand paradox, to another puzzling outcome with a sole price, the Diamond paradox.

The idea can be illustrated in a model where the buyers obtain their first price quote for free but, if they want more, they have to pay a search cost. The sellers may consider charging any price between consumer valuation and their own reservation value for the goods. Nonetheless, remarkably, even when there is a number of similar sellers, the unique equilibrium price is equal to consumer valuation: the monopoly price. This is so because the sellers have an incentive to exploit the holdup problem, that the buyers face when additional information is costly, by raising their price a bit above the price that the buyers expect to discover elsewhere. As every seller is doing so, the only price that can be supported in equilibrium is the maximal one.

This result that any positive search cost enables the sellers to charge the monopoly price is known as the Diamond paradox. Note particularly that, if the buyers have to pay for their first price quote as well, they have no incentive to search. The cost is positive and the benefit is zero. It other words, this revolutionizing idea by Diamond (1971) also demonstrates that, if sellers and buyers are identical, it is impossible to create a market where different sellers have different prices and the buyers search despite the cost.

Indeed, as highlighted by Burdett and Judd (1983) some form of \textit{ex post} heterogeneity is necessary to generate price variation among similar goods. The buyers could differ \textit{ex ante}, for example, in their search costs (Rob, 1985), storage capacity (Salop and Stiglitz, always gains if it, depending on the start point, increases or decreases its price until it is just below the lowest competing prices. This profitable deviation destroys all candidate equilibria except the mentioned one where the price is already as low as it can be.
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1982) or preferences (Sobel, 1984) or *ex post*, say, due to the randomness of their search paths. There is a multitude of different approaches to generating price dispersion in a market by now: in some of them it is spatial (a different price in a different store, pricing in mixed strategies), in others it is temporal (sale periods in between the business-as-usual times, pricing in pure strategies) etc. Generally, what is needed is just a method to introduce variation in the information under which the buyers are buying.

To give an example, in the classic papers by Varian (1980) and Stahl (1989) there are informed consumers (shoppers) and uninformed consumers (searchers). The former enjoy search and, thus, sample all the prices in the market whereas the latter dislike it and prefer to cut it off as early as possible. In such an environment, the sellers try to balance between attracting the former, by under cutting their competitor’s price, and exploiting the latter, through the holdup problem. The equilibrium is in randomized pricing strategies and converges to the Bertrand equilibrium, as the fraction of searchers vanishes, and to the Diamond equilibrium, as the fraction of shoppers vanishes. Capable of avoiding these extremes, the model is one of the main work-horses of industrial economics.

The Bertrand equilibrium and the Diamond equilibrium are still rather robust – reappearing for appropriate parametrizations in many price search models with an endogenous price distribution, including ours.

1.1.2 Models of search

There are numerous search models. The way the frictions and heterogeneity are introduced usually matters. Some research questions could be more naturally addressed by a specific approach, yet, some particularities of the results can usually be traced back to the approach directly. For instance, in models which build on the consumers’ holdup problem such as in Stahl (1989), the sellers use such pricing policies that the buyers typically search just once whilst, in models with horizontally differentiated products such as in Wolinsky (1986), the buyers search until they find a match value above a cutoff. Competitive search models tend to generate outcomes that are constrained efficient as the so called Hosios (1990) condition is satisfied. This list could go on.

Search models are applied in a number of fields. Sequential search models (optimal stopping problems, see, e.g. Weitzman (1979) for the Pandora box model and Robbins (1952) for the multi-armed bandit model) and non-sequential search models (fixed sample search, see, e.g., Stigler (1961)), and so called clearinghouse models (see, e.g., Baye and Morgan (2001)) are prevalent in micro and industrial economics. Random search and matching problems à la Diamond (1982); Mortensen and Pissarides (1994) and directed (finite economy) or competitive (infinite economy) search models à la Moen (1997); Peters

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3But see, e.g., Ellison and Wolitzky (2012)
4But see, e.g., Galenianos and Kircher (2009).
5See the excellent article by Baye et al. (2006) for a more detailed list of references.
Section 1.1

Burdett et al. (2001) are encountered, particularly, in macro and labor economics.\(^6\) Hence, while it is clear that frictions are essential in markets for various assets and durables – labor, houses, mates, consumer goods etc., the field is still in constant progress and there is no overarching, commonly accepted, unified approach as for how exactly the frictions should appear in a model.\(^7\) This work is no exception. Different research questions call for different approaches. In Chapter 2, we build on a model with random search and, in Chapters 3 and 4, we introduce a simple search model that features in-store frictions. To put them into a perspective, we next review some frequently used search models and extensions.

While ultimately our interest resides on dynamic price and search models with an endogenous price distribution and an endogenous search cost, the development of search models started with an exogenous price distribution and an exogenous search cost. Hence, we have touched upon models with an endogenous price distribution and models with an exogenous search cost. To cover the two other cases as well, we next take a look at some distinctive contributions to search theory: search with an exogenous price distribution in the classic Pandora box model and search with an endogenous search cost in the recent so called obfuscation literature.

Search from exogenous price distribution: the Pandora box model

Generally speaking, search from an exogenous payoff distribution refers to an optimal stopping problem where the distribution of prices is fixed. Consumers have to find the optimal way to sample from this distribution with free recall. They decide when to stop the exploration of various alternative options or, in other words, when to concentrate on the consumption – exploitation – of the best option they have so far discovered.

There are two especially noteworthy classes of such models: the Pandora box problem with immediate discovery of the prize (in each “box”) and the multi-armed bandit problem with gradual learning about the payoffs (of each “arm”). These problems got their first thorough treatise by Weitzman (1979) and by Robbins (1952), respectively. We next go through the basics behind the Pandora box model. For multi-armed bandit models, which could be regarded as an extension, we recommend the concise review by Bergemann and Välimäki (2006).

Various sequential problems of search can be cast into a setup where there is a number of opportunities or ”boxes”, each of them with an individual search (opening) cost, search (opening) time and expected reward inside. It is possible to open them only one-by-one and to take home one of the rewards only.

\(^6\)See the excellent review by Rogerson et al. (2005) for a more detailed list of references.

\(^7\)Note that, generally, search theory can be regarded as an attempt to develop further the Coasian argument for the significance of transactions costs for the institutional structure and the functioning of the economy. Without these costs and under clearly specified property rights, unlimited bargaining should result in a social optimum (Coase, 1937, 1960).
If the search cost and the search time for box $i$ are $c_i$ and $t_i$, respectively, and the distribution of rewards $x_i \sim F_i$, each box can be assigned an index $z_i$ such that

$$c_i = \beta^{t_i} \left( z_i \int_{-\infty}^{z_i} dF_i(x) + \int_{z_i}^{\infty} x dF_i(x) \right)$$

the decision-maker is exactly indifferent between opening the box and receiving a certain reward of size $z_i$, which could hence be taken as the value of the unopened box.

Thereafter, the solution to the problem has a simple form:

- **Choice across the closed boxes:** The closed boxes can be ordered by their index values $z_i$. The best box is then the one with the highest index value.

- **Choice across the opened boxes:** The opened boxes can be ordered by the realized rewards $x_i$. The best box is then the one with the highest reward.

- **Choice across the best closed box and the best opened box:** the boxes can be ordered by comparing the highest index value $z_i$, for closed boxes, and the highest reward $x_i$, for opened boxes.

This determines for the decision-maker what to open (the best closed box) and when to stop (when the best opened box is better than the best closed box). For the simplest problems with identical boxes $c_i = c$, $t_i = t$, $F_i = F$ for all $i$, the optimal solution has a threshold structure: stop if $x \geq z$ and continue if $x < z$.

The result has been in extensive use since its discovery and reappears also here.

**Search with endogenous cost: obfuscation and search costs inside a store**

In sequential search setups in the spirit of Weitzman (1979), it is important to specify where exactly the search cost lies - or, what the Pandora boxes stand for. Generally, there could be a friction to transfer from home to a store, from the store to the next one, and back home again (a box stands for a store) and frictions to navigate in a given store (a box stands for an item in a store). Both might have significant effects on search and prices. As a matter of history, price and search theory has, nevertheless, traditionally concentrated on the former case and only recently started to analyze the latter one.

In addition, while the literature has typically regarded the former kinds of costs mostly as exogenous as in Stahl (1989), the latter has been treated as endogenous from the very beginning. For instance, the seminal article by Ellison and Wolitzky (2012), that marks the birth of the so called obfuscation literature to be discussed right below, decomposes search frictions into two parts: in their model there is an exogenous time cost to travel to a store and an endogenous time cost to find the price in the store. In general, there could exist of course more than just these two possibilities (see Figure 1.1).
Before obfuscation literature gained popularity, the usual way to model sequential price search in a homogenous goods market was founded on Stahl (1989). In that seminal paper there is some fixed cost to reach a store, and this is then also the cost of discovering the price in the store. In other words, it is implicitly or explicitly assumed that once the buyer is in the store it is easy to find the price quote: it is either though to be immediate and costless or the idea is that the cost may be regarded as negligible in comparison to the much larger travel cost.

It is understandable that this might have seemed to be in accordance with experience regarding consumers’ usual shopping patterns in the past when search involved physically walking or driving to a store. However, today when online search is frequent, the situation is typically the opposite: the click paths from a search engine to a store may not be very long but it might take quite much clicking, scrolling and eying through the listings to gather, say, all the information necessary to calculate the total price. Indeed, the magnitude of frictions within the stores relative to those across the stores appears to be so much larger online than offline that in applications to the Internet it might no longer be warranted to ignore all the in-store costs.

These ideas are related to the expanding body of work analyzing endogenous frictions and, in particular, an individual seller’s incentive to increase the cost of search for the buyers. After the widely quoted papers by Ellison and Ellison (2009) and Ellison and Wolitzky (2012) were published this literature got associated with the term obfuscation, referring generally to the multitude of possible ways in which the sellers can make shopping time consuming, relevant price information hard to come by, or the properties of different products difficult to compare.

In an econometric contribution, Ellison and Ellison (2009) provide convincing evidence of obfuscation among a group of Internet retailers selling memory modules, differentiated by the quality of the product and contract terms, in an environment where a price search engine is the predominant channel of demand. As the price elasticities in this market are quite large, about -20, the stores have obviously strong incentives to come up with methods to curb down the price competition. The authors document various practices, at least, seemingly designed to make comparing prices more difficult, ranging from making the product descriptions complicated or creating multiple versions of a product to using a cheap low quality product to draw the consumers out of the search engine context to offer...
them a more expensive, higher quality upgrade in the firm’s own store. Based on their estimates, these kinds of obfuscation strategies are apparently quite successful indeed. Despite the very high elasticities, the markups are still about 12%.

In a complementing theory paper, Ellison and Wolitzky (2012) develop several models where firms have an incentive to hinder consumer search by elevating the costs of acquiring an additional price quote from another store. One model is based on the convexity of search cost in search time – say, a higher marginal return to leisure – whereby, if the start store can delay the search long enough, it can make the second search too costly. In their other model, consumers have imperfect knowledge of the cost of getting a price quote and, as they have to base their expectations to their past experiences, they become less willing to search if the cost is high in the first store because they then presume that it is high everywhere. In addition to these two widely known papers, there is by now a large number of other papers analyzing similar research problems of which very good examples would be, say, the papers by Ireland (2007), Wilson (2010), and Petrikaite (2012). We discuss this more in Chapters 3 and 4, that deal with in-store frictions.

A noteworthy comparison to obfuscation literature is advertizing literature (see Bagwell (2007) for a review) where, instead of making it costly for buyers to find additional information, sellers try to reduce these search costs. In practice, it appears safe to assume that firms use a mixture of retailing tricks: some aimed at herding in new consumers (“advertizing”), others to holding up old consumers (“obfuscation”).

1.2 Markets with quality uncertainty

Chapter 2 extends the setup for a new ingredient: asymmetric information, which is generally regarded as one of the main impediments to attaining efficient allocations. We analyze a large search market where different qualities are sold all together over time and, initially, only the seller is aware of the quality of his product. This general setup could be related to applications in markets for various different kinds of assets and durables – labor, houses, mates, consumer goods etc.

Classic papers on asymmetric information such as most notably the seminal article by Akerlof (1970) consider a static Walrasian market in which the players’ trading opportunities are extremely restricted as for when, with whom, and for what terms to trade and, specifically, in terms of revelation mechanisms which might be available in reality. We consider a richer market. To put this into a context, we proceed by summarizing some of the insights developed by the literature.
1.2.1 Asymmetric information in static Walrasian market

The literature begins with the classic sorting problem by Akerlof (1970)\(^8\) that is set up in the static Walrasian market with the requirement of immediate adjustment and without any strategic possibilities. One price should clear the market and buyers and sellers are simply price takers.

There is a number of buyers and sellers each interested in buying or selling a good. The problem is just that different qualities are sold in the same market and, while the seller does know the quality he is selling, it is not possible for the buyers to tell them apart before buying.

If there are sufficiently many high quality sellers, this asymmetric information need not cause a market failure though. When the the expected gain from purchasing a good from a random seller \(E(u)\) is above the high quality sellers’ reservation value \(c_h\), there actually exists a continuum of prices \(p \in [c_h, E(u)]\) that all the buyers and all the sellers are willing to accept.\(^9\) However, in the opposite case, \(c_h > E(u)\), the lowest price the high quality sellers are willing to accept, \(c_h\), is higher than the highest price the buyers are willing to pay for unknown quality, \(E(u)\). When that is so, it is impossible to trade in different qualities for the same Walrasian market price.

As an immediate corollary, this implies that when average quality in the market is low different qualities have to be traded for different terms of trade if at all and, hence, that some sort of information revelation has to occur before trade. Otherwise, it is impossible to trade in goods of high quality. We thus say that a “lemons market” arises.

The insight has spanned an extensive literature analyzing different sorting mechanisms which support trade also in high quality goods at least occasionally. Full efficiency is generally not attainable since these mechanisms come with a signaling cost or information rent to make the low quality sellers to opt out of the high quality sellers’ contract.

In some cases, it is possible to get quite close to full efficiency, however. Let’s take a look.

1.2.2 Revelation mechanisms in dynamic search markets

In the frictionless static Walrasian market, the only way to reduce excess demand or excess supply is to change the price. This implies that, if the average quality is low, high quality will remain unsold – period. A natural follow-up question would however seem to be: then what? Suppose the low quality goods have been sold and only the high remain. What should be done with them? Can the high quality sellers and the buyers abstain from trading now that it is known that all the low quality goods have been sold (is is rational

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\(^8\)One should also mention the seminal papers by Spence (1973) and Rothschild and Stiglitz (1976).

\(^9\)Observe that this may involve a subsidy \(p - u_l\) to low quality sellers.
not to trade in high quality \textit{ex post})? And what if they cannot (is it rational to trade in low quality \textit{ex ante})?

A very nice model that furthers this argument is provided by Bilancini and Boncinelli (2013), where the knowledge of supply (how many goods are sold? how many are left?) is allowed to play a role: The goods can be traded in two rounds. The idea is that the low quality goods are sold on the first one, for a lower price, and the high quality goods on the second one, for a higher price. This is possible without any trading frictions because in their model it is assumed that supply is common knowledge and what triggers the change from round one to round two is the event of reduction in supply. The high quality sellers pass the first round because the price is too low and the low quality sellers participate only in the first round since, otherwise, the second one never comes.

This is a very nice special way to separate the qualities. Generally, when such a round change triggering mechanism is not at work, however, similar-looking separation mechanisms with a tradeoff between price and liquidity have been described, for example, in dynamic search markets. That represents a rich environment to work with compared to the static Walrasian market where price is the only margin of adjustment. In dynamic search markets it is possible to have adjustment both in the intensive margin (price or the terms of trade) and in the extensive margin (liquidity, congestion, or the probability of trade). This has been applied in various different models in order to construct numerous different mechanisms to tackle the classic problem of adverse selection.

We review some of this work that goes under the heading of dynamic trading with common transaction values next. A full summary is not in our scope. We first take a look at models where the separation of different qualities is based on some sort of frictions and then cases where some additional information is provided to support trading.

Before setting off, we think it deserves to be remarked that, generally, the problem can be seen as that of competing mechanisms (see, e.g., Biais et al. (2000), McAfee (1993), Attar et al. (2011), Epstein and Peters (1999) for the literature), which bridges the gap between the mechanism design theory with one principal, one auctioneer, or one seller and one buyer engaged in a bilateral trading relation and the general equilibrium analysis of a market with many parties with private information. Usually, due to the complexity of the problem, quite heavy assumptions must be made.

Search theory is one of the main approaches to these kinds of problems as, by allowing for some sort of frictions in the meeting process between traders, it makes it possible to shift the focus out of the analysis of the entire set of market participants at once to that of smaller groups of matched buyers and sellers – with the bonus of explicit analysis of the strategic interaction between these players. Obviously, the way the frictions are introduced adds some restrictions of its own.\textsuperscript{10} This manifests also in the plethora of

\textsuperscript{10}See Garrett et al. (2014), for a general approach to model differences in consumers’ information sets in a competing mechanisms setting, and Eeckhout and Kircher (2010) for the effects of the available matching
different modeling traditions followed in the literature.

**Temporary separation mechanisms and the role of liquidity for durables**

Inderst and Muller (2002) were among the first ones to note that liquidity can act as a natural separation mechanism in markets for durables. The sellers of different qualities could sort themselves into different submarkets: the sellers of low quality to a market with lower prices and the sellers of high quality to a market with higher prices but more congestion and thus a longer circulation period. Surprisingly, they discover that welfare can be the same for the case of complete information and private information, in spite of the distorted liquidity for the latter. This is next to miraculous.

Interestingly, Inderst (2002) points out that the set of equilibria will converge to the least costly separating equilibrium (that is, the Riley equilibrium or the Rothschild-Stiglitz outcome) when analyzing in a game embedded in search market where the sender offers the contract to the receiver (which corresponds to cases where the seller makes the offer and search is random).

Inderst (2005) reiterates the idea that the least costly separating equilibrium is necessarily supported as an equilibrium when frictions approach zero. They consider a game that is embedded in a search market model where the principal offers the contract to the agent or the other way (which corresponds to cases where the buyer or the seller makes the offer and search is random).

Wambach and Inderst (2002) extend this further for the case of limited capacity in a competitive search environment. Guerrieri et al. (2010) present another such case where they find that equilibrium payoffs are unique but the equilibrium is generally not efficient. All these equilibria could be regarded as solutions to the Rothschild and Stiglitz (1976) nonexistence problem.\(^{11}\)

The particularities of the contracting environment can obviously have a strong effect on the outcome, yet, a general bargaining setting with asymmetric information is typically open to many equilibria. Although the multiplicity might be taken as an inherent feature of a market, some view this lack of unique prediction as disturbing. Various restrictions have thus been considered.

For instance, in Blouin (2003) the buyers and sellers play a bi-matrix game where their selected bargaining positions map into a given price. He finds that every unit is traded over time and every agent receives a positive value in expectation. This stands in stark contrast with the outcome in the static Walrasian market. Yet, as the frictions get smaller and smaller, Cho and Matsui (2013) show that, in a general random matching setup between

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\(^{11}\)In Wilson (1977) a pooling equilibrium survives if, after a deviation, the principals can withdraw contracts. In Riley (1979) a least costly separating equilibrium is sustained if, after a deviation, the principals can add new contracts.
privately informed buyers and sellers who form long term relationships that resolve from an idiosyncratic shock or at will, any stationary equilibrium where participants sometimes trade converges to that of the static Walrasian market.

Janssen and Roy (2002) analyze temporary sorting of different qualities: low quality is traded sooner than high quality such that average quality of the remaining goods is increasing over time. They show that everything is traded in finite time but there could be intermediate breaks of no trade. Janssen and Roy (2004) consider a fixed-entry model and show that, in addition to the unique stationary equilibrium – a repetition of the static Walrasian market, there exists a cyclical equilibrium with regular fluctuations. They also demonstrate that in any non-stationary equilibrium the marginal quality sold is non-monotonic over time as otherwise the buyers would ultimately face no uncertainty. Interestingly, the range of qualities traded in these cases is strictly wider than that in the static market.

Moreno and Wooders (2002) consider a market where trade is bilateral and prices are bargained by the buyer and the seller. They proof that, although there is delay in trade as the sellers try to distinguish between different buyers, prices are asymptotically competitive and the inefficiency vanishes with frictions – in spite of the persistent delay. The surplus realized in a decentralized market can be larger than that in a centralized market but the payoffs approach the competitive ones as the frictions get washed out. Frictions could be welfare improving. Buyer mix between lower prices (accepted by all) and higher prices (accepted by the low) such that low quality sellers trade faster and average market quality increases. This quality boost relaxes the adverse selection problems in a natural way.

The model by Moreno and Wooders (2010) is elegant and particularly easy to work with. We use it as the basic building block for our model in Chapter 2.

**Practical separation mechanisms and the role of additional information**

Temporary separation is costly, however. The delay in trade necessary to achieve sorting limits the gains from trading in high quality, in particular. If the gains from high quality trade are much larger than the gains from low quality trade, the realized surplus could be quite low. Moreover, if one thinks of any canonical market with adverse selection problems such as the labor market, it just does not ring true that high quality workers would have to idle long in the market to find work; indeed, that seems like a very inefficient revelation mechanism. Instead, there are interviews, internships, CV’s, etc. In many commonplace applications, buyers do obtain additional information that they can use to update their beliefs about the sellers. For example, Hendel and Lizzeri (2002, 1999) and Hendel et al. (2005) present several concrete revelation mechanisms that can help to mitigate the problems of adverse selection in practice: warranties, sorting by vintage, leasing etc. Other public or private signals could also be available in the market.
Daley and Green (2011) consider dynamic trading of an asset in a market with public news modeled by the Brownian diffusion process with some given drift, a function of the unknown asset quality. They observe that there could be periods of market freeze or no trade, that come to the end if there are enough good news to revive the market or enough bad news to induce a fire sale of some low quality assets. Better news quality does not necessarily improve welfare. The described equilibrium structure is the same whether the average market quality is above or below the high cost. In a sense, an endogenous market for lemons is thus generated. Daley and Green (2014) have a static model where the sellers face a tradeoff of relying on a costly signaling action (such as education) and a random free-of-cost signal (such as a performance test) that comes after. They find that separating equilibria do not survive a stability refinement; depending on weather the type distribution places enough weight on the high type, there is at least partial pooling.

The role of additional information provision and signaling is considered also in Taylor (1999), Hörner and Vieille (2009), Voorneveld and Weibull (2011) and Kaya and Kim (2013) to name just a few. Taylor (1999) considers a setup where buyers update their beliefs about house quality from the amount of time it spends on the market. He finds that high quality sellers are better off if the consumers’ inspection histories are public. Hörner and Vieille (2009) study observability in bargaining with correlated values. Buyers submit offers to a seller one by one. If earlier offers are unobservable bargaining is likely to result in an impasse; otherwise, usually agreement is ultimately reached but with delay. Voorneveld and Weibull (2011) considers a signaling game with a single seller and buyers who obtain a noisy quality signal; this is a game with two-sided asymmetric information: only the seller knows his quality and only the buyer knows her signal. Kaya and Kim (2013) consider trading dynamics with a single seller and a sequence of buyers who receive a signal of quality and make the seller a price offer; this is a screening game with two-sided asymmetric information.

Before we move into the analysis, we next give a brief summary of each paper.

1.3 Contributions

1.3.1 Dynamic trading with correlated information

Though a useful abstraction, information on common transaction values remains rarely purely private all the way up to the moment when the terms of trade are negotiated. In particular, when trading partners actually meet face-to-face, some information is necessarily transmitted under both traders’ eyes. This gives them a piece of correlated information, that could take their prospects of reaching an agreement either up or down.

We study the classic problem of dynamic trading with asymmetric information (e.g.,
Moreno and Wooders (2002, 2010); Inderst and Muller (2002); Inderst (2002, 2005); Wambach and Inderst (2002); Janssen and Roy (2002, 2004); Blouin (2003); Guerrieri et al. (2010) and Cho and Matsui (2013)). Buyers and sellers meet randomly and pairwise and, initially, only the seller knows the quality of his product. To explore a natural implication of the idea that the trading process is decentralized and trade truly takes place in small bilateral meetings within a larger market, we let matched trading partners share additional information prior to trade. First, to endow the seller with more bargaining power than usual in the literature, we let the (informed) seller offer the price to the (uninformed) buyer. That makes our model a signaling game. Second, to capture the idea that the buyer might be allowed to try the seller’s good under his watch, we let each pair get a shared signal of the seller’s quality. It is observed before the seller’s price offer is made. The price can hence be conditional on the signal. This additional information local in the sense that it is observed by the buyer and the seller but not by the others in the market as a whole. Its overall effects can still be far-reaching.

Surprisingly, we find that asymmetric information and the communication opportunities, added to alleviate the asymmetry, create a market that is necessarily inefficient. This is so even when the average quality is high in the larger market. An endogenous market for lemons arises because the traders have no incentive to trade if the shared signal is low. As the buyers’ beliefs go down and the terms of trade get worse, they are better off if they wait for a higher signal.

Since the seller makes the price offer conditional on the shared signal, our model makes it possible also to take a look at when a seller would prefer to rely in pricing on this costless signal (pool to as high a price as the buyer is willing to pay after seeing the signal) or whether to resort to costly signaling instead. The higher the signal, the higher the maximal price offer the buyers accept without any further revelation. We find that all sellers gain if they coordinate to an equilibrium where they refrain from full separation with a high signal and use it only if they happen to get a low signal.

Interestingly, there exist stationary Markovian equilibria where all sellers simply return to the market if the signal is low enough. This entails that, as a novelty, high quality is traded faster than low quality: lower signals came more often from low quality sellers. The result seems natural in markets with adverse selection issues, in markets for labor, houses or other assets and durables. However, to our current reading, only papers in which trading history or non-stationarities play a role (e.g., Taylor (1999); Vettas (1997); Kaya and Kim (2013)) have so far reached comparable results.

All in all, our paper finds that the signals can be a two-edged-sword. Most of the time they take the parties believes’ about the quality closer together. This eases the sorting problem. However, because the signals are noisy, they can take the parties’ beliefs quite far apart as well. When this disagreement is strong, the lemons problem shows up again.
1.3.2 Obfuscation by substitutes: Shopping frictions and equilibrium price dispersion within stores

The Internet is full of different online stores and almost all offer a lot of alternatives for exploration. Click on one of these, and be flooded by a visual stream of endless products where a lower price or a better matched product is always, seemingly, just a click away. Indeed, there could be so much to see at a single seller nowadays that, once you are done, there is not much time left for shopping in any other store. Déjà vu?

We show in this paper that the variability of alternatives within stores can be applied to amplify the existing search frictions and create new barriers to switching in an environment where none exist initially. This works even for simple price search. We find that sellers have an incentive to generate price variation across identical products in their store to keep the buyers searching longer in there; this leaves them less time for shopping in other competing stores.

Our paper hence contributes to literature analyzing retailer strategies to lock-in consumers (e.g., Ellison and Wolitzky (2012) on price obfuscation and Klemperer (1987) on switching costs) and to literature trying to explain price dispersion across homogenous goods (see Baye et al. (2006), Burdett and Judd (1983) and Butters (1977)). Yet, while the latter strand of literature has concentrated on price dispersion across stores we find it also within stores.

We consider a duopoly with two similar sellers and a unit mass of buyers. All items are of the same given type but a seller could carry them in multiple replicas and set a different price quote for each. In the base line case, both sellers have exactly two items in stock. This number is common knowledge but the prices are the sellers’ private information until the buyers find them.

We use a new dynamic model which abstracts from the hold-up problem present in many optimal stopping problems with endogenous price distribution (Diamond, 1971). Instead, we build on two novel features. First, the buyers search with a deadline. Second, the prices in the stores are not found immediately once a buyer enters a store but randomly and gradually one-by-one.

The buyers can switch the stores freely as long as they have time. There is no explicit switching cost. However, when there are different prices available in a store, we show that the buyers optimally switch the stores only when they have discovered the lowest one. We concentrate on a set of collusive equilibria where the sellers fix one of the two prices at a higher monopoly level but, for a probability strictly between zero and one, let their other price be a lower discount price. Since the buyers know this, if they first spot a monopoly price, they have an incentive to keep on searching in their start store in hope of finding another price at a discount.

This lock-in effect, that lengthens a consumer’s search time within a store, reduces the
sellers’ incentive to undercut each other’s prices compared to the case where both sellers have one item; for that case, there exists a unique mixed equilibrium à la Varian (1980) and Stahl (1989) where stores almost never use the monopoly price. Price variation helps the sellers also to discriminate better between buyers who end with more and less price information. Additionally, we show that, as the number of these similar items in stock expands, the sellers can extract more surplus. The limit equilibrium can look like the Diamond (1971) outcome.

1.3.3 Splitting consumers: Equilibria with endogenous shopping frictions

Managing the traffic of incoming and outgoing consumers is an important part of running an online store. As consumers are typically busy, it is not irrelevant in which order they sample the stores and how long a time they tend to stay in. The stores can affect this consumer turnover in many ways, particularly, figuratively, by putting some sand or oil in the wheels in terms of how the products are presented; the click paths could be made either long or short, for example.

We study the effects and origins of search frictions in a duopolistic price competition model featuring endogenous frictions, inspired by online search. To abstract from hold-up problems arising in sequential search setups with upfront payment of the search cost (Diamond, 1971), we use a model based on deadlines and gradual arrival of price information in every store; there is no explicit cost of searching nor switching.

This modification of the standard framework (Varian, 1980; Stahl, 1989) makes it possible to capture endogenous frictions in a new reduced way: we allow sellers to adjust the rates of the Poisson process that determine how fast a buyer finds a price in a store. These frictions affect both the number of trades and the shares of informed consumers and uninformed consumers in the market, appearing in the classic papers by (Varian, 1980; Stahl, 1989) that our model nests. This in turn enables us to put a number on the size of the loss generated by the frictions and comment on where the market is likely to stand in between the Diamond (1971) and Bertrand (1883) outcomes.

Specifically, we find that there exist precisely two pure equilibria. In both of them, one of the sellers – called prominent seller – has lower frictions and higher prices and the other seller has higher frictions and lower prices. Although the sellers compete in frictions, both generate positive frictions. This implies that some buyers always fail to find a price; under the Poisson process, this surplus loss amounts to 6%.

Interestingly, using the jargon from Stahl (1989), we also find that there are exactly equally many ”shoppers” (with two price quotes) and ”searchers” (with one price quote).

12 Where the sellers get all the surplus, MR=MC.
13 Where the buyers get all the surplus, p=MC.
This is a rather remarkable result because, arguably, our equilibria lie therefore precisely in between the Diamond and Bertrand outcomes. While it is well known that models like this where the buyers are divided into ”informed consumers” and ”uninformed consumers” span both outcomes for appropriate parametric assumptions (Varian (1980) and Stahl (1989)), despite the obvious interest in this division that dictates how competitive the market is, not much has been said about the actual shares before this.

It is noteworthy that both sellers have a strategic incentive to generate frictions, which does not arise, say, from a cost saving motive. Moreover, the universal incentive to generate frictions for sellers and the half-and-half division of consumers into to the informed and the uninformed, only depend on the existence of the deadline but not on what it is.

Our results are relevant especially for online search, where the greatest frictions are not exogenous (limited by the speed at which computers process information) but endogenous (limited by the speed at which consumers process information). They suggest that, though base line search technology is constantly improving, frictions never disappear.
Bibliography


Essays on Market Dynamics and Frictions


Chapter 2

Dynamic trading with correlated information
List of symbols

$\theta = h, l$  
seller index / seller’s quality

$b$  
buyer index

$t \in \mathbb{Z}$  
time index

$\delta \in [0, 1]$  
common discount factor

$c_\theta$  
seller’s valuation of quality $\theta$

$u_\theta$  
buyer’s valuation of quality $\theta$

$\lambda = u_l$  
the gains from trade for $\theta = l$

$1 - \lambda = 1 - c_h$  
the gains from trade for $\theta = h$

$\epsilon_\theta$  
the fraction of type $\theta$ sellers entering to the market

$\epsilon = (\epsilon_h, \epsilon_l)$  
sellers’ entry shares

$\tau_\theta$  
the probability of trade for type $\theta$ sellers

$\tau = (\tau_h, \tau_l)$  
sellers’ trading rates

$\gamma_\theta$  
the fraction of type $\theta$ sellers remaining in the market

$\gamma = (\gamma_h, \gamma_l)$  
sellers’ market shares

$s \in [0, 1]$  
shared signal

$F_\theta(s) \in [0, 1]$  
the distribution of shared signal for $\theta$

$p_\theta(s) \in [0, 1]$  
seller’s price offer

$a(p, s) \in [0, 1]$  
buyer’s acceptance probability of $p$

$q \in [0, 1]$  
seller’s quit rate

$E_\gamma(u|s, p)$  
buyer’s expected utility for $\gamma$, $s$, and $p$

$\sigma = (\sigma_b, \sigma_h, \sigma_l)$  
strategy profile

$V = (V_b, V_h, V_l)$  
market value profile
Dynamic trading with correlated information

Abstract

We investigate welfare and equilibrium trading in a decentralized search market with asymmetric information and bilateral communication opportunities. Sellers and buyers meet randomly and pairwise and view a shared signal of the seller’s quality. In the following signaling game, the sellers can either rely on this costless signal (pool) or costly signaling (separate). We observe that, although the average market quality is high, additional information is not generally welfare improving. All equilibria are inefficient. Contrary to the usual tradeoff between price and liquidity, we find that the signals can help sustaining stationary Markovian equilibria where higher quality is traded faster.

Keywords: Dynamic trading; Search; Asymmetric information; Learning; Signaling. JEL-codes: D82, D83.
2.1 Introduction

[In dynamic markets], the inefficiencies caused by asymmetric information manifest in the fact that sellers of higher qualities need to wait longer than sellers of lower quality in order to sell. The cost of waiting is an important factor that must be considered in any assessment of the loss in welfare caused by adverse selection.

[Nevertheless], observed market dynamics may differ considerably from our predictions. This is because in the real world, the price mechanism is augmented by other non-market institutions and technologies that enable agents to signal or screen information and they alter the behavior of agents as well as the pattern of trade.

Janssen and Roy (2004, pp. 567-568)

Though a useful abstraction, information on common transaction values remains rarely purely private all the way up to the moment when the terms of trade are negotiated. In particular, when trading partners actually meet face-to-face, some information is necessarily transmitted under both traders' eyes. This gives them a piece of correlated information, that could take their prospects of reaching an agreement either up or down.

We study the classic problem of dynamic trading with asymmetric information (e.g., Moreno and Wooders (2002, 2010); Inderst and Muller (2002); Inderst (2002, 2005); Wambach and Inderst (2002); Janssen and Roy (2002, 2004); Blouin (2003); Guerrieri et al. (2010) and Cho and Matsui (2013)). Buyers and sellers meet randomly and pairwise and, initially, only the sellers know the quality of the good they have. To analyze some natural consequences of decentralized trade, we let trading partners exchange additional bilateral information.

This information is observable by them but not by the others in the market at large. First, to endow the seller with more bargaining power than usual in the literature, we let the (informed) seller offer the price to the (uninformed) buyer. This makes our model a signaling game. Second, to capture the idea that the buyer might have an opportunity to see or try the seller’s good under his watch, we let each pair get a shared signal of the seller’s quality. It is viewed before the seller announces the price, which can hence be conditioned on the signal.

This constitutes a setup where it is possible to consider how the provision of additional correlated information affects dynamic trading in markets with quality uncertainty. Surprisingly, we find that asymmetric information and the communication opportunities, added to relax it, create a market that is necessarily inefficient, even when the average quality is high in the larger market. An endogenous market for lemons will arise because
the parties have no incentive to trade if the shared signal is low. They are better off if they resume their search instead and wait for a chance to trade under a higher signal.

Since the seller makes the price offer conditional on the signal, our model makes it possible also to take a look at when a seller would prefer to rely on this costless signal (pool to as high a price as the buyer is willing to pay after seeing the signal) or whether to resort to costly separation.¹ The higher the signal, the higher the maximal price offer the buyers are willing to accept without further revelation (if they think that both sellers offer that price after that signal). In particular, we find that all sellers gain if they refrain from full separation when the shared signal is high and use it for low signals only.

There exist many equilibria in this kind of game with signaling. We first characterize the set of Perfect Bayesian equilibria² in "stage games", played in the bilateral meetings, and then zoom in on some noteworthy equilibrium classes of the full game – seller maximal equilibria in particular. In "stage games", the sellers’ outside options are fixed. In the full game, the outside options are determined in equilibrium.

Interestingly, there exist also stationary Markovian equilibria where all sellers simply return to the market if the signal is low enough. This entails that, as a novelty, high quality is traded faster than low quality because these lower signals, that make the sellers willing to pass an opportunity to trade, came more often from low quality sellers than from high quality sellers.

The finding appears natural to us. However, to our current reading, only papers in which trading history or non-stationarities play a role (e.g., Taylor (1999); Vettas (1997); Kaya and Kim (2013)) have reached comparable results; mostly the literature has focused on the tradeoff between liquidity and price. In that standard case, higher quality is traded more slowly than lower quality but for higher prices.

We feel that our result pushes the literature to the right direction since it is obviously not the case that low quality is always more liquid than high quality, for example, in canonical adverse selection markets such as that for houses and used cars or the market for CEOs, GPs, APs, etc. An especially noteworthy feature of these markets is that the asymmetric information gets transmitted from sellers to buyers in many ways prior to trade and the gains from trade in high quality might be much larger than those in low quality. The same properties make the difference also in this paper.

We think these effects of correlated information on dynamic trading are unexpected. The signals can be a two-edged-sword. Most of the time they take the parties beliefs’ about the quality closer together. This is nice because it makes the sorting problem easier for the buyers and enables the high quality sellers to extract some returns to quality. This part is intuitive.

¹See Daley and Green (2014) for another paper with this feature. Again, our innovation is to condition actions on signals and embed the interactions into a dynamic search market.

²To make sure we cover all the equilibria, we consider maximal punishments off the path. All the equilibria are consistent with the intuitive criterion (Cho and Kreps, 1987).
Nevertheless, because the signals are noisy, sometimes they also take the parties’ beliefs quite far apart. From time to time, a high quality seller will give a very bad impression or a low quality seller a very good impression to a buyer. If this disagreement among a buyer and a high quality seller is strong, it might be best for both to split up since they cannot agree on the gains or the terms of trade (this basically gives us Proposition 1: the non-existence of an equilibrium where every meeting results in trade). Note however that the low quality sellers can also benefit from inflated beliefs to the extent that they might opportunistically refuse to trade under accurate beliefs (this basically gives us Proposition 2: the existence of stationary Markovian equilibria where high quality is more liquid).

Therefore, trade will partially or completely break down for the lowest signals. The findings can be traced back to the dispersion in buyers’ beliefs (the noise in the shared signal) and to the elevation of the traders’ outside options (dynamics and the possibility to wait for a better signal) suggesting that asymmetric information and variable buyer sentiment could be quite bad a mix, irrespective of average market quality. This has been observed by Daley and Green (2011) with a non-stationary public news flow. Our contributions show that similar issues hinder trading also in the microscopic anonymous encounters between trading partners in a large stationary market. The key ingredients are (i) dynamics and (ii) the partial revelation of quality uncertainty – public or private.

All in all, our findings thus highlight that the additional information is not generally welfare improving. All equilibria are inefficient once we add the shared signal. This is quite surprising given that we specialize in cases where the average quality is high. In other words, the static Walrasian market could still be efficient for the parameter values on which we focus.

Note that the presence of the shared signal enriches the dynamic trading patterns that can be supported in a stationary Markovian equilibrium. The sellers can propose a pooling price for one subset of signals, separate for another subset of signals, and execute their search option for the rest. Without noisy signals, the sellers would have pick just one of the actions or mix.

The underlying idea behind the shared signal is to allow the buyer to ”taste” or ”scent” the seller’s quality under his watch. As a result, the buyer obtains a signal and the seller obtains a signal of the signal. For instance, the seller of an old car would usually learn if the car, say, broke down while the buyer drove it, the job market candidate knows herself whether the job talk went well, and the sellers of a house can sense to a certain extent if the buyers are motivated in the property viewings. For concreteness, it is here assumed that the signal that the buyer sees and the signal that the seller sees are identical.

In a sense, the shared signal acts here as a meeting specific private coordination device, which the buyers cannot simply ignore because it is informative: the higher the signal,

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3See Voorneveld and Weibull (2011) for another paper with this feature. Again, our innovation is to condition actions on signals and embed the interactions into a dynamic search market.
the higher the probability that it comes from a high quality good rather than from a low quality good. Still, it is possible to have also a highly misleading signal, and albeit this is rare, it does pose a threat to high quality sellers (who might be taken for low quality sellers) and provides some benefits to low quality sellers (who might be taken for high quality sellers).

A high enough signal will increase the buyers’ belief above the market average and a low enough signal will decrease it below the market average. The former effect can temporarily boost trade but over time the prospect of shopping for a better signal can make the sellers also more picky; the latter effect obviously slows down trade. Overall, the total value of this additional information could hence be negative.4,5

Our setup is much like in the seminal article by Moreno and Wooders (2010); the key difference is that we have the shared signal and they let the (uninformed) buyer offer the price to the (informed) seller. Since different qualities are traded at their own rates, the average quality of the goods in the market is endogenous and so are, hence, the traders’ outside options if their price negotiations fail. This feature is present also, among others, in Moreno and Wooders (2002), where either party could make the price offer; they have no shared signal.

Specifically, Moreno and Wooders (2010) consider bilateral trading in a large decentralized market where the quality or the product is the seller’s private information all the way until trade takes place. There exists a unique equilibrium where the buyers mix between low prices (accepted by only low quality sellers) and high prices (accepted by both low and high quality sellers). As a result, high quality is traded slower than low quality, the average quality in the market gets higher, and the buyers become more willing to offer high prices. In other words, they find that in the long run the lemons problem is mitigated naturally by itself, through the implied boost in average quality (in a market where average quality is initially low). We find that adding the shared signal gives the inverse result instead: if the buyers and sellers have access to correlated information, the lemons problem arises by itself (in a market where average quality is initially high). Our results complement one another.

Generally, our paper falls within the body of work concerned with welfare and dynamic trading in a search market with uncertainty on common transactions values: Albrecht and Vroman (1992); Inderst and Muller (2002); Janssen and Roy (2002); Moreno and Wooders (2002); Blouin (2003); Janssen and Roy (2004) and Moreno and Wooders (2010) are some

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4The outcome seems natural in the light of the celebrated contribution by Rubinstein and Wolinsky (1990) which shows that the possibility of conditioning on extraneous information might have a strong effect on equilibrium play. Yet, in our case the information is not irrelevant but highly payoff relevant and history has no role to play.

5The outcome is also akin to the so called Hirshleifer effect, which refers to a decrease in overall welfare with more information as opportunities for risk sharing are removed (Hirshleifer, 1971). When the sellers use the optimal sorting method, there is risk sharing (pooling) between sellers of different qualities only for the highest signals.
of the earlier examples whereas Guerrieri et al. (2010) and Cho and Matsui (2013) are more recent ones. Most of this work is focused on finding solutions to the lemons problem, a part of that, specifically, motivated by liquidity problems in financial markets (see Camargo and Lester (2010); Chang (2011) and Guerrieri and Shimer (2012)).

One reoccurring, robust result in the literature is the tradeoff between price level and the probability of fast sale and the idea that it can be applied to support temporary sorting of qualities in dynamic search markets. This is based on a single crossing condition. Since low quality sellers are more willing to forgo a higher price for a higher probability of fast sale than high quality sellers, it is possible to trade the low quality for lower prices and the high quality for higher prices as long as the former are traded faster than the latter.

This represents a significant improvement to the static Walrasian market where only low quality sellers trade. The costs of delay might still be quite large, especially, when the gains from trade are much more pronounced for high quality sellers. Luckily, if the use of more elaborated sorting mechanisms is warranted, the outcome can typically be improved from this. The lemons problem is shown to get alleviated, for instance, by leasing and sorting by vintage (see Hendel and Lizzieri (2002) and Hendel et al. (2005), respectively).

On the other hand, it has also been remarked that additional information revelation could hinder trade. For example, Kultti et al. (2012) find that, with competition between the bidders in an auction, the trade may break down for a wider range of parameters than in a static Walrasian market. In Hörner and Vieille (2009), a public offer history leads to a complete break in trade but a private offer history to a delay in trade only. In Daley and Green (2011), with a public news flow, the market can freeze essentially for any parameters; the confidence is restored only with enough good news.

Lauermann and Wolinsky (2013) analyze information aggregation in a setup where an (informed) buyer meets a series of sellers who get private signals of the cost and the buyer and the seller bargain over prices. While the model appears to be quite close to what we have, the research questions are different, the payoff structure is different, the setup is static as it is presented, and only the buyer views the signal such that it cannot be utilized as a private coordination device.

Kaya and Kim (2013) explore a setup where an (informed) seller meets a series of randomly arriving buyers. The buyers make the seller a price offer after receiving a private quality signal and observing how long the asset has been for sale. They show that either quality could be traded faster, depending the buyers’ prior beliefs. If the prior is high from the start, high quality is more liquid. Our comparable result arises in a

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6 Another closely related body of work deals with information aggregation and convergence to competitive outcomes (see Wolinsky (1990) and Blouin and Serrano (2001)). Compared to this earlier work, our paper has a stronger emphasis on risk sharing and pooling.

7 This impact of trading history on the evolution of buyers’ beliefs and current trading prospects is analyzed also in the setups with a single seller by Taylor (1999); Bar-Isaac (2003); Kaya and Kim (2013); Gerardi and Maestri (2013) and Kremer and Skrzypacz (2007).
stationary equilibrium where trading history plays no role.

Last, outside of the realm of common transaction values, Lauermann (2012) studies a model where buyers have independent private or public valuations. Comparing these extremes of symmetric and asymmetric information, he finds that when bilateral trade is embedded into a well functioning market, asymmetric information may improve welfare. Without consumer privacy, the rents from price discrimination slow down trade because the sellers’ reservation prices get higher.

The paper is organized in the following way: The setup and notation are presented in Section 2.2. We prove the non-existence of efficient equilibria in Section 2.3. In Section 2.4.1, we characterize the full set of equilibria in "stage games" played in the bilateral meetings and, in Section 2.4.2, some particularly interesting equilibria of the full game. Section 2.5 offers some closing remarks. We develop an existence algorithm which stops at a seller maximal equilibrium in Appendix A. Most of the proofs are in Appendix B.

### 2.2 Model

We consider a large market with decentralized, uncoordinated trade. A unit mass of buyers and sellers enters the market over discrete time $t \in \mathbb{Z}$. For every point in time, the buyers and the sellers get matched with a random trader from the other market side. As they meet, the buyer and the seller are first given a shared signal $s \in [0, 1]$ that is informative of the quality of the product $\theta = h, l$ the seller has. Then, the seller makes a price offer, $p \in [0, 1]$, which the buyer either accepts or rejects, $a(p) = 0, 1$. Those who trade exit the market but others return to the pool of buyers and sellers, and get matched with a different buyer or a different seller the next time.

Note that our model comes quite close to that by Moreno and Wooders (2010), except for the shared signals, that we have, and the order of moves that we change from buyers move first to sellers move first, and certain other minor details. They make a difference, nonetheless, as shall be shown in short.

The sellers are endowed with products of different unobservable qualities. Before the shared signal is shown, information is perfectly asymmetric: the seller is fully informed about the quality of his product but the buyer has absolutely no information. Instead, after the signal is viewed by this pair, also the buyer has got somewhat informed. However, as the signals are inexact, the buyer’s belief $E(\theta|s)$ is likely to be inflated or deflated, and, as the signal is shared by the pair, the seller knows exactly what the bias $E(\theta|s) - \theta$ is.

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8 An earlier version of the paper was in continuous time. After a suggestion from a discussant, we decided to present the results in discrete time to get rid of a matching rate parameter and to get a more concrete feel on the whole model; no substantial differences were involved.

9 We refer to Proposition 1 (our finding that efficient equilibria are non existent even when average market quality is high) and Proposition 5 (our finding that high quality is traded faster than low quality in a stationary Markovian equilibrium), which contrast nicely with theirs.
This makes each match special to the involved buyer and seller for the information and the incentives may change from a meeting to a meeting.

In consequence, the set of equilibria that might arise can also vary from a meeting to a meeting as the buyers and the sellers are weighing against one another the gains from trading under the current signal and those from trading under a future signal. Both of them discount expected payoffs by $\delta \in (0,1)$.

We next lay out some details of this game and, then, move on to the buyer’s problem and the seller’s problem. We concentrate on stationary Markovian equilibria in pure or randomized behavioral strategies $\sigma = (\sigma_h, \sigma_l, \sigma_b)$: (i) The sellers’ (mixed) strategies $\sigma_h = p_h(s), \sigma_l = p_l(s) : [0,1] \rightarrow \Delta [0,1]$ attach a distribution of prices $p_h$ and $p_l$ to each signal $s$ for the high quality sellers and the low quality sellers, respectively. (i) The buyers’ (mixed) strategies $\sigma_b = a(s,p) : [0,1]^2 \rightarrow \Delta \{0,1\}$ map an acceptance probability $a$ to each pair $(s,p)$ consisting of a signal and a price.\(^{10}\)

The solution concept we apply is perfect Bayesian equilibrium (PBE). It is a pair $(\sigma, \pi)$ that consists of a strategy profile $\sigma$ and a belief system $\pi$ such that (i) the strategy profile $\sigma$ is consistent with sequential rationality given the belief system $\pi$ and (ii) the belief system $\pi$ is derived from the strategy profile $\sigma$ whenever possible. Recall also that we have a game of signaling: it is the seller who offers the price. All the PBE we consider satisfy the intuitive criterion (Cho and Kreps, 1987).\(^{11}\) To capture the full set, we consider maximal punishments off the equilibrium path.

### Payoffs

A fraction $\epsilon_l \in (0,1)$ of the entering sellers has a low quality product and a fraction $\epsilon_h \in (0,1)$ a high quality product, where $\epsilon := (\epsilon_l, \epsilon_h)$ and $\epsilon_l = 1 - \epsilon_h$. The products are different but not perishable nor divisible. Each buyer wants to get one, each seller wants to get rid of one.

The buyers and sellers have quasilinear preferences in money such that, if a buyer and a seller trade for price $p$, the buyer gets $u_\theta - p$ and the seller gets $p - c_\theta$. The buyer values, $u_l, u_h$ (utils), and the seller values, $c_l, c_h$ (costs), depend on whether the quality $\theta$ is low, $\theta = l$, or high, $\theta = h$.

Since different qualities may not trade at equal rates, the total market surplus is likely to depend on the relative gains from trade in high quality and in low quality. To capture those gains by a single parameter $\lambda$, we make the following assumption:

**Assumption 1** $\ 0 = c_l < u_l = \lambda = c_h < u_h = 1$.

This implies that the gains from trade in high quality are $1 - \lambda = u_h - c_h > 0$ and the gains from trade in low quality are $\lambda = u_l - c_l > 0$. Thus, the average quality is always so

\(^{10}\)Examples of non-stationary or non-Markovian equilibria are available by request.

\(^{11}\)Intuitive criterion (Cho and Kreps, 1987) is a standard refinement for these games.
high that it would be possible to sustain efficient trading in a static Walrasian market:\footnote{Most other papers start by the assumptions that $c_l < u_l < c_h$ and the values of $\gamma \in [0, 1]$ such that $E_\gamma(u) < c_h$. This implies that the static Walrasian market always fails. In this paper, we restrict our attention to parameters for which $E_\gamma(u) \geq c_h$, for any $\gamma \in [0, 1]$, such that the static Walrasian market need not fail. This choice allows us to parametrize the buyers and the sellers’ values in a parsimonious way and focus on liquidity problems arising, in particular, in dynamic markets with common values uncertainty and bilateral communication opportunities prior to trade. Some of our results go through also with more general payoffs.}

$$E_\gamma(u) := \gamma u_l + (1 - \gamma) u_h \geq c_h,$$

for any $\gamma \in [0, 1],$

where $\gamma (1 - \gamma)$ refers to the share of high quality sellers (low quality sellers) in the market.

### Signals

To recap, the extensive structure of a meetings is:

1. A shared signal, $s \sim F_\theta$, is drawn and shown to the buyer and the seller.
2. The seller makes the buyer a price offer, $p \in [0, 1]$.
3. The buyer either accepts the offer, $a(p) = 1$, or rejects the offer, $a(p) = 0$.

The signals $s$ are drawn independently across the meetings, according to continuous distribution functions $F_h(s), F_l(s) : [0, 1] \rightarrow [0, 1]$, supported on $[0, 1] = \text{cl}\{s | f_h(s) > 0\}$, where $f_h$ and $f_l$ are densities; $\mathbf{F} := (F_h, F_l)$. It is assumed that a higher signal is indicative of a higher quality and that extreme signals are perfectly revealing:

Assumption 2

$$\frac{\partial}{\partial s} \frac{f_h(s)}{f_l(s)} \geq 0$$

for all $s$.

$$\lim_{s \rightarrow 0} \frac{f_h(s)}{f_l(s)} = 0,$$

and

$$\lim_{s \rightarrow 1} \frac{f_h(s)}{f_l(s)} = \infty.$$

By the first part of this assumption, the signals $s$ satisfy the monotone likelihood ratio property (MLRP). By the second part of this assumption and by continuity, any positive likelihood ratio is attainable under an appropriate signal $s \in (0, 1)$.

Observe that both the shared signal about quality $s$ and the price $p$ may affect the buyer’s belief about the seller’s quality, $E(u|s, p)$, and, thus, whether the price offer is

\footnote{The closure of a set $A$ contains all the points $a$ whose every neighborhood $B(a)$ intersects with the set $A$: $\text{cl}(A) = \{a | \exists B(a) : A \cap B(a) \neq \emptyset\}$ where $B(a)$ is an arbitrary open set such that $a \in B(a)$.}

\footnote{This implies that it is possible for both high and low quality sellers to emit also a highly misleading signal, $E_\gamma(u|s) \approx u_l = \lambda$ for $\theta = h$ and $\gamma \in (0, 1)$ or $E_\gamma(u|s) \approx u_h = 1$ for $\theta = l$ and $\gamma \in (0, 1)$.}
accepted or rejected. In effect, it is possible to have quite much information revelation prior to trade in a model like this: first, the buyer and the seller could use the shared signal so as to coordinate their strategies and make the play of the game conditional on it and, second, the seller could signal his quality by his price offer. Note specifically that, as the signal is informative, the buyer cannot simply ignore it; hence, some equilibria that used to be supportable without the signal might cease to be so.

Additionally, the high quality sellers and the low quality sellers could, with no loss of generality, use a pooling pricing strategy for a subset of signals $S_p$, separating pricing strategies for a subset of signals $S_s$, and mix for the others $[0,1] - S_p - S_s$ for $S_p \cap S_s \neq \emptyset$. Given the usual flexibility with the off path beliefs, this partition is totally arbitrary because too high offers could simply be rejected. Yet, the focus of the paper is mostly on such cases where the sellers use pooling pricing strategies above a cutoff $s'$ and separating strategies below the cutoff $s'$. In other words, $S_p = [0,s')$ and $S_s = (s',1]$. As it turns out, this could also be defended as the seller maximal pricing pattern.

**Average market quality**

The price a buyer is willing to pay for a product depends on (i) the average quality in the market, (ii) the shared signal and (iii) the information that is carried by the price offer. Without further revelation by the shared signal of the price offer, the expected buyer value of a random product equals

$$E_\gamma(u) := \frac{\gamma_h}{\gamma_h + \gamma_l} + \frac{\gamma_l}{\gamma_h + \gamma_l} \lambda,$$

where $\gamma_h$ is the stock of high quality products and $\gamma_l$ is the stock of low quality products in the market. Note that, while the entry to the market is exogenous, i.e., given by the entry flows, $\epsilon = (\epsilon_h, \epsilon_l)$, the exit from the market is endogenous, i.e., given by the probabilities for which different qualities are traded in the bilateral meetings, $\tau = (\tau_h, \tau_l)$. Thus, in a stationary equilibrium where the inflow of each quality matches the outflow of that quality, the stocks are given by

$$\gamma_h = \frac{\epsilon_h}{\tau_h}, \quad (2.1)$$
$$\gamma_l = \frac{\epsilon_l}{\tau_l}, \quad (2.2)$$

By the Bayes’ law, after the shared signal is revealed, the expected buyer value of purchasing from that particular seller is given by

$$E_\gamma(u|s) := \frac{\gamma_h f_h(s)}{\gamma_h f_h(s) + \gamma_l f_l(s)} + \frac{\gamma_l f_l(s)}{\gamma_h f_h(s) + \gamma_l f_l(s)} \lambda,$$
It is noteworthy that both $E_{\gamma}(u|s) > E_{\gamma}(u)$ and $E_{\gamma}(u|s) < E_{\gamma}(u)$ are possible for whatever the average quality in the market.\footnote{This entails that it is possible to use the shared signal to support trade in the lemons case also.} In other words, if the signal is low enough (high enough), the maximum price the buyer is willing to pay, without further revelation, could be lower (higher) than it would have been for average market quality. Nevertheless, if average market quality is low (high), it does take a higher (lower) signal to raise the buyer’s belief to a given level.

**Value of search option**

Once matched with a pair, the buyers and the sellers each solve their respective optimal stopping problem: they could either trade with their current partner and get the related immediate payoff or search for better alternatives. The value of this search option is denoted by $V_b$, for the buyers, and by $V_h, V_l$, for the sellers, and determined in equilibrium.

A buyer, who has been made a price offer $p$, decides whether to accept it or reject it as an optimal solution to

$$V_b(p, s) := \max_{a \in \{0,1\}} a \left( E_{\gamma}(u|p, s) - p \right) + (1 - a) \delta V_b.$$ \hspace{1cm} (2.3)

A seller, who is endowed with a product of quality $\theta = h, l$, chooses the price offer as an optimal solution to

$$V_\theta(s) := \max_{p \in [0,1]} a(p|s) \left( p - c_\theta \right) + (1 - a(p|s)) \delta V_\theta,$$

or, equivalently, to

$$\max_{p \in [0,1]} a(p|s) \left( p - c_\theta - \delta V_\theta \right).$$ \hspace{1cm} (2.4)

Observe that both problems condition on the shared signal because the optimal actions can depend on what is the shared signal.

Note that the buyers are, in essence, sampling the sellers sequentially one by one. They draw new payoffs, $E_{\gamma}(u|p, s) - p$, for each new seller. These payoffs are distributed independently according to a given distribution with no recall option. It is well known that the solution to such an optimal stopping problem is characterized by a cutoff policy so that, if the expected utility net of the price is below the cutoff, the buyers accept the offer but, if the expected this is above the cutoff, the buyers reject the offer. At an optimum, the cutoff is equal to the buyer continuation value, $V_b$. 
Section 2.2

The seller’s problem is instead like that of a monopolist who is facing the demand \( a(p|s) \), i.e., the probability of trade for a given price, and has the cost function as given by \( c_\theta + V_\theta \), i.e., the seller reservation value plus the seller continuation value. Observe, however, that in contrast to the standard monopoly problem, the seller’s problem is not very well behaved in this case in which the price acts as a second signal of quality. In fact, even a slightest deviation from the anticipated price offer can make the buyer extremely suspicious of the quality and thus reject this price offer.\(^{16}\)

Observe next that any stationary equilibrium induces the buyers and the sellers continuation values. In particular, when \( s \mapsto p_h(s) \), \( s \mapsto p_l(s) \) are functions, i.e., when the sellers do not use randomized pricing strategies (as in the semi-pooling equilibria described in Ch. 2.4.1) but, rather, a fixed price for a fixed signal (as in the pooling and in the separating equilibria described in Ch. 2.4.1), market (continuation) values are given by

\[
V_b = \frac{\gamma_h}{\gamma_h + \gamma_l} \int_0^1 V_b(p(s), s) dF_h(s) + \frac{\gamma_l}{\gamma_h + \gamma_l} \int_0^1 V_b(p(s), s) dF_l(s), \tag{2.5}
\]

\[
V_h = \int_0^1 V_h(s) dF_h(s), \tag{2.6}
\]

\[
V_l = \int_0^1 V_l(s) dF_l(s), \tag{2.7}
\]

and the probabilities of trade are given by

\[
\tau_h = \int_0^1 a(p_h(s)|s) dF_h(s),
\]

\[
\tau_l = \int_0^1 a(p_l(s)|s) dF_l(s).
\]

Equations (2.6) and (2.7) can thus be rewritten as

\[
(1 - \delta(1 - \tau_h)) V_h = \int_0^1 a(p_h(s)|s) p_h(s) dF_h(s) - c_h \tau_h,
\]

\[
(1 - \delta(1 - \tau_l)) V_l = \int_0^1 a(p_l(s)|s) p_l(s) dF_l(s) - c_l \tau_l.
\]

As explained, the acceptance probability, \( a(p|s) \), is pinned down by the buyers’ beliefs, \( E_\gamma(u|p, s) - p \), and by their continuation value, \( V_b \). On the equilibrium path, the beliefs are derived directly from the equilibrium strategies (our solution concept is PBE) but, off

\(^{16}\)For instance, under the usual flexibility with beliefs off the equilibrium path in games of signaling, the price elasticity of the demand, \( \frac{\partial a(p|s)}{\partial p} \frac{p}{a(p|s)} \), can get infinite for some \( p \). This suggests that it is, typically, not possible to resort to, say, basic tools of calculus to tackle the seller’s problem.
the equilibrium path, we let the beliefs collapse to as negative as possible (to delineate the 
full set of PBE); the punishments are maximal. Observe that all this is consistent with the 
ituitive criterion (Cho and Kreps, 1987) because, usually, if the low quality sellers gain 
from a deviation when taken for high quality sellers, then also the high quality sellers gain 
from the deviation when taken for high quality sellers, yet, neither quality would gain from 
it if taken for low quality sellers. It is, therefore, very easy to discipline any deviations 
$p(s)$ that sellers would be tempted to make by hard off path beliefs, $E(u|p(s), s) = u_1$, 
without violating the refinement.

Outline of results

There exist a multiplicity of equilibria in this game. However, in Section 2.4.1, we show 
that a seller never mixes between more than two prices. This implies that it is possible to 
associate, with no loss of generality, for any shared signal $s$ a vector of strategies 

$$s \rightarrow (p_h, p_{h,2}, p_l, p_{l,2}, \rho_{h,2}, \rho_{l,2}, a(p_h), a(p_{h,2}), a(p_l), a(p_{l,2}))(s) \in [0, 1]^{10}. $$

where $p_h$ and $p_{h,2}$ are the two prices that high quality sellers may use and $p_l$ and $p_{l,2}$ are 
the two prices that high quality sellers may use and $(a(p_h), a(p_{h,2}), a(p_l), a(p_{l,2}))$ are their 
acceptance rates. When a seller is mixing, $\rho_{h,2}$ is the frequency of using the price $p_{h,2}$ and 
$\rho_{l,2}$ is the frequency of using the price $p_{l,2}$. Note that some of these might be redundant 
or irrelevant in a particular equilibrium. For example:

(i) For pooling strategies, we choose $\rho_{h,2} = \rho_{l,2} = 0$ and we need some $p := p_h = p_l$ 
and $a(p) := a(p_h) = a(p_l)$ such that

$$a(p) = 1, \text{ for } E_{\gamma}(u|s) - p > \delta V_b $$
$$a(p) \in [0, 1], \text{ for } E_{\gamma}(u|s) - p = \delta V_b $$
$$a(p) = 0, \text{ for } E_{\gamma}(u|s) - p < \delta V_b. $$

to make the buyers accept and reject in an optimal way.

(ii) For separating strategies, we choose $\rho_{h,2} = \rho_{l,2} = 0$ and we need some $p_h, p_l$ and 
$a(p_h), a(p_l)$ for which is should hold that

$$p_h = u_h - \delta V_b,$$

to keep the buyers indifferent between accepting and rejecting the higher price, and

$$p_l = u_l - \delta V_b, a(p_h)(p_h - c_l - \delta V_l) \leq a(p_l)(p_l - c_l - \delta V_l), a(p_l) = 1$$

to stop the low quality sellers from deviation to the higher price or to a lower price.
(iii) For strategies in which a seller mixes between using a pooling pricing strategy and separating, we may choose either $\rho_{l,2} = 0$ (only the high mix) and $p_l = p_{h,1} < p_{h,2}$ and, thus, $a(p_l) = a(p_{h,1}) > a(p_{h,2})$ or $\rho_{h,2} = 0$ (only the low mix) and $p_h = p_{l,1} > p_{l,2}$ and, thus, $a(p_h) = a(p_{l,1}) < a(p_{l,2})$. Also, some additional natural constraints must hold for this particular case.

We note that, as long as no additional refinements are introduced, this mapping $\mu$ between the shared signal and the associated strategies, whether related to cases (i), (ii) or (iii), could be chosen freely as long as everybody is given at least his or her continuation value and the low quality sellers are never put to offer a price that gives them less than a separating price offer would. Hence, there could be multiple stationary equilibria.

As our first key contribution in Section 2.3, we find that all equilibria are necessarily inefficient as they involve a delay in trade. This is somewhat unexpected because, for the payoffs we consider, there always exist an equilibrium in a static Walrasian market in which both high and low quality are traded for certain.\(^{17}\) Moreover, if we take away the shared signal, there exists a unique equilibrium where every meeting results in trade and where the products are, therefore, sold once they come to the market; interestingly, this pooling equilibrium will break down by any perturbation in the continuation values.\(^{18}\)

As the second major result, we show that two different dynamic trading patterns might arise: either high quality is sold faster or low quality is sold faster than the other quality. This depends on whether the equilibrium involves separation or not. If it does, the low quality is sold faster than high quality, if it does not, it is the other way.

The literature has so far concentrated on equilibria with separation and, thus, just one dynamic trading pattern. Yet, in Section 2.4.2, we find that the possibility to pool with high quality sellers may raise the low quality sellers continuation values so much that separation becomes infeasible, which leads to the other trading order. We can link this finding with the relative gains from trade and show that, when the gains from trade in low quality dominate, we can expect the low quality to be more liquid whereas, when the gains from trade in high quality dominate, we can expect the high quality to be more liquid.

### 2.3 Efficiency

This section considers the efficiency of dynamic trading with a shared signal and hence the value of this additional information for the traders. Note that, with positive but not necessarily equal-sized gains from trade in low quality and in high quality, $\lambda > 0$ and $1 - \lambda > 0$ respectively, the total surplus depends on the length of time it takes to trade in different qualities. Since the traders are paired each period, the market is efficient if every

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\(^{17}\) The average quality is "high"; we have not got a lemons market for the high cost $c_h = \lambda$ is equal to the low utility $u_l = \lambda$.

\(^{18}\) This result is robust to change in the order of the moves: it arises whether we have a signaling game or a screening game.
match results in trade; inefficiency is manifested in decreased liquidity. The standard welfare measure for stationary equilibria is the weighted sum of the values to a cohort of buyers and sellers who enters the market at any given point in time $t \in \mathbb{Z}^{19}$

$$S := V_b + \epsilon_h V_h + \epsilon_l V_l \leq \epsilon_h (1 - \lambda) + \epsilon_l \lambda.$$ 

Now, one of our most striking findings is the non-existence of efficient equilibria that obtains when the shared signal is added to the game form; otherwise, there does exist a continuum of efficient equilibria for the payoffs we consider. The reason is bifid: First, it is not possible to trade everything at once, as required by efficiency, without sharing any of the surplus with the buyers or with the high quality sellers. In dynamic markets, one of these groups is hence bound to have a positive search option. Second, what the buyers or what the high quality sellers thus make would have to vary according to the shared signal; the lower the signal, the less they can expect to get. For a low enough signal realization, they should expect to make less if they trade than if they execute their positive search option. Consequently, there is no trade for a low enough signal. This is so irrespective of who is making the price offer as long as the buyer and the seller have this piece of correlated information:

**Proposition 1** Consider any stationary or non-stationary\(^{20}\) equilibrium.

1. If $V_h = 0$, then $V_b > 0$. If $V_b = 0$, then $V_h > 0$.

2. Suppose $V_b > 0$. Then, $\exists s' : \forall s < s' : E(u|s) - p < \delta V_b$ even for (the minimal price the high quality sellers can sell for) $p = c_h$ and the buyers cannot trade for these prices.

3. Suppose $V_h > 0$. Then, $\exists s' : \forall s < s' : p - c_h < \delta V_h$ even for (the maximal price the buyer can buy for) $p = E(u|s)$ and the sellers of high quality cannot trade for these prices.

**Corollary 1** Any equilibrium is inefficient.

**Proof.** Note that, after a signal $s \in [0, 1]$ is viewed but without any further revelation as that would cost, the maximal price a buyer is willing to accept is $E_{\gamma}(u|s) - \delta V_b$ (to compensate the buyer for the loss of the search option) and the minimal price a seller of high quality would be willing to offer is $c_h + \delta V_h$ (to compensate the seller for the cost and the loss of the search option). Therefore, to guarantee that there would exist such a price $p(s) \in [c_h + \delta V_h, E_{\gamma}(u|s) - \delta V_b]$, for almost all $s$, even for the case in which $E_{\gamma}(u|s)$ is close to $c_h$ it must be that (i) the search option for the buyers $\delta V_b = 0$ and, thus,

\(^{19}\)Note that, although it is standard and oft-used, the measure ignores the surplus related to the potential transition period needed to reach the stationary equilibrium.

\(^{20}\)For simplicity, we omit the time indexes.
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$p(s) = E_{\gamma}(u|s)$, for almost all $s$, as the use of any price below it would raise $V_b$ over zero and (ii) the search option for the high quality sellers $\delta V_h = 0$ and, thus, $p(s) = c_h$, for almost all $s$, as the use of any price above it would raise $V_h$ over zero. But (i) and (ii) are clearly incompatible because $c_h < E_{\gamma}(u|s)$ for all $s \in (0,1)$. Observe also that, if we are to trade all goods in the first match, we cannot change the buyers’ beliefs by signaling or screening from where they are taken by the signal, $E_{\gamma}(u|s)$, because that would necessarily create some waste and delay in trade. Hence, there exists no equilibrium where everything is traded in the first match. Any equilibrium is inefficient. ■

Since it is impossible to trade all goods in the first match without giving a fraction of the rents to the buyers or to the high quality sellers, it is also impossible to trade all goods for the lowest signals, which would give them almost no rents. With a positive search option, it is better to wait for higher signals than to trade for the lowest signals. Someone always becomes too picky to trade for whatever the signal. Thus, some rationing must occur for the lowest signals. We do not even have to specify which party becomes too picky to trade; that is likely to depend on the particularities of the game form. It suffices to know that it is impossible to maintain mutually beneficial trade for all the signals over time. Some rents must to be payed in terms of delay in trade to shroud the signal information when it is unfavorable. This is noteworthy as the primitives of this game are such that all qualities could be traded efficiently in a static Walrasian market; it would not be a lemons market:

**Remark 1** Consider a static Walrasian market in which the fraction $\gamma$ of the sellers has a high quality good and the fraction $1 - \gamma$ of the sellers has a low quality good. Then, there exist a continuum of efficient Walrasian equilibria $p \in [\lambda, E_{\gamma}(u)]$ where every buyer and every seller trades instantaneously. The buyer value is $V_b = E_{\gamma}(u) - p$ and the seller values are $V_h = p - \lambda$ and $V_l = p$.\(^{21}\)

Also, with either fully asymmetric or fully symmetric information, all the gains could be realized:

**Lemma 1** Consider a market as described in Section 2.2 but where the shared signal is white noise. Then, there exist a continuum of efficient equilibria $p \in [\lambda, E_{\gamma}(u)]$ in which everything is traded in the first match. The buyer value is $V_b = E_{\gamma}(u) - p \geq 0$ and the seller values are $V_h = p - \lambda \geq 0$ and $V_l = p > 0$.

**Lemma 2** Consider a market as described in Section 2.2 but where the shared signal is perfectly revealing. Then, there exist a unique of efficient equilibrium $p_h = 1$ and $p_l = \lambda$ in which everything is traded in the first match. The buyer value is $V_b = 0$ and the seller

\(^{21}\)To specify, any price below the average quality could be the Walrasian equilibrium, the prices above high cost sustain only low quality trade, the prices below high cost sustain trade in both low and high quality. In other words, there would exist a continuum of efficient Walrasian equilibria and a continuum of inefficient Walrasian equilibria.
values are $V_h = 1 - \lambda$ and $V_l = \lambda$.

Corollary 2 A shared signal can be welfare-reducing.

2.4 Equilibria

2.4.1 Equilibria in "stage games"

In this section we characterize the full set of equilibria which arise in the "stage games" to be played in the bilateral meetings. This is basically a static problem because the search options depend on what is done in the continuation equilibrium, which is fixed when the actions are made,\textsuperscript{22} but not on what is done in the bilateral meetings, that have a zero size. Hence, we can take the average market quality, $\gamma := \frac{\gamma_h}{\gamma_h + \gamma_l} \in (0,1)$, the value of the search option, $V := (V_b, V_h, V_l) \in (0,1)^3$, and the signal, $s \in (0,1)$, as our data and study what kinds of equilibria are sustainable with this data.\textsuperscript{23}

Definition 1 A meeting-specific equilibrium is defined by data $d = (V, \gamma, s)$, where $s$ is the shared signal that has been observed, $V = (V_b, V_h, V_l)$ are the values of the search option induced by the full game and $\gamma$ is the share of high quality sellers in the market.

These meeting-specific equilibria come here in three different types: pooling, separating, and semi-pooling in which a seller is mixing between pooling and separting. Within each type, there usually exists a continuum of equilibria that are consistent with the intuitive criterion. Most deviations can be attributed to low quality sellers as, when the sellers of high quality would benefit from a deviation, then the sellers of low quality would also benefit from it. We later on suggest a way to refine the equilibrium set based on how the sellers would prefer to coordinate their strategies.\textsuperscript{24}

To be clear about the use of words, we add the following definition:

Definition 2 Consider a meeting-specific equilibrium with data $d = (V, \gamma, s)$.

1. A profile of pricing strategies $p_h(s), p_l(s) \in \Delta[0,1]$ is pooling if both sellers only make pooling offers, i.e., if supp$(p_h(s)) = supp(p_l(s))$.

2. A profile of pricing strategies $p_h(s), p_l(s) \in \Delta[0,1]$ is separating if both sellers only make separating offers, i.e., if supp$(p_h(s)) \cap supp(p_l(s)) = \emptyset$.

Otherwise, this profile is semi-pooling.

\textsuperscript{22}This continuation equilibrium could be stationary or non-stationary. For simplicity, we omit the time indexes.

\textsuperscript{23}Note that, while some of the pricing patterns may not be possible in a stationary equilibrium, they could arise more generally in a non-stationary equilibrium.

\textsuperscript{24}See also the ideas presented by Nöldeke and Samuelson (1997) as we apparently end focusing on mixtures of what they call Riley equilibria and Hellwig equilibria.
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Note that data and the shared signal, in particular, determines which meeting-specific equilibria are supportable:

**Proposition 2** Consider a meeting-specific equilibrium with data \( d = (V, \gamma, s) \).

1. There exists a pooling equilibrium iff \( E(u|s) - \delta V_b \geq c_h + \delta V_h (IR - b, IR - h) \) and \( E(u|s) - \delta V_b \geq c_l + \delta V_l (IR - b, IR - l) \). The price offer \( p \) is between \( \max \{c_l + \delta V_l, c_h + \delta V_h\} \) and \( E(u|s) - \delta V_b \). If \( p < E(u|s) - \delta V_b \) and the acceptance probability is given by

\[
a(p) = 1 \text{ for } p < E(u|s) - \delta V_b, a(p) \in [0, 1] \text{ for } p = E(u|s) - \delta V_b.
\]

2. There exists a separating equilibrium iff \( u_l - \delta V_b \geq c_h + \delta V_h (IR - b, IR - h) \) and \( u_l - \delta V_b \geq c_l + \delta V_l (IR - b, IR - l) \). The price offers are \( p_h = u_h - \delta V_h \), for the high quality seller, and \( p_l = u_l - \delta V_b \), for the low quality seller, and the acceptance probabilities are given by \( (IC - l) \)

\[
a(p_h) \in \left[0, \frac{p_l - (c_l + \delta V_l)}{p_h - (c_l + \delta V_l)}\right] \text{ and } a(p_l) = 1.
\]

3. In a semi-separating equilibrium, there would be at maximum one pooling price \( p \) in use and at maximum one separating price \( p_l \) or \( p_h \) in use: If the high quality seller is mixing between \( p \) and \( p_l > p \) the low quality seller only using \( p \) whereas if the low quality seller is mixing between \( p \) and \( p_l < p \), the low quality seller is only using \( p \).

In a pooling equilibrium, both low quality sellers and high quality sellers use the same price. If the price leaves the buyer positive surplus, it is accepted for probability one; otherwise, the buyer can also mix between accepting and rejecting the price. The best (seller maximal) of pooling equilibria combines the best of both worlds as the price keeps the buyers at their outside options, yet, is accepted for probability one.

In a separating equilibrium, both sellers are offering a revealing price, a low price for the low quality sellers and a high price for the high quality sellers. The former is accepted for certain but has to be accepted for a probability less than one to stop the low quality sellers from mimicking the high quality sellers.\(^{25}\) Both prices must keep the buyers at their outside options to honor the buyers and sellers’ optimality conditions.

Note also that, the low quality sellers can separate whenever they want by offering a price below the high cost; the high quality sellers cannot due to the adverse off the equilibrium path beliefs that would arise. Furthermore, if a seller would rather to resume his search to get a better signal, there is always the option to quit and pass an opportunity of trading by making the buyer some unacceptable price offer, like \( p = 1 \).

\(^{25}\)Observe that in most applications there exist many natural ways to interpret or purify the randomized strategies, for example, by perturbing the players’ payoffs à la Harsanyi (1973).
In a semi-pooling equilibrium, either the low quality sellers mix between a low and a high price while the high quality sellers only use the high price or the high quality sellers mix between a low and a high price while the low quality sellers only use the low price. To stop the low quality sellers from mimicking the high price must be accepted less often than the low price. To keep the buyers mixing in accepting and rejecting it, they must be kept at their outside options. That is, several fixed point conditions, i.e., the revenue equivalence condition for the mixing buyer and the mixing seller, plus, the incentive condition for the low quality seller, have to hold at once.

Note that any meeting-specific equilibrium is consistent with the full one – trivially, due to its negligible size. However, what we work towards is indeed an equilibrium that is constructed out of the meeting-specific equilibria with certain desired properties.

We are interested in particular in meeting-specific equilibria that the sellers would prefer to play for the data they have. We find that if the signal is high, they rely on the costless signal but, if the signal is low, they sometimes opt for the costly signaling.

**Definition 3** Consider a meeting-specific equilibrium with data \( d = (V, \gamma, s) \). The equilibrium (with data \( d \)) is seller maximal if there exist no other equilibrium (with data \( d \)) that both the high quality sellers and the low quality sellers would strictly (weakly) prefer. Otherwise, the former equilibrium is strictly (weakly) defeated by the latter equilibrium.

Crucially, we find that, in search for seller maximal equilibria, it is possible to ignore as defeated the semi-pooling equilibria and zoom in on the best pooling equilibrium and the best separating equilibrium. This result arises as the seller who is mixing has to be indifferent between playing the high price or the low price, yet, the other seller is going to be better off if the ratio in which the first seller mixes is degenerate; the seller maximal equilibrium is, therefore, either fully pooling or fully separating.

**Proposition 3** Consider a meeting-specific equilibrium with data \( d = (V, \gamma, s) \).

1. Any pooling equilibrium is defeated by the best pooling equilibrium where the price offer is \( p = E(u|s) - \delta V_b \) and the acceptance probability is \( a(p) = 1 \).

2. Any separating equilibrium is defeated by the best separating equilibrium where the acceptance probabilities are

\[
\bar{a} := a(p_h) = \frac{p_l - (c_l + \delta V_l)}{p_h - (c_l + \delta V_l)} \text{ and } a(p_l) = 1.
\]

3. Any semi-pooling equilibrium is defeated by the best pooling equilibrium or by the best separating equilibrium.

---

26The definition has a flavor of the one shot deviation property, which is a necessary condition of an equilibrium, yet, we are now comparing an equilibrium to an equilibrium. Heuristically, one could think of a situation in which all the sellers who have got a signal \( s \) contact one another to decide what meeting-specific equilibrium to play, the old one or a new, and then communicate that information to the buyers.
4. The best separating equilibrium is defeated by the best pooling equilibrium as long as

\[ E(u|p) - \delta V_h \geq a(p_h)(u_h - c_h) + (1 - a(p_h))\delta V_h. \]

5. The best pooling equilibrium is not defeated by the best separating equilibrium.

**Corollary 3** In a seller maximal equilibrium, the sellers always play either the seller maximal pooling equilibrium, the seller maximal separating equilibrium, or just quit by making some unacceptable price offer.

As the sellers make the price offers and play the equilibrium that serves them the best, it comes as no surprise that the Diamond (1971) result arises and the buyer value is zero.\(^{27}\)

Also, what the low quality sellers get from pooling (with the high) cannot exceed what the high quality sellers get from pooling (with the low).

**Remark 2** Consider the full game. In a seller maximal equilibrium, the opportunity cost of trading is higher for high quality sellers than for low quality sellers, \(c_h + \delta V_h > c_l + \delta V_l\) for all \(t\); the buyer value is zero, \(V_b = 0\).

The following lemma presents the basic structure of seller maximal equilibria: the sellers pool for higher signals, \([s', 1]\), and separate or quit for lower signals, \([0, s']\).

**Lemma 3** Consider the full game.

1. **On existence of a cutoff signal and its characterization:**

   In a seller maximal equilibrium, there exist a cutoff \(s'\) such that, (i) if the signal is above the cutoff, i.e., for \(s \geq s'\), the sellers make the best pooling offer and, (ii) if the signal is below the cutoff and separation is feasible, i.e., for \(s < s'\) and \(\lambda - \delta V_h \geq \delta V_l\), the sellers make the best separating offer but, (iii) if the signal is below the cutoff and separation is infeasible, i.e. for \(s < s'\) and \(\lambda - \delta V_h < \delta V_l\), the sellers just resume their search.

2. **On uniqueness of the cutoff or multiplicity of cutoffs:**

   In a seller maximal equilibrium, if \(p_l \geq \delta V_l\) (when separation is feasible), the cutoff is between \(s^l\) and \(s^h\), where \(p(s^l) := E_\gamma(u|s^l) - \delta V_h = c_h + \delta V_h\) (high quality sellers are indifferent between pooling and quitting) and \(p(s^h) := E_\gamma(u|s^h) - \delta V_h = c_h + a(p_h)(p_h - c_h) + (1 - a(p_h))\delta V_h\) (high quality sellers are indifferent between pooling and separating). Otherwise, if \(p_l < \delta V_l\) (when separation is infeasible), the cutoff equals \(s^l\).

Note that the existence of a pooling equilibrium depends on buyers' beliefs and, thus, the shared signal (it is possible for the highest signals but not for the lowest signals) but

\(^{27}\)As specified elsewhere in this paper, there can exist pooling and semi-pooling equilibria, where buyers extract positive surplus; in purely separating equilibria where the buyers have to keep randomizing between accepting and rejecting the high offer, the buyers get no surplus, though.
the existence of a separating equilibrium depends on the value of the search option to the low quality sellers (it is either feasible for all signals or infeasible for all signals).

If the signal is high and, thus, the maximal price the buyer is willing to pay without additional costly revelation is high, both high quality and low quality sellers are better off if they play the best pooling equilibrium and not the best separating equilibrium. For such \( s \in [s^h, 1] \), any seller maximal equilibrium features pooling. Yet, for intermediate signal realizations, the high quality sellers are better off separating, whenever it is feasible, but the low quality sellers are better off pooling. For such \( s \in [s^l, s^h] \), a seller maximal equilibrium could either be the pooling one or the separating one as a gain in one seller’s surplus is a loss in the other seller’s surplus.

For very low signals \( s \in [0, s^l] \), individual rationality constrains, \( p(s) > c_\theta + \delta V_\theta \), for \( \theta = h, l \), start binding, however. As the high quality sellers are worse off if they pool to a low price that corresponds with a low signal than if they resume their search, there exist no pooling equilibria and, whenever it is the case that low quality sellers rather quit than reveal their quality, there exist no separating equilibria, either. The possibility to shop for the highest signals makes the sellers too picky to offer the lowest prices. As a result, if the signal is too low to sustain pooling, the sellers either resort to costly separation or, when they cannot, quit to get another try.\(^{28}\)

The cutoff is unique when separation is infeasible but, when that is not the case, we have some leeway as high quality sellers prefer a higher cutoff but low quality sellers a lower cutoff. We concentrate on seller maximal equilibria where the cutoff is as long as is feasible, \( s^l \).\(^{29}\) They come in three different types:

**Corollary 4** In a seller maximal equilibrium, the sellers are either (i) pooling for high signals and separating for low signals (if \( p_l > \delta V_l + c_l \)), (ii) pooling for high signals and quitting for low signals (if \( p_l < \delta V_l + c_l \)), or (iii) pooling for high signals and mixing between quitting and separating for low signals (if \( p_l = \delta V_l + c_l \)).

### 2.4.2 Equilibria in the full game

We find that seller maximal equilibria can feature two different trading patterns:

**Proposition 4** For any \((\delta, \epsilon, F)\), there exists a minimal cutoff \( \lambda \in (0, 1) \) for the gains from trade \( \lambda \) such that, for \( \lambda > \lambda \), high quality is traded slower than low quality in a seller maximal equilibrium.

**Proposition 5** For any \((\delta, \epsilon, F)\), there exists a maximal cutoff \( \lambda \in (0, 1) \) for the gains from trade \( \lambda \) such that, for \( \lambda < \lambda \), high quality is traded faster than low quality in a seller

\(^{28}\)As a side-remark, observe that decentralized trade accompanied with variability in signals is a natural way to get variability in prices across homogenous goods, something that has been observed in data.

\(^{29}\)These equilibria could hence be described as featuring maximal risk sharing or pooling.
maximal equilibrium.

**Proposition 6** For any \((\delta, \epsilon, F)\) and \(\lambda\), there exists a seller maximal, stationary Markovian equilibrium: \(\lambda \leq \bar{\lambda}\).

**Proofs.** See Appendix A and B. ■

In other words, for the gains from trade in high quality low enough, there exists a seller maximal equilibrium where the low quality is more liquid and, for the gains from trade in high quality high enough, there exists a seller maximal equilibrium where the high quality is more liquid. Thus, either of these trading patterns could arise. Note that, in the former case, the average market quality is better (in FOSD sense) than the entering quality whereas, in the latter case, the average market quality is worse (in FOSD sense).

### 2.5 Closing remarks

We provide an example of a market in which the opportunity to get additional information in the bilateral meetings, through which trading occurs, causes a reduction in liquidity in a setup where the static Walrasian market could remain perfectly efficient. Indeed, all goods could be traded in the first match if the shared signals were removed.

This outcome arises because the possibility to wait for higher signals and better prices makes either the buyers or the high quality sellers too picky to trade for the lowest signals. This effect works through an increase in their search option; it does not depend, for example, whether we have a signaling game as we do or a screening game.

Hence, we obtain the classic trade-off between efficiency and rent extraction, of a sort.

Note that it does not seem to be the case that the inefficiencies would be caused by the failure of information aggregation by price as, although imperfect revelation is essential for the findings, also the efficient equilibria without the signals are pooling. Instead, the availability of the shared signal makes it possible for the sellers to pool to a different price for a different shared signal; this is a costless sorting mechanism and, thus, improves welfare relative to the cases where the sellers use a costly separation mechanism only.

We characterize the full set of equilibria that can be played in the bilateral stage games between a seller and a buyer and zoom in on seller maximal, stationary Markovian equilibria, which always exist. If the gains from trade in high quality are large in comparison to the gains from trade in low quality, high quality can be traded faster; if it is the opposite, low quality is more liquid. This suggests that the availability of correlated information can let the market to endogenously adjust to trade faster the most valuable qualities.
Appendix A

Intuitive Criterion (Cho and Kreps, 1987)

Definition 4 Consider an equilibrium belief system \( \pi \). Denote by \( \Theta(p') \subset \{h, l\} \) the subset of sellers who prefer the disequilibrium price \( p' \) to the equilibrium price \( p \) when considered high quality sellers. The equilibrium fails the intuitive criterion if there is some seller \( \theta' \in \Theta(p') \) who prefers \( p' \) to \( p \) for whatever the buyers’ belief, as long as the buyer takes him for some seller \( \theta \in \Theta(p') \).

Here, the belief system fails the intuitive criterion if (a) \( \pi(p') = E(u|p') > u_l \) but the high quality seller would always lose by the deviation to \( p' \) whereas the low quality seller would gain from it even if known to be low or (b) \( \pi(p') = E(u|p') < u_h \) but the low quality seller would always lose by the deviation to \( p' \) whereas the high quality seller would gain from it even if known to be high.

Existence algorithm to find a seller maximal equilibrium

In this part, we juxtapose two different ways of dynamic trading with a shared signal: one is based on costly signaling (where the sellers separate as often as possible) and the other one is based on the costless signal (where they pool as often as possible). For concreteness, we present an algorithm by which we can move from the signaling based equilibrium to the seller maximal equilibrium that is also the maximal pooling equilibrium. We show that at least initially the sellers’ profits get higher as we decrease the use of costly signaling and increase the use of costless signal. As a by-product of the process, we find that there always exists a equilibrium not only of the former type and but also of the latter type.

To run the algorithm, we consider different equilibria associated with a cutoff \( s \in [0, 1] \) such that, for signals above the cutoff, i.e., for \( s > s' \), the sellers play the maximal pooling equilibrium and, for signals below the cutoff, i.e., for \( s < s' \), the sellers play the maximal separating equilibrium, or quit once they cannot. We start by the cutoff \( s' = 1 \), that corresponds with the separation based equilibrium, and lower or alter it in a specified way until we know we have discovered a seller maximal equilibrium.

As we proceed along, any cutoff defines a stationary Markovian equilibrium, which is initially a non seller maximal one but ultimately a seller maximal one. Depending on which one binds sooner and when, a high quality seller’s participation constraint that makes pooling impossible below the cutoff or a low quality seller’s participation constraint that makes separation impossible below the cutoff, we can get either the standard trading pattern, where the low quality is more liquid, or the reversed one.

We present this process by a progression of propositions. As we move on, we pass continuously\(^{30}\) from an equilibrium to an equilibrium until we stop by one that is maximal.

\(^{30}\) The strategies are continuous and the seller values are continuous as we go on.
for the sellers. We set out from $s' = 1$.

**Proposition 7** There exists a unique separation based equilibrium where the sellers play the seller maximal separating equilibrium as defined by Propositions 2.2 and 3.2, with some $\bar{a}(1)$, for any $s \in [0, 1]$. The probability of trade is $\tau_h = \bar{a}(1)$ for the high quality sellers and $\tau_l = 1$ for the low quality sellers. Thus, the low quality is more liquid than the high quality. The seller values are

$$V_h = \frac{(1 - \lambda)\bar{a}(1)}{1 - \delta(1 - \bar{a}(1))} \text{ and } V_l = \lambda$$

We then consider smaller cutoffs $s' < 1$ and, first, the associated non seller maximal equilibria:

**Lemma 4** There exist a cutoff $s' < 1$, and a unique equilibrium where the sellers play the seller maximal pooling equilibrium as defined by Propositions 2.1 and 3.1 for any $s \in [s', 1]$ and the seller maximal separating equilibrium, with some $\bar{a}(s') > 0$, for any $s \in [0, s')$.

This requires that the high quality seller’s participation constraint holds so that they are willing to pool for any $s \in [s', 1]$, i.e., that

$$p(s') = E(u|s') \geq V_h(s') + c_h, \text{ IR-h},$$

and, additionally, that the low quality seller’s participation constraint holds so that they are willing to separate for any $s \in [0, s')$, i.e., that

$$p_l = u_l \geq V_h(s') + c_l, \text{ IR-l}.$$  

A seller maximal equilibrium is an equilibrium for which IR-h binds; it is always found before $s' = 0$. This comes from continuity and the fact that, as $s' \to 0$, the price goes down to high cost, $p(s') \to c_h$, but the high quality sellers continue to have a positive search option, $V_h(s') > 0$. Yet, what exactly happens as we move down the cutoff, depends on the effects on the seller values, which may not be monotone all the way until IR-h or IR-l binds; the sellers are not only shifting away from separation but also changing the market and the prices. Still, at least a first, all sellers are better off in an equilibrium where the cutoff is lower:

**Lemma 5** As the cutoff is reduced, the seller values initially increase: $$dV_h := \frac{\partial V_h}{\partial s'} + \frac{\partial V_h}{\partial \bar{a}} \frac{\partial \bar{a}}{\partial s'} < 0$$ and $$dV_l := \frac{\partial V_l}{\partial s'} + \frac{\partial V_l}{\partial \bar{a}} \frac{\partial \bar{a}}{\partial s'} < 0$$ for all $s' > s^0$ with some $s^0 \in (0, 1)$; they could later again decrease and increase and so on.

Note that not all the results depend on the particular parameter structure $u_h - c_h = 1 - \lambda$ and $u_l - c_l = \lambda$. We use $u_h$, $c_h$ and $u_l$ when we are not evoking the additional restriction $c_h = u_l = \lambda$. 
Lemma 6 When IR-h and IR-l not yet bind, there exists a unique acceptance probability \( \bar{a}(s') \) for any cutoff \( s' \in [0,1] \); \( \bar{a}(s') \) is continuous and increasing in the cutoff \( s' \) so long as \( dV_l < 0 \). The cutoff and the acceptance probability have the similar effects on the low quality seller values, \( \partial V_l / \partial s' \cdot \partial a / \partial s' > 0 \).

To find a seller maximal equilibrium, we reduce the cutoff \( s' < 1 \) until either IR-h or IR-l binds:

Proposition 8 Suppose IR-h binds above IR-l, at \( s' = s_h \). Then, there exists a seller maximal equilibrium with a cutoff \( s' = s_h \in (0,1) \) where the sellers pool for \( s \in [s',1] \) and separate for \( s \in [0,s') \) with some \( \bar{a}(s') > 0 \). The probability of trade is \( \tau_h = F_h(s')\bar{a}(s') + (1 - F_h(s')) \) for the high quality sellers and \( \tau_l = 1 \) for the low quality sellers. Thus, the low quality is more liquid than the high quality.

Thus, if IR-h binds first, we could stop at once because we would have found a seller maximal equilibrium but, if IR-l binds first, we continue by adjusting the quit rate from zero to one and simultaneously moving the cutoff such that the IR-l still binds; if the quit rate reaches one, we just keep decreasing the cutoff further until IR-h binds. This is necessary for the algorithm to work well. It allows us to adjust continuously the average market quality as we move from equilibria where high quality is traded less often to equilibria where low quality is traded less often and, hence, to go through all the possible seller maximal equilibria.

Lemma 7 Suppose IR-l binds above IR-h, at \( s' = s_l \). Then, there exist a quit rate \( q > 0 \), a cutoff \( s'(q) < s_l \), and an equilibrium where the sellers play the seller maximal pooling equilibrium for any \( s \in [s'(q),1] \). For \( s \in [0,s'(q)) \), the high quality sellers quit for probability one; the low quality sellers quit for probability \( q \) and separate for probability \( 1 - q \).

This requires that the high quality seller’s participation constraint holds so that they are willing to pool for any \( s \in [s'(q),1] \), i.e., that

\[
p(s'(q)) = E(u|s'(q)) \geq V_h(s'(q)) + c_h, \text{ IR-h,}
\]

and, additionally, that the low quality seller’s participation constraint holds so that they are willing to randomize for any \( s \in [0,s'(q)) \), i.e., that

\[
p_l = u_l = V_h(s'(q)) + c_l, \text{ IR-l'}.
\]

Lemma 8 As long as IR-h and IR-l’ are satisfied, there exists a unique cutoff \( s'(q) \) for any quit rate \( q \in [0,1] \); \( s'(q) \) is continuous and increasing in the quit rate \( q \) so long as \( \partial V_l / \partial s' < 0 \). The quit rate and the cutoff have the opposite effects on the low quality seller values, \( \partial V_l / \partial q \cdot \partial s' / \partial q < 0 \).
Lemma 9 As the quit rate $q$ is elevated and the cutoff $s'(q)$ is adjusted such that the low quality seller values are constant, the high quality seller values could increase or decrease. After the cutoff $s'$ is reduced from the point $s'(1)$ where the quit rate is one, the seller values are increasing: $dV_h := \frac{\partial V_h}{\partial s'} < 0$ and $dV_l := \frac{\partial V_l}{\partial s'} < 0$ for all $s' < s'(1)$.

We continue like this by keeping IR-l binding, adjusting the cutoff and increasing the quit rate up to one and, from that point on, just continue by reducing the cutoff. We stop once IR-h binds:

Proposition 9 Suppose IR-l binds above IR-h, at $s' = s_l$. Then, there exists a seller maximal equilibrium with a quit rate $q > 0$ and a cutoff $s' = s_h \in (0, 1)$ where the sellers pool for $s \in [s', 1]$. For $s \in [0, s')$, the high quality sellers quit for probability one; the low quality sellers quit for probability $q$ and separate for probability $1 - q$. The probability of trade is $\tau_h = (1 - F_h(s'))$ for the high quality sellers and $\tau_l = F_h(s')(1 - q) + (1 - F_h(s'))$ for the low quality sellers. Thus, the low quality is more liquid than the high quality for a high quit rate.

Note that, before IR-l binds, a reduction in the cutoff increases the quality in the market as the high quality is traded more often in absolute terms but, after IR-l binds and the quit rate is sufficiently high, a reduction in the cutoff decreases the quality in the market as the high quality is traded less often in relative terms. In particular, as we are elevating the quit rate, we at some point come to a quit rate $q'$ after which the entry distribution of qualities first order stochastically dominates the market distribution of qualities, which indicates that high quality is traded faster than low quality:

Remark 3 There exists a cutoff $q' \in (0, 1)$ for the quit rate $q$ such that

$$\epsilon_h = \frac{\gamma_h(q', s'(q'))}{\gamma_h(q', s'(q')) + \gamma_l(q', s'(q'))}$$

and

$$\epsilon_l = \frac{\gamma_l(q', s'(q'))}{\gamma_h(q', s'(q')) + \gamma_l(q', s'(q'))}.$$

If IR-h binds below IR-l and $s(q')$, the high quality is more liquid than the low quality but, otherwise, the low quality is more liquid than the high quality.

Remark 4 If the quit rate is degenerate $q = 0, 1$, equilibria where the high quality is less liquid are based on mixed pricing strategies whereas equilibria where the high quality is more liquid are based on pure pricing strategies.

Proofs for the existence algorithm

Proof of Proposition 7. By Propositions 2.2 and 3.2, the prices are $p_h = 1$ and $p_l = \lambda$. The former is accepted for probability one, the latter for a probability less than one, $\bar{a}$. As the seller values are...
\[ V_h = (u_h - c_h)(\bar{a} + \delta(1 - \bar{a}) + \delta^2(1 - \bar{a})^2 + \ldots) = \frac{(u_h - c_h)\bar{a}}{1 - \delta(1 - \bar{a})}, \quad \text{and} \quad V_l = u_l - c_l, \]

the acceptance probability is given by
\[
\bar{a} = \frac{u_l - c_l - \delta(u_l - c_l)}{u_h - c_l - \delta(u_l - c_l)}. \]

**Proof of Lemma 4.** When the seller play the seller maximal pooling equilibrium above a cutoff \(s' \in (0, 1)\) and a seller maximal separating equilibrium below the cutoff \(s' \in (0, 1)\), the seller values are

\[
V_h = F_h(s')(\bar{a}(s')(u_h - c_h) + (1 - \bar{a}(s'))\delta V_h) + \int_{s'}^{1} (p(s) - c_h) dF_h(s),
\]

\[
V_l = F_l(s')(u_l - c_l) + \int_{s'}^{1} (p(s) - c_l) dF_h(s),
\]

where under pooling, by the Bayes’ law, the price offer is

\[
p(s) = E(u|s) = \frac{\gamma_h f_h(s)u_h + \gamma_l f_l(s)u_l}{\gamma_h f_h(s) + \gamma_l f_l(s)},
\]

\[
\gamma_h = \frac{\epsilon_h}{\tau_h} = \frac{c_h}{\bar{a}(s')F(s') + (1 - F(s'))},
\]

\[
\gamma_l = \frac{\epsilon_l}{\tau_l} = \epsilon_l.
\]

When the cutoff is almost one, the seller values are almost as in Proposition 7. Thus, it is easy to see that this equilibrium indeed exists as it only requires that the incentive conditions hold both for the high quality sellers (IR-h),

\[
p(s') \approx u_h > \delta V_h + c_h \approx \delta \frac{(u_h - c_h)\bar{a}}{1 - \delta(1 - \bar{a})} + c_h,
\]

and for the low quality sellers (IR-l),

\[
p_l = u_l > V_l + c_l \approx \delta(u_l - c_l) + c_l.
\]

**Proof of Lemma 5.** First, the price can be written as

\[
p(s, s', \bar{a}) = \frac{u_h + g^2(s, s', \bar{a})u_l}{1 + g^2(s, s', \bar{a})},
\]

where
\[ g^2(s, s', \bar{a}) := \frac{\gamma_l f_l(s)}{\gamma_h f_h(s)} = \frac{\tau_h \epsilon_l f_l(s)}{\tau_l \epsilon_h f_h(s)}. \]

The effect of the cutoff on the price level is different before IR-l binds and separation is feasible and after IR-l binds and separation is not feasible. In the former case, the prices are higher when the cutoff is higher; the higher the cutoff the larger the effect. This is because the high quality sellers trade less often both in absolute terms and in relative terms if the cutoff is higher. The market gets better and the price level increases. In the latter case, the prices are lower when the cutoff is higher. While both the high and the low quality sellers trade less often in absolute terms, the high quality sellers trade more often in relative terms when the cutoff is higher. The market gets worse and the price level decreases. To show this, consider the partials

\[ \frac{\partial p(s, s', \bar{a})}{\partial s'} = -\frac{u_h - u_l}{(1 + g^2(s, s', \bar{a}))^2} \frac{\epsilon_l f_l(s)}{\epsilon_h f_h(s)} \frac{\partial}{\partial s'} \tau_h(s'), \]

\[ \frac{\partial^2 p(s, s', \bar{a})}{\partial s'^2} = 2 \frac{u_h - u_l}{(1 + g^2(s, s', \bar{a}))^2} \left( \frac{\epsilon_l f_l(s)}{\epsilon_h f_h(s)} \frac{\partial}{\partial s'} \tau_h(s') \right)^2 \]

\[ -\frac{u_h - u_l}{(1 + g^2(s, s', \bar{a}))^2} \frac{\epsilon_l f_l(s)}{\epsilon_h f_h(s)} \frac{\partial^2}{\partial s'^2} \tau_h(s') < 0. \]

First, when IR-l is satisfied, the rates of trade are

\[ \frac{\tau_h}{\tau_l} = \frac{F_h(s') \bar{a} + (1 - F_h(s'))}{1}. \]

Later, when IR-l is not satisfied, the probabilities are

\[ \frac{\tau_h}{\tau_l} = \frac{(1 - F_h(s'))}{(1 - F_l(s'))}. \]

In between, we also deal with the probabilities

\[ \frac{\tau_h}{\tau_l} = \frac{(1 - F_h(s'))}{F_l(s')(1 - q) + (1 - F_l(s'))}. \]

When IR-l holds, the effect of the cutoff is given by

\[ \frac{\partial}{\partial s'} \frac{\tau_h(s')}{\tau_l} = -(1 - \bar{a}) f_h(s') < 0, \frac{\partial^2}{\partial s'^2} \frac{\tau_h(s')}{\tau_l} = -(1 - \bar{a}) f_h'(s') < 0. \]

Thus, the price level is increasing in the cutoff:

\[ \frac{\partial p(s, s', \bar{a})}{\partial s'} = -\frac{u_h - u_l}{(1 + g^2(s, s', \bar{a}))^2} \frac{\epsilon_l f_l(s)}{\epsilon_h f_h(s)} (1 - \bar{a}) f_h(s') > 0, \]

and
\[ \frac{\partial^2 p(s, s', \bar{a})}{\partial s'^2} = 2 \frac{u_h - u_l}{(1 + g^2(s, s', \bar{a}))^3} \left( \frac{\epsilon_l f_l(s)}{\epsilon_h f_h(s)} (1 - \bar{a}) f_h(s') \right)^2 \]

\[ - \frac{u_h - u_l}{(1 + g^2(s, s', \bar{a}))^2} \frac{\epsilon_l f_l(s)}{\epsilon_h f_h(s)} (1 - \bar{a}) f_h(s') \]

When IR-l does not hold, the effect of the cutoff is given by

\[ \frac{\partial}{\partial s'} \tau_h(s') = \frac{-f_h(s') (1 - F_l(s')) + f_l(s') (1 - F_h(s'))}{(1 - F_l(s'))^2} \]

which is positive for any \( s' \) as the numerator is positive whenever \( \frac{f_h(s')}{f_l(s')} \leq \frac{1 - F_h(s')}{1 - F_l(s')} \), that is a direct result of MLRP. Hence, the price level is decreasing in the cutoff:

\[ \frac{\partial p(s, s', \bar{a})}{\partial s'} = -\frac{u_h - u_l}{(1 + g^2(s, s', \bar{a}))^2} \frac{\epsilon_l f_l(s) - f_h(s') (1 - F_l(s')) + f_l(s') (1 - F_h(s'))}{(1 - F_l(s'))^2} < 0. \]

For the last case with \( q \in [0, 1] \), the outcome is in between the earlier extrema: the direct effect of an increase in the cutoff is negative for low \( q \) and positive for high \( q \); it becomes more positive or less negative for a lower \( s' \)

\[ \frac{\partial p(s, s', \bar{a})}{\partial s'} = \frac{f_l(s') q (1 - F_h(s')) - f_h(s') (1 - F_l(s'))}{(1 - F_l(s'))^2}. \]

We can also see to the effect of an increase in the quit rate, though it is needed only later

\[ \frac{\partial p(s, s', \bar{a})}{\partial q} = \frac{F_l(s') (1 - F_h(s'))}{(1 - F_l(s'))^2} > 0. \]

Next, consider the direct effect of the cutoff on low quality seller values. When separation is feasible, the low quality seller value is

\[ V_l = F_l(s') (u_l - c_l) + \int_{s'}^{1} (p(s, s', \bar{a}) - c_l) dF_l \]

and, thus, the direct effect of the cutoff is

\[ \frac{\partial V_l}{\partial s'} = f_l(s') (u_l - c_l) - f_l(s') (p(s', s', \bar{a}) - c_l) + \int_{s'}^{1} \frac{\partial p(s, s', \bar{a})}{\partial s'} dF_l \]

\[ = f_l(s') (u_l - p(s', s', \bar{a})) + \int_{s'}^{1} \frac{\partial p(s, s', \bar{a})}{\partial s'} dF_l \]

which is negative for large \( s' \) and positive for small \( s' \).
\[
\frac{\partial^2 V_i}{\partial s'^2} = f_i'(s') (u_i - p(s', s', \bar{a})) - 2f_i(s') \frac{\partial p(s, s', \bar{a})}{\partial s'} + \int_{s'}^1 \frac{\partial^2 p(s, s', \bar{a})}{\partial s'^2} dF_i
\]

which is negative for large \(s'\) and positive for small \(s'\). Instead, when separation is infeasible, the low quality seller value is

\[
V_l = \int_{s'}^1 (p(s, s', \bar{a}) - c_l) dF_i
\]

and, thus, the direct effect of the cutoff is

\[
\frac{\partial V_l}{\partial s'} = -f_i(s') (p(s', s', \bar{a}) - c_l) + \int_{s'}^1 \frac{\partial p(s, s', \bar{a})}{\partial s'} dF_i
\]

\[
= f_i(s') (c_l - p(s', s', \bar{a})) + \int_{s'}^1 \frac{\partial p(s, s', \bar{a})}{\partial s'} dF_i
\]

which is negative for any \(s'\).

We show soon that the effect on the acceptance probability can be written as

\[
\frac{\partial \bar{a}}{\partial s'} = \frac{-(1 - \bar{a})\delta \frac{\partial V_l}{\partial s'} - (u_i - p(s')) f_i(s') + \int_{s'}^1 \frac{\partial p(s)}{\partial s'} dF_i(s)}{u_h - c_l - \delta V_l + (1 - \bar{a})\delta \int_{s'}^1 \frac{\partial p(s)}{\partial a} dF_i(s)},
\]

where \(\int_{s'}^1 \frac{\partial p(s)}{\partial s'} dF_i(s) > 0\) and \(\int_{s'}^1 \frac{\partial p(s)}{\partial a} dF_i(s) > 0\). We also prove that \(\frac{\partial V_l}{\partial s'} \frac{\partial V_l}{\partial a} > 0\) such that, \(\frac{\partial V_l}{\partial \bar{a}} > 0\) implies \(\frac{\partial V_l}{\partial s'} > 0\) and, consequently, \(dV_l > 0\) like we had to show.

We turn next to the high quality seller values. When IR-h and IR-l are satisfied the values are

\[
V_h = F_h(s')\bar{a}(u_h - c_h) + F_h(1 - \bar{a})\delta V_h + \int_{s'}^1 (p(s, s', \bar{a}) - c_h) dF_i(s).
\]

By differentiating totally we end up with

\[
dV_h = f_h(s')\bar{a}(u_h - c_h) + F_h(u_h - c_h) \frac{\partial \bar{a}}{\partial s'}
\]

\[
-(p(s') - c_l) f_h(s') + \int_{s'}^1 \frac{\partial p(s)}{\partial s'} dF_i(s) + \int_{s'}^1 \frac{\partial p(s)}{\partial \bar{a}} dF_i(s) \frac{\partial \bar{a}}{\partial s'}
\]

\[
+f_h(s')(1 - \bar{a})\delta V_h - F_h(s')\delta V_h \frac{\partial \bar{a}}{\partial s'} + F_h(s'(1 - \bar{a})\delta dV_h.
\]
As we consider small deviation from the highest cutoff $s' = 1$, the two integrals are negligible. Thus, the change in the seller values is negative as long as, for some tiny $\varepsilon > 0$ and the cutoff $s' = 1 - \varepsilon$

$$f_h(s') (\bar{a}(u_h - c_h) + (1 - \bar{a})\delta V_h - (p(s') - c_l)) + F_h(s') (u_h - c_h - \delta V_h) \frac{\partial \bar{a}}{\partial s'} < 0.$$ 

By continuity, if the cutoff is almost one, $\varepsilon \to 1$, the change in the seller values is negative whenever

$$f_h(1)(\bar{a}(1)(u_h - c_h) + (1 - \bar{a}(1))\delta V_h - (u_h - c_l)) - \delta(1 - \bar{a}(1)) \frac{u_l - c_l - \delta V_h}{u_h - c_h - \delta V_h} f_l(1) < 0$$

$$\iff$$

$$\left( (\bar{a}(1)(u_h - c_h) + (1 - \bar{a}(1))\delta V_h - (u_h - c_l)) - \delta(1 - \bar{a}(1)) \frac{u_l - c_l - \delta V_h}{u_h - c_h - \delta V_h} f_l(1) \right) < 0,$$ 

which is holds true for sure. 

\textit{Proof of Lemma 6.} This is a continuity based proof. In this equilibrium type, for any cutoff $s'$, the acceptance probability $\bar{a}$ has to be the root of the following auxiliary function

$$g^1(s', \bar{a}) := \bar{a} (u_h - c_l - \delta V_l(s', \bar{a})) - (u_l - c_l - \delta V_l(s', \bar{a})) = 0.$$ 

We demonstrate next that, for any cutoff $s'$, (i) $g^1$ is continuous in $\bar{a}$, (ii) $g^1(s', 0) < 0$ and $g^1(s', 1) > 0$ as long as IR-h and IR-l continue to hold true and (ii) $g^1$ is convex in $\bar{a}$. This implies that, indeed, there exists a unique $\bar{a}(s')$ such that $g^1(s', \bar{a}(s')) = 0$. We proceed step by step: First, note that the function $g^1 = g^1(s', \bar{a})$ is continuous as so are the functions $p(s, s', \bar{a})$, $V_h(s', \bar{a})$ and $V_l(s', \bar{a})$. Second, for any $s'$ such that IR-h and IR-l hold true, the bounds are

$$g^1(s', 0) = 0 \left( u_h - c_l - \delta V_l(s', 0) \right) - \left( u_l - c_l - \delta V_l(s', 0) \right) < 0,$$

$$g^1(s', 1) = 1 \left( u_h - c_l - \delta V_l(s', 1) \right) - \left( u_l - c_l - \delta V_l(s', 1) \right) > 0.$$ 

Third, to show that $g^1$ is convex in $\bar{a}$, we differentiate it first twice:

$$\frac{\partial g^1(s', \bar{a})}{\partial \bar{a}} = u_h - c_l - \delta V_l(s', \bar{a}) + (1 - \bar{a}) \delta \frac{\partial V_l(s', \bar{a})}{\partial \bar{a}},$$

and
\[
\frac{\partial^2 g^1(s', \bar{a})}{\partial \bar{a}^2} = -2\delta \frac{\partial}{\partial \bar{a}} V_i(s', \bar{a}) + (1 - \bar{a}) \delta \frac{\partial^2}{\partial \bar{a}^2} V_i(s', \bar{a}),
\]
and then prove that the second partial is negative:
\[
\frac{\partial V_i(s', \bar{a})}{\partial \bar{a}} = \int_{s'}^1 \frac{\partial p(s, s', \bar{a})}{\partial \bar{a}} dF_i(s) < 0,
\]
and
\[
\frac{\partial^2 V_i(s', \bar{a})}{\partial \bar{a}^2} = \int_{s'}^1 \frac{\partial^2 p(s, s', \bar{a})}{\partial \bar{a}^2} dF_i(s) > 0,
\]
for any interior \(\bar{a} \in (0, 1)\), which seems natural as a higher \(\bar{a}\) means that high qualities are traded more often. That generates a worse market and lowers the prices, yet, the effect would flatten out finally. More precisely, the price can be written as
\[
p(s, s', \bar{a}) = \frac{u_h + g^2(s, s', \bar{a})u_l}{1 + g^2(s, s', \bar{a})},
\]
where
\[
g^2(s, s', \bar{a}) := \frac{\gamma_l f_1(s)}{\gamma_h f_h(s)} = \frac{\tau_h \epsilon_l f_1(s)}{\tau_l \epsilon_h f_h(s)} = (\bar{a} F_h(s') + 1 - F_h(s')) \frac{\epsilon_l f_1(s)}{\epsilon_h f_h(s)}.
\]

By differentiating the price offer, we get
\[
\frac{\partial p(s, s', \bar{a})}{\partial \bar{a}} = - \frac{u_h - u_l}{(1 + g^2(s, s', \bar{a}))^2} \frac{\epsilon_l f_1(s)}{\epsilon_h f_h(s)} F_h(s') < 0
\]
and
\[
\frac{\partial^2 p(s, s', \bar{a})}{\partial \bar{a}^2} = 2 \frac{u_h - u_l}{(1 + g^2(s, s', \bar{a}))^3} \left( \frac{\epsilon_l f_1(s)}{\epsilon_h f_h(s)} F_h(s') \right)^2 > 0.
\]

Altogether, this implies that \(\frac{\partial V_i(s', \bar{a})}{\partial \bar{a}} < 0\), \(\frac{\partial^2 V_i(s', \bar{a})}{\partial \bar{a}^2} > 0\) and, thus, that \(\frac{\partial^2}{\partial \bar{a}^2} g^1(s', \bar{a}) > 0\) for all \(\bar{a} \in (0, 1)\). As a result, since \(g^1(s', 0) < 0\) and \(g^1(s', 1) > 0\), the continuity and the convexity of \(g^1\) in \(\bar{a}\) imply that for any \(s'\) there exists a unique root \(\bar{a}\) of \(g^1\) such that \(g^1(s', \bar{a}) = 0\) as long as \(\text{IR-h and IR-l remain to be satisfied}\). It is also clear that \(\bar{a}(s')\) is continuous in \(s'\) as so are the primitives.

To know when \(\bar{a}\) is increasing in \(s'\), we differentiate totally the function \(g^1\) so as to get
\[
\frac{\partial \bar{a}}{\partial s'} = \frac{-(1 - \bar{a}) \delta \frac{\partial V_i}{\partial s'} + (1 - \bar{a}) \delta \frac{\partial^2 V_i}{\partial s' \partial \bar{a}}}{u_h - c_1 - \delta V_i + (1 - \bar{a}) \delta \frac{\partial V_i}{\partial \bar{a}}}
\]
\[
= -(1 - \bar{a}) \delta \frac{(u_l - p(s')) f_1(s') + \int_{s'}^1 \frac{\partial p(s)}{\partial s} dF_i(s)}{u_h - c_1 - \delta V_i + (1 - \bar{a}) \delta \int_{s'}^1 \frac{\partial p(s)}{\partial s} dF_i(s)} =: -(1 - \bar{a}) \delta \frac{D}{N} \quad (2.8)
\]
where \( \frac{\partial V_i}{\partial s} = (u_l - p(s')) f_i(s') + \int_{s'}^1 \frac{\partial p(s)}{\partial s} dF_i(s) \) is negative for large \( s' \) and positive for small \( s' \); \( u_h - c_l - \delta V_i \) is positive and \( \frac{\partial V_i}{\partial a} = \int_{s'}^1 \frac{\partial p(s)}{\partial a} dF_i(s) \) is negative as long as IR-h and IR-l are satisfied. The symbols D and N stand for the numerator and the denominator, respectively. For the cutoff high enough, the final terms \( \int_{s'}^1 \frac{\partial p(s)}{\partial a} dF_i(s) \) in N and \( \int_{s'}^1 \frac{\partial p(s)}{\partial a} dF_i(s) \) in D are negligible and the acceptance probability goes down as the cutoff is reduced, \( \frac{\partial a}{\partial s'} > 0 \).

Note generally that the total effect of a change in the cutoff in the low quality seller value is

\[
dV_i = \frac{\partial V_i}{\partial s'} + \frac{\partial V_i}{\partial a} \frac{\partial a}{\partial s'}.
\]

If the total effect is negative, the effect of a cutoff on the acceptance probability is positive

\[
\frac{\partial a}{\partial s'} = -(1 - \bar{a}) \delta dV_i.
\]

Next, take a look at the effects on D of (2.8),

\[
-\delta \left( dV_i + \frac{\partial V_i}{\partial a} \frac{\partial a}{\partial s'} - (1 - \bar{a}) \left( \frac{\partial^2 V_i}{\partial a^2} \frac{\partial a}{\partial s'} + \frac{\partial^2 V_i}{\partial a \partial s'} \right) \right).
\]

Note that

\[
\frac{\partial V_i}{\partial a} = - \int_{s'}^1 \left[ \frac{\partial p(s, s', \bar{a})}{\partial a} \right] dF_i = - \int_{s'}^1 \left[ \frac{u_h - u_l}{(1 + g^2(s, s', \bar{a}))^2} \epsilon_{\bar{a}} f_i(s) F_h(s') \right] dF_i < 0,
\]

\[
\frac{\partial^2 V_i}{\partial a^2} = \int_{s'}^1 \left[ \frac{\partial^2 p(s, s', \bar{a})}{\partial a^2} \right] dF_i = \int_{s'}^1 \left[ \frac{2 u_h - u_l}{(1 + g^2)^2} \left( \epsilon_{\bar{a}} f_i(s') F_h(s') \right)^2 \right] dF_i > 0,
\]

\[
\frac{\partial^2 V_i}{\partial a \partial s'} = \int_{s'}^1 \left[ \frac{\partial^2 p(s, s', \bar{a})}{\partial a \partial s'} \right] dF_i
\]

\[
= \int_{s'}^1 \left[ \frac{u_h - u_l}{(1 + g^2)^2} \epsilon_{\bar{a}} f_i(s') F_h(s') - 2 \frac{u_h - u_l}{(1 + g^2)^2} \left( \epsilon_{\bar{a}} f_i(s') \right)^2 (1 - \bar{a}) f_h(s') F_h(s') \right] dF_i,
\]

\[
\frac{\partial^2 V_i}{\partial s'^2} = - \frac{\partial p(s, s', \bar{a})}{\partial a} \frac{\partial a}{\partial s'} f_i(s') + \int_{s'}^1 \left[ \frac{\partial^2 p(s, s', \bar{a})}{\partial a \partial s'} \right] dF_i
\]

\[
= - \frac{\partial p(s, s', \bar{a})}{\partial a} f_i(s')
\]

\[
+ \int_{s'}^1 \left[ \frac{u_h - u_l}{(1 + g^2)^2} \epsilon_{\bar{a}} f_i(s') (1 - \bar{a}) f_h(s') + 2 \frac{u_h - u_l}{(1 + g^2)^2} \left( \epsilon_{\bar{a}} f_i(s') \right)^2 (1 - \bar{a}) f_h(s') F_h(s') \right] dF_i.
\]

We next show that, if N and D of (2.8) are positive, \( dV_i > 0 \) and \( u_h - c_l - \delta V_i + (1 - \bar{a}) \delta \int_{s'}^1 \frac{\partial p(s)}{\partial a} dF_i(s) > 0 \), such that \( \frac{\partial a}{\partial s'} < 0 \), D is bounded away from zero. The proof is by contradiction. Thereby, assume that for some cutoff \( s^0 \),

\[
u_h - c_l - \delta V_i + (1 - \bar{a}) \delta \int_{s'}^1 \frac{\partial p(s)}{\partial a} dF_i(s) \rightarrow 0 + \text{ as } s' \rightarrow s^0 + .
\]
Obviously, this implies that $D$ is increasing in $(s^o, s^o + \zeta)$, for some tiny $\zeta > 0$,

$$-\delta \left( dV_l + \frac{\partial V_l \partial \bar{a}}{\partial \bar{a} \partial s'} - (1 - \bar{a}) \left( \frac{\partial^2 V_l \partial \bar{a}}{\partial \bar{a}^2 \partial s'} + \frac{\partial^2 V_l}{\partial \bar{a} \partial s'} \right) \right) > 0.$$

But now, since everything is continuous, $\frac{\partial^2 V_l}{\partial \bar{a} \partial s'}$ is bounded in $(s^o, s^o + \eta)$. As $N$ and $D$ are positive and, thus, $dV_l > 0$ and $\frac{\partial \bar{a}}{\partial s'} < 0$ and it holds true that $\frac{\partial V_l}{\partial s'} < 0$ and $\frac{\partial^2 V_l}{\partial \bar{a} \partial s'} > 0$, there would always be such a tiny $\xi$ that, for all $(s^o, s^o + \xi)$,

$$-\delta \left( dV_l + \frac{\partial V_l \partial \bar{a}}{\partial \bar{a} \partial s'} - (1 - \bar{a}) \left( \frac{\partial^2 V_l \partial \bar{a}}{\partial \bar{a}^2 \partial s'} + \frac{\partial^2 V_l}{\partial \bar{a} \partial s'} \right) \right) < 0.$$

This demonstrates that $D$ is bounded away from zero for $N$ and $D$ positive.

By a very similar argument we can show that if $N < 0$ and $D > 0$ such that $\frac{\partial \bar{a}}{\partial s'} > 0$, if $D \to 0$ that would take place because IR-l is starting to bind and not just because it should happen that $-\delta V_l + (1 - \bar{a})\delta \int_{s'}^1 \frac{\partial p(s)}{\partial \bar{a}} dF_l(s) \to -(u_h - c_l)$ for some other cause whereas IR-l remains lax.

In conclusion, the relevant cases for us are those in which $D > 0$ such that the sign of $\frac{\partial \bar{a}}{\partial s'}$ is determined by $N$. This means that, if $\frac{\partial V_l}{\partial s'} > 0$, then $\frac{\partial \bar{a}}{\partial s'} < 0$ and, if $\frac{\partial V_l}{\partial s'} > 0$, then $\frac{\partial \bar{a}}{\partial s'} > 0$. Thereby, the cutoff and the acceptance probability have similar effects on low quality seller values:

$$\left( \frac{\partial V_l}{\partial s'} + \frac{\partial V_l \partial \bar{a}}{\partial \bar{a} \partial s'} \right) \frac{\partial \bar{a}}{\partial s'} < 0 \text{ and } \frac{\partial V_l}{\partial \bar{a} \partial s'} (\frac{\partial V_l \partial \bar{a}}{\partial \bar{a} \partial s'}) > 0$$

$\blacksquare$

Proof of Proposition 8. This holds true by the construction. See the algorithm presented. $\blacksquare$

Proof of Lemma 8. Take a look at the constraint IR-l’ and define a new function $g^3$:

$$g^3(s', q) = u_l - c_l - \delta V_l(s', q),$$

such that $g^3 = 0$ if IR-l’ is satisfied. Next, differentiate this function totally to obtain

$$\frac{\partial V_l}{\partial q} + \frac{\partial V_l \partial s'}{\partial s' \partial q} = 0$$

$\blacksquare$

Proof of Lemma 7. The proof of Lemma 7 is similar to the proof of Lemma 4. $\blacksquare$

Proof of Lemma 9. Before the quit rate is one the change in high quality seller values is given by

$$dV_h = \left( -(p(s') - c_h) f_h(s') + \int_{s'}^1 \frac{\partial p(s)}{\partial s'} dF_h(s) \right) \frac{\partial s'}{\partial q} + \int_{s'}^1 \frac{\partial p(s)}{\partial q} dF_h(s).$$
By construction, the low quality seller values are constant

\[ dV_l = \left( (1 - q)(u_l - c_l)F_l(s') - (p(s') - c_l) f_l(s') + \int_{s'}^{1} \frac{\partial p(s)}{\partial s} dF_l(s) \right) \frac{\partial s'}{\partial q} \]

\[ + \int_{s'}^{1} \frac{\partial p(s)}{\partial q} dF_l(s) - (u_l - c_l)F_l(s') = 0. \]

After the quit rate is one the changes in both low and high quality seller values are given by

\[ dV_\theta = -(p(s') - c_\theta) f_\theta(s') + \int_{s'}^{1} \frac{\partial p(s)}{\partial s} dF_\theta(s) < 0, \]

since \( p(s') > c_h > c_l \) and it has been proofed earlier that \( \frac{\partial p(s)}{\partial s} < 0 \) after \( q = 1 \). ■

Proof of Proposition 9. This holds true by the construction. See the algorithm presented. ■

Appendix B

Proofs for Subsection 2.3

Proofs of Lemmata 1 and 2. Due to the discounting, it is better to trade at once than wait for the repetition of this stationary equilibrium. In Lemma 1, the seller is willing to offer \( p \) provided that any other price offer is rejected for the low off path beliefs that would arise. In Lemma 2, such adverse beliefs present no threat as both parties know the quality and the sellers can use \( p_h = u_h \) and \( p_l = u_l \) to extract full surplus. ■

Proofs for Subsection 2.4.1

Pooling equilibria in meetings

As both sellers offer the price \( p \) for the associated signal realization, \( s \), its use prompts no updating of buyers’ beliefs, \( E_\gamma(u|s, p) = E_\gamma(u|s) \). Consequently, to make the buyers accept a pooling price \( p(s) \), it is necessary that \( p \leq E_\gamma(u|s) - \delta V_h \) and, to make both low and high quality sellers offer the pooling price, that \( p \geq \max \{ \delta V_l, c_h + \delta V_h \} \). Otherwise, any price would do as deviations can be kept at bay by attributing them to low quality sellers. If \( p < E_\gamma(u|s) - \delta V_h \), the buyers must accept the pooling price for probability one but, if \( p = E_\gamma(u|s) - \delta V_h \), they can also randomize between accepting and rejecting the pooling price. ■
Separating equilibria in meetings

As different sellers use different prices, \( p_l \) for the low and \( p_h \) for the high, in a separating equilibrium given the signal, \( s \), the buyers’ beliefs become degenerate after the price offer has been made, \( E_\gamma(u|s,p_l) = u_l \) and \( E_\gamma(u|s,p_h) = u_h \). Thus, to make the buyers accept the price, it must be so that \( p_l \in [c_l,u_l] \) and \( p_h \in [c_h,u_h] \), and to keep the low quality sellers from mimicking the high quality sellers that

\[
a(p_l)p_l + (1 - a(p_l)) \delta V_l \geq a(p_h)p_h + (1 - a(p_h)) \delta V_l.
\]

This implies that, first, \( p_l \geq \delta V_l \) (the low price must be above the quality sellers’ outside options to make it worthwhile to offer it) and, second, given that is must hold that \( p_l < p_h \), also \( a(p_l) > a(p_h) \) must hold (the low price must be accepted more frequently than the high price to satisfy the incentive condition for the low). As a result, to make the buyers randomize between accepting and rejecting the high price, it is necessary that the high price keeps the buyers at their continuation values, \( p_h = E_\gamma(u|p_h,s) - \delta V_h \). Obviously, the high quality sellers have no incentive to mimic the low quality sellers as the low price \( p_l \) is below the high cost \( c_h \).

Observe also that the low price must keep the buyers at their continuation values, \( p_h = E_\gamma(u|p_h,s) - \delta V_h \), but they do have to accept it for probability one, \( a(p_l) = 1 \). Namely, if the former requirement would not hold, there would be a profitable deviation from \( p_l \) to \( p_l - \eta \) to make the buyer accept the price for certain and, if the latter would not hold, there would be a profitable deviation from \( p_l \) to \( p_l + \eta \) to keep the acceptance rate the same but to increase the price offer, for some \( \eta > 0 \). As buyers’ beliefs are already the harshest possible for \( p_l \) it is impossible to discipline such deviations by out of equilibrium path beliefs, which could not get worse still. ■

Semi-pooling equilibria in meetings

To the contrary, suppose the sellers mix between two pooling prices, \( p^1 \) and \( p^2 \) such that \( a^1 := a(p^1) \) and \( a^2 := a(p^2) \). With no loss of generality, \( p^1 < p^2 \) such that \( a^1 > a^2 \). Note that, by individual rationality, \( p^j - c_\theta \geq \delta V_\theta \) if price \( p^j \) is to be used by sellers of quality \( \theta = h,l \). Now, to keep the high quality sellers mixing between \( p^1 \) and \( p^2 \), it must hold that

\[
a^1(p^1 - \lambda) + (1 - a^1) \delta V_h = a^2(p^2 - \lambda) + (1 - a^2) \delta V_h
\]

\[
a^1p^1 = (a^1 - a^2)(\delta V_h + \lambda) + a^2p^2
\]

and, to keep the low quality sellers mixing between \( p^1 \) and \( p^2 \), it must hold that
\[ a^1 p^1 + (1 - a^1) \delta V_l = a^2 p^2 + (1 - a^2) \delta V_l \]

\[ a^1 p^1 = (a^1 - a^2) \delta V_l + a^2 p^2. \]

Clearly, both cannot hold simultaneously so long as \( \delta V_h + \lambda \neq \delta V_l \). Rather, for \( \delta V_h + \lambda > \delta V_l \), if the high quality mixes between \( p^1 \) and \( p^2 \), then the low quality prefers to use \( p^1 \) only and, if the low quality mixes between \( p^1 \) and \( p^2 \), then the high quality prefers to use \( p^2 \) only. In the former case, \( p^2 \) is a separating price and \( p^1 \) is a pooling price; in the latter case, it is the other way. We show later that, indeed, \( \delta V_h + \lambda > \delta V_l \) for any \( t \).

Note that separating prices are perfectly revealing such that \( E(u|s,p_l) = \lambda \) and \( E(u|s,p_h) = 1 \) for any \( s \). This implies that \( p_l \in [0,\lambda] \) and \( p_h \in [1-\lambda,1] \) for the sellers to offer them and for the buyers to accept them. The pooling price has to lie within \( [1-\lambda,1] \). Furthermore, we can show that the separating prices are unique for both low and high quality sellers: (i) By the intuitive criterion, \( p_l = \lambda - \delta V_h \) (otherwise, there is a profitable deviation to \( p_l + \epsilon \) for a higher price) and \( a(p_l) = 1 \) (otherwise, there is a profitable deviation to \( p_l - \epsilon \) for a higher acceptance rate) since any price deviations in below \( \lambda \) must be attributed to low quality sellers. (ii) To stop the low quality sellers from mimicking the high quality sellers, it must be that \( a(p_h) \leq \frac{p_h - \delta V_l}{p_h - \delta V_h} \) and, thus, to keep the buyers mixing, it must be that \( p_h = 1 - \delta V_h \) such that the buyers are indifferent between accepting and rejecting \( p_h \). As a result, \( p_l = \lambda - \delta V_h \) and \( p_h = 1 - \delta V_h \).

We show next that \( \delta V_h + \lambda > \delta V_l \). This finding follows as the profit of the low quality sellers cannot exceed the profit of the high quality sellers by more than \( \lambda \) whether the sellers pool, semi-pool or separate. For any shared signals \( s \) for which the sellers pool or semi-pool, the high quality sellers get \( p(s) - \lambda \cdot (a(p(s) - \lambda)) \) and the low quality sellers get \( p(s) \cdot a(p(s)) \) and, for any shared signals \( s \) for which the sellers separate, the high quality sellers may receive no payoffs but the the low quality sellers do not get more than \( \lambda \), either.

**Proof of Proposition 3.3.** As we are looking for a seller maximal equilibrium, to simplify the notation, it is without loss to assume already that \( V_h = 0 \). Otherwise, we ought to scale down each price \( p \) by subtracting from them \( V_h \). To satisfy the individual rationality constraints, the prices must also be such that \( p - c_0 \geq \delta V_h \).

**Case 1:** Suppose both sellers use a pooling price \( p \) and the low quality sellers mix between the pooling price \( p \) and a separating price \( p_l \). The maximal pooling price \( E(u|s,p) \in [E(u|s),1] \) clearly depends on the ratio \( \beta = Pr(p|l) \) in which the low quality sellers mix. To keep the low quality sellers indifferent between offering the pooling price and the separating price, as \( p_l = \lambda \) and \( a(p_l) = 1 \), the acceptance probability of the higher price offer \( p \) must be equal to \( a(p) = \frac{\lambda - \delta V_l}{p - \delta V_l} \).
Observe that, for any suitable $\gamma$, $s$, and $V_{t+dt}(\sigma_{t+dt})$, there can exist many such equilibria with different $\beta$ and, hence, $p$. As low quality sellers are mixing between $p_l$, that is accepted for sure, and $p$, what they get, $V_t$, is constant over all such equilibria. However, as high quality sellers are pooling to $p$, what they expect to obtain equals

$$V_h(s) = \frac{\lambda - \delta V_l}{p - \delta V_l} (p - \lambda - \delta V_h) + \delta V_h,$$

which is maximized by the highest feasible price $p = 1$ ($\beta = 0$), whenever $\frac{p - \lambda - \delta V_h}{p - \delta V_l} < 1$, and by the lowest feasible price $p = E(u|s)$ ($\beta = 1$), whenever $\frac{p - \lambda - \delta V_h}{p - \delta V_l} > 1$. That is, any semi pooling equilibrium of this type would be defeated by either the best separating equilibrium with $p = 1$ or by the best pooling equilibrium with $p = E(u|s)$ and $a(p) = 1$. As $V_h + c_h > V_l$, it is rather the former than the latter.

Case 2: Suppose both sellers use a pooling price $p$ and the high quality sellers mix between the pooling price $p$ and a separating price $p_h$. The maximal pooling price $E(u|s,p) \in [\lambda, E(u|s)]$ clearly depends on the ratio $\beta = Pr(p|h)$ in which the high quality sellers mix. To keep the high quality sellers indifferent between offering the pooling price and the separating price, as $p_h = 1$, the acceptance probability of the higher price offer $p_h$ must be equal to $a(p_h) (1 - c_h - \delta V_h) = a(p) (p - c_h - \delta V_h)$. Note also that, if $a(p_h) (1 - c_h - \delta V_h) = a(p) (p - c_h - \delta V_h)$ is satisfied, then $a(p_h) (1 - \delta V_l) \leq a(p) (p - \delta V_l)$ is satisfied, by $\delta V_h + c_h > \delta V_l$, such that the low quality seller have no incentive to mimic the high quality sellers.

Observe that, for any suitable $\gamma$, $s$, and $V$, there can exist many such equilibria with different $\beta$ and, hence, $p$. As high quality sellers are mixing between $p_l$ and $p$ and low quality sellers are pooling to $p$, what they expect to obtain equals

$$V_l(s) = a(p)p + (1 - a(p)) \delta V_l,$$

$$V_h(s) = a(p) (p - \lambda) + (1 - a(p)) \delta V_h,$$

which are maximized by the highest feasible price $p = E(u|s)$ ($\beta = 1$) and the highest feasible acceptance probability $a(p) = 1$. That is, any semi pooling equilibrium of this type would be defeated by the best pooling equilibrium with $p = E(u|s)$ and $a(p) = 1$.

As semi-pooling equilibria are either as in Case 1 or as in Case 2, any semi pooling equilibrium is defeated by either the best pooling equilibrium or by the best separating equilibrium, or the sellers’ individual rationality constraints hold and those sellers prefer to resume their search.

Proof of Lemma 3. As $p = E(u|s) > p_l$, the low quality sellers are always better off if they play the best pooling equilibrium than if they play the best separating equilibrium. This is not the case for high quality sellers, however. Denote by $s^h \in (0,1)$
the signal that solves \( p(s) - \lambda = a(p_h) (p_h - \lambda) + (1 - a(p_h)) \delta V_h \) and by \( s^l \in (0,1) \) the signal that solves \( p(s) - \lambda = \delta V_h \). Thus, for \( s \geq s^h \), the high quality sellers are better off if they play the best pooling equilibrium than if they play the best separating equilibrium, for \( s < s^h \) but for \( s \geq s^l \), it is the opposite. Notice the individual rationality constraint, for the low quality sellers or for the high quality sellers, will always bind for sufficiently low signal values. If \( p(s) - \lambda < \delta V_h \), pooling is not feasible and, \( p_l < \delta V_l \), separation is impossible. In the former case the high quality sellers would rather quit than pool, in the latter case the low quality sellers would rather quit than separate.

**Proofs for Subsection 2.4.2**

*Proof of Proposition 4.* We have already proofed that for any cutoff there exists a unique acceptance probability \( a(s') \) and it defines a function \( s' \mapsto a(s') \) that is continuous in \( s' \).

Now, we show that, first, there is always a cutoff \( s' \) such that IR-h binds and then that, for \( \lambda \), IR-l does not bind for this cutoff \( s' \). This proofs the result.

To proceed, consider a mapping \( g^4 \)

\[
g^4(s') = p(s', s', a(s')) - \lambda - \delta V_h(s', a(s')).
\]

Clearly, IR-h is satisfied iff \( g^4 \geq 0 \) and binding iff \( g^4 = 0 \). As \( g^4 \) is continuous in \( s' \) and \( g^4(0) < 0 \) and \( g^4(1) > 0 \), is must have a root at some \( s' \in (0, 1) \).

To be more specific, we calculate the bound values as to see that one is positive and the other one is negative to show that the fixed point is in between

\[
g^4(0) = p(0, 0, a(0)) - c_h - \delta \left[ \int_0^1 (p(s, 0, a(0)) - c_h) f_h(s) ds \right]
\]

\[
= -\delta \left[ \int_0^1 (p(s, 0, a(0)) - c_h) f_h(s) ds \right] < 0
\]

and

\[
g^4(1) = p(1, 1, a(1)) - c_h - \frac{a(1)}{1 - (1 - a(1))\delta} (u_h - c_l)
\]

\[
= u_h - c_h - \frac{a(1)}{1 - (1 - a(1))\delta} (u_h - c_l) > 0.
\]

Now, what remains is to show that, for sufficiently large gains from trade in the low quality, \( \lambda = c_h = u_l \), where \( u_h = 1 \) and \( c_l = 0 \), there exists a cutoff \( s' \) for which IR-h binds and IR-l does not bind. This is very simple. If \( \lambda = u_l - c_l > \delta \) it is impossible to raise the value \( \delta V_l \leq \delta(u_h - c_l) = \delta \) above \( u_l - c_l \) no matter what is played; the low quality sellers are always better of by separating than by returning to the market for search.
Note that this equilibrium type can only exist if there are gains from trade in low quality, i.e., $\lambda > 0$. ■

Proof of Proposition 5. Consider a situation where the sellers are forced to pool for signals above a cutoff $s'$ and quit for signals below a cutoff $s'$. At the beginning, we do not pay attention to the fact that the sellers may not have no incentive to do so but, ultimately, we are interested to find a pair $(\lambda, s'(\lambda))$ of exogenous $\lambda$ and endogenous $s'$ for which this would be a seller maximal equilibrium such that IR-h would be satisfied as an equality at $s'$

$$p(s', s') - c_h = \delta V_h = \frac{\delta \int_{s'}^1 (p(s', s) - c_h) dF_h(s)}{1 - \delta F_h(s')} \iff p(s', s') - \lambda = \delta V_h = \frac{\delta \int_{s'}^1 (p(s', s) - \lambda) dF_h(s)}{1 - \delta F_h(s')}$$

and IR-l for separation would not be satisfied at $s'$

$$u_i - c_l < \delta V_l = \frac{\delta \int_{s'}^1 (p(s', s) - c_l) dF_l(s)}{1 - \delta F_l(s')} \iff \lambda < \frac{\delta \int_{s'}^1 (p(s', s) dF_l(s))}{1 - \delta F_l(s')}.$$ 

We refer to equation 2.9 as IR-h' and to equation 2.10 as IR-l''.

For later use, note that IR-h' can be rewritten as

$$p(s', s') - \lambda = \delta V_h = \delta \int_{s'}^1 (p(s', s)) dF_h(s)$$

This is applied to show the monotonicity of $\pi$ and $s$.

We consider also function $g^5$

$$g^5(s') = p(s', s') - \lambda - \delta V_h(s').$$

Note that, for any $\lambda \in [0, \lambda]$, such that $\lambda \in (0, 1)$, $g^5$ is continuous in $s'$, positive for $s' = 1$ as

$$g^5(1) = 1 - \lambda - \delta V_h(1)$$

and negative for $s' = 0$ as
\[ g^5(0) = 0 - \lambda - \delta V_5(0), \]

This implies that for any \( \lambda \in [0, \overline{\lambda}] \) there always exists a fixed point \( s'(\lambda) \in (0,1) \) such that IR-h’ holds. We have not got a proof showing that the fixed point is unique but it appears safe to assume that the number or roots is finite such that a maximal one and a minimal one exist. They are denoted by \( s(\lambda) \) and \( g(\lambda) \), respectively. Observe that both \( s(\lambda) \in (0,1) \) and \( g(\lambda) \in (0,1) \) are increasing in \( \lambda \):

\[ g^5(s(\lambda), \lambda) = p(s(\lambda), s(\lambda)) - M(s(\lambda))\lambda - \delta W_h(s(\lambda)) = 0 \]

\[ \Rightarrow g^5(s(\lambda), \lambda + \varepsilon) = p(s(\lambda), s(\lambda)) - M(s(\lambda))(\lambda + \varepsilon) - \delta W_h(s(\lambda)) < 0 \]

but as \( g^5(1, \lambda + \varepsilon) > 0 \) the largest root for \( \lambda + \varepsilon \) must lie between \( s(\lambda) \) and 1,

\[ s(\lambda) < s(\lambda + \varepsilon), \]

and

\[ g^5(g(\lambda), \lambda) = p(g(\lambda), g(\lambda)) - M(g(\lambda))\lambda - \delta W_h(g(\lambda)) = 0 \]

\[ \Rightarrow g^5(g(\lambda), \lambda - \varepsilon) = p(g(\lambda), g(\lambda)) - M(g(\lambda))(\lambda - \varepsilon) - \delta W_h(g(\lambda)) > 0 \]

but as \( g^5(0, \lambda - \varepsilon) < 0 \) the smallest root for \( \lambda - \varepsilon \) must lie between 0 and \( g(\lambda) \),

\[ g(\lambda) > g(\lambda - \varepsilon). \]

This implies that all those pairs \( (\lambda, s') \) that we take interest in are in \( [0, \overline{\lambda}] \times [g(0), s(\overline{\lambda})] \).

Next, we define two sets for each arbitrary \( \overline{\lambda} \in (0,1) \) and point out that one is included in the other one

\[ S(\overline{\lambda}) = \{ (\lambda, s')|\lambda \in [0, \overline{\lambda}], s' = s'(\lambda) \} \subset \overline{S(\overline{\lambda})} = \{ (\lambda, s')(\lambda, s') \in [0, \overline{\lambda}] \times [g(0), s(\overline{\lambda})] \}. \]

Then, we consider the two related minimization problems where the value of the latter is bounded by the value of the former as

\[ \min_{(\lambda, s') \in S(\overline{\lambda})} V_I(\lambda, s') \geq \min_{(\lambda, s') \in \overline{S(\overline{\lambda})}} V_I(\lambda, s') =: V_{I_\overline{\lambda}}. \]

Last, we note that the value \( V_{I_\overline{\lambda}} \) is well-defined as the function \( V_I \) is continuous in \( (\lambda, s') \) and the set \( \overline{S(\overline{\lambda})} \) is compact. The value \( V_{I_\overline{\lambda}} \) is also positive as \( V_I(\lambda, s') \) is positive for any
pair \((\lambda, s') \in S(\lambda)\). As a result, we have a positive minimum for low quality seller values in this equilibrium type; in particular, the low quality seller values do not get smaller and smaller as \(\lambda\) does. This permits us to conclude that both IR-h’ and IR-l’” are satisfied for \(\lambda < \delta V_i\).

Note that this equilibrium type can only exist if there are gains from trade in high quality, i.e., \(\lambda < 1\). ■
Bibliography


Chapter 3

Obfuscation by substitutes: Shopping frictions and equilibrium price dispersion within stores
List of symbols

\[ B = [0, 1] \] the set of buyers
\[ t \in [0, 1] \] time index
\[ i \in \{1, 2\} \] seller/store index
\[ n^i \in \{1, 2\} \] the number of items in store \( i \)
\[ n \in \{1, 2\} \] index of items available in a store
\[ k \in \{1, 2, 3, 4\} \] index of items observed by a buyer

\[ \theta \in (0, \infty) \] the Poisson arrival/finding rate of the prices in a store
\[ \sigma(n) \in (0, \infty) \] the multiplier of \( \theta \) for \( n \) (for economies of scale in search)

\[ F^i \in \Delta [0, 1] \] the joint price distribution in store \( i \)
\[ F^i_n \in \Delta [0, 1] \] the marginal price distribution of item \( n \) in store \( i \)
\[ p^i_n \in [0, 1] \] (realized) price of item \( n \) in store \( i \)
\[ E(p|F^i_n) \in [0, 1] \] expected price of item \( n \) in store \( i \)

\( \omega \) buyer’s search outcome
\( \omega_0 \) ‘no price from store 1 nor from store 2’
\( \omega^i_m \) ‘a monopoly price from store \( i \), no price from store \( -i \)’
\( \omega^{1,2}_m \) ‘a monopoly price from store 1 and from store 2’
\( \omega^i_d \) ‘a discounted price from store \( i \), no price or a higher price from store \( -i \)’
\( \omega^{1,2}_d \) ‘a discounted price from store 1 and from store 2’
\( \omega^{1,2}_A \) ‘all prices from store 1 and from store 2’
\[ E(p|\omega) \] expected price given outcome \( \omega \)

\[ P = \{p^i_n\} \] the set of prices in the market
\[ B_K \in [0, 1] \] the mass of buyers who end with prices \( K \subset P \)
\[ B_k \in [0, 1] \] the mass of buyers who end with \( k \in \mathbb{N} \) prices
\[ B^i \in [0, 1] \] the mass of buyers who end with seller \( i \)’s price
Obfuscation by substitutes: Shopping frictions and equilibrium price dispersion within stores

Abstract

We provide a novel search model that features in-store frictions and equilibrium price dispersion both within and across stores. The frictions originate from the gradual arrival of price information within stores and the existence of deadlines for buyers. We show that sellers have an incentive carry several similar items and generate price variation among these items to amplify the existing search frictions and create barriers to switching in an environment where none exist initially. It also helps them to discriminate better between buyers, who end with diverse degrees of price information. As the number of items in stock expands, sellers can extract more profits.

Keywords: Obfuscation; Substitutes; Search frictions in-store; Price variation in-store; Deadlines. JEL-codes: D43, D83.
3.1 Introduction

The Internet is full of different online stores and almost all offer a lot of alternatives for exploration. Click on one of these, and be flooded by a visual stream of endless products where a lower price or a better matched product is always, seemingly, just a click away. Indeed, there could be so much to see at a single seller nowadays that, once you are done, there is not much time left for shopping in any other store. Déjà vu?

We show in this paper that such variability of alternatives can be applied to amplify the existing search frictions and create new barriers to switching in an environment where none exist initially. This works even for simple price search; it is not necessary to introduce product differentiation or any kind of ex ante consumer heterogeneity in the model to tackle the research question.\footnote{Obviously, this is not to say that product differentiation would not matter in search. Our aim is rather to point out more elementary search efficiency related mechanisms, which maybe arise as a positive side product of other benefits of deepening variety provision but can affect search, prices and profit all the same. It is to this aim that we assume a more abstract simpler approach where the items in stock are homogeneous. To transfer this idea back to practice of retailing, there could naturally be various superficial differences between the products as long as most buyers regard them as perfect substitutes essentially.} We find that sellers have an incentive to generate price variation across identical products in their store to keep buyers searching longer in there; this leaves the buyers with less time for shopping in the other competing store. To put the idea bluntly, the sellers gain if they all add to their original stock more items, of the same kind they already have but with higher prices, and then spread them around to let the buyers find then one-by-one. The underlying assumptions are that the buyers search under time-pressure and the items available in a given store are found randomly and gradually.

Our paper hence contributes to literature analyzing retailer strategies to lock-in consumers (e.g., Ellison and Wolitzky (2012) on price obfuscation and Klemperer (1987) on switching costs) and to literature trying to explain price dispersion among homogenous items (see Baye et al. (2006), Burdett and Judd (1983) and Butters (1977)). Yet, while the latter strand of literature has concentrated on price dispersion across stores we find it also within stores.

We consider a duopoly with two similar sellers and a unit mass of buyers. All items are of the same given type but a seller could carry them in multiple replicas and tag a different price quote on each. In the base line case, both sellers have exactly two items in stock. This number is common knowledge but the prices are the sellers’ private information until the buyers find them.

We use a new dynamic model which abstracts from the hold-up problem present in many optimal stopping problems with endogenous price distribution (Diamond, 1971). Instead, we build on two novel features. First, the buyers search with a deadline: their search costs jump from zero to infinity at the deadline. Second, the prices in the stores are not found immediately once a buyer enters a store but gradually one-by-one, after a
random wait time.

The buyers can switch the stores freely as long as they have time. There is no explicit switching cost. However, when there are different prices available in a store, we find that the buyers optimally switch the stores only when they have discovered the lowest one. We concentrate on a set of collusive equilibria where the sellers fix one of the two prices at a higher monopoly level but, for a probability strictly between zero and one, let their other price be a lower discount price. Since the buyers know this, if they first spot a monopoly price, they have an incentive to keep on searching in their start store in hope of finding another price at a discount. This lock-in effect, that postpones the consumer’s switching time away from the start store, reduces the sellers’ incentive to undercut each other’s prices compared to the setup in which both sellers have one item; for that case, there exists a unique mixed equilibrium à la Varian (1980) and Stahl (1989) where stores almost never use the monopoly price.

The generated price variation helps the sellers also to discriminate better between buyers with more and less price information. While the buyers are similar *ex ante*, their search outcomes differ *ex post* due to the random arrival of price information. The sellers can extract more profits with an additional price instrument: they can use the monopoly price to tax some of the least informed buyers and the discount price to compete for the more informed ones.

Interestingly, we also prove that, as the number of these similar items in stock expands, the sellers can extract more and more surplus. They can then use a combination of a single discount price and a larger number of monopoly prices. In an extension, we allow for economies of scale in search (faster or slower information arrival rate with a larger number of items) and find that our results remain robust at least to moderate positive economies of scale.

Surprisingly, although the buyers search dynamically rationally – they always go for the store where they are most likely to find the best prices – and take no interest in any other characteristics but prices, we thus show that the stores can drastically reduce search efficiency by colluding, not in higher prices directly but, indirectly, in a higher number of expensive items in stock.

A store’s equilibrium pricing pattern could be very rich. Generally, there are three different regimes even for two prices in a store: in the most typical case, the store has a monopoly price and a slightly discounted price (in the hi-lo regime). Yet, in some cases, both prices are monopoly prices (in the hi-hi regime), or both prices are strongly discounted prices (in the lo-lo regime). The buyers switch the seller later when it is in the hi-hi regime than when it is in the hi-lo regime or in the lo-lo regime. As a result, the sellers are best sheltered from competition when they have worst prices.

Note that, if the sellers tag similar items in stock with different prices, they implicitly also commit to improving their best offer to a buyer as time goes on. This resembles a
bit the effect of hiring sales people who lower the price for the consumers little by little to lock them in. However, when items have different fixed prices, this can be implemented in an entirely passive way from the part of the seller. No sales people are needed in the bargaining and the commitment issues as for when to give the promised discount – now, later or never – can be totally avoided. It is the buyer doing all the work.

Our model is inspired by the observed inventory expansion in retail sector. Nowadays, wherever one does his or her shopping there is usually just a huge number of different items available. For instance, in UK the average number of items in stock that a supermarket carries was 38 718 in 2010 and 42 686 in 2012 (Food Marketing Institute, 2014). Clearly, a bulk of this is related to product differentiation, inventory competition, and vertical relations.² Anyhow, a typical store has also got a number of products the buyers probably regard as nearly perfect substitutes.³⁴⁵ Of course, that would not matter for search if the buyers entering the stores had an ability to somehow instantaneously scan in all the information about the items in the store and then head for the best ones right on. Unfortunately, that is not so. There exist frictions.

To give an example, Reutskaja et al. (2011) used an experimental setup to mimic consumers’ experience in a supermarket under heavy time pressure. They recorded the subjects eye fixations as they sampled through different alternatives that were presented to them to see how people actually search. They found that the subjects were good at optimizing within the set of products they had time to see but not otherwise. It always took some time to fixate on an alternative.⁶ For another example, Pinna and Seiler (2013) estimated from consumers’ walk path data that shopping in a grocery store takes on average ten minutes and an additional search minute lowers the expenditures by $1.5, per category per consumer. This suggests the presence of quite heavy in-store frictions and price variation across easily substitutable items.

As our sellers use additional inventories to delay search within stores and, thereby, switching from store to store, our paper is quite closely related to the expanding obfuscation literature, which focuses on sellers’ incentives for increasing the time cost of search.

²See, for example, the seminal papers by Wolinsky (1987) and Klemperer and Padilla (1997), Mahajan and Van Ryzin (2001), and Avenel and Caprice (2006).
³Auchan in France offers, for instance, just ordinary milk, “lait demi-écrémé stérilisé UHT, 1 l”, at least, under names “GrandLait”, “la Vache au bon lait”, “’J’♥ le lait d’ici”, and its own basic brand, placed seemingly scattered such that, even when there is no congestion, the time cost of comparing the prices would be counted in minutes (see Dreze et al. (1995)).
⁴Anupindi et al. (1998) find that consumers often switch brands if their favorite brand is unavailable on a vending machine.
⁵See Reutskaja et al. (2011) on the buying process in a modern supermarket under an overflow of alternatives and binding time budgets, and Huberman et al. (1998) on the browsing process on the internet and the problems of slow access rate and inability to find relevant information, or literature on retailing, say, Donovan et al. (1994); Puccinelli et al. (2009).
⁶This suggests that there is a natural upper bound on the potential economies of scale in search (for a larger number of items) given by this minimal fixation time. In other words, additions to available items might slow down search even when the products are put side by side and there is no need to do more clicking on walking to read the next price quote.
The seminal papers by Ellison and Ellison (2009), Ellison and Wolitzky (2012) and Wilson (2010) provide both anecdotal and empirical evidence on obfuscation in retailing and e-retailing and propose several mechanisms to rationalize this obfuscation.\footnote{Note that alike what we have, the model of costly obfuscation by Ellison and Wolitzky (2012) begins from the idea that it is impossible or not worthwhile for the buyers to continue searching ad infinitum: In their model, this is made operational by postulating that search costs \( c \) are unbounded, increasing and convex in accrued search time \( t \), i.e. \( c(t) \to \infty \) as \( t \to \infty \), \( c' \geq 0 \) and \( c'' \geq 0 \). In our model, this happens instead because the finite search horizon generates a form of extreme convexity of search costs \( c \) in search time \( t \), as buyers do not mind shopping before deadlines \( t = 1 \), i.e. \( c(t) = 0 \), for \( t < 1 \), but could not search any longer, i.e. \( c(t) = \infty \), for \( t \geq 1 \). As another important modeling difference, Ellison and Wolitzky (2012) look at buyers’ stopping rule. We make stopping decisions trivial and look at buyers’ switching rule.} Other related papers include Ireland (2007), Petrikaite (2014) and Armstrong and Zhou (2011). The articles by Ireland (2007) and Petrikaite (2014) are quite close to ours, specifically.

Ireland (2007) allows the stores to quote multiple prices in order to reduce the effectiveness of search engine search. The buyers get a sample of prices, one or two each, but cannot see if the prices come from a single seller or from two different, competing sellers. In equilibrium, every store sends out two perfectly correlated duplicate prices, two to frustrate or obfuscate the consumers who get this fixed number of prices and perfectly correlated to avoid undercutting their own price. Instead, we find that, if buyers search under a deadline and there are frictions within stores, the sellers use a high-price-low-price strategy even when the buyers are well aware of where each price is originating from.

Petrikaite (2014) considers a monopoly with several differentiated products. The monopolist has an incentive to raise search costs of some varieties to control their search order. This will help to reveal information about match values for the earlier sampled varieties and, thus, allow the store to cash on the externalizes that the abundance of varieties yields to buyers. As buyers search less, from a consumers’s standpoint, just like in our case, the model has a sense of redundancy in the seemingly rich variety present in-store. Apart from this, she and our models are quite different overall although both deal with the case of explicit frictions within a store; usually a store is treated as a black box.

Related problems are addressed also in the papers by Eliaz and Spiegler (2011b) and Hagiu and Jullien (2014), looking at markets with intermediaries and their incentives to divert the demand, and by the behavioral approaches, for example, in the papers Ellison (2005); Gabaix and Laibson (2006); Eliaz and Spiegler (2011a).

In their seminal article on inventory expansion under differentiation, Klemperer and Padilla (1997) note that stores have an incentive to provide excessive variety in order to steal more buyers from the competitors if the consumers appreciate more choice, yet, prefer to patronize same providers. In another classic paper, Wolinsky (1987) establishes the possibility that a store can price discriminate by the selective use of differentiating brand labels put side by side with unlabeled products. The motive to increase the inventory is present also in the "newsboy problem" and the related inventory competition literature.
(see, e.g., Mahajan and Van Ryzin (2001)). Our results in this paper work through different channels but, in a broad sense, generalize this inventory expansion incentive to homogenous commodities.

Given that stores choose the joint price distribution for the items they carry in stock, our paper comes also quite near to some papers that concentrate explicitly on multiproduct retailing in which, as the key distinction, buyers search for a bundle of items and not a single item like here. In one of the first papers to analyze rigorously multiproduct equilibrium price dispersion, McAfee (1995) characterizes the joint price distributions for two classes of equilibria in symmetric pricing strategies, constant profit equilibria and frontier equilibria. Shelegia (2012) derives the optimal pricing strategy for two items which are interdependent and typically purchased together. He observes a negative price correlation for complements, their total is kept constant, but no correlation or a positive price correlation for substitutes.

In an independent study, Menzio and Trachter (2015) develop another elegant price search model that generates price dispersion within and across stores. In contrast to our model where the buyers differ only \textit{ex post}, they have buyers who differ \textit{ex ante} in their ability to shop around, at different stores and at different times. As is the case also here, they find that price dispersion helps the sellers to better discriminate between different buyers. Otherwise, their setup is different and they do not consider the delaying lock-in effect that availability of multiple different prices has on consumers during a single search spell. In their paper, a pre-requisite for price dispersion within stores is a particular correlation structure between different buyer groups: the buyers who are able to shop at uncomfortable times should also be more likely to shop both nearby and further.

The paper is structured in the following way. The setup is shown in Section 2. Section 3 analyzes mainly symmetric equilibria, starting from the one item case and some trivial multi-price equilibria and thereby working towards the collusive equilibria that are the main interest in this paper. We first give a good treatment of the two item case (in Section 3.3) and then move on to the general $k$ item case (in Section 4.1). In Section 4.2 we consider an extension for economies of scale in search. Section 5 offers some closing remarks. A more complex variant of the equilibrium is derived in Appendix A. Our proofs are mainly in Appendix B, and tables are in Appendix C.

### 3.2 Model

We consider a model of duopolistic price competition in a market that features shopping frictions in-store. There are two stores $i = 1, 2$ and a unit mass of buyers $B = [0, 1]$, each buyer with a unit of time $t \in [0, 1]$.

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8Note that it does not really matter for the results whether this deadline $d = 1$ is five seconds or five decades. Note additionally that, instead of the unit mass of buyers we could have just one representative.
substitutable products or "items in stock" each tagged with its own individual price quote \( p_i^n \), \( n \leq n_i \).

Every buyer is interested in purchasing any one unit of these items, for as low a price as it is possible to find in the limited search horizon. The sellers have unlimited capacities in each item they carry; it does not cost to stock additional items or units or, if it does, it is assumed this cost is sunk.

The payoffs are linear in prices. If a seller trades a unit of an item for price \( p_i^n \) with every buyer in the market, the payoff to a buyer is given by \( 1 - p_i^n \) and the payoff to the seller by \( p_i^n \). In other words, the buyer value of a product is normalized to one and a seller’s reservation value to zero.

The numbers of items in stock, \( n_i \), are fixed and common knowledge across everybody. Our main results concern the case where each seller has two items in stock and, thus, the possibility to have two different prices within its store. Extensions to a larger (symmetric) number of items in stock \( n_i \) are analyzed in Sections 3.3.2 and 3.4.1.

For every point in time \( t \in [0, 1] \), the buyers choose afresh whether to search at seller \( i = 1 \) or at seller \( i = 2 \). They incur no costs of search during this time, that they are prepared to devote to shopping. They can also switch the seller free of cost, as often as they want.\(^{10,11,12}\)

To put it another way, the buyers search under a deadline. Their search costs jump from zero to infinity at the deadline. They have no switching costs.

Search is a random gradual process taking place inside the stores. The items in stock are found one-by-one,\(^{13}\) not immediately once a buyer enters a store.
We use the continuous time Poisson process to model these in-store frictions and capture them in reduced form. When a buyer is in a store, the items in stock are revealed to her at rate $\theta \in (0, \infty)$. Thus, as long as the buyer has not found all the available items, in any tiny (infinitesimal) time interval, $dt := t^2 - t^1 > 0$, for $t^1, t^2 \in [0, 1]$, her probability of discovering a new price quote is $\theta dt$; her probability of discovering more than one new price quote at a time is an event of order $(\theta dt)^2$ or smaller – negligible. During this search, the available items are drawn in random order without replacement until the buyer has exhausted the inventory in the given store. In other words, when the seller $i$ has two items in stock, a buyer who starts to search in the store finds the first one at rate $\theta$. For probability $1/2$ it is price $p^1_i$ and for probability $1/2$ it is price $p^1_i$. Thereafter, if the buyer continues searching with the same seller, the remaining price quote is found at rate $\theta$ as well. If the buyer first found $p^1_i$, the second one is $p^2_i$ – or the other way. After the second price is found, the buyer cannot find anything new in the store.

In the base line model, this rate $\theta$ at which buyers learn new price information is independent of the number of items in stock $n$. In an extension presented in Section 3.4.2, we allow for the possibility that search becomes easier (harder) with a larger number or items in stock and let $\theta$ be an increasing (decreasing) function of $n$.

This game has the following sequential structure:

1. The stores $i = 1, 2$ set the prices, $p^*_n$, for each item they have in stock. The prices are unobservable to the other store and also to the buyers before they find them.

2. The buyers search, for every point in time $t \in [0, 1]$, either at store $i = 1$ or at store $i = 2$ and, when no time is left at $t = 1$, buy for the lowest price they have found.

To determine how the buyers behave when they are indifferent, we add the following assumption:

**Assumption 3**

(i) If the buyers are indifferent between the sellers at $t = 0$, half of them go to seller $i = 1$ and half of them go to seller $i = 2$. (ii) If the buyers are indifferent between the two sellers at $t > 0$, they stay with their current store. (iii) If the buyers are indifferent between stopping the searching and continuing, they stop. (iv) If a buyer’s lowest price from store one equals the buyer’s lowest price from store two, she purchases half the times from store one and half the times from store two.

We analyze subgame perfect equilibria of this game. We begin with some general remarks on them. To facilitate the exposition, we first introduce some helpful notation:

and the buyers search offline) or walking and looking around within the store (this wording seems more appropriate when the time horizon is quite long – say, $1 - 2$ hours – and the buyers search offline) to spot them and to take in the related information content.

Unless it is specified otherwise, it is assumed that beliefs are passive: they are the same on the equilibrium path and off the equilibrium path.
In the most general case conceivable, there could be any finite number of sellers $i \in \mathbb{N}$ in the market, each carrying some finite number of items in stock is $n_i \in \mathbb{N}$. We denote by $F = (F^i, F^{-i})$ their expected prices and by $P = (P^i, P^{-i})$ their realized prices, where $F^i$ gives the distribution of prices at seller $i$ (a marginal of $F$) and $P^i$ the realization of prices at seller $i$ (a subset of $P$). The family of all subsets $A$ of $P$ is denoted by $2^P$, the minimum of $A \in 2^P$ is denoted by $p_{\min A}$, and its distribution function by $F_{\min A}$.

**General analysis: fixed point**

In this paper we determine three equilibria for the case where a seller has more than one product: two trivial multi-price equilibria, the Diamond equilibrium and the Ireland equilibrium, and a class of equilibria that we call hi-lo equilibria because they involve price variation within each store: higher prices (for hi) and lower prices (for lo). We do not try to characterize the equilibrium set in full. However, to give some idea of how the model works in general, we next take a look at the underlying fixed point conditions.

**Buyer’s problem**

The buyers search optimally. They know the numbers of the items in stock $n = (n^i, n^{-i})$ and have some expectations $F = (F^i, F^{-i})$ about their prices. In an equilibrium, it is required that the buyers’ beliefs are correct. Since the items are sampled without replacement, the buyers update their expectations about the remaining, unsampled prices each time a new price is found.

As a result, which seller they select at a given point in time $t \in [0, 1]$ can depend both on the expected prices $F$ and on the realized prices $P = (P^i, P^{-i})$ they have seen. We denote by $p(t) = (p_1, \ldots, p_k) \in P = (P^i, P^{-i})$ the vector of prices that a buyer has found by time $t$. The buyer’s problem can thus be described by the following Bellman equation where the state variable is $p = p(t)$:

$$
V(p(t)) = \max_i V^i(p(t)) := \max_i \left( \theta dt E_{\theta}(V^i(p'(t + dt))) + (1 - \theta dt) V(p(t + dt)) \right) \quad (3.1)
$$

with the terminal condition

$$
V(p(1)) = 1 + \| -p(1) \|_\infty, \text{ for } p(0) \neq p(1), \\
V(p(1)) = 0, \text{ for } p(0) = p(1),
$$

where $\|p\|_\infty = \max(p)$ is the (element-by-element) maximum-norm of $p$.

By the principle of optimality, the buyer’s problem has a solution for any $n, F,$ and $P$ as long as the induced probability distribution $F(p)$ is defined in an appropriate measurable
space for any $p$. For simplicity of exposition, we assume for now this is so.\footnote{This is more than we need, of course, since this only has to hold for the on the equilibrium path and off the equilibrium path beliefs \textit{that we have specified}.} \footnote{As the set of prices is finite and the prices are bounded from above and from below, there should not be any measurability problems with a well defined prior.} Therefore, any $n$, $F$, and $P$ uniquely partition the set of buyers as

$$B(n, F, P) = \sum_{A \in 2^P} B_A(n, F, P),$$

where $B_A \subset B$ denotes the buyers who end with prices $A \subset P$.

If $F$ is continuous, the seller’s profits can now be decomposed as

$$\Pi^i(n, F, P) = \sum_{A \in 2^P} B_A(n, P, F) \left(1 - F_{\min A^{-i}}(p_{\min A^i})\right) p_{\min A^i},$$

where it is assumed that $p_{\min A} = \infty$ and $F_{\min A} \equiv 0$ if $A = \emptyset$. The buyers $B_A$ who have seen prices $A$ buy for $\min A$. In other words, they purchase from seller $i$ if seller $i$’s best price in set $A$ is lower than seller $-i$’s best price in set $A$, $p_{\min A^i} \leq p_{\min A^{-i}}$.

**Seller’s problem**

Therefore, to constitute an equilibrium, for any prices the seller is using $P^i \in \text{supp}(F^i) \in [0, 1]^n$ and for any prices the seller could be using $P^i \in [0, 1]^n$, it must hold that the profit to the seller is not higher, in expectation, with the latter prices than with the former

$$E_{P^{-i}} \left( \Pi^i(n, F, P^i, P^{-i}) \right) \geq E_{P^{-i}} \left( \Pi^i(n, F, P^i, P^{-i}) \right).$$

Otherwise, the seller has a profitable deviation from $P^i \in \text{supp}(F^i) \in [0, 1]^n$ to $P^i \in [0, 1]^n$. Seller $i$’ pricing strategy $P^i \in \text{supp}(F^i)$ has to be the best response to seller $-i$’ pricing strategy, $F^{-i}$, and buyers search, $(B_A(n, F, P))_{A \in 2^P}$, for $P^i$, $F^i$ and $P^{-i} \sim F^{-i}$.

Note especially that, instead of the standard exogenous partition of the set of buyers into the ”informed consumers” and the ”uninformed consumers” like in Varian (1980) or ”shoppers” (no search costs) and ”searchers” (search costs) like in Stahl (1989), in this case the buyers are partitioned endogenously into several subsets, some with more price information than others. Moreover, the way the buyers search and what they find depends
on the number of items in stock and both the expected and realized prices. Some of the usual properties of equilibrium price distributions still hold, though in a weak form:  

\textbf{Lemma 10}  
1. If the buyers search with non zero probability, all sellers mix in one price at least, over the same support \( S := \text{supp}(F^i) = \text{cl}\{p|f^i(p) > 0\} \subset [0, 1] \).

2. There are no atoms in the interior of \( S \) nor at the lower bound of \( S \).

3. If a buyer’s probability of switching away from store \( i \) after discovering price \( p^i \) is decreasing in \( p^i \), (i) there are no gaps in \( S \) and (ii) one is the upper bound of \( S \).

\textit{Proof.} In Appendix B. ■

Therefore, as long as the buyers search, we find that any equilibrium features randomized pricing strategies. Moreover, assuming that buyers are more likely to switch the seller if they find a lower price than a higher price, the prices are mixed over an interval support \([p, 1]\) for some \( p \in (0, 1) \). This seems like a natural assumption because, the lower the price, the more convinced the buyers should be that it is the best price in that store. This entails that, as is standard in previous work related to Varian (1980) and Stahl (1989), a seller’s profit is given by its ”captive buyers”, to whom it sells even for price one.

3.3 Equilibria

3.3.1 Benchmark: equilibrium for one price

For a benchmark, we next go through the case where both sellers have one item. The solution is symmetric and unique. The equilibrium boils down to the well known case developed by Varian (1980) and Stahl (1989).

As the sellers have the same number of items in stock, the buyers first approach the seller who has a lower expected price or, by Assumption 3, pick a random seller for the case of ties. This rules out the possibility of asymmetric pricing strategies. If one seller had a lower price than the other one, it would attract more captive buyers. But this entails that the seller has a higher price because prices are higher for a larger number of captive buyers: a contradiction.

Instead, if the sellers use symmetric pricing strategies, the buyers approach the sellers in random. They search in one store til they find its price and thereafter in the other store.

\footnote{So far we have not been able to get stronger general results for any finite number of stores and for any finite number of items in stock. Standard thinking may not always go through. This is because a store’s prices can have a different role each: some of them could be used mainly to encourage the buyers to search for the other available prices, which could then be used for the purpose of selling. A deviation in such an environment can prompt a harsh punishment. For example, the buyers who observe the deviation could immediately switch the seller and never come back.}
Section 3.3

... till they find its price – or until no time is left.\(^\text{19}\)

This implies that, when it is time to stop,

\[ B^1 := B_{\{p^1\}} = \frac{\theta e^{-\theta}}{2} \]

buyers have found only price \(p^1\) of seller \(i = 1\),

\[ B^2 := B_{\{p^2\}} = \frac{\theta e^{-\theta}}{2} \]

buyers have found only price \(p^2\) of seller \(i = 2\), and

\[ B_\emptyset = e^{-\theta} \]

buyers have found neither price. The residual

\[ B^{1,2} := B_{\{p^1, p^2\}} = 1 - B_\emptyset - B^1 - B^2 \]

of the buyers has found both of them.

In other words, the buyers are partitioned into uninformed consumers and informed consumers as in \textit{Varian} (1980) or into shoppers and searchers as in \textit{Stahl} (1989) – plus some buyers who fail to find anything, who did not feature in \textit{Varian} (1980) nor \textit{Stahl} (1989). The main difference here is that our partition is parametrized by the Poisson process that governs the arrival rate of price information. The stronger the frictions, the larger the fraction of uninformed consumers to informed consumers \(\frac{B_i}{B_1,2}\) and the larger the number of frustrated consumers \(B_\emptyset\) who do not discover anything. As anticipated, in this standard case where there is one price in one store, the equilibrium is thus essentially equivalent with that in \textit{Varian} (1980) and \textit{Stahl} (1989) except for the scaling that we need to account for the buyers who fail to find a price, \(B_\emptyset\).

\textbf{Proposition 10} (\textit{Varian}, 1980; \textit{Stahl}, 1989) If each sellers has one item, there exists a unique equilibrium price distribution:

\[ F^i : \left[ \frac{B_i}{B^1 + B^{1,2}}, 1 \right] \to [0, 1], \]

\[ F^i(p) = \frac{B^i + B^{1,2}}{B^{1,2}} \cdot \frac{1}{B^1} \cdot \frac{1}{p}. \]

A seller’s profit \(\Pi^i(1) = B^i = \frac{\theta e^{-\theta}}{2}\) equals the number of buyers, ”captive buyers”, who buy for \(p^i\) even when it is one.

\(^{19}\)Note that with passive beliefs off the path and one price per one store, what matters for search is expected prices not realized prices. The discovery of the first price does not give any additional information on the remaining price in the competing store – their strategies are independent in (Nash) equilibrium.
Proof. Just like in Varian (1980) and Stahl (1989) once we replace the shoppers (or the informed consumers) by \(B_{1,2}\) and the searchers (or the uninformed consumers) by \(B_1 + B_2\) and note that the sum \(B_{1,2} + B_1 + B_2\) is not equal to one. ■

3.3.2 Trivial multi-price equilibria

Note that existence is generally never an issue with this model. As pointed out earlier, there exists a unique equilibrium with one item in stock but, with more than one in stock, a multitude of equilibria.

**Remark 5** (Diamond, 1971) For any \(n \geq 1\), there exists a trivial stay-home equilibrium, where the sellers use only the monopoly price, \(p_i^m = 1\) for all \(n \leq n^i\), and the buyers do not search at all.

This is the Diamond (1971) equilibrium essentially, the famous result that proves the non-existence of an equilibrium with both costly search and endogenous price dispersion. If additional price information is costly and the goods alike, the sellers have an incentive to exploit the buyers’ hold-up problem by raising their price over their competitor’s price. Hence, the monopoly price is the unique equilibrium price irrespective of the number of sellers in the market. The buyers thus refuse to search.

Since the buyers’ hold-up problem appears here in a weak form only, the existence of this type of equilibrium hinges solely on Assumption 3 (iii): If the sellers charge the monopoly price, the buyers have no incentive to search and, if the buyers do not search, the sellers have no incentive to charge a discount price.

**Remark 6** (Ireland, 2007) For any \(n \geq 1\), there exists a trivial many-item equilibrium, where the sellers use only identical prices, \(p_i^n = p_i^m\) for all \(n, m \leq n^i\), and the buyers search for one price for one store.

This equilibrium is reminiscent of the one described by Ireland (2007) where the buyers obtain a sample of prices via a price search engine but do not distinguish if the prices in the sample come from several sellers or a single seller. Like in here, the sellers can offer the same good for a number of prices. However, as a buyer might sample two prices from one seller, the sellers have an incentive to avoid price variation as they would risk undercutting their own price. Instead, they send out two identical random prices.

The existence of this equilibrium is can be proved most easily by reference to Assumption 3 (iii): Clearly, no buyer has an incentive to stay to find another item in a given store if the items have the same price. Moreover, no seller has an incentive to charge different prices for two items if the buyers search for one item per one store.

Remarks 5 and 6 show that, while the focus of this paper lies on equilibrium price dispersion within and across the stores, it is possible to maintain also (i) a multi-price
equilibrium with no price variation (Remark 5) and (ii) a multi-price equilibrium with price variation across stores but not within stores (Remark 6). We refer to these equilibria as trivial multi-price equilibria. Although the sellers have several items which could in principle have each a different price, there is just one price in one store at any given point in time.

### 3.3.3 Simple hi-lo equilibrium for two prices

We now turn to our main contribution in this paper. We show that there exists a hi-lo equilibrium, where the sellers have sometimes two monopoly prices and, at other times, one monopoly price and one discount price. The size of this discount is random. In the first case, we say the seller is in the hi-hi regime and, in the second case, we say the seller is in the hi-lo regime. Even more complex patterns are possible, though. We develop an example of that in Appendix A.

An interesting implication of this hi-lo pricing pattern is that the buyers switch the stores after they have found a discount price or after they have found all. In other words, if the buyer first finds a seller’s monopoly price, she optimally continues with her start store in order to find also the seller’s discount price. This makes using a monopoly price valuable to a store. It helps to delay switching. The sellers have thus an incentive to sometimes use two monopoly prices (in hi-hi regime) instead of one monopoly price and one discount price (in hi-lo regime).

Specifically, denoting by $a$ the probability that a seller is in the hi-hi regime and by $b = 1-a$ the probability a seller is in the hi-lo regime, the chances that a buyer will switch the store after finding one price are less than half, $b/2 < 1/2$. The chances that a buyer will switch the store after finding two prices are larger, $1-b/2 > 1/2$. As shown in Figure 3.1, the expected switching time could thus be significantly delayed compared to the case of one price for one store. This demonstrates that price variation within stores acts here as an implicit switching barrier.

After the buyer has switched, the process of finding another competing discount price is also postponed. The probability that a buyer has found two discount prices by time $t$ is given by

$$b^2 \left( \frac{1}{2} \right)^2 e^{-\theta t} + \frac{b^2}{3!} \left( \frac{1}{2} \right)^2 e^{-\theta t} + \sum_{k=4}^{\infty} \frac{b^k}{k!} \left( \frac{\theta t}{2} \right)^k e^{-\theta t}$$

compared to $1 - e^{-\theta t} - \theta t e^{-\theta t}$ for the case in which there is just one price for one store. This consequence of the lock-in or delay effect of playing hi-lo equilibrium is illustrated by Figure 3.1.

Furthermore, additional prices enable the sellers also to discriminate better between
buyers, who end with diverse degrees of price information. The expected prices that are paid by buyers who find one, two, three or four prices, respectively, are juxtaposed in Figure 3.2. It shows a clearly decreasing pattern, which testifies to the fact that the sellers charge different buyers different prices. From ex ante perspective, the lowest price the average buyer has so far discovered, \(-\|p(t)\|_\infty\), is decreasing in time \(t \in (0, 1)\). This phenomenon is visible in Figure 3.2.

Our main contribution is the following:

**Proposition 11** If each seller has two items and \(\theta \leq \theta^o \approx 713\), there exists a simple
hi-lo equilibrium. The equilibrium price distribution is given by:

\[ F^i_1(p) = 0, \text{ for } p < 1, \]
\[ F^i_1(p) = 1, \text{ for } p = 1, \]
\[ F^i_2(p) = \left[ \frac{\Pi^i(1,1) - A^i_1}{(1-a)A^{1,2}_2 + A^{1,2}_1}, 1 \right] \rightarrow [0,1], \]
\[ F^i_2(p) = \frac{A^i_2 + (1-a)A^{1,2}_2}{A^{1,2}_2} - \frac{\Pi^i(1,1) - A^i_1}{A^{1,2}_2} \frac{1}{p}. \]

where

\[ A^i_1 := \frac{1}{4}B_1 + a/8B_3, \]
\[ A^i_2 := \frac{1}{4}(B_1 + B_2) + (1-a)/8(B_2 + B_3) + a/4(B_2 + 3B_3 + 4B_4), \]
\[ A^{1,2}_2 := \frac{1}{4}(B_2 + 3B_3 + 4B_4), \]

and where the atom size is

\[ a = Pr(p^i_2 = 1) = \frac{B_2}{B_2 + 3B_3 + 4B_4}. \]

A seller’s profit is given by the expected number of “captive buyers”, who are willing to buy for \( p^i_1 \) or \( p^i_2 \) even when both are one,

\[ \Pi^i(1,1) = \frac{1}{2} \frac{B_2}{B_2 + 3B_3 + 4B_4} (B_1 + B_2 + B_3 + B_4) \]
\[ + \frac{1}{2} \frac{3B_3 + 4B_4}{B_2 + 3B_3 + 4B_4} (B_1 + B_2 + 1/4B_3), \]

where

\[ B_1 = \theta e^{-\theta}, \quad B_2 = \theta e^{-\theta}, \quad B_2^2 = \frac{\theta^2}{2} e^{-\theta}, \quad B_4 = \sum_{k=4}^{\infty} \frac{\theta^k}{k!} e^{-\theta}. \]

Notice in particular that, due to the lock-in or delay effect, a seller’s profit is larger here than with just one price (see Figure 3.3).
Proof: We proceed through 8 steps.

Step 1: Noting that the joint price distribution of \((p_1^i, p_2^i)\) can be obtained by first deriving the marginal distribution of the lower price \(p_2^i\) and then deriving the conditional distribution of the higher price \(p_1^i\) given the lower price \(p_2^i\). Listing what we need for the proof.

It is clearly without loss of generality to assume that price \(p_1^i\) is weakly larger than price \(p_2^i\), \(p_1^i \geq p_2^i\). Thus, in this equilibrium we have to construct, \(p_1^i \equiv 1\) and \(F_2^i(p \to 1−) = b = 1−a \in (0,1)\). By Lemma 10, we know that the marginal distribution \(F_2^i\) has an interval support \(\text{supp}(F_2^i) = [p, 1]\) with the lower bound \(p \in (0, 1)\).

To show that Proposition 11 holds, we also have to determine a seller’s profit \(\Pi^i\) and the marginal distribution for the lower price \(F_2^i\), with \(b = 1−a\) and \(p\), such that there exist no profitable deviations in the lower price \(p_2^i\) to \(p_2^i' \in [0, p]\) for \(p_1^i \equiv 1\).

To end our proof, we also have to need sure there exist no profitable deviations in the higher price \(p_1^i\) to \(p_1^i' \in [p_2^i, 1]\) for any \(p_2^i \in [p, 1]\). When this holds, there clearly exist no profitable joint deviations in the lower price and the higher price \((p_1^i, p_2^i)\) to \((p_1^i', p_2^i') \in [0, p]^2\) since those deviations are dominated by the one to \((p, p)\).

Step 2: Proving that the buyers switch the store after they have discovered their first price if and only if it is lower than unity. Otherwise, the buyers switch the store only after they discover both of the two prices available in their start store.

Suppose a buyer has found a price from a store at time \(t\). Now, the buyer can either switch the seller immediately or postpone switching until she has found both prices from the start store.
If the price is lower than unity, the buyer will switch the stores. The probability of finding a discount price in the start store is zero but the probability of finding a discount price in the other store is positive.

Instead, if the price is unity, for probability \( (\theta (1 - t)) \frac{1}{e^{\theta (1 - t)}} \), the buyer finds one more price in the time that remains. In that case, the probability of finding a discount price in the start store is \( \frac{1}{2b+a} \) whereas the probability of finding a discount price in the other store is \( \frac{1}{2b} \). The former is clearly larger than the latter.

For probability \( (\theta (1 - t))^2 \frac{2}{e^{\theta (1 - t)}} \), the buyer finds two more prices. Then, if the buyer will switch the store after the first price, her chances of finding one discount price and two discount price are, respectively,

\[
\frac{\frac{1}{2b}}{\frac{1}{2b} + a} (1/2b + a) + \frac{\frac{1}{2b}}{\frac{1}{2b} + a} \quad \text{and} \quad \frac{\frac{1}{2b}}{\frac{1}{2b} + a} \left( \frac{1}{2b} \right) \quad \text{and} \quad \frac{\frac{1}{2b}}{\frac{1}{2b} + a} \frac{1}{2b}.
\]

and, if the buyer postpones switching, her chances of finding one discount price and two discount price are, respectively,

\[
\frac{\frac{1}{2b}}{\frac{1}{2b} + a} \quad \text{and} \quad \frac{\frac{1}{2b}}{\frac{1}{2b} + a} \frac{1}{2b}.
\]

These are equal. It is also immediate that, if the buyer finds zero or three additional prices, it does not matter for her payoffs at which point she switches. In conclusion, if the buyer finds a discount price first, she will switch the stores immediately but, if the buyer finds a monopoly price first, she will postpone switching.

**Step 3: Deriving a seller’s profits on the equilibrium path (both in the hi-hi regime and in the hi-lo regime).**

Since \((1, 1) \in \text{supp}(F)\), seller i’s profit can be determined from that case. The seller’s profit depends on whether the other seller has two monopoly prices \( p_{1i} = p_{2i} = 1 \) or a monopoly price \( p_{1i} = 1 \) and a discount price \( p_{2i} < 1 \). The former case occurs with probability \( a \) and the latter case with probability \( b = 1 - a \).

If both sellers have two monopoly prices \( p_1^i = p_2^i = 1 \), the sellers clearly share the market. Both thus make \( 1/2 (B_1 + B_2 + B_3 + B_4) \).

Instead, if seller i has two monopoly prices \( p_1^i = p_2^i = 1 \) but seller \(-i\) has a monopoly price \( p_{1i} = 1 \) and a discount price \( p_{2i} < 1 \), we also have to take into account how the buyers optimally search in that case.

Half the buyers start from store i and the rest start from store \(-i\). The buyers who start from store \(-i\) find \( p_{2i} < 1 \) before they switch to store i. Hence, they have no incentive to buy for \( p_1^i = 1 \) nor \( p_2^i = 1 \).
Instead, the buyers who start from store \( i \) switch to store \(-i\) only after they have found both \( p_i^1 \) and \( p_i^2 \) = 1. Thus, if they only find one or two prices in total, they buy for \( p_i^1 = 1 \) or \( p_i^2 = 1 \). Otherwise, if they discover three prices in total, there is half the chance those prices are \( p_i^1 = 1, p_i^2 = 1 \) and \( p_i^{-i} = 1 \) and half the chance the prices are \( p_i^1 = 1, p_i^2 = 1 \) and \( p_i^{-i} < 1 \). In the former case, the buyers select the seller in random. In the latter case, they buy for \( p_i^{-i} < 1 \). Clearly, if they find all the prices, they also buy for \( p_i^{-i} < 1 \). As a result, the profit to store \( i \) is given by \( 1/2 (B_1 + B_2 + 1/4B_3) \).

Altogether, this shows that the profit to seller \( i \) in the hi-hi regime is (see Appendix B and Appendix C Tables 3.1 and 3.2)

\[
\Pi^i(1,1) = a/2 (B_1 + B_2 + B_3 + B_4) + (1-a)/2 (B_1 + B_2 + 1/4B_3). 
\]

To get the marginal for the lower price \( F_i^2 \), we also need to determine seller \( i \)'s profit in the hi-lo regime where the seller has a monopoly price \( p_i^1 = 1 \) and a discount price \( p_i^2 < 1 \).

In this other case, the seller’s profit is given by (see Appendix B and Appendix C Tables 3.2 and 3.4 for the distribution of expected search outcomes in this case)\(^{20}\)

\[
\Pi^i(1,p_i^2) = 1/4B_1 + a/8B_3 + 1/4 (B_1 + B_2) p_i^2 + (1-a)/8 (B_2 + B_3) p_i^2 \\
+ a/4 (B_2 + 3B_3 + 4B_4) p_i^2 + (1-a)/4 (B_2 + 3B_3 + 4B_4) \left( \frac{1-a-F_i^2(p_i^2)}{1-a} \right) p_i^2 \\
=: A_1^i + A_2^i p_i^2 + A_2^{1,2} (1-a-F_i^2(p_i^2)) p_i^2. 
\]

Note that, to avoid dealing with overly long expressions for the seller’s profit, we have defined some new auxiliary constants above, \( A_1^i, A_2^i \) and \( A_2^{1,2} \).

\[
A_1^i := 1/4B_1 + a/8B_3, 
\]

There are \( 1/4B_1 \) buyers who start from store \( i \) and find only \( p_i^1 = 1 \) and \( a/8B_3 \) buyers who start from store \(-i\) and find only \( p_i^1 = p_i^{-i} = p_i^{-i} = 1 \).

\(^{20}\) Of course, the seller’s equilibrium profit must be the same in both regimes.
Here our notation attempts to parallel the case with one price per one store, replacing $B$’s by $A$’s. $A_i^1$’s refer to seller $i$’s captive buyers who pay the monopoly price, $A_i^2$’s refer to seller $i$’s captive buyers who pay a discount price, and $A_i^{1,2}$ refers to buyers who find two discount prices. They buy for the lower one.

*Step 4: Showing that, if the buyers use the above given switching rule, the sellers have an incentive to set two monopoly prices with a probability larger than zero. Determining this atom $a > 0$ and, thus, the probability that a store gives a discount $b > 0$. 

In equilibrium, a seller’s profit must be the same both in the hi-hi regime and the hi-lo regime. In particular, the profit to the seller must be the same when it has prices $(1, 1)$ and prices $(1, 1 - \varepsilon)$ for any small $\varepsilon > 0$. Taking the limit $\varepsilon \to 0^+$, yields

$$a/2(B_1 + B_2 + B_3 + B_4) + (1 - a)/2(B_1 + B_2 + 1/4B_3)$$

$$=1/4B_1 + a/8B_3 + 1/4(B_1 + B_2) + (1 - a)/8(B_2 + B_3) + a/4(B_2 + 3B_3 + 4B_4)$$

We can now solve this equality for the atom $a = Pr(p_2^1 = 1)$

$$a = \frac{B_2}{B_2 + 3B_3 + 4B_4},$$

which is strictly between zero and one for $\theta \in (0, \infty)$.

Basically, this means that there exist no equilibrium where the sellers have always one monopoly price $p_1^1 = 1$ and one discount price $p_2^i < 1$. The sellers would then have a profitable deviation to two monopoly prices $p_1^i = p_2^i = 1$ because the buyers switch the
store if they find a discount price first but continue with their start store if they find a monopoly price first.

In consequence, to compensate for the loss of captive buyers when the lower price is reduced from unity just slightly below, the sellers’ chances of attracting the buyers who find prices from both stores must slump down at unity. This is exactly what happens if the competitor has two monopoly prices for a non zero probability. The two regimes are both necessary here.

**Step 5: Deriving the marginal distribution for the lower price** $F_i^2$ **and the lower bound of the support** $p$.

As all the price pairs $(p_i^1, p_i^2) \in \text{supp}(F)$ must generate an equally much profit to the seller, we can use this profit equivalence condition to derive the marginal distribution for the lower price $F_i^2$.

$$\Pi^i(1,1) = A_i^1 + A_i^2 p_i^2 + A_i^{1,2} (1 - a - F_i^2(p_i^2)) p_i^2$$

implies

$$F_i^2(p_i^2) = \frac{A_i^1 + (1 - a)A_i^{1,2}}{A_i^{1,2}} - \frac{\Pi^i(1,1) - A_i^1}{A_i^{1,2}} \frac{1}{p_i^2}.$$  

The lower bound $p$ is given by the price where the marginal $F_i^2$ vanishes

$$F_i^2(p) = \frac{A_i^1 + (1 - a)A_i^{1,2}}{A_i^{1,2}} - \frac{\Pi^i(1,1) - A_i^1}{A_i^{1,2}} \frac{1}{p} = 0,$$

yielding

$$p = \frac{\Pi^i(1,1) - A_i^1}{(1 - a)A_i^{1,2} + A_i^2}.$$  

Quoting any lower price $p_i^2 < p$ yields less profit: the number of buyers who buy for $p_i^2$ equals to number of buyers who buy for $p$ but, since $p_i^2$ is smaller than $p$, the seller’s profit is reduced.

**Step 6: Deriving a seller’s profits off the equilibrium path (in a lo-lo regime).**

Suppose a seller deviates to some $p_i^1 \in [p_i^2, 1)$ for $p_i^2 \in [p, 1)$. Then, the seller’s profits are given by (see Appendix B and Appendix C Tables 3.3 and 3.5 for the diffusion of information to consumers in this case)
\[ \Pi' = \frac{1}{2} \left( \frac{1}{2}B_1p_1^i + a/2 (B_2 + 2B_3 + 2B_4) + b/4B_2p_1^i \right) + 1/2 (B_2 + 2B_3 + 2B_4) (1 - a - F_2(p_1^i)) p_1^i \\
+ 1/2 (1/2B_1p_2^i + a/2 (B_2 + 2B_3 + 2B_4) p_2^i + b/4B_2p_2^i) \\
+ 1/2 (B_2 + 2B_3 + 2B_4) (1 - a - F_2(p_2^i)) p_2^i \\
= : C_1p_1^i + D_1 (1 - a - F_2(p_1^i)) p_1^i + C_2p_2^i + D_2 (1 - a - F_2(p_2^i)) p_2^i, \quad (3.5) \]

where the auxiliary constants \( C_1, C_2, D_1 \) and \( D_2 \) are defined to abbreviate the exposition. Thus, we can solve for \((1 - a - F(p))\) from (3.3) and plug it into (3.5) to obtain an expression for a deviating seller’s profit

\[ \Pi' = C_1p_1^i + D_1 \frac{\Pi'(1,1) - A_1^i}{A_1^i} + D_1 \frac{A_2^i}{A_2^i} p_1^i + C_2p_2^i + D_2 \frac{\Pi'(1,1) - A_1^i}{A_1^i} + D_2 \frac{A_2^i}{A_2^i} p_2^i. \quad (3.6) \]

**Step 7:** Observing that the profit to the seller who deviates by lowering the higher price from \( p_1^i = 1 \) to \( p_1^i \in [p_2^i, 1] \) is linear in the deviation: the extremes \( p_1^i = p_2^i \) and \( p_1^i = 1 - \epsilon, \) for \( \epsilon > 0 \) small, are the best or worst. Showing the absence of a profitable deviation to \((1 - \epsilon, p^i)\) or \((1 - \epsilon, 1 - \epsilon)\) for \( \epsilon \to 0+ \).

It is thus clear from (3.6) that, by the linearity of seller’s profit \( \Pi' \) in \((p_1^i, p_2^i)\), the profit to the seller who deviates to some \((p_1^i, p_2^i) \in [p, 1]^2\) is the largest for extreme price choices. The maximum or supremum of the deviating seller’s profit \( \Pi' \) can hence be found by considering the limit where the higher price is either (right below) unity or equal to the lower price and the lower price is either (right below) unity of equal to the lower bound: \((p_1^i, p_2^i) = (1 - \epsilon, 1 - \epsilon), (p_1^i, p_2^i) = (1 - \epsilon, p)\) or \((p_1^i, p_2^i) = (p, p)\) for some tiny \( \epsilon > 0 \). We start by checking when it is the case that the seller has a profitable deviation to \((1 - \epsilon, 1 - \epsilon)\) or \((1 - \epsilon, p)\). To make comparisons easy, we place on-the-path profit on the left hand side (lhs) and off-the-path profit on the right hand side (rhs):

**Case 1:** A deviation to \((1 - \epsilon, 1 - \epsilon)\), where \( \epsilon \to 0+ \), is not worthwhile if

\[
\frac{1/2 (B_1 + B_2) + a/2 (B_3 + B_4) + b/8B_3 \geq 1/2B_1 + a/2 (B_2 + 2B_3 + 2B_4) + b/4B_2}{B_2 + 1/2B_3} \geq \frac{B_2}{B_2 + 5/2B_3 + 2B_4}.
\]

This is obviously a tautology.

**Case 2:** A deviation to \((1 - \epsilon, p)\), where \( \epsilon \to 0+ \), is not worthwhile if
\[
1/4B_1 + a/8B_3 \\
+ (1/4B_1 + 1/4B_2 + a/4(B_2 + 3B_3 + 4B_4) + b/8(B_2 + B_3))p \\
+ 1/4(B_2 + 3B_3 + 4B_4) \left(1 - a - F(p)\right)p
\]

This inequality is equivalent to

\[
\]

which is a necessary condition for

\[
\frac{B_2 + 5/3B_3 + 8/3B_4}{B_2 + 3B_3 + 8/3B_4} \geq \frac{B_2}{B_2 + 3B_3 + 4B_4},
\]

This is also a tautology.

Case 3: Last, there a profitable deviation to \((p, \bar{p})\) if:

\[
\frac{1}{2} (B_1 + B_2) + a \frac{1}{2} (B_3 + B_4) + \frac{b}{8} B_3 \\
< (1/2B_1 + a/2(B_2 + 2B_3 + 2B_4) + b/4B_2)p + 1/2(B_2 + 2B_3 + 2B_4) \left(1 - a - F(p)\right) p
\]

\[
= (1/2B_1 + b/4B_2 + 1/2(B_2 + 2B_3 + 2B_4))p.
\]

This inequality is equivalent with

\[
1/2 (B_1 + B_2) (1 - \bar{p}) + (B_3 + B_4) \left(\frac{a}{2} - \bar{p}\right) + b/4B_3 \left(1/2 - \bar{p}\right) < 0.
\]

**Step 8:** Confirming numerically the absence of a profitable deviation to \((p, \bar{p})\) for strong enough search frictions, \(\theta \leq \theta^o \approx 713\).

Our results are documented in Table 3.7.

Note that, even when the simple hi-lo equilibrium fails to exist because the sellers have
a profitable deviation to two discount prices, there may exist a more complex variant of a \textit{hi-lo} equilibrium, where this would not be a deviation. There could be three regimes instead of two: \textit{hi-hi} regime and \textit{hi-lo} regime like here and additionally a \textit{lo-lo} regime where the sellers indeed use two discount prices. This equilibrium is constructed and the conditions for its existence are determined in Appendix A. Our simple \textit{hi-lo} equilibrium is a special case of this more complex \textit{hi-lo} equilibrium; the latter nests the former.

### 3.4 Extensions

#### 3.4.1 Simple \textit{hi-lo} equilibrium for \( n \) prices

We analyze here a more general variant of a simple \textit{hi-lo} equilibrium where there are \( n > 1 \) prices in every store. In the \textit{hi-hi} regime, the sellers have \( n \) monopoly prices, \( p_i^1 = ... = p_{n-1}^i = 1 \), and, in the \textit{hi-lo} regime, \( n - 1 \) monopoly prices and just one discount price, \( p_n < 1 \).

As before, in this kind of an equilibrium where a store has never more than one discount price, a buyer has an incentive to switch the seller immediately after she finds one. The buyer’s search problem is non-trivial only if that has not occurred yet. We only need to cover that case.

To simplify notation, we denote the buyers’ possible search outcomes at \( t = 1 \) by:

\[
\omega_0 = \text{‘no price from store 1 nor from store 2’} \\
\omega_m^i = \text{‘a monopoly price from store i, no price from store -i’} \\
\omega_{1,2}^m = \text{‘a monopoly price from store 1 and from store 2’} \\
\omega_d^i = \text{‘a discount price from store i, no price or a higher price from store -i’} \\
\omega_{1,2}^d = \text{‘a discount price from store 1 and from store 2’} \\
\omega_A = \text{‘all prices from store 1 and from store 2’}
\]

Note that, if a seller is in the \textit{hi-lo} regime, the probability that a buyer who finds \( p < n \) prices at this seller does not find a single discount price is \( \frac{n - p}{n} \). If a seller is in the \textit{hi-hi} regime, it is one. Thus, when \( n - p \) prices remain in the start store and \( n \) in the other store, the probability of next finding a discount price is \( \frac{1}{n - p} \) in the start store and \( \frac{1}{n} \) in the other store if the stores are in the \textit{hi-lo} regime. It is zero if the store is in the \textit{hi-hi} regime. Therefore, assuming the buyer has so far found a total of \( p \) monopoly prices in her start store but none from the other store, the expected gain of finding one more price in the start store is
\[ \frac{n-p b}{a + \frac{n-p b}{n} n - p} (1 - E(p|\omega_d)) \]

and the expected gain of finding one more price in the other store is

\[ b \frac{1}{n} (1 - E(p|\omega_d)). \]

Obviously, the former exceeds the latter. The buyer is closer to finding a discount price is she remains in her start store than if she changes to the other store. By an inductive argument, it can be shown that it is this difference in the expected gain from the next price that drives the buyers’ search incentives. Therefore, the buyers switch the seller only when they find a discount price or nothing remains.

The seller’s profit is an immediate extension of the two-items-per-one-store case. To construct a tentative equilibrium, we have to derive the profit in four cases on the path (two regimes for two sellers) and in two cases off the path (the other store could be in either regime) since, as we have seen, the sellers might prefer to have more than one discount price. Generically, the sellers’ profit is given by

\[ \Pi_i = Pr(\omega^i_m) + Pr(\omega^i_d)p^i_n + Pr(\omega^{1,2}_m)1/2 + Pr(\omega^{1,2}_d) \left(1 - F_n(p^i_n)\right) p^i_n, \]

If the best prices the buyers find are seller $i$’s monopoly price or seller $i$’s discount price, they pay them. Instead, if the buyers find some monopoly prices from seller $i$ and some monopoly prices from seller $-i$, they purchase from a random seller. If they find two discount prices, they buy for the lower one.

If neither of the two sellers has a discount price, the profit to seller $i$ is

\[ \Pi^i = Pr(\omega^i_m) + Pr(\omega^{1,2}_m)1/2 \]

\[ = 1/2 \sum_{i=1}^{n} B_i + \sum_{i=1}^{n} B_n+1/2. \]

If only seller $i$ has a discount price, the profit to seller $i$ is
\[ \Pi^i = Pr(\omega^i_m) + Pr(\omega^i_d)p + Pr(\omega^{1,2}_m)1/2 \]
\[ = 1/2 \sum_{i=1}^{n} B_i \frac{n-i}{n} \]
\[ + 1/2 \sum_{i=1}^{2n} B_i \min\left\{ \frac{i}{n}, 1 \right\} p^i_n \]
\[ + 1/2 \sum_{i=1}^{n} B_{n+i} \frac{i}{n} p^i_n \]
\[ + 1/2 \sum_{i=1}^{n} B_{n+i} \frac{n-i}{n-1}/2. \]

If \( i \) prices are found, \( \frac{n-i}{n} = \frac{n-1}{n} \frac{n-2}{n} \cdots \frac{n-i}{n-(i-1)} \) is the probability of not finding a discount and \( \frac{i}{n} = \frac{1}{n} + \frac{n-1}{n} \frac{1}{n-1} + \cdots + \frac{n-(i-1)}{n} \frac{1}{n-(i-1)} \) is thus that of finding a discount.

If only seller \(-i\) has a discount price, the profit to seller \( i \) is

\[ \Pi^i = Pr(\omega^i_m) + Pr(\omega^{1,2}_m)1/2 \]
\[ = 1/2 \sum_{i=1}^{n} B_i + 1/2 \sum_{i=1}^{n} B_{n+i} \frac{n-i}{n-1}/2. \]

If both of the two sellers have a discount price, the profit to seller \( i \) is

\[ \Pi^i = Pr(\omega^i_m) + Pr(\omega^i_d)p^i_n + Pr(\omega^{1,2}_m) \left( 1 - F^i_n(p^i_n) \right) p^i_n \]
\[ = 1/2 \sum_{i=1}^{n} B_i \frac{n-i}{n} \]
\[ + 1/2 \sum_{i=1}^{2n} B_i \min\left\{ \frac{i}{n}, 1 \right\} \sum_{p=1}^{\min\{i,n\}} \frac{1}{n} \max\left\{ \frac{n-(i-p)}{n}, 0 \right\} p^i_n \]
\[ + \sum_{i=1}^{2n} B_i \min\{i,n\} \sum_{p=1}^{\min\{i,n\}} \frac{1}{n} \min\left\{ \frac{i-p}{n}, 1 \right\} \left( 1 - F^i_n(p^i_n) \right) p^i_n \]

where \( \frac{n-(i-p)}{n} \) is the probability of finding a discount on the \( p \)'th draw. If altogether \( i \) prices are found, the likelihood of not finding a discount at the second seller is \( \frac{n-(i-p)}{n} \) or zero (for \( i \) large) whereas the likelihood of finding a discount at the second seller is \( \frac{i-p}{n} \) or one (for \( i \) large).

Instead, the seller’s profit after a deviation to two identical discount prices is given by
\[ \Pi^i = Pr(\omega^i_m) + Pr(\omega^i_d)p + Pr(\omega^{1,2}_m)1/2 \]
\[ = 1/2 \sum_{i=1}^{2n} B_i \max \left\{ \frac{(n-i)(n-1-i)}{n(n-1)}, 0 \right\} \]
\[ + 1/2 \sum_{i=1}^{2n} B_i \min \left\{ 1 - \frac{(n-i)(n-1-i)}{n(n-1)}, 1 \right\} p_n^i \]
\[ + 1/2 \sum_{i=1}^{2n} B_{n+i} \min \left\{ 1 - \frac{(n-i)(n-1-i)}{n(n-1)}, 1 \right\} p_n^i \]
\[ + 1/2 \sum_{i=1}^{2n} B_{n+i} \max \left\{ \frac{(n-i)(n-1-i)}{n(n-1)}, 0 \right\} 1/2 \]

if the other seller does not have a discount price and

\[ \Pi^i = Pr(\omega^i_m) + Pr(\omega^i_d)p + Pr(\omega^{1,2}_d) \left( 1 - F_n^i(p_n^i) \right) p_n^i \]
\[ = 1/2 \sum_{i=1}^{2n} B_i \max \left\{ \frac{(n-i)(n-1-i)}{n(n-1)}, 0 \right\} \]
\[ + 1/2 \sum_{i=1}^{2n} B_i \sum_{p=1}^{\min(i,n-1)} 2(n-p) \frac{n-(i-p)}{n(n-1)} \max \left\{ \frac{n-(i-p)}{n}, 0 \right\} p_n^i \]
\[ + 1/2 \sum_{i=1}^{2n} B_i \sum_{p=1}^{\min(i,n-1)} 2(n-p) \frac{n-(i-p)}{n(n-1)} \min \left\{ \frac{i-p}{n}, 1 \right\} \left( 1 - F_n^i(p_n^i) \right) p_n^i \]
\[ + 1/2 \sum_{i=1}^{2n} B_i \sum_{p=1}^{\min(i,n-1)} \frac{1}{n} \min \left\{ 1 - \frac{(n-(i-p))(n-1-(i-p))}{n(n-1)}, 1 \right\} \left( 1 - F_n^i(p_n^i) \right) p_n^i \]

if the other seller does have a discount price.\textsuperscript{21}

If a store has two discount prices, \( \frac{(n-i)(n-i-1)}{n(n-1)} = \frac{n-2}{n-1} \frac{n-3}{n-2} \ldots \frac{n-(i-1)-2}{n-(i-1)} \) is the probability of not finding a discount from that store when \( i \) prices are found (for \( i \leq n \)) and \( \frac{2(n-p)}{n(n-1)} = \frac{(n-(p-1))(n-(p-1)-1)}{n-(p-1)} \frac{2}{n-(p-1)} \) is thus the probability of finding a discount from that store on the \( p \)'th draw (for \( p \leq n \)). A sum over the \( p \)'s, from the 1'st draw to the \( i \)'th draw, is \( \frac{2n-i-1}{n(n-1)} \) when the (deviating) store has two discount prices and \( \frac{1}{n} \) when the (non-deviating) store has one discount price.

It is now straightforward to confirm that this formulation is equivalent to the one derived earlier for \( n = 2 \). When each seller has two items, we know that the simple hi-lo equilibrium is sustained for \( \theta < \theta^0(2) \approx 713 \). To see what is the effect of additional items in stock beyond two, we determine this boundary also for \( n = 3 \). When sellers

\textsuperscript{21} Clearly, if a one-price deviation is not profitable, a two-price deviation is not profitable.
have three items, we find numerically that the simple *hi-lo* equilibrium is sustained for $\theta < \theta^o(3) \approx 719$. This is more relaxed.

**Proposition 12** If each sellers has three items and $\theta \leq \theta^o \approx 719$, there exists a simple *hi-lo* equilibrium.

Interestingly, if the sellers have two items in stock, a seller’s profit is

$$\Pi^i(1, 1) = \frac{a^{(2)}}{2} (B_1 + B_2 + B_3 + B_4) + (1 - a^{(2)})/2 (B_1 + B_2 + 1/4B_3) > 1/2 (B_1 + B_2)$$

where

$$a^{(2)} = \frac{B_2}{B_2 + 3B_3 + 4B_4}$$

whereas if the sellers have three items in stock, the profit to a seller is

$$\Pi^i(1, 1, 1) = \frac{a^{(3)}}{2} (B_1 + B_2 + B_3 + B_4 + B_5 + B_6) + (1 - a^{(3)})/2 (B_1 + B_2 + B_3 + 2/3B_4 + 1/3B_5) > 1/2 (B_1 + B_2 + B_3)$$

where

$$a^{(3)} = \frac{B_2 + 3B_3 + 3B_4 + 2B_5}{B_2 + 3B_3 + 9B_4 + 19/3B_5 + 9B_6}.$$  

In words, the seller’s profit is larger for $n = 3$, where it has two monopoly prices and one discount, than for $n = 2$, where it has one monopoly price and one discount. When there is one price in one store, the seller’s profit is $B_1$.

Although we are not sure what happens for $n > 3$ exactly, this suggests that inventory expansion and in-store price variation might represent an avenue for the sellers to raise the expected price towards the monopoly price, as in the Diamond (1971) outcome, yet give the buyers a reason to search. The finiteness of items in stock can help the sellers to commit to tremble away from the monopoly price level so that the stay-home outcome can be avoided.

However, especially for comparisons with a larger number of items in stock, we think it is important to take into account the possibility that additional items in stock can increase or decrease search efficiency. We consider that next.

### 3.4.2 Economies of scale in search

Here we analyze the idea that search could become either easier or more difficult with additional items in stock. To capture this idea in our model, we suppose an increase in
the number of items in stock \( n_i \) modifies the base line search frictions \( \theta \) by a multiplier \( \sigma(n_i) \) that could be either above one (for positive economies of scale) or below one (for negative economies of scale). To facilitate the exposition, we normalize \( \sigma(1) = 1 \) and introduce the following definition:

**Definition 5**

(A) There are positive economies of scale in search if \( \sigma(2) > 1 \) (or, generally, if \( \sigma(n_i + 1) \geq \sigma(n_i) \geq 1 \) for all \( n_i \in \mathbb{N} \) and \( \sigma(n_i + 1) > \sigma(n_i) \) for some \( n_i \in \mathbb{N} \)), i.e., if the search cost goes down with more items in stock. (B) There are negative economies of scale in search if \( \sigma(2) < 1 \) (or, generally, if \( \sigma(n_i + 1) \leq \sigma(n_i) \leq 1 \) for all \( n_i \in \mathbb{N} \) and \( \sigma(n_i + 1) < \sigma(n_i) \) for some \( n_i \in \mathbb{N} \)), i.e., if the search cost goes up with more items in stock.

Note that it is not immediate from the outset whether search should have positive or negative economies of scale: it can be easier to find an item, when there are more of them, but the buyers can also get overwhelmed by the larger number of items in stock. Still, the range for \( \sigma(2) \) that we find the most reasonable is \([1, 2]\), the one that lies between no economies of scale and moderate positive economies of scale. To narrow down to this range, suppose for a moment that each item is associated with a rate \( \phi \), for which it is found on a page or in a room (representing a store). This rate is specific to this particular item and, thus, independent of the other items’ rates. Then, (i) if we model a two-item store as one page or one room with two items on it, then the first is found at rate \( 2\phi \) and the second at rate \( \phi \), but, (ii) if we model a two-item store as two pages or two rooms with one item on each, then both are found at rate \( \phi \). Thus, the average finding rate should be within \( \phi \) and \( 2\phi \).

With this new notation, the profit to the seller with one price is given by

\[
\Pi^i(1) = B^i = \frac{\theta e^{-\theta}}{2}.
\]

The profit to the seller in the hi-lo equilibrium with two prices is

\[
\Pi^i(1, 1) = \frac{a^{(2)}}{2}(B_1 + B_2 + B_3 + B_4) + \frac{1 - a^{(2)}}{2}(B_1 + B_2 + 1/4B_3)
\]

where

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22 At an extreme, we could think of maintaining the finding rate constant for a larger number of items in stock by replacing the old store with one item in by multiple replicas with one item in each. This idea is a modification of the standard replica argument for constant returns to scale.

23 Note that it is a reasonable assumption that different prices are found in random order because the seller would prefer one finding order (high first) and the buyer another finding order (low first). This thus gives the seller an incentive to introduce randomness in its product placing strategy.

24 We consider only constant finding rates \( \sigma(n_i) \) because it is neither clear whether the first items are easier or harder to find that the last ones: The first ones could be harder to find in search where a buyer is checking the possible spots one by one in a systematic way (the location of the item gets narrowed down as fewer spots remain). The last ones could be harder to find in search where a buyer is starting from the most promising spots (the most prominent items are found the first, the most remote spot is left for last).
\[ a^{(2)} = \frac{B_2}{B_2 + 3B_3 + 4B_4} \text{ and } \]
\[ B_k = \frac{(\sigma(2)\theta)^k}{k!} e^{-(\sigma(2)\theta)} \text{ for } k < 4 \text{ and } B_4 = \sum_{k=4}^{\infty} \frac{(\sigma(2)\theta)^k}{k!} e^{-(\sigma(2)\theta)}. \]

The profit to the seller in the hi-lo equilibrium with three prices is

\[ \Pi^i(1, 1, 1) = \frac{a^{(3)}}{2} (B_1 + B_2 + B_3 + B_4 + B_5 + B_6) \]
\[ + \frac{1 - a^{(3)}}{2} (B_1 + B_2 + B_3 + 2/3B_4 + 1/3B_5) \]

where

\[ a^{(3)} = \frac{B_2 + 3B_3 + 3B_4 + 2B_5}{B_2 + 3B_3 + 9B_4 + 19/3B_5 + 9B_6} \text{ and } \]
\[ B_k = \frac{(\sigma(3)\theta)^k}{k!} e^{-(\sigma(3)\theta)} \text{ for } k < 6 \text{ and } B_6 = \sum_{k=6}^{\infty} \frac{(\sigma(3)\theta)^k}{k!} e^{-(\sigma(3)\theta)}. \]

It is now clear based on the previous analysis that, as long as the economies of scale in search \( \sigma(2) \) and \( \sigma(3) \) are not too large, the seller’s profits are larger for more items in stock: \( \Pi^i(1) < \Pi^i(1, 1) < \Pi^i(1, 1, 1) \). Our findings about the sellers’ incentive to generate price dispersion among similar items thus continue to hold true. Additionally, we think it might be possible to establish the following even stronger claim:

**Claim 1** For any \( \theta \) and for any sequence \( \sigma(n) \) that is bounded from upwards, there exists \( n' \in \mathbb{N} \) such that, a simple hi-lo equilibrium can be supported for all \( n > n' \). As the number of items in stock is increased, this will eventually lead to full extraction: \( \Pi_n^i \to \frac{1 - B_6}{2} \) as \( n \to \infty \).

This claim or hypothesis boldly states that as long as the economies of scale in search are bounded from upwards, the sellers can divide the market peacefully and extract full surplus when the number of items in stock explodes. Claim 3.4.2 rests on the observation that the only motive for the seller to reduce its numerous monopoly prices is to try to attract earlier the buyers who come from the other store. It seems clear that, when the number of items in stock is increased, \( n \to \infty \), the mass of this buyer group diminishes. Search in the start store is thus consuming more and more time because it is more and more difficult to find the one and only discount price among the \( n - 1 \) monopoly prices. Ultimately so few of the buyers actually switch the seller that the motive to randomize in prices completely disappears. The buyers who start the search from the seller itself, from their part, never switch the seller before they find the discount price such that there is no motive to please those buyers by offering a reduction in any of the monopoly prices – this
would only make them switch sooner. Thus, wherever the consumers are shopping, they almost always pay the monopoly price.

3.5 Closing remarks

We develop a novel obfuscation model that features search frictions within stores and, thus, equilibrium price variation both within and across stores. Everything is homogeneous ex ante. The search frictions originate from the gradual arrival of price information within stores and the existence of deadlines for buyers. This is all we need.

We find that stores can have an incentive to generate price variation across identical products to make search for better prices less effective to consumers. The general problem on the part of the buyers is that they cannot commit to shop around to play the stores against one another but, instead, tend to grow a stronger and stronger preference for their start store as time goes on.

To put it differently, our model shows a new way in which the retailers can use inventory expansion to generate barriers to switching even in an environment where switching is basically free of cost, like with online search. For this to work, it is important that it is focal in the economy that usually a seller indeed offers a discount price. This might give one explanation to why sellers often picture themselves in adverts as having discount prices everyday.

Interestingly, the effects on search and surplus sharing can be achieved totally passively from an individual seller’s viewpoint, who can just fix the prices and wait for the buyers to search in the optimal way. The seller’s best price offer to the buyer gets ”bargained” down over time as the buyer keeps finding lower and lower prices; no sales men are needed to make it happen.

While this paper concentrates on lock-in effects arising from inventory expansion in a simple price search model, similar effects are likely to arise under differentiation as well; modeling this is a straightforward research question for the future.\textsuperscript{25}

Obviously, additional alternatives are just one way to readjust search frictions within stores. To analyze the seller’s incentives more directly and generally, in a companion paper Hämäläinen (2015), we let the sellers choose the $\theta^i$’s entirely freely.\textsuperscript{26}

As for a simple concrete policy recommendation, one way to diminish the frictions within stores is to put all the prices of closely related items side by side in order to allow for immediate comparison at a glimpse. The number of steps or clicks or just, more

\textsuperscript{25}One interesting way to try would be to let the sellers to choose the average match quality as in Bar-Isaac et al. (2010) when the consumers sample the match values one by one for some cost or with some time pressure. Also, the use of Bandit models (see Bergemann and Välimäki (2006) for a concise review) could be one natural way to proceed, to let the consumers learn about the frictions within the stores during their search.

\textsuperscript{26}The model is similar to what we have here but there is just one item in every store.
generally, the ”distance” between different products, as measured in time to switch from one to the next, may not be irrelevant or innocuous. In fact, people have been shown to be quite sensitive to even apparently small time costs (see Dreze et al. (1995) and Huberman et al. (1998)).

There could exist quite subtle frictions related to, for instance, what the consumers fix their eyes on along their search paths (see Reutskaja et al. (2011) and Pinna and Seiler (2013)). Hence, since there is apparently a limit to how efficiently the sellers can put the products on display and how efficiently the buyers can eye through these products, any dramatic enough increase in the number of items in stock is likely to generate some frictions of its own an thus pave the way for the kinds of obfuscation strategies we have described here.

Appendix A

Complex hi-lo equilibrium for two prices

We next show in detail how to construct a more complex variant of the hi-lo equilibrium. It exists in a positive interval of parameters where the simplex variant of the hi-lo equilibrium fails to exist. It is of interest also on its own also because it shows that quite rich pricing patterns are possible even when sellers have only two prices. The exposition here is self contained and demonstrates how the simple hi-lo equilibrium arises as a natural special case of the complex hi-lo equilibrium. In the complex variant of the hi-lo equilibrium, the sellers have once again sometimes two monopoly prices and, at other times, a monopoly price and a discount price. As a novelty, however, here they sometimes have also two discount prices. Indeed, there is a cutoff such that, if the random discount price is above it, the other price is the monopoly price but, if the random discount price is below it, the two prices are identical.

To distinguish between these three regimes, we denote the probability that a sellers has just high monopoly prices by

$$a := Pr(A = 'hi-hi') = Pr(p^i_1 = 1, p^i_2 = 1) = Pr(p^{-i}_1 = 1, p^{-i}_2 = 1),$$

the probability that the sellers have a high monopoly price and a low discount price by

$$b := Pr(B = 'hi-lo') = Pr(p^i_1 = 1, p^i_2 < 1) = Pr(p^{-i}_1 = 1, p^{-i}_2 < 1),$$

and the probability that the sellers have just low discount prices by

$$c := Pr(C = 'lo-lo') = Pr(p^i_1 < 1, p^i_2 < 1) = Pr(p^{-i}_1 < 1, p^{-i}_2 < 1).$$

We assume further, with no loss of generality, that price one is larger than price two in
every store, $p^1_i \geq p^2_i$.

Specifically, we focus on joint price distributions $F(p_1, p_2)$, which could be analyzed by deriving, first, the marginal for the lower price $F(p_2)$ and, then, the conditional for the higher price $F(p_1 | p_2)$. Heuristically, the seller who fixes the prices can first draw the lower price and then the higher price. In the equilibrium we now concentrate on, after the lower price $p^2_i$ is drawn, the higher price $p^1_i$ is obtained in the following way:

1. If $p^2_i = 1$, then $p^1_i = 1$. This is the hi-hi regime.

2. If $p^2_i \in (p', 1)$, then $p^1_i = 1$. This is the hi-lo regime.

3. If $p^2_i \in [p, p']$, then $p^1_i = p^2_i$. This is the lo-lo regime.

In other words, if the lower price is the monopoly price, the higher price is the monopoly price, obviously. If the lower price is a discount price, the higher price is the monopoly price for the other price is above a threshold and a discount price if the other price is below the threshold. The threshold price $p' \in [p, 1]$ that distinguishes a slight discount, $p \in (p', 1)$, from a strong discount, $p' \in [p', 1]$, is determined in equilibrium.

Pay attention also to the fact that, while is is without loss to assume that the sellers use only high prices for probability $a \geq 0$, a high price and a low price for probability $b \geq 0$ and only low prices for probability $c \geq 0$, assuming a threshold of the kind that we have in $p'$ places already rather a lot of structure on equilibria. We later show that this is indeed the only symmetric candidate for a hi-lo equilibrium for two prices.

Next, note that as the sellers have the same number of items in stock, two, the positive economies of scale or the negative economies of scale, $\sigma$, affect the stores identically. Thus, the number of buyers, $B_k := \frac{(\sigma \theta)^k e^{-\sigma \theta}}{k!}$, who find a particular number of items by the end, $k = 0, ..., 4$, is essentially independent of search: by Assumption 3, the buyers do stop when have a reason to believe that they cannot find better prices anywhere, yet, their equilibrium buying choices would have been the same had they continued until the very end. Quite conveniently, this implies that it is easiest to conduct the analysis by tracking how each set of buyers $B_k$, who finds a given number of items $k = 0, ..., 4$, is divided between the two sellers in the end or what are the lowest prices they find from each store. We denote the possible search outcomes by
\( \omega_0 \) = 'no price from store 1 nor from store 2'
\( \omega_{im} \) = 'a monopoly price from store i, no price from store -i'
\( \omega_{1,2m} \) = 'a monopoly price from store 1 and from store 2'
\( \omega_{id} \) = 'a discounted price from store i, no price or a higher price from store -i'
\( \omega_{1,2d} \) = 'a discounted price from store 1 and from store 2'
\( \omega_A \) = 'all prices from store 1 and from store 2'

In some of the cases we also need to distinguish between

\( \omega_{is} \) = 'a slightly discounted price from store i, no price or some higher price from store -i'
\( \omega_{1,2s} \) = 'a slightly discounted price from store 1 and from store 2'
\( \omega_i^S \) = 'a strongly discounted price from store i, no price or some higher price from store -i'
\( \omega_{1,2s}^S \) = 'a strongly discounted price from store 1 and from store 2'

The search outcomes are ordered: a buyer is better off in the latter cases than in the former cases. Also, there is price competition and some uncertainty about the purchase decision just for the cases \( \omega_{1,2m} \) and \( \omega_{1,2l} \); otherwise, it is clear for which price the buyers are buying the product. With no loss of generality, the expected price given a search outcome is denoted by

\[
1 = p_0 = E(p|\omega_0) \\
1 = p_m = E(p|\omega_{1,2m}) \\
1 = p_{1,2} = E(p|\omega_{1,2}^S) \\
\geq p_s = E(p|\omega_{is}) \\
\geq p_{1,2}^S = E(p|\omega_{1,2s}^S) \\
\geq p_S = E(p|\omega_i^S) \\
\geq p_{1,2} = E(p|\omega_{1,2}^S)
\]

(3.7)

Furthermore, to make it easier to track the flow of buyers from one store to the other, we introduce the following auxiliary notation to denote the residual set of buyers \( R^k \) who find more than a given number \( k \) of items in stock.
\[ R^k = \sum_{j=k+1}^{4} B_j \] such that, say, \( R^1 = B_2 + B_3 + B_4 \) and \( R^2 = B_3 + B_4 \).

This notation is helpful to shorten the otherwise lengthy expressions and to highlight how some pricing policies postpone the switch of the stores and, thus, expose the sellers earlier to price competition, while others would not.

We first start with the buyer’s problem and then move on to the seller’s problem.

**Buyer’s problem**

In the equilibrium we are constructing, buyers choose a random seller and search there until the first price is discovered. Then, if the first price is a discount price, they switch the store after the discovery but, if it is the monopoly price, they should keep looking for the other price because that should most likely be a discount price.

This kind of search behavior places, clearly, some restrictions on pricing policies. As the sellers are using symmetric strategies, it is natural that buyers select the start store in random. It is also clear that the buyers must switch the seller at latest when they have found two price from the start store, as no more are to be found.

However, it is crucially important for the existence of this equilibrium that the buyers update their beliefs about the remaining price *upward* when they find the monopoly price and *downward* when they find a discount price. This implies that the *hi-lo* regime should dominate the *hi-hi* and *lo-lo* regimes in the sense that it is focal in the economy that most sellers are offering both a monopoly price and a discount price. In other words, for the equilibrium to work, the buyers should, first, not be too dismayed when they find that both prices are not strongly discounted and, second, remain confident enough that the monopoly price they have just found is coupled by a slightly discounted price they should now search for. To see when this is the case, if the first price is one, we have to compare the value to the buyer who stays in the start store and the value to the buyer who does not. Observe that this might vary a bit depending on how ties are broken. Assumption 3 states that if a buyer’s lowest price from store one is the same as the buyer’s lowest price from store two, she purchases from both equally often.

The value of sticking to the start store \( i \) for the second price can be written as (given that one item has been found at \( t \))\(^{27}\)

\(^{27}\)Here, \( Pr(k = p + 1) \) is an abbreviation for \( Pr(k_1 = p + 1|k_1 = 1) \), where \( k_1 \) denotes the number of items found by \( t \). To keep the notation as short as possible we suppress this.
\[ V^i_t = Pr_t(k = 1)V^i_t(\omega^i_1) + Pr_t(k = 2) \left( \frac{a}{a + b/2}0 + \frac{b/2}{a + b/2} (1 - p^i_s) \right) \]
\[ + Pr_t(k = 3) \times \left( \frac{a}{a + b/2} (a0 + b/20 + 1/2(1 - p^i_s)) \right) \]
\[ + \frac{b/2}{a + b/2} \left( a(1 - p^i_s) + b/2(1 - p^i_s) + b/2(1 - p^{i1}_s) \right) \]
\[ + Pr_t(k = 3) c(1 - p^{i-1}_S) + Pr_t(k = 4)V^i_t(\omega^{1,2}_A) \] (3.8)

The value of switching to the other store \(-i\) for the second price can be written as (given that one item has been found at \(t\))

\[ V^{-i}_t = Pr_t(k = 1)V^{-i}_t(\omega^{-i}_1) + Pr_t(k = 2) \left( a0 + b/20 + b/2(1 - p^{-i}_s) + c(1 - p^{-i}_S) \right) \]
\[ + Pr_t(k = 3) \times \left( a0 + b/2(1 - p^{-i}_s) + b/2 \left( \frac{a}{a + b/2} (1 - p^{-i}_s) + \frac{b/2}{a + b/2} (1 - p^{i1}_s) \right) \right) \]
\[ + Pr_t(k = 3) c(1 - p^{-i}_S) + Pr_t(k = 4)V^{-i}_t(\omega^{1,2}_A) \] (3.9)

Note first that the buyer could find either zero, one, two, or three additional prices for probabilities \(Pr_t(k = 1 + 0)\), \(Pr_t(k = 1 + 1)\), \(Pr_t(k = 1 + 2)\) and \(Pr_t(k = 1 + 3)\), respectively. If she finds none of the remaining prices or all of the remaining prices, the outcome of search is clearly just the same for whichever the chosen search order: in the first case the buyer value is \(V^i_t(\omega^i_1) = V^{-i}_t(\omega^{-i}_1) = 0\) and in the second case the buyer value is \(V^i_t(\omega^{1,2}_A) = V^{-i}_t(\omega^{1,2}_A) = 1 - \min\{p^{i2}_2, p^{-i2}_2\}\).

Interestingly, the buyer value is the same also conditional on the case that she finds two additional prices, independent of whether she goes for store \(i\) or for store \(-i\). To see why, notice that these extra prices could be two monopoly prices, which occurs for probability \(a\) or

\[ \frac{a}{a + b/2} (a + b/2) = (a + b/2) \frac{a}{a + b/2} \]

in both cases, one slightly discounted price and a higher price, which occurs for probability

\[ \frac{a}{a + b/2} b/2 + \frac{b/2}{a + b/2} (a + b/2) = b/2 + b/2 \frac{a}{a + b/2} \]

in both cases, two slightly discounted prices, which occurs for probability

\[ \frac{b/2}{a + b/2} b/2 = b/2 \frac{b/2}{a + b/2} \]

in both cases, or one strongly discounted price and a higher price, which occurs for probability \(c\) in both cases. Above, the lhs denotes the probability when the buyer would stick
to seller \( i \) and the rhs denotes the probability when the buyer would switch to seller \(-i\).

Note specifically that, after the buyer has found the monopoly price from store \( i \), she updates her beliefs about the remaining price in store \( i \). Her prior was that store is in the hi-hi regime for probability \( a \), in the hi-lo regime for probability \( b \), and in the lo-lo regime for probability \( c \) but now that she has observed the monopoly price she can be sure that the store is not in the lo-lo regime and her posterior for the hi-hi regime is \( \frac{a}{a+b/2} \) and for the hi-lo regime is \( \frac{b/2}{a+b/2} \).

Therefore, to guarantee that the buyers prefer to switch the stores if they find a discount price but not if they find the monopoly price, as required, it is sufficient that the expected buyer value is higher if the buyer goes for store \( i \) than if the buyer goes for store \(-i\), conditional on the case that she finds one additional price:

**Lemma 11** The buyers’ switching strategies are consistent with the hi-lo equilibrium if

\[
\frac{1}{2b} \left( \frac{1}{a} + \frac{1}{2b} \right) (1 - p_s) \geq \frac{1}{2b} (1 - p_s) + c (1 - p_S).
\]

**Proof.** A sketch of the proof is above. ■

This holds true if \( c = 0 \) or, otherwise, if \( b \) is ”high” in comparison to \( a \) and \( c \). Observe also that this is a condition for the variables \( a \), \( b \) and \( c \), which are endogenous; it is something we have to check once we have derived them. Basically, it states that the buyers have an incentive to search and switch as required as long as, either, the lo-lo regime is not at use, or, the hi-lo regime is the dominant one. This appears very natural as the motive to continue with the same seller after one price is found stems from the very expectation that most sellers keep available a mix of different prices, higher and lower.

**Seller’s problem**

We now move from the buyer’s problem to the seller’s problem. Remember that the key contribution of our paper is to show that the sellers can use a combination of two price instruments in stead of one price instrument to affect the search for identical products. This entails that the buyers must change the way they search in reaction to the prices they find; some prices make them stay while other prices make them go.

It is foreseeable although unavoidable that a problem like this results in quite heavy combinatorics as we have to keep track of where the buyer starts and in which order she finds the prices, the higher first or the lower first, because that affects the point at which she wants to switch the stores. To derive the equilibrium pricing strategies, we also have to go through all the combinations of the pricing regimes, altogether \( 3 \times 3 \).

Basically, a seller’s profit is still quite simple and mechanical to calculate. We only have to recall that half the buyers start from each seller, sample the two available prices
randomly one by one, and switch the seller if they find a discount price, \( p < 1 \), but stick
to their start store if they find a monopoly price, \( p = 1 \). Then we just count how many
of each buyer group \( B_1, B_2, B_3 \) and \( B_4 \) purchase from each seller in each case. To recall
how this model works, consider how the buyers would search if they were able to search
as long as they wanted. In that case, their search paths would be described for essential
parts as follows:

(A) Inside the start store

1. If the start store is in the \( hi-hi \) regime, the buyers switch the seller after the second
price is found and they all walk out of the store with a monopoly price \( p_m = 1 \).

2. If the start store is in the \( hi-lo \) regime, half the buyers switch the seller after the first
price is found and half the buyers switch the seller after the second price is found
and they all walk out of the store with a slightly discounted price \( p_s \in (p', 1) \).

3. If the start store is in the \( lo-lo \) regime, the buyers switch the seller after the first
price is found and they all walk out of the store with a strongly discounted price
\( p_s \in [p, p'] \).

(B) After the switch of stores

1. If the other store is in the \( hi-hi \) regime, there is no price competition from the second
seller (there are only monopoly prices).

2. If the other store is in the \( hi-lo \) regime, price competition from the second seller can
set in once the buyers have found the slightly discounted price (there is one such
price).

3. If the other store is in the \( lo-lo \) regime, price competition from the second seller can
set in once the buyers have found a strongly discounted price (there are two such
prices).

Recall also that, if the start store is in the \( hi-hi \) regime, the stores divide equally the
buyers who have only found a monopoly prices from both; otherwise, the start store gets
them all. All in all, the distribution of consumer search outcomes is represented most
conveniently by Tables 1 - 6 for all the six possible cases, respectively: (i) both sellers in
the \( hi-hi \) regime (Table 1), (ii) one seller in the \( hi-hi \) regime and one seller in the \( hi-lo \)
regime (Table 2), (iii) one seller in the \( hi-hi \) regime and one seller in the \( lo-lo \) regime (Table
3), (iv) both sellers in the \( hi-lo \) regime (Table 4), (v) one seller in the \( hi-lo \) regime and
one seller in the \( lo-lo \) regime (Table 5) and (vi) both sellers in the \( lo-lo \) regime (Table 6).
To emphasize, notice that this information in Tables 1 - 6 is both necessary and sufficient
to determine the seller’s profit in each of these three regimes. To support mixed pricing strategies, the seller’s profit should of course be the same over all the regimes.

In the hi-hi regime, the seller has only monopoly prices, \( p^i_2 = p^i_1 = 1 \). The profit is given by (see Tables 3.1, 3.2 and 3.3 in Appendix C)

\[
\Pi_{two}^i = \frac{1}{2} (B_1 + B_2) + \frac{a}{2} (B_3 + B_4) + \frac{b}{8} B_3. \tag{3.10}
\]

In the hi-lo regime, the seller has the monopoly price, \( p^i_1 = 1 \), and a discount price, \( p^i_2 \in (p', 1) \). The seller’s profit is (see Tables 3.2, 3.4 and 3.5 in Appendix C)

\[
\Pi_{two}^i = 1/4B_1 + a/8B_3 + b/8 (B_2 + B_3) p^i_2 + 1/4B_1 p^i_1 + 1/4B_2 p^i_2
\]
\[
+ a/4 (R^1 + 2R^2 + R^3) p^i_2 + b/4 (R^1 + 2R^2 + R^3) (1 - F(p^i_2 | b)) p^i_2
\]
\[
=: d_b + e_b p^i_2 + s_b (1 - a - F(p^i_2)) p^i_2. \tag{3.11}
\]

In the lo-lo regime, the seller has only discount prices, \( p^i_2 = p^v_1 \in [p, p'] \). The profit is given by (see Tables 3.3, 3.5 and 3.6 in Appendix C)

\[
\Pi_{two}^i = 1/2B_1 p^i_2 + b/4B_2 p^i_2 + (a + b)/2 (R^1 + R^2) p^i_2 + c R^1 (1 - F(p^i_2 | c)) p^i_2
\]
\[
=: e_c p^i_2 + s_c (1 - a - b - F(p^i_2)) p^i_2. \tag{3.12}
\]

Note that, to avoid overly long expression for the seller’s profit, we have defined some new constants, \( d_b, e_b, s_b, e_c, s_c \), where \( d \)’s refer to captive buyers who pay the monopoly price, \( e \)’s refer to captives who pay a discount price and \( s \)’s refer to shoppers; the subindex refers to regime, \( b \) stands for hi-lo and \( c \) stands for lo-lo. Also, as the marginal for the lower price \( p^i_2 \) is denoted by \( F \), \( 1 - F(p^i_2 | b) = \frac{1-a-F(p^i_1)}{b} \) and \( 1 - F(p^i_2 | c) = \frac{1-a-b-F(p^i_1)}{c} \) stand, respectively, for the conditionals of undercutting the other store’s discount price when it plays the hi-lo regime or the lo-lo regime.\(^{28,29}\)

It is clear from these expressions for a seller’s profit that the generated price variation keeps the buyers searching longer in every store. Since the buyers have little time, the sellers with higher prices (in the hi-hi and hi-lo regime) may clearly gain from this. However, as a backstop to the surplus extraction process, offering two low prices (in the lo-lo regime) may not always be a very bad idea either because, although the switch of the stores is swifter then, the prices are likely to be lower than what the other store has. Indeed, quite conveniently for the sellers, by mixing between these three different regimes,\(^{28}\)

\(^{28}\)The seller’s profit is derived in Appendix B.

\(^{29}\)To economize on the notation, we use \( F^v \) to refer to the marginal of \( p^v_1 \) in stead of referring to the joint price distribution of \( p^i_1 \) and \( p^i_2 \) as it was applied earlier on in Ch. 3.2.
the stores can decelerate the search when they have high prices to offer and they need a shelter from competition and accelerate the search when they have low prices to offer and they have less need for a shelter.

Now that we know how the seller’s profit is determined in each of the regimes, we can immediately make some interesting observations about the equilibrium pricing policies. First, at least two pricing regimes, the hi-hi and the hi-lo, are needed:

**Proposition 13** For any hi-lo equilibrium, the probability that $p^1_i = p^2_i = 1$ is given by

$$a = \frac{B_1 (1 + c)}{B_1 + 3B_3 + 4B_4} \in (0, 1).$$

In other words, any hi-lo equilibrium features an increase in the price level: the lower price $p^2_i$ is the monopoly price for probability $a > 0$ and the higher price $p^1_i$ is the monopoly price for probability $a + b = 1 - F(p') = 1 - c > 0$. These equilibria could be compared to the equilibrium with just one price where every price is a discount price for probability one.

Next, recall that discount prices above a threshold $p'$ are called slightly discounted and discount prices below a threshold $p'$ are called strongly discounted. Note also that, for $c = 0$, all discount prices are slightly discounted but, for $c > 0$, they come in both slightly and strongly discounted specimens. Their marginal distribution functions can be solved in closed form:

**Proposition 14** The slight discount prices, $p^2_i \in (p', 1)$, are drawn from

$$F(p^2_i) = 1 - a - \frac{\Pi_{two} - d_b + e_b p^2_i}{s_b p^2_i},$$

the strong discount prices, $p^2_i \in [p, p']$, are drawn from

$$F(p^2_i) = 1 - a - b - \frac{\Pi_{two} - e_c p^2_i}{s_c p^2_i}.$$  

If $c = 0$, the lowest price is $p = \frac{\Pi_{two} - d_b}{s_b(1 - a) + e_b}$ and, if $c > 0$, the lowest price is $p = \frac{\Pi_{two}}{s_c(1 - a - b) + e_c}$.

**Sketch of proof:** The atom size is pinned down by the requirement that the seller’s profit must be the same in the hi-hi regime with no discount price and in the hi-lo regime with a discount price near the monopoly price, which leads to a loss of some captive buyers because the buyers stick to the start store when they find a monopoly price but switch the stores when they find a discount price. To compensate for this drastic loss of captive buyers by a seller who offers a discount price $p_2 \approx 1$ in stead of the monopoly price $p_2 = 1$, there has to be a jump in the seller’s profit from the informed consumers: an atom at the upper bound and, thus, a drastic gain from undercutting it slightly.

The marginal distribution of the lower price, $p^2_i \mapsto F(p^2_i)$, comes as a direct result of
the requirement that the seller’s profit must be the same, \( \Pi_{\text{two}} \), across all the regimes to make it optimal for the sellers to randomize in pricing. The lower bound \( p \) is the price where the distributions have to vanish for each case, \( F(p) = 0 \), because, if an even lower price were used, the profit that the seller can obtain from the captive buyers at the upper bound one would be above the profit that the seller could obtain by selling for that low a price to the captive buyers and to all the informed consumers. For further details on how the equilibrium price distributions are calculated, see Appendix B. ■

Proposition 14 characterizes the two possibilities for the joint price distribution in a hi-lo equilibrium for two prices. Proposition 15 gives a condition as for when each of these cases would arise and when a hi-lo equilibrium for two prices exists. For the next existence result, recall that a hi-lo equilibrium has two regimes if \( c = 0 \) and three regimes if \( c > 0 \).

**Proposition 15** If there are two prices in two stores,

(i) there exists a simple hi-lo equilibrium where \( c = 0 \), for \( \sigma(2) \theta \in (0, \theta^o) \). The upper bound of this interval is such that \( \theta^o \approx 713 \)

(ii) there exists a simple hi-lo equilibrium where \( c > 0 \), for \( \sigma(2) \theta \in (\theta^o, \theta^x) \). The upper bound of this interval is such that \( \theta^x > \theta^o \).

**Proof.** See Appendix B. ■

For our model where the frictions are higher for lower Poisson rates \( \sigma(2) \theta \), there exists a simple hi-lo equilibrium without the lo-lo regime, i.e., such that \( c = 0 \), as long as the frictions are strong enough, i.e., for \( \sigma(2) \theta \in (0, \theta^o) \), and with the lo-lo regime, i.e. such that \( c > 0 \), as long as the frictions are a bit lower but nevertheless strong enough, i.e., for \( \sigma(2) \theta \in (\theta^o, \theta^x) \). With two prices in two stores, the cutoffs for the frictions are \( \theta^x > \theta^o \approx 713 \).

**Remark 7** Out of our model, there exists a simple hi-lo equilibrium with two items also under some other circumstances. To give an example, it is sustainable also in an environment where the the number of prices the buyers find is uniform on \( \{0, 1, \ldots, k\} \), for \( 4 \leq k \leq 99 \), or \( B_n = 1/(k + 1) \), for \( 0 \leq n \leq k - 1 \) or beyond. In addition, for instance, the following partitions (ad hoc) would be appropriate:

\[
\begin{align*}
B_\emptyset &= 0.96, B_1 = 0.01, B_2 = 0.01, B_3 = 0.01, B_4 = 0.01, \\
B_\emptyset &= 0.01, B_1 = 0.96, B_2 = 0.01, B_3 = 0.01, B_4 = 0.01, \\
B_\emptyset &= 0.01, B_1 = 0.01, B_2 = 0.01, B_3 = 0.01, B_4 = 0.96,
\end{align*}
\]

but not the following
\[ B_\emptyset = 0.01, B_1 = 0.01, B_2 = 0.96, B_3 = 0.01, B_4 = 0.01, \]
\[ B_\emptyset = 0.01, B_1 = 0.01, B_2 = 0.01, B_3 = 0.96, B_4 = 0.01. \]

Proof. See Table 3.7. □

We consider in the main text also hi-lo equilibria with \( k \) prices. Similar results hold for them:

**Remark 8** Out of our model, there exists a simple hi-lo equilibrium with three items also under some other circumstances. To give an example, it is sustainable also in an environment where the number of prices the buyers find is uniform on \{0, 1, \ldots, n\}, for \( 5 \leq n \leq 99 \), or \( B_p = 1/(n+1) \), for \( 0 \leq p \leq n - \) or beyond. In addition, for instance, the following partitions (ad hoc) would be appropriate:

\[
\begin{align*}
B_\emptyset &= 0.94, B_1 = 0.01, B_2 = 0.01, B_3 = 0.01, B_4 = 0.01, B_5 = 0.01, B_6 = 0.01, \\
B_\emptyset &= 0.01, B_1 = 0.94, B_2 = 0.01, B_3 = 0.01, B_4 = 0.01, B_5 = 0.01, B_6 = 0.01, \\
B_\emptyset &= 0.01, B_1 = 0.01, B_2 = 0.94, B_3 = 0.01, B_4 = 0.01, B_5 = 0.01, B_6 = 0.01, \\
B_\emptyset &= 0.01, B_1 = 0.01, B_2 = 0.01, B_3 = 0.01, B_4 = 0.94, B_5 = 0.01, B_6 = 0.01, \\
B_\emptyset &= 0.01, B_1 = 0.01, B_2 = 0.01, B_3 = 0.01, B_4 = 0.01, B_5 = 0.94, B_6 = 0.01, \\
B_\emptyset &= 0.01, B_1 = 0.01, B_2 = 0.01, B_3 = 0.01, B_4 = 0.01, B_5 = 0.01, B_6 = 0.94,
\end{align*}
\]

but not the following

\[ B_\emptyset = 0.01, B_1 = 0.01, B_2 = 0.01, B_3 = 0.94, B_4 = 0.01, B_5 = 0.01, B_6 = 0.01. \]

Proof. See Table 3.8. □

This shows that, generally, it is possible to support a simple hi-lo equilibrium even when the frictions are quite low as in the uniform case with large \( k \).\(^{30}\) However, there exists is a profitable deviation from these candidate equilibria if almost every buyer finds more items than a single seller is carrying. This appears natural for the only motive to deviate from a simple hi-lo equilibrium is that of attracting the buyers who start from the

\(^{30}\)It appears natural to claim that a partition \( \{B_k\}_{k=1}^{\infty} \) features lower frictions than partition \( \{C_k\}_{k=1}^{\infty} \) if \( B_k \geq C_k \) for all \( k \) because, in that case, there are more buyers who find less prices in partition \( \{B_k\}_{k=1}^{\infty} \) than in partition \( \{C_k\}_{k=1}^{\infty} \).
other store; the buyers who start from the seller itself do not switch the store until they have found a discount price.

So far we have not excluded the existence of other kinds of multi-price equilibria. Yet, while the perhaps simplest way to make the buyers search longer is offering two independent and identically distributed prices, we find that this cannot be an equilibrium but in a very restrictive environment where, coincidentally, \( B_3 = 1/4 \). Otherwise, the sellers have a profitable deviation to a monopoly price and a discount price or two perfectly correlated discount prices. In other words, they have an incentive to shift their pricing strategies towards a hi-lo equilibrium.

**Proposition 16** If there are two prices in two stores, there exists an equilibrium where each seller has two independent and identically distributed prices iff \( B_3 = 1/4 \).

*Proof.* See Appendix B. ■

**Appendix B: Proofs**

*Proof of Lemma 10.* We start by showing that there exist no pure equilibrium in pricing policies, whatever the number of stores and the number of items in stock in each of them. The proof is by contradiction. Assume the sellers use some pure pricing policies. Focus on the lowest prices and the second lowest prices. If only one store is using the lowest prices or faces no competition for those prices, it has an incentive to increase them to extract more revenue from the buyers and, if two competing sellers are using the lowest prices, they have an incentive to decrease them to undercut the other one; note that since these are the lowest prices it does not matter if the change in prices triggers a change in search because the consumers who find the lowest prices will anyway return to buy for them. It is also without loss to assume that the lowest prices are positive because, as long as the buyers search, there are always some captive buyers in the market from whom the sellers can extract positive revenue such that the Bertrand outcome is not an equilibrium. Note that every store must have a randomized price. Otherwise, its competitors can undercut its price too easily. Obviously, every store must face some competition because, if this was not the case, it has either only lower prices than others (which it would, thus, have an incentive to increase to tax the captive buyers) or only higher prices than others (which would not draw any traffic to the store and, in particular, no captive buyers willing to buy for those higher prices).

To organize our thoughts when the consumer’s response to different prices is unknown, we introduce the notion of competing prices: price \( p^n_i \) competes with price \( p^{m}_{i-1} \) if some group of buyers who finds them both would sometimes purchase for \( p^n_i \) and, at other times, purchase for \( p^{m}_{i-1} \).
Now, consider the support of a seller’s lowest prices \( \text{supp}(F_{\text{min}}^i) \) and the support for all other competing prices \( \text{supp}(F_{\text{min}}^{-i}) \). We next derive some elementary properties of those prices. First, if there is no overlap in the supports, \( \exists S(q,r) = (q-r, q+r) \) for \( (q,r) \in [0,1] \times (0,\infty) : S(q,r) \cup \text{supp}(F_{\text{min}}^i) = S(q,r) \) and \( S(q,r) \cup \text{supp}(F_{\text{min}}^{-i}) = \emptyset \), there is a profitable deviation to adjust the pricing policies such that all weight in the gap \( S(q,r) \) is put on the upper bound \( p = q + r \). This implies that the support of a seller’s lowest prices must overlap with those in other competing stores.

Note that as these deviations are on the path, so to say, we need not worry about the effects on buyers search behavior which might otherwise undermine the profitability of a change in a price. If we assume out such off path punishments, we can easily get additional results.

Assume that the probability that the buyer would switch the seller after seeing a price is continuous in the price. Then it is immediate to show that, if there is an atom \( a \) in some \( \exists T(q,r) = [q-r, q+r) \) for \( (q,r) \in [0,1] \times (0,\infty) : T(q,r) \cup \text{supp}(F_{\text{min}}^i) \cup \text{supp}(F_{\text{min}}^{-i}) \neq \emptyset \), there is a profitable deviation from \( p \in [a, a+\epsilon] \) to \( p \in (a-\epsilon, a) \) for some tiny \( \epsilon > 0 \) as the probability of having a lower price than the competition would go up discontinuously but the selling price would go down only continuously. This shows that the interior and the lower bound of the support must be clear from atoms.

Observe that, for symmetric strategies, this implies that competing prices should come from an interval support. In addition, as long as there is no countervailing effect through an atom at the upper bound or through the switching response that is triggered, the upper bound of this support must be equal to one because those prices cannot be utilized to sell but to captive buyers and, thus, should be raised as high as possible to extract maximal revenue from them.

Deriving the seller’s profit in the hi-lo equilibrium for two prices. As both of the two stores play three different regimes, we have to go through nine cases in total to determine a seller’s profit. Each price could be either the monopoly price \( p_m \) or a discount price \( p_d \).

Case 1: Seller \( i \)’s profit when it is in the hi-hi regime and has two monopoly prices.

Subcase 1.1: If seller \(-i\) is in the hi-hi regime, seller \( i \)’s profit can derived from Table 1.
The buyers who start from seller \( i \) or from seller \(-i\) switch after finding the two monopoly prices, but do not find only monopoly prices from the second seller as well and, thus, buy half the times from seller \( i \) and half the times from seller \(-i\):

\[
\Pi^i = \frac{1}{2} (B_1 + B_2 + 1/2B_3 + 1/2B_4) + \frac{1}{2} (1/2B_3 + 1/2B_4) = 1/2R^0.
\]

Subcase 1.2: If seller \(-i\) is in the hi-lo regime, seller \( i \)'s profit can derived from Table 2. The buyers who start from seller \( i \) switch after they have found the two monopoly prices. If they only find the monopoly price from the other store they buy half the times from seller \( i \) and half the times from seller \(-i\) but if they also find the discount price, they buy from seller \(-i\). The buyers who start from seller \(-i\) switch after they have found the discount price but, given that store \(-i\)'s discount price obviously beats store \( i \)'s monopoly prices, it is the start store \(-i\) where they buy from ultimately:

\[
\Pi^i = \frac{1}{2} (B_1 + B_2 + 1/2B_3) = 1/2 \left( R^0 - 3/4R^2 - 1/4R^3 \right).
\]

Subcase 1.3: If seller \(-i\) is in the lo-lo regime, seller \( i \)'s profit can derived from Table 3. The buyers who start from seller \( i \) switch after they have found the two monopoly prices and, once they find a discount price, they buy from seller \(-i\). The buyers who start from seller \(-i\) switch right after they have found a discount price but, given that store \(-i\)'s discount price obviously beats store \( i \)'s monopoly price, it is the start store \(-i\) where they buy from ultimately:

\[
\Pi^i = 1/2 (B_1 + B_2) = 1/2 \left( R^0 - R^2 \right).
\]

Summing up over these three cases, we obtain the seller’s profit for the hi-hi regime as:

\[
\Pi^i = \frac{1}{2} (B_1 + B_2) + \frac{a}{2} (B_3 + B_4) + \frac{b}{8}B_3.
\]

Case 2: Seller \( i \)'s profit when it is in the hi-lo regime and has a monopoly price and a discount price \( p_d^i \).

Subcase 2.1: If seller \(-i\) is in the hi-hi regime, seller \( i \)'s profit is given by Table 2 as we just did for Subcase 1.2 except that we have to change seller \( i \) into seller \(-i\) and, thus, seller \(-i\) into seller \( i \). The buyers who start from seller \( i \) switch after they have found the discount price but, given that store \( i \)'s discount price obviously beats store \(-i\)'s monopoly prices, it is the start store \( i \) where they buy from ultimately. The buyers who start from seller \(-i\) switch after they have found the two monopoly prices. If they only find the
monopoly price from the other store they buy half the times from seller \(-i\) and half the times from seller \(i\) but if they also find the discount price, they buy from seller \(i\):

\[
\Pi^i = \frac{1}{2} \left( \frac{1}{2} B_1 + 1 \frac{1}{2} B_1 p_d^i + (B_2 + B_3 + B_4) p_d^i \right) + \frac{1}{2} \left( \frac{1}{4} B_3 + 1 \frac{1}{2} B_3 p_d^i + B_4 p_d^i \right) + \frac{1}{2} \left( \frac{1}{4} B_3 + 1 \frac{1}{2} B_3 p_d^i + B_4 p_d^i \right) = \frac{1}{4} B_1 + \frac{1}{8} B_3 + \frac{1}{4} \left( R^0 + R^1 + R^2 + R^3 \right) p_d^i.
\]

Subcase 2.2: If seller \(-i\) is in the hi-lo regime, seller \(i\)'s profit can derived from Table 4. The buyers who start from seller \(i\) or from seller \(-i\) switch after finding the discount price. If they only find the monopoly price from the other store, they buy from their start store but, if they also find the discount price from the other store, they buy from the store whose price is the lowest one:

\[
\Pi^i = \frac{1}{2} \left( \frac{1}{2} B_1 + 1 \frac{1}{2} B_1 p_d^i + 3 \frac{1}{4} B_2 p_d^i + 1 \frac{1}{4} B_3 p_d^i \right) + \frac{1}{2} \left( \frac{1}{4} B_2 \left( 1 - F(p_d^i | b) \right) p_d^i + 3 \frac{1}{4} B_3 \left( 1 - F(p_d^i | b) \right) p_d^i + B_4 \left( 1 - F(p_d^i | b) \right) p_d^i \right) + \frac{1}{2} \left( \frac{1}{4} B_2 \left( 1 - F(p_d^i | b) \right) p_d^i + 3 \frac{1}{4} B_3 \left( 1 - F(p_d^i | b) \right) p_d^i + B_4 \left( 1 - F(p_d^i | b) \right) p_d^i \right) = \frac{1}{4} B_1 + \frac{1}{8} B_2 + \frac{1}{8} B_3 \left( 1 - F(p_d^i | b) \right) p_d^i.
\]

Subcase 2.3: If seller \(-i\) is in the lo-lo regime, seller \(i\)'s profit can derived from Table 5. The buyers who start from seller \(i\) switch after they have found the discount price and, once they find a discount price from seller \(-i\), they compare them and buy for the lowest one. The buyers who start from seller \(-i\) switch after they have found a discount price and, once they find the discount price from seller \(i\), they compare them and buy for the lowest one:

\[
\Pi^i = \frac{1}{2} \left( \frac{1}{2} B_1 + 1 \frac{1}{2} B_2 p_d^i + 1 \frac{1}{2} B_2 p_d^i + 1 \frac{1}{2} B_2 p_d^i + (1 \frac{1}{2} B_2 + B_3 + B_4) \left( 1 - F(p_d^i | c) \right) \right) + \frac{1}{2} \left( \frac{1}{2} B_2 + B_3 + B_4 \right) \left( 1 - F(p_d^i | c) \right) p_d^i = \frac{1}{4} B_1 + \frac{1}{4} B_2 p_d^i + \frac{1}{4} B_2 p_d^i + \frac{1}{4} B_2 p_d^i + \left( 1 \frac{1}{2} R^1 + 1 \frac{1}{2} R^2 \right) \left( 1 - F(p_d^i | c) \right) p_d^i.
\]
Summing up over these three cases, we obtain the seller’s profit for the hi-lo regime as:

\[ \Pi^i = 1/4B_1 + 1/4B_1 p^i_d + 1/4B_2 p^i_d + a/8B_3 + a/4 \left( R^1 + 2R^2 + R^3 \right) p^i_d + b/8 \left( B_2 + B_3 \right) \]

\[ + b/4 \left( R^1 + 2R^2 + R^3 \right) \left( 1 - F(p^i_d | b) \right) p^i_d + c/2 \left( R^1 + R^2 \right) \left( 1 - F(p^i_d | c) \right) p^i_d. \]

If we take into account the equilibrium restriction that discount price is lower in the lo-lo regime than in the hi-lo regime, we get

\[ \Pi^i = 1/4B_1 + 1/4B_1 p^i_d + 1/4B_2 p^i_d + a/8B_3 + a/4 \left( R^1 + 2R^2 + R^3 \right) p^i_d + b/8 \left( B_2 + B_3 \right) \]

\[ + b/4 \left( R^1 + 2R^2 + R^3 \right) \left( 1 - F(p^i_d | b) \right) p^i_d \]

\[ = 1/4B_1 + a/8B_3 + \left( 1/4B_1 + 1/4B_2 + a/4 \left( R^1 + 2R^2 + R^3 \right) + b/8 \left( B_2 + B_3 \right) \right) p^i_d \]

\[ + 1/4 \left( R^1 + 2R^2 + R^3 \right) \left( 1 - a - F(p^i_d) \right) p^i_d. \]

Yet, the earlier profit expression can also be useful to analyze profitability of different deviations.

*Case 3*: Seller *i*’s profit when it is in the lo-lo regime and has two discount prices \( p^i_d(1) \) and \( p^i_d(2) \).

Subcase 3.1: If seller \(-i\) is in the hi-hi regime, seller *i*’s profit is given by Table 3 as we just did for Subcase 1.3 except that we have to change seller \( i \) into seller \(-i\) and, thus, seller \(-i\) into seller \( i \). The buyers who start from seller \( i \) switch right after they have found a discount price but, given that store *i*’s discount prices obviously beat store \(-i*\) monopoly prices, it is the start store *i* where they buy from ultimately. The buyers who start from seller \(-i\) switch after they have found the two monopoly prices and, once they find a discount price, they buy from seller \( i \).

\[ \Pi^i = 1/2 \left( 1/2 \left( B_1 + B_2 + B_3 + B_4 \right) p^i_d(1) + 1/2 \left( B_1 + B_2 + B_3 + B_4 \right) p^i_d(2) \right) \]

\[ + 1/4 \left( B_3 + B_4 \right) p^i_d(1) \]

\[ + 1/4 \left( B_3 + B_4 \right) p^i_d(2) \]

\[ = (1/4R^0 + 1/4R^2) p^i_d(1) + (1/4R^0 + 1/4R^2) p^i_d(2). \]
Subcase 3.2: If seller $-i$ is in the hi-lo regime, seller $i$’s profit is given by Table 5 as we just did for Subcase 2.3 except that we have to change seller $i$ into seller $-i$ and, thus, seller $-i$ into seller $i$. The buyers who start from seller $i$ switch after they have found a discount price and, once they find the discount price from seller $i$, they compare them and buy for the lowest one. The buyers who start from seller $-i$ switch after they have found the discount price and, once they find a discount price from seller $i$, they compare them and buy for the lowest one:

$$\Pi^i = \frac{1}{2} \left( \frac{1}{2} B_1 p_{d(1)}^i + \frac{1}{4} B_2 p_{d(1)}^i + \frac{1}{2} \left( \frac{1}{2} B_2 + B_3 + B_4 \right) \left( 1 - F(p_{d(1)}^i | b) \right) p_{d(1)}^i \right)_{\text{start store } i}$$

$$+ \frac{1}{2} B_1 p_{d(2)}^i + \frac{1}{4} B_2 p_{d(2)}^i + \frac{1}{2} \left( \frac{1}{2} B_2 + B_3 + B_4 \right) \left( 1 - F(p_{d(2)}^i | b) \right) p_{d(2)}^i_{\text{start store } i}$$

$$+ \frac{1}{4} \left( \frac{1}{2} B_2 + B_3 + B_4 \right) \left( 1 - F(p_{d(1)}^i | b) \right) p_{d(1)}^i_{\text{start store } -i}$$

$$+ \frac{1}{4} \left( \frac{1}{2} B_2 + B_3 + B_4 \right) \left( 1 - F(p_{d(2)}^i | b) \right) p_{d(2)}^i_{\text{start store } -i}$$

$$= \frac{1}{4} (B_1 + \frac{1}{2} B_2) p_{d(1)}^i + \frac{1}{4} \left( R^1 + R^2 \right) \left( 1 - F(p_{d(1)}^i | b) \right) p_{d(1)}^i$$

$$+ \frac{1}{4} (B_1 + \frac{1}{2} B_2) p_{d(2)}^i + \frac{1}{4} \left( R^1 + R^2 \right) \left( 1 - F(p_{d(2)}^i | b) \right) p_{d(2)}^i.$$
\[ \Pi^i = 1/2 \left( 1/2 B_1 p^i_{d(1)} + 1/2 (B_2 + B_3 + B_4) \left( 1 - F(p^i_{d(1)}|c) \right) p^i_{d(1)} \right)_i \\
+ 1/2 B_1 p^i_{d(2)} + 1/2 (B_2 + B_3 + B_4) \left( 1 - F(p^i_{d(2)}|c) \right) p^i_{d(2)} \\
+ 1/4 (B_2 + B_3 + B_4) \left( 1 - F(p^i_{d(1)}|c) \right) p^i_{d(1)}_{\text{start store } i} \\
+ 1/4 (B_2 + B_3 + B_4) \left( 1 - F(p^i_{d(2)}|c) \right) p^i_{d(2)}_{\text{start store } i} \\
= 1/4 B_1 p^i_{d(1)} + 1/2 R^1 \left( 1 - F(p^i_{d(1)}|c) \right) p^i_{d(1)} \\
+ 1/4 B_1 p^i_{d(2)} + 1/2 R^1 \left( 1 - F(p^i_{d(2)}|c) \right) p^i_{d(2)}. \]

Summing up over these three cases, we obtain the seller’s profit for the \textit{lo-lo} regime as:

\[ \Pi^i = 1/2 B_1 p^i_d + a/2 \left( R^1 + R^2 \right) p^i_d \\
+ b/4 B_2 p^i_d + b/2 \left( R^1 + R^2 \right) \left( 1 - F(p^i_d|b) \right) p^i_d \\
+ c R^1 \left( 1 - F(p^i_d|c) \right) p^i_d. \]

If we take into account the equilibrium restriction that discount price is lower in the \textit{lo-lo} regime than in the \textit{hi-lo} regime, we get

\[ \Pi^i = 1/2 B_1 p^i_d + b/4 B_2 p^i_d + (a + b)/2 \left( R^1 + R^2 \right) p^i_d + c R^1 \left( 1 - F(p^i_d|c) \right) p^i_d \]

\[ = \left( 1/2 B_1 + b/4 B_2 + (a + b)/2 \left( R^1 + R^2 \right) \right) p^i_d + \underbrace{R^1}_{=:\epsilon_c} \left( 1 - a - b - F(p^i_d) \right) p^i_d. \]

Yet, the earlier profit expression is also useful to analyze profitability of different deviations.

\textit{Proofs of Proposition 13 and Proposition 14.} To support randomized pricing over the regimes, the profit to the seller in the \textit{hi-hi} regime must be the same as the the profit to the seller in the \textit{hi-lo} regime. In particular, the seller’s profit must be the same when it has the prices \((1, 1)\) (\textit{hi-hi}) and the prices \((1, 1 - \epsilon)\) (\textit{hi-lo}) for some tiny \(\epsilon > 0\). At the limit where \(\epsilon \to 0\), this results in the following condition:
\[
\frac{1}{2} (B_1 + B_2) + \frac{a}{2} (B_3 + B_4) + \frac{b}{8} B_3 \\
= \frac{1}{4} B_1 + \frac{1}{4} B_2 + a/8 B_3 + a/4 (R^1 + 2 R^2 + R^3) + b/8 (B_2 + B_3),
\]

which immediately gives the positive atom size
\[
a = \frac{B_2 (1 + c)}{B_2 + 3B_3 + 4B_4} \in (0,1).
\]

The equilibrium price distribution can be solved from
\[
\Pi_{two} = d_b + e_b p^i_2 + s_b (1 - a - F(p^i_2)) p^i_2,
\]
for the hi-lo regime, and from
\[
\Pi_{two} = e_c p^i_2 + s_c (1 - a - b - F(p^i_2)) p^i_2,
\]
for the lo-lo regime.

Thus, if the lower bound \(p\) is attained in the hi-lo regime already, it is given by
\[
\Pi_{two} = d_b + e_b p + s_b (1 - a)p,
\]
but, if the lower bound is attained in the lo-lo regime instead, it is given by
\[
\Pi_{two} = e_c p + s_c (1 - a - b)p. \quad \blacksquare
\]

**Proof of Proposition 15.** Since we now have shown that \(a > 0\) and that every active store randomizes at least one price, we know that a hi-lo equilibrium features at least two regimes, the hi-hi regime and the hi-lo regime. We start by constructing an equilibrium with just these two regimes first. This enables us to derive existence conditions for this simple two regime equilibrium. For the cases where the existence fails, we then move on to analyze more complex the equilibrium with also the lo-lo regime and the existence conditions associated with that one.

**Hi-lo equilibrium with two regimes:** Gathering the results derived so far, we know that if \(c = 0\), the atom size is given by
\[
a = \frac{B_2}{B_2 + 3B_3 + 4B_4} \in (0,1),
\]
the probability that seller \(i\)'s discount price beats seller \(-i\)'s discount price, if they both have one discount price, is given by
(1 − F(p_i^2 | b)) = \frac{\Pi_{two} - d_b}{s_b p_i^2} - \frac{e_b}{s_b},

and the lowest price quoted in an equilibrium is

p = \frac{\Pi_{two} - d_b}{e_b + s_b} = \frac{1}{2} (B_1 + B_2) + \frac{a}{2} (B_3 + B_4) + \frac{b}{8} B_3 - \frac{1}{4} B_1 + a/8 B_3

(1/4 B_1 + 1/4 B_2 + 1/4 (R^1 + 2 R^2 + R^3)) + b/8 (B_2 + B_3)

Note that, since it is with no loss of generality to assume that a seller has a (weakly) lower price \( p_i^2 \) and a (weakly) higher price \( p_i^1 \) and since the lower price is supported on \([p, 1]\), we have quite much on the equilibrium path already. Note additionally that a deviation in both prices to some \( p_1 < p \) and to some \( p_2 < p \) is dominate by a deviation to \( p \) in both of them since that the price is higher but the gain is the same in terms of the buyers. In consequence, to look for profitable deviations, we only need to consider the seller’s profit if he, for some lower price \( p_i^2 \in [p, 1] \), deviates to some higher price \( p_i^1 \in [p_2, 1) \) that is a discount price in stead of the monopoly price.

The seller’s profit is

\[ \Pi^i = \frac{1}{2} (B_1 + B_2) + \frac{a}{2} (B_3 + B_4) + \frac{b}{8} B_3 \]

or, if expressed in the other way,

\[ \Pi^i = 1/4 B_1 + 1/4 B_1 p_i^2 + 1/4 B_2 p_i^2 + a/8 B_3 + a/4 (R^1 + 2 R^2 + R^3) p_i^2 + b/8 (B_2 + B_3) + b/4 (R^1 + 2 R^2 + R^3) (1 - F(p_i^2 | b)) p_i^2 + c/2 (R^1 + R^2) (1 - F(p_i^2 | c)) p_i^2 \]

\[ = 1/4 B_1 + 1/4 B_1 p_i^2 + 1/4 B_2 p_i^2 + a/8 B_3 + a/4 (R^1 + 2 R^2 + R^3) p_i^2 + b/8 (B_2 + B_3) + 1/4 (R^1 + 2 R^2 + R^3) (1 - a - F(p_i^2)) p_i^2. \] (3.14)
\[ \Pi' = \frac{1}{2} \left( \frac{1}{2} B_1 p_2^i + \frac{a}{2} (R_1 + R_2) p_2^i + b/4 B_2 p_2^i + b/2 (R_1 + R_2) \right) (1 - F(p_2^i | b)) p_2^i + c R_1^2 \left( 1 - F(p_2^i | c) \right) p_2^i + \frac{1}{2} \left( \frac{1}{2} B_1 p_1^i + \frac{a}{2} (R_1 + R_2) p_1^i + b/4 B_2 p_1^i + b/2 (R_1 + R_2) \right) (1 - F(p_1^i | b)) p_1^i + c R_1^2 \left( 1 - F(p_1^i | c) \right) p_1^i \]

\[ = \frac{1}{2} \left( \frac{1}{2} B_1 + \frac{a}{2} (R_1 + R_2) + b/4 B_2 \right) p_1^i + \frac{1}{2} \left( R_1 + R_2 \right) \right) (1 - a - F(p_1^i)) \right) p_1^i \]

\[ + \frac{1}{2} \left( \frac{1}{2} B_1 + \frac{a}{2} (R_1 + R_2) + b/4 B_2 \right) p_2^i + \frac{1}{2} \left( R_1 + R_2 \right) \right) (1 - a - F(p_2^i)) \right) p_2^i \right) \],

\[ (1 - a - F(p_2^i)) p_2^i = \frac{\Pi' - \frac{1}{4} B_1 - \frac{a}{8} B_3 - \frac{b}{8} (B_2 + B_3)}{1/4 (R_1 + 2R_2 + R_3)} \]

\[ - \frac{1/4 B_1 + 1/4 B_2 + a/4 (R_1 + 2R_2 + R_3)}{1/4 (R_1 + 2R_2 + R_3)} \] p_2^i.

This entails that, the seller’s profit after a deviation is expressed as

\[ \Pi' = \frac{1}{2} \left( k_d p_1^i + l_d (u_d - v_d p_1^i)) + \frac{1}{2} \left( k_d p_2^i + l_d (u_d - v_d p_2^i)) \right) \].

Since the expression is linear in \( p_1^i \) and \( p_1^i \) it is clear that the maximum or the supremum is located on the boundaries such that the higher price is either (right below) unity or equal to the lower price and the lower price is either (right below) unity of equal to the lower bound: \((p_1^i, p_2^i) = (1 - \epsilon, 1 - \epsilon)\), \((p_1^i, p_2^i) = (1 - \epsilon, p)\) or \((p_1^i, p_2^i) = (p, p)\) for some tiny \( \epsilon > 0 \). We start by checking when it is the case that the seller has a profitable deviation to \((1 - \epsilon, 1 - \epsilon)\) or to \((1 - \epsilon, p)\). For the comparisons, we place on-the-path profit on the lhs and off-the-path profit on the rhs:

As the case number one, if the deviation is to \((1 - \epsilon, 1 - \epsilon)\) where \( \epsilon \to 0^+ \), it is not worthwhile if

\[ \ldots \]
\[
\frac{B_2 + 5/2B_3 + 2B_4}{B_2 + 3B_3 + 2B_4} \geq \frac{B_2}{B_2 + 3B_3 + 4B_4}
\]

which is an identical truth.

As the case number two, if the deviation is to \((1 - \epsilon, p)\) where \(\epsilon \to 0^+\), it is not worthwhile if

\[
\frac{1}{4}B_1 + \frac{a}{8}B_3
\]
\[
+ \left( \frac{1}{4}B_1 + \frac{1}{4}B_2 + \frac{a}{4}(B_2 + 3B_3 + 4B_4) + \frac{b}{8}(B_2 + B_3) \right) p
\]
\[
+ \frac{1}{4}(B_2 + 3B_3 + 4B_4) \left( 1 - a - F(p) \right) p
\]
\[
\geq \frac{1}{4}B_1 + \frac{a}{4}(B_2 + 2B_3 + 2B_4) + \frac{b}{8}B_2
\]
\[
+ \left( \frac{1}{4}B_1 + \frac{a}{4}(B_2 + 2B_3 + 2B_4) + \frac{b}{8}B_2 \right) p
\]
\[
+ \frac{1}{4}(B_2 + 2B_3 + 2B_4) \left( 1 - a - F(p) \right) p
\]

which is equivalent to

\[
a/8B_3 + 1/4(3/4B_2 + 5/4B_3 + 2B_4) \geq a/4(3/4B_2 + 9/4B_3 + 2B_4)
\]

and a necessary condition of

\[
\frac{B_2 + 5/3B_3 + 8/3B_4}{B_2 + 3B_3 + 8/3B_4} \geq \frac{B_2}{B_2 + 3B_3 + 4B_4}
\]

Again, this is an identical truth.

Finally, consider when there is a profitable deviation to \((p, p)\):
\[ \frac{1}{2} (B_1 + B_2) + \frac{a}{2} (B_3 + B_4) + \frac{b}{8} B_3 < (1/2B_1 + a/2 (B_2 + 2B_3 + 2B_4) + b/4B_2) p + 1/2 (B_2 + 2B_3 + 2B_4) (1 - a - F(p)) = (1/2B_1 + b/4B_2 + 1/2 (B_2 + 2B_3 + 2B_4)) p, \]
which holds whenever
\[ 1/2 (B_1 + B_2) (1 - p) + (B_3 + B_4) (a/2 - p) + b/4B_3 (1/2 - p) < 0. \]

For our Poisson model, we find numerically that there is no such profitable deviation \( \text{iff} \sigma \theta \leq \theta^0 \) where \( \theta^0 \approx 713.31 \). Otherwise, we have to go for the more complex case where there is also the lo-lo regime. We analyze that one next.

**Hi-lo equilibrium with three regimes:** Note that, for any \( \sigma \theta > 0 \), we could always start the construction of a hi-lo equilibrium by deriving cdf \( F \) and \( p \) of a candidate for the unique, simple hi-lo equilibrium as we just did. If there is no profitable deviation from prices \( (1, 1) \) to prices \( (p, p) \), we now know we are done since there cannot be profitable deviations to prices \( (1 - \epsilon, p) \) or \( (1 - \epsilon, 1 - \epsilon) \) for any tiny \( \epsilon \) or, thus, to any mixtures of these by what has been proofed above. However, if there is a profitable deviation from prices \( (1, 1) \) to prices \( (p, p) \), we proceed as follows:

We start to raise \( c \) continuously. To any \( c \) we map a price \( p' \) which is a candidate for a lower bound of the hi-lo regime and thus a candidate for a upper bound of the lo-lo regime. In other words, \( p'(c) \) is such that it equates the seller’s profit in the hi-hi regime for the given \( c \) and the seller’s profit in the hi-lo regime for the given \( c \), and the price \( p'(c) \) is better than any discount price \( p > p' \) in the hi-lo regime but worse than any discount price \( p < p' \) in the lo-lo regime:

\[
\Pi' \left( 1; \begin{pmatrix} 1; c \end{pmatrix} \right) := \frac{1}{2} (B_1 + B_2) + \left( \frac{3a}{8} + \frac{1}{8} \right) B_3 + \frac{a}{2} B_4
\]
\[
\Pi' \left( 1; \begin{pmatrix} p'; c \end{pmatrix} \right) := 1/4B_1 + 1/4B_1p' + 1/4B_2p' + a/8B_3 + a/4 \left( R^1 + 2R^2 + R^3 \right) p'
+ b/8 (B_2 + B_3) p' + b/4 \left( R^1 + 2R^2 + R^3 \right) (1 - a - c) p'.
\]

where \( a = (1 + c)A := (1 + c) \frac{B_2}{B_2 + 3B_3 + 4B_4} \) is increasing in \( c \). This shows that, since \( \Pi' \left( 1; \begin{pmatrix} 1; c \end{pmatrix} \right) \) is increasing in \( c \) and \( \Pi' \left( 1; \begin{pmatrix} p'; c \end{pmatrix} \right) \) is decreasing in \( c \), a larger \( c \) implies a larger \( p'(c) \). This shows that, in particular, \( p'(c) > p'(0) \) for \( c > 0 \), where \( p'(0) = \underline{p} \) for the lower bound in the simple hi-lo equilibrium that was our starting point.

\( ^{31}\)The numerical findings are reported in Table 3.7.
Notice also that as, by construction,

\[
\Pi'(1,1;c) := \frac{1}{2}(B_1 + B_2) + \left(\frac{3a}{8} + \frac{1}{8}\right)B_3 + \frac{a}{2}B_4 = 0
\]

\[
\Pi'(1,1-c;1-a) := \frac{1}{4}B_1 + \frac{1}{4}B_2 + \frac{1}{4}B_2 + \frac{a}{8}B_3 + \frac{a}{4}(R^1 + 2R^2 + R^3) + \frac{b}{8}(B_2 + B_3) + \frac{b}{4}(R_1 + 2R_2 + R_3)(1 - a - (1 - a)) < 0
\]

\[
\Pi'(1,1-c;c) := \frac{1}{4}B_1 + \frac{1}{4}B_1 + \frac{1}{4}B_2 + \frac{a}{8}B_3 + \frac{a}{4}(R^1 + 2R^2 + R^3) + \frac{b}{8}(B_2 + B_3) + \frac{b}{4}(R^1 + 2R^2 + R^3)(1 - a - c) > 0
\]

\[
\Pi'(1,p';c) := \frac{1}{4}B_1 + \frac{1}{4}B_1p' + \frac{1}{4}B_2p' + \frac{a}{8}B_3 + \frac{a}{4}(R^1 + 2R^2 + R^3)p' + \frac{b}{8}(B_2 + B_3)p' + \frac{b}{4}(R^1 + 2R^2 + R^3)(1 - a - c)p'.
\]

(3.16)

for \(c > 0\) as \(c \to 0^+\). Specifically, \(\Pi'(1,1;c) \in (\Pi'(1,p'(0);c),\Pi'(1,1-c;c))\). This implies that, by continuity of \(\Pi'(1,p;c)\) in \(p\), it is indeed possible to always assign to any \(c < 1-a\) a unique price \(p'(c) < 1\) such that \(\Pi'(1,1;c) = \Pi'(1,p';c)\). The mapping that this produces, \(c \mapsto p'(c)\), is continuous and increasing in \(c\).

Next, we have to find \(c\) such that the seller is indifferent between quoting the prices \((1,p'(c))\) and the prices \((p'(c),p'(c))\). This is possible iff there is a profitable deviation \(\Pi'(p'(0),p'(0);0) > \Pi'(1,1;0)\) in the candidate for a simple hi-lo equilibrium. In addition, we can use the finding that \(\Pi'(1,1;0) \leq \Pi'(1 - c,1 - c;0)\). As \(\Pi'(1,1;c)\) is increasing in \(c\) but \(\Pi'(p,p;c)\) is decreasing in \(c\), these inequalities hold also as we start to raise \(c\); they hold for any \(c \in (0, 1-a)\).

Therefore, it is clear that, as we trace the continuous path \(c \mapsto p'(c)\) from \((c,p'(c)) = (0,p'(0))\) to \((c,p'(c)) = (1 - a, 1)\), we necessarily go through a fixed point \((c,p'(c))\) such that the seller’s profit is the same for \((p'(c),p'(c))\) and for \((1,p'(c))\), By monotonicity of the deviating seller’s profit along the map \(c \mapsto p'(c)\), the fixed point is unique.

We could also just say that we have two equations \(\Pi(1,1;c) = \Pi(1,p'(c);c)\) and \(\Pi(1,p'(c);c) = \Pi(p'(c),p'(c);c)\) for two unknowns \(p'(c)\) and \(c\). While the equations are nonlinear in \(p'(c)\) and \(c\), due to the cross terms of form \(pc\), we show that there always exists a solution and it is unique.

To recap, we have shown that, assuming that the buyers search as postulated, it is always possible to find pricing policies consistent with a hi-lo equilibrium. These pricing policies are unique by construction. This implies that, as long as the condition in Lemma 11,

\[
\frac{1}{2b} \geq \frac{1}{a + 2b}(1 - p_s) \geq 1/2b(1 - p_s) + c(1 - p_s),
\]

(3.17)

is satisfied such that the buyers have an incentive to stick to the start store if they first
discover a price \( p = 1 \) and switch the stores if they first discover a price \( p < 1 \), there exists a hi-lo equilibrium for two prices for whatever the level of frictions \( \sigma \).

By now we know that, for \( \sigma \in (0, \theta^o] \), the pricing policies are such that \( c = 0 \) and, since (3.17) holds, the buyers are willing to search as required. We have also seen that, for any frictions \( \sigma \in (\theta^o, \infty] \), the candidate pricing policies consistent with hi-lo equilibrium have \( c > 0 \), where the probability of the lo-lo regime \( c \) is continuous and increasing in the frictions \( \sigma \). The continuity then implies that there exist a rate \( \theta^x \) larger than the rate \( \theta^o \) such that, for any frictions \( \sigma \in (\theta^o, \theta^x] \), since (3.17) still continues to hold, the buyers are willing to search as required.

Furthermore, since the candidate pricing policies consistent with the hi-lo equilibrium are unique by construction, for any \( \sigma \leq \theta^x \), there exists a unique hi-lo equilibrium for two prices. If \( \sigma \in (0, \theta^o] \), it is the unique, simple hi-lo equilibrium with the hi-hi regime and the hi-lo regime but no lo-lo regime and, if \( \sigma \in (\theta^o, \theta^x] \), it is the unique hi-lo equilibrium with all these three regimes. □

Proof of Proposition 16. Suppose the sellers have both two prices \((p_1^1, p_2^1)\) that are independent and identically distributed such that \( p_n^i \sim F \) for \( n = 1, 2 \). By Lemma 10, the support has to be an interval, \( \text{supp}(F) = [p, 1] \), and there could not be atoms in it. The buyers contact a random seller and, if they have time, search the two prices and, then, switch the seller. For any two random draws \( p_1^i \) and \( p_2^i \) from \( F \), denote by \( p_{\text{max}} \) the larger an by by \( p_{\text{min}} \) the smaller. Thus, the profit to the seller is

\[
\Pi = \frac{1}{4} B_1 p_{\text{max}} + \frac{1}{4} B_3 (1 - (p_{\text{max}}))^2 p_{\text{max}} + (\frac{1}{4} B_1 + \frac{1}{2} B_2) p_{\text{min}} + \frac{1}{4} B_3 (1 - (p_{\text{min}}))^2 p_{\text{min}} + \frac{1}{2} B_3 (1 - (p_{\text{min}}))^2 p_{\text{min}} + \frac{1}{2} B_3 (1 - (p_{\text{min}}))^2 p_{\text{min}} + B_4 (1 - (p_{\text{min}}))^2 p_{\text{min}},
\]

(3.18)

which could be decomposed as \( \Pi = \Pi_{\text{max}} + \Pi_{\text{min}} \)

\[
\Pi_{\text{max}} = \frac{1}{4} B_1 p_{\text{max}} + \frac{1}{4} B_3 (1 - (p_{\text{max}}))^2 p_{\text{max}}
\]
\[
\Pi_{\text{min}} = (\frac{1}{4} B_1 + \frac{1}{2} B_2) p_{\text{min}} + \frac{1}{4} B_3 (1 - (p_{\text{min}}))^2 p_{\text{min}} + \frac{1}{2} B_3 (1 - (p_{\text{min}}))^2 p_{\text{min}} + B_4 (1 - (p_{\text{min}}))^2 p_{\text{min}}.
\]

(3.19)

Observe that due to the additive structure \( \Pi_{\text{max}} \) and \( \Pi_{\text{min}} \) have to be constant or else there would be a profitable deviation in maximizing both of them. To keep the seller indifferent between following the recommended realizations \( p_{\text{max}} \) and \( p_{\text{min}} \) and some other feasible draws \( p_{\text{max}} \in [p_{\text{min}}, 1] \) and \( p_{\text{min}} \in [p, 1] \) the distribution \( F \) has to be such that

\[
\Pi_{\text{max}} = \frac{1}{4} B_1 p + \frac{1}{4} B_3 (1 - F(p))^2 p
\]
for all $p \in [p, 1]$ and

$$\Pi_{\text{min}} = (1/4B_1 + 1/2B_2) p + 1/4B_3 (1 - F(p))^2 p$$
$$+ 1/2B_3 (1 - F(p))^2 p + B_4 (1 - F(p))^2 p$$

(3.20)

for all $p \in [p, 1]$. Thus,

$$(1 - F(p))^2 = \frac{\Pi_{\text{max}}}{1/4B_3 p} - \frac{1/4B_1}{1/4B_3}$$

and

$$(1/4B_3 + B_4) (1 - F(p))^2 + 1/2B_3 (1 - F(p)) = \frac{\Pi_{\text{min}}}{p} - 1/4B_1 - 1/2B_2.$$  

(3.21)

Both hold only for $B_3 = 1/4$. ■

*Seller’s profit in a simple hi-lo equilibrium for three prices.* Applying directly the formulas for the seller’s profit in Section 3.4.1 for $n = 1$, we find that in the hi-hi regime the profit is

$$\Pi(1, 1) = a/2 (B_1 + B_2 + B_3 + B_4 + B_5 + B_6)$$
$$+ b/2 (B_1 + B_2 + B_3 + 2/3B_4 + 1/3B_5)$$

and in the hi-lo regime the profit is

$$\Pi(1, p) = a/2 (2/3B_1 + 1/3B_2)$$
$$+ a/2 (1/3B_1 + 2/3B_2 + B_3 + B_4 + B_5 + B_6) p$$
$$+ a/2 (1/3B_4 + 2/3B_5 + B_6) p$$
$$+ a/2 (2/3B_4 + 1/3B_5) 1/2$$
$$+ b/2 (2/3B_1 + 1/3B_2)$$
$$+ b/2 (1/3B_1 + 5/9B_2 + 6/9B_3 + 3/9B_4 + 1/9B_5) p$$
$$+ b (1/9B_2 + 3/9B_3 + 6/9B_4 + 8/9B_5 + B_6) (1 - F(p|b)) p$$

In comparison, if the seller deviates by lowering one of the two monopoly prices into a discount price, the seller’s profit is
\( \Pi(p, p) = a/2 \left( 1/3B_1 \right) \\
+ a/2 \left( 2/3B_1 + B_2 + B_3 + B_4 + B_5 + B_6 \right) p \\
+ a/2 \left( 2/3B_4 + B_5 + B_6 \right) p \\
+ a/2 \left( 1/3B_4 \right) \frac{1}{2} \\
+ b/2 \left( 1/3B_1 \right) \\
+ b/2 \left( 2/3B_1 + 7/9B_2 + 4/9B_3 + 1/9B_4 \right) p \\
+ b/2 \left( 2/9B_2 + 5/9B_3 + 8/9B_4 + B_5 + B_6 \right) (1 - F(p|b)) p \\
+ b/2 \left( 2/9B_2 + 5/9B_3 + 8/9B_4 + B_5 + B_6 \right) (1 - F(p|b)) p

The atom size \( a \) is the solution to equation

\[
\Pi(1, 1) = \Pi(1, 1 - \epsilon)
\]

for \( \epsilon \to 0^+ \) and, after we have derived the atom size, the lower bound \( p \) can be solved from

\[
\Pi(1, 1) = \Pi(1, p)
\]

where it should be noted that \( (1 - F(p|b)) = 0 \).

As a result,

\[
a = \frac{B_2 + 3B_4 + 3B_4 + 2B_5}{B_2 + 3B_3 + 9B_4 + 19/3B_5 + 9B_6}
\]

\[
p = \frac{\Pi - A}{B}, \text{ where}
\]

\[
\Pi = a/2 \left( B_1 + B_2 + B_3 + B_4 + B_5 + B_6 \right) + (1 - a)/2 \left( B_1 + B_2 + B_3 + 2/3B_4 + 1/3B_5 \right)
= 1/2 \left( B_1 + B_2 + B_3 + 2/3B_4 + 1/3B_5 + a \left( 1/3B_4 + 2/3B_5 + B_6 \right) \right)
\]

(3.22)
\[ A = \frac{a}{2} \left( \frac{2}{3}B_1 + \frac{1}{3}B_2 \right) \\
+ \frac{a}{2} \left( \frac{2}{3}B_4 + \frac{1}{3}B_5 \right) \frac{1}{2} \\
+ \frac{(1 - a)}{2} \left( \frac{2}{3}B_1 + \frac{1}{3}B_2 \right) \\
= \frac{1}{2} \left( \frac{2}{3}B_1 + \frac{1}{3}B_2 + a \left( \frac{2}{3}B_4 + \frac{1}{3}B_5 \right) \right) \tag{3.23} \]

\[ B = \frac{a}{2} \left( \frac{1}{3}B_1 + \frac{2}{3}B_2 + B_3 + B_4 + B_5 + B_6 \right) \\
+ \frac{a}{2} \left( \frac{1}{3}B_4 + \frac{2}{3}B_5 + B_6 \right) \\
+ \frac{(1 - a)}{2} \left( \frac{1}{3}B_1 + \frac{5}{9}B_2 + \frac{6}{9}B_3 + \frac{3}{9}B_4 + \frac{1}{9}B_5 \right) \\
+ \frac{(1 - a)}{2} \left( \frac{1}{9}B_2 + \frac{3}{9}B_3 + \frac{6}{9}B_4 + \frac{8}{9}B_5 + B_6 \right) \\
= \frac{1}{2} \left( \frac{1}{3}B_1 + \frac{2}{3}B_2 + B_3 + B_4 + \frac{5}{3}B_5 + 2B_6 \right) \\
+ \frac{1}{2} \left( 1 - a \right) \left( \frac{1}{9}B_2 + \frac{1}{3}B_3 + \frac{2}{3}B_4 + \frac{2}{9}B_5 \right) \]

This information can now be used to find out numerically when there is a profitable deviation from prices \((1, 1)\) to prices \((p, \overline{p})\). For our Poisson model, we find numerically that there is no such profitable deviation \(\text{iff } \sigma \theta \leq \theta^o\) where \(\theta^o \approx 719.32\).\textsuperscript{32}

\textsuperscript{32}The numerical findings are reported in Table 3.8.
### Appendix C: Tables

#### Table 3.1: Search outcomes: store i in hi-hi and store -i in hi-hi regime

<table>
<thead>
<tr>
<th>start store</th>
<th>$Pr(k = 1) = B_1$</th>
<th>$Pr(k = 2) = B_2$</th>
<th>$Pr(k = 3) = B_3$</th>
<th>$Pr(k = 4) = B_4$</th>
</tr>
</thead>
<tbody>
<tr>
<td>i: hi-hi</td>
<td>$Pr(\omega_i^m</td>
<td>k = 1) = \frac{1}{2}$</td>
<td>$Pr(\omega_i^m</td>
<td>k = 2) = \frac{1}{2}$</td>
</tr>
<tr>
<td>-i: hi-hi</td>
<td>$Pr(\omega_{-i}^m</td>
<td>k = 1) = \frac{1}{2}$</td>
<td>$Pr(\omega_{-i}^m</td>
<td>k = 2) = \frac{1}{2}$</td>
</tr>
</tbody>
</table>

#### Table 3.2: Search outcomes: store i in hi-hi and store -i in hi-lo regime

<table>
<thead>
<tr>
<th>start store</th>
<th>$Pr(k = 1) = B_1$</th>
<th>$Pr(k = 2) = B_2$</th>
<th>$Pr(k = 3) = B_3$</th>
<th>$Pr(k = 4) = B_4$</th>
</tr>
</thead>
<tbody>
<tr>
<td>i: hi-hi</td>
<td>$Pr(\omega_i^m</td>
<td>k = 1) = \frac{1}{2}$</td>
<td>$Pr(\omega_i^m</td>
<td>k = 2) = \frac{1}{2}$</td>
</tr>
<tr>
<td>-i: hi-lo</td>
<td>$Pr(\omega_{-i}^m</td>
<td>k = 1) = \frac{1}{2}$</td>
<td>$Pr(\omega_{-i}^m</td>
<td>k = 2) = \frac{1}{2}$</td>
</tr>
</tbody>
</table>

#### Table 3.3: Search outcomes: store i in hi-hi and store -i in lo-lo regime

<table>
<thead>
<tr>
<th>start store</th>
<th>$Pr(k = 1) = B_1$</th>
<th>$Pr(k = 2) = B_2$</th>
<th>$Pr(k = 3) = B_3$</th>
<th>$Pr(k = 4) = B_4$</th>
</tr>
</thead>
<tbody>
<tr>
<td>i: hi-lo</td>
<td>$Pr(\omega_i^l</td>
<td>k = 1) = \frac{1}{4}$</td>
<td>$Pr(\omega_i^l</td>
<td>k = 2) = \frac{1}{4}$</td>
</tr>
<tr>
<td>-i: lo-lo</td>
<td>$Pr(\omega_{-i}^l</td>
<td>k = 1) = \frac{1}{4}$</td>
<td>$Pr(\omega_{-i}^l</td>
<td>k = 2) = \frac{1}{4}$</td>
</tr>
</tbody>
</table>

#### Table 3.4: Search outcomes: store i in hi-lo and store -i in hi-hi regime

<table>
<thead>
<tr>
<th>start store</th>
<th>$Pr(k = 1) = B_1$</th>
<th>$Pr(k = 2) = B_2$</th>
<th>$Pr(k = 3) = B_3$</th>
<th>$Pr(k = 4) = B_4$</th>
</tr>
</thead>
<tbody>
<tr>
<td>i: hi-lo</td>
<td>$Pr(\omega_i^l</td>
<td>k = 1) = \frac{1}{4}$</td>
<td>$Pr(\omega_i^l</td>
<td>k = 2) = \frac{1}{4}$</td>
</tr>
<tr>
<td>-i: hi-hi</td>
<td>$Pr(\omega_{-i}^m</td>
<td>k = 1) = \frac{1}{4}$</td>
<td>$Pr(\omega_{-i}^m</td>
<td>k = 2) = \frac{1}{4}$</td>
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#### Table 3.5: Search outcomes: store i in hi-lo and store -i in lo-lo regime

<table>
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<tr>
<th>start store</th>
<th>$Pr(k = 1) = B_1$</th>
<th>$Pr(k = 2) = B_2$</th>
<th>$Pr(k = 3) = B_3$</th>
<th>$Pr(k = 4) = B_4$</th>
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</thead>
<tbody>
<tr>
<td>i: hi-lo</td>
<td>$Pr(\omega_i^l</td>
<td>k = 1) = \frac{1}{4}$</td>
<td>$Pr(\omega_i^l</td>
<td>k = 2) = \frac{1}{4}$</td>
</tr>
<tr>
<td>-i: lo-lo</td>
<td>$Pr(\omega_{-i}^l</td>
<td>k = 1) = \frac{1}{4}$</td>
<td>$Pr(\omega_{-i}^l</td>
<td>k = 2) = \frac{1}{4}$</td>
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#### Table 3.6: Search outcomes: store i in lo-lo and store -i in lo-lo regime

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<th>start store</th>
<th>$Pr(k = 1) = B_1$</th>
<th>$Pr(k = 2) = B_2$</th>
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</tr>
</thead>
<tbody>
<tr>
<td>i: lo-lo</td>
<td>$Pr(\omega_i^l</td>
<td>k = 1) = \frac{1}{4}$</td>
<td>$Pr(\omega_i^l</td>
<td>k = 2) = \frac{1}{4}$</td>
</tr>
<tr>
<td>-i: lo-lo</td>
<td>$Pr(\omega_{-i}^l</td>
<td>k = 1) = \frac{1}{4}$</td>
<td>$Pr(\omega_{-i}^l</td>
<td>k = 2) = \frac{1}{4}$</td>
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Table 3.7: Profitable deviations ($\Delta(\Pi) < 0$) and non-profitable deviations ($\Delta(\Pi) > 0$)
Table 3.8: Profitable deviations ($\Delta(\Pi) < 0$) and non-profitable deviations ($\Delta(\Pi) > 0$)
Bibliography


Chapter 4

Splitting consumers:
Equilibria with endogenous shopping frictions
List of symbols

\[ i \in \{1, 2\} \quad \text{seller/store index} \]
\[ B = [0, 1] \quad \text{the set of buyers} \]
\[ t \in [0, 1] \quad \text{time index} \]
\[ d \equiv 1 \quad \text{buyers’ common deadline} \]

STAGE 1
\[ \theta^i \in [0, \infty] \quad \text{the Poisson arrival/finding rate of price } p^i \text{ in store } i \]

STAGE 2
\[ F^i \in \Delta [0, 1] \quad \text{store } i \text{'s (mixed) pricing strategy} \]
\[ p^i \in [0, 1] \quad \text{(realized) price in store } i \]
\[ E(p|F^i) \in [0, 1] \quad \text{expected price in store } i \]

STAGE 3
\[ t_i \in [0, 1] \quad \text{the fraction of buyers who start from store } i \]
\[ B_0 \in [0, 1] \quad \text{the mass of buyers who have not found a price by } t = d \]
\[ B_i \in [0, 1] \quad \text{the mass of buyers who have found only } p^i \text{ by } t = d \]
\[ B_{1,2} \in [0, 1] \quad \text{the mass of buyers who have found both } p^1 \text{ and } p^2 \text{ by } t = d \]
Splitting consumers: 
Equilibria with endogenous shopping frictions

Abstract

We develop a price search model that features endogenous frictions in a duopolistic environment. These frictions originate from the gradual arrival or price information within stores and the existence of deadlines for buyers. We show that both sellers have a strategic incentive to generate frictions. There exists exactly two equilibria with a unique asymmetric pattern: a prominent seller, whose expected price is higher but the in-store frictions lower, and a non-prominent seller. The buyers are divided exactly equally into informed and uninformed consumers, and into those who fail to find anything. Under the Poisson process, this surplus loss is 6%.

Keywords: Deadlines; Frictions; Prices; Search; Poisson process; Obfuscation. JEL-codes: D43, D83.
4.1 Introduction

Managing the traffic of incoming and outgoing consumers is an important part of running an online store. As consumers are typically busy, it is not irrelevant in which order they sample the stores and how long a time they tend to stay in. The stores can affect this consumer turnover in many ways, particularly, figuratively, by putting some sand or oil in the wheels in terms of how the products are presented; the click paths could be made either long or short, for example.

We study the effects and origins of search frictions in a duopolistic price competition model featuring endogenous frictions, inspired by online search. To abstract from hold-up problems (Diamond, 1971) arising in sequential search setups with upfront payment of the search cost, we use a model based on deadlines and gradual arrival of price information in every store; there is no explicit cost of searching nor switching.

This modification of the standard framework (Varian, 1980; Stahl, 1989) makes it possible to capture endogenous frictions in a new reduced way: we allow sellers to adjust the rates of the Poisson process that determine how fast a buyer finds a price in a store. These search frictions affect both the number of trades and the shares of informed consumers and uninformed consumers in the market, appearing in the classic papers by (Varian, 1980; Stahl, 1989) that our model nests. This in turn enables us to put a number on the size of the loss generated by the frictions and comment on where the market is likely to stand in between the Diamond (1971)\(^1\) and Bertrand (1883)\(^2\) outcomes.

Specifically, we find that there exist precisely two equilibria with pure frictions. In both of them, one of the stores – called the prominent one – has lower frictions and higher prices in comparison to the other store. Although the sellers are competing in frictions, they both generate positive frictions. This implies that some buyers always fail to find a price; under the Poisson process, this surplus loss amounts to 6%.

Interestingly, using the jargon from Stahl (1989), we also find that there are exactly equally many shoppers (with two price quotes) and searchers (with one price quote). This is a rather remarkable result because, arguably, our outcome is therefore precisely in between the Diamond and Bertrand outcomes. While it is well known that models like this where the buyers are divided into informed consumers and uninformed consumers span both the outcomes for appropriate parametric assumptions (Varian (1980) and Stahl (1989)), despite the obvious interest in this division that dictates how competitive the market is, not much has been said about the actual shares before.

We study a setup with a unit mass of buyers and two similar sellers competing in frictions, by which we refer to the intensities of the Poisson process that determine how

\(^1\)Where the sellers get all the surplus, MR=MC.

\(^2\)Where the buyers get all the surplus, p=MC.
fast it is for the buyers to find a price in a store.\(^3\)\(^,\)\(^4\) The price is not found right away once a buyer enters a store but according to this Poisson process. The sellers set their arrival rates in the beginning of the game (stage 1 of the game) and they are common knowledge immediately thereafter affecting, thus, prices (stage 2) and search (stage 3). Buyers search under a deadline. Their search costs are zero before the deadline and infinity after the deadline.

Apart from being a theory contribution, this paper could hence be regarded as an attempt to put together some elements pertinent to online search, namely, (i) the negligible time cost of "traveling" to a store or switching from store to store, (ii) the positive, endogenous time cost of finding payoff relevant information within a store, and (iii) the significance of deadlines in managing our daily lives.

While the buyers are free to switch the seller at any point as long as they have time, we find that in equilibrium they do so only after they have found a price. This entails that the rates at which price information arrives play a role of an implicit endogenous switching cost. If the frictions are weaker in the first store, there is more time to discover the price in the second one. That intensifies price competition. As a result, there exist no efficient equilibria. Although one store could serve the entire market if it chose to play down all the frictions, it has no incentive to do so because that would also eliminate the switching cost.

It is noteworthy that both sellers have a strategic incentive to generate frictions, which does not arise, say, from a cost saving motive. In the unique equilibrium pattern, the in-store wait time, until the price is found, is drawn from \(\text{Exp}(2.76)\)\(^5\) for the prominent seller and from \(\text{Exp}(1.03)\)\(^6\) for the non-prominent seller. About 6 %\(^7\) of the cake is now lost due to the frictions. Of what is remaining, 50 % goes to the prominent seller, 25 % to the non-prominent seller and 25 % to the buyers. As it turns out, the size of the loss is related to the prominent seller’s incentives and the surplus division to the non-prominent sellers incentive’s.

The prominent seller’s profit is given by the number of uninformed, captive buyers it gains. Since consumers switch the store once they find a price, the prominent seller has a tradeoff between maximizing the number of consumers who find its own price (by decreasing the frictions, increasing the inflow) and minimizing the number of consumers who find the other seller’s price (by increasing the frictions, decreasing the outflow). It is hence optimal for it to avoid extremes and generate intermediate frictions. This implies that the number of trades is non-optimal. To summarize, as the seller does not fully reap

\(^3\)This gives our model a slight flavor of a Poisson bandit problem (see Bergemann and Välimäki (2006) for a compact review).
\(^4\)Apart from endogenous frictions, our model is essentially the same as in Hämäläinen (2015) for the case of one item in stock.
\(^5\)Rounded to percentiles.
\(^6\)Rounded to percentiles.
\(^7\)Rounded to percentiles.
(bear) the positive (negative) externality that faster (slower) search has on the consumers, it has no incentive to serve every buyer. As a result, any equilibrium is inefficient.

We observe that the two sellers’ incentives become generally more and more aligned when price competition intensifies. In the neighborhood of the equilibrium, the non-prominent seller’s profit is essentially a product of informed consumers and uninformed consumers. Its demand comes only from the former but it wins them over more frequently if the other seller has higher prices – if there is a larger number of the latter. Therefore, the non-prominent seller’s profit is maximized when the number of informed consumers equals the number of uninformed consumers. This is also reflected in the surplus sharing: 50% - 25% - 25%.

Note that our main findings, the universal incentive to generate frictions for sellers and the half-and-half division of consumers into the informed and the uninformed, only depend on the existence of the deadline, \(d < \infty\), but not on what it is – it could be a second or a decade.\(^8\) However, as the deadline vanishes, \(d = \infty\), every buyer finds every price and the Bertrand outcome uniquely obtains. There is hence a discontinuity in the equilibrium set at \(d = \infty\).

Our results are relevant especially for online search, where the greatest frictions are not exogenous (limited by the speed at which *computers* process information) but endogenous (limited by the speed at which *consumers* process information). They suggest that, although the base line search technology is constantly improving, frictions may not disappear. This point has been made also by, for instance, Ellison and Ellison (2009) and Ellison and Wolitzky (2012).

In their seminal article, Ellison and Wolitzky (2012) show that the sellers have a universal incentive to raise their prices and, to keep this from affecting the search, increase the time cost of shopping by various confusing selling practices, coined with the term obfuscation. Obfuscation aggravates the hold-up problem arising in sequential search setups. It enables the sellers to raise their prices without losing demand.

In another interesting contribution, Wilson (2010) demonstrates that the incentive to obfuscate is sustained even when the sellers could attract the buyers by committing to a lower obfuscation level: while one of the sellers has an incentive play down the frictions to boost its demand, the others are willing to maintain a high friction position to relax price competition. There exists a continuum of asymmetric equilibria.

However, although this previous work and the complementing empirical evidence (Ellison and Ellison, 2009) do make a case for the incentive to obfuscate, given the assumed approach, since the number of trades is fixed in this earlier work, it is difficult to assess whether the effects of obfuscation are restricted to surplus sharing or whether it impinges

---

\(^8\)If the deadline one is scaled up by \(c \in \mathbb{R}_+\), the intensities \(\theta^1 \approx 2.76\) and \(\theta^2 \approx 1.03\) just have to be scaled down by \(c \in \mathbb{R}_+\). A similar pattern is likely to arise for cases where the wait time is not exponentially distributed as long as the deadline is finite. It is also noteworthy that our model has no parameters except for this deadline that can be freely scaled up and down; it is essentially parameter-free.
on efficiency as well. Ellison and Wolitzky (2012) and Wilson (2010) are both founded on a classic model à la Varian (1980) and Stahl (1989) where the number of trades and the ratio of informed consumers to uninformed consumers is not responsive to obfuscation. That is a very nice, basic model to work with in extensions. Nevertheless, since a consumer’s decision to become informed or remain ignorant is likely to depend on frictions, that ratio is one of the most obvious adjustment margins to obfuscation and, thus, crucial for welfare analysis.

To take this properly into account, we use a new modeling approach motivated by the particularities of online search as described. As a key distinction from what has been done, it is assumed that the buyers can search free of cost, as long as they have time.\(^9\)\(^10\)\(^11\) This entails that the buyer’s optimal stopping problem is trivial (stop when time is up); instead, we focus on the optimal switching problem.\(^12\)

The results we thereby obtain extend the previous findings by demonstrating that obfuscation is inefficient and, thus, its adverse effects are not restricted to surplus sharing. Yet, the welfare loss seems small relative to the consequences on the division. It came also somewhat as a surprise that in our model the frictions get adjusted so as to equate the numbers of informed consumers and uninformed consumers. Our results also proof that welfare is not linked in any obvious way with symmetry (as of Ellison and Wolitzky (2012)) or asymmetry (as of Wilson (2010)) of equilibrium obfuscation, which is shown to be a matter of degree, rather than of kind: we find that both sellers generate frictions. With its new take on online search with deadlines, our model has obviously also some interest of its own.

Our findings can also be juxtaposed with those concerning market prominence. We find that a strict prominence order will arise. As buyers search efficiently in equilibrium, it is the frictions and not the price that determine the order in which the buyers search. The prominent seller is faster and has, therefore, higher prices and higher profit. In Armstrong

\(^9\)The assumption is in accordance with the observation that many buyers are quite happy to search for a bit, but not in extreme amounts: they tire out or run out of time. To put it differently, the search cost is zero for \(t \in [0, 1]\) and infinity for \(t > 1\).

\(^10\)Note that this extreme form of search cost \(g(t) \in \{0, \infty\}\) is a special case of convex search cost \(g(t)\) in search time \(t\) as in Ellison and Wolitzky (2012). The mechanism that renders obfuscation worthwhile is not the same, however. In their case, if the first price is found slowly, the expected cost of the second search is higher since the change is the cost that is due, \(g(t + \Delta) - g(t)\), increases with time, for any given \(\Delta\). In our case, if the first price is found slowly, the expected gain of the second search is lower since the chance of finding another price, \(1 - e^{\theta(1-t)}\), decreases with time.

\(^11\)Observe the 0-∞ search cost also strips the stores from, what some might feel is, too strong control for the firm over how many times a “searcher” (with positive cost of search) pays a visit to a store, which would, otherwise, frequently, be one; the sellers lose profits if they let the buyers search more. Yet, see an example in Ellison and Wolitzky (2012) for a more complex search pattern where, as a novelty to this model class, some of the “searchers” conduct a single search but others, who obtain a price quote from the intermediate range of price distribution, search multiple stores.

\(^12\)It was pointed out to us that our model has some familiarity not only with sequential search models but also with non-sequential search models; yet, it is not nested any of these model classes. See Baye et al. (2006) for a paper with both of these approaches.
et al. (2009) and Rhodes (2011), the non-prominent seller has a higher price whereas, in Arbatskaya (2007); Wilson (2010), it is the other way.

Our model has also connections with competitive search models à la Moen (1997); Peters (2000, 1991) and Burdett et al. (2001). We analyze a market where the sellers commit to frictions, that indirectly advertize the price. Competitive search models explore a market where the sellers commit to prices, that indirectly advertize the frictions. The frictions are modeled by a Poisson process in both. Yet, competitive search equilibria are usually constrained efficient but our model is not.\footnote{The so called Hosios (1990) condition is typically satisfied. Yet, see the models by Albrecht et al. (2006) and Galenianos and Kircher (2009) in which each worker can apply for many jobs. The authors find that the equilibrium is not constrained efficient. Note that also our model would be trivially efficient if we required that each buyer can visit one store at maximum, as typical in competitive search models. The Diamond (1971) outcome would arise but every buyer would find a good. This reiterates the idea that the inefficiency in our case arises from the fact that the seller cannot fully extract the benefits of low frictions ∼ with full monopoly power he could. It also suggests that, as frictions matter both for consumer inflow and for consumer outflow, to get the big picture right, it is important to use a model with some consumer turnover. Moreover, as the shortage or appropriate instruments is one possible source of inefficiency, it might be interesting to extend the setting by allowing the sellers to use two instruments, one to manage the inflow (“advertizing” instrument), the other one to manage the outflow (“obfuscation” instrument). In our present model there is one instrument for both, the intensity of the Poisson process.}

The paper is organized in the following way: The model is given in Section 2. Section 3 derives the equilibrium: a buyer’s search problem, a seller’s pricing problem and, finally, a seller’s problem of choosing the frictions. Section 4 offers some closing remarks.

4.2 Model

We analyze equilibria with endogenous frictions in a duopolistic market of simple price search. To give an application, the frictions could be taken to represent the sellers’ long-term investments in a particular search technology in their store (for consumer search offline) or webpage (for consumer search online).\footnote{The scrolls and the click-paths can be made either long or short by adjusting some elementary design parameters and the complexity of the webpage content. The general navigation experience might be controlled by other means such as a registration requirement, videos, ads etc.}

There are two sellers $i \in \{1,2\}$ and a unit mass of buyers $B = [0, 1]$, each buyer with a unit of free time $t \in [0, 1]$. Both sellers have an infinite capacity in the products of the kind the buyers search for. Every buyer is willing to buy exactly one product.

The products are homogenous.\footnote{Or, if the products are different, it is assumed that the (fixed) quality differences are offset by (fixed) price differences such that the search is about the residual price variation.} The unit production cost to a seller is normalized to zero and the buyer unit value to one. Payoffs are linear in prices: if a mass $B_i$ of buyers trades with a seller for price $p_i$, each buyer gets $1 - p_i$ and the seller $B_i p_i$.

The sellers choose their prices $p_i \in [0, 1]$ or, as equilibria are in randomized pricing strategies here, their price distributions $F^i \in \Delta [0, 1]$. The sellers have also full control
over the "frictions" in their store \( \theta^i \in [0, \infty] \). These frictions determine how the buyers optimally search and how long it takes for a buyer on average to find a product in a given store.

Search is a random gradual process. It goes on in the store where the buyer is at the moment. As the key difference to previous models of search frictions, (i) the buyers search under a deadline and (ii) the prices in the stores, \( p^i \), are not found immediately once a buyer enters a store but after an uncertain wait time which is given by the exponential distribution, \( \text{Exp}(\theta^i) \); the intensity parameter \( \theta^i \) is the seller’s decision variable.

A buyer’s search cost is zero for \( t \leq 1 \) (before the deadline \( d = 1 \)) and infinite for \( t > 1 \) (after the deadline \( d = 1 \)). For every point in time \( t \in [0,1] \), the buyers can thus decide afresh whether to search at seller \( i = 1 \) or at seller \( i = 2 \). At seller one, the price \( p^1 \) is found at Poisson rate \( \theta^1 \) whereas, at seller two, the price \( p^2 \) is found at Poisson rate \( \theta^2 \). No other restrictions apply: the buyers can switch the seller freely on the go.\(^{16,17}\)

The precise timing is as follows:\(^{18}\)

1. **In stage one**, the sellers (simultaneously) fix the rates \( \theta = (\theta^1, \theta^2) \). They become common knowledge immediately thereafter.

2. **In stage two**, the sellers (simultaneously) fix the prices \( p = (p^1, p^2) \). They are the sellers’ private information until discovered.

3. **In stage three**, the buyers search the prices while time runs from \( t = 0 \) to \( t = d \). For any point in time \( t \in [0, d] \), a buyer chooses whether to search at seller one or at seller two, in other words, whether to switch the seller or to continue with the same one. The final purchase decisions will be made at time \( t = d \).

Thus, altogether, we have a three stage extensive game with a dynamic program embedded in the final stage. Or, equivalently for this case, a two stage game where, first, the sellers publicly commit to the frictions and, then, the sellers choose their prices (their randomized pricing strategies) and the buyers select their seller (their sequential search strategies).\(^{19,20}\)

\(^{16}\)As a result, they can also recall the prices they have so far observed without any further cost, as long as they have time.

\(^{17}\)The exponential distribution \( \text{Exp}(\theta) \) is the standard way to model an uncertain wait time. If the wait time is exponential, the number of occurrences taking place during any interval can be modeled by the Poisson process \( P(\theta) \).

\(^{18}\)Note that our model is quite similar to the one we have in Hämäläinen (2015) but now there is one price for one store – not either one, two, or beyond – and here the in-store frictions are a decision variable instead of a parameter.

\(^{19}\)Observe that like in Wilson (2010) it is important for the sellers to commit to the frictions (as said, they represent here a long-term investment in a particular search technology within store). Namely, if it was feasible, a non-prominent seller would prefer to serve immediately all the buyers who visit its store. If the buyers knew this, the seller would no longer be the last in the line. For that case, there might not exist an equilibrium in pure strategies for frictions; a solution might involve mixed frictions.

\(^{20}\)The model is not equivalent with a strategic game in which the sellers choose a distribution of prices
Next, this game is solved by *backwards induction*, starting from the prices and the search and, then, moving on to the first stage in which the sellers determine their frictions.

### 4.3 Equilibria

#### 4.3.1 Buyer’s problem: Search

A buyer’s problem is captured by the following Bellman equation:

\[
V_t := \max_{i=1,2} V_t^i = \max_{i=1,2} \left( \theta^i dt \left( (1 - e^{-\theta^i(1-t)})(1 - E(p|F_{min})) + e^{-\theta^i(1-t)}(1 - E(p|F^i)) \right) + (1 - \theta^i dt) V_{t+dt} \right). \tag{4.1}
\]

Recall the buyers can switch the seller free of cost whenever they want. Thus, when no price is found, a buyer will search at seller \(i\) at moment \(t\) if the continuation value \(V_t^i\) (of being at seller \(i\) at time \(t\)) is at least as large as the continuation value \(V_t^{-i}\) (of being at seller \(-i\) at time \(t\)).\(^{21}\) When a price is found, the buyer will obviously switch the seller because that is the only store where they could still find a lower price. In continuation, for probability \(1 - e^{-\theta^{-i}(1-t)}\), she will find it (and buys for the minimum of \(p^i\) and \(p^{-i}\)) and, for probability \(e^{-\theta^{-i}(1-t)}\), she will fail to find it (and buys for \(p^i\)).

Note that it is without loss of generality to assume that \(\theta^1 \geq \theta^2\). In other words, we can refer to seller one as the faster seller and seller two as the slower seller from here on. The expected price at the faster seller is denoted by \(E(p|F^1)\), the expected price at the slower seller is denoted by \(E(p|F^2)\), and the expected minimum of the two prices by \(E(p|F_{min})\).

Generally, three cases are possible: \(i\) the faster seller could have a lower expected price, \(ii\) both stores could look the same or \(iii\) the faster seller could have a higher expected price. The buyer’s search problem is trivial in the first case and in the second one; the interesting case in the third one where we have a tradeoff between frictions and the price.

To determine how ties in the buyer’s problem are handled, we make the following assumption:

**Assumption 4** The buyers do not switch if they are indifferent between the sellers.

This implies that we only need to focus on where the buyers start their search and

\[\Delta[0,1]\] and a distribution of rates \(\Delta[0,\infty]\) and the buyers choose a search plan. Indeed, the reason why the Bertrand equilibrium is eliminated is that we let the sellers first choose the rates and only then the prices. Note however that, even in this modification with simultaneous moves, the Bertrand outcome is not robust to a slight tremble in a price \(p^i\); that would make \(\theta^{-i} > 0\) and \(p^{-i} > 0\) a profitable deviation.\(^{21}\) Hence, \(V_t^i\) is the continuation value of search at seller \(i\) at moment \(t\) assuming no price is found.
whether they have an incentive switch the seller at some time point \( t \in (0, 1) \) before the deadline, provided they have not found a price yet. Once they discover a price, they of course switch the seller to find the only remaining price quote. Surprisingly, we find that the following holds:  

**Lemma 12** The buyers switch the seller only once a price is found.

(i) If \( \theta^i (1 - E(p|F^i)) > \theta^{-i} (1 - E(p|F^{-i})) \), the buyers start from seller \( i = 1, 2 \); they search there until they find its price.

(ii) If \( \theta^1 (1 - E(p|F^1)) = \theta^2 (1 - E(p|F^2)) \), the buyers could start from either seller; they search there until they find its price.

The buyer continuation value is given by

\[
V_t = \frac{\theta^i}{\theta^i - \theta^{-i}} \left( e^{-\theta^{-i}(1-t)} - e^{-\theta^i(1-t)} \right) (E(p|F_{\min}) - E(p|F^i)) + \left( 1 - e^{-\theta^i(1-t)} \right) (1 - E(p|F_{\min})),
\]

for \( \theta^i \neq \theta^{-i} \), and

\[
V_t = \theta^i e^{-\theta^i(1-t)} (E(p|F_{\min}) - E(p|F^i)) + \left( 1 - e^{-\theta^i(1-t)} \right) (1 - E(p|F_{\min})),
\]

for \( \theta^i = \theta^{-i} \).

*Proof.* See Appendix. ■

The interplay of frictions and search will ultimately partition the set of buyers as

\[
B = B_0 \cup B_1 \cup B_2 \cup B_{1,2},
\]

or, with the usual abuse of notation equating the subsets of \( B \) with their mass, as

\[
1 = B_0 + B_1 + B_2 + B_{1,2},
\]

where the buyers \( B_0 \) fail to find any price, the buyers \( B_i \) ("captive buyers" or "uninformed consumers") find price \( p^i \) from store \( i \), and the buyers \( B_{1,2} \) ("shoppers" or "informed consumers") find both of the two prices.

These sets are easy to determine now. We start by denoting the fraction of buyers who start from seller \( i = 1 \) by \( t^1 \) and the rest who start from seller \( i = 2 \) by \( t^2 = 1 - t^1 \). The two of them are captured by \( t = (t^1, t^2) \).

Thus, the number of buyers who fail to find any price is

\[22\]
\[
B_0 = t^1 e^{-\theta_1} + t^2 e^{-\theta_2},
\]
and the number of trades is equal to
\[
1 - B_0 = 1 - t^1 e^{-\theta_1} - t^2 e^{-\theta_2}.
\]
The numbers of captive buyers to the sellers are
\[
B_1 = t^1 \theta e^{-\theta} \quad \text{and} \quad B_2 = t^2 \theta e^{-\theta}, \quad \text{for } \theta = \theta_1 = \theta_2, \quad (4.2)
\]
and
\[
B_1 = t^1 \int_0^1 e^{-\theta_1 (1 - \tau)} \theta^1 e^{-\theta_1 \tau} \, d\tau = t^1 \frac{\theta^1}{\theta^2 - \theta^1} (e^{-\theta_1} - e^{-\theta_2}),
\]
\[
= \frac{t^1 \theta^1}{\theta^1 - \theta^2} (e^{-\theta_1} - e^{-\theta_2}) = \frac{\theta^1}{\theta^1 - \theta^2} (e^{-\theta_1} - B_0), \quad \text{for } \theta^1 \neq \theta^2 \quad (4.3)
\]
and
\[
B_2 = t^2 \int_0^1 e^{-\theta_2 (1 - \tau)} \theta^2 e^{-\theta_2 \tau} \, d\tau = t^2 \frac{\theta^2}{\theta^1 - \theta^2} (e^{-\theta_2} - e^{-\theta_1}),
\]
\[
= \frac{t^2 \theta^2}{\theta^1 - \theta^2} (e^{-\theta_2} - e^{-\theta_1}) = \frac{\theta^2}{\theta^1 - \theta^2} (B_0 - e^{-\theta_1}), \quad \text{for } \theta^2 \neq \theta^1 \quad (4.4)
\]
where the terms in the integrands are \(e^{-\theta_i \tau}\), the probability that the buyer does not find store \(i\)'s price during \([0, \tau]\), \(\theta^i\), the probability that the buyer succeeds to discover this price exactly at moment \(t = \tau\), and \(e^{-\theta_i (1 - \tau)}\), the probability that the buyer does not find store \(-i\)'s price during \([\tau, 1]\). The shoppers are just the residual
\[
B_{1,2} = 1 - B_0 - B_1 - B_2. \quad (4.5)
\]

These notions will be used repeatedly in the sellers’ problem. It is clear from above that \(\frac{\partial B_1}{\partial t^1} = 0, \frac{\partial B_2}{\partial t^1} < 0, \frac{\partial B_1}{\partial \theta^1} > 0 \) and \(\frac{\partial B_2}{\partial \theta^1} < 0\). Thus, if buyers search more efficiently, the number of shoppers does not change but the number of trades increases and the faster store gets more and the slower store gets less of captive buyers.

To maximize the number of trades, the buyers should search in the faster store at least until one price is found. That is, if store \(i = 1\) is faster than store \(i = 2\), \(\theta^1 > \theta^2\), buyers search efficiently iff they all start from store \(i = 1\), \(t^1 = 1 - t^2 = 1\).
4.3.2 Seller’s problem: Prices

Now, for any partition \( \{ B_0, B_1, B_2, B_{1,2} \} \), the profit \( \Pi^i \) to the seller \( i \) is decomposed as

\[
\Pi^i(p^i) = (B_i + B_{1,2}(1 - F^{-i}(p^i))) \cdot p^i.
\]

Given a way the buyers search (as captured by the partition into \( B_0, B_1, B_2 \) and \( B_{1,2} \)), the demand has thus a price-insensitive part (the 1st term in the brackets on the rhs) and a price-sensitive part (the 2nd term in the brackets on the rhs). The captive buyers \( B_i \), who have only the price quote \( p^i \), buy from seller \( i = 1, 2 \) whatever the price \( p^i \). The shoppers \( B_{1,2} \), who have discovered both price \( p_1 \) and price \( p_2 \), buy from seller \( i = 1 \) iff \( p_2 \) is above \( p_1 \); it takes place for probability \( (1 - F_2(p_1)) \).

Note particularly that, as in Varian (1980); Stahl (1989) and, say, as in Wilson (2010), search affects profit only through the number of captive buyers \( B_i \), where \( i = 1, 2 \), and the number of shoppers \( B_{1,2} \). Moreover, the numbers of captive buyers \( B_i \) and shoppers \( B_{1,2} \) are affected only by the expected prices \( F = (F^1, F^{-i}) \) but not by the realized prices \( p = (p^1, p^{-1}) \); the buyer’s search problem is non-trivial only when no price is found yet. Once that happens, they of course to switch the seller.

Hence, after the rates \( \theta \) have been set, the equilibrium of the subgame that follows is a fixed point between the maximal sequential search strategies \( t \in BR(F) \) for the buyers and the maximal randomized pricing strategies \( F \in BR(t) \) for the stores. This entails that the analysis of equilibrium pricing strategies goes along the same lines as in Varian (1980) and Stahl (1989) and, basically, as in Wilson (2010). The difference is only that now the numbers of captive buyers and shoppers are determined in equilibrium: In the preceding literature, they were given by a parameter.\(^{23}\)

**Lemma 13** Assume \( B_i > 0 \), either for seller \( i = 1 \) or seller \( i = 2 \), and \( B_{1,2} > 0 \). Then, the following hold true in any equilibrium:

1. The sellers use randomized pricing strategies: \( F^1 \) and \( F^2 \).
2. Both \( F^1 \) and \( F^2 \) have the same interval support \( \text{supp}(F) = [p, \bar{p}] \), where \( 0 < p < \bar{p} = 1 \).
3. Neither has an atom at \( p \in [p, 1) \): \( \lim_{x \to p} F^i(x) = F^i(p) \) for all \( p < 1 \) and \( i = 1, 2 \).
4. If \( F^1 \) has an atom at \( p = 1 \), \( F^2 \) has not and, if \( F^2 \) has an atom at \( p = 1 \), \( F^1 \) has not.

**Proof.** See Appendix. \(\square\)

\(^{23}\)The support is defined as \( \text{supp}(F) = \text{cl} \{ x | f(x) > 0 \} \), where \( \text{cl} \) denotes a closure of a set and \( f \) is a point probability or a density function.
Observe also that, if \( B_{1,2} = 0 \) (no shoppers; would arise under \( \theta = (0, 0), \theta = (a, 0) \) and \( \theta = (0, a) \) for \( a \geq 0 \)), the sellers use a pure strategy \( p^i = 1 \) (\( p^i : MR(p) = MC(p) \)), this is basically the Diamond outcome) or, if \( B_{1,2} > 0 \) but \( B_1 = B_2 = 0 \) (no captive buyers; would arise under \( \theta = (\infty, \infty), \theta = (a, 0) \)), the sellers use a pure strategy \( p^i = 0 \) (\( p = MC \), this is basically the Bertrand outcome). That is, our model nests the Diamond outcome and the Bertrand outcome as special cases for appropriate \( \theta \).

**Lemma 14** Consider \( \theta = (\theta^1, \theta^2) \) and \( t = (t^1, t^2) \) such that \( B^1 \geq B^2 \) and \( B_{1,2} > 0 \). Then, there exists a unique equilibrium price distribution \( F = (F^1, F^2) \) where

\[
F^1(p) = \frac{B_2 + B_{1,2}}{B_{1,2}} - \frac{\Pi^2}{B_{1,2}p} \text{ for all } p \in [p, 1],
\]

with an atom \( \alpha := \frac{B_1 - B_{1,2}}{B_1 + B_{1,2}} \leq p \) at the highest price \( p = 1 \), and

\[
F^2(p) = \frac{B_1 + B_{1,2}}{B_{1,2}} - \frac{\Pi^1}{B_{1,2}p} \text{ for all } p \in [p, 1].
\]

The lowest price is given by \( p = \frac{B_1}{B_1 + B_{1,2}} \) and the sellers’ profits by

\[
\Pi^1 = B_1 \text{ and } \Pi^2 = pB_2 + (1 - p)B_1 \leq B_1.
\]

**Proof.** See Appendix. ■

The store with more captive buyers has higher profit and prices. It mixes between using random discount prices \( p^1 < 1 \), to compete with the other store over the shoppers, and the monopoly price \( p^1 = 1 \), to tax its numerous captive buyers. The other store, who has fewer captive buyers, is randomizing only the size of the discount, \( p^2 < 1 \).

In other words, the stores’ equilibrium pricing strategies are wired so as to let them specialize in different groups of buyers. This aligns the sellers’ payoffs and helps to relax the price competition. The profit to the high-profit seller, \( \Pi^1 \), equals the number of captive buyers it attracts, \( B_1 \), whereas the profit to the low-profit seller, \( \Pi^2 \), is a weighted average of its own captive buyers, \( B_2 \), and the other store’s captive buyers, \( B_1 \).

The weights, \( p = \frac{B_1}{B_1 + B_{1,2}} \) and \( 1 - p = \frac{B_{1,2}}{B_1 + B_{1,2}} \), could be taken as a measure of how close the market is to the Bertrand outcome (arising for \( B_{1,2} > 0, B_1 = B_2 = 0 \)) and to the Diamond outcome (arising for \( B_{1,2} = 0, B_1 > 0, B_2 \geq 0 \)) – or the competitiveness and the relative standing of the sellers and the buyers in the market.

This entails that, if the sellers’ have high ”bargaining power”, as captured by a high \( p \), the sellers’ have less aligned preferences (they compete more fiercely) but, if the sellers’ have low ”bargaining power”, a low \( p \), they have more aligned preferences (they compete less fiercely). As it later turns out, the outcome that obtains can therefore be regarded as a compromise of some sort between the two stores and the buyers.\(^{24}\)

\(^{24}\)In particular, we show that in equilibrium \( p = 1/2 \).
It is now straightforward to calculate the expected prices that we need:

\[ E(p|F^1) = \int_0^1 pf^1(p) \, dp + \alpha = \frac{\Pi^2 \ln \left( \frac{1}{\frac{p}{B}} \right) + \alpha}{B_{1,2}} \]

\[ \geq E(p|F^2) = \int_0^1 pf^2(p) \, dp = \frac{\Pi^1 \ln \left( \frac{1}{\frac{p}{B}} \right)}{B_{1,2}}. \]

The firm who is capable of attracting more captive buyers extracts a higher profit and has an incentive to set higher prices in expectation. While it tends to offer a lower discount price, when it does so, \( \Pi_2 B_{1,2} \ln \left( \frac{1}{p} \right) \leq \Pi_1 B_{1,2} \ln \left( \frac{1}{p} \right) \), it also uses the monopoly price more often, \( \alpha \geq 0 \). It is next an easy three line homework to show that the latter effect offsets the former. To calculate the expected minimum of the two prices \( E(p|F_{\text{min}}) \), note that the distribution function \( F_{\text{min}} \) is given by

\[ 1 - F_{\text{min}} = (1 - F^1)(1 - F^2) \]

and the density function by

\[ f_{\text{min}} = f^2(1 - F^1) + f^1(1 - F^2). \]

A direct calculation results in\(^{25}\)

\[ E(p|F_{\text{min}}) = \int_0^1 p \left( f^2(p)(1 - F^1(p)) + f^1(p)(1 - F^2(p)) \right) \, dp \]

\[ = \frac{B_1 \Pi^1 + B_2 \Pi^2}{B_{1,2}^2} \ln \left( \frac{1}{\frac{p}{B}} \right) + \frac{\Pi^1 \Pi^2}{B_{1,2}^2} \left( \frac{1 - \frac{p}{B}}{\frac{p}{B}} \right). \]

It is also noteworthy that, by Equations 4.2, 4.3, 4.4 and 4.5, \( B_1, B_2 \) and \( B_{1,2} \) are determined by \( \theta \) and \( t \) uniquely whereas, by Lemma 14, \( F \) is dependent on \( \theta \) and \( t \) only through \( B_1, B_2 \) and \( B_{1,2} \). This allows us to construct a hypothetical price distribution \( F(\theta, t) \) for any \( (\theta, t) \) by first calculating the associated \( B_1(\theta, t), B_2(\theta, t) \) and \( B_{1,2}(\theta, t) \) and then the induced \( F(B_1, B_2, B_{1,2}) \). We next use this property to characterize the fixed point between optimal search and optimal prices.

**Proposition 17** For any \( \theta \), there exists a unique fixed point in search and prices \((t, F)\) where \( F = F(\theta, t) \) and \( t = t(\theta, F) \).

In particular,

1. if \( \theta^1 \left( 1 - E(p|F^1(\theta, (1,0))) \right) \geq \theta^2 \left( 1 - E(p|F^2(\theta, (1,0))) \right) \), then \( t^1 = 1 - t^2 = 1 \),

\(^{25}\)Note that a linear approximation of \( \ln(p) \) around \( p = 1 \) yields \( \frac{1-p}{p} \geq \ln \left( \frac{1}{p} \right) \). This implies that \( \alpha \geq \frac{\Pi^1 - \Pi^2}{B_{1,2}^2} \ln \left( \frac{1}{\frac{p}{B}} \right) \iff E(p|F^1) \geq E(p|F^2). \)
$B_1 > B_2 = 0$ and $E(p|F^1) > E(p|F^2)$, and

2. if $\theta^1 \left(1 - E(p|F^1(\theta, (1, 0))) \right) < \theta^2 \left(1 - E(p|F^2(\theta, (1, 0))) \right)$, then $t^1 = 1 - t^2 < 1$, $B_1 \geq B_2 > 0$.

In this latter case, $t = t(\theta, F)$ is the unique solution to

$$\frac{\theta^2}{\theta^1} = \frac{1 - E(p|F^1(\theta, t))}{1 - E(p|F^2(\theta, t))} = 1 - \alpha(\theta, t).$$

**Proof.** See Appendix. □

**Corollary 5** (Effects of frictions on search efficiency) The buyers search efficiently if the sellers are either distinctly different in terms of their frictions, $\theta^1 \left(1 - E(p|F^1(\theta, (1, 0))) \right) \geq \theta^2 \left(1 - E(p|F^2(\theta, (1, 0))) \right)$, or exactly similar, $\theta^1 = \theta^2$.

**Corollary 6** (Effects of frictions on seller prominence) Lower frictions grant a seller more prominent position in search and thus higher prices and profit: if $\theta^i \geq \theta^{-i}$, then $B_i \geq B_{-i}$ implying $\Pi^i \geq \Pi^{-i}$ and $E(p|F^i) \geq E(p|F^{-i})$.

In other words, candidate equilibria are of two kinds: If the sellers are equally fast, the equilibrium is symmetric but, if seller one is faster than seller two, the equilibrium is asymmetric.

### 4.3.3 Seller’s problem: Frictions

This section carries our main results. We first prove that there exists no equilibrium in symmetric pure strategies for in-store frictions. This also rules out the Diamond equilibrium and the Bertrand equilibrium and makes it possible to divide our analysis into the prominent seller’s problem and the non-prominent seller’s problem. Moreover, we find that in an equilibrium, frictions are significantly lower at the former seller than at the latter seller. This entails that buyers always search efficiently, from the prominent seller to the non-prominent seller. Despite this finding, we show that (i) any equilibrium features inefficient frictions (this comes from the prominent seller’s problem) and (ii) the featured frictions are such that the shares of informed consumers and uninformed consumers are precisely the same in the market (this comes from the non-prominent seller’s problem). Finally, we show that there exist two pure equilibria, that we then describe.

We begin with a strong result:

**Lemma 15** There exist no equilibrium in pure strategies for frictions, $\theta^2 \leq \theta^1 < \infty$, where the buyers are indifferent between the sellers, $t^1 = 1 - t^2 < 1$.

In other words, there is a clear prominence order between the sellers:
Corollary 7 In an equilibrium, the stores differ so much in frictions, $\theta^2 \ll \theta^1 < \infty$, that all buyers start from the faster (prominent) seller, $t^1 = 1 - t^2 = 1$.

Corollary 8 There exist no equilibrium in symmetric pure strategies for frictions $\theta \in (0, \infty)^2$.

We can therefore also rule out the Bertrand equilibrium and the Diamond equilibrium.

Lemma 16 There exist no Bertrand equilibrium, where either of the two sellers generates no frictions and the market price equals zero.

Proof. The Bertrand equilibrium requires that both seller choose zero frictions $\theta = (\infty, \infty)$. But now both sellers gain if one deviates to some finite rate $\theta$ because it raises their profit up from zero to $\frac{B_i B_{1,2}}{2B_i + B_{1,2}} = (1 - e^{-\theta}) e^{-\theta}$ (to the deviator, who has $t^{-1} = 0$ for its markedly higher frictions $\theta^{-1} < \infty$) and $B_i = 1 - e^{-\theta}$ (to the non-deviator, who has $t^i = 1$ for its markedly lower frictions $\theta^i = \infty$).

Lemma 17 There exist no Diamond equilibrium, where at least one seller generates infinite frictions and the market price equals one.

Proof. Since the buyers always search, the Diamond equilibrium requires that at least one seller is practically out of the market due to its infinite frictions, $\theta = (\theta^i, 0)$, $(0, \theta^{-i})$. As the seller serves no buyers, its profit equals zero. Yet, for any lower level of frictions, the seller’s profit is positive, $\Pi^i = B_i > 0$ or $\Pi^{-i} = pB_{i^{-1}} + (1 - p) B_i > 0$. There is hence a profitable deviation to a lower $\theta’ < \infty$.

Since we are interested primarily in situations in which frictions represent the sellers’ public long-term choices, we focus here on pure equilibria where the sellers use a fixed level of frictions in stead of mixed equilibria where the sellers randomize between different levels. In an equilibrium in pure strategies for frictions, we just proved that we must have a prominent seller and a non prominent seller in the market. Next, we analyze their problems one by one.

The frictions $\theta^1 \in [0, \infty]$ chosen by the prominent (non prominent) seller have to be the best response to the frictions chosen by the non prominent (prominent) seller $\theta^2 \in [0, \infty]$.

Additionally, we need to make sure the prominent seller wants to be prominent and the non-prominent wishes to remain non-prominent. This will be checked after getting the best responses conditional on the assumption that one seller is prominent and the other one is non prominent.

Prominent seller’s problem

To analyze the tradeoffs that the prominent seller is facing, we derive in this subsection seller $i = 1$’s conditional best response to frictions $\theta^2$ assuming that seller $i = 1$ is so much

\[26\text{ The proof of Lemma 15 covers also this case but, as this is much shorter, we display it also.}\]
faster than seller $i = 2$, $\theta^1 \gg \theta^2$, that $t^1 = 1 - t^2 = 1$. To facilitate the exposition, we find it useful to introduce the following reparametrizations: $\rho = \theta^2 / \theta^1 \leq 1$ and $\delta = \theta^1 - \theta^2 \geq 0$.

Thus, the prominent seller maximizes its profit $B_1$

$$\max_{\theta^1} \theta^1 e^{-\theta^1} e^{\delta} - 1$$

such that the prominence order stays the same:

$$\rho \leq 1 - \alpha(\theta^1, \theta^2).$$

We show in the Appendix that whether the constraint is slack or binds, the prominent seller’s problem has a unique solution $\theta^1 < \infty$ which satisfies the following complementary slackness constraints (one binds and the other one is slack)

$$\frac{e^{\delta} - 1}{\delta} \geq \rho^{-1} \geq \frac{1}{1 - \alpha(\theta^1, \theta^2)}.$$

Moreover, if the latter constraint is slack, the solution $\theta^1(\theta^2)$ is decreasing in $\theta^2$ whereas, if the latter constraint binds, the solution $\theta^1(\theta^2)$ is increasing in $\theta^2$. The finiteness of $\theta^1(\theta^2)$ has the following noteworthy implication:

**Proposition 18** There exist no efficient equilibria.

*Proof.* Appendix. ■

This result arises because the prominent seller is facing a tradeoff between lowering the frictions to increase “inflow” to the store (the number of buyers who have found its own price $p^1$) and raising the frictions to decrease the ”outflow” form the store (the number of buyers who have found both prices $p^1$ and $p^2$). Stronger inflow is beneficial, stronger outflow is detrimental. Since the store has just one instrument to affect this turnover rate, it is best off with moderate frictions. It has no incentive to get rid of them altogether. In consequence, some of the buyers necessarily fail to find a price before their deadline; all the gains from trade are not commensurated.

It is noteworthy that the prominent seller’s profit is a product of $\theta^1 e^{-\theta^1}$ and $\frac{e^{\delta} - 1}{\delta}$. The first one, $\theta^1 e^{-\theta^1} \geq 0$, is the number of buyers who would find just one price quote if the frictions were the same in both stores, $\theta^1 = \theta^2$. The second one, $\frac{e^{\delta} - 1}{\delta} \geq 1$, represents the additional frictions in discovering the second price quote, which arise from the fact that the prominent store has lower frictions than the non prominent store, $\theta^1 > \theta^2$. The former factor is maximized by $\theta^1 = 1$, minimized by $\theta^1 = 0$ and approaches its minimum for $\theta^1 \to \infty$ whereas the latter one is the larger for a larger difference $\theta^1 - \theta^2$. The seller’s profit is thus maximized by some $\theta^1(\theta^2) \in (1, \infty)$.

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27See Appendix for the proof.
Non-prominent seller’s problem

To analyze now the tradeoffs that the non prominent seller is facing, we derive in this subsection seller $i = 2$’s conditional best response to frictions $\theta^1$, assuming that seller $i = 1$ is so much faster than seller $i = 2$, $\theta^1 \gg \theta^2$, that $t^1 = 1 - t^2 = 1$.

The non prominent seller maximizes its profit $(1 - p)B_1$

$$\max_{\theta^2} \Pi^2 \frac{B_1B_{1,2}}{B_1 + B_{1,2}} = \max_{\theta^2} \Pi^2 \frac{B_1B_{1,2}}{1 - B_0}$$

such that the prominence order stays the same:

$$\rho \leq 1 - \alpha(\theta^1, \theta^2).$$

Note that the non prominent seller’s profit is of the following very simple form

$$\frac{a(\theta)b(\theta)}{a(\theta) + b(\theta)}$$

where $a(\theta)$ and $b(\theta)$ are non negative constants. The maximum of this expression is reached by choosing $\theta$ with the largest feasible $a(\theta)$ and $b(\theta)$ such that $a(\theta) = b(\theta)$.

For our particular case, where $a(\theta) = B_1$ and $b(\theta) = B_{1,2}$, the non prominent seller’s problem has a unique solution $\theta^2 > 0$ which satisfies the following complementary slackness constraints (one binds and the other one is slack)

$$B_1 \geq B_{1,2} \text{ and } \rho \leq 1 - \alpha(\theta^1, \theta^2)$$

This is so because $B_1 > 0$ and $B_{1,2} = 0$ for $0 = \theta^2 < \theta^1$ and because a reduction in frictions at the non-prominent seller decreases the number of uninformed buyers $B_1$ and decreases the number of informed buyers $1 - B_1 - B_0$

$$\frac{\partial B_1}{\partial \theta^2} < 0, \frac{\partial B_0}{\partial \theta^2} = 0, \frac{\partial B_{1,2}}{\partial \theta^2} > 0.$$

Thus, any equilibrium is characterized by the following property:

**Proposition 19** If $\rho < 1 - \alpha(\theta^1, \theta^2)$, the number of informed buyers, $B_{1,2}$, is equal to the number of uninformed buyers, $B_1$.

---

28 Consider function $f : f(a, b) = \frac{ab}{a + b}$, and maximize it subject to the constraint that $a(\theta) = B_1(\theta)$ and $b(\theta) = B_{1,2}(\theta)$ and $B_0(\theta) + B_1(\theta) + B_{1,2}(\theta) = 1$ with $\theta^2$ as the choice variable and $\theta^1$ as a fixed parameter. The first order conditions are $\frac{\partial f}{\partial \theta^2} = \frac{a^2}{(a + b)^2}$ and $a + b = 1 - B_0$. Solving these gives as a solution $a = b = (1 - B_0)/2$. The Hessian is negative semi-definite

$$H(f) = \left[ \frac{1}{(a + b)^2} \right] ^T \begin{bmatrix} -2a^2 & 2ab \\ 2ab & -2a^2 \end{bmatrix}.$$
If \( \rho = 1 - \alpha(\theta^1, \theta^2) \), the number of informed buyers, \( B_{1,2} \), is weakly smaller than the number of uninformed buyers, \( B_1 \).

A sketch of a proof: Above.

The non prominent seller has mixed incentives in choosing the frictions \( \theta^2 \): in prefers to increase both the number of informed consumers \( B_{1,2} \) and that of uninformed consumers \( B_1 \). If it reduces the frictions by elevating \( \theta^2 \), the first one increases but the second one goes down.

To balance these effects, the non prominent seller has thus an incentive to make sure the outcome is exactly in between the Bertrand equilibrium and the Diamond equilibrium as measured by the relative numbers of informed consumers \( p = \frac{B_1}{B_1 + B_{1,2}} \) and uninformed consumers \( 1 - p = \frac{B_{1,2}}{B_1 + B_{1,2}} \).

As with the prominent seller’s problem, we again find that, if the constraint \( \rho \leq 1 - \alpha(\theta^1, \theta^2) \) is slack, the solution to the non prominent seller’s problem \( \theta^2(\theta^1) \) is decreasing in \( \theta^1 \) whereas, if the constraint \( \rho \leq 1 - \alpha(\theta^1, \theta^2) \) binds, the solution \( \theta^2(\theta^1) \) is increasing in \( \theta^1 \).

Claim 2 The frictions are determined as a fixed point of the sellers’ best response mappings \( BR_i(\theta^{-i}) := \sup_{\theta^i \in [0, \infty]} \Pi^i(\theta^i, \theta^{-i}) \) for which the following hold:

1. There exist a unique cutoff for the frictions, \( \theta^* \approx 2.33 \) such that: if the other seller is faster, \( \theta^{-i} < \theta^* \), seller i’s best response is to become the prominent seller, i.e., \( BR_i(\theta^{-i}) > \theta^{-i} \), and, if the other seller is slower, \( \theta^{-i} > \theta^* \), seller i’s best response is to become the non prominent seller, i.e., \( BR_i(\theta^{-i}) < \theta^{-i} \).

2. The sellers’ best responses are single valued a.e., discontinuous only at the unique cutoff \( \theta^* \), continuous decreasing when the constraint \( \rho \leq 1 - \alpha(\theta^1, \theta^2) \) is slack and continuous increasing when the constraint \( \rho \leq 1 - \alpha(\theta^1, \theta^2) \) binds. They are also convex, at least of the constraint \( \rho \leq 1 - \alpha(\theta^1, \theta^2) \) is slack.

Proof. As for now, we rely on the numerical results presented in Figure 4.1 and the knowledge of the best responses as derived in Subsections 4.3.3 and 4.3.3. We are confident that it is possible to get also a full analytical proof.

Next, we present an important existence result and lay out the key properties of the equilibrium.

\[ \text{See Appendix for the proof.} \]
Proposition 20  There exist two equilibria in pure strategies for frictions, with the same unique form: \( \theta^* \approx (1.03, 2.76) \) and \( \theta^* \approx (2.76, 1.03) \).

Proof. It is easy to ascertain that conditions (4.6) and (4.7) are satisfied and the constraint \( \rho \leq 1 - \alpha(\theta^1, \theta^2) \) is slack for some \( \theta^* \) in the neighborhood of \( (2.76, 1.03) \). Otherwise, we rely on Figure 4.1 and Claim 2. ■

Corollary 9 Both equilibria have the same unique form:

1. Frictions: there is a prominent seller with frictions \( \theta^i = 2.76 \) and a non-prominent seller with frictions \( \theta^{-i} = 1.03 \). The expected wait time at the former is about 36\% of the total time and the expected wait time at the latter is about 97\% of the total time.

2. Search: The buyers search in the prominent seller until they find a price quote, \( t^i = 1 \) and \( t^{-i} = 0 \). 47 per cent of the buyers find a price from both the prominent and the non-prominent seller, \( B_{1,2} \approx 0.47 \), and 47 per cent of the buyers find a price from the prominent seller only, \( B_i \approx 0.47 \). 6 per cent of the buyers fail to find a price, \( B_\emptyset \approx 0.06 \).

3. Prices: The prominent seller offers the monopoly price \( (p = 1) \) and a discount price \( (p < 1) \) equally often, \( \alpha = 0.5 \); the non prominent seller always offers a discount price. Given that a seller offers a discount, the expected discount is 31 per cent of the monopoly price at either seller; the largest such regularly used discount is 50 per cent, \( p = 0.5 \).

4. Surplus sharing: The prominent seller is making the double of what the non-prominent seller is making, \( \Pi^i = B_i \approx 0.47, \Pi^{-i} = \alpha B_{1,2} \approx 0.5 \cdot 0.47 \). The prominent seller
gets half the surplus, the non-prominent seller gets a quarter and the buyers get a quarter; 6 per cent of the cake is wasted.

Proof. An elementary calculation that uses the fact that \( \theta \approx (2.76, 1.03) \) and the expressions that we have provided above for \( B_i(\theta), B_{1,2}(\theta), B_{\emptyset}(\theta), \) and \( E(p|F) \).

This pattern of frictions is the unique one even if we increase or decrease the deadline. In other words, the outcome is just the same in terms of prices and search if the buyers can search for a decade or a minute. The sellers have an incentive to adjust the frictions such that the number of trades and the informed consumers and the uninformed consumers is constant. However, if there were no deadline, the Bertrand equilibrium would obtain and, if the buyers had no time whatsoever, the Diamond equilibrium could obtain.

Remark 9 An identical equilibrium outcome arises whatever the deadline \( d < \infty \) is as long as it is finite: if \( (\theta^i, \theta^{-i}) \) is an equilibrium when the search horizon is \( t \in [0, 1] \), then \( (\frac{\theta^i}{d}, \frac{\theta^{-i}}{d}) \) is an equilibrium when the search horizon is \( t \in [0, d] \), and the other way.

Remark 10 There is a discontinuity in the equilibrium set as \( d \to \infty \) because, at \( d = \infty \), the Bertrand equilibrium with \( p \equiv 0 \) is the unique equilibrium.

Remark 11 There could be a discontinuity in the equilibrium set as \( d \to 0 \) because, at \( d = 0 \), the Diamond equilibrium with \( p \equiv 1 \) is another equilibrium.

Thus, the set of equilibria is invariant to finite translations in the deadline, which is the only exogenous parameter in our model. The Bertrand equilibrium is possible only if the buyers are extremely patient and the Diamond equilibrium if the buyers are extremely impatient. Otherwise, the outcome is precisely in between these extremes in the sense that there are exactly as many informed consumers as there are uninformed consumers.

4.4 Closing remarks

We introduce a novel model of price search that features endogenous frictions in-store, modeled by the gradual arrival of price information within stores and deadlines. Assuming that frictions represent a seller’s long-term investment in a particular search technology, we find that there exists a unique inefficient equilibrium pattern. There is a prominent seller, a non-prominent seller, and exactly equally many informed and uninformed consumers in the market. The surplus loss amounts to 6 per cent of the cake, approximately.

A similar result arises as long as there is a deadline by which a buyer must stop. It could be two seconds or two decades; that does not matter. It is because of this deadline that the sellers gain if they slow down the searching consumers a bit – yet, not in extreme amounts: If the frictions are very high, the buyers fail to find anything but, if the frictions are very low, the buyers become perfectly informed, which drives the stores into a price war. Interestingly, as the deadline vanishes, the Bertrand equilibrium reappears.
Our model is quite flexible and appears well suited to many setups where consumers are doing their shopping in an exploring, relaxed fashion but constrained by some sort of a schedule. This is pertinent to online search: most people seem to enjoy it for the first bit but, unequivocally, not for ever. In Hämäläinen (2015), we develop another variant of this same model to analyze the retailers’ incentives to expand the number of items they have in stock.

Appendix

Proof of Lemma 12. Note that, as the buyers can switch the seller freely any moment, their continuation value conditional on not having found a price is the same whether the buyer is currently at seller $i=1$ or at seller $i=2$. In other words, $V_{t+dt}$ in equation (4.1) is independent of $i=1,2$. This implies that, to maximize the buyer value, $V_t$, the buyer should search in the store who is offering the largest marginal descent in buyer value, $\dot{V}_t$:

$$\text{argmax}_i V^i_t = \text{argmin}_i \dot{V}^i_t.$$  

Now, provided the buyer stays in store $i$ during the next short time interval $[t,t+dt]$, based on (4.1) the change in the buyer value can be written as follows:

$$\frac{V_{t+dt} - V^i_t}{dt} = -\theta^i \left( e^{-\theta^i(1-t-dt)}(1 - E(p|F^i) - V_{t+dt}) + \left(1 - e^{-\theta^i(1-t-dt)}(1 - E(p|F_{min}) - V_{t+dt})\right) \right)$$

$$\rightarrow \dot{V}^i_t = -\theta^i \left( e^{-\theta^i(1-t)}(E(p|F_{min}) - E(p|F^i)) + (1 - E(p|F_{min}) - V_t) \right).$$

Obviously, the buyer value is positive, $V^i_t \geq 0$, and the change in buyer value is negative, $\dot{V}^i_t \leq 0$, for any $t$ and $i$. Otherwise, it would pay off to stay idle. Altogether, this entails that, for any point in time $t \in [0,1]$, a buyer who has not yet discovered a price chooses store $i=1$ over store $i=2$ iff

$$\theta^1 e^{-\theta^2(1-t)}(CS^1 - V_t) + \theta^1(1 - e^{-\theta^2(1-t)})(CS_{min} - V_t) \geq \theta^2 e^{-\theta^1(1-t)}(CS^2 - V_t) + \theta^2(1 - e^{-\theta^2(1-t)})(CS_{min} - V_t),$$

or, iff

$$\theta^1 e^{-\theta^2(1-t)}(CS^1 - V_t) + \theta^1(1 - e^{-\theta^2(1-t)})(CS_{min} - V_t) \geq \theta^2 e^{-\theta^1(1-t)}(CS^2 - V_t) + \theta^2(1 - e^{-\theta^2(1-t)})(CS_{min} - V_t),$$

(4.8)

---

Observe that this time derivative is well defined as long as the buyer does not change the store at $t$. Furthermore, even if the buyer does switch the store at $t$, as long as the buyer does not change the stores infinitely often, we can still use these same expressions which then refer to the right derivative. It is the right derivative that matters for buyers’ search incentives.
\[\theta e^{-\theta (1-t)}(1 - E(p|F^1) - V_i) - \theta e^{-\theta (1-t)}(1 - E(p|F^2) - V_i) \geq 0\]

where

\[CS^1 := 1 - E(p|F^1), CS^2 := 1 - E(p|F^2), CS_{\min} := 1 - E(p|F_{\min}).\]

To see which store the buyers actually prefer, we next analyze three cases, from simpler to more complex:

**Case 1.** Suppose the (faster) seller \(i = 1\) has lower prices than the (slower) seller \(i = 2\): \(\theta^1 \geq \theta^2\) and \(E(p|F^2) \geq E(p|F^1) \geq E(p|F_{\min})\) implying \(CS_{\min} \geq CS^1 \geq CS^2\). Then, by reference to condition (4.8), the buyer prefers seller one to seller two as

\[\theta^1 e^{-\theta^1 (1-t)}(1 - E(p|F^1) - V_i) - \theta^2 e^{-\theta^2 (1-t)}(1 - E(p|F^2) - V_i) \geq 0\]

and

\[\left(\theta^1 (1 - e^{-\theta^1 (1-t)}) - \theta^2 (1 - e^{-\theta^2 (1-t)})\right) (1 - E(p|F_{\min}) - V_i) \geq 0.\]

The latter one is always satisfied because the function

\[f : f(\theta) = \frac{\theta}{1 - e^{-\theta (1-t)}}\]

is increasing in \(\theta \in [0, \infty)\) for any \(t \in [0, 1)\).

**Case 2.** Suppose the sellers are equally fast but seller \(i = 1\) has lower prices than seller \(i = 2\), \(\theta^1 = \theta^2\) and \(E(p|F^2) \geq E(p|F^1) \geq E(p|F_{\min})\) implying \(CS_{\min} \geq CS^2 = CS^1\). Again, by reference to condition (4.8), the buyers prefer seller \(i = 1\) over seller \(i = 2\) as

\[\theta e^{-\theta (1-t)}(1 - E(p|F^1) - V_i) - \theta e^{-\theta (1-t)}(1 - E(p|F^2) - V_i) \geq 0\]

and

\[\left(\theta (1 - e^{-\theta (1-t)}) - \theta (1 - e^{-\theta (1-t)})\right) (1 - E(p|F_{\min}) - V_i) = 0.\]

**Case 3.** Suppose the (faster) seller \(i = 1\) has higher prices than the (slower) seller \(i = 2\): \(\theta^1 \geq \theta^2\) and \(E(p|F^1) \geq E(p|F^2) \geq E(p|F_{\min})\) implying \(CS_{\min} \geq CS^2 \geq CS^1\).

This last case is next handled by showing that, if a buyer prefers one store over the
other at a given point in time, $t'$, this is her preference order also later, for any $t > t'$.

To proceed, suppose that the buyer prefers store one to store two at moment $t$:

$$
\theta_1 e^{-\theta_2 (1-t)} (CS^1 - CS_{min}) + \theta_1 (CS_{min} - V_t) \\
- \theta_2 e^{-\theta_1 (1-t)} (CS^2 - CS_{min}) - \theta_2 (CS_{min} - V_t) \geq 0
$$

and

$$
\dot{V}_t = -\theta_1 e^{-\theta_2 (1-t)} (CS^1 - CS_{min}) - \theta_1 (CS_{min} - V_t).
$$

Now, to see whether the buyer's preference for store $i = 1$ over store $i = 2$ becomes stronger or weaker over time, we differentiate (4.9) with respect to time to obtain

$$
\theta_1 \theta_2 e^{-\theta_2 (1-t)} (CS^1 - CS_{min}) - \theta_1 \dot{V}_t \\
- \theta_1 \theta_2 e^{-\theta_1 (1-t)} (CS^2 - CS_{min}) + \theta_2 \dot{V}_t \\
= \theta_1 \theta_2 e^{-\theta_2 (1-t)} (CS^1 - CS_{min}) - \theta_1 \theta_2 e^{-\theta_1 (1-t)} (CS^2 - CS_{min}) \\
+ (\theta_1 - \theta_2) \left( \theta_1 e^{-\theta_2 (1-t)} (CS^1 - CS_{min}) + \theta_1 (CS_{min} - V_t) \right) \\
+ \theta_1 \theta_2 (CS_{min} - V_t) - \theta_1 \theta_2 (CS_{min} - V_t) \\
= \theta_1 \left( \theta_1 e^{-\theta_2 (1-t)} (CS^1 - CS_{min}) + \theta_1 (CS_{min} - V_t) \right) \\
- \theta_1 \left( \theta_2 e^{-\theta_1 (1-t)} (CS^2 - CS_{min}) + \theta_2 (CS_{min} - V_t) \right) \geq 0
$$

This implies that, if the buyer is in store $i = 1$, then the buyer also stays in store $i = 1$. A similar calculation demonstrates that, if the buyer is in store $i = 2$, then the buyer also stays in store $i = 2$. For this case, suppose that the buyer prefers store two to store one at moment $t$:

$$
\theta_1 e^{-\theta_2 (1-t)} (CS^1 - CS_{min}) + \theta_1 (CS_{min} - V_t) \\
- \theta_2 e^{-\theta_1 (1-t)} (CS^2 - CS_{min}) - \theta_2 (CS_{min} - V_t) \leq 0
$$

and

$$
\dot{V}_t = -\theta_2 e^{-\theta_1 (1-t)} (CS^2 - CS_{min}) - \theta_2 (CS_{min} - V_t).
$$

Again, to see whether the buyer’s preference for store $i = 1$ over store $i = 2$ becomes stronger or weaker over time, we differentiate (4.9) with respect to time to obtain

---

31 One could say that the stores are *absorbing.*
\[
\begin{align*}
\theta_1\theta_2 e^{-\theta_2 (1-t)} (C_1 - C_{\text{min}}) & - \theta_1 \dot{V}_i \\
- \theta_1 \theta_2 e^{-\theta_1 (1-t)} (C_2 - C_{\text{min}}) + \theta_2 \dot{V}_i \\
= \theta_1 \dot{V}_i - \theta_2 \dot{V}_i \\
&= \theta_1 \theta_2 e^{-\theta_1 (1-t)} (C_1 - C_{\text{min}}) - \theta_1 \theta_2 e^{-\theta_1 (1-t)} (C_2 - C_{\text{min}}) \\
&+ (\theta_1 - \theta_2) \left( \theta_2 e^{-\theta_1 (1-t)} (C_2 - C_{\text{min}}) + \theta_2 (C_{\text{min}} - V_i) \right) \\
&+ \theta_1 \theta_2 (C_{\text{min}} - V_i) - \theta_1 \theta_2 (C_{\text{min}} - V_i) \\
&= \theta_2 \left( \theta_1 e^{-\theta_2 (1-t)} (C_1 - C_{\text{min}}) + \theta_1 (C_{\text{min}} - V_i) \right) \\
&- \theta_2 \left( \theta_2 e^{-\theta_1 (1-t)} (C_2 - C_{\text{min}}) + \theta_2 (C_{\text{min}} - V_i) \right) \leq 0
\end{align*}
\]

In other words, if the buyer is in store \( i = 2 \), then the buyer also stays in store \( i = 2 \). Note also that the derivative \( \dot{V}_i \) is well defined in both cases since the buyer has no incentive to switch the seller: by continuity of (4.1), there exist no kink in \( V_i \) unless the buyer changes the store.

Altogether, this implies that the buyers have no incentive to switch the store before they find a price. They start their search from the store which they would choose at the very last moment, had they not found a price by that time. They continue with that store until they have found its price.

To identify this store where the buyers first search, note that, at the deadline \( t = 1 \), buyers prefer store \( i = 1 \) over store \( i = 2 \) iff the following condition holds

\[
\theta_1 (1 - E(p|F^1)) \geq \theta_2 (1 - E(p|F^2)).
\]

Observe that this condition is satisfied automatically for Case 1 and Case 2 in which the buyers always prefer store \( i = 1 \) over store \( i = 2 \). It thus covers them all.

We next need to solve explicitly for the buyer value. We start by assuming that buyers prefer store \( i \) over store \(-i\). Note first that

\[
\dot{V}_i = -\theta_i \left( e^{-\theta_i (1-t)} (C_1 - C_{\text{min}}) + C_{\text{min}} - V_i \right)
\]

defines a linear first order differential equation

\[
\dot{V}_i - \theta_i V_i = -\theta_i \left( e^{-\theta_i (1-t)} (C_1 - C_{\text{min}}) + C_{\text{min}} \right)
\]

A solution to the related homogenous equation is

\[
V_i = ce^{\theta_i t},
\]

where \( c \) is a constant. To solve the non-homogenous equation, we can use the variation of
the constants in which we let the constants \(c(t)\) be dependent on time such that

\[
V_t = c(t)e^{\theta t}, \quad \dot{V}_t = c(t)\theta e^{\theta t} + c'(t)e^{\theta t}.
\]

This implies that

\[
\dot{V}_t + \theta V_t = c'(t)e^{\theta t} = -\theta \left(e^{-\theta^{-i}(1-t)}(CS^i - C_{S_{min}}) + C_{S_{min}}\right)
\]

and

\[
c(t) = -\int \theta^i e^{-\theta t} e^{-\theta^{-i}(1-t)}(CS^i - C_{S_{min}})dt - \int \theta^i e^{-\theta t} C_{S_{min}}dt + d,
\]

where \(d\) is a constant. As a result, the buyer value is given by

\[
V_t = \left(\frac{\theta^i}{\theta^i - \theta^{-i}}e^{-\theta^{-i}(1-t)}(CS^i - C_{S_{min}}) + e^{-\theta t} C_{S_{min}} + d\right)e^{\theta t},
\]

where the constant \(d\) is determined by the terminal condition

\[
V_1 = \frac{\theta^i}{\theta^i - \theta^{-i}}(CS^i - C_{S_{min}}) + C_{S_{min}} + de^{\theta t} = 0
\]

implying

\[
de^{\theta t} = -\frac{\theta^i}{\theta^i - \theta^{-i}}(CS^i - C_{S_{min}}) - C_{S_{min}}.
\]

A general solution to the terminal value problem is given by

\[
V_t = V^i_t = \frac{\theta^i}{\theta^i - \theta^{-i}}\left(e^{-\theta^{-i}(1-t)} - e^{-\theta^i(1-t)}\right) (CS^i - C_{S_{min}}) + \left(1 - e^{-\theta^i(1-t)}\right) C_{S_{min}}
\]

\[
= B^i_1 (CS^i - C_{S_{min}}) + (1 - B^i_0) C_{S_{min}} = B^i_1 C^i + B^i_{1,2} C_{S_{min}}, \quad (4.10)
\]

where

\[
B^i_1 = \frac{\theta^i}{\theta^i - \theta^{-i}}\left(e^{-\theta^{-i}(1-t)} - e^{-\theta^i(1-t)}\right), \text{ for } \theta^i \neq \theta^{-i},
\]

\[
B^i_1 = e^{\theta^i(1-t)}, \text{ for } \theta^i = \theta^{-i},
\]

\[
B^i_{1,2} = \left(1 - e^{-\theta^i(1-t)}\right).
\]

Note that, the last expression (4.10) is applicable to generalize the buyer value from case
\( \theta^i \neq \theta^{-i} \) also to the other case where \( \theta^i = \theta^{-i} \). Indeed, when the frictions are identical in both stores, it is particularly easy to see that the buyer value must be just a weighted average of buyer value if she finds one price (which occurs for probability \( B^1_{\epsilon} \)) and that if she finds two prices (which occurs for probability \( B^1_{1,2} \)).  

**Proof of Lemma 13.** We assume in this proof that \( B_{1,2} > 0 \) (there are shoppers) and \( B_1 > 0 \) or \( B_2 > 0 \) (there are captive buyers). We also take \( \epsilon > 0 \) to represent some tiny (infinitesimal) number.

First, we analyze three cases to prove by contradiction that both sellers mix in equilibrium. In doing so, we make the assumption that one of the sellers uses a pure strategy \( p^i \). Case 1: \( p^i < \min supp(F^{-i}) \). As the demand \( B_i + B_{1,2} > 0 \) is unchanged as long as \( p^i \) stays below \( \min supp(F^{-i}) \), there is a profitable deviation for seller \( i \) from price \( p^i \) to price \( p^i + \epsilon \). Case 2: \( p^i > \max supp(F^{-i}) \). As the demand \( B_{-i} + B_{1,2} > 0 \) is unchanged as long as \( p^i \) stays above \( \max supp(F^{-i}) \), there is a profitable deviation for seller \(-i\) from a price \( p \in supp(F^{-i}) \) to a price \( p + \epsilon \). Case 3a: \( p^i > 0 \) and \( p^i \in supp(F^{-i}) \). As the demand \( B_{-i} + B_{1,2}(1 - F^i(p)) \) jumps up at \( p = p^i \), there is a profitable deviation for seller \(-i\) from price \( p^{-i} \) to price \( p^{-i} - \epsilon \). Case 3b: \( p^i = 0 \) and \( 0 \in supp(F^{-i}) \). Note that there are some captive buyers but, as both of the sellers use the price zero, both of them are making zero profit. Thus, the seller who has captive buyers has a profitable deviation up from zero to extract some profit from the captive buyers. Altogether, Cases 1, 2, 3a and 3b show that (i) both stores use randomized pricing strategies, (ii) both stores’ prices are bounded away from zero and (iii) both stores’ profits are bounded away from zero.

Next, we consider the supports \( supp(F^i) \) and \( supp(F^{-i}) \) of the seller’s randomized strategies \( F^i \) and \( F^{-i} \). Suppose that \( supp(F^i) \neq supp(F^{-i}) \). This implies that there is some open set \( U \neq \emptyset \) such that, with no loss of generality, \( U \subset supp(F^i) \) and \( U \cap supp(F^{-i}) = \emptyset \). But now, as the demand is unchanged for all \( p^i \in U \) there is a profitable deviation up from the lower prices in \( U \) to the higher prices in \( U \). This shows that the seller mix over the same set of prices \( supp(F) := supp(F^i) = supp(F^{-i}) \).

Last, we examine the support for possible gaps and jumps/atoms and delineate its boundaries. Gaps: Suppose the support is not connected but has a gap \([\underline{g}, \bar{g}] \cup supp(F) = \emptyset \) but for some \([\underline{g} - \epsilon, \bar{g}] \cup supp(F) \neq \emptyset \) and \([\bar{g}, \bar{g} + \epsilon] \cup supp(F) \neq \emptyset \). Then, as the demand is unchanged for all \( p \in [\underline{g}, \bar{g}] \), there is a profitable deviation from some price \( p \in [\underline{g} - \epsilon, \bar{g}] \) to some price \( p \in [\underline{g}, \bar{g} + \epsilon] \). Atoms: Suppose the strategy \( F^i \) is not continuous but has an atom \( \alpha^i > 0 \) at \( p^i_\alpha \in supp(F) \). Then, as the demand from shoppers, \((1 - F^i(p)) B_{1,2} \) is reduced by \( \alpha^i \) at \( p^i_\alpha \), there is a profitable deviation for seller \(-i\) from a price \( p^i_\alpha \) or some price \( p^i_\alpha + \epsilon \) to some price \( p^i_\alpha - \epsilon \). This implies that there can be an atom at the upper bound only and used by a single seller only; this makes sure the other seller does not use \( p_\alpha \) or any \( p^i_\alpha + \epsilon \), from which it would have a profitable deviation. Boundaries: (i) Consider the highest price \( p \) the sellers use. Note that the seller who has that price is only selling to its captive buyers \( B_i > 0 \). Hence, there is a profitable deviation up in \( p \) unless it
equals 1. (ii) Consider the lowest price $p$ the sellers use. As both of the stores make some profit, there is a profitable deviation up in price $p$ unless it is bounded away from 0. ■

**Proof of Lemma 14.** Based on Lemma 13, we only need to determine the sellers’ profits $\Pi^i$, the lower bound $\bar{p} > 0$ of the support, whether we need an atom $\alpha^i > 0$ at the upper bound $\bar{p} = 1$ of the support for seller $i = 1$ or $i = 2$, and the cumulative distribution functions $F^1$ and $F^2$.

Note first that, if seller $i$ prices right below the upper bound, it sells for $p = 1$ to its captive buyers and, provided seller $-i$ sometimes prices at the upper bound, for that case also to shoppers. Thus, the seller’s profit is given by $\Pi^i = B_i + \alpha^i B_{1,2}$ for $i = 1, 2$.

Instead, if a seller prices at the lower bound, it sells for $p = 1$ to its captive buyers and all shoppers such that the seller’s profit is $\Pi^i = (B_i + B_{1,2}) \bar{p}$ for $i = 1, 2$. As the profit has to be the same over the whole support to sustain randomized pricing strategies, equating $\Pi^i = B_i + \alpha^i B_{1,2} = (B_i + B_{1,2}) \bar{p}$ for $i = 1$ and $i = 2$ gives us the lower bound

$$\bar{p} = \frac{B_i + \alpha^i B_{1,2}}{B_i + B_{1,2}} = \frac{B_{-i} + \alpha^i B_{1,2}}{B_{-i} + B_{1,2}}.$$

For $B_i \geq B_{-i}$, this is solved only if $\alpha^i = \frac{B_i - B_{-i}}{B_i + B_{1,2}} \geq 0$ such that $\alpha^i = 0$. In consequence, the lower bound is $\bar{p} = \frac{B_i}{B_i + B_{1,2}}$ and the sellers’ profits are $\Pi^i = B_i$ and $\Pi^{-i} = B_{-i} + \alpha^i B_{1,2}$.

To simplify, we shorten $\alpha^i \rightarrow \alpha$.

The cumulative distribution functions $F^1$ and $F^2$ can now be obtained in closed-form by observing that the profit has to be invariant everywhere in the support. Therefore, a seller $i = 1, 2$ whose price is $p$ makes

$$\Pi^i = (B_i + (1 - F^{-i}(p)) B_{1,2}) p,$$

which gives

$$F^i(p) = \frac{B_{-i} + B_{1,2}}{B_{1,2}} - \frac{\Pi^{-i} 1}{B_{1,2} \bar{p}},$$

for $p \leq 1$, as required.

Observe that the profit $\Pi^{-i} \leq \Pi^i$ can be rewritten as

$$\Pi^{-i} = B_{-i} + \alpha B_{1,2} = B_{-i} + \frac{B_i - B_{-i}}{B_i + B_{1,2}} B_{1,2} B_{1,2}$$

$$\Pi^{-i} = \left(1 - \frac{B_{1,2}}{B_i + B_{1,2}}\right) B_{-i} + \frac{B_{1,2}}{B_i + B_{1,2}} B_i$$
\[ \Pi^{-i} = pB_{-i} + (1 - p)B_i, \]

which shows that it is a convex combination of seller \( i \)'s and seller \(-i\)'s captive buyers: the higher the lower bound \( p = \frac{B_i}{B_i + B_{1,2}} \), the more weight put on own captive buyers \( B_i \).

Also, if we continue from here,

\[
\Pi^{-i} = -p(B_i - B_{-i}) + B_i \\
= -\frac{B_i}{B_i + B_{1,2}}(B_i - B_{-i}) + B_i \\
= \left(1 - \frac{B_i - B_{-i}}{B_i + B_{1,2}}\right)B_i \\
= (1 - \alpha)\Pi^i.
\]

This will be needed a bit later. ■

**Proof of Proposition 17.** By Lemma 12, if

\[ \theta^1 (1 - E(p|F^1)) > \theta^2 (1 - E(p|F^2)) \]

then \( t^1 = 1 - t^2 = 1 \), and, if

\[ \theta^1 (1 - E(p|F^1)) < \theta^2 (1 - E(p|F^2)) \]

then \( t^1 = 1 - t^2 = 0 \). Otherwise, any \( t^1 = 1 - t^2 \in [0,1] \) and \( t^2 \in [0,1] \) such that \( t^1 = 1 - t^2 \) would do.

Note first that, if \( \theta^1 (1 - E(p|F^1(\theta, (1,0)))) \geq \theta^2 (1 - E(p|F^2(\theta, (1,0)))) \), then we have \( F(\theta, (1,0)) \) and \( t = (1,0) \) is clearly a fixed point. As a result, the buyers are willing to start from store \( i = 1 \) even for the price ratio \( E(p|F^1(\theta, (1,0)))/E(p|F^2(\theta, (1,0))) \).

It is also clear that, if we increase \( t^1 \) from the level \( t^* \) where \( B_1 = B_2 \) up to one, by continuity of \( E(p|F^1(\theta, t)) > E(p|F^2(\theta, t)) \) in \( t \), we must span all the values of \( E(p|F^1) > E(p|F^2) \) between one and \( E(p|F^1(\theta, (1,0))) > E(p|F^2(\theta, (1,0))) \). Hence, assuming that

\[ \theta^1 (1 - E(p|F^1(\theta, (1,0)))) < \theta^2 (1 - E(p|F^2(\theta, (1,0)))) \]

and recalling that

\[ \theta^1 (1 - E(p|F^1(\theta, (t^*, 1 - t^*)))) > \theta^2 (1 - E(p|F^2(\theta, (t^*, 1 - t^*)))) \]

there necessarily exist a fixed point \( F(\theta, t) \) and \( t \) in between \( t = (1,0) \) and \( t = (t^*, 1 - t^*) \), where
\[ \theta^1 \left( 1 - E(p|F^1(\theta, t)) \right) = \theta^2 \left( 1 - E(p|F^2(\theta, t)) \right). \]

When the price ratio is exactly \( E(p|F^1(\theta, t))/E(p|F^2(\theta, t)) \), the buyers are indifferent between the sellers. They are thus willing to be assigned to any store. If they are assigned according to \( t \), the sellers willing to price in accordance with \( F(\theta, t) \). We have a fixed point. For uniqueness, we use the monotonicity of \( E(p|F^1(\theta, t))/E(p|F^2(\theta, t)) \) in \( t \):

\[
1 - E(p|F^1(t)) \quad 1 - E(p|F^2(t)) = 1 - \frac{\Pi^2}{B_{1,2}} ln \left( \frac{1}{p} \right) - \alpha \\
1 - \frac{\Pi^1}{B_{1,2}} ln \left( \frac{1}{p} \right) = 1 - \frac{\Pi^1}{B_{1,2}} ln \left( \frac{1}{p} \right) - \alpha \\
1 + \frac{\Pi^1}{B_{1,2}} ln \left( \frac{1}{p} \right) - \frac{\Pi^1}{B_{1,2}} ln \left( \frac{1}{p} \right) - \alpha \\
= 1 - \frac{\Pi^1}{B_{1,2}} ln \left( \frac{1}{p} \right) \\
= 1 - \alpha,
\]

where

\[
\frac{\partial \alpha}{\partial t^1} = \frac{\partial}{\partial t^1} \frac{B_1 - B_2}{B_1 + B_{1,2}} \geq 0,
\]

because \( \frac{\partial B_1}{\partial t^1} \geq 0, \frac{\partial B_2}{\partial t^1} \leq 0 \) and \( \frac{\partial B_{1,2}}{\partial t^1} = 0 \). This implies that, if we begin with \( t^1 = t^* \) such that \( B_1 = B_2 \) and start to raise it up to one, \( E(p|F^1(\theta, t))/E(p|F^2(\theta, t)) \) either decreases or remains constant: the fixed point is unique. ■

**Proof of Lemma 15.** We just proved that, if \( \theta^1 \geq \theta^2 \) and \( t^1 = 1 - t^2 \in (0, 1) \), then
\[
\frac{\theta^2}{\theta^1} = 1 - \alpha \\
\frac{\theta^2}{\theta^1} = \frac{B_2 + B_{1,2}}{B_1 + B_{1,2}} \\
\frac{\theta^2}{\theta^1} = 1 - B_0 - B_1 \\
\frac{\theta^2}{\theta^1} = 1 - B_0 - B_2 \\
\frac{\theta^2}{\theta^1} = 1 - B_0 \left( 1 - \frac{\theta^1}{\theta^1 - \theta^2} \right) - \frac{\theta^1}{\theta^1 - \theta^2} e^{-\theta^2} \\
\frac{\theta^2}{\theta^1} = \frac{1}{1 - B_0 \left( 1 + \frac{\theta^2}{\theta^1 - \theta^2} \right) + \frac{\theta^1}{\theta^2} e^{-\theta^1}} \\
\frac{\theta^2}{\theta^1} = \frac{\theta^1 - \theta^2 + \theta^2 B_0 - \theta^1 e^{-\theta^2}}{\theta^1 - \theta^2 + \theta^2 e^{-\theta^1} - \theta^1 B_0}.
\]

We can hence solve for \( B_0 \) as

\[
B_0 = -\frac{1}{2} \frac{\theta^2}{\theta^1} \left( 1 - e^{-\theta^1} \right) - \frac{1}{2} \frac{\theta^1}{\theta^2} \left( 1 - e^{-\theta^2} \right) + 1.
\]

From here on, it is useful to work with the reparametrization \( \rho = \frac{\theta^2}{\theta^1} \) under which

\[
B_0 = -\frac{1}{2} \rho \left( 1 - e^{-\theta^1} \right) - \frac{1}{2} \rho^{-1} \left( 1 - e^{-\theta^2} \right) + 1
\]

and, since \( \frac{\partial \theta^1}{\partial \rho} = -\frac{\theta^1}{\rho} \) and \( \frac{\partial \theta^2}{\partial \rho} = \frac{\theta^2}{\rho} \),

\[
\frac{\partial B_0}{\partial \rho} = -\frac{1}{2} \left( 1 - e^{-\theta^1} \right) + \frac{1}{2} \theta^1 e^{-\theta^1} + \frac{1}{2} \rho^{-1} \left( 1 - e^{-\theta^2} \right) + \frac{1}{2} \rho^{-2} \theta^2 e^{-\theta^2}
\]

or, returning to the original variables,

\[
\frac{\partial B_0}{\partial \rho} = \frac{1}{2} \left( - \left( 1 - e^{-\theta^1} - \theta^1 e^{-\theta^1} \right) + \frac{\theta^1}{\theta^2} \left( 1 - e^{-\theta^2} + \theta^1 e^{-\theta^2} \right) \right).
\]

This is positive for all \( \theta^1 \geq \theta^2 > 0 \) because

\[
\frac{1 - e^{-\theta^1} - \theta^1 e^{-\theta^1}}{\theta^1} < \frac{1 - e^{-\theta^1}}{\theta^1} < \frac{1 - e^{-\theta^2}}{\theta^2} < \frac{1 - e^{-\theta^2} + \theta^1 e^{-\theta^2}}{\theta^2}
\]

and the function \( \frac{1 - e^{-x}}{x} \) is decreasing in \( x \).

We can now revert to \( \rho = \frac{\theta^2}{\theta^1} = \frac{1 - B_0 - B_1}{1 - B_0 - B_2} \) to solve it for \( B_1 \) and \( B_2 \) as a function of \( \rho \)

\[
B_1 = (1 - \rho) (1 - B_0) + \rho B_2,
\]

\[
B_2 = (1 - \rho^{-1}) (1 - B_0) + \rho^{-1} B_1.
\]
Their partials with respect to \( \rho \) are given by

\[
\frac{\partial B_1}{\partial \rho} = -(1 - B_0 - B_2) - (1 - \rho) \frac{\partial B_0}{\partial \rho} + \rho \frac{\partial B_2}{\partial \rho},
\]

\[
\frac{\partial B_2}{\partial \rho} = \rho^{-1} (1 - B_0 - B_1) - (1 - \rho^{-1}) \frac{\partial B_0}{\partial \rho} + \rho^{-1} \frac{\partial B_1}{\partial \rho}.
\]

In other words, as we can take \( \rho = \rho(\theta^1, \theta^2) \) as a seller’s choice variable, the first order conditions are

\[
\frac{\partial \Pi^1}{\partial \rho} = 0 \iff \frac{\partial B_1}{\partial \rho} = 0 \iff \rho = 1 - B_0 - B_2 + (1 - \rho) \frac{\partial B_0}{\partial \rho} > 0
\]

and

\[
\frac{\partial \Pi^2}{\partial \rho} = 0 \iff (1 - \alpha) \Pi^1 = 0 \iff \frac{\partial \rho B_1}{\partial \rho} = 0 \iff B_1 + \rho \frac{\partial B_1}{\partial \rho} = 0.
\]

Both or them cannot be satisfied for the same \( \rho \) because a seller’s profit is positive, \( \Pi^1 = B_1 > 0 \).

This implies that it cannot be optimal for both the sellers to use \( \theta^1 \) and \( \theta^2 \) such that \( t^1 < 1 \). ■

**Proof of Proposition 18.**

Consider seller \( i \)’s best response \( \theta^i \) to seller \( -i \)’s frictions \( \theta^{-i} \).

**Case 1:** \( \theta^{-i} = 0 \).

As seller \( -i \) is essentially out of the market, seller \( i \) will act like a monopolist and set \( \theta^i = \infty \) and \( p^i = 1 \).

**Case 2:** \( \theta^{-i} \in (0, \infty) \).

First, if the seller chooses an extremely slow rate \( \theta^i = 0 \) it serves nobody and extracts no revenue, \( \Pi^i = 0 \).

Second, if the seller chooses an extremely fast rate \( \theta^i = \infty \) such that \( t^i = 1 \), the seller’s profit is given as

\[
\Pi^i = e^{-\theta^i}.
\]

Third, if the seller chooses a finite but sufficiently fast rate \( \theta^i \gg \theta^{-i} \) such that \( t^i = 1 \), the seller’s profit is given as

\[
\Pi^i = \frac{\theta^i}{\theta^i - \theta^{-i}} \left( 1 - e^{-\theta^{-i}} \right) e^{-\theta^{-i}}.
\]

\[32\text{For } \theta^i = 0, B_{-i} = 1 - e^{-\theta^{-i}} \text{ and } B_0 = e^{-\theta^{-i}} \text{ while } B_i = B_{1,2} = 0.\]

\[33\text{For } \theta^i = \infty, B_i = e^{-\theta^{-i}} \text{ and } B_{1,2} = 1 - e^{-\theta^{-i}} \text{ while } B_{-i} = B_0 = 0.\]
Section 4.4

It is now easy to show that
\[ \frac{\theta^i}{\theta^i - \theta^{-i}} \left( 1 - e^{-(\theta^i - \theta^{-i})} \right) > 1 \]
as long as \( |\theta^i - 1| > |\theta^{-i} - 1| \).

This implies that, by choosing a large enough finite \( \theta^i \), the seller is guaranteed to extract more revenue than by choosing \( \theta^i = 0 \) or \( \theta^i = \infty \).

**Case 3:** \( \theta^{-i} = \infty \).

Note first that, if both sellers have an infinite rate, \( \theta^i = \infty \), all buyers find all prices and both sellers’ profits go to zero, \( \Pi^i = 0 \).

Instead, if seller \(-i\) has an infinite rate and seller \(i\) has a finite rate, \( \theta^{-i} = \infty \) and \( \theta^i < \infty \) such that \( \Gamma^{-i} = 1 \), seller \(i\)’s profit is\(^{34} \)
\[ \Pi^i = \frac{B^{-i} - B_i}{B^{-i} + B_{1,2}} B_{1,2} = e^{-\theta^i} \left( 1 - e^{-\theta^i} \right), \]
It is maximized by \( \theta^i = \ln(2) < \infty \).

It is thus clear from Cases 1 to 3 that \( \theta^i = \infty \) cannot arise in equilibrium. ■

**The prominent seller’s problem**

*Proof that there is a unique solution \( \theta^i(\theta^2) \) to the seller’s problem:*

Start by considering a relaxed unconstrained problem

\[ \max_{\theta^1 \geq \theta^2} P^1, \]
where the prominent seller’s profit is represented by

\[ P^1(\theta^1, \theta^2) := \frac{\theta^1}{\theta^1 - \theta^2} \left( e^{-\theta^2} - e^{-\theta^1} \right). \]

The first partial with respect to \( \theta^1 \) is

\[ \frac{\partial P^1(\theta^1, \theta^2)}{\partial \theta^1} = \frac{\theta^1}{\theta^1 - \theta^2} e^{-\theta^i} - \frac{\theta^2}{(\theta^1 - \theta^2)^2} \left( e^{-\theta^2} - e^{-\theta^1} \right). \]

Hence, an increase in \( \theta^1 \) increases \( P^1 \) if

\[ f(\theta^1, \theta^2) := -\frac{e^\delta - 1}{\delta} + \rho^{-1} \geq 0. \]

\(^{34}\)Here, \( B_{-i} = e^{-\theta^i} \) and \( B_{1,2} = 1 - e^{-\theta^i} \) whereas \( B_i = B_\emptyset = 0 \).
Therefore, the existence and uniqueness of the solution to this relaxed problem basically come from the fact that the growth rate of \( e^{\delta - 1} = \frac{e^{(\theta^1 - \theta^2)} - 1}{\theta^2 - \theta^1} \) is exponential in \( \theta^1 \) whereas the growth of \( \rho^{-1} = \frac{\theta^3}{\theta^2} \) is linear in \( \theta^1 \). Let us consider this in more detail.

We can now differentiate this function \( f \) we just defined with respect to \( \theta^1 \) to obtain

\[
f'(\theta^1, \theta^2) := -\frac{\delta e^\delta - e^\delta + 1}{\delta^2} + \frac{1}{\theta^2} < -e^\delta + e^\delta - 1 + \rho^{-1}.
\]

This implies that, if \( f = -\frac{e^\delta - 1}{\delta} + \rho^{-1} \) is negative, the change in \( f' = -\frac{e^\delta - 1}{\delta} + \rho^{-1} \) is negative (strictly negative for \( \delta > 0 \) and zero for \( \delta = 0 \)) because clearly

\[
e^\delta = 2 \frac{e^\delta - 1}{\delta^2}, \text{ for } \delta = 0,
\]

\[
e^\delta > 2 \frac{e^\delta - 1}{\delta^2}, \text{ for } \delta > 0.
\]

In consequence, \( \frac{e^\delta - 1}{\delta} \) and \( \rho^{-1} \) cannot cross more than once. The solution to the first order condition for an interior optimum \( \frac{e^\delta - 1}{\delta} = \rho^{-1} \) is thereby unique.

Regarding existence, note that, if we start with \( \theta^1 \) just above \( \theta^2 \) and, thus, with \( \rho \approx 1 \) and \( \delta \approx 0 \),

\[
\lim_{\theta^1 \to \theta^2} f(\theta^1, \theta^2) = 0 \text{ and } \lim_{\theta^1 \to \theta^2} f'(\theta^1, \theta^2) = \frac{1}{\theta^2} - \frac{1}{2}.
\]

The existence of interior solution \( \theta^1 > \theta^2 \) to this relaxed problem thus hinges on the condition that the other store has strong enough frictions, \( \theta^2 < 2 \). Otherwise, the seller would prefer to raise its own frictions by setting a smaller \( \theta^1 \), as \( \frac{e^\delta - 1}{\delta} > \rho^{-1} \) for all \( \theta^1 \geq \theta^2 \).

Returning back to the original problem, it is hence clear that, if the constraint \( \theta^1 \geq \theta^2 \frac{1}{\Gamma - \alpha} \) binds,

\[
e^\delta - 1 > \rho^{-1}
\]

(without the constraint, the seller would choose a lower \( \theta^1 \)) but, if the constraint \( \theta^1 \geq \theta^2 \frac{1}{\Gamma - \alpha} \) is slack,

\[
e^\delta - 1 = \rho^{-1}
\]

(without the constraint, the seller would choose the same \( \theta^1 \)). □

Proof that the solution \( \theta^1(\theta^2) \) is decreasing if the constraint is slack:

Differentiating totally the condition
\[
\frac{e^\delta}{\delta} = \theta^1_1
\]
gives
\[
\frac{d\theta^1}{d\theta^2} = \frac{\xi(\delta) - \frac{\theta^1}{(\theta^2)^2}}{\xi(\delta) - \frac{1}{\theta^2}},
\]
where
\[
\xi(x) := \frac{e^x}{x} - \frac{e^x - 1}{x^2} \geq 0.
\]
This is negative if \( \xi(\delta) \in \left[\frac{1}{\theta^2}, \frac{\theta^1}{(\theta^2)^2}\right]\).

(I) We first prove that \( \xi(\delta) \geq \frac{1}{\theta^2} \), when \( e^\delta = \rho^{-1} \) holds:

\[
\begin{align*}
\frac{e^\delta}{\delta} - \frac{e^\delta - 1}{\delta^2} &\geq \frac{1}{\theta^2} \\
\frac{e^\delta}{\delta} - \rho^{-1} &\geq \frac{1}{\theta^2} \\
\frac{e^\delta}{\delta} - 2\rho^{-1} &\geq -1
\end{align*}
\]

which holds because,
\[
e^\delta - 2\rho^{-1} = e^\delta (\delta - 2) + 2 geq 0 \geq -1.
\]

(II) We then prove that \( \xi(\delta) \leq \frac{\theta^1}{(\theta^2)^2} \), when \( e^\delta = \rho^{-1} \) holds:

\[
\begin{align*}
\frac{e^\delta}{\delta} - \frac{e^\delta - 1}{\delta^2} &\leq \frac{\theta^1}{(\theta^2)^2} \\
\frac{e^\delta}{\delta} - \delta \rho^{-1} &\leq \frac{1}{\theta^2} \rho^{-1} \\
e^\delta &\leq \rho^{-2}
\end{align*}
\]

which holds because,
\[
e^\delta \leq \left(\frac{e^\delta - 1}{\delta}\right)^2.
\]

Altogether, this implies that \( \frac{d\theta^1}{d\theta^2} \leq 0 \) where \( e^\delta = \rho^{-1} \) binds.
The non prominent seller’s problem

Proof that the solution $\theta^2(\theta^1)$ is decreasing if the constraint is slack:

When $t^1 = 1 - t^2 = 1$, $B_1 = B_{1,2}$ is equivalent to

$$2 \frac{e^\delta}{\delta} = \frac{e^{\theta^1}}{\theta^1}.$$

Differentiating it totally results in

$$\frac{d\theta^1}{d\theta^2} = \frac{2\xi(\delta) - \xi(\theta^1)}{2\xi(\delta)},$$

where

$$\xi(x) := \frac{e^x}{x} - \frac{e^x - 1}{x^2} \geq 0.$$

This is negative if

$$2\left(\frac{e^\delta - e^\delta - 1}{\delta^2}\right) - \left(\frac{e^{\theta^1}}{\theta^1} - \frac{e^{\theta^1} - 1}{(\theta^1)^2}\right) \leq 0.$$

$$2\frac{e^\delta - 1 + 1}{\delta} - 2\frac{e^\delta - 1}{\delta^2} - \frac{e^{\theta^1} - 1 + 1}{\theta^1} + \frac{e^{\theta^1} - 1}{(\theta^1)^2} \leq 0.$$

$$2\frac{1}{\delta} - 2\frac{1}{\delta} \frac{e^\delta - 1}{\delta} - 1 \frac{1}{\theta^1} + 1 \frac{e^{\theta^1} - 1}{\theta^1} \leq 0.$$

$$2 \left(\frac{1}{\delta} - \frac{1}{\theta^1}\right) - \left(\frac{1}{\delta} - \frac{1}{\theta^1}\right) 2\frac{e^\delta - 1}{\delta} \leq 0.$$

$$\frac{1}{\delta} + \left(\frac{1}{\delta} - 1 \frac{1}{\theta^1}\right) 1 - 2\frac{e^\delta - 1}{\delta} \leq 0.$$

$$1 + \rho \left(1 - 2\frac{e^\delta - 1}{\delta}\right) \leq 0.$$

We can now solve $\rho$ from $2 \frac{e^\delta}{\delta} = \frac{e^{\theta^1}}{\theta^1}$ as

$$\rho = 1 - 2 \frac{e^\delta - 1}{e^{\theta^1} - 1}$$

and then continue with the above calculation.
$$\left(1 - 2e^\delta - 1 \right) \left(1 - 2e^\theta - 1 \right) \leq -1$$

$$\frac{-2e^\delta - 1}{e^{\theta_1} - 1} - \frac{2e^\delta - 1}{\delta} + 4 \left(\frac{e^\delta - 1}{\delta} \right)^2 \leq -2 \frac{e^{\theta_1} - 1}{e^{\delta} - 1}$$

$$\frac{-(e^\delta - 1)\delta - (e^\delta - 1)(e^{\theta_1} - 1)}{(e^{\theta_1} - 1)\delta} + 2 \frac{(e^\delta - 1)^2}{(e^{\theta_1} - 1)\delta} + \frac{(e^{\theta_1} - 1)\delta}{(e^{\theta_1} - 1)\delta} \leq 0$$

$$\frac{-(e^\delta - 1)\delta - (e^\delta - 1)(e^{\theta_1} - 1)}{\delta} + 2 \frac{(e^\delta - 1)^2}{\theta_1^2} + \frac{(e^{\theta_1} - 1)\delta}{\theta_1^2} \leq 0$$

$$\frac{(e^\delta - 1)\left(\frac{e^\delta - 1}{\delta} - 1\right)}{(e^\theta - 1)} \geq \left(\frac{e^{\theta_1} - 1}{\theta_1} - 1\right)$$

$$\frac{(e^{\theta_1} - 1)}{\theta_1 - 1} \geq \frac{(e^\delta - 1)}{\delta - 1}.$$
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