COMPACTNESS PROPERTIES OF VOLterra-TYPE INTEGRAL OPERATORS ON ANALYTIC FUNCTION SPACES

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Academic dissertation

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Santeri Miihkinen
List of included articles

This dissertation consists of an introductory part and the three articles listed below.


The joint articles [I] and [III] contain meaningful contribution by the author. The article [II] consists of the author’s independent research.

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1 Introduction

The topic of this dissertation lies at the intersection of analytic function theory and operator theory. A central theme in this so-called function-theoretic operator theory is the investigation of the properties of linear transformations between spaces of analytic functions. Extensively studied classes of operators are e.g. composition operators, Toeplitz operators and Hankel operators to name a few. In our case the principal interest is the class of Volterra-type integral operators.

Let $D$ be the open unit disc of the complex plane $\mathbb{C}$ and $\mathcal{H}(D)$ be the algebra of all analytic functions in $D$. For a fixed $g \in \mathcal{H}(D)$, we define the Volterra-type integral operator

$$T_g f(z) = \int_0^z f(t)g'(t)\,dt, \quad z \in D$$

for $f \in \mathcal{H}(D)$. Here the function $g$ is called the symbol of $T_g$. Typically, one seeks to characterize the behaviour of $T_g$ in terms of the function-theoretic properties of its symbol. As the name suggests, the first example is the classical and well-studied Volterra operator, which is obtained by choosing $g(z) = z$. Another important example is the Cesàro operator $\frac{1}{z}T_g$ induced by the choice $g(z) = \log \frac{1}{1-z}$.

Example 1.1. The Cesàro operator $C$ is usually defined on the sequence spaces $\ell^p$, $1 < p < \infty$, in the following way

$$C : \ell^p \to \ell^p, \quad (Cx)_n = \frac{1}{n+1} \sum_{k=0}^n x_k,$$

where $x = (x_n) \in \ell^p$. If we identify the space $\ell^2$ with the Hardy space

$$H^2 = \left\{ f \in \mathcal{H}(D) : f(z) = \sum_{n=0}^\infty a_n z^n \text{ with } (a_n) \in \ell^2 \right\}$$

and denote again by $C$ the Cesàro operator on $H^2$, i.e.

$$Cf(z) = \sum_{n=0}^\infty \left( \frac{1}{n+1} \sum_{k=0}^n a_k \right) z^n$$

for $f \in H^2$, $f(z) = \sum_{n=0}^\infty a_n z^n$, we observe by choosing $g(z) = \log \frac{1}{1-z}$ that

$$\frac{1}{z}T_g f(z) = \frac{1}{z} \int_0^z f(t)g'(t)\,dt = \frac{1}{z} \int_0^z \frac{f(t)}{1-t} \,dt$$

$$= \frac{1}{z} \int_0^z \left( \sum_{n=0}^\infty a_n t^n \right) \left( \sum_{n=0}^\infty t^n \right) \,dt = \sum_{n=0}^\infty \left( \frac{1}{n+1} \sum_{k=0}^n a_k \right) z^n$$

$$= Cf(z),$$
where the second last equality follows from the Cauchy product of the series \( \sum_{n=0}^{\infty} a_n t^n \) and \( \sum_{n=0}^{\infty} t^n \) and termwise integration.

The boundedness of the Cesàro operator on \( l^2 \) and hence on \( H^2 \) was first observed by Hardy [20] in 1918; see also [21]. The systematic research of this operator on Hardy spaces \( H^p \) (see below) was initiated by Siskakis [38], [39], [15] (joint with Danikas), who showed its boundedness on \( H^p \) for \( 1 \leq p < \infty \). The case \( 0 < p < 1 \) was solved by Miao in [31].

The general operator \( T_g \) was introduced by Pommerenke [35]. He proved the deep John-Nirenberg inequality with a novel method exploiting the characterization of the boundedness of \( T_g \) on the Hardy space \( H^2 \). Before elaborating on this, let us introduce some notions and definitions relevant to this context.

### 1.1 Classical spaces of analytic functions

The Hardy spaces \( H^p \) of the unit disc \( \mathbb{D} \) are defined as follows.

**Definition 1.2.** Let \( 0 < p < \infty \). Then

\[
H^p = \left\{ f \in \mathcal{H}(\mathbb{D}) : \|f\|_p = \sup_{0 < r < 1} \left( \frac{1}{2\pi} \int_0^{2\pi} |f(re^{it})|^p dt \right)^{1/p} < \infty \right\}.
\]

In connection to Hardy spaces, it should be mentioned that for every function \( f \in H^p \) the limit \( \tilde{f}(e^{it}) = \lim_{r \to 1^-} f(re^{it}) \) exists for a.e. \( t \in [0, 2\pi) \). By identifying \( f \) with \( \tilde{f} \), it can be shown that the space \( H^p \) is isometrically isomorphic to the closed subspace

\[
\left\{ f \in L^p(\mathbb{T}) : f(n) = 0 \text{ for } n < 0 \right\}
\]

of \( L^p(\mathbb{T}) \), where \( \mathbb{T} = \partial\mathbb{D} \) and \( \tilde{f}(n) \) is the \( n \)th Fourier coefficient of \( f \).

The space \( BMOA \) and its closed subspace \( VMOA \) consist of those \( g \in H^2 \) with boundary values \( \tilde{g} \) in \( BMO \) (functions of bounded mean oscillation) and \( VMO \) (functions of vanishing mean oscillation) respectively, or equivalently

**Definition 1.3.**

\[
BMOA = \left\{ g \in \mathcal{H}(\mathbb{D}) : \|g\|_{BMOA} = |g(0)| + \|g\|_* < \infty \right\}
\]

\[
VMOA = \left\{ g \in BMOA : \lim_{|a| \to 1^-} \|g \circ \sigma_a - g(a)\|_2 = 0 \right\},
\]

where \( \|g\|_* = \sup_{a \in \mathbb{D}} \|g \circ \sigma_a - g(a)\|_2 \) and \( \sigma_a(z) = (a - z)/(1 - \overline{a}z) \) is the Möbius automorphism of \( \mathbb{D} \) that interchanges the origin and \( a \in \mathbb{D} \).
Functions in $BMOA$ satisfy the following “reverse Hölder inequality”: see also comments after [10, Corollary 3].

**Lemma 1.4.** Let $0 < p < q < \infty$ and $g \in BMOA$. Then there exists a constant $C_{p,q} > 0$ such that

$$\sup_{a \in \mathbb{D}} \|g \circ \sigma_a - g(a)\|_q \leq C_{p,q} \sup_{a \in \mathbb{D}} \|g \circ \sigma_a - g(a)\|_p. \tag{1}$$

It is a consequence of Lemma 1.4 that for any $0 < p < \infty$ one can replace $H^2$–norm with $H^p$–norm in Definition 1.3 and obtain an equivalent norm on $BMOA$; see e.g. [10, Corollary 3] or [19, Theorem 5.4]. Both $BMOA$ and $VMOA$ admit a characterization in terms of Carleson measures. Let us write $|I| = m(I)$ for any arc $I \subset \mathbb{T}$, where $m$ is the normalized Lebesgue measure on $\mathbb{T}$. A positive Borel measure $\mu$ on $\mathbb{D}$ is a Carleson measure, if there exists a positive constant $C$ such that $\mu(S(I)) \leq C |I|$ for all arcs $I \subset \mathbb{T}$, where

$$S(I) = \{re^{it} \in \mathbb{D} : 1 - |I| < r < 1, \ e^{it} \in I\}$$

is the Carleson window associated with $I$. If

$$\lim_{|I| \to 0} \frac{\mu(S(I))}{|I|} = 0,$$

we say that $\mu$ is a vanishing Carleson measure. We write $A \simeq B$ if each side is bounded above by a constant multiple of the other and say that quantities $A$ and $B$ are equivalent. We have the following characterization for $BMOA$ and $VMOA$; see e.g. Theorems 6.5 and 6.6 in [19].

**Theorem 1.5.** Let $g \in \mathcal{H}(\mathbb{D})$. Then $g$ is in $BMOA$ if and only if the measure $d\mu_g(z) = |g'(z)|^2(1 - |z|^2) dA(z)$ is a Carleson measure, where $A$ is the normalized Lebesgue area measure on $\mathbb{D}$. Correspondingly, $g$ is in $VMOA$ if and only if $d\mu_g$ is a vanishing Carleson measure. The same result holds, if we replace $d\mu_g(z)$ with $d\nu_g(z) = |g'(z)|^2 \log \frac{1}{|z|} dA(z)$. Moreover,

$$\|g\|_* \simeq \sup_{I \subset \mathbb{T}} \left( \frac{\mu_g(S(I))}{|I|} \right)^{1/2} \simeq \sup_{I \subset \mathbb{T}} \left( \frac{\nu_g(S(I))}{|I|} \right)^{1/2}$$

with the interpretation that $g \in BMOA$ if and only if any of the quantities is finite.

The classical Littlewood-Paley identities relate the $H^2$–norm of a function $f \in \mathcal{H}(\mathbb{D})$ to integrals which involve the integral means of order two of the derivative $f'$; see [19, Theorem 6.1]. These identities are key tools in the study of $T_g$, since they can be used to eliminate the integral in its definition; see e.g. [7]. One of them is given below.
Theorem 1.6. If \( f \in H(D) \), then

\[
\| f \|_2^2 = |f(0)|^2 + 2 \int_D |f'(z)|^2 \log \frac{1}{|z|} \, dA(z).
\]

From now on, we use the notation \( \| S \| \) for the operator norm of a bounded operator \( S \) between (quasi-)Banach spaces \( X \) and \( Y \). Let us also denote by \( X^* \) the dual space of a Banach space \( (X, \| \cdot \|_X) \), i.e. the space of all continuous linear functionals \( L : X \to \mathbb{C} \) endowed with the norm

\[
\| L \| = \sup\{|Lx| : \| x \|_X \leq 1\}.
\]

Fefferman’s duality theorem states that the dual space \( (H^1)^* \) can be identified with \( BMOA \) in the following sense; see [19, Theorem 7.1].

Theorem 1.7. Let \( g \in BMOA \). Define

\[
L_g : H^1 \to \mathbb{C}, \quad L_g(f) = \lim_{r \to 1} \frac{1}{2\pi} \int_0^{2\pi} f(re^{it}) \overline{g(e^{it})} \, dt, \quad f \in H^1.
\]

Then \( L_g \in (H^1)^* \) and \( \| L_g \| \simeq \| g \|_{BMOA} \). Conversely, if \( L \in (H^1)^* \) then there exists a unique \( g \in BMOA \) such that \( L = L_g \).

Theorem 1.7 can be complemented by stating that \( (VMOA)^* \simeq H^1 \); see [19, Theorem 7.3]. For general accounts on the spaces \( BMOA \) and \( VMOA \), we mention [10, 18, 19].

Let us also recall the definitions of the (unweighted) Bergman spaces \( A^p \), the Bloch space \( B \) and its closed subspace the little Bloch space \( B_0 \).

Definition 1.8. Let \( 0 < p < \infty \). Then

\[
A^p = \left\{ f \in H(D) : \| f \|_{A^p} = \left( \int_D |f(z)|^p dA(z) \right)^{1/p} < \infty \right\};
\]

\[
B = \left\{ f \in H(D) : \| f \|_B = |f(0)| + \| f \|_{B_0} < \infty \right\},
\]

where \( \| f \|_{B_0} = \sup_{z \in D} (1 - |z|^2) |f'(z)| \);

\[
B_0 = \left\{ f \in B : \lim_{|z| \to 1} \sup_{z \in D} (1 - |z|^2) |f'(z)| = 0 \right\}.
\]

1.2 Operator \( T_g \) on \( H^p \)

The boundedness and compactness of \( T_g \) acting on the Hardy spaces was characterized by Pommerenke [35], Aleman and Siskakis [7] and Aleman and Cima [3]. Their combined result reads as follows.
1.2 Operator $T_g$ on $H^p$

**Theorem 1.9.** Let $0 < p < \infty$. Then the operator $T_g$ is bounded (compact) on $H^p$ if and only if $g \in BMOA$ ($g \in VMOA$). Moreover, $\|T_g\| \simeq \|g\|_*$.

As mentioned above, Pommerenke introduced the operator $T_g$ and characterized its boundedness on $H^2$, see [35, Lemma 1], and provided a short proof for the John-Nirenberg inequality. However, a systematic research of the operator $T_g$ in general was not started until the late 1990’s. Then Aleman and Siskakis extended Pommerenke’s result by characterizing the boundedness and compactness of $T_g$ for $1 \leq p < \infty$ (in Theorem 1.9); see [7, Theorem 1, Corollary 1]. The Hardy spaces $H^p$ induced by these values of $p$ are Banach spaces and therefore the duality techniques of functional analysis are available. In the proof of Theorem 1.9 for $1 \leq p < \infty$, Aleman and Siskakis utilized the duality $BMOA \simeq (H^1)^*$ (Theorem 1.7), Littlewood-Paley identity (Theorem 1.6) and the characterization of $BMOA$ and $VMOA$ in terms of Carleson measures (Theorem 1.5) among other techniques. Later Aleman and Cima extended the boundedness and compactness characterizations of $T_g$ acting on $H^p$ to the scale $0 < p < 1$; see [3, Theorem 1, Corollary 1]. The Hardy spaces $H^p$ for these values of $p$ are quasi-Banach spaces and therefore the tools used in [3] are more function-theoretic in nature than the ones in [7].

Let us now prove the case $p = 2$ in Theorem 1.9.

**Proof of the case $p = 2$ in Theorem 1.9.** We use the Littlewood-Paley identity

$$\|f\|_2^2 = |f(0)|^2 + 2 \int_{\mathbb{D}} |f'(z)|^2 \log \frac{1}{|z|} dA(z)$$

for all $f \in H^2$; see Theorem 1.6. Then

$$\|T_g f\|_2^2 = 2 \int_{\mathbb{D}} |f(z)|^2 d\nu_g(z) = 2 \|f\|_2^2 \nu_g^{1/2}(dnu_g),$$

where $d\nu_g(z) = |g'(z)|^2 \log \frac{1}{|z|} dA(z)$. Now the inclusion mapping

$$i: H^2 \hookrightarrow L^2(d\nu_g)$$

is bounded (compact) if and only if $d\nu_g$ is a Carleson measure (vanishing Carleson measure); see [19, Theorem 6.4]. This is equivalent to the fact that $g \in BMOA$ ($g \in VMOA$) by Theorem 1.5. Moreover,

$$\|T_g\| \simeq \|i\| \simeq \sup_{I \subset \mathbb{T}} \left( \frac{\nu_g(S(I))}{|I|} \right)^{1/2} \simeq \|g\|_*;$$

see [19, Theorem 6.4].}

The proof of the John-Nirenberg inequality discovered by Pommerenke is very elegant and we reproduce it here [35, Theorem 2].
Theorem 1.10 (The John-Nirenberg inequality). Let \( g \in \text{BMOA} \) and define \( g_a = g \circ \sigma_a - g(a) \) for each \( a \in \mathbb{D} \), where \( \sigma_a(z) = (a - z)/(1 - \bar{a}z) \), \( z \in \mathbb{D} \) is a Möbius automorphism of the unit disc. Then there exist positive constants \( C_1 \) and \( C_2 \) such that
\[
\|e^{C_1 g_a}\|_2 \leq C_2
\]
for all \( a \in \mathbb{D} \).

Proof. Since \( g \in \text{BMOA} \), the operator \( T_g \) is bounded on \( H^2 \) by Theorem 1.9 and therefore there exists \( C > 0 \) so that
\[
\|T_g f\|_2 \leq C \|g\|_* \|f\|_2
\]
for all \( f \in H^2 \). Define \( C_1 = \frac{1}{2C\|g\|_*} \) and \( h = e^{C_1 g_a} \). We observe that
\[
T_{C_1 g_a} h(z) = \int_0^z C_1 g_a(t) h(t) dt = h(z) - 1, \quad z \in \mathbb{D}.
\]
By a change of variables, we have
\[
\|g_a\|_* = \|g\|_* \quad \text{(4)}
\]
for all \( a \in \mathbb{D} \). Using identity (3) and estimates (2) and (4), we obtain
\[
\|h\|_2 \leq \|h - 1\|_2 + 1 = \|T_{C_1 g_a} h\|_2 + 1 \leq C_1 C \|g\|_* \|h\|_2 + 1 = \frac{1}{2} \|h\|_2 + 1
\]
and consequently
\[
\|h\|_2 \leq 2.
\]

As a remark, we point out that the reverse Hölder inequality (1) in Lemma 1.4 is a consequence of the John-Nirenberg inequality; see comments after [10, Corollary 3].

Special cases of the operator \( T_g \) appear in many different fields of applications. For example, one can come across the operator \( T_g \) in a classical theorem of Hardy and Littlewood [16, Theorem 5.12] which states that for \( 0 < p < 1 \) and for an analytic function \( f \) in \( H^p \) it holds that any primitive of \( f \) is in \( H^{1 - p} \). In other words, the operator \( T_g : H^p \to H^{1 - p} \) with the choice \( g(z) = z \), is bounded. See also [3, Theorem 1].

Another instance of the operator \( T_g \) is in the theory of the semigroups of composition operators. The infinitesimal generator \( S \) of this semigroup is connected to the operator \( T_g \) through its resolvent operator \((\lambda I - S)^{-1}\), \( \lambda \in \mathbb{C} \). To be more specific, the resolvent operator is of the form
\[
(\lambda I - S)^{-1} = \left( \frac{1}{\lambda} - T_g \right)^{-1} - \frac{1}{\lambda}
\]
for certain \( g \in \mathcal{H}(\mathbb{D}) \); see [2]. For more background on research concerning \( T_g \); see also surveys [40] and [2].
1.3 Further results on $T_g$

The characterization of those symbols $g \in \mathcal{H}(\mathbb{D})$ which induce bounded or compact $T_g$ acting on the space $H^\infty$ of bounded analytic functions in $\mathbb{D}$ is still an open question. However, there are some partial results; see e.g. [42], [9], and [14].

In addition to the Hardy spaces, there are boundedness and compactness results on $T_g$ acting on other spaces of analytic functions. Let us briefly mention some of them here. Aleman and Siskakis characterized the boundedness and compactness of $T_g$ on a large class of weighted Bergman spaces in [8, Theorem 1]. Their result in the unweighted case reads as follows.

**Theorem 1.11.** Let $1 \leq p < \infty$. Then the operator $T_g$ is bounded (compact) on $A^p$ if and only if $g \in B$ ($g \in B_0$). Moreover, $\|T_g\| \simeq \|g\|_{B^p}$.

See also Theorem 4.1. The case of $T_g$ acting on $BMOA$ (or $VMOA$) is due to Siskakis and Zhao [41, Theorem 3.1, Corollary 3.3]; see Theorem 2.4.

**Theorem 1.12.** Let $g \in \mathcal{H}(\mathbb{D})$. Then

$$T_g : BMOA \to BMOA$$

is bounded (compact) if and only if $g \in LMOA$ ($g \in LMOA_0$). Moreover, we have $\|T_g\| \simeq \|g\|_{*,\log}$, where $\|g\|_{*,\log} = \sup_{a \in \mathbb{D}} (\lambda(a) \|g \circ \sigma_a - g(a)\|_2)$ with $\lambda(a) = \log \frac{2}{1-|a|}$.

Here $LMOA$ and $LMOA_0$ are the logarithmic versions of $BMOA$ and $VMOA$ respectively; see Definition 2.3 in Section 2. The boundedness and compactness characterizations of $T_g$ acting on the Bloch space $B$ was discovered by Yoneda [45, Theorem 2.1, Theorem 2.3].

A direction in contemporary research on $T_g$ is the study of its spectral properties. The spectrum of $T_g$ is investigated in weighted Bergman spaces and the Hardy spaces in [4], [6], and [5]. The cases of weighted spaces of entire functions and generalized Fock spaces are considered in [12] and [13]. Also, the investigation of $T_g$ acting on different spaces of analytic functions is still of interest. For instance, Peláez and Rättyä characterize the boundedness and compactness of $T_g$ on weighted Bergman spaces induced by radial weights from the classes of rapidly increasing and regular weights or from even a larger class of weights satisfying a doubling property; see [33], [32] and Section 4.

1.4 Overview of the included articles

Now we give a brief overview of the research carried out in this dissertation. The common theme of the included articles is results on compactness properties of the Volterra-type integral operator. In article [1], we consider the essential and weak essential norms of $T_g$ on the Hardy spaces $H^p$ and the
spaces $BMOA$ and $VMOA$ extending Rätyyö’s result for the $H^2$ case [36].

We obtain the same quantitative estimate for the essential norm as for the weak essential norm in terms of the symbol $g$. Consequently, the compactness and weak compactness of $T_g$ coincide on these spaces. In article [II], we investigate the strict singularity of $T_g$ acting on $H^p$. We show that a non-compact operator $T_g$ on $H^p$, $0 < p < \infty$, fixes an isomorphic copy of $\ell^p$. This implies that the compactness and strict singularity are equivalent for $T_g$: $H^p \to H^p$. We also point out that it follows from the results in [I] that the strict singularity of $T_g$ coincides with the compactness in the cases of $BMOA$ and $VMOA$. The same phenomenon persists on the Bloch space.

The framework of article [III] is the weighted Bergman spaces $A^p_\omega$ induced by weights $\omega$ satisfying a doubling condition. We establish estimates for the essential norms of $T_g$ on these spaces as well.

In the rest of the introductory part, Sections 2–4, we give more detailed summaries of the articles included in this thesis.

2 Summary of [I]: Essential and weak essential norms of $T_g$ on $H^p$ and $BMOA$

2.1 Preliminaries

A classical setting for several different classes of operators is the Hardy spaces of the unit disc. The boundedness and compactness of the Volterra-type integral operator acting on these spaces were characterized by Aleman, Siskakis and Cima; see Theorem 1.9. A natural direction is then to seek quantitative versions of the compactness result, namely to estimate the distance of a given operator to the ideal of compact operators. Also, the notion of weak compactness of a bounded operator is worth investigating.

**Definition 2.1.** Let $X$ and $Y$ be Banach spaces and $S: X \to Y$ be a bounded operator. We say that the operator $S$ is **weakly compact** if the closure of $S(B_X)$ is weakly compact set. Here $B_X$ is the closed unit ball of $X$.

The weak compactness of an operator $S: X \to Y$ is relevant only on non-reflexive spaces $X$ and $Y$, since bounded operators on reflexive spaces are weakly compact.

**Definition 2.2.** Let $X$ and $Y$ be Banach spaces and $S: X \to Y$ be a bounded operator. Then the **essential norm** and **weak essential norm** of $S$ are the distances of $S$ to the ideals $\mathcal{K}(X,Y)$ of compact operators and weakly compact operators $\mathcal{W}(X,Y)$ respectively, i.e.

$$
\|S\|_e = \inf\{\|S - K\| : K \in \mathcal{K}(X,Y)\};
$$

$$
\|S\|_w = \inf\{\|S - W\| : W \in \mathcal{W}(X,Y)\}.
$$
2.1 Preliminaries

Let us also recall a few function-theoretic notions. The logarithmic \( BMOA \) and \( VMOA \) spaces, denoted here by \( LMOA \) and \( LMOA_0 \) respectively, are defined as follows.

**Definition 2.3.**

\[
LMOA = \left\{ g \in \mathcal{H}(D) : \|g\|_{LMOA} = |g(0)| + \|g\|_{*, \log} < \infty \right\};
\]

\[
LMOA_0 = \left\{ g \in LMOA : \lim_{|a| \to 1^-} \lambda(a)\|g \circ \sigma_a - g(a)\|_2 = 0 \right\},
\]

where \( \|g\|_{*, \log} = \sup_{a \in \mathbb{D}} \lambda(a)\|g \circ \sigma_a - g(a)\|_2 < \infty \) with \( \lambda(a) = \log \frac{2}{1-|a|} \).

The spaces \( LMOA \) and \( LMOA_0 \) characterize the boundedness and compactness of \( T_g \) on \( BMOA \), respectively; see [41, Theorem 3.1, Corollary 3.3, Theorem 3.6].

**Theorem 2.4.** Let \( g \in \mathcal{H}(D) \). Then

\[ T_g : BMOA \to BMOA \]

is bounded (compact) if and only if \( g \in LMOA \) \((g \in LMOA_0)\). Moreover, we have \( \|T_g\| \simeq \|g\|_{*, \log} \). The same boundedness and compactness characterizations hold for the restriction \( T_g : VMOA \to VMOA \).

We also need normalized standard test functions in \( H^p, 1 \leq p < \infty \), which are defined by

\[
f_{a,p}(z) = f_a(z) = \left[ \frac{1 - |a|^2}{(1 - \bar{a}z)^2} \right]^{1/p}, \quad z \in \mathbb{D}
\]

for each \( a \in \mathbb{D} \). A computation shows that we have \( \|f_a\|_p = 1 \) and for \( p = 2 \) these are the normalized reproducing kernels of \( H^2 \).

The following result proved by Aleman and Cima in [3, Theorem 3] establishes an interesting connection between the operator \( T_g \), the functions \( f_a \) and the \( H^p \)-norms of the hyperbolic translates \( g \circ \sigma_a - g(a) \) of the symbol \( g \).

**Proposition 2.5.** Let \( 0 < p < \infty, a \in \mathbb{D}, \) and \( g \in H^p \). Then for \( 0 < t < p/2 \) there exists a constant \( C \) depending only on \( p \) and \( t \) such that

\[
\|T_g f_a\|_p \geq C\|g \circ \sigma_a - g(a)\|_t,
\]

where \( \sigma_a(z) = (a - z)/(1 - \bar{a}z) \).

To prove Proposition 2.5, Aleman and Cima consider for a fixed symbol \( g \in BMOA \) and \( \gamma \in \mathbb{C} \), the operator

\[
R(g, \gamma) : H^p \to H^p, \quad R(g, \gamma)f(z) = e^{-\gamma g(z)} \int_0^z e^{\gamma g(t)} f'(t) dt,
\]
which is a right inverse of \( I + \gamma T_g \) on the subspace \( H_0^p = z H^p \) of those \( H^p \)-functions vanishing at the origin. From this, it follows that one has the estimate
\[
\| R(g, \gamma) f \|_p \geq \frac{\| f \|_p}{1 + |\gamma| C \| g \|_*},
\]
where \( \| g \|_* = \sup_{a \in \mathbb{D}} \| g \circ \sigma_a - g(a) \|_2 \) and the constant \( C > 0 \) comes from the inequality \( \| T_g \| \leq C \| g \|_* \); see also (11) in [3]. The estimate (7) is then applied to a situation when \( R \) is operating to the function \( g \circ \sigma_a - g(a) \in H_0^1 \) and the function \( \log f_a \circ \sigma_a \) is acting as its symbol.

### 2.2 Essential and weak essential norms of \( T_g \) on Hardy spaces \( H^p \)

The aim is to estimate the essential norm of \( T_g \) in terms of the symbol \( g \). The first author to consider the essential norm of \( T_g \) was Rättyä [36]. He obtained estimates for the essential norm of \( T_g \) on the classical weighted Bergman spaces \( A_p^\alpha \) and the Hardy space \( H^2 \). For instance, in the case of \( T_g \) acting on \( H^2 \):
\[
\| T_g \|_e \simeq \limsup_{|a| \to 1^-} \left( \int_{\mathbb{D}} |g'(z)|^2 (1 - |\sigma_a(z)|^2)^2 dA(z) \right)^{1/2};
\]
see [36, Theorem 1]. A motivation for the article [I] was to extend his results to \( H^p \) for \( 1 \leq p < \infty \) and investigate the weak compactness of \( T_g \). We show that the essential norm is equivalent to the distance of the symbol \( g \) to \( VMOA \) measured in the \( BMOA \) seminorm and it is equivalent to the weak essential norm in the case of \( H^1 \). Our first main result reads as follows.

**Theorem 2.6 ([I]).** Let \( g \in BMOA \). Then, for \( T_g: H^p \to H^p \), \( 1 \leq p < \infty \),
\[
\| T_g \|_e \simeq \text{dist}(g, VMOA),
\]
and for \( T_g: H^1 \to H^1 \),
\[
\| T_g \|_w \simeq \text{dist}(g, VMOA),
\]
where the distance is measured in the \( BMOA \) seminorm \( \| \cdot \|_* \). In particular, \( T_g \) is weakly compact on \( H^1 \) if and only if it is compact, or equivalently, \( g \in VMOA \).

An interesting consequence of estimates (8) and (9) is that the operator \( T_g \) acting on \( H^1 \) is rigid in the sense that its weak compactness coincides with the compactness. This is not true for an arbitrary bounded operator on \( H^1 \). There exist weakly compact operators acting on \( H^1 \) that are not compact. This is due to the fact that there exist complemented copies of \( l^2 \) inside \( H^1 \); see [19, Section 9]. The projection \( P: H^1 \to M \) onto a closed
subspace $M \subset H^1$ isomorphic to $\ell^2$ provides an example of a weakly compact operator, which is not compact.

Establishing the upper estimate for the essential and weak essential norms in Theorem 2.6 is straightforward after using the fact $\|T_g\| \simeq \|g\|_*$ (Theorem 1.9) proved by Aleman and Siskakis [7, Theorem 1] and the linearity of $T_g$ with respect to its symbol $g$. Main ingredients in the proof of the lower estimate are Proposition 2.5 and a formula for the distance of a general $BMOA$-function to $VMOA$. The case $p = 1$ also requires utilizing a localization argument for $T_g$ combined with the Dunford-Pettis criterion, which are discussed in more detail below.

There exist different function-theoretic estimates for the distance of a general $BMOA$-function to the space $VMOA$. Usually, they involve a limsup version of an expression defining (or equivalent to) the $BMOA$ norm. One version of the distance formula is the following (see Lemma 3 in [I]).

**Lemma 2.7 ([I]).** Let $g \in BMOA$ and $0 < p < \infty$. Then

$$
\text{dist}(g, VMOA) \simeq \limsup_{|a| \to 1^-} \|g \circ \sigma_a - g(a)\|_p,
$$

where the constants of comparison depend on $p$.

The proof of Lemma 2.7 is based on the Littlewood subordination theorem [16, Theorem 1.7]. Both estimates (6) (in Proposition 2.5) and (10) are used to establish the lower estimate for the essential norm of $T_g$.

In estimating the weak essential norm of $T_g$ in the Hardy space $H^1$ case, we utilize the classical Dunford-Pettis criterion [1, Theorem 5.2.9]. Let $m$ be the normalized Lebesgue measure on the unit circle $\mathbb{T}$. The criterion states that a set $A \subset L^1(m)$ is relatively compact in the weak topology of $L^1(m)$ precisely when it is uniformly integrable, i.e.

$$
\sup_{f \in A} \int_E |f| dm \to 0,
$$
as $m(E) \to 0$. The application of the criterion in $H^1$-setting relies upon the following localization argument; see Lemma 5 in [I].

**Lemma 2.8 ([I]).** Let $g \in BMOA$. For all non-zero $a \in \mathbb{D}$, let

$$
I(a) = \left\{ e^{i\theta} : |\theta - \arg a| < (1 - |a|)^{1/6} \right\}
$$

and $f_a(z) = (1 - |a|^2)/(1 - \bar{a}z)^2$. Then

$$
\lim_{|a| \to 1} \int_{T \setminus I(a)} |T_g f_a| dm = 0.
$$
Lemma 2.8 in combination with the Dunford-Pettis criterion is used to establish the lower bound for the weak essential norm (Theorem 2.6). Lemma 2.8 basically states that the images $T_g f_a$ behave similarly as $f_a$ in the sense that “the mass of $T_g f_a$ peaks at a point $\omega \in \mathbb{T}$ when $a \to \omega$”. In particular, we have that
\[
\limsup_{|a| \to 1} \int_{I(a)} |T_g f_a| dm = \limsup_{|a| \to 1} \|T_g f_a\|_1,
\]
where $I(a)$ is the arc from Lemma 2.8. The Dunford-Pettis criterion in turn is applied to the images $W f_a$ of the test functions $f_a$ under a weakly compact operator $W$ to conclude that $\int_{I(a)} |W f_a| dm \to 0$ as $|a| \to 1$. In other words, the mass of $W f_a$ does not peak at the boundary $\mathbb{T}$ in the case of a weakly compact operator $W$. These observations together establish the lower bound for the weak essential norm in the following way.

\[
\|T_g - W\| \geq \int_{I(a)} |(T_g - W) f_a| dm \geq \int_{I(a)} |T_g f_a| dm - \int_{I(a)} |W f_a| dm, \tag{11}
\]

where $W : H^1 \to H^1$ is a weakly compact operator. Letting $|a| \to 1$ in the estimate (11) and taking the infimum over all weakly compact operators $W$, we have
\[
\|T_g\|_w \geq \limsup_{|a| \to 1} \|T_g f_a\|_1,
\]
which in the light of the estimate (6) and Lemma 2.7 establishes the lower bound for the weak essential norm of $T_g$.

\section{Essential and weak essential norms of $T_g$ on BMOA and VMOA}

Our result concerning the $BMOA$ case reads as follows.

\textbf{Theorem 2.9 ([II]).} Let $g \in LMOA$. Then $T_g : BMOA \to BMOA$ satisfies
\[
\|T_g\|_e \simeq \|T_g\|_w \simeq \text{dist}(g, LMOA_0).
\]
In particular, $T_g$ is weakly compact on $BMOA$ if and only if it is compact, or equivalently, $g \in LMOA_0$. The same estimates hold for the restriction $T_g : VMOA \to VMOA$.

It should be noted that the equivalence of weak compactness and compactness for $T_g$ on $BMOA$ was proved independently by Blasco et al. in [11]. However, the techniques used in [11] were more function-theoretic in nature than ours.

To elaborate on the proof of Theorem 2.9, the upper bound for the essential and weak essential norms is obtained analogously to the Hardy space case using the facts that $\|T_g\| \simeq \|g\|_{* \log}$, see Theorem 2.4, and the linearity of $T_g$ with respect to $g$. 
Establishing the lower bound for the weak essential norm is the non-trivial part of the proof of Theorem 2.9. Again, a difficulty with the distance quantity \( \text{dist}(g, \text{LMOA}_0) \) is that it is abstract to work with. To remedy the situation, we prove the following analogue (Lemma 4 in [I]) of (10):

**Lemma 2.10** ([I]). For \( g \in \text{LMOA} \), we have

\[
\text{dist}(g, \text{LMOA}_0) \simeq \limsup_{|a| \to 1-} \| g \circ \sigma_a - g(a) \|_2,
\]

(12)

where \( \lambda(a) = \log \frac{2}{1-|a|} \).

The proof of Lemma 2.10 is based on a weighted version of the Littlewood subordination theorem proved by Laitila in [24, Proposition 2.3]. Moreover, the proof of [41, Lemma 3.4] implies the following.

**Lemma 2.11.** Let \( g \in \text{LMOA} \). Then

\[
\limsup_{|a| \to 1-} \| g \circ \sigma_a - g(a) \|_2 \simeq \limsup_{|I| \to 0} \left( \log \frac{2}{|I|} \right)^{1/2} \left( \frac{\mu_g(S(I))}{|I|} \right)^{1/2},
\]

(13)

where \( \mu_g(S(I)) = \int_{S(I)} |g'(z)|^2 (1 - |z|^2) dA(z) \).

Estimates (12) and (13) provide us concrete function-theoretic expressions we need to establish the lower bound for the weak essential norm. In particular, estimate (13) in Lemma 2.11 is a Carleson-type estimate (cf. Theorem 1.5) for the distance of a general \( \text{LMOA} \)-function to \( \text{LMOA}_0 \).

However, a key tool in proving Theorem 2.9 is a result of Leibov [27], which states that one can construct isomorphic copies of the sequence space \( c_0 \) inside \( \text{VMOA} \); see also [25, Proposition 6] and [I, Lemma 6].

**Lemma 2.12.** Let \( (f_n) \) be a sequence in \( \text{VMOA} \) such that \( \|f_n\|_* \simeq 1 \) and \( \|f_n\|_{H^2} \to 0 \) as \( n \to \infty \). Then there is a subsequence \( (f_{n_j}) \) which is equivalent to the natural basis of \( c_0 \); that is, the map \( \iota : (\lambda_j) \to \sum_j \lambda_j f_{n_j} \) is an isomorphism from \( c_0 \) into \( \text{VMOA} \).

In [25, Proposition 6], this result was used in connection to composition operators. In [I], the idea is to construct a weakly converging sequence of \( \text{VMOA} \)-functions satisfying the conditions in Lemma 2.12 and then to use the Dunford-Pettis property of \( c_0 \) stating that any weakly compact operator \( T : c_0 \to Y \), where \( Y \) is a Banach space, maps weakly convergent sequences into norm convergent sequences.

### 3 Summary of [II]: Strict singularity of \( T_g \)

A bounded operator is **strictly singular**, if its restriction to any infinite-dimensional closed subspace is not a linear isomorphism onto its range, i.e.
it is not bounded from below. Strictly singular operators acting on a Banach space $X$ form a closed two-sided ideal in the class of bounded operators, i.e. for any strictly singular operator $S: X \to X$ operators $ST$ and $TS$ are strictly singular for all bounded operators $T: X \to X$. The notion of strict singularity is a generalization of the compactness of an operator: All compact operators are strictly singular.

The principal goal of the article [II] is to study the strict singularity of $T_g$ acting on the Hardy spaces $H^p$. As far as we know, the strict singularity has not been considered in the case of $T_g$ before in contrast to e.g. composition operators; see a recent work of Laitila et al. [26].

Let us recall that the condition $g \in BMOA \setminus VMOA$ is equivalent to the fact that $T_g: H^p \to H^p$ is not compact; see Theorem 1.9. Our main result reads as follows.

**Theorem 3.1** ([II]). Let $g \in BMOA \setminus VMOA$ and $1 \leq p < \infty$. Then the operator $T_g: H^p \to H^p$ fixes an isomorphic copy of $\ell^p$ inside $H^p$, that is, there exists a subspace $M \subset H^p$ which is isomorphic to $\ell^p$ and such that the restriction of $T_g$ to $M$ is an isomorphism onto its range. In particular, $T_g$ is not strictly singular.

It should be noted that Theorem 3.1 states a stronger result than the equivalence of strict singularity and compactness for $T_g$: A non-compact $T_g$ fixes an isomorphic copy of the sequence space $\ell^p$. As a consequence, we obtain a new proof for the equivalence of compactness and weak compactness of $T_g$ acting on $H^1$: A non-compact $T_g$ on $H^1$ fixes a copy of non-reflexive space $\ell^1$. Hence it is not weakly compact. Moreover, Theorem 3.1 also holds on the scale $0 < p < 1$ when the corresponding Hardy spaces $H^p$ are quasi-Banach.

Another way to state Theorem 3.1 is that a non-compact $T_g$ does not belong to the class $S_p(H^p)$ of $\ell^p$—singular operators on $H^p$. This class consists of those bounded operators on $H^p$, which do not fix any isomorphic copy of $\ell^p$. In this sense, the operator $T_g$ is rigid in comparison to a general bounded operator, since it holds that

$$K(H^p) \subset S(H^p) = S_p(H^p) \cap S_2(H^p)$$

for $1 < p < \infty$ and $p \neq 2$, where $S(H^p)$ is the class of strictly singular operators on $H^p$. The identity in (14) follows from an analogous characterization of $S(L^p(0,1))$ by Weis [43] in combination with the classical fact that $L^p$ is isomorphic to $H^p$; see e.g. [29, Proposition 2.c.17]. Our work raises the open question whether and when a non-compact operator $T_g$ on $H^p$ fixes an isomorphic copy of the sequence space $\ell^2$, i.e. $T_g \notin S_2(H^p)$.

### 3.1 Extrapolation result for strictly singular operators

In the context of strict singularity, we should mention a striking extrapolation result due to Hernández, Semenov, and Tradacete [22, Theorem 3.3]
concerning strictly singular operators.

**Theorem 3.2.** Let $1 < q < r < \infty$. If an operator $T$ is bounded on $L^q$ and $L^r$, and strictly singular on $L^p$ for some $p \in (q, r)$, then $T$ is compact on $L^s$ for all $s \in (q, r)$.

Here $L^p$ stands for the $L^p$ space of real-valued functions on a finite measure space. Provided the complex-valued counterpart of this result is true, the equivalence of strict singularity and compactness of $T_g$ on $H^p$ for $1 < p < \infty$ would follow by using the Riesz projection. However, this argument does not give any information on whether a non-compact $T_g$ on $H^p$ does not belong to the class $S_p(H^p)$.

### 3.2 Strategy proving the main result

The proof of Theorem 3.1 is based on constructing bounded operators $U$ and $V$ from $\ell^p$ into $H^p$ such that $U = T_g V$ and $U$ is bounded from below; see Figure 1.

Figure 1: Operators $U, V$ and $T_g$

$$
\begin{array}{c}
\ell^p \\
V \downarrow \\
H^p \rightarrow \\
T_g \\
H^p
\end{array}
$$

To this end, we utilize the normalized standard test functions $f_a \in H^p$; see (5). The idea is to choose a sequence $(a_n)$ in the unit disc satisfying condition (15) in Proposition 3.3 with $|a_n|$ converging to 1 fast enough. Then for this reason in combination with the assumption $g \in BMOA \setminus VMOA$, the corresponding test functions $f_{a_n}$ and their images $T_g f_{a_n}$ approximate disjointly supported peaks in $L^p(\mathbb{T})$ close to some point in $T$ (cf. Lemma 2.8). This ensures that the operators

$$
V : (\alpha_n) \mapsto \sum_{n=1}^{\infty} \alpha_n f_{a_n}
$$

and

$$
U : (\alpha_n) \mapsto \sum_{n=1}^{\infty} \alpha_n T_g f_{a_n}
$$

from $\ell^p$ into $H^p$ are bounded and $U$ is also bounded from below. Consequently, $T_g$ fixes an isomorphic copy of $\ell^p$.

To deduce the fact that $U$ is bounded from below, we need the following observation (by Aleman and Cima), which is a corollary to Proposition 2.5 in Section 2.
Proposition 3.3. Let \( g \in BMOA \setminus VMOA \) and \( 1 \leq p < \infty \). Then
\[
c := \limsup_{|a| \to 1} \| T_g f_a \|_p > 0.
\] (15)

In particular, there exists a sequence \( (a_k) \subset \mathbb{D}, a_k \to \omega \in \mathbb{T} \) so that
\[
\lim_{k \to \infty} \| T_g f_{a_k} \|_p = c.
\]

Proposition 3.3 basically states that we can identify the non-compactness of \( T_g \) utilizing the test functions \( f_a \).

### 3.3 Strict singularity of \( T_g \) on other spaces of analytic functions

It is also natural to consider the relation between the strict singularity and compactness of \( T_g \) on other spaces of analytic functions in addition to \( H^p \).

In this section, we consider the spaces \( BMOA \) and \( VMOA \), the Bergman spaces \( A_p \), and the Bloch space \( B \).

The proof of Theorem 2 and Lemma 6 in the article \([I]\) imply that the strict singularity of \( T_g \) acting on \( BMOA \) (or on \( VMOA \)) coincides with the compactness.

The comparison between the strict singularity and compactness of \( T_g \) acting on the Bergman spaces \( A_p \), \( 1 \leq p < \infty \), is straightforward, since spaces \( A_p \) are isomorphic to \( \ell^p \) \([44, \text{Chapter 3.A, Theorem 11}\) and strictly singular operators on \( \ell^p \) are compact \([28, \text{Proposition 2.c.3}\).

In the case of the Bloch space, we can deduce the equivalence of strict singularity and compactness of \( T_g \) in the following way. We have that \( T_g(B_0) \subset B_0 \) and \( B_0 \) is dense in \( B \) with respect to the weak-star topology. Now
\[
\langle T_g h, f \rangle = \langle h, T_g^* f \rangle = \langle T_g^* f, h \rangle = \langle f, T_g^{**} h \rangle = \langle T_g^{**} h, f \rangle
\] (16)
for all \( f \in A^1 \) and all \( h \in B_0 \), where \( \langle \cdot, \cdot \rangle \) is the integral pairing used to identify the dual \( (A^1)^* \) with \( B \) \([46, \text{Theorem 5.3}\). The operator \( T_g \) acting on \( B \) is continuous with respect to the weak-star topology of \( B \) and \( B_0 \) is weak-star dense in \( B \). Hence it follows from (16) that
\[
\langle T_g h, f \rangle = \langle T_g^{**} h, f \rangle
\]
for all \( f \in A^1 \) and all \( h \in B \). Therefore the operator \( T_g: B \to B \) can be identified with the biadjoint \( (T_g|B_0)^{**} \). Finally, \( B_0 \) is isomorphic to the space \( c_0 \) of null-sequences \([37, \text{Theorem 7}\), on which strictly singular operators are compact; see \([28, \text{Proposition 2.c.3}\).

It would be interesting to find an example of a Banach space \( X \subset H(\mathbb{D}) \) and a symbol \( g \in H(\mathbb{D}) \) so that the strict singularity of \( T_g: X \to X \) does not coincide with its compactness.
4 Summary of [III]: Weighted Bergman spaces $A^p_\omega$ and $T_g$

Our framework for the article [III] is the weighted Bergman spaces $A^p_\omega, 0 < p < \infty$. By a weight function (or a weight) we mean an integrable function $\omega: \mathbb{D} \to (0, \infty)$. A weight $\omega$ is radial if $\omega(z) = \omega(|z|)$ for all $z \in \mathbb{D}$. The spaces $A^p_\omega$ are defined as

$$A^p_\omega = \left\{ f \in \mathcal{H}(\mathbb{D}) : \|f\|_{A^p_\omega} = \left( \int_{\mathbb{D}} |f|^p \omega dA \right)^{1/p} < \infty \right\},$$

where $dA$ is the normalized Lebesgue area measure and the weight function $\omega$ belongs to the class $\hat{D}$ of radial weights satisfying the doubling property:

$$\int_r^1 \omega(s) ds \leq C \int_{(1+r)/2}^1 \omega(s) ds, \quad 0 \leq r < 1 \quad (17)$$

for some constant $C = C(\omega) > 0$; see [32] and [34]. As usual, we write $A^p_\alpha$ for the classical weighted Bergman space induced by the standard radial weight $\omega(z) = (1 - |z|^2)^\alpha, -1 < \alpha < \infty$. We also use the notation $\omega(B) = \int_B \omega dA$ whenever $B \subset \mathbb{D}$ is measurable.

Important subclasses of $\hat{D}$ are the class $\mathcal{I}$ of rapidly increasing weights and the class $\mathcal{R}$ of regular weights (see below). A pioneering work in this field is the monograph of Peláez and Rättyä [33]. Among other things, they describe Carleson measures for the spaces $A^p_\omega$ and study zeros and factorization of functions in these spaces. They also characterize the boundedness and compactness of $T_g$ on spaces $A^p_\omega$, where $\omega \in \mathcal{I} \cup \mathcal{R}$.

The first results concerning $T_g$ acting on weighted Bergman spaces were obtained by Aleman and Siskakis. They characterized the boundedness and compactness of $T_g$ for a large class of weights including the regular weights [8, Theorem 1]. Their result in the case of the classical weighted Bergman spaces reads as follows.

**Theorem 4.1.** Let $1 \leq p < \infty$. Then the operator $T_g$ is bounded (compact) on $A^p_\alpha, \alpha > -1$, if and only if $g \in \mathcal{B}$ (if $g \in \mathcal{B}_0$). Moreover, $\|T_g\| \simeq \|g\|_{\mathcal{B}, \alpha}$.

In [III], we estimate the essential norm of the operator $T_g$ on weighted Bergman spaces $A^p_\omega, 0 < p < \infty$, where the weight function $\omega$ is in the class $\hat{D}$. A motivation behind the article [III] was to extend the result of Rättyä [36, Theorem 6] on the essential norm of $T_g$ acting on the classical weighted Bergman spaces $A^p_\omega$ to the spaces $A^p_\omega$, where $\omega \in \hat{D}$. We also give a quantitative version of boundedness characterization of $T_g: A^p_\omega \to A^q_\omega$, where $0 < p \leq q < \infty$. Before stating our results, let us introduce some relevant notions and definitions in our context.
A radial weight \( \omega: [0, 1) \to (0, \infty) \) is extended to the unit disc \( \mathbb{D} \) by defining \( \omega(z) = \omega(|z|) \) for all \( z \in \mathbb{D} \). The weight class \( \mathcal{I} \) consists of those continuous radial weights \( \omega \) such that
\[
\lim_{r \to 1^-} \frac{\int_r^1 \omega(s)ds}{(1-r)\omega(r)} = \infty
\]
and the class \( \mathcal{R} \) in turn contains those continuous radial weights such that there exist constants \( C_1, C_2 > 0 \) depending on the weight \( \omega \) so that
\[
C_1 \frac{1}{1-r} \int_r^1 \omega(s)ds \leq \omega(r) \leq C_2 \frac{1}{1-r} \int_r^1 \omega(s)ds, \quad 0 \leq r < 1.
\]

**Example 4.2.** Concrete examples of rapidly increasing weights are
\[
\omega(r) = \left( (1-r) \prod_{n=1}^N \frac{\log(n)}{1-r} \left( \frac{\exp((N+1)^{1-\alpha} \log(n))}{1-r} \right)^{\alpha} \right)^{-1}
\]
for all \( 1 < \alpha < \infty \) and \( N \in \{1, 2, \ldots\} \). Here we define
\[
\exp_n(x) = \exp(\exp_{n-1}(x)), \quad \exp_1(x) = e^x
\]
and
\[
\log_n(x) = \log(\log_{n-1}(x)), \quad \log_1(x) = \log(x).
\]
The standard radial weights \( \omega(r) = (1-r^2)^\beta, \beta > -1 \), are regular.

The spaces \( A^p_\omega \), where \( \omega \in \mathcal{I} \), lie closer to the Hardy spaces than any classical weighted Bergman space \( A^p_\alpha \) in the sense that
\[
H^p \subset A^p_\omega \subset \bigcap_{-1 < \alpha < \infty} A^p_\alpha;
\]
see [33, Lemma 1.1] and comments thereafter. The standard test functions of the spaces \( A^p_\omega, \omega \in \mathcal{I} \), are defined to be
\[
f_{a,p,\gamma}(z) = f_{a,p}(z) = \left( \frac{1-|a|^2}{1-\bar{a}z} \right)^{\gamma+1} \frac{1}{\omega(S(a))^{1/p}}, \quad z \in \mathbb{D}, \quad (18)
\]
where \( a \in \mathbb{D} \setminus \{0\} \), and \( \gamma > 0 \) is chosen to be large enough so that
\[
\sup_{a \in \mathbb{D}} \|f_{a,p}\|_{A^p_\omega} < \infty,
\]
and \( S(a) = S(I_a) \) is the Carleson window associated with the interval
\[
I_a = \{ e^{i\theta} : |\arg(ae^{-i\theta})| \leq \pi(1-|a|) \};
\]
see [32, Lemma 3.1].

When describing those symbols \( g \) that induce a bounded operator
\[
T_g: A^p_\omega \to A^q_\omega;
\]
one encounters the following spaces of analytic functions.
4.1 Spaces $C^\alpha(\omega^*)$

Let $\alpha \geq 1$ and $\omega \in \hat{D}$. Then $g \in \mathcal{H}(\mathbb{D})$ is in the space $C^\alpha(\omega^*)$ if and only if the seminorm

$$
\|g\|_{*,\alpha,\omega} = \sup_{a \in \mathbb{D}} \left( \frac{\int_{S(a)} |g'(z)|^2 \omega^*(z) dA(z)}{\omega(S(a))^{\alpha}} \right)^{1/2}
$$

is finite. The subspace $C^\alpha_0(\omega^*) \subset C^\alpha(\omega^*)$ consists of those $g$ for which

$$
\lim_{|a| \to 1} \int_{S(a)} |g'(z)|^2 \omega^*(z) dA(z) \omega(S(a))^{\alpha} = 0.
$$

Here

$$
\omega^*(z) = \int_1^{|z|} \log \left( \frac{s}{|z|} \right) \omega(s) ds, \quad z \in \mathbb{D} \setminus \{0\},
$$

is the associated weight of $\omega$ that arises from the Littlewood-Paley formula for $A_2^\omega$; see the identity (25) in Lemma 4.6. The spaces $C^1(\omega^*)$ and $C^1_0(\omega^*)$ have the following properties.

- $BMOA \subset C^1(\omega^*) \subset B$ when $\omega \in \hat{D}$, where the inclusions can be strict at the same time; see [32, Proposition 6.1].
- $VMOA \subset C^1_0(\omega^*) \subset B_0$.
- Unlike the spaces $BMOA$ and $B$, the space $C^1(\omega^*)$ is not necessarily conformally invariant; see [32, Proposition 6.2].

Moreover, for those weights $\omega \in \hat{D}$ for which $C^1(\omega^*) \subset B$ the space $C^1(\omega^*)$ can not be characterized in terms of a simple growth condition on the maximum modulus of $g'$. Let us prove this. Assume the contrary, i.e. that there exist a weight $\omega \in \hat{D}$ and a function $F: [0,1) \to \mathbb{R}_+$ such that $C^1(\omega^*) \subset B$ and $g \in C^1(\omega^*)$ whenever

$$
\sup_{|z|=r} |g'(z)| \leq CF(r), \quad 0 \leq r < 1
$$

for some constant $C = C_g > 0$. Since $\log(1-z) \in BMOA \subset C^1(\omega^*)$, it follows from the estimate (20) that

$$
\frac{1}{1-r} = \sup_{|z|=r} \left| \frac{d}{dz} \log(1-z) \right| \leq CF(r)
$$

for some constant $C > 0$ independent of $r$. Now if $g \in B$, then using the estimate (21) we have

$$
\sup_{|z|=r} |g'(z)| \leq \frac{C'}{1-r} \leq C'' F(r),
$$
where constants $C' > 0$ and $C'' > 0$ are independent of $r$. Consequently, $g \in C^1(\omega^*)$ and we obtain $C^1(\omega^*) = \mathcal{B}$, which is in contradiction to the assumption that $C^1(\omega^*) \subsetneq \mathcal{B}$.

However, the situation is different in the case of $\alpha > 1$. Then one can show [32, Theorem 6.5] that $g \in C^\alpha(\omega^*)$ if and only if

$$
\sup_{|z|=r} |g'(z)| \leq C \frac{\omega^*(r)^{\frac{\alpha-1}{2}}}{1-r}, \quad r \to 1-,
$$

where $C = C(\alpha, g, \omega) > 0$.

The spaces $C^\alpha(\omega^*)$ appear in [33, Chapter 4] and [32, Chapter 6], where they are defined in terms of Carleson measures for the spaces $A^p_\omega$. If $p \leq q$, then $g \in C^{\frac{\alpha}{2}}(\omega^*)$ if and only if $d\mu(z) = |g'(z)|^2 \omega^*(z) dA(z)$ is a $q$-Carleson measure for $A^q_\omega$, i.e. the identity operator $I : A^q_\omega \to L^q(\mu)$ is bounded. In the light of Theorem 3.3 in [32], this is equivalent to the condition

$$
\sup_{a \in \overline{D}} \frac{\mu(S(a))}{\omega(S(a))^{\frac{1}{p}}} < \infty.
$$

The spaces $C^\alpha(\omega^*)$ are used to characterize the boundedness and compactness of $T_g$ on the spaces $A^p_\omega$. We have the following result; see [32, Theorem 6.4, Theorem 6.5]

**Theorem 4.3.** Let $0 < p \leq q < \infty$ be such that $\frac{1}{p} - \frac{1}{q} < 1$ and let $\omega \in \hat{D}$ and $g \in \mathcal{H}(\mathbb{D})$. Then $T_g : A^p_\omega \to A^q_\omega$ is bounded if and only if $g \in C^\alpha(\omega^*)$, where $\alpha = 2 \left( \frac{1}{p} - \frac{1}{q} \right) + 1$.

**Theorem 4.9** in [33] reads as follows.

**Theorem 4.4.** Let $\omega \in \mathcal{I} \cup \mathcal{R}, 0 < p \leq q < \infty, \frac{1}{p} - \frac{1}{q} < 1$ and $\alpha = 2 \left( \frac{1}{p} - \frac{1}{q} \right) + 1$. Then $T_g : A^p_\omega \to A^q_\omega$ is compact if and only if $g \in C^\alpha_0(\omega^*)$.

In the case $\omega \in \mathcal{R}$, we have $C^1(\omega^*) = \mathcal{B}$; see [32, Proposition 6.1]. This reflects the fact that $T_g$ is bounded on $A^p_\omega$, $\omega \in \mathcal{R}$, precisely when $g \in \mathcal{B}$, see [8], where a larger class of weights containing regular weights is considered.

The technical condition $\frac{1}{p} - \frac{1}{q} < 1$ in Theorems 4.3 and 4.4 is due to the fact that otherwise the only bounded $T_g : A^p_\omega \to A^q_\omega$ is the zero operator, which can be deduced from the growth estimate (22).

### 4.2 Equivalent norms

In this section, we introduce some tools needed to establish our main results. It was shown in [32, Proposition 3.7] (and [33, Proposition 4.3]) that there does not exist a Littlewood-Paley formula in general for the spaces $A^p_\omega$, $p \neq 2$, in the following sense.
Lemma 4.5. Let $0 < p < \infty$, $p \neq 2$. Then there exists a weight $\omega \in \hat{D}$ such that for any function $\varphi : [0, 1) \to (0, \infty)$ the relation

$$\|f\|_{A_p^\omega}^p \simeq \int_{\mathbb{D}} |f'(z)|^p \varphi(|z|)^p \omega(z) dA(z) + |f(0)|^p$$

can not be valid for all $f \in \mathcal{H}(\mathbb{D})$.

The proof of Lemma 4.5 is based on using rapidly increasing weights

$$\omega(r) = (1 - r)^{-1} \left( \log \frac{e}{1 - r} \right)^{-\alpha}, \quad \alpha > 1,$$

and lacunary series $\sum_{k=0}^{\infty} z^{2^k}$. To delete the integral in the definition of $T_g$, we follow the ideas in [33] and [32] and utilize other norms, which are equivalent to the norm $\| \cdot \|_{A_p^\omega}$ and involve the derivative $f'$. These norms are inherited from the theory of the Hardy spaces; see [33, Chapter 4] and [32, Section 3].

A non-tangential lens-type region is defined to be

$$\Gamma(u) = \left\{ z \in \mathbb{D} : |\theta - \arg z| < \frac{1}{2} \left( 1 - \frac{|z|}{r} \right) \right\}, \quad u = re^{i\theta} \in \overline{\mathbb{D}} \setminus \{0\}.$$

Lemma 4.6. Let $0 < p < \infty$ and $f \in \mathcal{H}(\mathbb{D})$, and let $\omega$ be a radial weight. Then

$$\|f\|_{A_p^\omega}^p = p^2 \int_{\mathbb{D}} |f(z)|^{p-2} |f'(z)|^2 \omega^*(z) dA(z) + \omega(\mathbb{D}) |f(0)|^p, \quad (23)$$

and

$$\|f\|_{A_p^\omega}^p \simeq \int_{\mathbb{D}} \left( \int_{\Gamma(u)} |f'(z)|^2 dA(z) \right)^{\frac{p}{2}} \omega(u) dA(u) + |f(0)|^p, \quad (24)$$

where the constants of comparison depend only on $p$ and $\omega$. In particular,

$$\|f\|_{A_2^\omega}^2 = 4\|f\|_{A_2^{\omega^*}}^2 + \omega(\mathbb{D}) |f(0)|^2. \quad (25)$$

For instance, the identity (23) in Lemma 4.6 follows by applying the classical Hardy-Stein-Spencer estimate [18]

$$\|f\|_p^p = \frac{p^2}{2} \int_{\mathbb{D}} |f(z)|^{p-2} |f'(z)|^2 \log \frac{1}{|z|} dA(z) + |f(0)|^p$$

to dilatations $f_r(z) = f(rz)$, $0 < r < 1$, and integrating it with respect to the weight $\omega_r(z) r^2 dr$; see the proof of Theorem 4.2 in [33]. Correspondingly, the estimate (24) stems from the classical extension of the Littlewood-Paley formula for $H^2$ obtained by Fefferman and Stein [17, Section 7, Theorem 8].

Also, a non-tangential maximal function related to the regions $\Gamma(u)$ is needed. Let $f \in \mathcal{H}(\mathbb{D})$. Then we define

$$N(f)(u) = \sup_{z \in \Gamma(u)} |f(z)|, \quad u \in \overline{\mathbb{D}} \setminus \{0\}.$$
Lemma 4.7. Let $0 < p < \infty$ and let $\omega$ be a radial weight. Then
$$\|N(f)\|_{L^p(\omega)} \simeq \|f\|_{A^p_\omega}$$
for all $f \in A^p_\omega$.

The proof of Lemma 4.7 follows by using dilatations $f_r$ and integrating the well-known inequality [18, Theorem 3.1]
$$\|f^*\|_{L^p(T)} \leq C\|f\|_p,$$
where $f^*(\zeta) = \sup_{z \in \Gamma(\zeta)} |f(z)|$, $\zeta \in T$, with respect to the weight $\omega(r)rdr$.

The weighted maximal operator
$$M_\omega(\psi)(z) = \sup_{a : z \in S(a)} \frac{1}{\omega(S(a))} \int_{S(a)} |\psi(u)|\omega(u)dA(u),$$
where $\omega \in \hat{D}$, $\psi \in L^1(\omega)$, introduced by Hörmander [23] is also utilized. Note that the operator $M_\omega$ is sublinear, but its norm is defined like in the case of a linear operator. The following lemma [32, Theorem 3.3, 3.4] establishes a connection between Carleson measures for the spaces $A^p_\omega$ and the boundedness of $M_\omega$; see also Theorem 2.1 and Corollary 2.2 in [33].

Lemma 4.8. Let $0 < p \leq q < \infty$ and $\omega \in \hat{D}$, and let $\mu$ be a positive Borel measure on $D$. Then $\mu$ is a $q$-Carleson measure for $A^p_\omega$ if and only if
$$G = \sup_{a \in \hat{D}} \frac{\mu(S(a))}{\omega(S(a))^{\frac{1}{p}}} < \infty. \quad (26)$$

Moreover, in this case
$$\|f\|_{L^q(\mu)} \leq CG\|f\|_{A^p_\omega}^q$$
for all $f \in A^p_\omega$, where $C > 0$ is a constant. Furthermore, if $\alpha \in (0, \infty)$ such that $p\alpha > 1$, then the sublinear operator
$$[M_\omega((\cdot)^{\frac{1}{p}})^{\alpha}] : L^p(\omega) \to L^q(\mu)$$
is bounded if and only if $\mu$ satisfies (26). In this case, the norm of this operator satisfies
$$\|\|M_\omega((\cdot)^{\frac{1}{p}})^{\alpha}\|_{L^q(\mu)} \simeq G.$$
4.3 Essential norm of \( T_g \) on \( A^p_\omega \)

**Theorem 4.9.** Let \( 1 < p \leq q < \infty \) and \(-1 < \alpha, \beta < \infty \) such that \( (2 + \alpha)/p - (2 + \beta)/q \leq 1 \) and let \( T_g : A^p_\omega \rightarrow A^q_\omega \) be bounded. Then

\[
\| T_g \|_e \simeq \limsup_{|z| \to 1^-} |g'(z)|(1 - |z|^2)^{1+(2+\beta)/q-(2+\alpha)/p}. \tag{27}
\]

A natural endeavour is to try to establish an estimate for the essential norm of \( T_g \) in the case of the weighted Bergman spaces \( A^p_\omega \), where the weight satisfies the doubling property (17) and further show that this estimate is equivalent to the distance of \( g \) to \( C_0^\alpha(\omega^*) \).

In order to estimate the essential norm from above, we prove the following quantitative version of Theorem 4.3.

**Theorem 4.10 ([III]).** Let \( 0 < p \leq q < \infty \) be such that \( 1/p - 1/q < 1 \) and let \( \omega \in \hat{D} \). Let \( g \in C^\alpha(\omega^*) \) with \( \alpha = 2(\frac{1}{p} - \frac{1}{q}) + 1 \) and \( T_g : A^p_\omega \rightarrow A^q_\omega \). Then the operator norm \( \| T_g \| \) is equivalent to \( \| g \|_{*, \alpha, \omega} \). Moreover, if \( p < q \), then \( \| T_g \| \) is also equivalent to the quantity

\[
\sup_{z \in \mathbb{D}} |g'(z)|(1 - |z|)\omega^*(z)^{(1-\alpha)/2}.
\]

The constants of comparison are independent of \( g \).

Comparability \( \| T_g \| \simeq \| g \|_{*, \alpha, \omega} \) in Theorem 4.10 follows directly from the closed graph theorem applied to the mappings \( g \mapsto T_g \) and \( T_g \mapsto g \), where \( g \) is identified with its equivalence class \([g]\) obtained by dividing out the constant functions.

Our main result is a quantitative generalization of the compactness characterization in Theorem 4.4.

**Theorem 4.11 ([III]).** Let \( 0 < p \leq q < \infty \) be such that \( 1/p - 1/q < 1 \) and let \( \omega \in \hat{D} \). Let \( g \in C^\alpha(\omega^*) \) with \( \alpha = 2(\frac{1}{p} - \frac{1}{q}) + 1 \). Then \( T_g \) as an operator from \( A^p_\omega \) to \( A^q_\omega \) satisfies

\[
\| T_g \|_e \simeq \text{dist}(g, C_0^\alpha(\omega^*)),
\]

where

\[
\text{dist}(g, C_0^\alpha(\omega^*)) \simeq \limsup_{|a| \to 1} \left( \int_{S(a)} |g'(z)|^2 \omega^*(z) \, dA(z) / \omega(S(a))^{\alpha} \right)^{1/2}, \tag{28}
\]

and if \( p < q \),

\[
\text{dist}(g, C_0^\alpha(\omega^*)) \simeq \limsup_{|z| \to 1} |g'(z)|(1 - |z|)\omega^*(z)^{(1-\alpha)/2}.
\]
One may observe that the essential norm of \( T_g \) is equivalent to a \( \limsup \) version (28) of an expression defining the seminorm (19) of the space \( C^\alpha(\omega^*) \).

The basic ideas of the proof of Theorem 4.11 are similar to the ideas used in [33, Chapter 4] and [32, Section 6]. To elaborate on the proof of Theorem 4.11, the upper estimate for the essential norm is straightforward using Theorem 4.10. For the lower estimate

\[
\|T_g\|_e \geq C \text{dist}(g, C^\alpha_0(\omega^*)) ,
\]

where \( C > 0 \), we utilize the standard test functions \( f_{a,p} \) of the spaces \( A^p_\omega \); see (18). Namely, we are led to estimate the upper limit of norms of the images \( T_g f_{a,p} \). Our goal is to establish a lower bound for this upper limit in terms of the quantity in (28) (see the estimate (31) below). To achieve this, one seeks to cancel the integral present in the definition of \( T_g \). For this purpose, a Littlewood-Paley-type formula is often used. However, the spaces \( A^p_\omega \) do not admit such a formula in general unless \( p = 2 \); see Lemma 4.5. Instead, equivalent norms stemmed from the theory of the Hardy spaces are used; see Lemmas 4.6 and 4.7. Let us illustrate this by presenting the proof of the lower bound

\[
\|T_g\|_e \geq C \limsup_{|a| \to 1} \left( \frac{\int_{S(a)} |g'(z)|^2 \omega^*(z) \, dA(z)}{\omega(S(a))^{\alpha}} \right)^{1/2} .
\]

in Theorem 4.11 in the case \( T_g: A^p_\omega \to A^p_\omega \), i.e. \( \alpha = 1 \), where \( p > 2 \).

Let \( K: A^p_\omega \to A^p_\omega \) be compact. Then

\[
\|T_g - K\| \geq C \|(T_g - K)f_{a,p}\|_{A^p_\omega} \geq C (\|T_g f_{a,p}\|_{A^p_\omega} - \|K f_{a,p}\|_{A^p_\omega}) ,
\]

where \( C > 0 \). Since the functions \( f_{a,p} \) converge weakly to zero as \( |a| \to 1 \), we have that \( \|K f_{a,p}\|_{A^p_\omega} \to 0 \) as \( |a| \to 1 \). Therefore we are led to the estimate

\[
\|T_g\|_e \geq C \limsup_{|a| \to 1} \|T_g f_{a,p}\|_{A^p_\omega} .
\]

The next step is to establish the estimate

\[
\limsup_{|a| \to 1} \|T_g f_{a,p}\|_{A^p_\omega} \geq C \limsup_{|a| \to 1} \left( \frac{\int_{S(a)} |g'(z)|^2 \omega^*(z) \, dA(z)}{\omega(S(a))} \right)^{1/2} .
\]

(31)

To this end, we utilize Lemma 4.7 and the estimate (24) in Lemma 4.6 in the following way.

Let \( 1/2 < r < |a| < 1 \). We observe that \( |f_{a,p}(z)| \simeq \omega(S(a))^{-1/p} \) for all \( z \in S(a) \). Moreover, the associated weight \( \omega^* \) satisfies \( \omega^*(z) \simeq \omega(T(z)) \) when
\[ |z| \geq 1/2, \text{ where } T(z) = \{ u \in \mathbb{D} | z \in \Gamma(u) \}; \text{ see Lemma 2.1 in [32]. Now} \]
\[
\frac{1}{\omega(S(a))} \int_{S(a)} |g'(z)|^2 \omega^*(z) dA(z) \leq C \int_{S(a)} |f_{a,p}(z)|^p |g'(z)|^2 \omega^*(z) dA(z)
\]
\[
\leq C \int_{\mathbb{D}} |f_{a,p}(z)|^p |g'(z)|^2 \int_{\Gamma(u)}(z) \omega(u) dA(u) dA(z)
\]
\[
= \int_{\mathbb{D}} N(f_{a,p})(u) \left( \int_{\Gamma(u)} |f_{a,p}(z)|^2 \omega(u) dA(u) dA(z) \right) \leq C \left( \int_{\mathbb{D}} N(f_{a,p})(u)^p \omega(u) dA(u) \right)^{(p-2)/p}
\]
\[
\times \left( \int_{\mathbb{D}} \left( \int_{\Gamma(u)} |f_{a,p}(z)|^2 \omega(u) dA(u) \right)^{p/2} \omega(u) dA(u) \right)^{2/p}
\]
\[
\leq C \| N(f_{a,p}) \|_{L^p(\mathbb{D})}^2 \| T_g f_{a,p} \|_{A^\infty_p}^2 \leq C \| f_{a,p} \|_{A^\infty_p}^2 \| T_g f_{a,p} \|_{A^\infty_p} \leq C \| T_g f_{a,p} \|_{A^\infty_p}^2
\]
for all \( r < |a| < 1 \), where constants \( C > 0 \) may change from one instance to another and we utilized the estimate (24), Lemma 4.7, and the fact \( \sup_{a \in \mathbb{D}} \| f_{a,p} \|_{A^\infty_p} < \infty \) in the last three inequalities. From the last inequality, we obtain
\[
\| T_g f_{a,p} \|_{A^\infty_p} \geq C \left( \frac{1}{\omega(S(a))} \int_{S(a)} |g'(z)|^2 \omega^*(z) dA(z) \right)^{1/2}
\]  
(32)
for all \( r < |a| < 1 \) and the estimate (31) follows. The lowerbound (29) follows from the estimates (30) and (31). The proof of (29) in the general case \( 0 < p \leq q < \infty \) is slightly different and requires a localization argument; see Lemma 9 in [111]. This is due to the fact that when \( p = 1 \) the functions \( f_{a,p} \) do not converge weakly to zero and consequently it does not necessarily hold that \( \| K f_{a,p} \|_{A^\infty_p} \to 0 \) as \( |a| \to 1 \).

The proof of the equivalence (28) in Theorem 4.11 is technical and we do not repeat it here.

**Remark.** In contrast to \( T_g \) acting on e.g. the Hardy space \( H^1 \), weak compactness of \( T_g : A^1_\omega \to A^1_\omega \) is not a relevant question. This can be deduced from a result of Lusky [30, Thm 1.2], which states that the space \( A^1_\omega \) is either isomorphic to \( \ell^1 \) or to the direct \( \ell^1 \)-sum
\[
\sum_{n=1}^\infty (\oplus P_n) = \left\{ (p_n) : p_n \in P_n, \sum_{n=1}^\infty \| p_n \|_{L^1(T)} < \infty \right\},
\]
where \( P_n \) is the space of analytic polynomials of degree at most \( n \) endowed with the \( L^1(T) \) norm. Hence \( A^1_\omega \) has the Schur property, which implies that every weakly compact operator is compact.
References


