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RESCALING PRINCIPLE FOR ISOLATED ESSENTIAL SINGULARITIES OF QUASIREGULAR MAPPINGS

YÚSUEKE OKUYAMA AND PEKKA PANKKA

Abstract. We establish a rescaling theorem for isolated essential singularities of quasiregular mappings. As a consequence we show that the class of closed manifolds receiving a quasiregular mapping from a punctured unit ball with an essential singularity at the origin is exactly the class of closed quasiregularly elliptic manifolds, that is, closed manifolds receiving a non-constant quasiregular mapping from a Euclidean space.

1. Introduction

A continuous mapping \( f: M \to N \) between oriented Riemannian \( n \)-manifolds is \( K \)-quasiregular if \( f \) belongs to the Sobolev space \( W_{\text{loc}}^{1,n}(M,N) \) and satisfies the distortion inequality

\[
\|Df\|^n \leq K J_f \quad \text{a.e.,}
\]

where \( \|Df\| \) is the operator norm and \( J_f \) is the Jacobian determinant of the differential \( Df \) of \( f \).

The main result of this paper is the following rescaling theorem. We denote the open unit ball about the origin in \( \mathbb{R}^n \) by \( B^n \). We say that a quasiregular mapping \( f \) from \( B^n \setminus \{0\} \) to a closed and oriented Riemannian \( n \)-manifold \( N \) has an essential singularity at the origin if the limit \( \lim_{x \to 0} f(x) \) does not exist in \( N \).

**Theorem 1.** Let \( N \) be a closed and oriented Riemannian \( n \)-manifold, \( n \geq 2 \), and let \( f: \mathbb{R}^n \setminus \{0\} \to N \) be a \( K \)-quasiregular mapping with an essential singularity at the origin, \( K \geq 1 \). Then there exist a non-constant \( K \)-quasiregular mapping \( g: X \to N \), where \( X \) is either \( \mathbb{R}^n \) or \( \mathbb{R}^n \setminus \{0\} \), and sequences \( (x_k) \) and \( (\rho_k) \) in \( \mathbb{R}^n \) and \( (0,\infty) \), respectively, such that \( \lim_{k \to \infty} x_k = 0 \), \( \lim_{k \to \infty} \rho_k = 0 \) and

\[
\lim_{k \to \infty} f(x_k + \rho_k v) = g(v)
\]

locally uniformly on \( X \).

Theorem [1] bears a close resemblance to Miniowitz’s Zalcman lemma for quasiregular mappings; see Miniowitz [10] and Zalcman [16]. It seems, however, that this version for isolated essential singularities has gone unnoticed.
in the quasiregular literature although the heuristic idea behind this rescaling principle is well known in the classical holomorphic case ($n = 2$ and $K = 1$); see, e.g., Bergweiler [1] and Minda [9].

Theorem 1 readily yields the following characterization of closed and oriented Riemannian manifolds receiving a quasiregular mapping with an isolated essential singularity.

**Theorem 2.** Let $N$ be a closed and oriented Riemannian $n$-manifold, $n \geq 2$. If there exists a $K$-quasiregular mapping $f : \mathbb{B}^n \setminus \{0\} \to N$ having an essential singularity at the origin, $K \geq 1$, then there exists a non-constant $K'$-quasiregular mapping $g : \mathbb{R}^n \to N$ satisfying $g(\mathbb{R}^n) \subset f(\mathbb{B}^n \setminus \{0\})$.

Conversely, if there exists a non-constant $K$-quasiregular mapping $g : \mathbb{R}^n \to N$ having an essential singularity at the origin such that $f(\mathbb{B}^n \setminus \{0\}) \subset g(\mathbb{R}^n)$.

Here $K' = K'(n, K) \geq 1$ depends only on $n$ and $K$, and $K'(2, K) = K$.

Having Theorem 2 at our disposal, we readily obtain “big” versions of Varopoulos’s theorem [15, pp. 146-147] and the Bonk–Heinonen theorem [2, Theorem 1.1], which respectively give a bound of the fundamental group and the de Rham cohomology ring of a closed quasiregularly elliptic manifold. Recall that a connected and oriented Riemannian $n$-manifold $N$, $n \geq 2$, is called quasiregularly elliptic if there exists a non-constant quasiregular mapping from $\mathbb{R}^n$ to $N$.

**Corollary 1.** Let $N$ be a closed, connected, and oriented Riemannian $n$-manifold, $n \geq 2$, with a $K$-quasiregular mapping $\mathbb{B}^n \setminus \{0\} \to N$ having an essential singularity at the origin, $K \geq 1$. Then the fundamental group $\pi_1(N)$ of $N$ has polynomial growth of order at most $n$, and the de Rham cohomology ring $H^*(N)$ of $N$ satisfies

$$\dim H^*(N) := \sum_{k=0}^{n} \dim H^k(N) \leq C,$$

where $C = C(n, K) > 0$ depends only on $n$ and $K$.

Although the former half of Corollary 1, the big Varopoulos theorem, is well-known to the experts, we have been unable to find it in the literature. For a direct proof of the big Bonk–Heinonen theorem, i.e., the bound (1.1), see [12].

We would also like to note that together with the Holopainen–Rickman Picard theorem for quasiregularly elliptic manifolds [6], we obtain a big Picard type theorem for quasiregular mappings into closed manifolds; see also [5].

**Corollary 2.** Let $N$ be a closed, oriented, and connected Riemannian $n$-manifold, $n \geq 2$, and $f : \mathbb{B}^n \setminus \{0\} \to N$ be a $K$-quasiregular mapping with an essential singularity at the origin, $K \geq 1$. Then for every $x \in N$, except for at most $q - 1$ points, it holds that $\# f^{-1}(x) = \infty$, where $q = q(n, K) \in \mathbb{N}$ depends only on $n$ and $K$.

We conclude this introduction with an application of Theorem 1 to the Ahlfors five islands theorem; see, e.g., Bergweiler [1] or Nevanlinna [11, XII §7, §8] for a detailed discussion.
Let $f$ be a quasimeromorphic function on a domain $U$ in $\mathbb{S}^2$, i.e., a quasiregular mapping from $U$ to $\mathbb{S}^2$. We say that $f$ has a simple island $\Omega$ over a Jordan domain $D^\prime$ in $\mathbb{S}^2$ if $\Omega$ is a subdomain in $U$ and is mapped univalently onto $D^\prime$ by $f$. The Ahlfors five islands theorem states that given five Jordan domains in $\mathbb{S}^2$ with pair-wise disjoint closures, any non-constant quasimeromorphic function on $\mathbb{R}^2$ has a simple island over one of these Jordan domains.

**Corollary 3.** Let $f$ be a quasimeromorphic function on $\mathbb{B}^2 \setminus \{0\}$ having an essential singularity at the origin. Then given five Jordan domains $D_1, \ldots, D_5$ in $\mathbb{S}^2$ with pair-wise disjoint closures, $f$ has a simple island over one of $D_1, \ldots, D_5$.

**Proof.** Applying Theorem 1 to $f$, we obtain sequences $(x_k)$ and $(\rho_k)$, and a non-constant quasimeromorphic function $g$ on $\mathbb{R}^2 \setminus \{0\}$, where $f_k$ is the mapping $v \mapsto f(x_k + \rho_k v)$ and $g$ is the locally uniform limit of $(f_k)$, as in Theorem 1. We may fix Jordan domains $D_{1}', \ldots, D_{5}'$ in $\mathbb{S}^2$ satisfying $D_j \Subset D_{j}'$ for every $j \in \{1, 2, 3, 4, 5\}$ and having pair-wise disjoint closures. By the Ahlfors five islands theorem, the quasimeromorphic function $g \circ \exp$ on $\mathbb{R}^2$ has a simple island $\Omega'$ over one of these Jordan domains, say $D_{j}'$. Hence $g$ has a simple island $\Omega$ over $D_{j}'. \quad \Box$

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### 2. Preliminaries

Let $\mathbb{B}^n(x, r)$ be the open ball in $\mathbb{R}^n$ about $x \in \mathbb{R}^n$ of radius $r > 0$. Set $\mathbb{B}^n(r) := \mathbb{B}^n(0, r)$ for each $r > 0$ and set $\mathbb{B}^n := \mathbb{B}^n(1)$.

The corresponding closed balls are denoted by $\overline{\mathbb{B}}^n(x, r)$, $\mathbb{B}^n(r)$, and $\mathbb{B}^n$, respectively.

Let $M$ be an oriented Riemannian $n$-manifold, $n \geq 2$. We denote by $|x - y|$ the distance between $x$ and $y$ in $M$, and by $B(x, r)$ the Riemannian ball $\{y \in M : |x - y| < r\}$ about $x \in M$ of radius $r > 0$ in $M$. Similarly, we denote by $\overline{B}(x, r)$ the corresponding closed ball about $x \in M$ of radius $r > 0$.

By [14] III.1.11, every $K$-quasiregular mapping from an open set $U \subset \mathbb{R}^n$ to $\mathbb{R}^n$ is locally $\alpha$-Hölder continuous with $\alpha = (1/K)^{1/(n-1)}$. We refer to [13] and [7] for the Euclidean theory of quasiregular mappings.

For every $x \in M$, there exist $r > 0$ and a 2-bilipschitz chart $B(x, r) \rightarrow \mathbb{R}^n$. Thus every $K$-quasiregular mapping from an open set $U \subset \mathbb{R}^n$ to $M$ is locally $\beta$-Hölder continuous with $\beta = \beta(n, K)$ depending only on $n$ and $K$.

The local Hölder continuity plays a key role in the proof of the following manifold version ([7] Theorem 19.9.3) of Miniowitz’s Zalcman lemma [10] §4]. Recall that a family $\mathcal{F}$ of $K$-quasiregular mappings from a domain $\Omega$ in $\mathbb{R}^n$ to $M$ is normal at $a \in \Omega$ if $\mathcal{F}$ is normal on some open neighborhood of $a$.

**Theorem 2.1.** Let $\Omega$ be a domain in $\mathbb{R}^n$, and $N$ be a closed, connected, and oriented Riemannian $n$-manifold, $n \geq 2$. Let $\mathcal{F}$ be a family of $K$-quasiregular mappings from $\Omega$ to $N$, $K \geq 1$. If $\mathcal{F}$ is not normal at $a \in$
respectively, and a non-constant $K$-quasiregular mapping $g : \mathbb{R}^n \to N$ such that $\lim_{j \to \infty} x_j = a$, $\lim_{j \to \infty} \rho_j = 0$ and
\[
\lim_{j \to \infty} f_j(x_j + \rho_j v) = g(v)
\]
locally uniformly on $\mathbb{R}^n$.

3. PROOF OF THEOREM \[ \square \]

We begin the proof of Theorem \[ \square \] by showing a manifold version of a classical lemma on isolated essential singularities due to Lehto and Virtanen \[ \$ \]: see also Heinonen and Rossi \[ \cite{4}, Theorem 2.3 \] and Gauld and Martin \[ \cite{3} \].

**Lemma 3.1.** Let $M$ be a closed and oriented Riemannian $n$-manifold, $n \geq 2$, and $f : \mathbb{R}^n \setminus \{0\} \to M$ be a quasiregular mapping with an essential singularity at the origin. Then
\[
\limsup_{r \to 0} \operatorname{diam}(f(\partial \mathbb{B}^n(r))) > 0.
\]

**Proof.** The proof follows the argument of Heinonen and Rossi in \[ \cite{4} \].

Since the origin is an isolated essential singularity of $f$, there exist sequences $(z_k)$ and $(w_j)$ in $\mathbb{B}^n \setminus \{0\}$ such that $\lim_{k \to \infty} z_k = \lim_{j \to \infty} w_j = 0$ and that both limits
\[ a := \lim_{k \to \infty} f(z_k) \quad \text{and} \quad b := \lim_{j \to \infty} f(w_j) \]
exist in $M$ and are distinct.

We choose $R \in (0, |b - a|/4)$ so small that $B(a, 4R)$ is 2-bilipschitz equivalent to an open ball in $\mathbb{R}^n$. Note that, in particular, $b \notin B(a, 4R)$.

Suppose that \[ \text{(3.1)} \] does not hold. Then there is $r_0 > 0$ such that, for every $r \in (0, r_0)$, $\operatorname{diam}(f(\partial \mathbb{B}^n(r))) < R/2$.

We fix $k_1 \in \mathbb{N}$ so large that $|z_{k_1}| < r_0$ and that $|z_{k_1} - a| \leq R/2$. Let
\[ r_1 := |z_{k_1}|. \]
Since $\operatorname{diam}(f(\partial \mathbb{B}^n(r_1))) < R/2$, we have
\[
\operatorname{diam}(f(\partial \mathbb{B}^n(r_1))) < B(a, R).
\]

Since $f(w_j) \to b \notin B(a, 4R)$ as $j \to \infty$, the continuity of $f$ implies the existence of the maximal element, say $r_2 \in (0, r_1)$, of the subset
\[
\{ r \in (0, r_1] : f(\partial \mathbb{B}^n(r)) \not\subset B(a, 2R) \}.
\]

By the maximality, $f(\partial \mathbb{B}^n(r_2)) \cap \partial B(a, 2R) \neq \emptyset$. Let $c \in f(\partial \mathbb{B}^n(r_2)) \cap \partial B(a, 2R)$. Then $\operatorname{diam}(f(\partial \mathbb{B}^n(r_2))) < R/2$ and
\[
\operatorname{diam}(f(\partial \mathbb{B}^n(r_2))) < B(c, R/2).
\]

We join $\partial \mathbb{B}^n(r_1)$ and $\partial \mathbb{B}^n(r_2)$ by a line segment $\ell$, which is contained, except for the end points, in the ring domain
\[
A(r_2, r_1) := B^n(r_1) \setminus \overline{\mathbb{B}^n(r_2)}.
\]

Then the path $f(\ell)$ in $M$ joins $f(\partial \mathbb{B}^n(r_1))$ and $f(\partial \mathbb{B}^n(r_2))$, and by \[ \text{(3.2)} \], \[ \text{(3.3)} \] and the choice of $r_1$ and $r_2$, we may fix $y_0 \in \ell$ such that
\[ f(y_0) \in B(a, 2R) \setminus (B(a, R) \cup B(c, R/2)). \]
Since \( B(a,4R) \) is bilipschitz equivalent to a Euclidean \( n \)-ball, \( M \setminus (\bar{B}(a,R) \cup \bar{B}(c,R/2)) \) is connected. Thus \( f(y_0) \) can be joined with \( b \notin \bar{B}(a,4R) \) by a path \( \beta: [0,1] \to M \) such that \( \beta([0,1]) \cap f(\partial(A(r_2,r_1))) = \emptyset \).

Let \( \alpha: I_0 \to \mathbb{B}^n \setminus \{0\} \), where \( I_0 = [0,1] \) or \( [0,t_0) \) for some \( t_0 \in (0,1] \), be a maximal lift of \( \beta \) under \( f \) starting at \( y_0 = \alpha(0) \in A(r_2,r_1) \). If \( I_0 = [0,1] \), then \( f(\alpha(1)) = b \notin f(A(r_2,r_1)) \), so \( \alpha(1) \notin A(r_2,r_1) \). If \( I_0 \neq [0,1] \), then \( \text{dist}(\alpha(t), \partial \mathbb{B}^n \cup \{0\}) \to 0 \) as \( t \to t_0 \). In both cases, \( \beta(I_0) \cap f(\partial(A(r_2,r_1))) = f(\alpha(I_0) \cap \partial A(r_2,r_1)) \neq \emptyset \). This is a contradiction and (3.1) holds. \( \square \)

**Proof of Theorem 1** Define a function \( Q_f: \mathbb{B}^n \setminus \{0\} \to [0,\infty) \) by

\[
Q_f(x) := \sup_{y,y' \in \mathbb{B}^n(x,|x|/2),y \neq y'} \frac{|f(y) - f(y')|}{|y - y'|^\beta},
\]

where \( \beta = \beta(n,K) \) as in Section 2. Put

\[
M_f := \limsup_{x \to 0} Q_f(x)|x|^\beta.
\]

Suppose first that

(3.4) \[ M_f = \infty, \]

or equivalently, that there exists a sequence \( (y_k) \) in \( \mathbb{B}^n \setminus \{0\} \) satisfying \( y_k \to 0 \) and \( Q_f(y_k)|y_k|^\beta \to \infty \) as \( k \to \infty \).

Fix \( \delta \in (0,1) \) and, for each \( k \in \mathbb{N} \), define a mapping \( g_k: \mathbb{B}^n(1 + \delta) \to N \) by

\[
g_k(z) := f(y_k + \frac{|y_k|}{2}z).
\]

By (3.21), there exist sequences \( (z_k) \) and \( (w_k) \) in \( \mathbb{B}^n \) satisfying

\[
\limsup_{k \to \infty} \frac{|g_k(z_k) - g_k(w_k)|}{|z_k - w_k|^\beta} \geq \limsup_{k \to \infty} \frac{1}{2} Q_f(y_k) \left( \frac{|y_k|}{2} \right)^\beta = \infty.
\]

Hence the family \( \{g_k: k \in \mathbb{N}\} \) is not normal on \( \mathbb{B}^n(1 + \delta) \). Indeed, otherwise, there exists a locally uniform limit point of \( \{g_k: k \in \mathbb{N}\} \), which is \( K \)-quasiregular on \( \mathbb{B}^n(1 + \delta) \) but not \( \beta \)-Hölder continuous on \( \mathbb{B}^n \). This is impossible.

By Theorem 2.1 there exist sequences \( (z_j), (\rho_j) \) and \( (k_j) \) in \( \mathbb{B}^n(1 + \delta) \), \( (0,\infty) \), and \( \mathbb{N} \), respectively, and a non-constant \( K \)-quasiregular mapping \( g: \mathbb{R}^n \to N \) such that \( \lim_{j \to \infty} \rho_j = 0 \), \( \lim_{j \to \infty} k_j = \infty \), and

\[
\lim_{j \to \infty} g_{k_j}(z_j + \rho_j v) = g(v)
\]

locally uniformly on \( \mathbb{R}^n \). This completes the proof in this case.

Suppose next that

(3.5) \[ M_f < \infty. \]

Let \( \{g_k: \mathbb{B}^n(e^k) \setminus \{0\} \to N; k \in \mathbb{N}\} \) be the family of \( K \)-quasiregular mappings defined as

\[
g_k(v) := f(e^{-k}v).
\]
By (3.5), we have for every \( v \in \mathbb{R}^n \setminus \{0\} \),
\[
\limsup_{k \to \infty} \sup_{w \in \mathbb{B}^n(v, |v|/2), w \neq v} \frac{|g_k(w) - g_k(v)|}{|w - v|^\beta} \leq M_f|v|^{-\beta} < \infty,
\]
so the family \( \{g_k : k \in \mathbb{N}\} \) is locally equicontinuous on \( \mathbb{R}^n \setminus \{0\} \).

By Lemma 3.1, there exists a subsequence \( (g_{k_j}) \) of \( (g_k) \) satisfying
\[
(3.6) \quad \lim_{j \to \infty} \text{diam}(g_{k_j}(\mathbb{B}^n \setminus \mathbb{B}^n(e^{-1}))) > 0.
\]

By passing to a further subsequence of \( (g_{k_j}) \) if necessary, we may assume, by
the Arzelà–Ascoli theorem, that \( (g_{k_j}) \) converges locally uniformly on \( \mathbb{R}^n \setminus \{0\} \)
to a mapping \( g : \mathbb{R}^n \setminus \{0\} \to N \). Since \( (g_{k_j}) \) is a sequence of \( K \)-quasiregular mappings, \( g \) is \( K \)-quasiregular, and by (3.6), non-constant. This completes
the proof. □

Example 3.2. To see that both cases in the proof of Theorem 1 actually
occur, we give two examples, which are similar to Examples 23 and 24 in [13].

For \( M_f < \infty \), we may take the conformal mapping \( f : \mathbb{B}^n \setminus \{0\} \to S^{n-1} \times S^1 \), \( x \mapsto (x/|x|, e^{-\log |x|}) \). Then \( f \) is the composition \( \psi \circ h \) of a conformal homeomorphism \( h : \mathbb{B}^n \setminus \{0\} \to S^{n-1} \times \mathbb{R} \), \( x \mapsto (x/|x|, -\log |x|) \), and a locally isometric covering map \( \psi : S^{n-1} \times \mathbb{R} \to S^{n-1} \times S^1 \). Since \( |h(sx) - h(ty)| \leq |\log s - \log t| + |x - y| \) for all \( s, t \in (0, 1) \) and all \( x, y \in S^{n-1} \), we easily observe
that \( M_f < \infty \).

For \( M_f = \infty \), we construct \( f : \mathbb{B}^n \setminus \{0\} \to S^n \) using the winding map
\( h : S^n \to \mathbb{R}^n \),
\[
(x_1, \ldots, x_{n-2}, re^{i\theta}) \mapsto (x_1, \ldots, x_{n-2}, re^{i\theta}),
\]
which is a quasiregular endomorphism on \( S^n \); we identify \( \mathbb{R}^{n+1} \) with \( \mathbb{R}^{n-1} \times \mathbb{C} \).

Let \( \sigma : S^n \setminus \{e_{n+1}\} \to \mathbb{R}^n \) be the stereographic projection, and \( S \) the lower hemisphere \( \{(x_1, \ldots, x_{n+1}) \in S^n : x_{n+1} \leq 0\} \) of \( S^n \). We note that \( h|\partial S \) is the identity.

Let \( (r_k) \) be a sequence tending to \( 0 \) in \( (0, 1/2) \) and put \( x_k := r_ke_1 \) and \( B_k := \mathbb{B}^n(x_k, r_k/2) \) for each \( k \in \mathbb{N} \). We may assume that balls \( B_k, k \in \mathbb{N}, \) are mutually disjoint. For each \( k \in \mathbb{N} \), put \( x_k' := \sigma^{-1}(x_k) \) and \( B_k' := \sigma^{-1}(B_k) \) and let \( \rho_k \) be the the Möbius transformation on \( S^n \) defined as
\[
\rho_k(y) = \begin{cases} 
\sigma^{-1} \circ \alpha_k \circ \sigma(y), & y \neq e_{n+1} \\
\epsilon_{n+1}, & y = e_{n+1}
\end{cases}
\]
where \( \alpha_k \) is the affine transformation \( x \mapsto r_k^{-1}(x-x_k) \) on \( \mathbb{R}^n \). Then \( \rho_k(B_k') = S \) and \( \rho_k(x_k') = -\epsilon_{n+1} \), so the mapping \( f : \mathbb{B}^n \setminus \{0\} \to S^n \) defined by
\[
f(x) = \begin{cases} 
\sigma^{-1}(x), & x \in (\mathbb{B}^n \setminus \{0\}) \setminus \bigcup_{k=1}^\infty B_k \\
(\rho_k^{-1} \circ h \circ \rho_k) \circ \sigma^{-1}(x), & x \in B_k
\end{cases}
\]
is quasiregular with the same distortion constant as \( h \). We observe that,
for every \( k \in \mathbb{N} \), there exists a unique \( y_k \in B_k \) satisfying \( f(y_k) = e_1 \) and
\( |x_k - y_k| \leq Cr_k^2 \), where \( C > 0 \) is independent of \( k \). Since \( f(x_k) = e_{n+1} \) for
every $k \in \mathbb{N}$, by a direct computation, there is $C' > 0$ such that for every $k \in \mathbb{N}$, $Q_f(x_k)|x_k|^\beta \geq C'r_k^{-\beta}$, so $f$ satisfies $M_f = \infty$.

4. Proof of Theorem 2

Let $Z_n : \mathbb{R}^n \to \mathbb{R}^n \setminus \{0\}$, $n \geq 2$, be the Zorich mapping (see [17] or [14, I.3.3] for the construction of $Z_n$), which is $K_n$-quasiregular for some $K_n \geq 1$ and an analogue of the exponential function $Z_2 : \mathbb{C} \to \mathbb{C} \setminus \{0\}$. The mapping $Z_2$ is $K_2$-quasiregular for $K_2 = 1$. Set $K' = K \cdot K_n \geq 1$ for each $n \geq 2$ and each $K \geq 1$.

Let $N$ be a closed, connected, and oriented Riemannian $n$-manifold.

Suppose there exists a $K$-quasiregular mapping $f : \mathbb{B}^n \setminus \{0\} \to N$ with an essential singularity at the origin. By Theorem 1 and a manifold version of Hurwitz’s theorem (cf. the proof of [10, Lemma 2]), there exists a non-constant $K$-quasiregular mapping $g : X \to N$, where $X$ is either $\mathbb{R}^n$ or $\mathbb{R}^n \setminus \{0\}$, satisfying $g(X) \subset f(\mathbb{B}^n \setminus \{0\})$. If $X = \mathbb{R}^n$, then the mapping $g$ has the desired properties. If $X = \mathbb{R}^n \setminus \{0\}$, then the mapping $g \circ Z_n : \mathbb{R}^n \to N$ has the desired properties.

Suppose now that $g : \mathbb{R}^n \to N$ is a non-constant $K$-quasiregular mapping. Let $\iota$ be an orientation preserving conformal involution of $\mathbb{R}^n \setminus \{0\}$ satisfying $\iota(\mathbb{B}^n \setminus \{0\}) = \mathbb{R}^n \setminus \mathbb{B}^n$. Then the mapping $f : \mathbb{B}^n \setminus \{0\} \to N, x \mapsto g \circ Z_n(\iota(x))$, has the desired properties.

References


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