On Heavy-tailed Risks with Applications to Insurance and Finance

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for the degree of Doctor of Philosophy

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Preface

The idea of writing this work emerged during the finishing phase of my master's thesis in 2011. Its topic concerning the interplay between two types of heavy-tailed risks was a promising starting point for more dedicated research efforts.

At an early stage of the research it became apparent that heavy-tailed random variables are central to understanding extremal events. Despite the extensive literature on both light- and heavy-tailed variables in general, there seemed to be a gap between the two extremes. This led to the discovery of a central concept of the work, the so called concave natural scale of a heavy-tailed random variable. The book is about heavy-tailed random variables and the phenomena they induce to mathematical models.

This thesis would not have been possible to make without the diligent guidance of Harri Nyrhinen during the writing process. The continuous support of Esa Nummelin as well as the numerous discussions with him also played an important part in its success. In addition, I want to thank my fellow students of the doctoral program, colleagues and other friends whose support and friendship has made the work immensely more enjoyable.

In addition, I want to thank the pre-examiners Tommi Sottinen and Lasse Leskelä for careful reading of the manuscript.

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Finally, I wish to express my most sincere thanks to my parents, Sirkka and Ahti, as well as my grandmothers Maire and Anna, who have patiently supported me on my quest in the academic world.

Helsinki, 18.1.2016
Jaakko Lehtomaa

Contents

1 Included Articles

The dissertation consists of this introduction and the following four articles. The aim of the introductory part is to give a clear overview of the new results with minimal emphasis on technicalities. The articles are referenced in the text by Roman numerals [I]-[IV][1].

[I] Lehtomaa, J., 2015. Asymptotic behaviour of ruin probabilities in a general discrete risk model using moment indices. *J. Theoret. Probab.* 28(4), 1380-1405. **26 pp**. DOI:10.1007/s10959-014-0547-y.

[II] Lehtomaa, J., 2015. Limiting behaviour of constrained sums of two variables and the principle of a single big jump. *Statist. Probab. Lett.* 107, 157-163. **7 pp**. DOI:10.1016/j.spl.2015.08.017.

[III] Lehtomaa, J., 2015. Large deviations of means of heavy-tailed random variables with finite moments of all orders. *Submitted.* **27 pp**.

[IV] Lehtomaa, J., 2015. Logarithmic asymptotics of tails of independently stopped random walks. *Preprint.* **21 pp**.

1.1 Contribution of the Author

All of the articles [I]-[IV] are independent work of the author. The work has been done entirely in the University of Helsinki.

1.2 Interrelations of Articles [I]-[IV]

The articles [I], [II] and [III] have no mathematical dependencies from each other. The article [IV] is largely based on the concept of natural scale, whose existence is constructively shown in [III].

[1]Articles [I] and [II] are reprinted by the kind permissions of their publishers Springer and Elsevier, respectively.

2 Prologue

Modern financial agents participate in market operations whose adverse outcomes have dramatic consequences. The maximal conceivable loss typically exceeds the amount of liquid assets by multiple magnitudes, making agents vulnerable to large market fluctuations. From the mathematical viewpoint these fluctuations are caused by extremal events in cash flows, which are sensitive to the tail distributions of the underlying random variables.

It has become increasingly clear that the classical models employing light-tailed variables have an inherent tendency to underestimate the magnitude of extremal events. Recent developments in the financial world (financial crises) suggest that such shortcomings are not merely theoretical curiosities, but game-changing phenomena that can solely determine the fate of an agent.

To overcome the obstacles set by the classical lines of thought an agent must consider ways to improve the way *risk* is modelled and assessed. The work proposes two ways to do this. Firstly, the existing models should be reconfigured to include the effects of different types of risks. Secondly, the used models should be compatible with heavy-tailed effects such as *the principle of a single big jump*.

The combination of different types of risks into the same category increases uncertainty. Although some decision between the level of sophistication and simplicity of calculations must be made, the used model should be able to separate the fundamentally distinct risks. The context of insurance and finance offers some natural candidates for these.

An insurance company has two common risk types: *insurance risks* and *financial risks*. Insurance risks originate from the aggregate claims, whereas the financial risks are due to the uncertainties involved with the investment of the available assets at a given time. The insurance risk may further consist of a random number of claims of random sizes.

This work aims to understand the nature of risks by splitting them into components. In this analysis, one wishes to discover the dominating, or most hazardous, risks. That is, to find out if large losses are due to a single big claim or many small claims, or if the claims themselves are not a source of problems at all, but the large potential losses stem from the investment activities alone.

Once the components of the model have been established, the marginal distributions of the risks need to be specified. For this, the work proposes the use of heavy-tailed random variables. Since the heavy-tailed variables play such a central role in the story of realistic risks, their general properties seem to be worth pursuing in their own right as well.

The work includes a novel idea to complement the current classification of heavy-tailed risks. It is based on the explicit tail decay speed of a single variable rather than the implicit large scale behaviour of several variables. The benefit of the approach is that the principles of dominance can be understood in a great generality. Each heavy-tailed variable is assigned a function called a *natural scale* by which the study of heavy-tailed phenomena is made possible.

To summarise, wrongly assessed risks may lead to catastrophic chain reactions and insolvency. Thus, the efforts should be aimed to mitigate the possibility of harmful extremal events by deepening the understanding of the underlying random variables. The current work, hopefully, contributes to the improvement of accuracy of the best estimates concerning riskiness in both insurance and finance.

3 Introduction

The mathematical theory of ruin and solvency is fundamentally based on the theory of random walks and their generalisations. *A random walk* is the process (S_n) defined as

$$S_n = X_1 + X_2 + \ldots + X_n, \tag{3.1}$$

where $X, X_1, X_2, \ldots$ is an i.i.d. sequence of random variables. In the early 1900s, behaviour of random walks had already been studied in various contexts, most notably in the form of laws of large numbers and central limit theorems. The study of random walks in insurance contexts originated during this time from the works of H. Cramér and F. Lundberg.

The random walk as acts as a starting point for the more advanced models. Specifically, the following two stochastic generalisations based on random walks are discussed in this book. The first one is a randomly weighted random walk, defined using an i.i.d. sequence $(A, B), (A_1, B_1), (A_2, B_2), \ldots$ as

$$Y_n = B_1 + A_1B_2 + A_1A_2B_3 + \ldots + A_1 \ldots A_{n-1}B_n. \tag{3.2}$$

The second one is the random walk, which is stopped at a random time and defined by

$$S_N = X_1 + X_2 + \ldots + X_N, \tag{3.3}$$

where N is an integer valued random variable.

In classical analysis all of the variables involved in (3.1), (3.2) and (3.3) are assumed to be *light-tailed*. This means that there exists some finite exponential moment, i.e. $E(e^{sX}) < \infty$ for some $s > 0$. The condition is usually called the *Cramér's condition*. If the Cramér's condition does not hold, then the variable is *heavy-tailed*.

It turns out that the key function in asymptotic analysis of heavy-tailed variables is not the *tail function* $\overline{F}(x) = P(X > x)$, but the *hazard function*

$$R_X(x) = R(x) = -\log P(X > x) \tag{3.4}$$

or a suitable approximation of R. For light-tailed distributions we define the *exponential index* $\mathcal{E}(X)$ by

$$\mathcal{E}(X) = \liminf_{x\to\infty} \frac{R_X(x)}{x} = \sup\{s \geq 0 : E(e^{sX}) < \infty\}, \tag{3.5}$$

where the last equality can be found from e.g. [17] or [18]. For light-tailed variables the rate at which the tail function $\overline{F}$ decreases is at least exponential, that is, $\overline{F}(x) \leq e^{-ax}$ for some $a > 0$ and all x large enough. If X is not light-tailed, it is heavy-tailed. In this situation one can use the *moment-index* defined by

$$\mathbb{I}(X) = \liminf_{x\to\infty} \frac{R_X(x)}{\log x} = \sup\{s \geq 0 : E((X^+)^s) < \infty\} \tag{3.6}$$

instead. If $0 < \mathbb{I}(X) < \infty$, polynomial bound $\overline{F}(x) \leq x^{-b}$ for some $b > 0$ and all large enough x may be obtained, whereas the exponential bound is not possible. It is important to note that there exist distributions such that both of their indices (3.5) and (3.6) share the same value zero or infinity.

The aim of the book is to investigate what happens in the mentioned models (3.1), (3.2) and (3.3) when some of the variables are not assumed to be light-tailed, but heavy-tailed. This is done mostly in the asymptotic sense. More specifically, the results concern the logarithmic asymptotics of tails of random variables. The benefit of this approach is that complicated models can be investigated under the most general possible assumptions.

Usually, the starting point for the analysis is to find out the moment indices of the relevant quantities. More generally, similar analysis can be performed using *asymptotic scales* which are analogous to the indices (3.5) and (3.6) but can be applied to a wider class of heavy-tailed distributions.

3.1 General Ruin Model and Review of [I]

Suppose the sequence (U_n) describes the capital available to an insurance company at the end of the year n. The initial capital is a fixed number $U_0 > 0$. The rest of the sequence is then defined by the recursive relation

$$U_n = (1 + r_n)(U_{n-1} - B_n), \quad n = 1, 2, \ldots,$$

where r_n represents the stochastic return on investment over the period $n - 1$ to n and (B_n) is the aggregate claim process.

The process (U_n) is used to define the time of ruin T by

$$T_{U_0} = \inf\{n : U_n < 0\}.$$

Direct calculation shows that by setting $A_n = 1/(1 + r_n)$ it holds under the assumption $r_n > -1$ that

$$T_{U_0} = \inf\{n : Y_n > U_0\},$$

where (Y_n) is defined as in (3.2). This means that in order to study the probability of ruin event, one can study the level crossing probabilities of the weighted random walk (Y_n). In this sense, the process (Y_n) acts as a general ruin model. Two auxiliary variables are defined using the process (Y_n) by

$$\bar{Y}_n = \sup_{1 \le k \le n} Y_k \text{ and } \bar{Y} = \sup_k Y_k. \tag{3.7}$$

This allows one to form the ruin probabilities

$$P(\bar{Y}_N > U_0) \text{ and } P(\bar{Y} > U_0) \tag{3.8}$$

corresponding to finite and infinite time intervals, respectively. We are mainly interested in how the quantities of (3.8) respond to changes in initial capital. The aim is to discover, in an asymptotic sense, how a perturbation in the initial capital affects the ruin probabilities.

In many cases, especially when heavy-tailed random variables are involved, these probabilities decay at a polynomial rate. Quantities of interest are then

$$\limsup_{U_0 \to \infty} \frac{\log P(\bar{Y}_N > U_0)}{\log U_0} \text{ and } \limsup_{U_0 \to \infty} \frac{\log P(\bar{Y} > U_0)}{\log U_0}. \tag{3.9}$$

The quantities of (3.9) can be interpreted as the polynomial rates at which the ruin probability decays as the initial capital grows. For the insurance application it is important to note that they specify an eventual upper limit for the ruin probability.

The article [I] is devoted to the study of (3.9) under minimal assumptions when the sequence $((A_i, B_i))$ is an i.i.d. sequence. The main contribution of [I] is to prove the following theorem. The proof is based on a new operational characterisation of the condition $P(\bar{Y} > y) > 0$ for all $y > 0$.

Theorem 3.1. *Suppose the sequences (A_i) and (B_i) satisfy the following:*

1. *The sequence $((A_i, B_i))$ consists of i.i.d. random vectors.*
2. *The members of (A_i) are strictly positive and A is not the constant* 1.
3. *The members of (B_i) are real valued and $P(B > 0) > 0$.*

Assume further that $P(\bar{Y} > y) > 0$ for all $y > 0$.

Then

$$\limsup_{U_0 \to \infty} \frac{\log P(T_{U_0} < \infty)}{\log U_0} = -\min(\mathbb{I}^1(A), \mathbb{I}(B)), \tag{3.10}$$

that is, $\mathbb{I}(\bar{Y}) = \min(\mathbb{I}^1(A), \mathbb{I}(B))$, *where* $\mathbb{I}^1(Y) = \sup\{s \ge 0 : E(Y^s) \le 1\}$.

3.2 Principle of a Single Big Jump and Review of [II]

The random walk presented in (3.1) is one of the simplest models that exhibits the *principle of a single big jump*. The term refers to the phenomenon where a large deviation of the sum of random variables is realized by only one of them taking a very large value. This behaviour is usually associated with variables that have heavy tails, and is in contrast to the Gibbs' conditioning principle when the Cramer's condition holds.

This principle is a recurring theme of the book. In [II], the principle is studied in the simplest conceivable environment where there are only two variables. We study the asymptotic properties of the variable

$$Z_d := \frac{X_1}{d}\Big|\{X_1 + X_2 = d\}, \tag{3.11}$$

as $d \to \infty$. Here X_1 and X_2 are non-negative i.i.d. variables with a common twice differentiable density function f. The variable Z_d is the relative contribution of the first component X_1 to the sum $X_1 + X_2$. The following convergence types are then defined for the laws of random variables in the limit $d \to \infty$.

1. $\mathcal{L}(Z_d) \to \frac{1}{2}\delta_0 + \frac{1}{2}\delta_1$ and
2. $\mathcal{L}(Z_d) \to \delta_{\frac{1}{2}}$.

Behaviour 1 resembles the way many heavy-tailed variables are known to behave: if the sum $X_1 + X_2$ is large then one of the variables is large. Behaviour 2 is related to a phenomenon encountered within the class of light-tailed distributions: both of the variables X_1 and X_2 contribute equally.

The density function f_{Z_d} of the variable Z_d is concentrated in the interval $[0, 1]$ and given by formula

$$f_{Z_d}(x) = \frac{f(dx)f(d(1-x))}{\displaystyle\int_0^1 f(dy)f(d(1-y))\,\mathrm{d}y}, \quad x \in [0,1]. \tag{3.12}$$

It can be directly obtained from the conditional distribution of X_1/d given $X_1 + X_2$ by a transformation of variables in the resulting integral. Formula (3.12) may not be defined for all d. It is well-defined under general assumptions eventually, as d grows.

In [II], general results concerning the distributional limits of Z_d are discussed with various examples. Eventual log-convexity or log-concavity of f turns out to be the key ingredient that determines how the variable Z_d behaves. As a consequence, two surprising discoveries are presented: Firstly, it is noted that the distributional limit is not strictly determined by the decay rate of the tail function. Secondly, it is shown that there exists a light-tailed distribution exhibiting behaviour that is commonly associated with heavy-tailed distributions i.e. the principle of a single big jump.

The main contributions of [II] are based on the simple observation that links eventual log-concavity or log-convexity of f to the behaviour of f_{Z_d}.

Lemma 3.2. *Suppose*

$$L := \lim_{x\to\infty} \operatorname{sign}\left(\frac{d^2}{dx^2}\log f(x)\right) \tag{3.13}$$

exists, where

$$\operatorname{sign}(x) := \begin{cases} 1 & : x > 0 \\ 0 & : x = 0 \\ -1 & : x < 0. \end{cases}$$

Then the function f_{Z_d} of Formula (3.12) *is eventually, in d, strictly convex with respect to the variable x at point $x = 1/2$ if and only if $L = 1$. Similarly, f_{Z_d} is eventually, in d, strictly concave with respect to the variable x at point $x = 1/2$ if and only if $L = -1$.*

Additional conditions need to be imposed in order to obtain Behaviour 1 or 2. Proposition 3.3 does exactly this, but it requires that $f_{Z_d}(x) \to 0$ for all $x \in (0, 1/2)$. This may be tedious to check unless the density is extremely simple. However, Theorem 3.4 provides a sufficient condition which guarantees the validity of the required property.

Proposition 3.3. *Suppose the limit L of Equation* (3.13) *exists. Assume further that $f_{Z_d}(x) \to 0$ for all $x \in (0, 1/2)$, as $d \to \infty$.*

If $L = 1$, then 1 holds. If $L = -1$, then 2 holds.

Theorem 3.4. *Suppose f is eventually strictly log-convex or log-concave. Assume further that*

$$\lim_{d\to\infty} d\left|\frac{f'(dx)}{f(dx)} - \frac{f'(d(1-x))}{f(d(1-x))}\right| = \infty \tag{3.14}$$

for every $x \in (0, 1/2)$.

Then $f_{Z_d}(x) \to 0$ for every $x \in (0, 1/2)$.

In summary, the new results create a link between the Behaviour types 1 and 2 and the eventual convexity or concavity of the function $\log f$.

3.3 Large Deviations of Means and Review of [III]

In [III], the mean process (S_n/n) is studied as $n \to \infty$. The topic is closely related to the laws of large numbers and central limit theorems. In this sense, statements about the large deviations of means can be seen as a natural continuation to the classical theme.

The aim is to study the logarithmic asymptotics of $P(S_n > an)$ for $a > E(X)$. More specifically, the asymptotic behaviour of the quantity

$$\frac{-\log P(S_n > an)}{g(n)} \tag{3.15}$$

is studied in a large class of increment distributions under different choices of the *scaling function* g, as $n \to \infty$. The function g will be the hazard function

$$R(x) = -\log P(X > x)$$

or an approximation of R.

The quantity (3.15) is extensively studied in a vast number of papers and books. However, the bulk of the research is focused into two major categories of increments: light-tailed and extremely heavy-tailed. Between the distributions with polynomial or heavier tails and light-tailed distributions lies a rich but relatively little understood set of distributions. They share the following two properties:

1. Members are heavy-tailed
2. Members have finite moments of all orders, i.e. $E((X^+)^\alpha) < \infty$, for all $\alpha > 0$.

The paper [III] is meant to narrow the research gap between light- and seriously heavy-tailed distributions. Specifically, the paper studies the asymptotics of the quantity (3.15) when the increment of the random walk S_n satisfies Requirements 1-2.

All hazard functions corresponding to a heavy-tailed distribution admit a concave approximation called *natural scale*. It is a smooth function that approximates R from below in an asymptotic sense. Related ideas have been previously presented in [34, 20]. However, the concavity seems to be the property that truly makes the concept of natural scale operational.

Theorem 3.5. *Let R be the hazard function of X. Suppose X is heavy-tailed i.e.*

$$\liminf_{x\to\infty} \frac{R(x)}{x} = 0. \tag{3.16}$$

Then there exists a concave function $\underline{R}\colon [0,\infty) \to [0,\infty)$ satisfying:

1. $\underline{R} \geq 0$
2. $\underline{R}(0) = 0$
3. $\underline{R}(x) = o(x)$, *as* $x \to \infty$ *and*
4. $$\liminf_{x\to\infty} \frac{R(x)}{\underline{R}(x)} = 1. \tag{3.17}$$

Condition (3.17) *is equivalent to*

$$\sup\left\{s \geq 0 : E(e^{s\underline{R}(X)}) < \infty\right\} = 1. \tag{3.18}$$

A concave function $\underline{R}$ satisfying properties 1-4 of the Theorem 3.5 is called a natural scale of X. Note that the concept of natural scale does not fix a unique function. A good initial guess for finding some natural scale in practice is $R = -\log P(X > x)$ itself or a dominating component of R.

The result of Theorem 3.5 is analogous to indices (3.5) and (3.6). The twist is that the function $\underline{R}$ is chosen based on the properties of the hazard function R instead of using fixed choices $\underline{R}(x) = x$ or $\underline{R}(x) = \log x$. This enables one to choose from different shapes of scale functions and expand the idea of exponential and moment indices.

Under 1-2 and some additional general assumptions specified in [III], it holds that

$$\limsup_{n\to\infty} \frac{-\log P(S_n > n)}{R(n)} = 1. \tag{3.19}$$

and

$$\liminf_{n\to\infty} \frac{-\log P(S_n > n)}{\underline{R}(n)} = 1. \tag{3.20}$$

Equations (3.19) and (3.20) are the key to proving more advanced results, say large deviations principles for suitable heavy-tailed mean processes. The following theorem illustrates this fact by presenting a result for regularly varying hazard functions. This class is important, because it includes the famous class of *streched exponential distributions*.

Theorem 3.6. *Suppose* $E(X) = 0$, $X^- = \max(0, -X)$ *is light-tailed and* $R \in \mathcal{R}_\alpha$ *with* $\alpha \in (0, 1)$. *Then*

$$\lim_{n\to\infty} \frac{-\log P(S_n > an)}{R(n)} = \begin{cases} 0 & : a < 0 \\ a^\alpha & : a \geq 0. \end{cases} \tag{3.21}$$

Furthermore, the process $\{S_n/n\}$ satisfies full large deviations principle with good rate function

$$I(x) = \begin{cases} \infty & : x < 0 \\ x^\alpha & : x \geq 0 \end{cases} \tag{3.22}$$

and scale R, i.e.

$$-\inf_{y \in A^\circ} I(y) \leq \liminf_{n\to\infty} \frac{\log P(S_n/n \in A)}{R(n)} \leq \limsup_{n\to\infty} \frac{\log P(S_n/n \in A)}{R(n)} \leq -\inf_{y \in \mathrm{cl}(A)} I(y)$$

for all Borel sets $A \subset \mathbb{R}$.

The class $\mathcal{R}_\alpha$ mentioned in Theorem 3.6 is the class of regularly varying functions with index α, see e.g. [12] for details.

3.4 Tails of Randomly Stopped Sums and Review of [IV]

Randomly stopped sums (3.3) are some of the most studied randomly stopped processes. Typically, one is interested to know how the tail of S_N is affected by its increments and the stopping variable. Identification of the dominant variable of the stopped sum has applications in e.g. insurance, where the compounded variable S_N is used to model the aggregate loss of a company. In this setting, the aim is to find out if large losses are mainly caused by few big claims or unusually large amounts of small claims.

Asymptotic form of the tail of S_N has been studied extensively under different distributional assumptions. Specifically, the central topic has been to study if the tail behaviour of $P(S_N > x)$, as $x \to \infty$, corresponds to that of

1. $E(N)P(X > x)$ or
2. $P(E(X)N > x)$.

In 1, the increment X is dominating while in 2 the dominating variable is N. The article [IV] contributes to this topic by presenting the following new results. In [IV], it is assumed that the sequence of increments (X_i) is an i.i.d. sequence and the stopping variable N is independent of (X_i).

At the level of moments one can obtain the following result.

Proposition 3.7. *Assume $S_n \to \infty$ almost surely.*

If at least one of the expectations $E(|X|)$ or $E(N)$ is finite, then

$$\mathbb{I}(S_N) = \min(\mathbb{I}(X), \mathbb{I}(N)). \tag{3.23}$$

If $E(X^+) = E(N) = \infty$, it holds that

$$\mathbb{I}(X)\mathbb{I}(N) \leq \mathbb{I}(S_N) \leq \min(\mathbb{I}(X), \mathbb{I}(N)). \tag{3.24}$$

More generally, the following theorem holds for asymptotic scales of randomly stopped sums. The necessity of the generalisation becomes apparent in situations where the increment or stopping variable has a relatively light but still heavy-tailed distribution. Specifically, this is the case with Weibull and lognormal type variables, whose moments of all orders are finite. This is the class of distributions studied in [III].

Theorem 3.8. *Let X and N be heavy-tailed random variables. Assume* $0 < E(X) = \mu < \infty$.

1. *Suppose $E((X^+)^{1+\delta}) < \infty$ for some $\delta > 0$. Let h_X be a natural scale of X such that $h_X(x) \geq (1+\delta)\log x$ for all large enough x. If*

$$\liminf_{x\to\infty} \frac{h_N(x/c_1)}{h_X(x)} \in [1,\infty] \tag{3.25}$$

holds for some natural scale h_N of N and some $c_1 > \mu$, then h_X is a natural scale of S_N.

2. *Suppose $E(N^{1+\delta}) < \infty$ for some $\delta > 0$. Let h_N be a natural scale of N such that $h_N(x) \geq (1+\delta)\log x$ for all large enough x. If*

$$\liminf_{x\to\infty} \frac{h_X(x)}{h_N(x/\mu)} \in [1,\infty] \tag{3.26}$$

holds for some natural scale h_X of X, then $h_N(x/\mu)$ is a natural scale of S_N.

Theorem 3.8 tells the asymptotic scale of the stopped sum. This information can be used to analyse the moment determinacy of randomly stopped sums. Moment determinacy (in Stieltjes' sense) of an unbounded non-negative random variable X means that for all non-negative random variables Y:

$$E(X^k) = E(Y^k),\ \forall k \in \mathbb{N} \Longrightarrow X \stackrel{d}{=} Y. \tag{3.27}$$

If Implication (3.27) does not hold, the distribution is called *moment indeterminate*. The main application of [IV] is the fact that Theorem 3.8 can be used to give a new sufficient condition for moment determinacy randomly stopped sums. In contrast to previous results, the condition presented in [IV] is suitable also to the case of heavy-tailed stopping variable N.

4 Articles [I]-[IV]

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