Qualitative Theory of Autonomous Ordinary Differential Equation Systems

Master’s Thesis

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This work is about the qualitative theory of autonomous ordinary differential equation (ODE) systems. The purpose of the work is threefold. First, it is intended to familiarize the reader with the essential theory of autonomous systems in dimension $n$. Second, it is hoped that the reader will learn the importance of planar autonomous systems, such the beautiful result of the Poincaré-Bendixson theorem. Third, since the theory is utilised in applied science, considerably space has been devoted to analytical methods that are used widely in applications.

The fundamental theory of existence and uniqueness of solutions to ODE systems are presented in Chapter 2. Then, Chapter 3 treats with the essential theory of autonomous systems in dimension $n$, such as the orbits and the limit sets of solutions.

In Chapter 4 we consider planar autonomous systems. What makes planar systems different from higher dimensions is the existence of Jordan Curve theorem, which has made it possible for the theory to go much further. In particular, the Poincaré-Bendixson theorem, which is a statement about the long-term behavior of solutions to an autonomous system in the plane. Note that the Jordan Curve theorem is stated without proof, since the proof is terribly difficult but the result is obvious.

Lastly, in order not to lose sight of the applied side of the subject, Chapters 5 and 6 are devoted to analytical methods of autonomous systems. First, Chapter 5 treats with local stability analysis of an equilibrium. Then, in Chapter 6 we work with a relatively large study of an abnormal competing species model based on the science fiction movie The Terminator (1984), which should be taken with a pinch of salt. In its dystopian world there are two powerful forces of Men and the Terminator cyborgs trying to get completely rid of one another. Lack of space has, however, forced us to simplify some of the individual behaviour. These simplifications are partly justified by the fact that the purpose is to present how the theory can be applied even in a (hopefully) fictional situation and, of course, to answer the puzzling question whether the human race would stand a chance against the Terminators.
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Chapter 1

Introduction

This work is about the qualitative theory of autonomous ordinary differential equation (ODE) systems. The purpose of the work is threefold. First, it is intended to familiarize the reader with the essential theory of autonomous systems in dimension $n$. Second, it is hoped that the reader will learn the importance of planar autonomous systems, such the beautiful result of the Poincaré-Bendixson theorem. Third, since the theory is utilised in applied science, considerably space has been devoted to analytical methods that are used widely in applications.

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against the Terminators.

Independently of which phenomenon one is studying, be it a fictional war against the Terminators or just about any real-world phenomenon, it is generally of utmost importance to have knowledge of its long-term behavior. In nature, this might concern the populations of multiple species related to each other. It is fascinating how often the forces of nature remain uniform, as was noted by Darwin (1859, p.86) “– In the long-run the forces are so nicely balanced, that the face of nature remains uniform for long periods of time, though assuredly the merest trifle would often give the victory to one organic being over another.”

To take an example from nature; let us study the densities of hares and lynxes in some environment that remains the same during the process. Suppose there is a large stock of hares and that they find ample food at all times. Suppose furthermore that there is only a relatively small number of lynxes whose only food supply is the hare population. We have reason to believe that the lynxes will take advantage of the situation and feed on the hares in great numbers. In nature the reproduction of predators is typically proportional with the rate at which they catch the prey. Bearing such facts in mind, there is little doubt that lynxes will reproduce in high numbers. This consequently will make the hare population decline in number, since there will be more and more lynxes feeding on them. But after some time the feed on prey will become harder, with there being less and less hares to feed even larger stock of lynxes. No doubt many lynxes will find their contest soon decided with also the proportion of surviving offspring decreasing, thus leading to a decline in the lynx population. Eventually the density of hares will pick up, since their offspring will have higher chance of survival and coming of age. This will slowly lead to an increase in the hare density, and consequently the decline of the lynx population will come to an end. We began this series with a large stock of hares and a relatively small number of lynxes, and we have ended with them.

Generally these relations between the species, not necessary between a predator and its prey, are much more complex. These kind of forces that remain uniform for long periods of time interests us greatly, and in this work our attention will be mostly in the underlying mathematics, which traditionally means the use of autonomous ODE systems. In general we are not able to obtain analytic solutions to an arbitrary system, yet we are able to derive information about the behavior of a family of solutions. This is called qualitative analysis, which is a branch of mathematics created by Poincaré in 1882, the greatest mathematical genius at the time, who was indeed the first person to study in detail the behavior of solutions to a differential equation.

In applications perhaps the most important aspect of qualitative analysis is the stability or instability of an equilibrium: Is the merest trifle enough to break the balance? This question came up at the latest on 1889 when Poincaré won the prize established by the
King Oscar II of Sweden and Norway for solving Newtonian N-body problem.\(^1\) Poincaré had studied whether the Solar System is stable; does the state of motion between the objects in the Solar System remain uniform until the end of time?\(^2\) It turned out that the problem is far more complicated than what he had expected, and to this day it remains an open question.\(^1\)

As for an example of today’s research, we might consider an infective agent that spreads upon a contact between an infective and a susceptible. If this infective agent was to enter a ‘virgin’ population, then we may ask the questions, as was done by Diekmann et al. (2013, p. 3): "Does it cause an epidemic? If so, at what rate does the number of infected hosts increase during the rise of the epidemic? What proportion of the population will ultimately have experienced infection?"

The reader of this work should be familiar with the basic theory of ODEs, such that are generally covered over the first and second year courses. It is also recommended to be familiar with basic concepts of topology. For further reading, we refer to the work by Jordan and Smith (1987), which offers over 400 examples on the subject, and to the work by Hale (2009), in which the theory will go further than in this work. For an interesting discussion about the underlying dynamics of nature, see Darwin (1859).

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Chapter 2

Ordinary Differential Equations

The purpose of this chapter is to present a number of definitions and notations, and to also discuss existence and uniqueness properties of the solutions to an ODE on initial data. These are discussed in sections (2.1) and (2.2) accordingly.

2.1 Preliminaries

Definition 2.1. Let $\Omega$ be an open set in $\mathbb{R}^{n+1}$ with an element of $\Omega$ written as $(t,p)$, where $p = (p_1,...,p_n)$ is an element of $D$, which is an open subset of $\mathbb{R}^n$. Let $f : \Omega \to \mathbb{R}^n$, be continuous and let $\dot{x} = dx/dt$. An ordinary differential equation of the first order is a relation

\begin{equation}
\dot{x}(t) = f(t,x(t)) \quad \text{or, briefly} \quad \dot{x} = f(t,x).
\end{equation}

We refer to $f$ as a vector field on $D$.

Definition 2.3. Consider the initial value problem (IVP)

\begin{equation}
\dot{x} = f(t,x), \quad x(t_0) = p_0.
\end{equation}

We say that $x$ is a solution of (2.4) on an interval $I \subset \mathbb{R}$ containing $t_0$ if $x$ is a continuously differentiable function defined on $I$, $(t,x(t)) \in \Omega$, $x(t_0) = p_0$ and $x$ satisfies $\dot{x} = f(t,x)$ on $I$. The trajectory through $(t_0,p_0)$ is the set of points in $\mathbb{R}^{n+1}$ given by $(t,x(t))$ for $t \in I$.

Definition 2.5. A function $f(t,x)$ defined on a $(t,p)$-set $\Omega$, where $p \in D$, is said to be uniformly Lipschitz continuous on $\Omega$ with respect to $x$ if there exists a constant $L$ satisfying

\begin{equation}
\|f(t,x_1) - f(t,x_2)\| \leq L\|p_1 - p_2\|.
\end{equation}
for all \((t, p_i) \in \Omega\) and \(x_i\) are divergent solutions for initial conditions \(x_i(t_0) = p_i\) with \(i \in \{1, 2\}\). Any constant \(L\) satisfying (2.6) is called a Lipschitz constant for \(f\) on \(\Omega\).

A function \(f(t, x)\) is said to be locally Lipschitz on \(\Omega\) with respect to \(x\) if for all \((t, p) \in \Omega\) there exists a neighbourhood \(U_p\) where \(f\) is uniformly Lipschitz continuous.

**Definition 2.7.** Let \((X, d)\) be a metric space, with \(p\) as an element of \(X\). A sequence \((p_n)\) in \((X, d)\) is said to be a Cauchy sequence if for every \(\varepsilon > 0\) there is an \(N\) such that \(d(p_m, p_n) < \varepsilon\) if \(m, n \geq N\). The space \((X, d)\) is complete if every Cauchy sequence in \((X, d)\) is convergent.

**Definition 2.8.** A Banach space is a normed linear space that is a complete metric space with respect to the metric derived from its norm.

**Definition 2.9.** Suppose \((X, d)\) and \((Y, d')\) are metric spaces. A mapping \(f : X \to Y\) is called a contraction with a contraction constant \(\lambda\) if it is \(\lambda\)-Lipschitz for some \(0 \leq \lambda < 1\), that is,

\[
d'(f(p_1), f(p_2)) \leq \lambda d(p_1, p_2)
\]

for \(p_1, p_2 \in X\).

**Lemma 2.10 (Banach’s Fixed Point).** Let \(f : X \to X\) be a contraction of a complete metric space \(X \neq \emptyset\). Then \(f\) has a unique fixed point \(p_0\), that is, \(f(p_0) = p_0\). Moreover, if \(p \in X\) is arbitrary, then the sequence \(f(p), f(f(p)), f(f(f(p))), \ldots\) converges to \(p_0\).

**Proof.** Uniqueness. Suppose \(\lambda\) is the contraction constant for \(f\) on \(X\), \(p_1 = f(p_1)\), and \(p_2 = f(p_2)\) with \(p_1, p_2 \in X\). Then \(d(p_1, p_2) = d(f(p_1), f(p_2)) \leq \lambda d(p_1, p_2)\), which implies \(d(p_1, p_2) \leq 0\), and thus \(d(p_1, p_2) = 0\) and \(p_1 = p_2\).

Existence. Let \(p_1 \in X\), and write \(p_{n+1} = f(p_n)\). By hypotheses, each \(p_n\) is in \(X\). For \(n > 1\),

\[
d(p_{n+1}, p_n) \leq \lambda d(p_n, p_{n-1}) \leq \ldots \leq \lambda^n d(p_2, p_1) \leq 0.
\]

Thus, for \(m > n\),

\[
d(p_m, p_n) \leq d(p_m, p_{m-1}) + d(p_{m-1}, p_{m-2}) + \ldots + d(p_{n+1}, p_n)
\]

\[
\leq (\lambda^{m-1} + \lambda^{m-2} + \ldots + \lambda^n) d(p_2, p_1)
\]

\[
= \lambda^n (1 + \lambda + \ldots + \lambda^{m-n-1}) d(p_2, p_1)
\]

\[
= \frac{\lambda^n (1 - \lambda^{m-n})}{1 - \lambda} d(p_2, p_1)
\]

\[
\leq \frac{\lambda^n}{1 - \lambda} d(p_2, p_1).
\]

Since \(\lambda < 1\), \(\lambda^n \to 0\) as \(n \to \infty\), which implies that \((p_n)\) forms a Cauchy sequence. Since \(X\) is a complete metric space, \((p_n)\) converges to some point \(p_0 \in X\), and since \(f\) is continuous, then \(f(p_n) \to f(p_0)\) as \(n \to \infty\). On the other hand, \(f(p_n) = p_{n+1}\), and thus \(f(p_n) \to p_0\), that is, \(f(p_0) = p_0\). 

\[\square\]
2.2 Existence and Uniqueness Properties

The purpose of this section is to discuss existence and uniqueness of solutions on initial data. Peano (1890, pp. 182-228) proved the existence of a solution for a continuous system and its initial value problem:

**Proposition 2.11 (Peano).** Let \( p_0 \in \mathbb{R}^n \), and suppose \( f : [t_0 - \alpha, t_0 + \alpha] \times \overline{B(p_0, \beta)} \to \mathbb{R}^n \) is continuous and bounded by \( M \). Then the IVP

\[
\dot{x} = f(t, x), \quad x(t_0) = p_0
\]

possesses at least one solution \( x \) defined on \([t_0 - b, t_0 + b]\), where \( b = \min(\alpha, \beta/M) \).

Since the emphasis throughout is on the uniqueness, we do not go into detail here and simply refer to the literature\(^1\). The uniqueness of a solution to an ODE and its initial data was stated by Lindelöf (1894, pp. 454-457), who discussed a generalization of an earlier approach by Picard. This result is known as the Picard-Lindelöf theorem. Although the uniqueness comes with an additional requirement for the system to be uniformly Lipschitz continuous, in applications this is seldom a problem. The proof we are about to present uses the Banach’s Fixed Point theorem (1922, pp. 133-181). Readers who are interested in a more traditional proof obtained by successive approximations, a method known as the Picard iteration, are referred to, e.g., Hartman (1964, pp. 8-10).

**Theorem 2.13 (Picard-Lindelöf).** Let \( p_0 \in \mathbb{R}^n \), and suppose \( f : [t_0 - \alpha, t_0 + \alpha] \times \overline{B(p_0, \beta)} \to \mathbb{R}^n \) is continuous and bounded by \( M \). Suppose furthermore that \( f(t, x) \) is uniformly Lipschitz continuous with a Lipschitz constant \( L \) for every \( t \in [t_0 - \alpha, t_0 + \alpha] \). Then the IVP

\[
\dot{x} = f(t, x), \quad x(t_0) = p_0
\]

has a unique solution \( x \) defined on \([t_0 - b, t_0 + b]\), where \( b = \min(\alpha, \beta/M) \).

**Proof.** Let \( X \) be the set of continuous functions from \([t_0 - b, t_0 + b]\) to \( \overline{B(p_0, \beta)} \), where \( b = \min\{\alpha, \beta/M\} \). The space \( C([t_0 - b, t_0 + b], \mathbb{R}^n) \) under the norm

\[
||g||_\omega = \sup \{e^{-2L|t-t_0|}|g(t)| : t \in [t_0 - b, t_0 + b]\}
\]

is complete, since \( ||\cdot||_\omega \) is equivalent to the standard supremum norm. The set \( X \) endowed with this norm is a closed subset of this complete Banach space, so \( X \) equipped with the metric \( d(x_1, x_2) = ||x_1 - x_2||_\omega \) is a complete metric space.

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Given $x \in X$, define $Tx : [t_0 - b, t_0 + b] \to \mathbb{R}^n$ by the formula

$$Tx(t) = p_0 + \int_{t_0}^{t} f(s, x(s)) ds,$$

that is, $Tx$ is continuous and differentiable. Furthermore, for $t \in [t_0 - b, t_0 + b]$

$$|Tx(t) - p_0| = \left| \int_{t_0}^{t} f(s, x(s)) ds \right| \leq \int_{t_0}^{t} |f(s, x(s))| ds \leq Mb \leq \beta,$$

and hence $Tx \in X$. Then, for every $x, y \in X$ and $t \in [t_0 - b, t_0 + b]$ we obtain

$$e^{-2L|t-t_0|} \left| \int_{t_0}^{t} f(s, x(s)) - f(s, y(s)) ds \right| \leq e^{-2L|t-t_0|} \int_{t_0}^{t} |f(s, x(s)) - f(s, y(s))| ds$$

$$\leq e^{-2L|t-t_0|} \int_{t_0}^{t} L|x(s) - y(s)| ds$$

$$\leq Le^{-2L|t-t_0|} \int_{t_0}^{t} ||x(s) - y(s)|| \omega \ e^{2L|s-t_0|} ds$$

$$\leq ||x - y|| \omega (1 - e^{-2L|t-t_0|})$$

$$\leq \frac{1}{2} ||x - y|| \omega.$$

By taking the supremum over all $t \in [t_0 - b, t_0 + b]$, we find that $T : X \to X$ is a contraction with $\lambda = 1/2$, thus by Banach’s Fixed Point (2.10) $T$ has a unique fixed point in $X$. This means that the IVP (2.14) has a unique solution in $X$. □
Chapter 3

Autonomous ODE Systems

The purpose of this chapter is to familiarize the reader with the essential theory of autonomous ODE systems of the first order in dimension $n$. In Section (3.1) we discuss the maximal solution and its maximum interval of existence. Then, in Section (3.2) we discuss orbits of solutions. Lastly, Section (3.3) treats with the basic theory of limit sets. Later, the Chapter 4 continues from the results of this chapter, but the theory will be limited to dimension two.

**Definition 3.1.** An autonomous system of the first order is a relation

$$\dot{x}(t) = f(x(t)) \quad \text{or, briefly} \quad \dot{x} = f(x),$$

where $f : D \to \mathbb{R}^n$ is continuous and $D$ is an open set of $\mathbb{R}^n$.

To warm off we shall start with a few simple examples.

**Example 3.3.** Let $D = \mathbb{R}^2$, and consider the linear autonomous differential equation of second order:

$$\ddot{x}(t) - x(t) = 0.$$

It can be reverted to a first order system by putting $x_1(t) = x(t)$ and $x_2(t) = \dot{x}(t)$:

$$\begin{align*}
\dot{x}_1(t) &= x_2(t) \\
\dot{x}_2(t) &= x_1(t)
\end{align*}$$

Recall that if we have a linear differential equation of first order

$$\dot{z}(t) = Az(t),$$
where \( z : \mathbb{R} \to \mathbb{R}^n \) and \( A \in \mathbb{R}^{n \times n} \), and eigenvectors of \( A \) are real and eigenvalues of \( A \) are linearly independent, then the general solution is
\[
z(t) = c_1 e^{\lambda_1 t} u_1 + c_2 e^{\lambda_2 t} u_2 + \ldots + c_n e^{\lambda_n t} u_n,
\]
where \( \lambda_i \) are the eigenvalues of \( A \), \( u_i \) are the eigenvectors of \( A \), and \( c_i \in \mathbb{R} \) with \( i \in \{1, \ldots, n\} \). Write \( z = (x_1, x_2) \), then \( \dot{z}(t) = Az(t) \), where \( A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \). The eigenvalues of \( A \) are
\[
\det(A - \lambda I) = 0
\]
\[
\det \begin{bmatrix} -\lambda & 1 \\ 1 & -\lambda \end{bmatrix} = 0
\]
\[
\lambda = \pm 1.
\]
Denote these by \( \lambda_1 = 1 \) and \( \lambda_2 = -1 \). Next we solve the corresponding eigenvectors. For
\( \lambda_1 \) we get

\[
Au_1 = \lambda_1 u_1 \\
(A - \lambda_1 I)u_1 = 0 \\
\begin{bmatrix}
-1 & 1 \\
1 & -1
\end{bmatrix} u_1 = 0
\]

\[
u_1 = s \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad s \in \mathbb{R},
\]

Similarly for \( \lambda_2 \) we obtain \( u_2 = s \begin{bmatrix} 1 \\ -1 \end{bmatrix} \). Then general solution for the homogenous system of the first order (3.4) is

\[
z(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = c_1 e^t \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 e^{-t} \begin{bmatrix} 1 \\ -1 \end{bmatrix}
\]

for \( c_1, c_2 \in \mathbb{R} \).

Clearly the term \( e^t \) dominates the equation for large values of \( t \). This essentially means that if a solution, say \( y \) satisfies \( c_1 \neq 0 \), then \( \|y(t)\| \to \infty \) as \( t \to \infty \). On the other hand, if \( y \) satisfies \( c_1 = 0 \),

\[ y(t) = c e^{-t} \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \]

then \( y \) tends to \( (0,0) \) as \( t \to \infty \). We say that \( \{(0,0)\} \) is the limit set of \( y \). None of the solutions satisfying \( c_1 \neq 0 \) converge to any specific point or a set of points. These observations are depicted in the phase-plane figure 3.1. It is also worthwhile to note that at origin the derivatives of \( x_1 \) and \( x_2 \) are zero; \( \dot{x}_1(t) = \dot{x}_2(t) \equiv 0 \) for all values of \( t \). This special point, now origin, is called the equilibrium of the system, but this we leave for further discussions.

**Example 3.6.** Let \( D = \mathbb{R}^2 \), and consider the linear autonomous ODE of second order:

\[ \ddot{x}(t) + x(t) = 0. \]

This can be similarly reverted to a first order system by putting \( x_1(t) = x(t) \) and \( x_2(t) = \dot{x}(t) \):

\[
\begin{cases}
\dot{x}_1(t) = x_2(t) \\
\dot{x}_2(t) = -x_1(t)
\end{cases}
\]

(3.7)
Figure 3.2: The phase path for (3.7).

Write $z = (x_1, x_2)$. Then $z(t) = Az(t)$, where $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$. The eigenvalues of $A$ are

$$\det(A - \lambda I) = 0$$

$$\det \begin{bmatrix} -\lambda & 1 \\ -1 & -\lambda \end{bmatrix} = 0$$

$$\lambda^2 = -1$$

$$\lambda = \pm i.$$

Thus, the eigenvalues of $A$ are complex valued; $\lambda_1 = i$ and $\lambda_2 = -i$. Next we solve the corresponding eigenvectors. For $\lambda_1$ we get

$$Au_1 = \lambda_1 u_1$$

$$(A - \lambda_1 I)u_1 = 0$$

$$\begin{bmatrix} -i & 1 \\ -1 & -i \end{bmatrix} \begin{bmatrix} u_{1,1} \\ u_{1,2} \end{bmatrix} = 0,$$
where \( u_1 = \begin{bmatrix} u_{1,1} \\ u_{1,2} \end{bmatrix} \). This equals to the following system of equations:

\[
\begin{align*}
-\text{i}u_{1,1} + u_{1,2} &= 0 \\
-u_{1,1} - \text{i}u_{1,2} &= 0 \\
\end{align*}
\]

\[\iff \]

\[
\begin{align*}
u_{1,1} &= s \\
u_{1,2} &= is,
\end{align*}
\]

which equals to

\[ u_1 = s \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \text{i} \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \text{for } s \in \mathbb{R}. \]

Similarly for \( \lambda_2 \) we get the eigenvector \( u_2 = s \begin{bmatrix} 1 \\ 0 \end{bmatrix} - \text{i} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \) for \( u \in \mathbb{R} \). Recall that if a matrix \( A \in \mathbb{R}^n \times \mathbb{R}^n \) has two distinct complex conjugate eigenvalues \( \lambda_1 = \alpha + \text{i}\beta \) and \( \lambda_2 = \alpha - \text{i}\beta \), where \( \alpha, \beta \in \mathbb{R}, \beta \neq 0 \), with corresponding eigenvectors \( u_1 = a + \text{i}b \) and \( u_2 = a - \text{i}b \), where \( a, b \in \mathbb{R}^n \setminus \{0\} \), then the homogeneous system of the first order \( \dot{x} = Ax \) has the general solution

\[ x(t) = c_1 e^{\alpha t} (a \cos \beta t - b \sin \beta t) + c_2 e^{\alpha t} (a \sin \beta t + b \cos \beta t), \]

where \( c_1, c_2 \in \mathbb{R} \). In this example we have \( a = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, b = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \alpha = 0 \) and \( \beta = 1 \). Then the general solution for (3.7) is

\[ z(t) = \begin{bmatrix} c_1 \cos t + c_2 \sin t \\ -c_1 \sin t + c_2 \cos t \end{bmatrix} = c_1 \begin{bmatrix} \cos t \\ -\sin t \end{bmatrix} + c_2 \begin{bmatrix} \sin t \\ \cos t \end{bmatrix}, \]

where \( c_1, c_2 \in \mathbb{R} \). This implies that the limit set for any given solution is always non-empty: It is the path of the solution itself, a periodic circle. This is illustrated in the figure 3.2.

In both of these examples the solutions were defined for every \( t \in \mathbb{R} \). But this is not always the case; take for example \( \dot{x} = x^2 \), which has a solution \( x(t) = -1/t \). It is then clear that we need to be speak of the maximal interval of existence. This is elaborated in the following short section.

### 3.1 Maximal Solution and Interval

**Definition 3.8.** Consider the IVP

(3.9) \[ \dot{x}(t) = f(t, x(t)), \quad x(t_0) = p_0. \]
We say that $x(t)$ or, briefly $x$, is the maximal solution of (3.9) on an interval $I \subset \mathbb{R}$ containing $t_0$ if $x$ is a solution of (3.9) on $I$ and for every $I' \supset I$ there exists $t' \in I'$ such that $x$ does not satisfy $\dot{x} = f(t,x)$ at $t'$. We then say that $I$ is the maximal interval of existence of $x$.

**Theorem 3.10 (Maximal solution).** Let $f$ be locally Lipschitz on $D$. Then the autonomous system $\dot{x} = f(x)$ and its initial condition $x(t_0) = p_0$ holds a unique maximal solution

$$x : \Delta(x) \to D,$$

where $\Delta(x) = [t^-(x), t^+(x)]$, with $t^-(x), t^+(x) \in \mathbb{R} \cup \{\pm \infty\}$. We refer to $\Delta(x)$ as the maximum interval of $x$.

**Proof.** The uniqueness follows directly from the Picard-Lindelöf theorem (2.13).

Assume $x_i : \Delta_i \to D$ are all the possible solutions for the initial condition $x(t_0) = p_0$. An interval $\Delta_i$ may be open, half-open or closed. Put

$$\Delta = \bigcup_i \Delta_i, \quad \text{and} \quad x(t) = x_i(t) \text{ for } t \in \Delta_i.$$  

The uniqueness of the solution implies that $x$ is defined properly. Clearly $x$ satisfies $\dot{x} = f(x)$, and $\Delta$ is the maximal interval. Especially the interval $\Delta$ may not contain its boundary values, say $t^+(x)$, since then there would exist a solution $y$ to the initial condition $y(t^+(x)) = x(t^+(x))$, which by the Peano’s existence theorem (2.13) would then exceed over $t^+(x)$. Since $f$ is locally Lipschitz, it would follow that $x = y$, thus exceeding the interval $\Delta$. \[\square\]

**Remark 3.11.** We write $\Delta^+(x)$ when we are only interested in the positive values of $\Delta(x)$, that is, $\Delta^+(x) \subset ]0, t^+(x)[$. Similarly for the negative values of $\Delta(x)$ we write $\Delta^-(x)$.

**Theorem 3.12.** Let $f$ be locally Lipschitz on $D$, and let $K \subset \mathbb{R}$ be compact. Suppose $x : \Delta(x) \to D$ be the maximal solution to the initial condition $x(t_0) = p_0$, where $p_0 \in K$. Then $t^+(x) = \infty$, or there exists $t_k < t^+(x)$ such that $x(t) \notin K$ for all $t_k < t < t^+(x)$.

Similarly for the lower bound $t^-(x)$.

**Proof.** Suppose $t^+(x) < \infty$, and that there exists $(t_k)$ such that $t_k \to t^+(x)$ and $x(t) \in K$ for all $t > t_k$. Since $K$ is compact, we can assume that $x(t_k) \to p_1$ for some $p_1 \in K$. Choose $r > 0$ such that $B(x_1, r) \subset D$ and $\|x(t_k) - p_1\| < r/2$. Then $B(x(t_k), r/2) \subset B(p_1, r) \subset D$. By the Picard-Lindelöf theorem (2.13) the system $\dot{y} = f(y)$ and the initial condition $y(t_k) = x(t_k)$ holds at least one solution $y$ on the interval $[t_0 - \delta, t_0 + \delta]$, where $\delta = r/2M$ and $f$ is bounded by $M$ on $B(x(t_k), r)$. Since $f$ is locally Lipschitz, by uniqueness of solutions we must have $y \equiv x$. Choose now $k$ that satisfies $t_k + \delta > t^+(x)$. Then the solution $x$ will cross the domain of $K$, which brings us to a contradiction.

A similar deduction serves the lower bound $t^-(x)$. \[\square\]
3.2 Orbits of a Solution

In this section we study the orbits of a solution. Since it is very awkward to continually verify the uniqueness of a solution, we suppose for the rest of Chapters 3 and 4 that the function $f$ is at least locally Lipschitz in $D$. The outline of the theory follows the presentation of Lamberg’s lectures (2014) and the work by Hale (2009, pp. 37-50).

Recall that the most essential detail of autonomous systems is that they do not depend on variable $t$. So if $x$ is a solution on $\Delta(x) = [a, b]$, then for any real number $s$ also the function $x(t-s)$ is a solution on an interval $[a+s, b+s]$, and the dynamics of these solutions are equivalent. This essentially means that in the initial condition we can always choose $t_0 = 0$.

We shall henceforth assume that for a given autonomous system $\dot{x} = f(x)$ and for any $p \in D$ there is a unique solution $\phi(t, p)$ passing through $p$ at $t = 0$. The function $\phi(t, p)$ is defined on an open set $\Omega \subset \mathbb{R}^{n+1}$ and satisfies:

(i) $\phi(0, p) = p$,

(ii) $\phi(t, p)$ is continuous in $\Omega$,

(iii) $\phi(t + \tau, p) = \phi(t, \phi(\tau, p))$ on $\Omega$.

The first property is clear from discussions above. The third property holds since both functions satisfy the system, and we have assumed uniqueness. Next, we shall prove that $\phi$ is indeed continuous and that $\Omega$ is an open set.

**Lemma 3.13 (Grönwall’s Inequality).** Let $u$ and $v$ be continuous; $u(t) \geq 0$, $v(t) \geq 0$ for $t \in [a, b]$ and $c \geq 0$ is a constant. If

\begin{equation}
(3.14) \quad v(t) \leq c + \int_a^t u(\tau)v(\tau) \, d\tau, \quad \text{for } t \in [a, b],
\end{equation}

then

\begin{equation}
(3.15) \quad v(t) \leq c \exp \left( \int_a^t u(\tau) \, d\tau \right), \quad \text{for } t \in [a, b].
\end{equation}

**Proof.** Case (i): Suppose $c > 0$, and define $g$ by

$$
g(t) \leq c + \int_a^t u(\tau)v(\tau) \, d\tau \quad \text{for } t \in [a, b].
$$

By (3.14) $v(t) \leq g(t)$, and since $u$ and $v$ are non-negative, $g(t) \geq g(a) = c > 0$ for all $t \in [a, b]$. Since also $\dot{g}(t) = u(t)v(t) \leq u(t)g(t)$, we have

$$
\frac{\dot{g}(t)}{g(t)} \leq u(t) \quad \text{for } t \in [a, b].
$$

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Therefore

\[ \ln \left( \frac{g(t)}{c} \right) = \ln \left( \frac{g(t)}{g(a)} \right) = \ln g(t) - \ln g(a) = \left[ \ln g(\tau) \right]_a^t = \int_a^t \frac{\dot{g}(\tau)}{g(\tau)} \, d\tau \leq \int_a^t u(\tau) \, d\tau, \]

which implies that

\[ v(t) \leq g(t) \leq c \exp \left( \int_a^t u(\tau) \, d\tau \right) \]

for \( t \in [a, b] \).

Case (ii): Suppose now \( c = 0 \). If (3.14) holds with \( c = 0 \), then Case (i) implies that (3.15) holds for \( c > 0 \). The result follows by letting \( c \to 0 \). □

**Lemma 3.16.** Suppose \( \phi \) and \( \phi' \) are the solutions of \( \dot{x} = f(x) \) satisfying \( \phi(0, p) = p \) and \( \phi'(0, p') = p' \). Let \( t_0 \in \Delta^+(\phi) \cap \Delta^+(\phi') \). Suppose furthermore that there exists \( r > 0 \) such that \( f \) is locally Lipschitz on

\[ \bigcup_{t \in [0, t_0]} B(\phi(t, p), r) \subset D \]

with respect to \( \phi \) and with a Lipschitz constant \( L \), and that \( \phi'(t, p') \in B(\phi(t, p), r) \) for \( t \in [0, t_0] \). Then

\[ \|\phi(t, p) - \phi'(t, p')\| \leq \|p - p'\| e^{L|t_0|} \]

for \( t \in [0, t_0] \).

**Proof.** Clearly \( \phi \) and \( \phi' \) satisfy the integrals

\[ \phi(t, p) = p + \int_0^t f(\phi(\tau, p)) \, d\tau, \quad \phi'(t, p') = p' + \int_0^t f(\phi'(\tau, p')) \, d\tau \]

for \( t \in [0, t_0] \). The definition of \( t_0 \) implies that \( \phi'(t, p') \in \overline{B(\phi(t, p), r)} \) for \( t \in [0, t_0] \), thus \( \phi'(t, p') \in B(\phi(\tau, p), r) \) for \( \tau \in [0, t] \). Then

\[ \|\phi(t, p) - \phi'(t, p')\| \leq \|p - p'\| + \int_0^t \|f(\phi(\tau, p)) - f(\phi'(\tau, p'))\| \, d\tau \]

\[ \leq \|p - p'\| + \int_0^t \|f(\phi(\tau, p)) - f(\phi'(\tau, p'))\| \, d\tau \]

\[ \leq \|p - p'\| + L \int_0^t \|\phi(\tau, p) - \phi'(\tau, p')\| \, d\tau \]

for \( t \in [0, t_0] \). Here the last inequation follows from \( f \) being locally Lipschitz, that is,

\[ \|f(\phi(\tau, p)) - f(\phi'(\tau, p'))\| \leq L \|\phi(\tau, p) - \phi'(\tau, p')\| \]
for $\tau \in [0, t]$. Then, by choosing $u(t) \equiv L$, $v(t) = \|\phi(t, p) - \phi'(t, p')\|$, and a constant $c = \|p - p'\|$ we can apply the Grönwall’s inequality (3.13) to obtain

$$
\|\phi(t, p) - \phi'(t, p')\| \equiv \|p - p'\| \exp\left(\int_0^t L \, d\tau\right) \leq \|p - p'\|e^{Lt}|t|.
$$

With these preparations we can prove the (iii) property of the function $\phi$:

**Theorem 3.17.** The phase path $\phi : \Omega \to D$ is continuous, and $\Omega$ is an open subset of $\mathbb{R} \times D$.

**Proof.** Fix $t \in \Delta^+ (\phi)$, and $0 < \delta < t$ satisfying $t + \delta \in \Delta^+ (\phi)$. Now, since the interval $\phi([0, t + \delta], p)$ is compact and $f$ is locally Lipschitz on $D$, it follows that there exists $r > 0$ such that

$$
\overline{B(\phi([0, t + \delta], p), r)} = \bigcup_{\tau \in [0, t + \delta]} \overline{B(\phi(\tau, p), r)} \subset D,
$$

$f$ is locally Lipschitz on $\overline{B(\phi(\tau, p), r)}$ for $\tau \in [0, t + \delta]$, and the Lipschitz constant $L$ is independent on $\tau$.

Choose $p'$ that satisfies $\|p - p'\| < \delta' = re^{-L(t+\delta)}$, and suppose $\phi'$ satisfies $\phi'(0, p') = p'$. It will be now verified that $t + \delta \in \Delta^+ (\phi')$. Suppose that $t + \delta \notin \Delta^+ (\phi')$. It then follows that $t + \delta < t + \delta$, and since $\overline{B(\phi([0, t + \delta], p), r)}$ is compact, by Theorem (3.12) there exists $0 < t_0 < t + \delta$ satisfying $\|\phi(t_0, p) - \phi'(t_0, p')\| = r$, and that $\|\phi(\tau, p) - \phi'(\tau, p')\| < r$ for $0 \leq \tau < t_0$. Furthermore, $t_0 \in \Delta (\phi) \cap \Delta (\phi')$. Thus by Lemma (3.16) we obtain

$$
\|\phi(t_0, p) - \phi'(t_0, p')\| \leq \|p - p'\|e^{LT} \leq \|p - p'\|e^{LT} < r,
$$

which states a contradiction. We have now verified that $t + \delta \in \Delta^+ (\phi')$ and $\phi'(\tau, p') \in \overline{B(\phi(\tau, p), r)}$ for $\tau \in [0, t + \delta]$. This essentially means that if $\|p - p'\| < \delta'$ and $|t - \tau| < \delta$, then $\tau \in \Delta^+ (\phi')$, that is, $(\tau, p') \in \Omega$. Hence $\Omega$ is an open set.

It will be now verified that the phase path $\phi$ is continuous. Let $\varepsilon > 0$, and choose $\delta > 0$ such that $\|\phi(\tau, p) - \phi(t, p)\| < \varepsilon/2$ for $|t - \tau| < \delta$. Choose also $\delta' > 0$ such that $\|\phi(\tau, p) - \phi'(\tau, p')\| < \varepsilon/2$ for $\tau \in [0, t + \delta]$ and $\|p - p'\| < \delta'$. This is possible, since by Lemma (3.16) we have $\phi'(\tau, p') \in \overline{B(\phi(\tau, p), r)}$. It then follows that

$$
\|\phi(t, p) - \phi'(\tau, p')\| \leq \|\phi(t, p) - \phi(t, p)\| + \|\phi(\tau, p) - \phi'(\tau, p')\| < \varepsilon/2 + \varepsilon/2 = \varepsilon,
$$

for $|t - \tau| < \delta$ and $\|p - p'\| < \delta'$, hence the phase path $\phi$ is continuous on $\Omega$. \qed
Definition 3.18. The orbit or path $\gamma$ through a given $p \in D$ is defined by

\begin{equation}
\gamma(p) = \{ \phi(t, p) : t \in \Delta(\phi) \}.
\end{equation}

The positive semiorbit through $p$ is

$\gamma^+(p) = \{ \phi(t, p) : t \in \Delta^+(\phi) \}$,

and the negative semiorbit through $p$ is

$\gamma^-(p) = \{ \phi(t, p) : t \in \Delta^-(\phi) \}$.

In applications we are often, if not always, interested in the positive semiorbits, which is why theory will be presented with the mindset that time is ticking towards the future. However, the nature of these definitions imply that the corresponding theory will hold while travelling 'back' in time.

Definition 3.20. A critical point of a vector field $f(x)$ is a point $p \in D$ such that $f(p) = 0$. If $p$ is critical, then $x \equiv p$ satisfies $\dot{x} = f(x)$. A regular point is a point which is not critical. In applications we refer to critical point as an equilibrium of the system.

Definition 3.21. The orbit $\gamma$ is called periodic, if there exists $t_0 > 0$ such that $\phi(t_0, p) = p$ for every $p \in \gamma$. The smallest such a value is denoted by $\tau$, and it is the period of $\gamma$. Moreover, by uniqueness $\phi(t + \tau, p) = \phi(t, p)$ for every $t \in \mathbb{R}$, since now $\Delta(\phi(p)) = \mathbb{R}$. In this work critical points are not considered periodic.

In the example (3.6) we had an interesting special case where all orbits were periodic. In applications this kind of behavior is extremely rare.

Definition 3.22. The orbit $\gamma$ is singularly closed if $\Delta(\phi(p)) = \mathbb{R}$ for every $p \in \gamma$ and there exists $p^+, p^- \in D$ satisfying $p^+ = \lim_{t \to \infty} \phi(t, p)$ and $p^- = \lim_{t \to -\infty} \phi(t, p)$. It then follows that $p^+$ and $p^-$ are critical, and $p^+, p^- \not\in \gamma(p)$.

Definition 3.23. Let $E$ be a open subset of $D$. We say that $E$ is invariant if

$\gamma(E) \subset E$.

Similarly, $E$ is positively invariant if

$\gamma^+(E) \subset E$,

and $E$ is negatively invariant if

$\gamma^-(E) \subset E$. 

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\textbf{Theorem 3.24.} Let $E \subset D$ be positively invariant, closure $\overline{E}$ be compact and $\overline{E} \subset D$. Then for every $p \in E$ we have $t^+(\phi(p)) = \infty$. If $E$ is invariant, then also $t^-(\phi(p)) = -\infty$ which implies that $\Delta(\phi(p)) = \mathbb{R}$.

\textit{Proof.} Since $E$ is positively invariant, it follows that $\gamma^+(\phi(p)) \subset E$. Clearly every $t_k$ satisfy $\phi(t_k, p) \in E$ for $t_k < t < t^+(\phi(p))$. By Theorem (3.12) we must have $t^+(\phi(p)) = \infty$.

Similarly for $t^-(\phi)$ when $E$ is invariant. \hfill \qed

\section{Limit Sets of an Orbit}

In this section we discuss the limit sets of an orbit. As before, most of the theory is presented for increasing time.

\textbf{Definition 3.25.} The \textit{positive limit set} of the path $\gamma$ through $p$ is

$$\omega(\gamma) = \bigcap_{t \in \Delta(\phi(p))} \gamma^+(t, p),$$

and similarly, the \textit{negative limit set} is

$$\alpha(\gamma) = \bigcap_{t \in \Delta(\phi(p))} \gamma^-(t, p).$$

As one might expect, there are other ways to define the limit sets that are equivalent with the above definition. For example, we could say that a point $p_0$ belongs to $\omega(\gamma)$,
where $\gamma$ is the path through $p$, if for every $\varepsilon > 0$ there exists a time sequence $(t_k)$ and $k_\varepsilon \in \mathbb{N}$ such that for $k > k_\varepsilon$, $\|\phi(t_k, p) - p_0\| < \varepsilon$. This implies that
\[
\omega(\gamma) = \bigcap_{k=1}^{\infty} \gamma^+(\phi(t_k, p))
\]
for all sequences $(t_k)$ satisfying $t_k > 0$ and $t_k \to \infty$ as $k \to \infty$.

It will be seen that it is actually convenient to have these two ways to define the (positive) limit set. However, the idea is the very same: a (positive) limit set $\omega(\gamma)$ is the set of points that are approached along $\gamma$ with increasing time. These definitions yield the following corollary:

**Theorem 3.26.** Let $p \in D$ and denote the path through $p$ by $\gamma$. Suppose $t^+ (\phi(p)) < \infty$. Then $\omega(\gamma) = \emptyset$.

**Proof.** Let $p_0 \in D$ be arbitrary, and fix $r > 0$ such that $\overline{B(p_0, r)} \subset D$. Since $\overline{B(p_0, r)}$ is compact, by Theorem (3.12) there exists $t > 0$ satisfying $\gamma^+(\phi(t, p)) \cap \overline{B(p_0, r)} = \emptyset$, and hence $p_0 \notin \gamma^+(\phi(t, p)) \supset \omega(\gamma)$. \qed

Although it is important to keep in mind that some solutions may not be defined properly when we let $t \to \infty$, it is sometimes considered so uninteresting, that the theory of maximum interval of existence have been completely left out.\(^1\) We do admit that with regards to the limit sets this is as far as the unbounded maximum interval can take us, and thus, unless stated otherwise, we shall henceforth assume that $t^+ (\phi) = \infty$.

**Lemma 3.27.** Suppose the sets $A_t \subset \mathbb{R}^n$, for $t > 0$, form a family of decreasing sets, that is, $A_t \subset A_s$ for $t > s$. Suppose furthermore that every $A_t$ is non-empty, compact and connected. Then also the intersection
\[
A = \bigcap_{t>0} A_t
\]
is non-empty, compact and connected.

**Proof.** Suppose $A$ is an empty set, and fix $t_0 > 0$. Let
\[
\bigcup_{t > 0} U_t = \mathbb{R}^n \setminus A_t
\]
be an open cover of $A_{t_0}$. Since $A_{t_0}$ is compact, there exists a finite subcover
\[
\{U_{t_1}, U_{t_2}, \ldots, U_{t_m}\}, \quad t_1 < t_2 < \ldots < t_m,
\]

\(^1\)E.g., Hale (2009, p. 46)
and essentially
\[ A_{t_0} \subset \bigcup_{k=1}^{m} U_{t_k}. \]

But this implies that
\[ A_{t_m} = \bigcap_{k=1}^{m} A_{t_k} = (\bigcup_{k=1}^{m} U_{t_k})^c \subset \mathbb{R}^n \setminus A_{t_0}, \]

which contradicts with \( A_t \) being a family of decreasing and non-empty sets.

Since \( A \) is closed and \( A \subset A_{t_0} \), it is also compact.

Suppose \( A = B_1 \cup B_2 \), where \( B_1 \) and \( B_2 \) are non-empty and compact disjoint sets. Since \( \mathbb{R}^n \) equipped with Euclidian metric is a Hausdorff space, we can find disjoint neighbourhoods \( U_1 \) and \( U_2 \) for \( B_1 \) and \( B_2 \) respectively. It will be now verified that there exists \( t > 0 \) satisfying
\[ A_t \subset U = U_1 \cup U_2. \]

Write \( U_t = \mathbb{R}^n \setminus A_t \) for \( t > 0 \). Since
\[ (\bigcup_{t \geq t_0} U_t)^c = \bigcap_{t \geq t_0} A_t = A \subset U \cup A_{t_0}^c = (U^c \cap A_{t_0})^c = (A_{t_0} \setminus U)^c, \]
then
\[ A_{t_0} \setminus U \subset \bigcup_{t \geq t_0} U_t. \]

This implies that the sets \( U_t \) for \( t \geq t_0 \) form an open cover for \( A_{t_0} \setminus U \). Since \( A_{t_0} \setminus U \) is a closed subset of \( A_{t_0} \), it is also compact and there exists a finite subcover
\[ \{U_{t_1}, U_{t_2}, ..., U_{t_m}\}, \quad t_1 < t_2 < ... < t_m. \]

Now since \( U_{t_k} \subset U_{t_m} \) for \( k \in \{1, 2, ..., m\} \), then
\[ A_{t_0} \setminus U \subset U_{t_m} = \mathbb{R}^n \setminus A_{t_m}. \]

Since also \( A_{t_m} \subset A_{t_0} \), then \( A_{t_m} \subset U = U_1 \cup U_2 \). But since \( A_{t_m} \) is connected, we must have either \( A_{t_m} \cap U_1 = \emptyset \) or \( A_{t_m} \cap U_2 = \emptyset \). Suppose for a moment \( A_{t_m} \cap U_2 = \emptyset \). Then \( A \subset A_{t_m} \subset U_1 \), that is, \( A = B_1 \) and \( B_2 = \emptyset \) and hence contradicting with our assumptions.

These preparations yield the following features of the (positive) limit set.

**Theorem 3.28.** Let \( p \in D \) and denote the path through \( p \) by \( \gamma \). Suppose \( t^+ (\phi(p)) = \infty \). Then:
Proof. (a) Let $p_0 \in \omega(\gamma)$. Then, for every $k \in \mathbb{N}_+$ we have $\gamma^+(\phi(k,p)) \cap B(p_0,1/k) \neq \emptyset$. Thus there exists values $s_k$ satisfying $s_k \geq 0$, $t_k = s_k + k$, and $\phi(t_k,p) = \phi(s_k + k,p) \in B(p_0,1/k)$ for every $k \in \mathbb{N}_+$. Clearly $t_k \to \infty$, and $\phi(t_k,p) \to p_0$ as $t_k \to \infty$. We have now found a sequence $(t_k)$ satisfying $\phi(t_k,p) \to p_0$ as $t_k \to \infty$.

Conversely, suppose $(t_k)$ is a sequence satisfying $t_k \to \infty$ and $\phi(t_k,p) \to p_0$. Fix $t > 0$. Then, for sufficiently large values of $k$ we have $t_k \geq t$, and thus $\phi(t_k,p) \in \gamma^+(p)$. Hence, the limit $\phi(t_k,p) \to p_0$ satisfy $p_0 \in \gamma^+(p)$, that is, $p_0 \in \omega(\gamma)$.

(b) Clearly $\gamma^+(p) \subset \gamma^+(p)$. By the definition of limit set for any given $t > 0$ we have

$$\omega(\gamma) \subset \gamma^+(\phi(t,p)) \subset \gamma^+(\phi(0,p)) = \gamma^+(p),$$

and hence $\omega(\gamma) \cup \gamma^+(p) \subset \gamma^+(p)$.

Conversely, if $p_0 \in \gamma^+(p)$ we are done. Suppose now $p_0 \in \overline{\gamma^+(p)} \setminus \gamma^+(p)$. For a given $k \in \mathbb{N}_+$ the interval $\phi([0,k],p)$ is compact and closed. Since $p_0 \notin \phi([0,k],p) \subset \gamma^+(p)$, yet $p_0 \in \gamma^+(p)$, it follows that there exists $r_k$ satisfying

$$0 < r_k \leq 1/k \quad \text{and} \quad B(p_0,r_k) \cap \phi([0,k],t) = \emptyset,$$

and there also exists $t_k \geq 0$ satisfying $\phi(t_k,p) \in B(p_0,r_k) \subset B(p_0,1/k)$. Clearly also $t_k > k$. From these $t_k$ we can form a sequence $(t_k)$ satisfying $t_k \to \infty$ as $k \to \infty$ and $\phi(t_k,p) \to p_0$. By section (a) we have $p_0 \in \omega(\gamma)$ and hence $\gamma^+(p) \subset \gamma^+(p) \cup \omega(\gamma)$.

(c) Clearly $\gamma^+(p)$ is closed in $D$, and by definition $\omega(\gamma)$ is closed in $D$. Assume $p_0 \in \omega(\gamma)$, and suppose $\phi'$ is the solution satisfying $\phi'(0,p_0) = p_0$. Fix $t \in \Delta(\phi')$. By

\[\begin{align*}
(a) \quad & \omega(\gamma) = \{p_0 \in D : \text{there exists } (t_k) \text{ satisfying } \phi(t_k,p) \to p_0 \text{ as } t_k \to \infty\}. \\
(b) \quad & \overline{\gamma^+(p)} = \gamma^+(p) \cup \omega(\gamma). \\
(c) \quad & \overline{\gamma^+(p)} \text{ and } \omega(\gamma) \text{ are both closed in } D, \ \omega(\gamma) \text{ is invariant, and } \gamma^+(p) \text{ is positively invariant.} \\
(d) \quad & \omega(\gamma) = \omega(\gamma(t)) \text{ for every } t \geq 0. \\
Suppose \text{ furthermore that there is a compact and invariant region } K \subset D \text{ with } p \in K. \ \text{Then:} \\
(e) \quad & \text{The limit set } \omega(\gamma) \text{ is non-empty, compact and connected.} \\
(f) \quad & \text{For every } p_0 \in \omega(\gamma) \text{ the maximum interval } \Delta(\phi(p_0)) \text{ is } \mathbb{R}. \\
(g) \quad & \phi(t,p) \to \omega(\gamma) \text{ as } t \to \infty.
section (a) there exists a sequence \((t_k)\) satisfying \(\phi(t_k, p) \to p_0\) as \(t_k \to \infty\). By continuity and translation of independent variable it follows that
\[
\phi(t + t_k, p) \to \phi'(t, p_0) \quad \text{as} \quad k \to \infty.
\]
Clearly \(t + t_k \to \infty\), and by section (a) we have \(\phi'(t, p_0) \in \omega(\gamma)\), i.e., \(\omega(\gamma)\) is invariant.

Suppose now \(p_0 \in \gamma^+(p)\) and that \(\phi'\) satisfies \(\phi'(0, p_0) = p_0\). Then \(p_0 = \phi(s, p)\) for some \(s \geq 0\), and so for \(t \geq 0\) we have
\[
\phi'(t, p_0) = \phi(s + t, p) \in \gamma^+(p),
\]
which implies that \(\gamma^+(p)\) is positively invariant. By section (b) we have \(\gamma^+(p) = \gamma^+(p) \cup \omega(\gamma)\), and hence \(\gamma^+(p)\) is positively invariant.

(d) By the definition of path \(\gamma\) we obtain
\[
\omega(\gamma(t)) = \bigcap_{s+t \in \mathbb{R}} \gamma^+(\phi(s + t, p)) \subset \bigcap_{t \in \mathbb{R}} \overline{\gamma^+(\phi(t, p))} = \omega(\gamma).
\]
Conversely, suppose \(p_0 \in \omega(\gamma)\) and fix \(t \geq 0\). Then \(p_0 \in \overline{\gamma^+(\phi(t, p))}\). By Lemma (3.29) we have
\[
p_0 \in \bigcap_{t \in \mathbb{R}} \overline{\gamma^+(\phi(t, p))} \subset \bigcap_{s+t \in \mathbb{R}} \gamma^+(\phi(s + t, p)) = \omega(\gamma(t)).
\]
This proves the section (d).

(e) The positive semi-orbit \(\gamma^+(p)\) is connected since it is a continuous image of a connected interval \([t, \infty]\), and the same holds for its closure. The non-emptiness follows from Lemma (3.27), since by definition \(\omega(\gamma)\) is an intersection of a family of decreasing sets, which are non-empty, compact and connected.

(f) From sections (c) and (e) it follows that \(\omega(\gamma)\) is non-empty, compact and invariant. Suppose \(p_0 \in \omega(\gamma)\). Since \(\omega(\gamma)\) is invariant, it follows that \(\gamma(p_0) \subset \omega(\gamma)\), and since \(\omega(\gamma)\) is also compact, the Theorem (3.12) implies \(t^+(\phi(p_0)) = \infty\). Similarly for the lower bound we get \(t^-(\phi(p_0)) = -\infty\), thus \(\Delta(\phi(p_0)) = \mathbb{R}\).

(g) A proof by contradiction: Suppose there exists a neighbourhood \(U \subset D\) of \(\omega(\gamma)\) and a sequence \((t_k)\) satisfying \(t_k \to \infty\) and \(\phi(t_k, p) \not\in U\). Since \(\phi(t_k, p) \in \gamma^+(p)\) and \(\gamma^+(p) \subset K\) is compact, the sequence \((\phi(t_k, p))\) has a limit \(p_0 \in \gamma^+(p)\). Choose now a subsequence of \((\phi(t_k, p))\) that converges to \(p_0\). By section (a) we have \(p_0 \in \omega(\gamma) \subset U\). On the other hand, \(p_0 \in D \setminus U\), since \(D \setminus U\) is closed on \(D\) and \(\phi(t_k, p) \in D \setminus U\). \(\square\)
Chapter 4

Theory of Poincaré and Bendixson

The purpose of this chapter is to discuss the global behaviour of solutions of autonomous systems in the plane. Namely, the Poincaré-Bendixson theorem, which is a statement about the long-term behavior of solutions originally discussed by Poincaré (1892) and later generalized by Bendixson (1901, pp. 1-88). The outline of this presentation follows the work by Teschl (2012, pp. 220-227), Hale (2009, pp. 51-56) and Lamberg’s lectures (2014).

As before, we assume that the function \( f \) is Lipschitz continuous in \( D \), which is now a subset of \( \mathbb{R}^2 \). Furthermore, whenever we speak of any familiar element, e.g., a point or path, they are assumed to be on \( D \).

Many of the results of this chapter have been made possible because of the Jordan Curve theorem:

Definition 4.1. A homeomorphic image of the circumference of a circle is called a **Jordan curve**.

Theorem 4.2. Every Jordan curve \( J \) in \( \mathbb{R}^2 \) separates the plane; \( \mathbb{R}^2 \setminus J = S_e \cap S_i \) where \( S_e \) and \( S_i \) are disjoint open sets, \( S_e \) is unbounded and called the exterior of \( J \), \( S_i \) is bounded and called the interior of \( J \) and both sets are arcwise connected.

The result may seem obvious at first, but it is actually rather difficult to prove. Although being one of the best known topological theorems, there are many, even among professional mathematicians who have never read a proof of it.\(^1\) We make no attempt to prove it here, and simply refer to the literature.\(^1,2\)

The definition of Jordan Curve yields the following corollary.

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Figure 4.1: Example of the flow on transversal $T(p)$.

**Corollary 4.3.** A periodic orbit corresponds to a closed path, a Jordan Curve.

*Proof.* If $\gamma$ is a closed path and $p$ is a point of $\gamma$, there exists $\tau > 0$ such that $\phi(\tau, p) = \phi(0, p) = p$. By uniqueness $\phi(\tau + t, p) = \phi(t, p)$ for every $t \in \mathbb{R}$, which implies that $\gamma$ has a period $\tau$.

Conversely, suppose $\phi(t, p) = \phi(t + \tau, p)$ for every $t \in \mathbb{R}$, and $\tau > 0$ is the period of $\gamma$. As $t$ varies in $[0, \tau]$, the phase path $\phi(t, p)$ describes a curve in $\mathbb{R}^2$ which is the homeomorphic image of the segment $[0, \tau]$ with $\phi(0, p) = \phi(\tau, p)$. On the other hand, the line segment $[0, \tau]$ with 0 and $\tau$ identified is homeomorphic to the unit circle, a Jordan Curve. \qed

**Lemma 4.4.** Suppose $p$ is regular. Then there exists a $p$-centered open segment $T(p)$ satisfying:

(a) The segment $T(p)$ is perpendicular to the vector $f(p)$.

(b) For every $p_T \in T(p)$ there exists a neighbourhood $U$ such that if $p_1 \in U$, then the intersection $\gamma(p_1) \cap T(p)$ contains at least one point.

(c) $f(p_T) \cdot f(p) > 0$ for every $p_T \in T(p)$.

We refer to $T$ simply as a transversal.

*Proof.* (a) & (c) Since $f(p) \neq 0$, there exists a segment $T(p)$ perpendicular to $f(p)$, which can be chosen to be sufficiently short to satisfy $f(p_T) \cdot f(p) > 0$ for every $p_T \in T(p)$.

(b) By Theorem (3.17) $\phi$ is defined in an open $\Omega \subset \mathbb{R} \times D$. Thus there exists $\alpha > 0$ and $r > 0$ such that

$$B(p, r) \times ] - \alpha, \alpha[ \subset \Omega,$$

that is, every $p_1 \in B(p, r)$ satisfies

$$] - \alpha, \alpha[ \subset \Delta(\phi(p_1)).$$

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Furthermore, we can choose $r$ to be small enough to satisfy

$$f(p_1) \cdot f(p) \geq \frac{1}{2} \| f(p) \|^2$$

for every $p_1 \in B(p,r)$, which is now a sufficiently small radius for $T(p)$.

Let $p_T \in T(p)$, and suppose $\beta = \beta(p_T)$ is such that $0 < \beta < \alpha/2$ and $\phi(t, p_T) \in B(p, r)$ for every $t \in [-2\beta, 2\beta]$. Choose now sufficiently small $U = U(p_T)$ to satisfy the following conditions:

1. $\phi(t, p_1) \in B(p, r)$ for every $p_1 \in U$ and $t \in [-\beta, \beta]$. This is possible, since $\phi$ is uniformly continuous in a compact region.

2. $\frac{1}{2} R \| f(p) \|^2 > |(p_1 - p) \cdot f(p)|$ for every $p_1 \in U$. By continuity this is possible, since $(p_T - p) \cdot f(p) = 0$ from section (a).

Let $p_1 \in U$ and suppose for instance $(p_1 - p) \cdot f(p) > 0$. Then

$$\left( \phi(0, p_1) - p \right) \cdot f(p) - \left( \phi(-\beta, p_1) - p \right) \cdot f(p) = \int_{-\beta}^{0} \phi(p_1) \cdot f(p) \, dt$$

$$= \int_{-\beta}^{0} f(\phi(p_1)) \cdot f(p) \, dt$$

$$\geq \frac{1}{2} \int_{-\beta}^{0} \| f(p) \|^2 \, dt$$

$$= \frac{\beta}{2} \| f(p) \|^2$$

$$> (p_1 - p) \cdot f(p),$$
which implies that \((\phi(-\beta, p_1) - p) \cdot f(p) < 0\). By the Bolzano’s theorem there exists \(u \in ] - \beta, 0[\) such that
\[(\phi(u, p_1) - p) \cdot f(p) = 0.\]
Then by (1) we must have \(\phi(u, p_1) \in B(p, r)\), and by (a) we have \(\phi(u, p_1) \in T(p)\), hence \(\phi(u, p_1) \in \gamma(p_1) \cap T(p)\). \(\Box\)

The idea behind the transversal \(T\) is to create a trapping zone for orbits, meaning that once an orbit enters this zone, it cannot escape. By uniqueness the orbit cannot intersect with itself and hence the only possibility to exit is through \(T\), a gateway, if you will. But this cannot happen, since by definition the ‘flow’ of \(f\) always points in the same direction.

Sometimes in the literature the transversal is simply defined by an arc that is defined to be perpendicular to the vector field in all of its points.\(^3\). But it will be seen that it is sufficient to be defined as we have done, where the vector field \(f\) is perpendicular to \(T\) in at least one point.

**Lemma 4.5.** Let \(T\) be a transversal containing a regular \(p_0\). Denote by \(p_n = \phi(t_n, p_0)\) the ordered sequence of intersections of \(\gamma(p_0)\) with \(T\). Then \(p_n\) is monotone with respect to \(T\).

**Proof.** If \(p_0 = p_1\), then \(\gamma(p_0)\) (and also \(\gamma(p_1)\) for that matter) is a periodic orbit and we are done. Suppose now \(p_0 \neq p_1\); and \(J\) is the curve from \(p_0\) to \(p_1\) along \(\gamma(p_0)\) and back from \(p_1\) to \(p_0\) along the transversal \(T\). This curve \(J\) is a homeomorphic image of a circle, making it by the definition a Jordan curve. This leads to two possible cases, which are depicted in the figure 4.3.

Denote the 'gateway' between $p_0$ and $p_1$ by $\tilde{T} \subset T$. By definition of transversal, $f$ points in the direction of either $S_i$ or $S_e$. This essentially means that $\gamma^+(p_1)$ enters either $S_i$ or $S_e$, and then is trapped since it cannot cross the orbit $\gamma(p_0)$, nor exit through $\tilde{T}$. Hence either $\gamma^+(p_1) \subset S_i$ or $\gamma^+(p_1) \subset S_e$. Suppose for a moment that $\gamma^+(p_1) \subset S_e$ (case (a) in the figure 4.3). Then $\gamma^+(p_1)$ remains trapped in $S_e$. Similarly if $\gamma^+(p_1) \subset S_i$, then $\gamma^+(p_1)$ remains trapped in the component $S_i$. Iterating this procedure proves the theorem.

**Corollary 4.6.** Let $T$ be a transversal, and suppose $\gamma$ is a path with $p \in \omega(\gamma)$. Then both $\omega(\gamma)$ and $\gamma(p)$ intersect $T$ in at most one point.

**Proof.** Suppose for a moment that $\omega(\gamma)$ intersects $T$ in at most one point. Since $\omega(\gamma)$ is invariant, then $\gamma(p) \in \omega(\gamma)$ and hence $\gamma(p)$ intersects $T(p)$ in at most one point.

Suppose $\omega(\gamma)$ intersects $T$ in at two points $p_1$ and $p_2$. Then there exists sequences $p_{1,n}, p_{2,n} \in T \cap \gamma$ converging to $p_1$ and $p_2$, respectively. But this is not possible since by Lemma (4.5) both $p_{1,n}$ and $p_{2,n}$ are subsequences of a monotone sequence $p_n$, which leads us to a contradiction. 

**Corollary 4.7.** Suppose $\gamma$ contains no critical points and $\omega(\gamma) \cap \gamma \neq \emptyset$. Then $\gamma = \omega(\gamma)$ is a periodic orbit.

**Proof.** By assumptions there exists a regular point $p \in \omega(\gamma) \cap \gamma$. Moreover, by invariance of $\omega(\gamma)$ we must have $\gamma = \gamma(p) \subset \omega(\gamma)$. Let $T(p)$ be a transversal, and denote by $p_n$ the ordered sequence of intersections of $\gamma$ with $T(p)$ converging to $p$. By Corollary (4.6) we have $p_n = p$ and hence $\gamma = \gamma(p)$ is a periodic orbit.

**Corollary 4.8.** Suppose $\omega(\gamma)$ contains a periodic orbit $\gamma(p)$. Then $\omega(\gamma) = \gamma(p)$.

**Proof.** By invariance of $\omega(\gamma)$ we must have $\gamma(p) \subset \omega(\gamma)$.

Suppose $\omega(\gamma) \not\subset \gamma(p)$, that is, $\omega(\gamma) \setminus \gamma(p) \neq \emptyset$. Then, by connectedness, there is a point $p_0 \in \gamma(p)$ such that we can find a sequence $(p_n) \in \omega(\gamma) \setminus \gamma(p)$ converging to $p_0$. Let $T(p_0)$ be a transversal. By Lemma (4.4) there exists $k$ such that the orbit $\gamma(p_k) \subset \omega(\gamma)$ intersects with $T(p_0)$ at some point $p_T$. But then we have $p_0, p_T \in \omega(\gamma) \cap T(p_0)$, and Corollary (4.6) implies $p_0 = p_T$ and hence $p_T \in \gamma(p)$ contradicting our assumption.

These preparations now yield our main theorem.

**Theorem 4.9 (Poincaré-Bendixson).** Suppose $K \subset D$ is a compact and invariant region, and let $\gamma \subset K$. If $\omega(\gamma)$ contains no critical points, it is a periodic orbit.
Proof. By Theorem (3.32) the limit set $\omega(\gamma)$ is non-empty, compact and connected. Suppose $p \in \omega(\gamma)$. Since $\gamma(p) \subset \omega(\gamma) \subset K$, then also $\omega(\gamma(p))$ is non-empty. Suppose $p_0 \in \omega(\gamma(p))$, and by assumptions $p_0$ is a regular point. Let $T(p_0)$ be a transversal. Now since $p_0 \in \omega(\gamma(p)) \cap T(p_0) \subset \omega(\gamma) \cap T(p_0)$, then by Lemma (4.6) we must have $\omega(\gamma) \cap T(p_0) = \{p_0\}$. Furthermore, by Lemma (4.4) the intersection $\gamma(p) \cap T(p_0)$ contains at least one point, and since $\gamma(p) \subset \omega(\gamma)$, we must have $\gamma(p) \cap T(p_0) = \{p_0\}$ and hence $p_0 \in \gamma(p)$. We have obtained that $p_0 \in \gamma(p) \cap \omega(\gamma)$. By Lemma (4.7) the path $\gamma(p) = \omega(\gamma)$ is a periodic orbit.

\[\square\]

Corollary 4.10. Suppose $K \subset D$ is a compact and invariant region. If $K$ contains no critical points, it must contain a periodic orbit.

**Lemma 4.11.** Suppose that $K \subset D$ is a compact and invariant region, with $p \in K$. Denote the path through $p$ by $\gamma$. Suppose furthermore that there are two distinct critical points $p^+, p^- \in \omega(\gamma)$. Then there exists at most one orbit $\gamma(p_0)$ contained in $\omega(\gamma)$ with $\omega(\gamma(p_0)) = p^+$ and $\alpha(\gamma(p_0)) = p^-$. 

Proof. Suppose there are two orbits $\gamma(p_1)$ and $\gamma(p_2)$ in $\omega(\gamma)$ with $\omega(\gamma(p_1)) = \omega(\gamma(p_2)) = \{p^+\}$. Let $T(p_1)$ and $T(p_2)$ be transversals for $p_1$ and $p_2$ respectively. By definition of $\omega(\gamma)$ there are points of $\gamma$ arbitrarily close to $\gamma(p_1)$ and $\gamma(p_2)$, since they belong to $\omega(\gamma)$. Hence we can find $t_1$ and $t_2$ with $\phi(t_1, p) \in T(p_1)$ and $\phi(t_2, p) \in T(p_2)$. Denote the curve $J$ from $p_1$ to $\phi(t_1, p)$ along $T(p_1)$, then to $\phi(t_2, p)$ along $\gamma^+$, then to $p_2$ along $T(p_2)$, and
Figure 4.5: Sketch of proof for Lemma (4.11).

then to $p^+$ along $\gamma(p_2)$, and finally back to $p_1$ along $\gamma(p_1)$ (see figure 4.5). This curve $J$ is by definition a Jordan curve. Denote the interior of $J$ by $S_i$ and the exterior by $S_e$. Hence $\gamma^+$ enters either $S_i$ or $S_e$, and then remains trapped in the component. Suppose for a moment that $\gamma^+$ enters $S_i$. Now since $p^- \notin S_i$, then $p^- \notin \omega(\gamma)$. The rest of the proof follows from these observations.

We can now generalize the Poincaré-Bendixson theorem:

**Theorem 4.12 (Generalized Poincaré-Bendixson).** Suppose $K \subset D$ is a compact and invariant region, and let $\gamma \subset K$. Then one of the following cases holds:

(i) $\omega(\gamma)$ is a critical point.

(ii) $\omega(\gamma)$ is a periodic orbit.

(iii) $\omega(\gamma)$ consists of finitely many critical points and singularly closed orbits connecting them.

**Proof.** If $\omega(\gamma)$ contains no critical points, it is by Theorem (4.9) a periodic orbit. If $\omega(\gamma)$ contains at least one critical point $p_0$, but no regular points, we must have $\omega(\gamma) = \{p_0\}$,
Figure 4.6: A limit set that consists of four critical points and four singularly closed orbits connecting them.

since critical points are isolated: $\gamma(p_0) \subset \omega(\gamma)$ and $\gamma(p_0) = \{p_0\}$, and by Theorem (3.28) the limit set $\omega(\gamma)$ is connected.

Suppose now $\omega(\gamma)$ contains both critical and regular points. Let $p \in \omega(\gamma)$ be regular. It is sufficient to show that $\omega(\gamma(p))$ consists of one critical point, that is, $\omega(\gamma(p))$ does not contain regular points. Let $T(p)$ be a transversal. By Corollary (4.6) the orbit $\gamma(p)$ intersects $T(p)$ in at most one point. Since $p$ is regular, $\gamma(p)$ contains no critical points. Hence by Corollary (4.7) the orbit $\gamma(p) = \omega(\gamma)$ is periodic, which is impossible since by assumption $\omega(\gamma)$ contains a critical point. □

Although the result may at first seem to solve most difficulties of planar systems, it is not the entire story: To this day the theory of planar autonomous systems is an evergoing process. Especially with periodic orbits there are many open questions. For instance, even when we are capable of proving the existence of a periodic orbit, actually finding this orbit is usually troublesome. Moreover, there are examples of planar systems of quadratic polynomials with four isolated periodic orbits, but as of the moment, no one has managed to prove nor disprove the existence of more than four isolated orbits.\footnote{[Online] Available from: http://www4.ncsu.edu/~schecter/ma_732_sp13/p-b.pdf [Accessed: 4th March 2016].} The general question of the number of isolated orbits for a polynomial system have been open ever since 1905, and it is known as the Hilbert’s 16th problem. (Ilyashenko 2002, pp. 301-354)

Nevertheless, there are vast amount of interesting results that follow from the theory of Poincaré and Bendixson, of which only a few will be presented in this work. For further
reading we refer to the literature.  

Corollary 4.13. The interior of every periodic orbit must contain a critical point.

Proof. By Corollary (4.3) the periodic orbit corresponds to a Jordan Curve and by the Jordan Curve theorem (4.2) the interior is arcwise connected. Clearly it is also compact. Since the orbits starting in the interior cannot escape, the interior is an invariant region. By generalized Poincaré-Bendixson theorem (4.12) the limit set of an orbit must contain either a periodic orbit or a critical point. (Recall that the theory also holds for negative limit set.) Iterating this procedure proves the theorem.

Since many open questions of planar systems are related to the periodic orbits, one useful result is the Dulac’s Criterion, which states a condition on which there can be no periodic orbits.

4.1 Dulac’s Criterion

This short section elaborates one useful consequence of the theory of Poincaré and Bendixon. Dulac’s Criterion or, the Bendixson-Dulac theorem states the condition on which there are no periodic orbits. It was first stated by Bendixson (1901, pp. 1-88), and then further refined by Dulac in 1933 using the Green’s theorem. (Perko 2006, p. 262)

Lemma 4.14 (Green’s Theorem). Let \( f(x) = (f_1(x), f_2(x)) : D \rightarrow \mathbb{R}^2 \), where \( x = (x_1, x_2) \), be a differentiable vectorfield on a closed and simply connected region \( G \subset \mathbb{R}^2 \) with smooth boundary \( \partial G \). Then

\[
\int_{\partial G} (f_1(x) \, dx_1 + f_2(x) \, dx_2) = \iint_G \left( \frac{\partial f_1(x)}{\partial x_1} - \frac{\partial f_2(x)}{\partial x_2} \right) \, dx_1 \, dx_2.
\]

For a proof we refer to, e.g., Riemann (1851, pp. 8-9).

Definition 4.15. The divergence of a vector field \( f : D \rightarrow \mathbb{R}^2 \) is

\[
\nabla \cdot f = \frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2}.
\]

Theorem 4.16. Suppose \( G \subset D \) is a simply connected region. If \( \nabla \cdot f \) does not change sign and does not vanish in \( G \), then there are no periodic orbits contained inside \( G \).

Proof. Suppose there is a periodic orbit $\gamma$ with a period $\tau$, and denote its interior by $K$. By Green’s theorem (4.14) we have
\[
\iint_K \nabla \cdot f \, dx = \iint_K \left( \frac{\partial f_1(x_1, x_2)}{\partial x_1} + \frac{\partial f_2(x_1, x_2)}{\partial x_2} \right) dx_1 dx_2
= -\oint_{\gamma} f_2(x_1, x_2) \, dx_1 - f_1(x_1, x_2) \, dx_2
= -\int_0^\tau \left( \frac{dx_2}{dt} \frac{dx_1}{dt} - \frac{dx_1}{dt} \frac{dx_2}{dt} \right) \, dx_1 dx_2 = 0,
\]
which implies that $\nabla \cdot f$ changes the sign on $K$, and hence on $G$. \qed

This can be generalised further to include every scalar function satisfying the same condition, which is then referred to as a Dulac function.

**Theorem 4.17 (Dulac’s Criterion).** Suppose $G \subset D$ is a simply connected region. Suppose furthermore that there is a scalar function $u$ such that $\nabla \cdot (uf) : G \to \mathbb{R}$ does not change sign and does not vanish in $G$. Then there are no periodic orbits contained inside $G$.

Proof. Suppose there is a periodic orbit $\gamma$ with a period $\tau$, and denote its interior by $K$. By Green’s theorem (4.14) we have
\[
0 < \iint_K \nabla \cdot (uf) \, dx = \int_\gamma uf \cdot n \, d\xi = 0,
\]
where $n$ is the outward normal to $\gamma$. \qed

In principle this a very useful theorem, since it allows us to rule out the existence of periodic orbits. But then again, there is no general method determining an appropriate Dulac function, and we feel that the methods used are often based on working with intuition.

**Example 4.18.** Suppose $G \subset D$ is a simply connected region, and consider the system
\[
\begin{align*}
\dot{x} &= a(y)x + b(y) \\
\dot{y} &= c(x)y + d(x),
\end{align*}
\]
where the functions $a, b, c$ and $d$ are differentiable, and $a$ and $c$ are both either negative or positive $G$. Then
\[
\nabla \cdot f(x, y) = a(y) + c(x)
\]
does not change sign in $G$. Choose now $u \equiv 1$. By Dulac’s Criterion (4.17) there are no periodic orbits contained inside $G$. 

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Chapter 5

Local Stability Analysis

In this chapter we discuss local stability analysis of autonomous ODE equations. We consider a small perturbation to an equilibrium, and analyze on what conditions does the perturbed system converge to the equilibrium.

Consider the system $\dot{x}(t) = f(x(t))$, where $x(t) = (x_1(t), ..., x_n(t))$, and $f$ is locally Lipschitz in $D \subset \mathbb{R}^n$, as before. Suppose that $\bar{x}$ is an equilibrium, that is, $f(\bar{x}) = 0$.

**Definition 5.1.** The equilibrium $\bar{x}$ is **stable** if for every $\varepsilon > 0$ there exists $\delta > 0$ such that $\|x(t) - \bar{x}\| < \varepsilon$ for $\|x(0) - \bar{x}\| < \delta$ and $t > 0$. Otherwise $\bar{x}$ is **unstable**.

Furthermore, the point $\bar{x}$ is **asymptotically stable** if it is stable and there exists $\mu > 0$ such that

$$\lim_{t \to \infty} x(t) = \bar{x}$$

for $\|x(0) - \bar{x}\| < \mu$.

If every non-trivial solution of a system corresponds to a circular periodic solution, with an equilibrium at the center, like in the example (3.4), then the equilibrium is stable, but not asymptotically stable. We could take a point arbitrarily close to the equilibrium, yet the positive semiorbit through this point would not converge to the equilibrium. In applications, however, this kind of behaviour is extremely rare, which is why we typically say that a point is stable, even if it is actually asymptotically stable.

Next, we analyze whether a given equilibrium $\bar{x}$ is stable by linearization around $\bar{x}$. By Taylor series expansion of functions we obtain:

$$\dot{x} = f(\bar{x}) + (x - \bar{x}) \cdot f'(\bar{x}) + O(\|x - \bar{x}\|),$$

where $f'(\bar{x})$ is the Jacobian matrix of $f$ evaluated at $\bar{x}$. If the eigenvalues of $f'(\bar{x})$ have negative real parts, then $\bar{x}$ is stable. If all eigenvalues have non-positive real parts, then $\bar{x}$ is asymptotically stable. If there exists an eigenvalue with positive real part, then $\bar{x}$ is unstable. If all eigenvalues have zero real parts, then the stability of $\bar{x}$ depends on higher-order terms in the Taylor expansion.
where \( O(\|x - \bar{x}\|) \) denotes the higher order terms, \( f(\bar{x}) = 0 \), and
\[
f'(\bar{x}) = \begin{bmatrix} \frac{\partial f_1(\bar{x})}{\partial x_1} & \cdots & \frac{\partial f_1(\bar{x})}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_n(\bar{x})}{\partial x_1} & \cdots & \frac{\partial f_n(\bar{x})}{\partial x_n} \end{bmatrix}.
\]

Furthermore, for \( x \) sufficiently close to \( \bar{x} \) these higher order terms will vanish, and so we can drop them to obtain the approximation:
\[
\dot{x} \approx (x - \bar{x}) \cdot f'(\bar{x})
\]

**Theorem 5.2 (Hartman-Grobman).** Suppose that all eigenvalues of \( f'(\bar{x}) \) have a non-zero real part. Then the linear system
\[
\dot{x} = (x - \bar{x}) \cdot f'(\bar{x})
\]
is locally topologically equivalent to the system \( \dot{x} = f(x) \).

We are only interested in the result, thus for more details we simply refer to the literature.\(^1\) So it is sufficient to study the linear system, which we can always solve explicitly. Write \( u(t) = x(t) - \bar{x} \) and \( A = f'(\bar{x}) \). Then the linear system of the first order
\[
(5.3) \quad \dot{u} = Au
\]
has an equilibrium at origin. Suppose \( \lambda_1, ..., \lambda_n \) are the eigenvalues of \( A \), and the corresponding linearly independent eigenvectors are \( b_1, ..., b_n \). Then
\[
Ab_j = b_j \lambda_j \quad \text{for } j \in \{1, ..., k\}.
\]
Denote by \( B = (b_1, ... b_n) \in \mathbb{C}^{n \times n} \) the invertible matrix and \( \Lambda = \text{diag}(\lambda_1, ..., \lambda_n) \in \mathbb{C}^{n \times n} \). Then:
\[
AB = BA \\
A = B\lambda B^{-1},
\]
and now the system (5.3) can be rewritten as
\[
\dot{u} = B\lambda B^{-1}u \\
\frac{dB^{-1}u}{dt} = \Lambda(B^{-1}u).
\]

\(^1\)Grobman (1959, pp. 880-881) and Hartman (1960, pp. 610-620).
Write \( v(t) = B^{-1}u(t) \in \mathbb{C}^n \). Then \( \dot{v} = \Lambda v \) and

\[
\begin{cases}
\dot{v}_1 = \lambda_1 v_1 \\
\vdots \\
\dot{v}_n = \lambda_n v_n,
\end{cases}
\]

which is a system of uncoupled differential equations with the general solution

\[
\begin{cases}
v_1(t) = v_1(0) e^{\lambda_1 t} \\
\vdots \\
v_n(t) = v_n(0) e^{\lambda_n t}.
\end{cases}
\]

The conclusions of these calculations are the following:

First, suppose that \( \text{Re}(\lambda_j) < 0 \) for every \( j \in \{1, ..., n\} \). Then \( v_j(t) \to 0 \) as \( t \to \infty \).

This is clear since \( \lambda_j = \alpha_j + i B_j \), and

\[
e^{\lambda_j t} = e^{\alpha_j t} \cdot e^{i B_j t} = e^{\alpha_j t} \left( \cos(B_j t) + i \sin(B_j t) \right).
\]

This means that the origin is a stable equilibrium of \( \dot{u} = Au \).

Second, suppose that there exists \( j \in \{1, ..., n\} \) such that \( \text{Re}(\lambda_j) > 0 \). Then \( \|v_j(t)\| \to \infty \) as \( t \to \infty \), which implies that origin is an unstable equilibrium of \( \dot{u} = Au \).

Thus, if every eigenvalue of \( A \) has a negative real part, then the equilibrium is stable. Furthermore, if there exists at least one eigenvalue of \( A \) with positive real part, then the equilibrium is unstable. If the real part is zero for some eigenvalue, we cannot fully characterize the stability of the equilibrium.

We now consider planar systems. Define the matrix \( A \) as before, and write

\[
A = \begin{bmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{bmatrix} \in \mathbb{R}^{2 \times 2}.
\]

The eigenvalues of \( A \) are

\[
\begin{align*}
0 &= \det(A - \lambda I) \\
0 &= (a_{1,1} - \lambda)(a_{2,2} - \lambda) - (a_{1,2}a_{2,1}) \\
0 &= \lambda^2 - \lambda(a_{1,1} + a_{2,2}) - a_{1,2}a_{2,1} + a_{1,1}a_{2,2} \\
\lambda &= \frac{1}{2}(a_{1,1} + a_{2,2}) \pm \frac{1}{2} \sqrt{(a_{1,1} + a_{2,2})^2 + 4 a_{1,2}a_{2,1} - 4a_{1,1}a_{2,2}}.
\end{align*}
\]

Thus the eigenvalues are both negative if

\[
\text{Re} \left( \frac{1}{2}(a_{1,1} + a_{2,2}) \pm \frac{1}{2} \sqrt{(a_{1,1} + a_{2,2})^2 + 4 a_{1,2}a_{2,1} - 4a_{1,1}a_{2,2}} \right) < 0,
\]

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that is, 
\[ a_{1,1} + a_{2,2} < 0 \]
and 
\[ a_{1,2}a_{2,1} - a_{1,1}a_{2,2} < 0. \]
Recall that \( A = f'(x) \), which means that 
\[
\text{trace } A = a_{1,1} + a_{2,2}, \\
\text{det } A = a_{1,1}a_{2,2} - a_{1,2}a_{2,1}.
\]
We have obtained that an equilibrium \( \mathbf{x} \) of a planar system \( \dot{x} = f(x) \) is locally stable if trace \( A < 0 \) and det \( A > 0 \), where \( A = f'(x) \) is a linearization near \( \mathbf{x} \).

Moreover, from the complex parts of the eigenvalues we could determine the flow of the paths around the equilibria to find out the type of stability, i.e., a node or a focus.

For further reading on stability of not only equilibrium, but also of periodic orbits we refer to Jordan & Smith (1987, pp. 212-261).
Chapter 6

The Terminator Model

“There was a nuclear war. A few years from now, all this, this whole place, everything, it’s gone. Just gone. There were survivors. Here, there. Nobody even knew who started it. It was the machines — Defense network computers. New... Powerful... hooked into everything, trusted to run it all. They say it got smart, a new order of intelligence. Then it saw all people as a threat, not just the ones on the other side. Decided our fate in a microsecond: extermination.”

(The Terminator 1984)

We now study a fictional situation based on the science fiction movie The Terminator (1984), that should be taken with a pinch of salt. First, we cover the necessary details of the situation and name the essential individual states (i-state) and individual level processes (i-level process). Then, we build the autonomous system describing the situation, which at first is not a planar system, but will eventually become one after using a clever technique of time-scale separation. Finally, we do a phase-plane analysis of the planar system and determine its long-term behaviour.

6.1 Individual States and Processes

Consider a dystopian world in near future, where a highly advanced artificial intelligence called Skynet had concluded that all of humanity must be exterminated. First, it launched an extensive nuclear attack in which most of the humankind was killed. Then, in order to finish its objective, Skynet developed a cyborg called the Terminator, which have the sole purpose to hunt down and kill the rest of scattered humanity. In the interest of self-preservation, humans learned to fight them and sometimes they even manage to reprogram a Terminator to fight alongside them.
Assuming that every human is equally fighting against the Terminators, we have three i-states: \( x \) denotes a human individual, and for the Terminators we have \( y \) and \( y_c \), respectively, a Terminator hunting down humans and a Terminator reprogrammed to fight against the Terminators of a different type. Here we have used lowercase letters to denote a single individual of a population. The corresponding population densities are denoted by uppercase letters (e.g. \( X \) denotes the density of the human population).

Since the Terminators do not reproduce in a natural way, a feature that makes this an exceptional model, we should take into account the factories and their corresponding dynamics. The problem, however, is that then we would have three major parties: the humans, the Terminators, and the factories. Humans would have to destroy both the Terminators and the factories, that some of the Terminators would build, so we would also have to distinguish between these two types of Terminators, and it would simply become a far too complicated model for our purposes. So instead of having an i-state for the factories, we simply assume that the number of automated factories is always proportional to the Terminator population, and hence the reproduction of Terminators resembles a per capita birth rate.

Suppose that the 'Terminator-free' dynamics of humans are described by the logistic equation:

\[
r_1 X \left(1 - \frac{X}{K_1}\right),
\]

where \( r_1 \) is the rate of maximum population growth and \( K_1 \) is the so-called carrying capacity; the maximum sustainable population. Suppose furthermore that the corresponding 'human-free' dynamics of the Terminators are described by:

\[
r_2 Y \left(1 - \frac{Y}{K_2}\right).
\]

Here we have assumed that the maximum capacity of Terminators that Skynet can control simultaneously is \( K_2 \), and \( r_2 \) is the maximum production rate. It is also reasonable to assume that \( r_2 \) is larger than \( r_1 \), since human reproduction is a slow process whereas automatized factories are very efficient at producing Terminators. But on the other hand, \( K_2 \) is smaller than \( K_1 \), since it would be unreasonable to assume that Skynet could control a very large number of Terminators simultaneously.

Assume \( \beta \) is the rate at which a human and a Terminator make a contact. Upon a contact we assume three possible outcomes:

(i) The Terminator kills the human with a probability \( p_1 \).

(ii) The human successfully destroys the Terminator with a probability \( p_2 \).

(iii) The human manages to disarm the Terminator and reprogram it to fight alongside humans with a probability \( p_3 = 1 - p_2 - p_1 \).
The first two outcomes are self-explanatory. In the third outcome the reprogrammed Terminators are considered to be rather similar to the opposing Terminators in the sense that without a threat they sustain for long periods of time. However, now they are reprogrammed to have a whole new purpose: to hunt down and destroy Skynet’s troops, the Terminators. Assume they make contacts quickly at a rate $\eta$. Upon a contact, this reprogrammed Terminator has a probability $q$ of getting destroyed, and consequently the probability $1 - q$ of surviving the fight and destroying the opposing Terminator.

We now arrive at the following autonomous ODE system:

\begin{equation}
\begin{aligned}
\dot{X} &= r_1 X \left(1 - \frac{X}{K_1}\right) - p_1 \beta XY \\
\dot{Y} &= r_2 Y \left(1 - \frac{Y}{K_2}\right) - (p_2 + p_3) \beta XY - (1 - q)\eta YY_c \\
\dot{Y}_c &= p_3 \beta XY - q\eta YY_c.
\end{aligned}
\end{equation}

Since this is an autonomous ODE system of three equations, the effective dimension is $n = 3$, and we cannot use the results obtained in Chapter 4. In the next section we introduce a way to effectively reduce the three-dimensional system into a planar system, which then allows us to apply the theory of Poincaré and Bendixson to obtain the long-term behaviour without solving the system or even the equilibria explicitly.

### 6.2 Time-Scale Separation

_Time-scale separation_ or, _singular perturbation theory_ is a simple yet very useful method to reduce a system into a planar system: If we assume that a part of the system operates sufficiently fast compared to the rest so that the fast components achieve a steady quasi-equilibria (i.e. a stable equilibria). Then we can eliminate the fast components and replace their expressions with the equilibria, which results in a simplified expression for the system that is topologically equivalent to the original system. For further reading we refer to the literature.\(^1\)

Suppose $\varepsilon > 0$ is a small dimensionless scaling parameter, and suppose that $Y_c \ll Y$, that is, the number of reprogrammed Terminators is small when compared to the number of Terminators controlled by Skynet. Suppose furthermore that the rate $\eta$ is relatively large when compared to the other rates. Denote these by:

\[ Y_c = \varepsilon Y_c^* \quad \text{and} \quad \eta = \eta^*/\varepsilon. \]

\(^1\)On singular perturbation theory, see: Lomov (1992). For a discussion on time-scale separation, see: Geritz (2010) and Metz & Diekmann (1986, pp. 6-7).
Then, by rewriting the system (6.1) gives us:

\[
\begin{align*}
\dot{X} &= r_1 X \left(1 - \frac{X}{K_1}\right) - p_1 \beta X Y \\
\dot{Y} &= r_2 Y \left(1 - \frac{Y}{K_2}\right) - \left(p_2 + p_3\right) \beta X Y - (1 - q) \frac{\eta^*}{\varepsilon} \varepsilon Y Y_c^* \\
\varepsilon \dot{Y}_c^* &= p_3 \beta XY - q \frac{\eta^*}{\varepsilon} \varepsilon YY_c
\end{align*}
\]

which simplifies into

\[
\begin{align*}
\dot{X} &= r_1 X \left(1 - \frac{X}{K_1}\right) - p_1 \beta X Y \\
\dot{Y} &= r_2 Y \left(1 - \frac{Y}{K_2}\right) - \left(p_2 + p_3\right) \beta X Y - (1 - q) \eta^* \varepsilon Y Y_c^* \\
\dot{Y}_c^* &= p_3 \beta XY - q \frac{\eta^*}{\varepsilon} YY_c^*
\end{align*}
\]

(6.2)

This implies that \(Y_c^*\) is a fast variable, while \(X\) and \(Y\) are slow. So essentially the dynamics of \(Y_c^*\) change very fast when compared to \(X\) and \(Y\). We now introduce the so-called fast-time: \(t^* = t/\varepsilon\),

\[
\frac{d}{dt^*} = \frac{d}{dt} \cdot \frac{dt}{dt^*} = \frac{d}{dt} \cdot \varepsilon,
\]

The slow variables act so slowly in fast time, that we can consider them as constants, and then solve the quasi-equilibrium of the fast dynamics.

\[
\begin{align*}
\frac{dX}{dt^*} &= \varepsilon \left[ r_1 X \left(1 - \frac{X}{K_1}\right) - p_1 \beta X Y \right] \\
\frac{dY}{dt^*} &= \varepsilon \left[ r_2 Y \left(1 - \frac{Y}{K_2}\right) - \left(p_2 + p_3\right) \beta X Y - (1 - q) \eta^* \varepsilon Y Y_c^* \right] \\
\frac{dY_c^*}{dt^*} &= p_3 \beta XY - q \eta^* \varepsilon YY_c^*.
\end{align*}
\]

Let now \(\varepsilon \to 0\) to obtain

\[
\begin{align*}
\frac{dX}{dt^*} &= 0 \\
\frac{dY}{dt^*} &= 0 \\
\frac{dY_c^*}{dt^*} &= p_3 \beta XY - q \eta^* YY_c^*.
\end{align*}
\]

The fast-dynamics are given by ODE of \(Y_c^*\), with the quasi-equilibrium:

\[
\frac{dY_c^*}{dt^*} = 0 \quad \iff \quad Y_c^* = \frac{p_3 \beta X}{q \eta^*},
\]

and from the figure (6.1) it is clearly stable.

We have obtained that the dynamics of \(Y_c\) function so quickly, that their population density achieves the quasi-equilibrium quickly, and then the rest of the dynamics will move slowly according to the changes in \(X\) and \(Y\).
6.3 Phase-Plane Analysis

By rewriting the slow-dynamics with the quasi-equilibrium we obtain the following planar system:

\[
\begin{align*}
\dot{X} &= r_1 X \left(1 - \frac{X}{K_1}\right) - p_1 \beta XY \\
\dot{Y} &= r_2 Y \left(1 - \frac{Y}{K_2}\right) - p_2 \beta XY - \frac{p_3 \beta}{q} XY.
\end{align*}
\]

It can be further rewritten as

\[
\begin{align*}
\dot{X} &= r_1 X \left(1 - \frac{X + a_{12} Y}{K_1}\right) \\
\dot{Y} &= r_2 Y \left(1 - \frac{Y + a_{21} X}{K_2}\right),
\end{align*}
\]

which is the competitive Lotka-Volterra model, where

\[
\begin{align*}
a_{12} &= \beta K_1 p_1 / r_1 \\
a_{21} &= \beta K_2 (p_2 + p_3 / q) / r_2
\end{align*}
\]

resemble the (negative) effect the species have over one another. We now ask question which specific conditions give the victory to one species being over another, or even a co-existence?

First, we determine the isoclines or zeroclines of the planar system, i.e., the regions where the derivates are zero with respect to \( t \).

The zero-cline for \( X \) is given by:

\[
\dot{X} = 0 \iff X \equiv 0 \quad \text{or} \quad Y = \left( K_1 - X \right) / a_{12}.
\]
Figure 6.2: X-isoclines and the flow of the dynamics with respect to $X$.

Now, since $\dot{X}$ is continuous, the sign of $\dot{X}$ can change only on the zero-clines. Consider a point $(K_1, Y)$ that is above the isocline. Then, from the equation of $\dot{X}$ it is clear that $\dot{X} < 0$. This means that the sign of $\dot{X}$ is always negative above the $X$-isocline. This is indicated in the figure (6.2) by the arrow facing left. Consider now a point $(X, Y)$ below the isocline, and assume $Y \approx 0$. Then the competition term $a_{12}Y$ becomes negligible, and since below the isocline $X < K_1$, we have $\dot{X} > 0$. This is indicated with the arrow facing right. So essentially this means that above the $X$-isocline the human population is always decreasing independently of the Terminator population, and similarly below the $X$-isocline the human population is always increasing.

The zero-cline for $Y$ is given by:

$$\dot{Y} = 0 \iff Y \equiv 0 \text{ or } Y = K_2 - a_{21}X.$$ 

Then, by doing similar analysis as with $\dot{X}$, which we do not repeat here, we find that the behaviour of the dynamics with respect to $Y$ follows the figure (6.3). By combining these results together we get a set of four different cases.

Case 1 (figure 6.4): Suppose that

$$K_1 > K_2/a_{21} \text{ and } K_2 > K_1/a_{12}.$$ 

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Figure 6.3: Y-isoclines and the flow of the dynamics with respect to Y.

Figure 6.4: Case 1
Here we have four (non-negative) equilibria: three trivial equilibria \((0, 0)\), \((0, K_2)\) and \((K_1, 0)\), and the non-trivial equilibrium \((\bar{X}, \bar{Y})\). The interest is now in the local stability properties of these equilibria. We could study the Jacobian matrix of the planar system at the equilibria, and determine their stability or instability from their trace and determinant, similar to what was done in Chapter 5. But it turns out that the theory of Poincaré and Bendixson allows us to obtain these properties without any further calculations. By simply observing the figure (6.4) one can deduce that both \((0, 0)\) and \((\bar{X}, \bar{Y})\) are unstable, since the arrows indicate that even a slight perturbation is enough to break the balance.

The other two equilibria \((K_1, 0)\) and \((0, K_2)\) are stable attractors, since the arrows indicate that the flow of the dynamics does not allow paths to escape from either of them. Moreover, the two cones between the two isocline curves are trapping regions, which makes the unstable equilibrium \((\bar{X}, \bar{Y})\) a so-called saddle point. Since the cones are isolated, and the arrows indicate that every path will eventually enter one of the cones, we cannot have periodic nor singularly closed orbits contained in \(D\). Hence by Poincaré-Bendixson theorem (4.12) the limit set is a critical point. This means that for any given initial condition, one of the species will take the victory over the other one.

However, it is not entirely clear which one of the outcomes a given path would have. But there exists theory of stable and unstable manifolds, which state that there are two unique unstable orbits that join together at \((\bar{X}, \bar{Y})\). This unstable manifold, indicated by the dotted line, is then referred to as a separatrix, since it separates \(D\) into two disjoint invariant regions with each containing one stable attractor. Then depending on the initial condition, one of the populations will be victorious over its rival.

Case 2 (figure 6.5): Suppose that

\[
K_1 < K_2/a_{21} \quad \text{and} \quad K_2 < K_1/a_{12}.
\]

Here we have the same equilibria as in the Case 1, but the dynamics have turned around: \((\bar{X}, \bar{Y})\) is now a global attractor, whereas \((0, 0)\), \((0, K_2)\) and \((K_1, 0)\) are repellors. The two cones are once again trapping regions, but the direction of the flow has changed entirely. Even when it is still not clear which one of the cones a given path would enter, the eventual outcome is same: the populations converge to the globally steady state \((\bar{X}, \bar{Y})\). However, it is not a peaceful co-existence, on the contrary, the war against the Terminators will last until the end of time, with neither of the forces ever attaining victory over the other one.

Case 3 (figure 6.6): Suppose that

\[
K_1 < K_2/a_{21} \quad \text{and} \quad K_2 > K_1/a_{12}.
\]

In this case there are only three equilibria \((0, 0)\), \((K_1, 0)\) and \((0, K_2)\), of which the first two are clearly unstable. The tube between the isoclines is a trapping zone, and the flow on the tube is towards the \('wall\)' where \(X = 0\). Paths will either reach the equilibrium \((0, K_2)\) right away or hit the wall. There, the only way is up, leaving no other possibility than to go towards the attractor. This means that, for any given path, the outcome is the same: human race will go extinct and the Terminators shall roam the Earth.

Case 4 (figure 6.7): Suppose that \(K_1 > K_2/a_{21}\) and \(K_2 < K_1/a_{12}\).

In this final case the behaviour is very similar to the Case 3, but now the flow on the trapping tube has turned around, which has made the equilibrium \((K_1, 0)\) an attractor. Once a path enters the tube, it will eventually either reach the attractor or the wall where \(Y = 0\). There, as before, there is only way to go, which is now right and towards the attractor. Thus the outcome is the same for any given initial condition: the Terminators will get exterminated and the triumph of humanity is secured.

In conclusion, the dynamics of the planar system (6.3) are characterized by four relatively simple phase-plane figures. The clever method of time-scale separation reduced the three-dimensional system into a planar system, which then allowed us to apply the theory of Poincaré and Bendixson and considerably amount of information was obtained without performing any troublesome calculations nor computational modelling.
Figure 6.6: Case 3

Figure 6.7: Case 4
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