

Quasiconformal and p -energy minimizing maps between metric spaces

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Academic dissertation

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How does a Finnish and Greek kid go from gangsta rap and skateboarding (my childhood future career) to the frontlines of science?

I am not sure but I suspect my family shares the majority of the blame. Which is incredible, because none of them have personal experience of any of those three things (apart from my brother, my beloved hip hop head).

What you have done for me – all of you, whether near or afar, present or passed on or not yet here – is, in all honesty, more incredible than anything I've accomplished here.

Äiti, Μπαμπά, Γιαννάκο, I shall not endeavour to explain your role in my life; I just hope I make you proud because it is ultimately why I did any of it.

By the time I entered university I had fallen in love with mathematics even though, looking back, I realise I understood next to nothing about it. Through the inevitably thinning ranks of students I and a few fellow companions – whom I am happy to call my friends – survived to glimpse what mathematics really is.

I want to thank Eero Saksman for my first glimpse, my advisor Ilkka Holopainen for giving me the freedom to explore the mathematical universe all the while guiding me in the seemingly endless aspects of academia, and Pekka Pankka for being the best academic big brother one can have. You three shaped me into the mathematician I am today. Also, I am indebted to Ilkka Holopainen, the Academy of Finland and the Finnish Academy of Science and Letters (Vilho, Yrjö ja Kalle Väisälän rahasto) for financial support.

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To keep the acknowledgements shorter than the rest of the dissertation I won't express all the thanks each of you deserve by name, but I will say this: thank you for the beers, the wine, the fond memories, the backgammon (and chess victories), the food, the good and bad jokes, the discussions on philosophy, culture, religion, psychology, economics, mankind, mathematics, the music, the trips domestic and abroad, the randomness, and some other things.

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LIST OF INCLUDED PUBLICATIONS

This doctoral dissertation consists of an introductory part and three scientific articles, labeled [A,B,C]. The author had a central role in the research and writing of article [A].

[A] Elefterios Soultanis, Marshall Williams, *Distortion of quasiconformal maps in terms of the quasihyperbolic metric*. J. Math. Anal. Appl., 402 (2013), 626 – 634.

[B] Elefterios Soultanis, *Homotopy classes of Newtonian spaces*. Submitted for publication in Rev. Mat. Iberoam. (2016), arXiv:1309.6472

[C] Elefterios Soultanis, *Existence of p -energy minimizers in homotopy classes and lifts of Newtonian maps*. To appear in J. Anal. Math. (2016), arXiv:1506.07767

The author acknowledges J. Math. Anal. Appl. as the original publisher of article [A].

SUMMARY

Of the three papers, [A,B,C], comprising this dissertation, the first is on the connection of quasiconformal maps and the quasihyperbolic metric, while the remaining two concern notions of homotopy classes of Sobolev type maps between metric spaces, comparison with the manifold case, and the existence of minimizers of a p -energy in these homotopy classes.

The unifying theme of all three papers is analysis on metric spaces. That is, all three papers deal with questions concerning maps between metric spaces. The particular type of metric spaces involved is generally referred to as PI-spaces. These satisfy conditions allowing one to extend a large part of classical first order calculus, such as the theory of Sobolev maps [69], and *à posteriori*, differentiability of Lipschitz functions [14].

In the first part of the introduction we give a very brief historical background and summarise what have come to be the standard assumptions in analysis on metric spaces – Poincaré inequalities and doubling measures – and discuss some of their implications. The second part concerns quasiconformal maps. It explains the setting and results of the first publication [A]. The last part is devoted to the last two articles [B,C] and in it we discuss the background and describe the main question of minimizing a p -energy functional in a homotopy class.

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1. ANALYSIS ON METRIC SPACES

Finnish mathematics has a long tradition in geometric function theory, starting from Lindelöf, Nevanlinna and continuing with Ahlfors. Geometric function theory studies *analytic*, or more generally, *meromorphic* functions in the complex plane. Subsequent generations have followed the natural path into quasiconformal and quasiregular mappings, a generalization of conformal and analytic functions in the complex plane, creating a rich tradition in the field what is now known as geometric analysis.

Although originally defined by Grötzsch [31] and studied by Ahlfors, Lehto, Virtanen, Lavrentiev and many others in the complex plane [55, 2, 56], quasiconformal and quasiregular maps turned out to have a rich theory in domains of n -space. (See for instance [70, 65].)

In higher dimensions the theory of quasiconformal maps was initiated by Väisälä and Gehring, [26] and further developed by Väisälä, Rickman, Martio, Gehring [58, 59, 60, 23, 24] to mention only a few. See also [64] for a treatment of both quasiconformal and quasiregular maps. In [25] Gehring and Osgood discovered that there is a connection between quasiconformal maps and distortion of the quasihyperbolic metric. This connection is the subject of [A] and will be discussed in more detail in section 2.

Subsequently quasiconformal maps have taken life between manifolds and, coming to the turn of the millenium, in more general metric spaces. While the analytic definition of quasiconformality readily generalizes to manifolds the metric case requires new techniques to make sense of the concepts needed in the definition.

Two notable pioneers in this respect were Heinonen and Koskela who generalized the notion of (the norm of) the gradient [43] and gave rise to the definition of the Newtonian spaces [69] over a metric measure space – function spaces of Sobolev-type functions.

Given an almost everywhere real valued function $u : X \rightarrow \overline{\mathbb{R}}$ from a metric measure space (X, d, μ) , a nonnegative Borel function $\rho : X \rightarrow [0, \infty]$ is an *upper gradient* of u if for any rectifiable curve $\gamma : [a, b] \rightarrow X$ the inequality

$$(UG) \quad |u(\gamma(b)) - u(\gamma(a))| \leq \int_{\gamma} \rho$$

is valid. If either of the numbers $u(\gamma(b)), u(\gamma(a))$ is $+\infty$ or $-\infty$ then the lefthand side of (UG) is interpreted to be infinity.

Besides the theory of quasiconformal and quasiregular maps there were other factors motivating the study of Sobolev type function spaces on nonriemannian spaces. Slightly earlier Semmes [68] studied how Poincaré inequalities and existence of “thick” families of curves connecting any two points are related. His ideas shed light on the geometry of such spaces.

Among other motivators were subriemannian geometry, analysis on fractals, Sobolev spaces for Hörmander vector fields, Muckenhoupt weights and variational problems with respect to different underlying measures [53, 62, 57, 29, 39, 76, 40, 6, 46]. The latter subject particularly is connected to p -harmonic functions. Yet another strong influence is the connection between analysis on metric spaces and

geometric group theory, [11, 28, 27, 30, 52, 7]. This topic will be briefly revisited in section 3.

From the beginning it was clear that Poincaré inequalities play an important role in the theory of Sobolev spaces. For quasiconformal maps Heinonen and Koskela required the spaces to have an Ahlfors Q -regular measure ($Q > 1$) while Hajłasz's embedding theorems [32] essentially rely on the doubling property of the measure.

Today Poincaré inequalities and doubling measures have become the standard assumptions on *analysis on metric spaces*. Spaces satisfying these are very general, encompassing the Euclidean as well as many Riemannian manifolds and subriemannian spaces, yet a large part of first order calculus is available in them.

A Borel regular measure μ on a metric space X is said to be *doubling* if the measure of any open ball is positive and finite and there is a constant $C < \infty$ so that

$$(D) \quad \mu(B(x, 2r)) \leq C\mu(B(x, r))$$

for every ball $B(x, r) \subset X$.

A metric measure space (X, d, μ) is said to support a *weak $(1, p)$ -Poincaré inequality*, $1 \leq p < \infty$ if there are constants $C > 0$ and $\lambda \geq 1$ so that

$$(p\text{-PI}) \quad \int_{B(x,r)} |u - u_{B(x,r)}| d\mu \leq Cr \left(\int_{B(x,\lambda r)} g^p d\mu \right)^{1/p}$$

for every ball $B(x, r)$, every locally integrable function u and any upper gradient g of u .

1.1. The Poincaré inequality and short paths. Nowadays an extensive literature exists exploring the connections of Poincaré inequalities to modulus estimates of path families [45, 38, 68, 51]. The *p -modulus of a path family* is a potential theoretic concept, introduced originally by Beurling and Ahlfors [1] for $p = 2$ in their study of conformal maps.

Given a family of (locally) rectifiable paths Γ in a metric measure space (X, d, ν) , and a number $p \geq 1$, the p -modulus of Γ is defined to be the quantity

$$(M) \quad \text{Mod}_p(\Gamma; \nu) = \inf \left\{ \int_Y \rho^p d\nu : \rho \text{ nonnegative Borel, } \int_\gamma \rho \geq 1 \forall \gamma \in \Gamma \right\}.$$

If there is no ambiguity on the underlying measure ν it is customary to drop it from the notation and denote by $\text{Mod}_p(\Gamma)$ the modulus $\text{Mod}_p(\Gamma; \nu)$ of Γ . Modulus estimates provide a tool for establishing the existence of curves in a metric space since $\text{Mod}_p(\Gamma) > 0$ implies, in particular, that Γ must be a nonempty collection.

More quantitatively, modulus estimates can be used to express the fact that a space admits a Poincaré inequality. The following characterization of the $(1, p)$ -Poincaré inequality, due to Keith, quantifies the idea of Semmes on the existence of “thick” curve families in the metric setting.

Theorem. ([51, Theorem 2]) *Let (X, d) be a complete metric space with a doubling measure μ . Then (X, d, μ) admits a weak $(1, p)$ -Poincaré inequality, $1 \leq p < \infty$, if and only if there is a constant C so that for all $x, y \in X$ the modulus estimate*

$$(S) \quad d(x, y)^{1-p} \leq C \text{Mod}_p(\Gamma_{xy}; \mu_{xy}^C)$$

holds.

Here Γ_{xy} is the family of rectifiable paths joining x and y in X , and μ_{xy}^C is the measure given by

$$\int_X f d\mu_{xy}^C = \int_{B_{xy}^C} f(z) \left[\frac{d(x,z)}{\mu(B(x,d(x,z)))} + \frac{d(y,z)}{\mu(B(y,d(y,z)))} \right] d\mu(z), \text{ where}$$

$$B_{xy}^C = B(x, Cd(x,y)) \cup B(y, Cd(x,y)).$$

To illustrate how one can obtain short paths from such modulus estimates we sketch the proof of the following statement. The statement represents a typical phenomenon in PI-spaces and was known and proved by many people in the case $E = \emptyset$; it demonstrates the philosophy of Semmes – that PI-spaces enjoy good connectivity properties, quantitatively.

Theorem. *Let (X, d, μ) be a complete metric space with a doubling measure supporting a weak $(1, p)$ -Poincaré inequality, and suppose $E \subset X$ is a set with $\text{Cap}_p(E) = 0$. Then $X \setminus E$ is quasiconvex, with quasiconvexity constant depending only on the data of X , and in particular not on E .*

Recall that the p -capacity of a set $A \subset X$ is given by

$$\text{Cap}_p(A) = \inf \left\{ \int_X (|u|^p + g^p) d\mu : \chi_A \leq u, g \text{ an upper gradient of } u \right\}.$$

Key to the proof is the maximal function estimate

$$(E) \quad \int_X \rho^p d\mu_{xy}^C \leq Cd(x,y)[\mathcal{M}\rho^p(x) + \mathcal{M}\rho^p(y)],$$

where $\mathcal{M}f$ is the maximal function of $f \in L^1_{loc}$. This can be proven using standard methods (see for instance [38, pp.70-72]).

Proof of Theorem. Let $E \subset X$ be a zero p -capacity set, and fix $x, y \in X \setminus E$. Recall that the path family

$$\Gamma_0 = \{\gamma : \gamma^{-1}(E) \neq \emptyset\}$$

of nonconstant curves *which meet* E has zero p -modulus, [5, Proposition 1.48]. Thus there is a p -integrable nonnegative Borel function ρ such that $\int_\gamma \rho = \infty$ for all $\gamma \in \Gamma_0$.

Step 1. We show first that, for all $z, w \notin \{\mathcal{M}\rho^p = \infty\} =: N$, there is a curve $\gamma \in \Gamma_{xy} \setminus \Gamma_0$ with

$$\ell(\gamma) \leq Ad(z, w),$$

where A depends only on the data;

Indeed, using estimate (E) we see that $\text{Mod}_p(\Gamma_{xy} \cap \Gamma_0; \mu_{xy}^C) = 0$. Now if $\ell(\gamma) > Ad(x, y)$ for all $\gamma \in \Gamma_{xy} \setminus \Gamma_0$ then the constant function $g = 1/(Ad(x, y))$ is admissible for $\Gamma_{xy} \setminus \Gamma_0$. Using (S) and estimate (E) we obtain the estimate

$$d(x, y)^{1-p}/C \leq \text{Mod}_p(\Gamma_{xy}; \mu_{xy}^C) = \text{Mod}_p(\Gamma_{xy} \setminus \Gamma_0; \mu_{xy}^C) \leq \frac{Cd(x, y)}{A^p d(x, y)^p}.$$

Step 2. Applying Step 1 we find points $x_1 \in B(x, d(x, y)) \setminus N$, $y_1 \in B(y, d(x, y)) \setminus N$ and a curve $\gamma_1 \in \Gamma_{x_1 y_1} \setminus \Gamma_E$ of length

$$\ell(\gamma_1) \leq Ad(x_1, y_1) \leq 3Ad(x, y).$$

Next we concentrate on joining x_1 to x . Proceeding by induction we may, for each $k \geq 2$, find a point $x_k \in B(x, 2^{-1}d(x, y)) \setminus N$ and a curve $\gamma_k \in \Gamma_{x_{k-1} x_k} \setminus \Gamma_E$ of length

$$\ell(\gamma_k) \leq Ad(x_{k-1}, x_k) < 2^{-k+2} Ad(x, y).$$

Step 3. A standard concatenation argument yields a curve α joining x_1 and x now of length

$$\ell(\alpha) \leq \sum_{k \geq 2} \ell(\gamma_k) \leq 4Ad(x, y)$$

and, analogously, a curve β joining y_1 and y with

$$\ell(\beta) \leq 4Ad(x, y).$$

Thus

$$\gamma := \alpha^{-1} \gamma_1 \beta$$

is a curve joining x and y of length

$$\ell(\gamma) \leq 11Ad(x, y).$$

Since all of the curves avoid E it follows that γ avoids E . We have constructed a curve γ joining x and y in $X \setminus E$ of length comparable to $d(x, y)$, and this finishes the proof. \square

1.2. Newtonian spaces. Poincaré inequalities, and especially nonsmooth analysis, gave rise to a theory of Sobolev-type function spaces on metric spaces – in fact many. The first definition of Sobolev space defined for metric measure spaces, nowadays called *Hajłasz space*, was given by Hajłasz [32]. In addition to this there were spaces defined by Cheeger, [14], and Shanmugalingam, [69]. The survey paper [33] gives an extensive review of some of the many existing notions of Sobolev spaces and their relations.

Among these the *Newtonian space* $N^{1,p}(X)$ defined by Shanmugalingam [69] through upper gradients came to be the background for all the papers in this thesis.

Given a metric measure space (X, d, μ) a measurable function $u : X \rightarrow \overline{\mathbb{R}}$ is called a *Newtonian function* if it is p -integrable and possesses a p -integrable upper gradient g . A seminorm on the class $\tilde{N}^{1,p}(X)$ of Newtonian maps is given by

$$\|u\|_{1,p} = (\|u\|_p^p + \inf \|g\|_p^p)^{1/p},$$

where the infimum is taken over all upper gradients of u . The *Newtonian space* $N^{1,p}(X)$ is defined to be the quotient

$$N^{1,p}(X) = \tilde{N}^{1,p}(X) / \sim,$$

where the equivalence relation is given by $u \sim v$ iff $\|u - v\|_{1,p} = 0$. Note that this equivalence relation is stricter than the L^p -equivalence, i.e. agreeing almost everywhere. In fact it turns out that $u \sim v$ if and only if

$$\mu(\{u \neq v\}) = \text{Cap}_p(\{u \neq v\}) = 0.$$

A Newtonian function $u \in N^{1,p}(X)$ also possesses a *minimal p -weak upper gradient* $g_u \in L^p(X)$, i.e. a nonnegative p -integrable function g_u such that

- (1) (UG) is satisfied for all curves except for a family Γ_0 of zero p -modulus, and
- (2) $g_u \leq g$ almost everywhere for any locally integrable upper gradient g of u .

From the point of view of this thesis it is important to consider Newtonian spaces with values in a Banach space, since this provides the standard way to define Newtonian maps with values in a metric spaces. Another approach was taken by Ohta [61] who defined Sobolev maps into metric spaces directly, without embeddings into a Banach space. See [44, Proposition 7.1.38] for more details on the relation between the two approaches. Their definition is similar, one simply replaces $\overline{\mathbb{R}}$ by a given Banach space V . For Poincaré spaces the (Banach valued) Newtonian space has basically all of the nice properties the classical Sobolev space has. They are reflexive when $1 < p < \infty$ [14], the local Sobolev, Nirenberg and Morrey embeddings hold [33, 5, 44] and, importantly, Newtonian maps (on locally complete Poincaré spaces) are *p -quasicontinuous*. That is, for every $\varepsilon > 0$ there is an open set $E \subset X$ with $\text{Cap}_p(E) < \varepsilon$ such that the restriction $u|_{X \setminus E}$ is continuous.

This latter fact forms the basis of the notion of p -quasihomotopy and thus of the articles [B,C]. Some excellent textbooks on the subject include [44, 5, 38] as well as the survey paper [33].

2. QUASICONFORMAL MAPS

A homeomorphic map $f : X \rightarrow Y$ between metric spaces X and Y is called *H -quasiconformal* if

$$(QC) \quad H_f(x) = \limsup_{r \rightarrow 0} \frac{L(x, r)}{l(x, r)} \leq H \text{ for all } x \in X,$$

where

$$\begin{aligned} L(x, r) &= \max\{d(f(x), f(y)) : d(x, y) \leq r\}, \\ l(x, r) &= \min\{d(f(x), f(y)) : d(x, y) \geq r\}. \end{aligned}$$

Classically, a homeomorphism $f \in W_{loc}^{1,n}(\Omega, \Omega')$ between Euclidean domains $\Omega, \Omega' \subset \mathbb{R}^n$ is said to be K -quasiconformal if the distortion inequality

$$(QC^n) \quad \|Df\|^n \leq KJ_f$$

is satisfied almost everywhere in Ω . The equivalence of these definitions for Euclidean domains was proven by Gehring [23] in two dimensions and by Gehring and Väisälä [26] for $n \geq 2$. Condition (QC) is often referred to as the *metric definition* while (QCⁿ) is known as the *analytic definition* for quasiconformality. See [70] for a detailed discussion.

Quasiconformal maps between domains in \mathbb{R}^n enjoy some “local-to-global” properties. For example they are locally quasisymmetric, [71]. This means, intuitively, that the infinitesimal information given by (QC) extends to quantitative control on the way the map distorts a neighbourhood of each point. This point is important as we shall see: a quantitative version of this fact, known as the *egg yolk*

principle can be used to study the distortion of the quasihyperbolic metric under a quasiconformal map.

A map $f : X \rightarrow Y$ between metric spaces is said to be η -*quasisymmetric*, for an increasing function $\eta : [0, \infty) \rightarrow [0, \infty)$ with $\eta(0) = 0$, if for any triple of points $x, a, b \in X$ the estimate

$$(QS) \quad \frac{d(f(x), f(a))}{d(f(x), f(b))} \leq \eta \left(\frac{d(x, a)}{d(x, b)} \right)$$

is valid (assuming $x \neq b$). A map $f : X \rightarrow Y$ is *quasisymmetric* if it is η -quasisymmetric for some η .

The pioneering work of Heinonen and Koskela [38], see also [41, 42, 43], gives an extension of this local-to-global principle into the setting of metric spaces. For this purpose a class of nonsmooth spaces was introduced in [43] that would allow a quasiconformal theory. This class now goes by the name of *spaces of locally Q -bounded geometry*.

2.1. Spaces of Q -bounded geometry. A locally compact metric measure space (X, d, μ) has Q -bounded geometry, $Q > 1$ if

(AR) The measure is Ahlfors Q -regular, i.e. there is a number $0 < C < \infty$ so that

$$r^Q/C \leq \mu(B(x, r)) \leq Cr^Q, \quad 0 \leq r < \text{diam } X, \quad x \in X$$

and

Q -PI it satisfies the weak $(1, Q)$ -Poincaré inequality, i.e. the condition (p -PI) with $p = Q$.

Spaces of Q -bounded geometry already encompass a large subclass of Riemannian manifolds. The localization of this notion, *spaces of locally Q -bounded geometry*, covers all Riemannian manifolds as well as many nonriemannian examples of interest, such as Carnot groups. Heinonen, Koskela, Shanmugalingam and Tyson proved the equivalence of the different characterizations of quasiconformal maps. In this context they proved the following theorem.

Theorem. ([45, Theorem 9.8]) *Let (X, μ) and (Y, ν) be spaces of locally Q -bounded geometry and $f : X \rightarrow Y$ a homeomorphism. Then the following conditions are equivalent.*

- (1) f is H -quasiconformal;
- (2) f is locally η -quasisymmetric;
- (3) $f \in N_{loc}^{1, Q}(X; Y)$ and $\text{Lip } f(x)^Q \leq CJ_f(x)$ holds for almost every $x \in X$;
- (4) The relation

$$(2.1) \quad \frac{1}{K} \text{Mod}_Q \Gamma \leq \text{Mod}_Q f\Gamma \leq K \text{Mod}_Q \Gamma$$

holds for each family Γ of curves in X .

The theorem is quantitative in that H , K , C , and η depend only on each other, and the data of X and Y .

This can be seen as a geometric application of the Poincaré inequality. In fact the Q -Ahlfors regularity of the measure and a $(1, Q)$ -Poincaré inequality imply the *Loewner condition* which is a quantitative estimate of the *modulus* of path families connecting two given compact continua in terms of the distance and diameter of said continua. The paper of Heinonen and Koskela [43] originally discusses the theory of quasiconformal maps on metric spaces in terms of the Loewner condition.

2.2. Distortion estimates for the quasihyperbolic metric. After extending the basic theory of quasiconformal maps to metric spaces it is natural to ask which of the Euclidean (or Riemannian) results transfer to metric spaces. The first publication [A] of the thesis deals with this question in case of the following distortion estimate for the quasihyperbolic distance, due to Gehring and Osgood [25]. Recall that the quasihyperbolic distance in a rectifiably connected proper domain Ω of a metric space X is given by

$$k_{\Omega}(x, y) = \inf_{\gamma} \int_{\gamma} \frac{ds}{\text{dist}(z, X \setminus \Omega)} \text{ for } x, y \in \Omega,$$

where the infimum is taken over all rectifiable curves joining x and y in Ω .

Let $\Omega, \Omega' \subset \mathbb{R}^n$ be proper domains and $f : \Omega \rightarrow \Omega'$ a K -quasiconformal homeomorphism. Denote by k_{Ω} and $k_{\Omega'}$ the quasihyperbolic metric of Ω and Ω' , respectively. The theorem of Gehring and Osgood states that there exists a constant $C < \infty$ depending only on n and K so that

$$(QH) \quad k_{\Omega'}(f(x), f(y)) \leq C \max\{k_{\Omega}(x, y), k_{\Omega}(x, y)^{\alpha}\} \text{ for } x, y \in \Omega$$

where $\alpha = K^{1/(1-n)}$. Anderson, Vamanamurthy and Vuorinen proved in [4] that the dependence of the constant C on n can be removed.

The fact that \mathbb{R}^n is an *unbounded* space of n -bounded geometry plays a crucial role in the validity of the result. Even for proper domains between manifolds the result need not be true; this is easily seen by taking $\Omega = \Omega' = S^2 \setminus \{e_3\}$, where e_1, e_2, e_3 are the standard orthogonal unit vectors spanning \mathbb{R}^3 . If $x \in S^2 \setminus \{e_3\}$ is a point then there is a conformal map $f = f_x : S^2 \setminus \{e_3\} \rightarrow S^2 \setminus \{e_3\}$ fixing e_1 and sending $-e_1$ to x . However the lefthand side of (QH),

$$k_{\Omega'}(f(e_1), f(-e_1)) = k_{S^2 \setminus \{e_3\}}(e_1, x),$$

increases without bound as x is moved closer to e_3 while the quasihyperbolic distance,

$$k_{\Omega}(e_1, -e_1),$$

remains constant.

The proof of the theorem of Gehring and Osgood result is based on two ingredients: the so called *egg yolk principle* and Schwartz's lemma for quasiconformal maps (see [38, 4]). In the case of metric spaces of Q -bounded geometry we do not, in general, have an egg yolk principle. To obtain one, one needs to make an extra assumption on the proper domain on the target side preventing situations like the example above. The additional assumption turns out to be precisely that $\partial\Omega'$ contains at least two points, if Y is bounded. The key lemma in [A] is the following.

Lemma. ([A, Lemma 5.2]) Suppose $f: \Omega \rightarrow \Omega'$ is K -qc. Then for each $x \in \Omega$ the map f restricted to $B(x, \text{dist}(x, X \setminus \Omega)/(4\lambda + 1))$, where λ is as in (1, Q), is quasisymmetric. The quasisymmetry data depends only on K, Q , and the data associated with X and Y and, if Y is bounded, the value of $\text{diam}(\Omega')/\text{diam}(\partial\Omega')$.

With the egg yolk principle at hand Schwartz's lemma is easy to replace with standard estimates for quasisymmetric maps. Doing this, however, sacrifices the sharpness of the exponent α in the claim. Thus the result obtained in [A] is quantitative in the data but not sharp.

Theorem 1. ([A, Theorem 1.1]) Let $f: \Omega \rightarrow \Omega'$ be a K -quasiconformal homeomorphism between two proper domains $\Omega \subset X$ and $\Omega' \subset Y$ of spaces X and Y of globally Q -bounded geometry ($Q > 1$) and suppose that either Y is unbounded, or $\partial\Omega'$ contains at least two points. Then there are constants $c > 0$ and $0 < \alpha < 1$, depending only on K , the data of the spaces X and Y and in case Y is bounded, the quantity

$$\frac{\text{diam}(\Omega')}{\text{diam}(\partial\Omega')}$$

such that

$$k_{\Omega'}(f(x), f(y)) \leq c \max\{k_{\Omega}(x, y)^{\alpha}, k_{\Omega}(x, y)\}$$

for any $x, y \in \Omega$. Here k_{Ω} denotes the quasihyperbolic metric of Ω .

A notable feature of Theorem 1 is the asymmetry of the condition; there is no a priori requirement on Ω . In fact the condition that $\partial\Omega'$ contains at least two points is a quasiconformal invariant, but not quantitatively so. That is to say, the quantity

$$\frac{\text{diam}(\Omega)}{\text{diam}(\partial\Omega)}$$

does not depend on

$$\frac{\text{diam}(\Omega')}{\text{diam}(\partial\Omega')}.$$

3. p -ENERGY MINIMIZING MAPS

One might say that p -energy minimizing maps, in contrast to quasiconformal maps, are not topological in nature. That is to say, locally energy minimizing maps need not preserve any topological properties, let alone be homeomorphisms. Yet questions on the topology of manifolds have been central in the development of the theory of p -energy minimizing maps. The homotopy groups of manifolds, information of groups acting on them by isometries, as well as Liouville type theorems have all motivated research on harmonic and p -energy minimizers between manifolds.

Given a class of maps $\mathcal{C} \subset N^{1,p}(X; Y)$ and an energy functional $E_p: N^{1,p}(X; Y) \rightarrow \mathbb{R}$ satisfying

- (1) $E_p(u) \leq C \|g_u\|_p^p$, and
- (2) E_p is sequentially lower semicontinuous with respect to L^p -convergence,

we say that a map $u \in \mathcal{C}$ is a p -energy minimizer in the class \mathcal{C} if

$$E_p(u) = \inf\{E_p(v) : v \in \mathcal{C}\}.$$

Going back 50 years, the seminal paper of Eells and Sampson [20] addressed the question of existence of harmonic maps between Riemannian manifolds, the target having nonpositive curvature. Starting with a C^1 -map f_0 between compact manifolds M and N , Eells and Sampson used the heat flow method to establish a flow of maps $f_t : M \rightarrow N$, starting at f_0 and converging to a C^1 harmonic map $f_\infty : M \rightarrow N$. Incidentally, nonpositive curvature in the target was needed in order to employ the heat flow method and so was an essential assumption in obtaining the existence of a smooth harmonic map.

This means that, given a C^1 -map, they demonstrated the existence and regularity of a harmonic map *in the same homotopy class as the original map*. Taking the variational point of view they proved the existence of a minimizer of a given energy functional, in this case the Dirichlet integral

$$(E) \quad E(f) = \frac{1}{2} \int_M |df|^2 dx$$

in a given homotopy class of maps.

Consider, for example, a map f_0 from the circle to a given manifold N of nonpositive curvature. Minimizers of the Dirichlet integral in the same homotopy class are closed geodesic loops homotopic to f_0 . Thus they provide nice representatives of the free conjugacy class of an element in the fundamental group. In higher dimensions harmonic maps can represent (among other things) submanifolds of minimal volume. See the survey articles [18, 16, 19].

After Eells and Sampson, many researchers have studied harmonic maps between manifolds and utilized them to study the topology of manifolds, their geometry, group actions, higher homotopy groups and so on, see for instance [18, 75, 66, 67, 12, 72].

After defining harmonic maps to be continuous minimizers of the Dirichlet integral (E) it is natural to consider *other* energies. A straightforward generalization is to allow for exponents $p \in (1, \infty)$. A p -harmonic map $M \rightarrow N$ is a continuous minimizer of the *Dirichlet p -energy*

$$(E_p) \quad E_p(f) = \frac{1}{p} \int_M |df|^p dx$$

in the homotopy class of a given continuous map in $W^{1,p}(M; N)$. They have been studied in connection with higher homotopy groups [73], but are also natural objects to study in themselves when moving from the Sobolev space $W^{1,2}$ to $W^{1,p}$, $p \in (1, \infty)$. See [74, 12, 72, 73, 36, 37] for more on p -harmonic maps between manifolds.

Harmonic and p -harmonic maps can also be studied between more general spaces; in [54] Schoen and Korevaar defined the notion of a Sobolev type space of maps between a manifold M and a general metric space Y , using a new notion of energy – the so called Korevaar-Schoen p -energy given by

$$E^2(f) = \limsup_{\varepsilon \rightarrow 0} \sup_{\varphi \in C_0(\Omega; [0,1])} \int_M \varphi e_\varepsilon(f) dx,$$

where

$$e_\varepsilon(f)(p) = \int_{B(p,\varepsilon)} \frac{d_Y^2(f(q), f(p))}{\varepsilon^2} dq,$$

and went on to study existence and regularity of harmonic maps $M \rightarrow Y$ under the assumption that Y is nonpositively curved in the sense of Alexandrov (see [11]). They extended the result of Eells and Sampson to this new setting.

Theorem. ([54, Theorem 2.7.1]) *Let M be a compact Riemannian manifold and Y a compact metric space of nonpositive curvature. Given a continuous map $f_0 \in C(M; Y)$ there is a Lipschitz-regular minimizer of the Korevaar-Schoen energy E^2 homotopic to f_0 .*

Thereafter, Fuglede and Eells [17, 21, 22] proved an analogous result for maps between *Riemannian polyhedra* and nonpositively curved spaces, and Jost [47, 48, 49, 50] considered maps between a compact 2-Poincaré space with doubling measure arising as a quotient of a group action and a nonpositively curved space.

The last two papers [B,C] comprising this thesis study the questions of homotopy and existence of p -energy minimizers in homotopy classes in the setting of maps between a compact p -Poincaré space with doubling measure, and a compact non-positively curved metric space.

3.1. Homotopy classes of Sobolev type maps. The notion of classical homotopy between (continuous) maps is not natural in the Sobolev setting, since in general Sobolev maps need not be continuous. This was well known and alternate notions of homotopy date back at least to [12]. In 1985 White introduced k -homotopy types of Sobolev maps between Riemannian manifolds as classes of maps for which the restrictions to a generic k -dimensional skeleton are homotopic, and proved that Sobolev maps have a well-defined $[p-1]$ -homotopy type, where $[a]$ denotes the largest integer $\leq a$.

General metric spaces, even ones supporting a p -Poincaré inequality, do not possess triangulations and so this definition does not readily extend to a metric setting. However the $[p-1]$ -homotopy types have a connection with connected components of $W^{1,p}(M; N)$. Brezis and Li [10] defined two maps $f_0, f_1 \in W^{1,p}(M; N)$ to be homotopic if there is a continuous path in $C([0, 1]; W^{1,p}(M; N))$ connecting f_0 to f_1 . This notion, termed *path-homotopy* generalizes to Newtonian spaces as soon as we have given them a topology.

It was shown by Hang and Lin [35] that two maps

$$f_0, f_1 \in W^{1,p}(M; N)$$

have the same $[p-1]$ -homotopy class if and only if they are path-homotopic. In the context of metric spaces, even path-homotopy is rather difficult to work with. Thus in [B] we introduced another definition of homotopy, relying on the fact that Newtonian maps are *p -quasi-continuous*.

Definition. *Two maps $u, v \in N^{1,p}(X; Y)$ are said to be p -quasihomotopic if there is a map*

$$H : X \times [0, 1] \rightarrow Y$$

(called a p -quasihomotopy) with the following property:

For every $\varepsilon > 0$ there is an open set $E \subset X$ with $\text{Cap}_p(E) < \varepsilon$ such that

$$H|_{X \setminus E \times [0,1]} : X \setminus E \times [0,1] \rightarrow Y$$

is a (continuous) homotopy between $u|_{X \setminus E}$ and $v|_{X \setminus E}$.

The second article of the thesis [B] explores the relationship of path-homotopy and p -quasihomotopy, and establishes some basic properties of the latter. The notion of p -quasihomotopy turns out to be stricter than path-homotopy. This can be seen with a simple example, where $M = B^2$ is the unit disc and $N = S^1$ is the unit circle.

The map

$$u : z \mapsto \frac{z}{|z|}$$

belongs to the Sobolev space $W^{1,p}(B^2; S^1)$ for every $1 < p < 2$. By [9, Theorem 0.2] the space $W^{1,p}(B^2; S^1)$ is path connected for these values of p , therefore u is path-homotopic to a constant map. On the other hand one can show that a map $w \in W^{1,p}(B^2; S^1)$ is p -quasihomotopic to a constant map if and only if it admits a lift $h \in W^{1,p}(B^2; \mathbb{R})$, see [B]. It is known that u does not admit a Sobolev lift [8], thus it is not p -quasihomotopic to a constant map.

The argument above establishes that path-homotopic maps need not be p -quasihomotopic. For manifolds the converse implication however always holds. In [B] we show that a p -quasihomotopy between two Sobolev maps defines a continuous homotopy when restricted to a generic skeleton of dimension $< p$. In particular it follows, using the theorem of Hang and Lin [35, Theorem 5.1] that

Theorem 2. (*[B, Theorem 5.6]*) *Two p -quasihomotopic maps $u, v \in W^{1,p}(M; N)$ between compact Riemannian manifolds are always path-homotopic.*

For Newtonian spaces the relationship is not as clear in part because, as demonstrated by Hajlasz [34], the topology of $N^{1,p}(X; Y)$ depends on the isometric embedding $Y \hookrightarrow V$ used to define the Newtonian class $N^{1,p}(X; Y)$. For general target spaces Y one can prove the following.

Theorem 3. (*[B, Theorem 2.11]*) *Let (X, d, μ) be a compact p -Poincaré space with doubling measure and Y a separable metric space. Suppose $u, v \in N^{1,p}(X; Y)$ can be connected by a path $h : [0, 1] \rightarrow N^{1,p}(X; Y)$ such that*

$$\|g_{d(h_t, h_s)}\|_{L^p} \leq C|t - s|, \quad s, t \in [0, 1].$$

Then u and v are p -quasihomotopic.

Note that the embedding $Y \hookrightarrow V$ does not play any role in the statement. The condition is basically a rectifiability requirement on the path h , and we may conclude that, for maps between manifolds

p -quasihomotopy lies in between path-homotopy and rectifiable path-homotopy, where the path used to connect given maps is rectifiable in $W^{1,p}(M; N)$.

Currently the precise relationship is not clear and it is possible (but not known) that two maps are p -quasihomotopic if and only if they are rectifiably path-connected.

3.2. Nonpositively curved targets. Another focus in [B] is p -energy minimizing maps into nonpositively curved targets. While the term *nonpositively curved metric space* nowadays more commonly refers to the notion defined by Alexandrov [3] there is in fact a more general concept, originated by Busemann [13], which proves a suitable framework for the questions studied in [B] and [C]. The former notion is also often referred to as (local) CAT(0)-space, and we shall adopt this terminology. A central theme in both definitions is convexity; good accounts on nonpositive curvature in metric spaces can be found in [63, 11].

We call a (path connected) metric space (Y, d) *locally complete and geodesic* if each point has a closed neighbourhood that is a complete geodesic space.

Definition (Locally Busemann convex spaces). (a) *A metric space Y is called a Busemann convex space if it is locally complete and geodesic, and for every pair of affinely reparametrized geodesics $\gamma, \sigma : [0, 1] \rightarrow Y$ the distance map*

$$t \mapsto d(\gamma(t), \sigma(t)) : [0, 1] \rightarrow \mathbb{R}$$

is convex.

(b) *A metric space Y is locally convex if each point has a neighbourhood that is a Busemann space with the induced metric. Such neighbourhoods are called Busemann convex neighbourhoods.*

Definition (Locally CAT(0)-spaces). (a) *A locally complete and geodesic space Y is said to be of global nonpositive curvature if, for all geodesic triangles Δ with comparison triangle $\bar{\Delta}$ and any two points $a, b \in \Delta$, the comparison points $\bar{a}, \bar{b} \in \bar{\Delta}$ satisfy*

$$d(a, b) \leq |\bar{a} - \bar{b}|.$$

(b) *A locally complete and geodesic space is said to be of nonpositive curvature (an NPC space for short) if each point has a neighbourhood that is a space of global nonpositive curvature when equipped with the inherited metric.*

A comparison triangle $\bar{\Delta}$ of a geodesic triangle Δ is a triangle in \mathbb{R}^2 with the same sidelengths as Δ , and a comparison point \bar{p} of $p \in \Delta$ is the point on the corresponding side of $\bar{\Delta}$ that is an equal distance away from the vertices that side joins.

Nonpositively curved *Riemannian manifolds* are known to be $K(\pi, 1)$ spaces, i.e. they admit a contractible universal cover – this is essentially the Cartan-Hadamard theorem [15]. A consequence of this fact is that two continuous maps $M \rightarrow N$ from a manifold M to a nonpositively curved manifold N are homotopic if and only the induced homomorphisms

$$\pi(M) \rightarrow \pi(N)$$

are conjugate. This connection between homotopy and induced homomorphisms is used by virtually all existence results of harmonic or p -harmonic maps. There also exists a metric version of the Cartan-Hadamard theorem (see [11]).

However in the generality considered in [B,C] the connection between homotopy and induced homomorphisms fails since Newtonian maps are not known to induce homomorphisms between the fundamental groups (even in the manifolds case Sobolev maps only induce such a homomorphism if $p \geq 2$). Instead the following characterization of homotopy turns out to be useful.

Proposition. (*[C, Proposition 3.2]*) *Let Y be a nonpositively curved metric space, and X a topological space. Two continuous maps $f, g : X \rightarrow Y$ are homotopic if and only if the product map*

$$(f, g) : X \rightarrow Y \times Y$$

has a continuous lift

$$h : X \rightarrow \widehat{Y}_{\text{diag}},$$

where

$$\widehat{Y}_{\text{diag}} = (\widetilde{Y} \times \widetilde{Y}) / \text{diag } \pi(Y)$$

is the quotient of the universal cover of $Y \times Y$ by the subgroup $\text{diag } \pi(Y) \leq \pi(Y \times Y)$.

This is a well known fact, and might seem like a cumbersome way of expressing homotopy. However it has the advantage of admitting a generalization to the Sobolev – and even the Newtonian – realm. In fact, for Newtonian maps it characterizes exactly p -quasihomotopy.

Theorem 4. (*[C, Theorem 1.2]*) *Suppose X is a compact p -Poincaré space with doubling measure, and let Y be a separable nonpositively curved metric space. Maps $u, v \in N^{1,p}(X; Y)$ are p -quasihomotopic if and only if the product map*

$$(u, v) \in N^{1,p}(X; Y \times Y)$$

admits a lift

$$h \in N^{1,p}(X; \widehat{Y}_{\text{diag}}).$$

3.3. Existence of energy minimizers. Theorem 4 reduces the question of p -quasihomotopy to existence of lifts. The third article [C] in this thesis focuses on existence of p -energy minimizers in p -quasihomotopy classes and attacks the problem from this point of view.

Traditionally the aforementioned connection between homotopy and induced homomorphisms has been used to study the existence of energy-minimizers. This was also done by Jost in [49], where he in fact studies local minimizers of a 2-energy in a given *equivariance class*. In general the two approaches lead to different notions of “homotopy.”

Theorem 5. (*[C, Theorem 1.1]*) *Let $p \in (1, \infty)$, (X, d, μ) be a compact p -Poincaré space with doubling measure and let Y be a compact nonpositively curved metric space with Noetherian or hyperbolic fundamental group. Given a map $v \in N^{1,p}(X; Y)$ there exists a map $u \in N^{1,p}(X; Y)$, p -quasihomotopic to v , with minimal p -energy*

$$e_p(f) = \int_X g_f^p d\mu$$

among all maps p -quasihomotopic to v .

Theorem 5 is purely an existence result; unlike the theorem of Korevaar and Schoen it does not guarantee any more regularity than what the setting already gives. Another notable feature is the presence of an extra assumption on the fundamental group of the target. A reason for both of these is the very general context – the domain space (X, d, μ) is not assumed to have any topological properties. Thus it can be fractal like, for instance.

The proof of Theorem 5 starts with the development of an analogue for the classical characterization of continuous maps $\varphi : X \rightarrow Y$ admitting a lift $\tilde{\varphi} : X \rightarrow \tilde{Y}$ to a covering space. Let $p : \tilde{Y} \rightarrow Y$ be a covering map. The characterization can be given elegantly in terms of the image of the fundamental group under the induced homomorphism; given $x_0 \in X$ and $y_0 \in p^{-1}(\varphi(x_0))$ the map φ admits a lift $\tilde{\varphi}$ with value $\tilde{\varphi}(x_0) = y_0$ if and only if

$$(L) \quad \varphi_{\#}\pi(X, x_0) \leq p_{\#}\pi(\tilde{Y}, y_0).$$

To avoid a number of technical difficulties with proving an analogue of (L) for Newtonian maps $\varphi \in N^{1,p}(X; Y)$ the structure of path-families as well as the analytic information of the domain space, given by the Poincaré inequality, is used. Note that the characterization does not require an induced homomorphism to exist, only an analogue of the object $\varphi_{\#}\pi(X, x_0)$, see [C, Theorem 1.3].

The crucial second step, one where the additional assumption of the *subconjugacy condition* emerges, is proving a “stability result” for the property of Newtonian maps admitting lifts. The subconjugacy condition for a subgroup H of a group G is the requirement, introduced in [C], that for any sequence $(g_j) \subset G$ there exists some $g \in G$ for which

$$(SC) \quad \liminf_{j \rightarrow \infty} H^{g_j} = \bigcup_{n \geq 1} \bigcap_{j \neq n} H^{g_j} \leq H^g.$$

Heuristically this simply means that any sequence of conjugates of H has a limit which is a subgroup of some conjugate of H .

Theorem 6. ([C, Theorem 1.4]) *Let $p \in (1, \infty)$, (X, d, μ) be a compact p -Poincaré space with doubling measure and Y a complete length space with a locally isometric covering $\phi : \tilde{Y} \rightarrow Y$ such that $\phi_{\#}\pi(\tilde{Y})$ satisfies the subconjugacy condition. Suppose (u_j) is a sequence in $N^{1,p}(X; Y)$ such that*

$$\sup_j \int_X g_{u_j}^p d\mu \leq C < \infty$$

and for each j there exists a lift $h_j \in N^{1,p}(X; \tilde{Y})$. If (u_j) converges in L^p to a map $u \in N^{1,p}(X; Y)$ then u admits a lift $h \in N^{1,p}(X; \tilde{Y})$.

As a direct corollary we obtain the following result.

Corollary. *Let $p \in (1, \infty)$, (X, d, μ) be a compact p -Poincaré space with doubling measure and Y a complete length space admitting a universal cover. Suppose (u_j) is a bounded sequence in $N^{1,p}(X; Y)$ with L^p -limit $u \in N^{1,p}(X; Y)$. If each u_j admits a lift $h_j \in N^{1,p}(X; \tilde{Y})$ to the universal cover then u admits a lift $h \in N^{1,p}(X; \tilde{Y})$ to the universal cover.*

At the onset this does not seem connected to hyperbolic or Noetherian groups. The link is given by the following observation.

Lemma. ([C, Lemma 5.2 and Lemma 5.3]) *Suppose G is a Noetherian or a torsion free hyperbolic group. Then $\text{diag } G \leq G \times G$ has the subconjugacy property.*

For Noetherian groups the statement is immediate while the hyperbolic case requires an argument. For CAT(0)-groups (which include fundamental groups of compact nonpositively curved spaces) this argument does not work and it is not known whether the claim holds for them. The validity of the claim for CAT(0)-groups would allow one to remove the extra hypothesis on the fundamental group of the target in Theorem 5.

The proof of the stability result above uses the analogue of the characterization (L) for Newtonian maps. After this Theorem 5 follows by piecing together Theorems 4 and 6.

For nonpositively curved manifolds the hyperbolicity of the fundamental group is implied by having strictly negative sectional curvature. In general, for a compact nonpositively curved metric space X , the fundamental group $\pi(X)$ is hyperbolic if and only if the universal cover \tilde{X} does not contain an isometric copy of a plane [11, Chapter III.Г, Theorem 3.1]. Thus the class of nonpositively curved spaces with hyperbolic fundamental group contains many interesting spaces.

The Noetherian property, on the other hand, combines rather poorly with non-positively curved spaces. A large subclass of Noetherian groups are the virtually nilpotent groups. The fundamental group of a nonpositively curved space is virtually nilpotent if and only if it is finite or virtually \mathbb{Z} . Nevertheless geometric group theory proves a powerful tool in the study of existence of energy minimizing maps.

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