Obfuscation by substitutes, drowning by numbers

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Abstract

We study the effects of price and product variety on competition in a new duopoly price search model that features intrafirm frictions and deadlines for consumers. We focus on a class of symmetric, collusive equilibria with price dispersion within stores and across stores. Firms tend to have one product variant on sale but the rest have monopoly prices. Consumers keep searching in a given store until they find its discount price or until they have sampled through all prices.

We find that carrying more variants boosts firms' profits: First, it makes it easier to price discriminate across consumers based on their price information. Second, it amplifies the existing search frictions and creates implicit barriers to switching, in an environment where none exist initially. Firms can extract full surplus at the limit where the number of variants explodes. Therefore, by stocking more variants it is possible to undo any pro-competitive effects of faster search technology.

JEL Classification: D43, D83

Keywords: online search, search frictions in-store, price variation in-store, deadlines

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1 Introduction

Coincidentally, lower search costs on the Internet have been accompanied by a proliferation of product variants with a rich spectrum of prices. While this implies more choice, it can also make it more time consuming to sort out the best alternative. In search markets say for electronics or other consumer goods, each product adds its own incremental time cost upon consumers, who have to look, touch, twist and turn the products to take in the relevant features or to read the small print of their contract terms to compare them with each other. Adding upon this effect, consumers are typically busy and cannot search for a very long time; at some point they hit their deadline or reach the limits of their patience. In markets like this, additional variability could thus have the unexpected non competitive side-effect of decreasing search efficiency.

In this paper we consider these ideas, that date back at least to Ireland (2007) and are analyzed more recently by Carlin and Ederer (2014). However, neither of these papers study explicitly the mediating role of intrafirm search frictions and price dispersion as natural causes of search fatigue in environments with a breathtaking number of variants. Indeed, due to the difficulty of modeling price competition under dynamic search in stores, it has not yet received satisfactory treatment in the literature. Problems arise mainly from complications pertinent to handling the $K$-dimensional joint price distribution, when competing firms may carry $K$ product variants in their store, and solving the related fixed point problem. Moreover, it is not immediate how to set up the model to highlight the potential effects of variety provision on search.\footnote{Standard models of consumer search seem as such incompatible with the observation that consumers keep shopping for prolonged times in different, individual stores and end finding numerous price quotes during a single search spell. Namely, in standard models of sequential search rational consumers usually stop when they discover their first price and in fixed sample models consumers are not able to readjust their strategies after each price is drawn (Baye et al., 2006). In our model this issue is solved by letting consumers search costlessly for a while and encounter the available products in a random dynamic manner.}

To fill this gap in the literature, this paper shows how the existing approaches can be modified to study these issues. Specifically, we develop a new search model which features intrafirm frictions and deadlines for consumers\footnote{Either there really is a deadline or consumers just get fed up with search at some random time point.} and focus on an interesting class of symmetric, collusive equilibria. In these equilibria firms’ profits go up with more variants because low prices get harder to find.

We consider a model where two firms set prices for $K$ variants and, then, consumers search costlessly for a while. It is also free to switch. Inside a given store, consumers discover the available variants one by one in mixed order according to some general counting process. A consumer’s search strategy thus simply specifies in which store to be at each time point, given what has been found so far.

Our analysis suggests a natural way for two competing firms to price multiple substitute products in their store, under optimal consumer search. It is this ingenious pricing strategy...
which allows firms to make search less efficient although the base line search costs would otherwise stay the same. Furthermore, in line with accumulated empirical evidence for offline and online search (Brynjolfsson and Smith (2000); Morgan et al. (2004); Orlov (2011) and Kaplan and Menzio (2015)), there is price dispersion between similar products not only across stores but also within stores.

To elaborate, we study a class of what we call obfuscation equilibria to underline the link to closely related work such as Ellison and Wolitzky (2012). In these obfuscation equilibria, both firms have $K$ product variants within their stores: $K - 1$ of them with the monopoly price, $p = 1$, and at most one with a discount price, $p < 1$. It turns out that this pricing strategy (i) helps each firm to maintain consumers' interest for a longer time with a locking effect, and (ii) helps it to price discriminate better across more and less informed consumers. Consumers have an incentive to keep searching in a given store until they find that one low price that could be there or until they have sampled through all the prices. Accordingly, both firms make a higher profit by coordinating to such an obfuscation equilibrium where each has more variants to engage or exhaust consumers with.

The key underlying assumptions in our novel search model are that consumers search under heavy time-pressure (subject to exogenous deadlines) and the variants in a given store are found one by one, randomly and gradually (according to an exogenous counting process). This new modeling approach enables us to capture a range of different search outcomes in a single basic model and vary the degree of exogenous search frictions: the higher the deadline or the intensity of the counting process, the better the available search technology. Due to the stochastic finding process, there would always be some informed consumers and some uninformed consumers just like in the standard workhorse model by Varian (1980). With more than one variety, however, there would also be some partially informed consumers, whose search paths then become relevant to track, to see where they buy from.

One of our main contributions is also to highlight conditions under which an industry wide increase in alternatives, clearly express in today’s retail sector, could aid firms to raise their profits or maintain them constant despite lower search costs. To this aim, we derive a (necessary and sufficient) condition of existence for our equilibrium class, which hinges on the details of the underlying search technology. It turns out that this condition is satisfied for any large enough number of variants as long as the tail probability of discovering extreme numbers of variants converges to zero sufficiently fast and the process variance is still adequately large.

If this sufficient condition holds, as is the case for example for the weighted Poisson process with enough variance relative to the mean, we can show that it is possible for firms to extract full surplus at the limit, where both firms sell infinitely many variants for the monopoly price. This limiting equilibrium resembles the one in Diamond (1971); with
just one variant, our model nests that by Varian (1980). Hence, if the sufficient condition holds for a range of search technologies like with our Poisson process, it is actually possible for firms to maintain their profits the same despite improving search technology just by providing additional variants.  

Figuratively, this finding portrays a search market as in a constant battle between consumers with steadily improving search technology and firms with larger numbers of redundant variants. More and more monopoly prices are added and spread all over to dampen consumers’ efforts to find a good price. As the number of variants explodes, searching for that one discount price becomes much like looking for a needle in a haystack because almost all prices are then just monopoly prices.

Nevertheless, as a distinction from the well known case of Diamond (1971), where nobody would have an incentive to search with positive cost and virtually zero gain, in our deadline based search model the residual price variation is enough to keep all consumers searching. We also find that our results remain robust to moderate economies of scale: as long as the sufficient condition still holds, our findings stay the same if we allow for the possibility that search becomes faster with more variants.

Our results may not map to reality one for one. However, some features do ring a bell if we think about consumer search nowadays. For example, the number of products offered by Amazon.com in 2014 is estimated to be 253 million (up by 21 million) and, by Amazon.co.uk, Amazon.de, Amazon.fr and Amazon.co.jp respectively, 153 million, 141 million, 119 million and 108 million. These items are categorized into more than 35 departments, that all include massive numbers of listings which could be considered substitutes. It is clear from these numbers that, if you wanted to check out all the relevant listings in the product category that interests you and, say, spend an average of 15 seconds looking at each, this would keep you browsing at Amazon for a very long time – anyway, longer than people usually can afford. The same can also be said about Amazon’s main competitor, eBay, and about many other e-retailers like Walmart, Staples, Alibaba, Target, IKEA, Fnac etc. or their various local competitors.

Our paper is related to several bodies of literature. The closest, recent branch or literature analyzes an individual firm’s or a platform firm’s incentive to increase the search costs faced by consumers. This work often goes under headings like ”add-on pricing”, ”shrouded attributes”, ”strategic complexity”, ”obfuscation”, ”diversion”, ”confusion”, ”framing”, or ”hiding” (Ellison, 2005; Gabaix and Laibson, 2006; Carlin, 2009; Wilson, 2009).
With rational consumers these practices can help to relax price competition by aggravating the consumers’ holdup problem (Ellison and Wolitzky, 2012), or help in targeting the right consumer group and in price discriminating between different consumers (Petrikaite, 2015; Taylor, 2015; Gamp, 2015). In behavioral approaches, firms try to take advantage of various forms of consumer bias (Spiegler, 2006; Eliaz and Spiegler, 2011; Piccione and Spiegler, 2012; Chioveanu and Zhou, 2013).

Ireland (2007) considers a market with a search engine and allows firms to quote multiple prices for the same good. Consumers get a sample of prices, one or two each, but cannot see if they come from a single seller or from different, competing sellers. Therefore, to profit from this consumer confusion, every firm sends out two perfectly correlated duplicate prices to the search engine.

Carlin and Ederer (2014) develop a dynamic model of search fatigue. Search costs depend on how many products were sampled last period. They find that one equilibrium is characterized by cycles where, every second period, firms flood consumers with numerous cheap products to tire them out. The next period, these tired consumers go to a random firm paying the monopoly price.

Ellison and Wolitzky (2012) extend the sequential price search model by Stahl (1989) to allow for the idea that search costs are composed of both an exogenous part (time to open a website) and an endogenous part (time to find the price in there). They assume that search cost is convex in search time: the more you search, the more costly is the next search. This gives firms an incentive to obfuscate. They show that, if obfuscation does not cost, a decrease in exogenous search costs is totally offset by an increase in endogenous search costs.

Similar ideas feature also in this paper. We consider extreme convexity of search cost in search time: before the deadline, search cost equals zero but, after the deadline, it is infinite. We also find that the adverse effects to profits by an improvement in exogenous search technology can be undone by an increase in variants. The key difference is however that instead of directly adjusting consumers’ search time and search cost we let the firms manipulate them indirectly, through their pricing strategy and the number of product variants.

In an independent study, Menzio and Trachter (2015) construct an elegant price search

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6 These actions taken by firms, making it harder for consumers to access product information, could be taken as opposite to advertizing, where firms are providing information.

7 Our model can be regarded as rational or behavioral depending how one views the idea that consumers have a deadline and use it as a heuristic to ration their search time; they search dynamically rationally during their finite time horizon.

8 Some models like Ayres and Nalebuff (2003) also consider add-on pricing as a behavioral phenomenon from the part of sellers, who fail to acknowledge the full value of long term commitment in building a reputation of having low prices.

9 As an important modeling difference, Ellison and Wolitzky (2012) look at consumers’ stopping rule. We make stopping decisions trivial and look at consumers’ switching rule.
model with price dispersion within stores. In contrast to our model, where buyers differ only *ex post*, their buyers differ *ex ante*: in their ability to shop in different stores and at different times. Also they find that price dispersion helps firms to discriminate between different buyers. Otherwise, their setup is different and they do not consider the locking in effect that variety has during a single search spell. In their paper, a pre-requisite for intrafirm price dispersion is a particular correlation structure: buyers who can shop at uncomfortable times should also be more likely to shop far away.

Petrikaite (2015) has another noteworthy setting with explicit search frictions and price variation in-store; most other search models treat a firm as a black box. She considers a monopoly with several differentiated variants, which can be placed and priced individually. For instance, the most expensive ones can be placed at the entrance and the cheaper to more remote shelves. Indeed, she finds that the firm has an incentive to raise search costs of some variants to control the order in which consumers find them and to reveal it information about consumers’ earlier match values. That also enables the firm to cash on the search externalizes that greater variety offers to consumers.10

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10 The classic paper closest to ours is Salop (1977), which considers a monopolist who offers many prices to tax the consumers with higher search costs. In another classic paper, Wolinsky (1987) observes that a store can price discriminate by offering brand label products side by side with unlabeled ones. For our case, price variation works in particular through consumer lock-in. McAfee (1995), Shelegia (2012), and Rhodes (2014) consider optimal pricing of a bundle of products in a store. In their seminal article, Klemperer and Padilla (1997) show that stores can also have an incentive to provide excess variety if consumers appreciate more choice but like to patronize same providers. Our results work through different channels but, in a broad sense, generalize this inventory expansion incentive to homogenous commodities. The motive to expand inventory is present also in the “newsboy problem” (Mahajan and Van Ryzin, 2001).
This paper is structured followingly. The model is set up in Section 2 where we also lay out the optimality conditions. In Section 3 we consider different benchmark equilibria, those with just one product variant and those without intrafirm price dispersion. Our main findings and examples of obfuscation equilibria are presented in Section 4. Section 5 offers some closing remarks. Most proofs appear in Appendix.

2 Model

We construct a new deadline based search model to study duopolistic price competition with intrafirm search frictions. Both firms have several product variants with individual prices available in their store. Consumers find them gradually over time by searching in one of the stores at a time; they are busy and have to stop by a deadline. To fix ideas we could view this as a model of online search (see the illustration in Figure 1) with two competing retailers.\(^\text{11}\) In this setup each of the available product variants adds its own incremental time cost on searching consumers, who have to take a good look at each product to see if it passes the initial relevance screening and check its price. The deadline could either be fixed or random: we could either think that a consumer has a finite time budget or that she will inevitably get fed up with searching, at some point when she is hit by a stochastic preference shock. As a simplification, we postulate that consumers regard all the variants as perfect substitutes.\(^\text{12}\)

\(^{11}\)To give an idea about the kinds of intrafirm frictions there might be, see for example Pinna and Seiler (2015) and Reutskaja et al. (2011). Pinna and Seiler (2015) estimated from consumers’ walk path data that an additional minute of search in a grocery store lowers the category wise expenditures by § 2.1. This suggest the presence of intrafirm frictions and price variation across substitutes. Often there are many alternatives in the consideration set. Auchan in France offers, for instance, just ordinary milk, “lait demi-écritéé stérilisé UHT, 1 l”, at least, under names “GrandLait”, “la Vache au bon lait”, “J‘♥ le lait d’ici”, and its own basic brand; some of them are placed scattered around. Dreze et al. (1995) observe in a self management experiment that product location had a large impact on sales. Anupindi et al. (1998) find that consumers often switch brands if their favorite brand is unavailable. Reutskaja et al. (2011) designed an experimental setup to mimic the experience in a supermarket, where consumers are searching under heavy time pressure. They recorded the subjects eye fixations as they were sampling through different alternative snacks, that were presented to them. They found that the subjects were good at optimizing within the set of products they had time to look but not otherwise; it always took some time to fixate on an alternative.

\(^{12}\)Obviously, this is not to suggest that vertical or horizontal product differences would not matter for search. Our aim is rather to point out more elementary price based mechanisms, which might arise only as a side-product of deepening variety provision but affect search, prices, and profit all the same. It is to this aim that we assume this more abstract simpler approach. Nonetheless, we presume that consumer specific match values \(\nu \sim \Delta \{0, 1\}\) could be incorporated into the model without changing its essence because consumers would still have the same incentive to switch the stores after they find a discount price; the only difference would then be that consumers might prefer to switch back in the end, which might slightly affect firm profit in different scenarios we consider. Furthermore, there could also naturally be common values type (quality) differences between variants, which would then show up in that different variants would have different monopoly prices and costs for firms; this would only amount to a rescaling of prices. In practice there could obviously also be various superficial product differences, which consumers might not really care about: sports equipment stores could offer, say, white, blue, green, and purple striped sneakers with different textured laces etc. These differences might, however, give the firm a pretext to price individually these variants.
Game, strategies, and equilibria

There are two firms \( i \in \{1, 2\} \), each with at least two product variants \( k, n, j \in \{1, ..., K\} \), and a unit mass of consumers, each consumer with a unit of time for searching. The number of variants is exogenous and they all look the same to consumers: they do not really mind which one they buy, they only care about the price. All variants have the same unit costs for firms, that we normalize to zero. Every consumer demands exactly one variant, whose value to her is one.

The firms set a price to each variant they provide and consumers are then free to search for a unit of time, in one firm’s store at a time, making their buying decisions in the end; we describe the specifics of this in a moment.

We note that, depending on whose perspective is taken, the prices set by firm \( i \) can be indexed either according to their size order (the firm knows this), \( p^i_{k,j} \geq p^i_{k,j+1} \), or according to their finding order (a consumer sees this): \( p^i_{n,j} \) is found prior to \( p^i_{n,j+1} \), where the last variant that the consumer has found at firm \( i \) by time \( t \) is indexed by \( N^i_t \).

Payoffs are linear in prices: if a mass \( B_{\{p_k\}} \) of consumers purchases from a firm for price \( p_k \), the firm’s payoff is the weighted sum \( \sum_{k=1}^{K} B_{\{p_k\}} p_k \) whereas the payoff to a consumer who purchased a variant for \( p_n \) equals \( 1 - p_n \).

The simplest way to think about consumer behavior in this model is to say that all consumers like searching for a while and they all have the same deadline \( t = 1 \) by which they must stop it: for \( t \in [0, 1] \), their search costs are zero and, for \( t \notin [0, 1] \), infinite. It is costless to switch between firms; this seems quite reasonable in a mall with many stores or for online search. Thus, for any \( t \in [0, 1] \) a consumer simply decides whether to search at firm \( i = 1 \) or at firm \( i = 2 \).

The timeline is the following:

1. Firms set prices \( (p^i_{k,1}, ..., p^i_{k,K}) \) for all their \( K \geq 2 \) product variants. These prices remain unobservable to consumers until they find them.

2. Consumers search optimally for a unit of time. For every point in time \( t \leq 1 \), they can search either in firm \( i = 1 \)’s or in firm \( i = 2 \)’s store.

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13For convenience, firms are here male and consumers are female.
14As we suggested, the primitive of the model being the partition \( B \) (to be defined), it is also possible to let the deadline be random.
15They can thus recall any of the products without a further cost.
16While searching under a deadline like this can be regarded as a behavioral assumption, consumers search rationally during the time they have. Given the flat search cost, we have been advised that our model could be interpreted not only as a sequential search model but also as a non-sequential search model. Notice, however, that here consumers’ search choices are sequentially rational. Further, the right order of search is crucial to sustain our obfuscation equilibria.
17Technically, all products are of the same type but a firm could carry them in multiple replicas, tagging a different price quote on each.
18The number of product variants \( K \) is common knowledge but the prices are the firms’ private information until consumers find them.
3. By the deadline $t = 1$ consumers make their purchase decisions and payoffs realize.

The set of prices chosen by firm $i \in \{1, 2\}$ is denoted by

$$P^i = (p^i_{k_1}, \ldots, p^i_{k_K}) \in [0, 1]^K,$$

and the firm’s (mixed) strategy is given by

$$F^i = (F^i_{k_1}, \ldots, F^i_{k_K}) \in \Delta [0, 1]^K.$$

A (pure) strategy for a consumer is given by

$$\sigma(h_t) \in \{1, 2\},$$

where $h_t$ denotes search history, which lists all the prices she has seen by time $t$,

$$h_t = \left(1; p^1_{n_1}, \ldots, p^1_{N^1_1}; p^2_{n_1}, \ldots, p^2_{N^2_1}\right).$$

If the consumer has found no price by $t$, it is assumed without loss that $h_t = 1$.

This is an extensive form game with complete but imperfect information where a consumer has a dynamic optimization program to solve. We analyze symmetric perfect Bayesian equilibria where both firms use the same pricing strategy, $F^1 = F^2$, and the consumer strategy $\sigma$ is the same for all consumers.

**Intrafirm frictions**

As a new modeling approach, it is assumed that a consumer does not find immediately all the variants available in a given firm’s store, and hence their prices, but rather that they are found one by one, randomly and gradually. This could mean either that it takes time to walk and look around the store or click and scroll about the website to detect a new product variant or that it takes time to process relevant information as illustrated in Figure 1.

We model these intrafirm search frictions in this paper with a (continuous time) counting process like a (continuous time) Poisson process. The primitive of our model is a partition of consumers

$$B := \{B_N\}_{N=0}^\infty,$$

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19Unless otherwise specified, it is assumed that beliefs are passive: they are the same on the equilibrium path and off the equilibrium path. This is very standard in consumer search models. Here this does not have much bite, though. As will be shown all prices $[p, 1]$ are on the path and since any price $p < p$ beats all other prices it would not really matter how consumers would search after discovering such a very low price; they would anyway end buying for it $p < p$. 

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where $B_N$ is the mass of consumers who manages to discover a total of $N = N_1^1 + N_1^2$ variants

$$\sum_{N=0}^{\infty} B_N = 1.$$  

For concreteness, we next describe the details of consumer search for the special case where $B$ comes from a Poisson process. However, these ideas generalize immediately to any other counting process.\(^{20}\)

For the Poisson case, the probability that a consumer observes an unseen variant within the next short time interval $dt > 0$ is fixed and given by $\theta dt$.\(^{21}\) We assume that the rate $\theta$ is the same for both firms $i$ and for different variants as long as there is something new to find in each, i.e. $N_i^1 < K$.\(^{22}\) Specifically, as a consumer keeps searching in a given store, the available variants are drawn in random order without replacement each time the Poisson shock hits, until nothing remains or the consumer switches.

For example, when firm $i$ has two variants, a consumer who starts to search in its store finds the first variant at rate $\theta$. For probability $1/2$ it is the (weakly) higher price, $p_{i1}^1$, and for probability $1/2$ the (weakly) lower price, $p_{i2}^1$. If the consumer then continues with firm $i$, she finds the other variant also at rate $\theta$. If the first price was $p_{i1}^1$, the second one is now $p_{i2}^1$, or the other way around. Then she is done with this firm. The consumer can discontinue or recommence this finding process by switching at any point.

To nail it down technically, when a consumer is searching in a given store, she could see the available variants in any order: all permutations \( n : k_j \mapsto n_j(k_j) \) (from the size index to the finding order index) are equally likely.\(^{23}\) One way to think of this is to say that the available variants are randomly scattered all over the store. As a motivation, note that, if the firm could have its way, the finding order would be the size order \( n = \text{id} \) whereas, if the consumer dictated it, it would be the reverse \( n = K + 1 - \text{id} \).\(^{24}\) Therefore, because of this conflict of interest, a random finding order seems like a reasonable prediction.\(^{25}\)

It is noteworthy that by the end, independent of how they search, the mass of consumers who has found $N = N_1^1 + N_1^2$ variants is\(^{26}\)

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\(^{20}\)A counting process \( \{N_t | t \in [0,1]\} \) is defined by: (i) $N_t \geq 0$, (ii) $N_t$ is an integer, and (iii) if $s \leq t$, then $N_s \leq N_t$.

\(^{21}\)Her probability of discovering more than one price at a time is an event of order \( (dt)^2 \) or smaller, i.e., negligible.

\(^{22}\)In the base line model, this rate $\theta$ at which buyers learn new price information is independent of the number of items in stock $K$. In an extension, we allow for the possibility that search becomes easier (harder) with a larger number of items in stock and let $\theta$ be an increasing (decreasing) function of $K$.

\(^{23}\)A permutation of set \( \{1, \ldots, K\} \) is a bijection \( \{1, \ldots, K\} \to \{1, \ldots, K\} \).

\(^{24}\)Here $\text{id}$ denotes the identity function $\text{id}(k) = k$ for all $k \in \{1, \ldots, K\}$.

\(^{25}\)We could think that a firm and its consumers are playing matching pennies in terms of where to place the cheapest variants and where to seek for them inside the store.

\(^{26}\)For a random impatience shock, hitting the consumer at time $s \sim E(\eta)$, the same model would
\[ B_N(\theta) = \frac{\theta^N}{N!} e^{-\theta}. \]

Search only affects where the consumer finds those \( N \) variants but not their total number (unless the consumer stops early).\(^{27}\)

This property is very helpful in the analysis, which could otherwise become prohibitively complex. It also enables us to move easily from the Poisson model \( B(\theta) = \{B_N(\theta)\}_{N=0}^\infty \) where \( B_N(\theta) = \frac{\theta^N}{N!} e^{-\theta} \) to a more general model \( B(\theta) = \{B_N(\theta)\}_{N=0}^\infty \) with arbitrary \( B_N \) adding up to one. Moreover, to discuss the effects of improving search technology, our formulation allows us to compare different models as follows:

**Definition 1** (Ranking search technologies) General setting: Search technology \( B = \{B_N\}_N \) is faster than search technology \( B' = \{B'_N\}_N \) if
\[
\sum_{N=N'}^\infty B_N > \sum_{N=N'}^\infty B'_N, \text{ for all } N' > 0.
\]

Poisson setting: Search technology \( B(\theta) = \{B_N(\theta)\}_N \) with \( \theta \) is faster than search technology \( B'(\theta') = \{B'_N(\theta')\}_N \) with \( \theta' \) if \( \theta > \theta' \) because
\[
\sum_{N=N'}^\infty B_N(\theta) > \sum_{N=N'}^\infty B_N(\theta'), \text{ for all } N' > 0.
\]

We also make the following assumption to deal with ties:

**Assumption 1** (i) If consumers are indifferent between searching within either store at time \( t = 0 \), half go to firm \( i = 1 \) and half go to firm \( i = 2 \). (ii) If consumers are indifferent between switching or not switching at some time \( t \in (0, 1) \), they do not switch. (iii) If consumers are indifferent between searching and not searching at some time \( t \in (0, 1) \), they do not search. (iv) If consumers are indifferent between purchasing from either firm at time \( t = 1 \), they buy from the firm where they were searching last time.

\(^{27}\)We can hence think of this game equivalently as the one where the nature draws for every consumer first the order in which the she discovers the variants available in each firm, \( n := (n^1(k), n^2(k))_{k=1}^K \), and, then, the number of variants that she will ultimately find, \( N = N^1 + N^2 \), and the exact finding times, \( t_1, \ldots, t_n \). (Note that the exact finding times \( t_1, \ldots, t_n \) do not carry any relevant information in this game: only their total \( N \) and the permutation \( n \) representing their finding orders matter.) The probability of some fixed pair of \( (n, N) \) is \( \frac{B_N(\theta)}{2^N} \). Initially, the consumer obviously does not see \( n \) nor \( N \) but she makes inferences about them along the way because they directly affect her search history; \( h_t \). Her optimal search behavior then determines, from which firm she finds those \( N \) variants she is to find, i.e., the decomposition of \( N \) into \( N^1 \) and \( N^2 \).
Optimality conditions

This part analyzes general optimality conditions that have to be met in an equilibrium; a busy reader can just skim this part for notation. In an equilibrium, it is required that consumers and firms’ strategies are best responses given beliefs and these beliefs are consistent with strategies.

To express this clearly for our model, it is useful to define some notation: \( \min A^i \) denotes the minimum price in set \( A^i \), and \( F_{\min A^i} \) denotes its marginal distribution; Also, each price set \( A \subset P^1 \cup P^2 \) can be partitioned into firm \( i = 1 \)'s prices, \( A^1 \subset P^1 \), and firm \( i = 2 \)'s prices, \( A^2 \subset P^2 \).

For this general part we do not use symmetry: all goes through even for \( F^1 \neq F^2 \).

Consumer problem

Consumers search optimally. They know the number of variants \( K \) and the underlying search technology \( B \), and they have some expectations about the price distributions \( F^1 \) and \( F^2 \). In an equilibrium, it is required that these beliefs are correct. As variants are sampled randomly without replacement, a consumer also updates her beliefs about the remaining prices each time a new price is found. In which firm’s store a consumer decides to search at time \( t \in [0,1] \) can thereby depend on both the expected prices \( F := (F^1, F^2) \) and on the realized prices \( P := (P^1, P^2) \) – through search history \( h_t \).

In a Poisson setting, for instance, consumer problem can be represented by the following Bellman equation

\[
V(h_t) := \max_i V^i(h_t) := \max_i \left( \theta dt E_{h_{t+dt}} \left[ V^i(h'_{t+dt}) \| h_t \right] + (1 - \theta dt) V(h_{t+dt}) \right),
\]

where \( V^i \) is her value of searching in firm \( i \) at time \( t \). The terminal condition for this consumer problem is

\[
V(h_1) = 1 - \| -h_1 \|_\infty,
\]

which simply restates the idea that a consumer buys for the best price she finds by the deadline \( t = 1; \| h_t \|_\infty \) gives the element-wise maximum-norm of \( h_t \).

By the principle of optimality, the problem has a solution for any \( F, P, K \) and \( B(\theta) \) as long as the expectations \( E_{h_{t+dt}} \left[ V^i(h'_{t+dt}) \| h_t \right] \) are well-defined in an appropriate measurable space for any history \( h_t \) (on-path and off-path). Therefore, assuming for the time being this is so, any \( F, P, K \) and \( B(\theta) \) uniquely partition the unit mass of consumers as

\[
\sum_{A \in 2^K} B_A(F, P; K) = 1,
\]

where \( B_A \) gives the mass of consumers ending with prices \( A \).
Firm problem
To take firm i’s point of view, we can now integrate out the effects of other firm’s prices $P^{-i}$ to concentrate on the effects of its own prices $P^i$:

$$B_A(F, P^i) = \int B_A(F, P)dP^{-i}(P^{-i}).$$

With this notation, if the functions $F^1$ and $F^2$ are continuous, the profit to firm i can now be decomposed as

$$\Pi^i(F, P^i; K, B) = \sum_{A \in \{0\}^K} B_A(F, P^i; K) \left(1 - F_{\min A^{-i}}(\min A^i)\right) \min A^i,$$

where it is assumed that $\min A > 1$ and $F_{\min A} = 0$ if $A = \emptyset$. Consumers $B_A$ who found prices $A$ buy for $p = \min A$. Thus, they purchase from firm $i$ if its lowest price within $A$ is lower than firm $-i$’s lowest price within $A$, i.e., $\min A^i \leq \min A^{-i}$.

Therefore, to constitute an equilibrium, for any prices a firm is using $P^i \in \text{supp}(F^i)$ and for any prices the firm could be using $P'^i \in [0, 1]$, it should hold that the firm’s profit is not higher in expectation for the latter than for former

$$\Pi^i(F, P^i; K, B) \geq \Pi^i(F, P'^i; K, B).$$

Otherwise, the firm has a profitable deviation.

Endogenous shoppers and searchers
Note especially that, instead of the standard exogenous partition, in our case the consumers are partitioned endogenously into several distinct subsets $B_A$, based on which prices they find. In Varian (1980) and Stahl (1989) the set of consumer consists, respectively, of (i) informed consumers who have seen all prices (here $B_{2K} + B_{2K+1} + \ldots$) or ”shoppers”, who have non-positive search costs, and (ii) uninformed consumers who have seen just one price (here $B_1$) or ”searchers”, who have positive search costs.

Sets $B_N$ in the partition $B = \{B_N\}_{N=0}^\infty$ correspond with the various degrees of price information that consumers could have in this model. The key distinction is that in our deadline based model consumers search costlessly for a while. In contrast to the earlier

28We assume the standard notation: $-i = 1 + T(i = 1)$, where $T$ is simply a truth function.

29If $F^i$ are not continuous, we have to specify how ties are broken. If ties are broken in favor of firm $i$,

$$\Pi^i(F, P^i; K, B) = \sum_{A \in \{0\}^K} B_A(F, P^i; K) \left(1 - F_{\min A^{-i}}(\min A^i)\right) \min A^i$$

$$+ \sum_{A \in \{0\}^K} B_A(F, P^i; K) \Pr\left(\min A^i = \min A^{-i}\right) \min A^i.$$

30Assume $f$ is the density function related to $F$. Then, the support of probability distribution $F$ is $\text{supp}(F) = \text{cl}\{p| f(p) > 0\}$; closure $\text{cl}$ denotes the smallest closed set which contains $\{p| f(p) > 0\}$. 12
setups, there are thus also some partially informed consumers, who have seen more than one price but less than all prices \( (B_2+B_3,...,B_{2K-1}) \). In addition, there are some unlucky consumers who do not manage to find even one price \( B_0 \).

We observe that some of the usual properties of equilibrium price distribution nevertheless continue holding, although generally in a weak form:

**Lemma 1**

1. Assume that consumers search in both stores with non zero probability. Then, each firm uses a mixed pricing strategy for \( p^i_{kk} \) and randomizes it over the same support, \( D := \text{supp}(F^i_{kk}) \subset [0,1] \).

2. Suppose in addition that consumers switch away from store \( i \) after finding \( p^i_{kj} \) weakly more often than after finding \( p^i_{kj}' \) if \( j < j' \). Then, the support is the interval \( D = [p, \overline{p}] \subset [0,1] \), where \( p \in (0,1) \) and \( \overline{p} = 1 \).

3. There are no atoms in the interior of \( D \) nor at the lower bound of \( D \).

**Notation**

Now that the general setting is defined, we introduce the following simplifications to be applied repeatedly throughout this paper: Since most action occurs here for the lowest prices \( p_{kk}^i \sim F_{kk}^i \) it is convenient to use their own, distinct notation for them: \( q^i := p_{kk}^i \) and \( G^i := F_{kk}^i \). Additionally, it will be necessary in the following analysis to distinguish between **monopoly prices** \( p = 1 \), and **discount prices** \( p < 1 \). We denote by \( a := \text{Pr}(q = 1) \) the point probability for which \( q \) is a monopoly price and denote by \( H := \frac{G}{1-a} \) its conditional distribution given the event that \( q \) is a discount price. Thus, \( H(p) = \text{Pr}(q < p | q < 1) \). We also often refer to a firms’ profit \( \Pi(F, P^i; K, B) \) simply by \( \Pi \), dropping the superindex and arguments.

### 3 Benchmarks

The existence of an equilibrium is never an issue with this model. As Benchmark I, we next go through the case where both firms have one product variant. There exists an equilibrium (basically similar to Diamond (1971)) where consumers do not search, a continuum of fixed price equilibria, and an equilibrium (essentially identical to Varian (1980) or Stahl (1989)) where consumers search. As Benchmark II, we then show that these kinds of equilibria exist also when firms have more than one product variant (then the last equilibrium type resembles that by Ireland (2007)). However, with many variants there generally exist also other equilibria, which yield more profit. Analyzing such equilibria in Section 4 is the main focus of this paper.
Benchmark I: the stay-home equilibrium and the search equilibrium with one product variant

Remark 1 (Diamond, 1971; Varian, 1980; Stahl, 1989) (i) If each firm has one product variant, there exists a (symmetric) stay-home equilibrium where all prices equal unity, $q = 1$.

(ii) If each firm has one product variant, there exists a continuum of (symmetric) search equilibria where all prices equal, $q \in \left[\frac{B_1}{1-B_0}, 1\right]$.

(iii) If each firm has one product variant, there exists a (symmetric) search equilibrium where the equilibrium price distribution is given by

$$G : \left[\frac{C}{C+S}, 1\right] \rightarrow [0, 1], G(q) = 1 + \frac{C}{S} \frac{\Pi}{q}.$$

A firm’s profit is there equal to $\Pi = C = \frac{\theta e^{-\theta}}{2}$ such that $\Pi(1, \theta) \rightarrow 0$ as $\theta \rightarrow 0$ or $\theta \rightarrow 0$; the maximal profit $\Pi(1, \theta) = \frac{1}{2}$ is attained at $\theta = 1$.

$$\text{supp}(G) = \left[\frac{C}{C+S}, 1\right] = \left[\frac{\theta e^{-\theta}}{1-e^{-\theta}}, 1\right] \rightarrow \begin{cases} [0, 1], & \text{as } \theta \rightarrow \infty \\ \{1\}, & \text{as } \theta \rightarrow 0 \end{cases}$$

The existence of the first two equilibria (i) and (ii) hinges solely on Assumption 1 (iii); any relaxation of this or an introduction of a tiny positive mass of shoppers would eliminate these equilibria. These fixed price equilibria are sustainable only because consumers stop their search early. They think that there are no gains from searching more.

For the last more robust equilibrium (iii), note that, if the firms use symmetric pricing strategies, by Assumption 1 (i) and (ii), searching consumers approach one of the firms in random. Then, they search in this firm’s store until they find its price and thereafter in the other firm’s store until they find its price – or until no time is left.

This entails that, when it is time to stop by the deadline $t = 1$,

$$C = C^1 := \frac{B_1}{2} = \frac{\theta e^{-\theta}}{2}$$

consumers have found only price $p^1$ of seller $i = 1$,

$$C = C^2 := \frac{B_1}{2} = \frac{\theta e^{-\theta}}{2}$$

---

31There is no profitable deviation from $q$ up to unity as long as $\frac{1-B_0}{2} q \geq \frac{B_1}{2} q$ because raising the price $q$ will make the consumers continue to the other store so that, instead of selling to half the consumers who find something, $\frac{1-B_0}{2}$, the firm only sells to consumers who solely find its own price, $\frac{B_1}{2}$.

32With passive beliefs off the path and one price per one store, what matters for search is expected prices not realized prices. The discovery of the first price does not give any additional information on the remaining price in the competing store – their strategies are independent in equilibrium.
consumers have found only price \( p^2 \) of seller \( i = 2 \), and

\[
F := B_0 = e^{-\theta}
\]

frustrated consumers have found neither price. The residual

\[
S := \sum_{N=2}^{\infty} B_N = 1 - F - 2C
\]

of the consumers has found both of them.

To summarize, consumers are here partitioned into "captives" \( 2C = C^1 + C^2 \) and "shoppers" \( S \) much like in Varian (1980) and Stahl (1989). As usual there is price competition only over shoppers not captives. Additionally, there are here also some "frustrated" consumers, \( F \), who fail to find any price prior to the deadline. They were absent from both Varian (1980) and Stahl (1989). Another novelty is that our partition can be parametrized conveniently by the Poisson process to consider the effects of faster search technology. For example, it is easy to see that the stronger the frictions (lower \( \theta \)), the higher the ratio of captives over shoppers \( \frac{C}{S} \) and the smaller the number of frustrated consumers \( F \). This implies that, as in Stahl (1989), it is possible also in this case to span continuously from Diamond-like monopoly pricing (obtains for extremely high frictions, \( \theta \to 0 \)) to Bertrand-like marginal cost pricing (obtains for extremely low frictions, \( \theta \to \infty \)) by varying \( \theta \). Still, firms’ profits are maximal for intermediate levels of search frictions because the number of captives \( C \) is the largest for \( \theta = 1 \). Overall, this equilibrium is nevertheless essentially equivalent with those in Varian (1980) and Stahl (1989).

**Benchmark II: Equilibria with multiple variants but without intrafirm price dispersion**

We next provide three examples where all product variants in a store have the same uniform price. Remarks 2, 3, and 4 show that, while the focus of this paper lies on equilibrium price dispersion within stores and across stores and its effects, it is possible to maintain also equilibria with multiple variants but (i) no price variation what so ever (Remark 2 and Remark 3) and (ii) price variation across stores but not within stores (Remark 4).

**Remark 2** (Diamond, 1971) *For any \( K > 1 \), there exists a stay-home equilibrium, where all variants in a store have the monopoly price, \( p^1_k = ... = p^1_{kK} = 1 \); consumers do not search at all.*

Again, this is the Diamond (1971) equilibrium essentially: it is the famous impossibility result, which basically shows the non-existence of an equilibrium with costly sequential search and endogenous price dispersion in homogenous environments.

The idea behind that result is simple. If additional price information is costly and
products are identical, then firms have an incentive to exploit the consumer hold-up problem, that arises when it costs to visit another firm, by raising their price slightly above their competitor’s price. Now, since all firms have this same incentive, the monopoly price is the unique equilibrium price irrespective of the number of firms in the market. Consumers thus refuse to search, because they would gain nothing from it.

Note that the hold-up problem appears here in a weak form only since it is costless to search before the deadline. Thus, the existence of this type of equilibrium hinges solely on Assumption 1 (iii). The proof for Remark 1 (i) is valid as such here.

**Remark 3** For any \( K > 1 \), there exists a continuum of search equilibria, where all variants in a store have the same fixed discount price, \( p_{k_1}^i = \ldots = p_{k_K}^i \in \left[ \frac{B_1}{1-B_0}, 1 \right] \); consumers search (at most) once, until they find one firm’s price.

This is the multiproduct counterpart of Remark 1 (ii). The existence relies as before on Assumption 1 (iii).

**Remark 4** (Ireland, 2007) For any \( K > 1 \), there exists a search equilibrium, where all variants in a store have the same random discount price, \( p_{k_1}^i = \ldots = p_{k_K}^i \sim G \) (below); consumers search (at most) twice, until they find both firms’ prices.

Again, the equilibrium price distribution is

\[
G : \left[ \frac{C}{C+S}, 1 \right] \to [0,1], G(q) = 1 + \frac{C}{S} \frac{\Pi 1}{S q},
\]

where a firm’s profit equals \( \Pi = C = \frac{B_1}{2} \).

This equilibrium is reminiscent of the standard case with just one variant we presented as Remark 1 (iii). It also looks like the equilibrium in Ireland (2007), where firms use a multitude of identical prices, that we discussed in Section 1.

As for Remarks 2 and 3, the existence of the equilibrium can be proved easily by reference to Assumption 1 (iii): Clearly, no consumer has an incentive to stay to find another variant in a given store if there are uniform prices within stores. Moreover, no firm has an incentive to charge different prices for its variants if consumers only search for one price for one store.

This equilibrium is no more robust than the earlier ones, however. It fails to exist if we relax Assumption 1 (iii) because the firm has then an incentive to introduce price variation to compete more fiercely over repeat customers.\(^{33}\)

We thus move on to analyze specific equilibria with intrafirm price dispersion. Before setting off, however, observe first that while randomizing all the prices independently

\(^{33}\)Suppose that firm \( i \) deviates to two prices \( p_1^i > p_2^i \in \left[ \frac{C}{C+S}, 1 \right] \). We know that equilibrium profit is

\[
2\Pi = B_1 p_1^i + \sum_{N=2}^{\infty} B_N (1 - G(p_1^i)) p_1^i = B_1 p_1^i + \sum_{N=2}^{\infty} B_N (1 - G(p_1^i)) p_1^i,
\]

whereas the deviation yields the firm strictly more
might seem like a good idea for the firms, because then consumers would search in every store until they have found all the product variants, it is an equilibrium only in some knife edge cases. Note also that, while Assumption 1 (iii) has so far had some bite, in the equilibrium class we study, it is more innocuous; this is so especially for the limiting cases where most consumers use all their time in their first store and the deadline is usually binding. We retain this assumption here mainly so that we avoid keeping track of consumers who loiter in the stores after serious search only because they have time.

4 Obfuscation equilibrium

In this section we analyze generally an interesting equilibrium class named obfuscation equilibria (OE).

Definition 2 (Obfuscation equilibrium (OE)) In an obfuscation equilibrium with $K$ variants $(OE(K))$:

1. (Search) Consumers start from a random firm and search in that firm’s store either until they find a price $p < 1$ or until they have found $K$ prices such that $p = 1$ for all; then they switch to the other firm’s store.

2. (Prices) With probability $1 - a > 0$ a firm has one (discount) price $p < 1$ and $K - 1$ (monopoly) prices such that $p = 1$ for all whereas with probability $a < 1$ the firm has $K$ (monopoly) prices such that $p = 1$ for all.

This combination of prices is particularly advantageous to firms. Specifically, as consumers know that each firm has one discount price with non zero probability, if they first spot a monopoly price, they have an incentive to keep on searching in their start store in hope of finding there another price at a discount. This locking effect, that postpones consumers’ switching away from their start store, lowers a firm’s incentive to undercut the other firm’s price relative to the usual setup. Competition becomes thus more relaxed.

$2\Pi' = 1/2B_1p_1^1 + 1/2B_2(1 - G(p_1^1))p_1^1 + 1/2B_1p_2^2 + 1/2B_2(1 - G(p_2^2))p_2^2 + \sum_{N=2}^{\infty} B_N(1 - G(p_2^2))p_2^2,$

as $\sum_{N=2}^{\infty} B_N(1 - G(p_2^2))p_2^2 > \sum_{N=2}^{\infty} B_N(1 - G(p_1^1))p_1^1$. For the case $K = 2$ it for instance requires that $B_1(1 - p_1^1) = 1/2B_3((1 - F_2^{-1}(p_2^2))p_2^2 - (1 - F_2^{-1}(p_1^1))p_1^1)$ for all $(p_1^1, p_2^2) \in supp(F)$ because, otherwise, the firm has a profitable deviation in its higher price $p_1^1$ to unity or to its lower price $p_2^2$ (details upon request); the first case looks a bit like a move towards an obfuscation equilibrium.

Repeat customers cannot find anything new on the second round because, in obfuscation equilibria, firms do not want to use multiple discount prices as that would make consumers switch too early. In addition, due to the linearity of the firm’s problem there is no reason for firms to offer two separate discount prices.
Also, the generated price variation within stores helps the firms to price discriminate better across different consumers, some of whom are better informed about prices than others. Although consumers are similar *ex ante*, their search outcomes differ *ex post* due to the random arrival of price information. Additional price instruments are thereby useful to firms who can use their monopoly prices to tax the least informed consumers while keeping a discount price available as well to compete for the more informed consumers.

We next show that the depicted consumer behavior is indeed a best response to firms pricing behavior.

**Theorem 1** (Optimal search in obfuscation equilibrium) *If firms price their variants as in Definition 2.2, then consumers optimally search as in Definition 2.1.*

The proof of this amounts to showing that, if a consumer has found more monopoly prices in store $i$ than in store $-i$, but not one discount price, her chances of finding one discount price are higher if she continues searching in store $i$ rather than switches into store $-i$. Her chances of finding two discount prices are, instead, independent of the order in which she searches, as long as the consumer switches after the first discount price is observed. This is so because she then anyway has to find a discount price from each store. The consumer can find either zero, one or two discount prices in total.

The intuition for this result is that, with a fixed number of variants in a store, at most one with a discount price, consumers become more and more optimistic about remaining prices for every monopoly price they find. This makes them continue in their chosen store until they indeed find a discount price or until they have found all there is available in the store. They revise their beliefs about remaining prices (in some given store) up each time they find a monopoly price (in that given store).

We next display a necessary and sufficient condition under which the described pricing behavior is the best reply to the other firm and consumers’ strategies. There the relevant question is whether a firm would have a profitable deviation to covertly include more than one discount price. Obviously, this would have different effects on consumers who start from this firm and on consumers who start from the other one. The condition we next show requires that the profit loss from captives who start from our firm and the other firm is larger than the profit gain from shoppers starting from the other one:

\[
|\Delta C_m^i| (1 - p) + |\Delta C_m^{-i}| (1 - p) \geq |\Delta S_d^{-i}| p
\]  

where $|\Delta C_m^i| = (C_m^i(d = 1) - C_m^i(d = d'))$ is the reduction in monopoly price paying consumers starting from our firm, $|\Delta C_m^{-i}| = (C_m^{-i}(d = 1) - C_m^{-i}(d = d'))$ is the decrease in monopoly price paying consumers starting from the other firm, and $|\Delta S_d^{-i}| = (S_d^{-i}(d = d') - S_d^{-i}(d = 1))$ is the increase in the discount price paying shoppers starting from the other firm when firm $i$ adds $d = d'$ more discount prices (e.g., raises the number of discount prices from $d = 1$ to $d = 2$). This (necessary an sufficient) condition for an OE to exist...
ought to hold for all }d^{i} \in \{2, \ldots, K\} .{36}

Here the key point to observe is that finding one of the discount prices becomes easier if there are more of them available in a given store. Since consumers switch immediately after they find the first discount price, this implies that consumers who start from our firm’s store switch faster to the other firm’s store. This clearly does not benefit our firm in any way. It just reduces the number of captives for this firm and exposes it earlier to price competition.

Still, offering more discount prices also entails that consumers starting from the other firm’s store find a discount price faster after switching to our firm’s store. If they have so far seen only monopoly prices, this again only harms our firm. By Assumption 1, these consumers would buy from our firm anyway as their last store. However, if the consumers have already found the other firm’s discount price, it helps to start the competition as soon as possible.

Condition (2) also captures our finding that the best possible deviation is to change some monopoly prices into the lowest price }p' = p .{ A comparison to the case with just one discount price }p = p makes it clear that the firm may not actually lose any consumers by the deviation: All consumers who start from its store or come there after finding only monopoly prices still purchase from the firm. The difference is only in that some of them now pay }p' = p instead of }p = 1 . The firm also gains some more demand from shoppers who now discover it discount price }p' = p earlier than before. The deviation is not profitable if the former negative effect dominates.

**Theorem 2** (Optimal prices in obfuscation equilibrium) If (2) holds and consumers are searching as in Definition 2.1, then firms optimally price as in Definition 2.2.

It is now possible to derive the equilibrium price distribution in closed form using essentially the same procedure as in the proof for Remark 1 (iii). In calculating it, we just have to take into account that, instead of just }C and }S, firm }i’s demand is coming from several different consumer groups:

- }C_{m}^{i} demand from (captive-like) consumers who start from store }i = 1, 2 and find no discount price,
- }C_{d}^{i} demand from (captive-like) consumers who start from store }i = 1, 2 and find a discount price only from store }i,
- }S_{d}^{i} demand from (shopper-like) consumers who start from store }i = 1, 2 and find a discount price from both stores }i and }−i.

It is natural to derive the demand of firm }i from these various demand sources by moving along consumers’ search paths (see Table 1 below). For example, if firm }i has }K

---

36It appears that a single-crossing condition does not hold generally so that we cannot simplify (2).
monopoly prices, consumers who start from its store are \( C_m \) until they switch after finding those \( K \) monopoly prices. Thereafter, the firm loses them by Assumption 1. (Row 1 in Table 1.) If firm \( i \) has \( K-1 \) monopoly prices and a discount price, consumers are \( C_m \) after finding a monopoly price and \( C_d \) after finding a discount price. They stay that way after switching until they find also firm \(-i\)'s discount price, which turns them into \( S_d \). (Row 2 in Table 1.) If the other firm \(-i\) has only monopoly prices, consumers who switch from there are by Assumption 1 immediately firm \( i \)'s captives. They become \( C_{-i} \) when they find firm \( i \)'s monopoly price and \( C_{-i} \) when they find firm \( i \)'s discount price. (Row 3 in Table 1.) If the other firm \(-i\) has also a discount price, consumers who switch from there become \( S_{-i} \) only if they manage to find also firm \( i \)'s discount price; before, they are firm \(-i\)'s captives. (Row 1 in Table 1.) As the parentheses are suggesting, these demands of firm \( i \) from different consumers can be read from left to right from Table 1; deriving them from the primitives \( K \) and \( \{B_N\}_{N=0}^{\infty} \) in Appendix is a taxing but rather straightforward combinatorial exercise:

<table>
<thead>
<tr>
<th>Consumer's start store</th>
<th>Finds in the start store: only ( p = 1 ) also ( p &lt; 1 )</th>
<th>Finds in the next one: only ( p = 1 ) also ( p &lt; 1 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( i ) with ( q = 1 )</td>
<td>( C^m )</td>
<td>- ( 0 ) ( 0 )</td>
</tr>
<tr>
<td>( i ) with ( q &lt; 1 )</td>
<td>( C^m ) ( C^i )</td>
<td>( S^i ) ( S_d )</td>
</tr>
<tr>
<td>( -i ) with ( q = 1 )</td>
<td>0</td>
<td>- ( C^m ) ( C_d )</td>
</tr>
<tr>
<td>( -i ) with ( q &lt; 1 )</td>
<td>0</td>
<td>0 ( 0 ) ( S_d )</td>
</tr>
</tbody>
</table>

Table 1: The demands of firm \( i \) from different consumers.

**Proposition 1** If there exists a OE with some fixed \( K \) and \( \{B_N\}_{N=0}^{\infty} \), then the equilibrium price distribution can be expressed as

\[
H : [p,1] \rightarrow [0,1], H(p) = Pr(q < p | q < 1) = 1 + \frac{(C^1_d + C^2_d)}{(S^1_d + S^2_d)} - \frac{\Pi - (C^1_m + C^2_m) 1}{p},
\]

\[
p = \frac{(C^1_d + C^2_d)}{(C^1_d + C^2_d + S^1_d + S^2_d)},
\]

\[
a = Pr(q = 1) = \max \left\{ \frac{\sum_{N=1}^{\infty} B_N \left( \sum_{p=1}^{\min(N,K)} \frac{1}{K} \min \left\{ \frac{N-p}{K}, 1 \right\} - T(N > K) \right)}{\sum_{N=1}^{\infty} B_N \left( \sum_{p=1}^{\min(N,K)} \frac{1}{K} \min \left\{ \frac{N-p}{K}, 1 \right\} \right)}, 0 \right\}
\]

and the equilibrium profit for firms is given by

\[
\Pi = C^1_m + C^1_d + C^2_m + C^2_d > \frac{B_1}{2}
\]

where the numbers of captives and shoppers are shown in Table 3 in the end of Appendix.
Moreover, if there exists an OE for $K^1$ and $K^2$ where $K^1 < K^2$, then $\Pi(K^1) < \Pi(K^2)$.

To sum up, the pricing pattern in an OE is wired so as to (i) alleviate price competition by locking in searching consumers and (ii) to price discriminate more effectively between different groups of consumers. Both effects are stronger as the number of variants increases. This helps to raise the firms’ profits. Benchmark equilibria in Remarks 2 and 4 provide a natural point of comparison. We elaborate this reasoning more in the context of the example in Section 4.1.

4.1 Example: OE with $K = 2$ in Poisson setting

As an illustration, we now show that in a Poisson setting there exists a $OE(2)$, where the firms may sometimes have two monopoly prices and, at other times, one monopoly price and one discount price. In the first case, we say that the firm is in the hi-hi regime and, in the second case, we say the firm is in the hi-lo regime. The size of this discount is random, $q \sim G$ and $supp(G) = [p, 1]$. \(^{37}\)

A noteworthy consequence of the hi-lo pricing pattern is that, if a consumer first finds a firm’s monopoly price, she optimally continues with this firm. She is then closer to finding a discount price in her start store than in the other firm’s store. Instead, if the consumer first finds a discount price, she obviously switches immediately in an attempt to find also the competing firm’s discount price.

This makes using monopoly prices rather valuable to a firm. It helps to delay switching. Indeed, the benefit from this could be so large that the firms have an incentive to sometimes use two monopoly prices (in hi-hi regime) instead of just always having one monopoly price and one discount price (in hi-lo regime). As a result, the equilibrium price distribution $G$ could have an atom $a > 0$ at the upperbound.

Specifically, denoting by $a$ the probability that a seller is in the hi-hi regime and by $b = 1 - a$ the probability a seller is in the hi-lo regime, the chances that a consumer will switch the store after finding one price are $b/2 < 1/2$ whereas the chances that a consumer will instead switch the store only after finding two prices are $1 - b/2 > 1/2$. As shown in Figure 2.a, the expected switching time could thus be significantly delayed compared to the case of one price for one store. This demonstrates that price variation within stores acts here as an implicit switching barrier.

After the consumer has switched, the process of finding another, competing discount price is also postponed. Namely, the probability that a consumer has observed two discount prices by $t$ can be expressed as

\(^{37}\)Note that, even when Poisson $OE(2)$ fails to exist because there is a profitable deviation to two discount prices, there may exist a more complex variant of an OE, where this would not be a deviation. There could indeed be three “regimes” over which firms would mix symmetrically: in hi-hi regime, they would have two monopoly prices, in hi-lo regime, one monopoly price and one discount price and, in lo-lo regime, two discount prices. The details appear in the author’s thesis.
\[ b^2 \left( \frac{1}{2} \right)^2 \frac{(\theta t)^2}{2!} e^{-\theta t} + b^2 \left( \frac{1}{2} \right)^2 + 2 \left( \frac{1}{2} \right)^2 \frac{(\theta t)^3}{3!} e^{-\theta t} + b^2 \sum_{N=4}^{\infty} \frac{(\theta t)^N}{N!} e^{-\theta t} \]

compared to \( 1 - e^{-\theta t} - (\theta t)e^{-\theta t} \) for the cases in Remarks 2 and 4 where there is just one price for one store and no intrafirm price dispersion. This consequence of a locking effect is illustrated by Figure 2.b.

Furthermore, additional prices enable the firms to price discriminate more effectively across different consumer groups, some of whom end having more information about prices than others. The expected prices that are paid by consumers who find one, two, three or four prices, respectively, are juxtaposed in Figure 3.a. It shows a clearly decreasing pattern, testifying to the fact that the firms charge different consumers different prices. From an \textit{ex ante} perspective, the lowest price the average consumer has discovered up to time \( t, \| - h_t \|_{\infty} \), is decreasing in this cumulative time \( t \in (0, 1) \). The phenomenon is visible in Figure 3.b.

Our general finding for this special case is the following:

\textbf{Proposition 2} In a Poisson setting, there exists a OE with \( K = 2 \) and \( \{ B(\theta_N) \}_{N=0}^{\infty} \) such that \( \theta \in \left( 0, \frac{3+\sqrt{105}}{2} \right) \), for which the following necessary and sufficient condition for existence is satisfied:

\[ \frac{1}{2} B_1 + \frac{a}{2} B_2 \geq \frac{1}{2} - \frac{a}{2} \left( \frac{1}{2} B_2 + \frac{1}{2} B_3 \right) . \]

The equilibrium price distribution can be expressed similarly as before. The numbers of captives and shoppers are shown in Table 2. Depending on these numbers there can be an atom at the upper end or not.

The atom size is

\[ a = Pr(q = 1) = \max \left\{ \frac{B_2 - B_3}{B_2 + 3B_3 + 4 \sum_{N=4}^{\infty} B_N}, 0 \right\} , \]

and the profit is

\[ \Pi = \begin{cases} \frac{1}{2} (B_1 + B_2) + a \frac{1}{2} (B_3 + \sum_{N=4}^{\infty} B_N) , & \text{for } a > 0 , \\ \frac{1}{2} (B_1 + \frac{3}{4} B_2 + \frac{1}{4} B_3) , & \text{for } a = 0 . \end{cases} \]

Notice in particular that, due to the strengthened price discrimination and the locking effect, a firm’s profit is larger in here than before in Remark 3 (iii) with just one product variant or in Remark 4 with a single uniform price in a store (see Figure 4). Observe also
Figure 2: Expected switching time as a function of $\theta$ for $K = 1, 2$ (left), the likelihood of having observed two discount prices as a function of $t$ for $K = 1, 2$ and $\theta = 3$ (right).

Figure 3: The lowest price the average consumer has found as a function of $N = 1, 2, 3, 4$ (left), the lowest price the average consumer has found as a function of $t$ (right); $\theta = 2$.

$$
\begin{align*}
C_m^1 &= \frac{1}{7} B_1 \\
C_d^1 &= \frac{1}{7} (\frac{1}{2} B_1 + (1 - (1 - a)\frac{1}{2}) B_2 + (1 - (1 - a)\frac{3}{4}) B_3 + a \sum_{N=4}^{\infty} B_N) \\
S_d^1 &= \frac{1}{7} ((1 - a)\frac{1}{2} B_2 + (1 - a)\frac{3}{4} B_3 + (1 - a) \sum_{N=4}^{\infty} B_N) \\
C_m^2 &= \frac{a}{2} B_3 \\
C_d^2 &= \frac{a}{2} (\frac{1}{2} B_3 + \sum_{N=4}^{\infty} B_N) \\
S_d^2 &= \frac{1 - a}{2} (\frac{1}{4} B_2 + \frac{3}{4} B_3 + \sum_{N=4}^{\infty} B_N)
\end{align*}
$$

Table 2: Captives and shoppers for a firm $i$ with a discount price.

that both $a > 0$ and $a = 0$ might arise depending on which is larger $B_2$ or $B_3$. If $B_2 > B_3$ (holds for $\theta < \frac{1}{3}$) consumers starting from our firm are relatively more important than consumers coming from the other store. Making them search longer (by carrying only monopoly prices) is thus more important than attracting as fast as possible those who come from the other store (by offering a discount price). Thus, $a > 0$. Inverse reasoning
applies if $B_3 \geq B_2$ (holds for $\theta \geq \frac{1}{3}$). In that case $a = 0$.

Figure 4: A firm’s profit for $K = 1, 2$.

4.2 Sufficient condition and limiting equilibria

In this section we present a sufficient condition that guarantees that an OE exists for all numbers of product variants $K$ that are larger than some cutoff $K^*$. The first part of this requires that the tail probability of observing extremely large numbers of variants converges to zero fast enough, $\sum_{N=K+1}^\infty B_N = O\left(\frac{1}{K}\right)$. Then also the low price $q$ is a monopoly price almost surely: $a \to 1$ as $K \to \infty$.\(^{38}\)

Intuitively, offering more discount prices is then not very beneficial since there are not many consumers switching from the other store to poach by a lower price. Additionally, the variance of the finding process $D^2[N] = E[N^2] - E[N] = \sum_{N=0}^\infty N^2B_N - \sum_{N=0}^\infty NB_N$ should not be too small relative to its mean $E[N]$; otherwise, it turns out that the firms may benefit from carrying only discount prices.

When this two-part sufficient condition holds, it is interesting to analyze the limiting equilibrium $OE(K)$ as $K \to \infty$. We find that firms can then extract full surplus and divide the market peacefully among themselves. Both conditions hold, for instance, when $B = \{B_N\}_{N=0}^\infty$ is given by an appropriately weighted Poisson process; we discuss this more in the following section.

**Theorem 3** (Sufficient condition) Assume the following conditions both hold:

1. $\sum_{N=K+1}^\infty B_N = O\left(\frac{1}{K}\right)$,

2. $E[N]/D^2[N] \leq 1 - B_0$.

Then, there exist a finite cutoff $K^*$ such that an $OE(K)$ exists for all $K \geq K^*$.

---

\(^{38}\)The Landau big O notation $O\left(\frac{1}{K}\right)$ puts an upper bound on the convergence rate of the sum $\sum_{N=K+1}^\infty B_N$ by requiring that there exists a fixed number $c$ such that $|\sum_{N=K+1}^\infty B_N| \leq \frac{c}{K}$ when $K$ gets sufficiently large.
The sufficient condition ascertains for (2) that
\[ |\Delta C_i^m| (1 - p) \to 0 \text{ as } K \to \infty \]
slower than
\[ |\Delta S_{d-i}^i| p \to 0 \text{ as } K \to \infty \]
because then
\[ a \to 1 \text{ as } K \to \infty \]
faster than \( O\left(\frac{1}{K}\right) \).

So, we analyze here whether a firm’s profits increase when it adds more discount prices. Recall that above, \( |\Delta C_i^m| (1 - p) \) denotes the change in profits from consumers who start from our firm and \( |\Delta C_i^m| p \) denotes the change in profits from consumers who start from the other firm, when that firm has a discount price.

Note generally that, as the number of variants increases, the event of finding one discount price is of order \( O\left(\frac{1}{K}\right) \) and the event of finding two discount prices is an order \( O\left(\frac{1}{K^2}\right) \) event. We find also that \( |\Delta C_i^m| \sim \frac{1}{K} \) and \( |\Delta S_{d-i}^i| \sim \frac{1}{K^2} \). The result then obtains as we show that \( \frac{1-p}{p} \sim \frac{2}{K} \); the details are in Appendix.\(^{39} \)

An additional technical requirement is that the variance of the process should not be too small relative to its mean for, otherwise, the firm has a profitable deviation to stocking only discount prices. Again, this depends in subtle ways on the convergence rates of \( |\Delta C_i^m| \), \( |\Delta S_{d-i}^i| \), and the price ratio \( \frac{1-p}{p} \). Namely, for a deviation where the firm has just discount prices, we can show that
\[
K \frac{1-p}{p} \to D^2[N] \text{ and } K \frac{|\Delta S_{d-i}^i|}{|\Delta C_i^m|} \to E[N] (1 - B_0) \text{ as } K \to \infty.
\]
The second condition in Theorem 3 is thereby necessary to make this kind of deviation not profitable.

The general message is that, when the number of variants gets very large, it becomes unattractive to lower prices to poach switching consumers because there are so few of them. Searching for the first discount price takes all their time for most consumers. Almost nobody has time to switch.

In effect, when the number of variants explodes, it becomes almost impossible to find even one discount price. Therefore, the number of captives \( C_i^m \) who only find monopoly prices converges to \( 1 - B_0 \). As a result, each firm gets an equal share of these consumers

\(^{39}\)Recall about orders of convergence that big \( O \) denotes an upper bound \( (f(K) = O(g(K)) \text{ if } |f(K)| \leq c |g(K)| \text{ for } K \text{ large}) \), big \( \Omega \) a lower bound \( (f(K) = \Omega(g(K)) \text{ if } |f(K)| \geq c |g(K)| \text{ for } K \text{ large}) \); \( f \sim g \) says that \( \frac{f(K)}{g(K)} \to 1 \text{ as } K \to \infty. \)
in the symmetric limiting OE:

**Corollary 1** (Limiting equilibrium) *If there exist a number \( K^* \) such that OE(\( K \)) exists for all \( K \geq K^* \), then firms extract full surplus in the limit where the number of variants explodes: \( \Pi(K) \to \frac{1-B_0}{2} \) as \( K \to \infty \).*

In other words, if the sufficient condition presented in Theorem 3 holds, firms can extract full surplus at the limit as the number of variants becomes infinitely large. This limiting equilibrium resembles the one in Diamond (1971) because both firms have infinitely many variants with monopoly prices. Finding the one discount price that could be there inside a given store, is hence much like looking for a needle in a haystack. Nevertheless, as a distinction from this well known case of Diamond (1971), where nobody would have an incentive to search when the gain is virtually zero, in our model where consumers tend to use all their time up to the deadline, the residual price variation is sufficient to keep all of them searching.

### 4.3 Example: OE with \( K > 2 \) in Poisson setting

In this section we return to the Poisson setting to analyze more specifically profits in OE with more than two variants. We observe that the sufficient condition that we just derived can be fulfilled for this case although some necessary modifications have to be made to satisfy the second part of the sufficient condition in Theorem 3. When this has been taken care by considering an appropriately modified, weighted Poisson setting, we observe that an OE always exists as long as the firms have sufficiently many product variants available.

**Definition 3** [Weighted Poisson process] In a weighted Poisson setting, the original intensity parameter \( \theta \) is modified randomly by another random variable \( w \), which turns it into \( w\theta \). It is assumed that the mean of \( w \) is \( E[w] = 1 \) and its variance \( D^2[w] > 0 \). With this modification, the probability of discovering \( N \) variants becomes \( B_N = \int \frac{(w\theta)^N}{N!} e^{-w\theta} dF \) where \( F \) is the probability distribution function of \( w \). Here, it is also assumed that \( \int (w\theta)^N e^{-w\theta} dF < \infty \) and \( D^2[w]\theta \geq \int \frac{e^{-w\theta}}{1-e^{-\theta}} dF \).

We have chosen this weighted Poisson setting mainly because it features \( E[N] = \theta \neq D^2[N] = \theta + \theta^2 D^2[w] \) instead of \( E[N] = D^2[N] = \theta \) in the standard Poisson setting. This allows us to illustrate our convergence results without changing the parameter family from which our examples come in this paper. Furthermore, the weighted Poisson setting is a natural choice to model a random finding process if search conditions change in a stochastic manner due to say broadband capacity, unexpected weather conditions, congestion etc.

**Remark 5** If \( \{B_N\}_{N=0}^\infty \) is given by the weighted Poisson process with \( B_N(w\theta) = \frac{w\theta^N}{N!} e^{-w\theta} \), then (i) \( \sum_{N=K+1}^\infty B_N(w\theta) = O\left(\frac{1}{K}\right) \) as long as \( \int (w\theta)^N e^{-w\theta} dF < \infty \) for all \( N \) and (ii) \( E[N]/D^2[N] \leq 1 - B_0 \) as long as \( D^2[w]\theta \geq \int \frac{e^{-w\theta}}{1-e^{-\theta}} dF \). For example, \( D^2[w] = 1 \) is enough for all \( \theta \geq 1 \).
In words, there is a weighted Poisson process satisfying the conditions of Theorem 3.

**Corollary 2** In a weighted Poisson setting, for any \( \theta \) there is a finite cutoff \( K^\star \) such that \( OE(K) \) exists for all \( K \geq K^\star \). For any \( K^2 \geq K^1 \geq K^\star \), we have \( \Pi(K^2) \geq \Pi(K^1) \).

This is very convenient for firms in the sense that it enables them to cover any profit loss from improving search technology by carrying sufficiently many additional variants:

**Corollary 3** Consider a change from a slower search technology \( \{B_N(w\theta^1)\}_{N=0}^\infty \) to a faster search technology \( \{B_N(w\theta^2)\}_{N=0}^\infty \) where \( \theta^2 - \theta^1 > 0 \). Then, for any original number of variants \( K^1 \) there exists a modified number of variants \( K^2 \) such that \( \Pi(\theta^2, K^2) \geq \Pi(\theta^1, K^1) \) where \( K^2 - K^1 > 0 \).

**Discussion and extensions**

It is good to remember that \( K \) is kept fixed in this paper. We are analyzing the possible, negative side-effects that a larger number of variants may have on consumers if they are busy and information arrives gradually within stores. We do not try to account for product expansion; we take it for granted.

In our symmetric model with fixed \( K \), it is still clear that if firms could they would have an incentive to coordinate to \( OE(K) \)’s with a larger number of variants, rather than smaller. Profit goes up with additional variants. However, generally, we do not know how the firms would price if they could also carry asymmetric numbers of product variants, say \( K^1 \) for firm \( i = 1 \) and \( K^2 \) for firm \( i = 2 \). Theorem 1 suggests that they might not find it optimal to copy as such the equilibrium pricing strategies of \( OE(K^1) \) and \( OE(K^2) \), respectively, because then consumers would always start from the firm with a lower number of variants. Nevertheless, we think that the firm with more variants could not be made worse off than the firm with less variants in an equilibrium that features shoppers because that would give it a profitable deviation by a lower price; it is thus not consistent for consumers to expect it to have higher prices and start from the other firm. It should therefore indeed be beneficial for a firm to hold a larger number of variants.

Recall also that there exist a multiplicity of possible equilibrium outcomes for this game. Technically, any deviation from the symmetric case could thus be prevented by postulating that a deviation from the same number of variants leads to equilibria in Remarks 2 or 4 with no intrafirm price dispersion. They arise also with asymmetric \( K^1 \) and \( K^2 \) and feature lower profits.

We hence leave the analysis of an asymmetric, endogenous number of variants for future. A realistic extension along these lines should perhaps feature product differentiation. If consumer utility from a product variant would be \( \mu = 1 \) or \( \mu = 0 \) where \( \mu \) would be consumer specific match value, firms would have an incentive to carry additional variants for efficiency reasons as well.

We end our analysis by studying one more interesting case. It is conceivable that search
could become either easier or more difficult with additional product variants. To capture
this idea in the Poisson setting, it is without loss of generality to suppose that an increase
in the number of variants \( K \) modifies the base line search frictions \( \theta \) by a multiplier \( \sigma(K) \)
such that \( \theta(K) = \sigma(K)\theta \). Generally this multiplier \( \sigma(K) \) could be either above one (for
positive economies of scale) or below one (for negative economies of scale). To facilitate
the exposition, we normalize \( \sigma(1) = 1 \) and introduce the following definition:

**Definition 4** (i) There are positive economies of scale in search if \( \sigma(K+1) \geq \sigma(K) \geq 1 \)
for all \( K \in \mathbb{N} \) and \( \sigma(K+1) > \sigma(K) \) for some \( K \in \mathbb{N} \). (i) There are negative economies
of scale in search if \( \sigma(K+1) \leq \sigma(K) \leq 1 \) for all \( K \in \mathbb{N} \) and \( \sigma(K+1) < \sigma(K) \) for some
\( K \in \mathbb{N} \).

It turns out that our analysis applies almost as such to this extended setting; the tail
probabilities just have to converge fast enough despite possible scale effects:

**Corollary 4** In a weighted Poisson setting, for any \( \theta(K) \) there is a number \( K^* \) such
that \( OE(K) \) exists for all \( K \geq K^* \) as long as \( \sum_{N=K+1}^{\infty} B_N(w\theta(N)) \rightarrow 0 \) at least at rate
\( O(\frac{1}{K}) \).

In consequence, as long as the economies of scale in search are not too large, an \( OE(K) \)
exists for any large enough \( K \) and firms’ profits increase with more variants.

Note that it is not immediate from the outset whether search should have positive or
negative economies of scale: it can become easier to find an individual product variant,
when there are more of them, but consumers can also get overwhelmed by the larger
number. A small number of product variants can also be displayed in a compact manner
but a larger number maybe spread around a wider space.

Still, the range for \( \sigma(2) \) that we find the most reasonable is \([1, 2]\): the one that lies
between no economies of scale and moderate positive economies of scale. To narrow down
to this range, suppose for a moment that we can associate each variant with a rate \( \phi \), for
which it is found on a page or in a room (representing a store). Since this rate is specific
to this given variant it is clearly independent of the other variants’ rates. As a result, (i) if
we model a store with two variants as one page or one room with two variants on it, then
the first is found at rate \( 2\phi \) and the second at rate \( \phi \), but, (ii) if we model a store with
two variants as two pages or two rooms with one variant in each, then both are found at
rate \( \phi \).

Thus, it is reasonable to think that the average finding rate per variant should be
within \( \phi \) and \( 2\phi \) for two variants (within \( \phi \) and \( K\phi \) for \( K \) variants). At extreme, we could
of course maintain a constant finding rate \( \theta \) for a larger number of product variants \( K \)
by replacing the old store, with say just one variant inside the store, by multiple replicas of
the old store, each of them with exactly one variant in it.40,41

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40Note that this idea is a modification of the standard replica argument for constant returns to scale.
41We consider here only constant finding rates \( \sigma(K) \) for all the variants in a given store because it is
5  Closing remarks

We provide a novel search model to analyze how a larger number of variants affects price competition. Our setup has explicit intrafirm frictions that originate from the gradual arrival of price information inside the stores and the existence of deadlines for consumers. These frictions make a firm’s problem of pricing its numerous variants a non-trivial one.

Our paper contributes to literature that analyzes retailer strategies to lock-in consumers (e.g., Ellison and Wolitzky (2012) for price obfuscation and Klemperer (1987) for switching costs) and to literature trying to explain price dispersion among identical goods (e.g., Baye et al. (2006), Burdett and Judd (1983) and Butters (1977)). Yet, while the latter strand of literature has concentrated on price dispersion across stores we find it also within stores.

We demonstrate that firms have an incentive carry several similar products and generate price dispersion in their store to amplify the existing search frictions and create implicit barriers to switching. Price dispersion keeps consumers searching longer in their first store, leaving them less time for the other competing store. It also helps firms to price discriminate more effectively. As the number of variants increases, firms can extract full surplus.

The general problem on the part of the consumers is that they cannot simply commit to shop around in random and play the stores against one another but, instead, tend to grow a stronger and stronger preference for their first store as time goes on. For this to work, it is important that it is focal in the economy that usually firms indeed offer a discount price. That might give one explanation to why firms often picture themselves as having discount prices everyday.

We note that, as the firms tag similar items with different prices, they implicitly also commit to improving their best price to a consumer as time goes on. This resembles a bit the effect of hiring sales people to reduce the prices little by little and play time so as to lock in busy consumers. However, when all product variants have their own fixed prices, this can be implemented in an entirely passive way on the part of the firm. No sales people are needed in the bargaining, and the commitment issues as for when to give the promised discount – now, later or never – can be totally avoided. It is the consumer who does all the work. This idea shows up in Figure 3.b.

As a follow-up project, it would be interesting to consider a model where increased not obvious whether the first ones that a consumer observes should be faster or slower to find than the last ones: One could think that a store is composed of different locations, rooms, pages, shelves, floors or department where each could have a product variant in it. Some of these locations could be easier or harder to check for a product variant. The first variants could thus be harder to find if a consumer is checking all the possible locations one by one in a random, sequential and systematic way as there remains fewer possible locations in the end. It could be the opposite, however. The first variants could be easier to find in a slightly more strategic way of searching where a consumer starts from the most promising locations and leaves the least promising or remote ones for the last.
product variety also has the effect of improving consumers’ match values. Then, the effects on consumer surplus might clearly be more positive than in this paper. We hypothesize that there might be a negative price effect and a positive match value effect to variety.

Appendix

PROOF OF LEMMA 1

We show first that, if consumers search in both firms with non zero probability, there exist no equilibrium where firms do not use randomized pricing strategies. The proof is by contradiction. Suppose instead that firms apply pure strategies and set a fixed price for each variant. Let us focus on the lowest price(s) \( q \) and the second lowest price(s) \( r \) in the market, such that \( q < r \) by assumption. Note that, if only one firm is using \( q \), it has an incentive to increase \( q \) almost up to \( r \) to extract more revenue from captives \( B_1 \). Also, if both firms are using \( q \), both have and incentive to decrease it slightly to undercut the firm’s price. Thus, every store must indeed have at least one price that is randomized.

Here are some remarks:

1. Since these prices \( q \) are the lowest prices, it does not matter if these suggested price changes trigger a change in search because consumers who find the lowest prices will anyway return to buy for them.

2. If consumers search, they search in both firms. Otherwise, if they only searched in one firm that firm would have an incentive to set its price equal to one. But this would imply that consumers do not search.

3. The lowest prices \( q \) are positive since, as long as consumers are searching, firms can extract positive revenue from captives \( B_1 \); marginal cost pricing where prices equal zero is not an equilibrium here.

Note that a consumer might respond to a given price by switching away the firm for good depending on the beliefs she holds about the joint price distribution. To organize our thoughts when this "punishing" switching response is unspecified, we introduce the notion of competing prices: price \( q^i \) (of some fixed variant) competes with price \( q^{-i} \) (of some fixed variant) if some mass of consumers who has found these variants sometimes purchases for \( q^i \) and, at other times, for \( q^{-i} \). With just two firms the lowest prices are always competing prices. We next derive some elementary properties of those lowest prices.

First, consider the support of a firm’s lowest prices \( \text{supp}(G^i) \) and the support of the other firm’s lowest prices \( \text{supp}(G^{-i}) \). Note particularly that, if these supports would not overlap, i.e. if \( \exists S(q,r) = (q - r, q + r) \) for \( (q,r) \in [0,1] \times (0,\infty) : S(q,r) \cap \text{supp}(G^i) = S(q,r) \) and \( S(q,r) \cap \text{supp}(G^{-i}) = \emptyset \), then the firm would have a profitable deviation to adjust the pricing strategy such that all the probability mass in the gap \( S(q,r) \) is put on the upper bound of the gap \( p = q + r \). This implies that \( \text{supp}(G^i) \subset \text{supp}(G^{-i}) \) and \( \text{supp}(G^i) \subset \text{supp}(G^{-i}) \) such that the supports of the lowest prices are the same and thus denoted by \( \text{supp}(G) \).

Next assume that the probability that a consumer would switch away from a firm after seeing a price is continuous in the price. Then it is immediate to show that, if there were an atom \( a \) below the upper bound of this support, i.e., if \( \exists T(q,r) = [q - r, q + r) \) for \( (q,r) \in [0,1] \times (0,\infty) : T(q,r) \cap \text{supp}(G) \neq \emptyset \), then a firm would have a profitable deviation from \( p \in [a, a + \epsilon] \) to \( p \in (a - \epsilon, a) \) for some tiny \( \epsilon > 0 \); namely, the probability of having a lower price than the other firm would go up discontinuously but the selling price would go down only continuously. This shows that it is not possible to have an atom in the interior nor at the lower bound of \( \text{supp}(G) \).

Observe also that these results imply that \( \text{supp}(G) \) is an interval. Furthermore, as long as there is no countervailing effect through an atom at the upper bound or through a triggered switching
response, the upper bound of this interval must be equal to unity. Only captives buy for those prices, and optimally they should thus be raised as high as possible. □

PROOF OF REMARK 1

(i) Since prices equal one consumers have no incentive to search. By Assumption 1 (iii), consumers stay at home. Firms have thus no incentive to lower their prices because no consumer would find them anyway. There exist (almost surely) no other kinds of stay-home equilibria because any tremble in consumers’ beliefs would restore their incentives to search and hence the firms’ incentives to randomize in prices. For (ii) see Footnote 31.

(iii) This equilibrium can be constructed like in Varian (1980) or Stahl (1989) if we replace the informed consumers by \( S = 1 - B_0 - B_1 \) and the uninformed consumers by \( 2C = 2B_1 \) and note that the sum is less than one, \( S + 2C = 1 - B_0 < 1 \).

Specifically, by Lemma 1 we know that \( \text{supp}(G) = \left[p, \bar{p}\right] \). A firm’s profit is simply \( \Pi(p) = (C + S(1 - G(p)))p \) because captives purchase from it for any price \( p < 1 \) but shoppers buy from it only if its price is lower than the other firm’s price. This can be evaluated at the upperbound \( \bar{p} = 1 \) to pin down the profit: \( \Pi = C \). The equilibrium price distribution can be obtained by requiring that price is the same all over \( \text{supp}(G) \): \( G \) can be thus solved from \( C = (C + S(1 - G(p)))p \). The lowerbound can then be derived by solving the equation \( G(p) = 0 \).

This symmetric solution is unique if some consumers search two prices. Since the firms have the same number of variants, consumers first approach the one with a lower expected price or, by Assumption 1, for the case of ties, pick a firm in random. Asymmetric pricing strategies are thus impossible. Namely, if one firm had a lower expected price than the other one, it would necessarily attract more captives. But this would also imply that the firm has actually a higher expected price because prices always increase with captives. This is a contradiction.42

Deriving the limits is a straightforward calculus exercise. ■

PROOF OF THEOREM 1

Step 1 Formulating the problem clearly

We introduce the following notations to help the exposition:

\[
\begin{align*}
    u_A &:= 1 - E \left[ \min \{p_K^1, p_K^2 \} | p_K^1 < 1 \land p_K^2 < 1 \right] > 0 \\
    u_B &:= 1 - E \left[ p_K^i | p_K^i < 1 \land i = 1, 2 \right] > 0
\end{align*}
\]

\( u_A \) is the expected consumer payoff for finding two discount prices whereas \( u_B \) is the expected consumer payoff for finding one discount price by the deadline.

\[
P_t(k) = \begin{cases} 
    \frac{(\theta(1-t))^k e^{-\theta(1-t)}}{k!}, & \text{for } k < K - N_1^t - N_2^t, \\
    \sum_{l=0}^{N_1^t + N_2^t} \frac{(\theta(1-t))^l e^{-\theta(1-t)}}{l!}, & \text{for } k \geq K - N_1^t - N_2^t,
\end{cases}
\]

is the probability of discovering \( k \) additional prices during the time that is remaining and

\[
\kappa(m^i) = \frac{(1-a) \frac{1}{K}}{a + (1-a) \frac{1}{K} m^i}
\]

is the probability that, after finding \( m^i \) monopoly prices, the next one is a discount price.

We consider a consumer’s optimal search problem when she has found \( m^i \) monopoly prices from store \( i = 1 \) and \( m^2 \) monopoly prices from store \( i = 2 \), where \( m^i \in \{0, 1, ..., K\} \) for \( i = 1, 2 \)

42 Asymmetric pricing strategies are analyzed more in a companion paper Hämäläinen (2015).
but not a discount price in either of these. Remember that we assumed that \( p_1^i = \ldots = p_{K-1}^i = 1 \) (\( K-1 \) monopoly prices) and with probability \( a \in (0, 1) \) \( p_K^i = 1 \) but with probability \( 1-a \in (0, 1) \) \( p_K^i < 1 \) (at maximum one discount price).

Note that the consumer’s problem is trivial after she finds a discount price because, since a store has only one discount price by assumption, it is then best for the consumer to switch the stores and spend the rest of the time in the other store to find also the other discount price. The store has only one discount price by assumption, it is then best for the consumer to switch the store where the consumer is at that very moment. Therefore, we are here only interested in cases where, with no loss, \( m^1 > m^2 \): consumer has found more monopoly prices from store \( i = 1 \) than from store \( i = 2 \).

Next, suppose we knew that the consumer will find \( k \in \{1, 2, \ldots \} \) more prices before the deadline \( t = 1 \). We want to show that, for any \((k, m^1, m^2)\) with \( m^1 > m^2 \), the consumer is at least weakly better off by continuing in store \( i = 1 \) than by switching to store \( i = 2 \). This increases her chances of finding two discount prices (one discount price), \( \phi_A (\phi_B) \), and decreases her chances of not finding any discount prices, \( 1 - \phi_A - \phi_B \).

Her payoff can thus be written as

\[
E[u] = u_A \phi_A + u_B \phi_B.
\]

We next want to determine \( \phi_A \) and \( \phi_B \) as functions of consumer strategy for some given tuple \((k, m^1, m^2)\). In other words, the consumer has so far drawn a total of \( m^1 + m^2 > 2m^2 \) monopoly prices and has now exactly \( k > 0 \) draws left to find also some discount prices. Denote by \( \rho(p) \) the probability that the first discount price is found at the \( p \)'th draw (\( p \in \{1, \ldots, k\} \)), and by \( \rho'(p) \) the probability that another discount price is found thereafter at some \( p'' \)'th draw (\( p'' \in \{p + 1, \ldots, k\} \)).

With this notation, it is now possible to express \( \phi_A \) and \( 1 - \phi_A - \phi_B \) as

\[
\phi_A = \rho(1) \rho'(1) (1 - \rho(1)) \rho(2) \rho'(2) (1 - \rho(1)) \cdots (1 - \rho(k-1)) \rho(k) \rho'(k) (1 - \rho(1)) \cdots (1 - \rho(k-1))
\]

\[
\phi_A = \sum_{i=1}^{k} \rho(i) \rho'(i) \prod_{j=1}^{i-1} (1 - \rho(j)) \tag{4}
\]

\[
1 - \phi_A - \phi_B = (1 - \rho(1)) + (1 - \rho(2)) + \cdots + (1 - \rho(k))
\]

\[
= \sum_{i=1}^{k} (1 - \rho(i)). \tag{5}
\]

It is thus clear that the vectors \( \rho = (\rho(1), \ldots, \rho(k)) \) and \( \rho' = (\rho'(1), \ldots, \rho'(k)) \) uniquely determine consumer payoffs. Note that both \( \rho \) and \( \rho' \) depend on consumer strategy and preset values \((k, m^1, m^2)\). We next consider how the vectors \( \rho \) and \( \rho' \) can be constructed of the \( \kappa \)'s we defined earlier. There are certain crucial restrictions on this process because \( m^1 (m^2) \) obviously increases by one each time the consumer finds a monopoly price from store \( i = 1 \) (\( i = 2 \)); it is necessary to make sure that we keep track of these dynamics.

This constraint, which is essentially a feasibility requirement, can be most conveniently satisfied by requiring that vector \( \rho \) has subvectors \( \rho^i := (\kappa(m^1), \ldots, \kappa(m^1 + k^1 - 1)) \) and \( \rho^j := (\kappa(m^2), \ldots, \kappa(m^2 + k^2 - 1)) \) where \( k^1 + k^2 = k \). A subvector is obtained from a vector by omitting certain elements but not changing their order. Another way to put this is saying that the index set \( I = \{1, \ldots, k\} \) can be partitioned into two disjoint sets, \( I^1 \cup I^2 = I \) and \( I^1 \cap I^2 = \emptyset \), such that \( (\rho(i))_{i \in I} = \rho \), \( (\rho(i))_{i \in I^1} = \rho^1 \), and \( (\rho(i))_{i \in I^2} = \rho^2 \).

The process is easiest to understand by thinking that vector \( \rho \) is constructed by stacking it up element for element from the beginning to the end, choosing for every element either the earliest unselected element of \( \kappa^1 \) or that of \( \kappa^2 \).
\[ \kappa^1 = (\kappa(m^1), ..., \kappa(K)) \]
\[ \kappa^2 = (\kappa(m^2), ..., \kappa(m^1), ..., \kappa(K)) \]

Note that \( \kappa^1 \) and \( \kappa^2 \) are increasing in the sense that \( \kappa(m) < \kappa(m+1) \) for all \( m \in \{m^2, ..., K-1\} \) as can be seen from (3). Also, \( \kappa^1 \) can be obtained as a subvector of \( \kappa^2 \) by omitting the first and therefore the lowest \( m^1 - m^2 \) elements of \( \kappa^2 \).

**Step 2** Showing that the probability of finding one more discount price is larger with this search order

It is hence clear that (5) is minimized by selecting

\[
\rho = \begin{cases} 
(k(m^1), ..., k(m^1 + k^1 - 1)), & \text{for } k \leq m^1 - K \text{ and } k^1 = k \\
(k(m^1), ..., k(m^1 + k^1 - 1), k(m^2), ..., k(m^2 + k^2 - 1)), & \text{for } m^1 - K < k, \\
 & \text{and } k^1 = K - m^1, \\
 & \text{and } k^2 = k^1 - 1,
\end{cases}
\]

where it is assumed without loss that \( k \leq m^1 + m^2 - 2K \).

**Step 3** Showing that the probability of finding two more discount prices does not depend on search order

We next want to show that the exact same choice is also a maximizer of (4). To do so note that each \( \rho \) with subvectors \( \rho^1 \) and \( \rho^2 \) induces a specific \( \rho' \). To see this, take any \( \rho(i) \) and define the numbers \( n^1(i) \) and \( n^2(i) \) as follows

\[
n^1(i) := \#\{ j \in I^1 | j \leq i \},
\]
\[
n^2(i) := \#\{ j \in I^2 | j \leq i \}.
\]

Then we have for the case of \( i \in I^1 \)

\[
\rho'(i) = \begin{cases} 
\sum_{s=m^2+n^2(i)}^{m^2+n^2(i)+k-i} \kappa(s) \prod_{t=m^2+n^2(i)}^{s-1} (1 - \kappa(t)), & \text{if } k - i < K - (m^2 + n^2(i)) > 0, \\
\sum_{s=m^2+n^2(i)}^{K} \kappa(s) \prod_{t=m^2+n^2(i)}^{s-1} (1 - \kappa(t)), & \text{if } k - i \geq K - (m^2 + n^2(i)) > 0, \\
0, & \text{if } K - (m^2 + n^2(i)) \leq 0.
\end{cases}
\]

The case with \( i \in I^2 \) is symmetric.

It is thus rather easy to see that (4) is independent of consumer strategy, the exact choice of \( \rho \). Due to any change in consumer strategy, what is gained in the probability of finding the first discount price is lost in the probability of finding the second one. Hence the search order makes no difference because the consumer should anyway search in both of these stores until a discount price is found. Thus, the minimizer of (5) is a maximizer of (4). \( \square \)

**PROOF OF THEOREM 2**

**Step 1** Deriving \( \Pi \) on-path and off-path
We first derive the profit to a firm who has \( d \) discount prices \( q < 1 \) and \( K - d \) monopoly prices \( p = 1 \). We assume the other firm has \( K \) monopoly prices and no discount price, with probability \( a \), and \( K - 1 \) monopoly prices and one discount price, with probability \( 1 - a \). Without loss of generality, let the former firm be firm \( i = 1 \) and let the latter one be \( i = 2 \).

Now consider a consumer who has found \( N \) prices, or all of them for \( N > 2K \). It is convenient to divide demand of firm \( i = 1 \) from the consumers into six possible consumer segments:

- \( C^a_m(N) \) demand from (captive-like) consumers who start from store \( i = 1, 2 \) and find no discount price,
- \( C^d_m(N) \) demand from (captive-like) consumers who start from store \( i = 1, 2 \) and find a discount price only from store \( i \),
- \( S^d_d(N) \) demand from (shopper-like) consumers who start from store \( i = 1, 2 \) and find a discount price from both stores \( i \) and \(-i\).

For concreteness, we assume the perspective of firm \( i = 1 \) next; this is clearly without loss. The profit to firm \( i = 1 \) from consumers who discover \( N \) prices can now be expressed as

\[
\Pi^1(N) = C^1_m(N) + C^2_m(N) + (C^1_d(N) + C^2_d(N)) q + (S^1_d(N) + S^2_d(N)) (1 - H(q))q.
\]

Firm \( i = 1 \)'s full profit is hence given by the following sum

\[
\Pi^1 = \sum_{N=1}^{\infty} B_N \Pi^1(N).
\]

If we write the above expressions of \( C^m_m, C^d_d \) and \( S^d_d \) such that \( d \) is kept a free variable, they immediately lend themselves for both on-path (\( d = 0 \) or \( d = 1 \)) and off-path analysis (\( d > 1 \)).

The demands from the defined consumer segments can now be written as

\[
2C^1_m(N) = T(N \leq K) \binom{K-d}{N} \binom{K}{N}
\]

\[
2C^1_d(N) = \sum_{p=1}^{\min\{N,K\}} \binom{K-d}{p-1} \binom{K}{p-1} \binom{d}{K-p+1} \left( a + (1 - a)T(N - p \leq K) \binom{K-1}{N-p} \right)
\]

\[
2S^1_d(N) = \sum_{p=1}^{\min\{N,K\}} \binom{K-d}{p-1} \binom{d}{K-p+1} \left( 1 - a - (1 - a)T(N - p \leq K) \binom{K-1}{N-p} \right)
\]

\[
2C^2_m(N) = aT(N - K \leq K) \binom{K-d}{N-K} \binom{K}{N-K}
\]

\[
2C^2_d(N) = a \left( 1 - T(N - K \leq K) \binom{K-d}{N-K} \right)
\]

\[
2S^2_d(N) = (1 - a) \sum_{p=1}^{\min\{N,K\}} \binom{K-1}{p-1} \binom{1}{K-p+1} \left( 1 - T(N - p \leq K) \binom{K-1}{N-p} \right) T(d > 0).
\]

Let us now briefly explain how these are obtained. First, the number two in front of each is there because half the consumers start from firm \( i = 1 \) and half of them start from firm \( i = 2 \). This implies that, for example, \( C^1_m + C^1_d + S^1_d = \frac{1 - B_0}{2} \) because these are those consumers who start from firm \( i = 1 \) and then change from \( C^1_m \) to \( C^1_d \) as they find the discount price from firm \( i = 1 \) and then from \( C^1_d \) to \( S^1_d \) as they find the discount price from firm \( i = 2 \). Of course, not all have time to go through those changes. Thus, we want to calculate how many end up being \( C^1_m, C^1_d \) and \( S^1_d \) respectively.
This depends when they would find each discount price and how many prices they manage to find in total. Consumers \( C_{m}^{i}(N) \) start from firm \( i = 1 \) and find only its monopoly prices. If a consumer finds a total of \( N \) prices, which takes place with probability \( B_{N} \), the likelihood that none of these is a discount price is \( \left( \frac{c_{m}^{i}}{c_{m}^{i}} \right)^{N} \) where \( N \leq K \). Consumers \( C_{n}^{i}(N) \) start from firm \( i = 1 \) and find only its discount price. In other words, we require that she does discover a discount price from store \( i = 1 \) but that she does not find a discount price from store \( i = 2 \). Since these events are interlinked because the consumer switches only after she finds the first discount price, we also have to take into account different possible timings as for her observing the first discount price in deriving for instance \( C_{d}^{1} \) and \( S_{d}^{1} \). We hence write \( C_{d}^{1}(N) \) as \( \sum_{p=1}^{\min(N,K)} \left( \frac{k-1}{c_{d}^{1}} \right)^{d} \frac{d}{K-p+1} \left( a + (1-a)T(N-p \leq K) \left( \frac{k-1}{(K-p)} \right) \right) \), where we sum over all possible finding times \( p \) and \( \frac{d}{K-p+1} \). The likelihood that consumer does not find a discount price before she finds variant \( p \), is the probability that she then finds it is \( \frac{d}{K-p+1} \) and the probability that she does not find another discount price after switching is \( a + (1-a)T(N-p \leq K) \left( \frac{k-1}{(K-p)} \right) \). In deriving \( S_{d}^{1} \) from \( C_{d}^{1} \) we can simply change the probability \( a + (1-a)T(N-p \leq K) \left( \frac{k-1}{(K-p)} \right) \) (of not finding \( q^{i} \) after switching) to the complementary probability \( 1 - a - (1-a)T(N-p \leq K) \left( \frac{k-1}{(K-p)} \right) \) (of finding \( q^{i} \) after switching). For \( C_{m}^{i-1} \) and \( C_{d}^{i-1} \) we note that these events arise only if the other firm does not have a discount price, occurring with probability \( a \) and for \( S_{d}^{i-1} \) that this requires that the other firm does have a discount price, occurring with probability \( 1 - a \).

Since unity is in \( \text{supp}(G) \) by Lemma 1, profit in OE can be obtained by evaluating it for \( d = 1 \) and \( q = 1 - \epsilon \) for \( \epsilon \to 0+ \):

\[
\Pi(K, B|d = 1, q = 1) = \sum_{N=1}^{\infty} B_{N} \left( C_{m}^{1}(N|d = 1) + C_{m}^{2}(N|d = 1) + C_{d}^{1}(N|d = 1) + C_{d}^{2}(N|d = 1) \right). \tag{7}
\]

Note that the firm could also raise \( q = 1 - \epsilon \) for \( \epsilon \to 0+ \) a little bit such that \( d = 0 \) and \( q = 1 \). That would yield a profit:

\[
\Pi(K, B|d = 0, q = 1) = \sum_{N=1}^{\infty} B_{N} \left( C_{m}^{1}(N|d = 0) + C_{m}^{2}(N|d = 0) + C_{d}^{1}(N|d = 0) + C_{d}^{2}(N|d = 0) \right). \tag{8}
\]

In either case the firm attracts only captives because it prices at unity. The probability of selling to shoppers is negligible. If the firm sets a lower discount price \( q < 1 \), however, its profit becomes

\[
\Pi(K, B|d = 0, q < 1) = \sum_{N=1}^{\infty} B_{N} \left( C_{m}^{1}(N|d = 1) + C_{m}^{2}(N|d = 1) \right.
\]

\[
+ \left( C_{d}^{1}(N|d = 1) + C_{d}^{2}(N|d = 1) \right) q
\]

\[
+ \left( S_{d}^{1}(N|d = 1) + S_{d}^{2}(N|d = 1) \right) (1 - H(q))q \right). \tag{9}
\]

**Step 2** Deriving \( a \) and \( H \) (i.e., \( G \), and \( p \)

We can now use these different ways of writing profits (7), (8), and (9) to derive \( a \) and \( H \) (i.e., \( G \), and \( p \). To support randomized pricing strategies, note that all discount prices \( q \in \text{supp}(G) \) should give the firm the same profit. In particular, if it is the case that (8) exceeds (7) for \( a = 0 \,
then the firm has a profitable deviation to using monopoly prices only unless the other firm also uses monopoly prices often enough to make it profitable to undercut those higher prices. In that case, \( a \) is defined by setting (7) and (8) equal:

\[
\begin{align*}
\sum_{N=1}^{\infty} B_N(C^1_m(N|d = 1) + C^2_m(N|d = 1) + C^1_d(N|d = 1) + C^2_d(N|d = 1)) = \\
\sum_{N=1}^{\infty} B_N(C^1_m(N|d = 0) + C^2_m(N|d = 0) + C^1_d(N|d = 0) + C^2_d(N|d = 0))).
\end{align*}
\]

By inserting the expressions for consumers (6) that we derived earlier and by rearranging we get:

\[
\begin{align*}
\sum_{N=1}^{\infty} B_N(T(N \leq K) + aT(N > K)) & = \sum_{N=1}^{\infty} B_N \left( T(N \leq K) \frac{K-N}{K} + aT(N > K) + \sum_{p=1}^{\min\{N,K\}} \frac{1}{K} \left( a + (1-a)T(N-p \leq K) \frac{K-N+p}{K} \right) \right) \\
\sum_{N=1}^{\infty} B_N \left( \min\left\{ \frac{N}{K}, 1 \right\} \right) & = \sum_{N=1}^{\infty} B_N \left( \sum_{p=1}^{\min\{N,K\}} \frac{1}{K} \left( a + (1-a) \max\left\{ \frac{K-N+p}{K}, 0 \right\} \right) \right)
\end{align*}
\]

\[
\begin{align*}
\alpha & = \frac{\sum_{N=1}^{\infty} B_N \left( \sum_{p=1}^{\min\{N,K\}} \frac{1}{K} \left( 1 - \max\left\{ \frac{K-N+p}{K}, 0 \right\} \right) \right) - \sum_{N=1}^{\infty} B_N \left( \sum_{p=1}^{\min\{N,K\}} \frac{1}{K} \min\left\{ \frac{N-p}{K}, 1 \right\} \right) - \sum_{N=1}^{\infty} B_N \left( \sum_{p=1}^{\min\{N,K\}} \frac{1}{K} \max\left\{ \frac{K-N+p}{K}, 0 \right\} \right)}{\sum_{p=1}^{\infty} B_N \left( \min\left\{ \frac{K-N+p}{K}, 1 \right\} \right) - \sum_{N=1}^{\infty} B_N \left( \sum_{p=1}^{\min\{N,K\}} \frac{1}{K} \min\left\{ \frac{N-p}{K}, 1 \right\} \right)} \\
\alpha & = \frac{\sum_{N=1}^{\infty} B_N \left( \sum_{p=1}^{\min\{N,K\}} \frac{1}{K} \left( \frac{N-p}{K}, 1 \right) \right) - \sum_{N=1}^{\infty} B_N \left( \sum_{p=1}^{\min\{N,K\}} \frac{1}{K} \min\left\{ \frac{N-p}{K}, 1 \right\} \right) - \sum_{N=1}^{\infty} B_N \left( \sum_{p=1}^{\min\{N,K\}} \frac{1}{K} \max\left\{ \frac{K-N+p}{K}, 0 \right\} \right)}{\sum_{N=1}^{\infty} B_N \left( \sum_{p=1}^{\min\{N,K\}} \frac{1}{K} \min\left\{ \frac{N-p}{K}, 1 \right\} \right) - \sum_{N=1}^{\infty} B_N \left( \sum_{p=1}^{\min\{N,K\}} \frac{1}{K} \max\left\{ \frac{K-N+p}{K}, 0 \right\} \right)}.
\end{align*}
\]

If the denominator is below zero, then \( \alpha = 0 \), but if the denominator is above zero, then \( \alpha \in (0, 1) \).

Next we can derive the equilibrium price distribution \( H \) conditional on assumption that \( q < 1 \) as a function of profits \( \Pi \) by requiring that (9) equals (7):

\[
\Pi = \sum_{N=1}^{\infty} B_N \left( C^1_m(N|d = 1) + C^2_m(N|d = 1) \right)
+ (C^1_d(N|d = 1) + C^2_d(N|d = 1))q
+ (S^1_d(N|d = 1) + S^2_d(N|d = 1)) \left( 1 - H(q) \right) q.
\]

which gives
is dominated by a deviation to \((h, g)\). Relevant deviation to consider is that to the other firm because they almost surely have already found a better price. Therefore, the only cases the firm loses a fraction of its captives but never gains in terms of shoppers starting from \(A\).

Observing that analyzing only deviations to the lower bound suffices leads to the necessary and sufficient condition

\[
H(q) = 1 + \frac{\sum_{N=1}^{\infty} B_{N} \left(C_{d}^{1}(N|d = 1) + C_{d}^{2}(N|d = 1)\right)}{\sum_{N=1}^{\infty} B_{N} \left(S_{d}^{1}(N|d = 1) + S_{d}^{2}(N|d = 1)\right)} - \frac{\Pi - \sum_{N=1}^{\infty} B_{N} \left(C_{m}^{1}(N|d = 1) - C_{m}^{2}(N|d = 1)\right)}{\sum_{N=1}^{\infty} B_{N} \left(S_{d}^{1}(N|d = 1) + S_{d}^{2}(N|d = 1)\right)} q. \tag{11}
\]

The lower bound it then the price for which the probability distribution function vanishes \(H(p) = 0\) and for which the firm thus attracts all the shoppers

\[
\sum_{N=1}^{\infty} B_{N} \left(C_{m}^{1}(N|d = 1) + C_{m}^{2}(N|d = 1)\right)
+ C_{d}^{1}(N|d = 1) + C_{d}^{2}(N|d = 1)
= \sum_{N=1}^{\infty} B_{N} \left(C_{m}^{1}(N|d = 1) + C_{m}^{2}(N|d = 1)\right)
+ (C_{d}^{1}(N|d = 1) + C_{d}^{2}(N|d = 1)) p
+ (S_{d}^{1}(N|d = 1) + S_{d}^{2}(N|d = 1)) p, .
\]

such that

\[
p = \frac{\sum_{N=1}^{\infty} B_{N} \left(C_{d}^{1}(N|d = 1) + C_{d}^{2}(N|d = 1)\right)}{\sum_{N=1}^{\infty} B_{N} \left(C_{d}^{1}(N|d = 1) + C_{d}^{2}(N|d = 1)\right) + S_{d}^{1}(N|d = 1) + S_{d}^{2}(N|d = 1)). \tag{12}
\]

**Step 3** Observing that analyzing only deviations to the lower bound suffices

With an extra discount price the firm’s profit is a linear function of form \(A_{1} + A_{2}q + A_{3}h + A_{4}(1 - H(q))q + A_{5}(1 - H(h))h\) where \((1 - H(q))q\) and \((1 - H(h))h\) are a linear functions of form \(A_{6} + A_{7}h\); this is easy to see by looking at (9) and (11). As a result, the optimal way to deviate involves choosing the old discount price \(q\) and the new discount price \(h\) so that they lie on the boundaries of \([p, 1]^2\). We can easily show that the firm never gains by deviating to \((h, g) = (1 - \epsilon, 1 - \epsilon)\) nor by deviating to \((h, g) = (1 - \epsilon, p)\), where \(\epsilon > 0\) is a small number. In both cases the firm loses a fraction of its captives but never gains in terms of shoppers starting from the other firm because they almost surely have already found a better price. Therefore, the only relevant deviation to consider is that to \((h, g) = (p, p)\). Obviously, any deviation to \((h, g) \in [0, p]^2\) is dominated by a deviation to \((h, g) = (p, p)\) so that we need not cover those. Essentially the same analysis applies for a larger number of discount prices.

**Step 4** Deriving the necessary and sufficient condition

Thus, firm \(i = 1\) has no profitable deviation from OE as long as

\[\Pi(K, B|d = 1, q = p) \geq \Pi(K, B|d = 2, q = p).\]

This boils down to the necessary and sufficient condition
\[
\sum_{N=1}^{\infty} B_N (C^1_m(N|d=1) + C^2_m(N|d=1)) \\
+ \left( C^1_m(N|d=1) + C^2_m(N|d=1) \right) p \\
+ \left( S^1_m(N|d=1) + S^2_m(N|d=1) \right) p \geq \\
\sum_{N=1}^{\infty} B_N (C^1_m(N|d=2) + C^2_m(N|d=2)) \\
+ \left( C^1_m(N|d=2) + C^2_m(N|d=2) \right) p \\
+ \left( S^1_m(N|d=2) + S^2_m(N|d=2) \right) p \ldots
\]

Note that \( C^1_m + C^2_m + S^1_m = \frac{1-B_N}{2} \) for both \( d = 1 \) and \( d = 2 \) and \( C^1_m + C^2_m = \frac{a \sum_{N=K+1}^B N^2}{2} \) for both \( d = 1 \) and \( d = 2 \). Since they are thus fixed and both \( C^1_m \) and \( S^2_m \) purchase for certain for \( p \), the loss from this deviation is

\[
\left( C^1_m(d=1) - C^1_m(d=2) \right) (1-p) + \left( C^2_m(d=1) - C^2_m(d=2) \right) (1-p),
\]

because those numbers of new consumers now buy for \( p \) whereas before they used to buy for 1. The gain of this deviation is

\[
\left( S^2_m(d=2) - S^2_m(d=1) \right) p,
\]

because that many new consumers, coming from the other firm after finding its discount price, discover \( p \) earlier and buy for that.

The cases where the firm deviates to a larger number of discount prices \( d > 2 \) must also be covered; the conditions are then basically the same: just replace each \( d = 2 \) with some \( d = d^* \).

**PROOF OF PROPOSITION 1**

This can be obtained directly from the above proof of Theorem 2 by inserting appropriate values of (6) into the expressions (10), (11), (12), and (7) that were derived for \( G \) and \( II \).

**PROOF OF PROPOSITION 2**

This is a special example of Proposition 1. The range of values \( \theta \) under which an \( OE(K) \) exists, \((0, -3+\sqrt{10}) = (0, \frac{1}{2}) \cup \left[ \frac{1}{3}, -3+\sqrt{10} \right] \), is obtained by assuming first zero atom size \( a = 0 \). With this we obtain that \( B_3 = \frac{\theta}{3} e^{-\theta} > B_2 = \frac{\theta^2}{2} e^{-\theta} \iff \theta \geq \frac{1}{3} \) (condition for zero atom size) and \( \frac{1}{2} B_1 = \frac{1}{2} \frac{\theta^2}{3} e^{-\theta} \geq \frac{1}{2} (\frac{1}{4} B_2 + \frac{1}{4} B_3) = \frac{1}{2} \left( \frac{\theta^2}{4} e^{-\theta} + \frac{1}{4} \frac{\theta^2}{3} e^{-\theta} \right) \iff -\theta^2 - 3\theta + 24 \geq 0 \iff \theta \in \left[ -3-\sqrt{10}, -3+\sqrt{10} \right] \) (condition for no profitable deviation). This gives us the first range of values \((0, \frac{1}{3}) \). We can then turn to the case of positive atom size \( a > 0 \). This itself implies that \( \theta < \frac{1}{3} \). We then observe that deviations are less profitable with \( a > 0 \) than with \( a = 1 \) because \( \frac{1}{2} B_1 - \frac{1}{2} (\frac{1}{4} B_2 + \frac{1}{4} B_3) + \frac{1}{2} (\frac{1}{4} B_2 + \frac{1}{4} B_3) > \frac{1}{2} B_1 - \frac{1}{2} (\frac{1}{4} B_2 + \frac{1}{4} B_3) \). In other words, all \( \theta < \frac{1}{3} \) will also do; there exist no profitable deviations for any of them because they are all included in \([-3-\sqrt{10}, -3+\sqrt{10}] \). This gives us the other range of values \((0, \frac{1}{3}) \).
PROOF OF THEOREM 3

So here we analyze only the first and the third terms of (2). The first term can be written as

\[ \left| \Delta C_m \right| (1 - p) = \sum_{N=1}^{\infty} B_N \left( \frac{(K-1)}{N} \right) - \frac{(K-2)}{N} \right) (1 - p), \]

and the third term can be written as

\[ \left| \Delta S_d - \sum w(K,N) \right| p = (1 - a) \sum_{N=1}^{\infty} B_N \sum_{p=1}^{N} \frac{1}{K} \left( \frac{(K-1)}{N-p} \right) - \frac{(K-2)}{N-p} \left) \right) p, \]

The price ratio can be written as

\[ \frac{1 - p}{p} = \frac{S_d + S_d^{-1}}{C_d + C_d^{-1}}, \]

\[ = 2(1 - a) \frac{\sum_{N=1}^{\infty} B_N \sum_{p=1}^{N} \frac{1}{K} \left( 1 - \frac{(K-1)}{(N-p)} \right)}{\sum_{N=1}^{\infty} B_N \sum_{p=1}^{N} \frac{1}{K} \left( a + (1 - a) \frac{(K-1)}{(N-p)} \right) + a \sum_{N=K+1}^{\infty} B_N \frac{2K-N}{K}}, \]

where we have used a newly defined function \( g(K,N,p) := 1 - \frac{(K-1)}{(N-p)} \). It is then easy to calculate that \( w(K,L) = \frac{(K-L)L}{(K-1)K} \) and simplify \( g \) slightly to obtain

\[ w(K,N) = \frac{(K-N)N}{(K-1)K} \text{ and } w(K,N-p) = \frac{(K-N+p)(N-p)}{(K-1)K} \]

\[ g(K,N,p) = \frac{N-p}{K} \text{ and } 1 - g(K,N,p) = \frac{K-N+p}{K}. \]

We want to show that, for large values of \( K \), the first term of (2) is at least as large as the third term of (2). This requirement can now be stated as

\[ \sum_{N=1}^{\infty} B_N \sum_{p=1}^{N} \frac{K}{N} w(K,N) \geq \sum_{N=1}^{\infty} B_N \sum_{p=1}^{N} \left( 1 - (1 - a) g(K,N,p) + aK \sum_{N=K+1}^{\infty} B_N \frac{2K-N}{K} \right) \]

\[ \geq \sum_{N=1}^{\infty} B_N \sum_{p=1}^{N} w(K,N-p) \]

We have inserted the derived expressions for \( w \) and \( g \), canceled out two \((1 - a)\)'s and multiplied

\[ 43 \text{We can from now on ignore } T(N \leq K) \text{ and } T(N - p \leq K) \text{ because we are doing this only for taking the limit } K \to \infty; \text{ both } T \text{ 's will equal one for larger values of } K. \]

\[ 44 \text{We have here replaced } 1 - \frac{(K-1)}{(N-p)} \text{ by } \frac{2K-N}{K}. \]
both numerators and denominators by $K$. Next, to take the limit of this, note that functions $w$ and $g$ appear both twice in symmetric positions there, after the double summations. These terms can now be rewritten as

\[
\frac{K}{N}w(K, N) = K \frac{(K - N)N}{(K - 1)K} \frac{1 - \frac{N}{K}}{1 - \frac{1}{K}} \to 1, \text{ as } K \to \infty,
\]

\[
1 - g(K, N, p) = \frac{K - N + p}{K} \frac{1 - \frac{N}{K} + p}{1 - \frac{1}{K}} \to 1, \text{ as } K \to \infty,
\]

\[
Kw(K, N - p) = K \frac{(K - N + p)(N - p)}{(K - 1)K} = (N - p) \frac{1 - \frac{N}{K} + \frac{p}{K}}{1 - \frac{1}{K}} \to N - p, \text{ as } K \to \infty,
\]

\[
Kg(K, N, p) = K \frac{N - p}{K} = N - p.
\]

This implies that, as long as $\sum_{N=K+1}^{N} B_N = O\left(\frac{1}{K}\right)$ such that $aK \sum_{N=K+1}^{N} B_N \frac{2K-N}{K} \to 0$ as $K \to \infty$ (we need this convergence) and $1 - a = O\left(\frac{1}{K}\right)$ (any $a$ would be just fine here), taking both sides of the inequality to the limit gives

\[
\frac{K}{N}w(K, N) \to 1 \
1 - g(K, N, p) \to 1 \
Kw(K, N - p) \to \frac{1}{2}
\]

Note that the same analysis applies for deviations to a larger number of discount prices $d > 2$ because the limits are just the same for those cases. We just replace $w(k, N)$ with $w(k, N, d)$, where $d$ would then be some larger fixed number:

\[
w(k, N, d) = \frac{(K-1)N}{K} \frac{K-d}{N} = \frac{K-N}{K} \left(1 - \frac{(K-1-N)!}{(K-d-N)!}\right)
\]

\[
= K - N \left(1 - \frac{(K-1-N)\cdots(K-d-N+1)}{(K-1)\cdots(K-d+1)}\right)
\]

\[
= K - N \left(1 - \frac{(K-1-N)K^{d-2}(1 - \frac{2}{K} - \frac{N}{K})\cdots(1 - \frac{d}{K} - \frac{N}{K} + \frac{1}{K})}{(K-1)K^{d-2}(1 - \frac{2}{K} - \frac{N}{K})\cdots(1 - \frac{d}{K} + \frac{1}{K})}\right)
\]

\[
= K - N \left(1 - \frac{(K-1-N)(1 - \frac{2}{K} - \frac{N}{K})\cdots(1 - \frac{d}{K} - \frac{N}{K} + \frac{1}{K})}{(K-1)(1 - \frac{2}{K} - \frac{N}{K})\cdots(1 - \frac{d}{K} + \frac{1}{K})}\right) \to \frac{a_{K,N,d}}{\pi^{d-1}} \text{ as } K \to \infty
\]

We also want to make sure that the firm has no profitable deviation to keeping infinite numbers of discount prices. In particular, if the firm had just discount prices the change in its profit from $C^m_i$ would be $\sum_{N=1}^{\infty} B_N \sum_{p=1}^{N} \frac{1}{K} \frac{K-N}{K}(1 - p)$ and $S^d_i$ would be $\sum_{N=1}^{\infty} B_N \sum_{p=1}^{N} \frac{1}{K} \left(\frac{K-N+p}{K}\right) p$. These can be compared in the similar manner as before.\(^{45}\)

\(^{45}\)Same kind of comparisons can be made for cases where there is a finite number $m$ of monopoly prices
\[
\frac{K}{1 - a} \frac{1 - p}{p} = \frac{2 \sum_{N=1}^{\infty} B_N \sum_{p=1}^{N}(N - p)}{\sum_{N=1}^{\infty} B_N \sum_{p=1}^{N}(1 - (1 - a)g(K, N, p)) + aK \sum_{N=K+1}^{\infty} B_N \frac{2K-N}{K}} \rightarrow D^2[N] \frac{1}{1},
\]

because

\[
2 \sum_{N=1}^{\infty} B_N \sum_{p=1}^{N}(N - p) = 2 \sum_{N=1}^{\infty} B_N(N^2 - N \frac{1 + N}{2})
= 2 \sum_{N=1}^{\infty} B_N(N^2 - \frac{N}{2}) = \sum_{N=1}^{\infty} B_N(N^2 - N)
= E[N^2] - E[N] = D^2[N],
\]

whereas

\[
\frac{K}{1 - a} \frac{|\Delta S_{m}^{-1}|}{|\Delta S_{m}|} = \frac{\sum_{N=1}^{\infty} B_N \sum_{p=1}^{N}(1 - \frac{N}{K} + \frac{p}{K})}{\sum_{N=1}^{\infty} B_N \sum_{p=1}^{N} \frac{1}{N}(1 - \frac{N}{K})} \rightarrow E[N] \frac{1}{1 - B_0}.
\]

So, particularly, this implies that Poisson distribution does not satisfy these requirements as such because for that case \(E[N] = D^2[N] = \theta.\) However, a weighted Poisson variable \(N \sim P(w\theta)\) where \(w > 0\) and \(E[w] = 1\) works fine since then \(E[N] = \theta\) but \(D^2[N] = \theta^2D^2[w].\) In that case we would also need to require \(E[N]D^2[w] \geq \frac{B_0}{1 - B_0}. \square\)

\[
\begin{align*}
C_m^{1} &= \frac{1}{2} \sum_{N=1}^{\infty} B_N T(N \leq K) \frac{1}{K-N} \theta^N \\
C_m^{2} &= \frac{1}{2} \sum_{N=1}^{\infty} B_N \sum_{p=1}^{\min\{N,K\}} \frac{1}{K-p+1} (a + (1 - a)T(N - p \leq K) \frac{1}{K-N-p}) \\
S_m^{1} &= \frac{1}{2} \sum_{N=1}^{\infty} B_N \sum_{p=1}^{\min\{N,K\}} \frac{1}{K-p+1} (1 - a - (1 - a)T(N - p \leq K) \frac{1}{K-N-p}) \\
C_m^{3} &= \frac{1}{2} \sum_{N=1}^{\infty} B_N T(N - K \leq K) \frac{1}{K-N-K} \\
C_m^{4} &= \frac{1}{2} \sum_{N=1}^{\infty} B_N \left(1 - T(N - K \leq K) \frac{1}{K-N-K}\right) \\
S_m^{2} &= \frac{1-a}{2} \sum_{N=1}^{\infty} B_N \sum_{p=1}^{\min\{N,K\}} \frac{1}{K-p+1} (1 - T(N - p \leq K) \frac{1}{K-N-p})
\end{align*}
\]

Table 3: Captives and shoppers for a firm \(i\) with a discount price.

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and an infinite number \(K - m\) discount prices; this is so because \(\frac{m}{K} \rightarrow 0\) and \(\frac{m}{K-N-p} \rightarrow 0\) as \(K \rightarrow \infty.\)
References


