Dynamic Management of R&D under Uncertainty and Competition

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Discussion Paper No. 174

ISSN 1795-0562
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Abstract

This paper studies the optimal management of R&D investments in a dynamic duopoly where the firms compete with homogeneous products and the progress of production technology is characterised by a leapfrogging dynamics. Two distinct forms of uncertainty are incorporated: technological uncertainty in the sense that the relationship between the rate of R&D investments and innovations is probabilistic and economic uncertainty in the sense that both the product market demand and the R&D costs fluctuate unpredictably. As one important result, we establish a U-shaped investment-uncertainty relationship by showing that higher economic uncertainty first renders R&D investments more attractive, but at a sufficiently high level becomes a deterrent for the investments. In addition, higher technological uncertainty, as manifested by a longer (expected) lag between engaging in R&D and leapfrogging the competitor, is found to always act as a deterrent.

JEL Classification: D81, O32

Keywords: R&D, investment, uncertainty

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*The author wishes to thank Erkki Koskela, Rune Stenbacka, Luis Alvarez, Staffan Ringbom and Timo Vesala for comments.
1 Introduction

When a firm engages in R&D for the production technology, it generally encounters two distinct forms of uncertainty. First, there is technological uncertainty in the sense that the relationship between the rate of R&D investments and innovations is probabilistic. This can be restated by saying that the knowhow leading to innovations takes ‘time to build’ in an unpredictable way. Second, there is economic uncertainty in the sense that both the product market demand and the investment rate required to maintain a given intensity of R&D (the cost of R&D) fluctuate unpredictably over time. Possible sources for the latter fluctuation are, for example, uncertainty in exogenous R&D input markets and unpredictable changes in government regulations.

A general notion of the extant innovation literature is that the attractiveness of R&D investments depends on the incremental product market profits accrued to a successful innovator relative to an unsuccessful one, not on the absolute profit levels per se (e.g. Aghion et al., 2001). Consequently, intensifying product market competition generally renders R&D investments more attractive though it is possible that the total industry profits decline. The mechanics behind this is known as the selection effect of competition, originating with Vickers (1995): as in a more competitive industry the profits earned by the technological leader are larger relative to the competitors, intensifying competition creates incentives for R&D by increasing the incremental profits. Assuming the polar case of perfect competition leads to the Schumpeterian model of innovation (Schumpeterian branch of the endogenous growth theory) in that the market is occupied by one insider firm at a time, with the market outsiders earning zero profits. Acting as an incumbent monopolist, the insider firm has lower incentives for R&D than a market outsider striving to capture the incumbency. Particularly, under relatively mild conditions, the Arrow replacement effect takes over, meaning that in equilibrium the insider will choose not to perform R&D and an innovating outsider always becomes the new monopolist (e.g. Aghion and Howitt, 1992).

The objective of this paper is to extend the theoretical boundaries of the innovation literature by studying the optimal management of R&D investments in a genuinely dynamic industry where the two forms of uncertainty coexist together with operational flexibility to adjust the rate of investments in response to the resolution of economic uncertainty. The context is a continuous time industry organised as a duopoly with respect to both production and R&D. The duopoly firms compete with homogenous products and engage in R&D with the aim of gaining competitive advantage in the product market through productivity-enhancing innovations. Intermediate competition intensities between perfect competition and pure Cournot competition is allowed for by the conjectural variations approach, as a simpler alternative to the popular product differentiation approach. Technological progress is characterised by a ‘quality ladder’ dynamics where, in contrast to the step-by-step dynamics (e.g. Harris and Vickers, 1987), the technological laggard does not have to catch up with the leader at the cutting edge before racing for the technological leadership in the future. In other words, upon innovating the laggard leapfrogs the leader
by advancing directly to the new cutting edge. Depending on the intensity of competition, leapfrogging may be of the strong type in that the product market is characterised by a persistent monopoly in the fashion of the Schumpeterian model, or of the weak type in that both firms are earning profits (Encaoua and Ulph, 2004). Along the lines of Aghion et al. (1997), it is assumed that the technological gap between the duopoly firms cannot exceed one innovation. The implied absence of the 'escape competition' motive rules out the incentives of the leader to perform R&D (see eg. Mookherjee and Ray, 1991, for discussion on the motive).

Arguably, industries most susceptible to the leapfrogging dynamics (and to the Schumpeterian process of creative destruction, see eg. Diamond, 2006) are those characterised by rapid technological progress and vigorous competition over the leadership. Recent examples include the leapfrogging of Dell’s direct retailing model over the previously dominant PC retailers (IBM, HP and Compaq), the leapfrogging of Walmart over the other US retailers in the use of information technology to manage the supply chain logistics, the leapfrogging of Nokia’s mobile phone technology and design over Motorola in the late eighties and early nineties, the leapfrogging of Apple’s IPod over the previous generation of mobile music devices, the leapfrogging of online security brokers over the traditional brokers, and the leapfrogging of online travel agencies (Expidia, Travelocity, Orbitz etc.) over the bricks-and-mortar agencies. As for ongoing competition of the weak leapfrogging type, where one firm takes the lead in one generation and the competitor in the next, illustrative examples are the race between the graphics card manufacturers ATI and NVIDIA for superior graphics processing unit (GPU) technology and the race between Intel and AMD for central processing unit (CPU) performance.

Besides the papers mentioned above, our paper is related to a wide range of other papers revolving around R&D. For example, Lee and Wilde (1980) study strategic R&D investments, focusing on the relationship between R&D and market structure. Reinganum (1982) studies strategic R&D, focusing on patent protection and imitation. Fudenberg et al. (1983) and Grossman and Shapiro (1987) study strategic R&D in multistage patent races with the emphasis on finding the conditions under which a race is characterised by vigorous competition and when it degenerates to a monopoly. Dixit (1988) provides a general framework for studying R&D races when the competing firms are heterogeneous in the efficiency of R&D. Pennings and Lint (1997) construct a jump-diffusion model to allow for the possibility that the arrival of strategic information (on new standards, for example) may have a drastic impact on the projected cash flow generated by a new product. Childs and Triantis (1999) study R&D investments when there is operational flexibility to alter between multiple R&D projects. Schwartz (2003) and Schwartz and Moon (2001) study R&D investments when there is an exogenous probability of what is called a catastrophic event, such as losing a patent race. Weeds (2002) considers both cooperation and non-cooperation in an R&D race that is subject to a winner-takes-all patent system. Huisman and Kort (2003) consider optimal adoption of new technology in a symmetric duopoly when a firm has operational flexibility
to decide both when to adopt and what technology to adopt.

Summing up, the questions of the analysis are:

1. Given the leapfrogging dynamics, what is the programme for the optimal management of R&D investments?
2. How does product market uncertainty affect the optimal management?
3. How does R&D cost uncertainty affect the optimal management?
4. How does technological uncertainty affect the optimal management?
5. How does the intensity of product market competition affect the optimal management?

The main findings are as follows. To begin with, the analysis is constructed in such a way (by use of suitable linearity assumptions) that the applicable stochastic control problems become optimal switching problems, as opposed to a continuous control problems. Particularly, as the leader has no incentives for R&D, it suffices to find the rule that determines whether or not it is optimal for the existing follower to perform R&D. We show that in the absence of hysteresis this rule derives from a unique switching trigger. In the one dimensional case where R&D cost uncertainty is excluded, the trigger is defined for a stochastic demand shock. In the two dimensional case, on the other hand, the trigger is defined for the ratio between the shock and the flow R&D cost, due to a particular linear homogeneity property. The comparative statics of the two triggers reveals a number of important properties. First, we reproduce the selection effect by showing that the triggers are inversely related to the intensity of product market competition. Second, we establish a U-shaped investment-uncertainty relationship by showing that the triggers first decrease and then increase as product market uncertainty (in the mean preserving sense) gets higher. Moreover, R&D cost uncertainty has the same effect. Making use of real options concepts, we derive the economic intuition from an interaction between an option value for delaying and an opportunity cost of delaying. The effect of technological uncertainty manifests itself through the (expected) lag between engaging into R&D and leapfrogging. We show that the triggers increase monotonously in the lag. In other words, higher 'time to build' renders R&D investments less attractive, a result by no means obvious a priori.

The paper is organised as follows. Section 2 sets up the duopoly by introducing the product market and the leapfrogging dynamics. Section 3 devises the stochastic control problems and solves for the optimal switching trigger in both the one dimensional case and in the two dimensional case. Section 4 carries out the comparative statics. Section 5 finally concludes and proposes a number directions towards which the analysis can be extended.
2 Basic setup

We consider an industry organised as a duopoly with respect to both production and R&D. Time is continuous and the planning horizon infinite. The duopoly firms are risk neutral and output homogeneous products with constant unit cost technologies. The firms utilise R&D as a means for gaining competitive advantage in the product market: by enhancing cost efficiency, an innovation enables a firm to expand its market share.

Product market. The inverse demand function in the product market is of the unit elastic form

\[ p_t = \frac{x_t}{q_t + \bar{q}_t}, \]  

where \((q_t, \bar{q}_t)\) is the pair of outputs at time \(t\) and \(x_t\) is a systematic shock driven by a geometric Brownian motion

\[ dx_t = \mu x_t dt + \sigma x_t dz_t, \quad x_0 = x, \]  

where \(dz_t\) is the increment of a standard Brownian motion, \(\mu\) is a drift coefficient \(\sigma\) and is a diffusion coefficient measuring the degree of product market volatility. Under risk neutrality it must be that \(\mu = r - \delta\) where \(r\) is the riskfree rate of return and \(\delta \geq 0\) is the equilibrium rate of return shortfall. It is also standard to show (eg. Øksendal 2003, p. 62) that

\[ x_t = x e^{\nu t + \sigma z_t}, \]

where the adjusted drift \(\nu = \mu - \frac{\sigma^2}{2}\). We assume

\(\sigma < \sigma_{\text{max}} = \sqrt{2\mu}\)

to make \(\nu > 0\). This assures that the shock will reach any finite value above \(x\) in finite time almost surely. For brevity, notation \(A_x\) is adopted for the differential operator associated with the shock:

\[ A_x = \frac{1}{2}\sigma^2 x^2 \frac{d^2}{dx^2} + \mu x \frac{d}{dx}. \]
The intensity of product market competition is measured by a conjectural variation parameter \( \theta \in [0, 1] \). The idea is that when choosing \( q \) so as to maximise profits, a firm conjectures that the competitor reacts according to

\[
\frac{dq}{dq} = -\theta. \tag{3}
\]

A larger \( \theta \) represents more intense competition with pure Cournot competition and pure Bertrand competition obtaining in the polar cases. For further discussion, we refer to Varian (1992, pp. 302-303). In the innovation literature, a popular alternative to conjectural variation is to assume product differentiation by way of specifying a CES utility function for the representative consumer (e.g. Aghion et al., 1997, and Aghion et al., 2001). In that setup \( \theta \) would enter as the elasticity of substitution between the products.

**Equilibrium profits.** Due to unit elastic demand an equiproportional change in the unit costs leaves the equilibrium profits unchanged since only the relative cost matters. Formally, let \( \kappa \) denote the unit cost of the more efficient firm relative to the less efficient firm. The total flow profits in equilibrium are then given by

\[
\Sigma(\theta, \kappa) = \begin{cases} \frac{(1+\theta^2)(1+\kappa^2)-4\theta\kappa}{(1-\theta)(1+\kappa)^2}, & 0 \leq \frac{\theta}{\kappa} \leq 1 \\ 1-\kappa, & 1 < \frac{\theta}{\kappa} \leq \frac{1}{\kappa}. \end{cases} \tag{4}
\]

The allocation of the less cost efficient firm is

\[
\pi(\theta, \kappa) = \begin{cases} \frac{1}{(1-\theta)(1+\kappa)^2} (\kappa - \theta)^2, & 0 \leq \frac{\theta}{\kappa} \leq 1 \\ 0, & 1 < \frac{\theta}{\kappa} \leq \frac{1}{\kappa} \end{cases} \tag{5}
\]

while the allocation of the more cost efficient firm is

\[
\Sigma(\theta, \kappa) - \pi(\theta, \kappa) = \begin{cases} \frac{1}{(1-\theta)(1+\kappa)^2} (1 - \theta\kappa)^2, & 0 \leq \frac{\theta}{\kappa} \leq 1 \\ 1-\kappa, & 1 < \frac{\theta}{\kappa} \leq \frac{1}{\kappa}. \end{cases} \tag{6}
\]

For \( 1 < \frac{\theta}{\kappa} \leq \frac{1}{\kappa} \) competition is sufficiently intense for the more cost efficient firm to monopolise the product market.
The firms are identical in terms of R&D. The flow cost function associated with R&D is a linear function of the instantaneous R&D input \( z \) as \( ze > 0 \) where \( e > 0 \). We assume \( z \) can be changed with no adjustment costs and, without loss of generality, restrict \( z \in [0, 1] \). As in Aghion et al. (1997), the technological gap between the firms cannot exceed one innovation. This assumption can be rationalised on two grounds. First, it may be prohibitively costly in terms of R&D effort to extend the gap beyond one innovation. Second, the technologies lagging the cutting edge may have zero patent protection due to the reluctance of the cutting edge firm to pay a renewal fee for the protection of an outdated technology. The expiration of patents then occurs in tandem with innovations. Particularly, when an innovation advances a firm to the cutting edge, the lapse of protection on its outdated technology makes it possible for the competitor to imitate or copy. The patent protection on the cutting edge technology, on the other hand, is perfect so there are no technological spillovers between the firms.

The firms are subject to technological uncertainty which means that the relationship between R&D inputs and innovations is probabilistic. Specifically, we assume there exists a Poisson innovation process allowing improvements in cost efficiency such that when a firm innovates its unit cost is scaled down to the fraction \( v < 1 \) of the current level. The hazard rate (mean arrival rate) of innovations is a linear function of the R&D input as \( z \lambda \) where \( \lambda > 0 \) measures the efficiency of R&D. The memoryless property of Poisson processes rules out cumulative effects and spillovers from past R&D.

Furthermore, the firms are subject to a leapfrogging dynamics which means that the technological follower can gain competitive advantage by directly adopting technology more cost efficient than the cutting edge. In other words, the product market never has neck-and-neck competition since the follower does not have to catch up with the leader before racing for technological dominance. This in contrast to the step-by-step dynamics which requires that the follower draw level with the leader first (see eg. Aghion et al., 2001, and Hoernig, 2003). Particularly, since the technological gap in the duopoly is a fixed at one innovation, there is only one competitive state in the product market, with the identities of the leader and the follower changing whenever leapfrogging takes place. Along the lines of Encaoua and Ulph (2004), we rationalise leapfrogging through the notion of successful R&D, by stating that the follower in the course of undertaking R&D learns to master the cutting edge and so is in the position to achieve the same technology as the leader would if it innovated, the new and improved cutting edge.

The above implies that the duopoly has \( \kappa \) is fixed at the leader-follower relative unit cost, given by \( v \). For \( 1 < \frac{\kappa}{v} \leq 1 \) the product market is characterised by a persistent monopoly, with the identity of the monopolist changing upon innovations. We then say that strong leapfrogging obtains. For \( 0 \leq \frac{\kappa}{v} < 1 \) the firms coexist in the product market instead and we say that weak leapfrogging obtains. Subtracting (5) from (6) with \( \kappa = v \) yields the flow profit spread coefficient
\[
\Delta = \Delta(\theta, v) = \begin{cases} 
(1 + \theta)\frac{1 - v}{\theta - v}, & 0 \leq \frac{\theta}{\theta - v} \leq 1 \\
1 - v, & 1 < \frac{\theta}{\theta - v} \leq \frac{1}{v}.
\end{cases}
\] (7)

The fact that \(\Delta\) increases in \(\theta\) under weak leapfrogging is an instance of the more general Boone reallocation criterion (Boone, 2001) which states that intensifying competition must reallocate profits from inefficient to more efficient firms. Due to unit elastic demand \(\Delta\) is path independent of the innovation process.

## 3 Optimal R&D Management

In the section we find the optimal lifetime R&D management programmes for the duopoly firms. We start the analysis by formulating the lifetime profit maximisation problems (valuation problems) as stochastic control problems. Having done that, we proceed to invoke standard techniques of stochastics and dynamic programming in order to find the rules governing the optimal instantaneous R&D inputs. Table 1 summarises the assumptions and their main implications for the analysis. Since there are no cumulative effects from past R&D and the planning horizon is infinite, the problems are time homogenous. The properties of Itô diffusions (eg. Øksendal 2003, pp. 115-116) moreover imply that the problems are Markov. In other words, the optimal R&D rules are independent of the evolution of calendar time and depend on the current shock value only.

<table>
<thead>
<tr>
<th>Assumption</th>
<th>Implication</th>
</tr>
</thead>
<tbody>
<tr>
<td>Unit elastic demand</td>
<td>(\Delta) path independent of the innovation process</td>
</tr>
<tr>
<td>Poisson innovation process</td>
<td>No spillovers/cumulative effects from past R&amp;D</td>
</tr>
<tr>
<td>Leapfrogging dynamics</td>
<td>No neck-and-neck product market competition</td>
</tr>
<tr>
<td>+ Technological gap at 1</td>
<td>Only 1 competitive state in the product market</td>
</tr>
</tbody>
</table>

Table 1: assumptions

We denote the instantaneous R&D inputs of the firms by \(z_t\) and \(\hat{z}_t\). Discarding the terms of second order in \(dt\), Table 2 displays the associated probabilities for the four innovation scenarios over \([t, t + dt]\).

<table>
<thead>
<tr>
<th>firm ((z_t))</th>
<th>competitor ((\hat{z}_t))</th>
<th>innovates</th>
<th>fails to innovate</th>
</tr>
</thead>
<tbody>
<tr>
<td>innovates</td>
<td>0</td>
<td>(z_t\lambda dt)</td>
<td>(II)</td>
</tr>
<tr>
<td>fails to innovate</td>
<td>(\hat{z}_t\lambda dt)</td>
<td>1 - ((z_t + \hat{z}_t)\lambda dt)</td>
<td>(FF)</td>
</tr>
</tbody>
</table>

Table 2: innovation scenarios over \([t, t + dt]\)
3.1 Problem formulation

Define a relative unit cost process $\kappa_t$ to track the technological position of a firm relative to the competitor in the passage of time. The R&D environment implies that $\kappa_t$ is driven by a Poisson jump process as follows. The process starts from either the leader position $v$ or the follower position $\frac{1}{v}$ and jumps to the reciprocal value $\frac{1}{v}$ in the event that leapfrogging takes place over $[t, t + dt]$. In the event that the duopoly finishes the infinitesimal period in the same configuration it started it, with the identities of the leader and the follower unchanged, the process remains static. Now suppose the tracked firm is in the leader position at time $t$. Since scenario (II) in Table 2 can be ruled out as an impossibility, the possibility of the process remaining static is then accounted for by (IF) and (FF), the total probability of which is $1 - \tilde{z}_t \lambda dt$. With the complementary probability $\tilde{z}_t \lambda dt$ associated with (FI), the process jumps upwards to $\frac{1}{v}$. Symmetrically, in the follower position the possibility of the process remaining static is accounted for by (FI) and (FF), the total probability of which is $1 - z_t \lambda dt$. Combining the two positions we have

$$d\kappa_t = \begin{cases} -\kappa_t + \frac{1}{\kappa_t}, & \text{with probability } \lambda_t(\kappa_t)dt \\ 0, & \text{with probability } 1 - \lambda_t(\kappa_t)dt \end{cases}$$

where

$$\lambda_t(\kappa) = \begin{cases} z_t \lambda, & \kappa = \frac{1}{v} \\ \tilde{z}_t \lambda, & \kappa = v. \end{cases}$$

Let $E_{s,x}$ denote the expectation operator associated with the probability law of the shock when it starts at $x$ for $t = s \geq 0$. Also define a profit coefficient process $\pi_t = \pi_{\kappa_t}$ that in the follower position takes on the value $\pi(\theta, v)$ and in the leader position the value $\Sigma(\theta, v) - \pi(\theta, v)$, with $\Sigma$ and $\pi$ defined as in (4) and (5). By the Markov property of Itô diffusions, the profit maximisation problems of the firms are then stated by the pair of optimal stochastic control problems

$$W^*(s, x) = \sup_{\pi_t} E^\pi \left\{ E_{s,x} \int_s^\infty e^{-rt}(\pi_{\kappa_t} x_t - z_t c)dt \right\},$$

and

$$9$$
\[ \hat{W}^*(s,x) = \sup_{\hat{z}_t} \mathbb{E}^x \left\{ \mathbb{E}_{s,x} \int_s^\infty e^{-rt}((\Sigma(\theta,v) - \pi(x)) x_t - \hat{z}_t c) dt \right\}, \quad (10) \]

where \( \mathbb{E}^x \) denotes the expectation operator associated with the probability law of \( \kappa_t \) when \( \kappa_s = \kappa \). We suppress the technical rigor of the analysis by assuming/guessing straight out there exist a unique pair of controls \( (\hat{z}_t^*, \hat{z}_t^*) \) attaining the suprema. These controls represent the optimal R&D programmes over the infinite planning horizon. As optimal performance functions, \( W^* \) and \( \hat{W}^* \) then represent the firm values. We also note that since (9) is implicitly conditioned on \( \hat{z}_t^* \) and (10) is conditioned on \( z_t^* \), the problem pair defines in general terms a stochastic differential game with \( (z_t^*, \hat{z}_t^*) \) constituting a Markovian Nash equilibrium. Now, however, genuine strategic interaction in terms of R&D is ruled out by construction: as investments into R&D do not have any potential to improve the competitive position under the one innovation limit on the technological gap, the leader will always opt out from R&D competition. Put differently, the leader has no incentives for R&D due to the absence of the 'escape competition' motive. Recalling that the firms are identical except for the unit cost, it is also clear that \( z_t^* \) and \( \hat{z}_t^* \) display symmetry in the sense that \( \hat{z}_t^* \) equals \( z_t^* \) conditional on the unit cost \( \frac{1}{\mu} \) while \( z_t^* \) equals \( \hat{z}_t^* \) conditional on \( \kappa_t \). Hence, when leapfrogging takes place the firm values are switched.

Below we reserve \( W^* \) for the time \( t = s \) follower. Decomposing \( W^* \) to the instantaneous profits and to the value of the future profits with

\[ g_t = \pi(x_t - z_t c), \]

yields the Bellman equation

\[ W^*(s,x) = \sup_{\hat{z}_t} \mathbb{E}^x \left\{ \mathbb{E}_{s,x} \left\{ \int_s^{s+ds} e^{-rt} g_t dt + \int_{s+ds}^\infty e^{-rt} g_t dt \right\} \right\} \]
\[ = \sup_{\hat{z}_t} \mathbb{E}^x \left\{ e^{-rs} (\pi(\theta,v)x - zc) dt + \mathbb{E}_{s,x} \int_{s+ds}^\infty e^{-rt} g_t dt \right\} \]
\[ = \sup_{\hat{z}_t} \left\{ e^{-rs} (\pi(\theta,v)x - zc) dt + \mathbb{E}^x \left\{ \mathbb{E}_{s,x} \int_{s+ds}^\infty e^{-rt} g_t dt \right\} \right\}. \quad (11) \]

The expectation in (11) necessitates distinguishing the scenarios where leapfrogging has and has not taken place over \([s, s + ds]\). Given that leapfrogging has not taken place, an application of Itô’s lemma together with the differential operator

\[ A_{s,x} = \frac{d}{ds} + A_x, \]

10
Given that leapfrogging has taken place, similarly

\[
\sup_{z_t} E_t^\infty \left\{ E_{s,x} \int_{s+s_d}^{\infty} e^{-rt} g_t dt \right\} = W^*(s, x) + E_{s,x} dW^*(s, x)
\]

\[= W^*(s, x) + (A_{s,x} W^*(s, x))dt. \quad (12)\]

by the symmetry argument. Weighing (12) and (13) by the respective probabilities and substituting the obtained expectation into (11), we find on discarding the terms of order higher than \(dt\) that the follower problem is stated by the Hamilton-Jacobi-Bellman (HJB) equation

\[
\sup_z \left\{ A_{s,x} W^*(s, x) + e^{-rs}(\pi(\theta, v)x - zc) + z\Delta^*(x, s) \right\} = 0, \quad (14)
\]

where \(\Delta^*\) is the leader-follower value spread:

\[\Delta^*(x, s) = \tilde{W}^*(x, s) - W^*(x, s).\]

By formal analogy, the leader value satisfies the Bellman equation

\[A_{s,x} \tilde{W}^*(s, x) + e^{-rs}(\Sigma(\theta, v) - \pi(\theta, v))x - z^*\lambda \Delta^*(x, s) = 0, \quad (15)\]

with \(z^*\) attaining the supremum in (14). The fact that the functional inside the supremum is affine in \(z\) implies that \(z^*\) is bang bang, constituting a solution to an optimal switching problem (sequential stopping problem) with two operating modes: the follower obtains the best performance by either undertaking R&D with the maximal input or temporarily suspending R&D. In economic terms, the dynamic decision between the operating modes is governed by an infinite sequence of switching options. Since the timing of exercise is totally free, these options allow similar operational flexibility as American-style financial options with infinite expiration (see e.g. Merton 1992, pp. 255-308, for the fundamentals on financial options). Moreover, since exercise activates the next option in the sequence, or the option to make the converse switch, the options are compound options in the sense of Geske (1979).
Remark 1 The assumption of no adjustments costs translates to costless switching. By ruling out hysteresis, this enables us to find a closed form solution for $z^*$. Specifically, since there is no range of inertia associated between switching out of the R&D mode and switching out of the suspension mode, $z^*$ will be defined in terms of a single condition representing indifference between the two operating modes. Switching costs would necessitate two distinct indifference conditions, one for each operating mode. For further discussion on hysteresis, we refer to Dixit (1992), Dixit and Pindyck (1994, pp. 213-244).

Since the planning horizon is infinite, the discounting factor $e^{-rs}$ is the only source of the $s$-dependence in (14) and (15). Hence, it is natural to impose solutions of the separated form $e^{-rs}V^*(x)$ and $e^{-rs}\hat{V}^*(x)$ (the method of separating variables for PDEs). Then letting $\Delta^*(x) = \hat{V}^*(x) - V^*(x)$, we find that (14) is equivalent to

$$\sup_z \{A_y V^*(x) + \pi(\theta, v)x + z(\lambda \Delta^*(x) - c)\} = rV^*(x),$$

while (15) is equivalent to

$$A_y \hat{V}^*(x) + \hat{\pi}(\theta, v)x - z^* \lambda \Delta^*(x) = r\hat{V}^*(x),$$

where $\hat{\pi}(\theta, v) = \Sigma(\theta, v) - \pi(\theta, v)$. We restrict attention to the special case $s = 0$ to simplify $W^* = V^*$ and $\hat{W} = \hat{V}^*$.

Solution conditions. The necessary and sufficient solution conditions for optimal switching problems are provided by Brekke and Øksendal (1993). Sometimes referred to as the impulse control verification theorem, these conditions relate the solution of an impulse control problem to solving a system of quasi-variational inequalities for the optimal performance function which now is $\hat{V}^*$, as characterised by (16). In conjunction with solving for $V^*$ (and $\hat{V}^*$), also $z^*$ is found. We note that the impulse controls pertaining to the follower problem are of the double sequence form $\{(\tau_j); (z_j)\}$ where the $\tau_j$ are shock adapted stopping times with $\tau_j \geq \tau_{j+1}$ (a.s.) and the $z_j$ are control actions taking values from $\{0, 1\}$. Assuming that the follower starts in the suspension mode at time $t = 0$, all odd controls equal unity while all even controls equal zero. Hence, the optimal impulse control associated with $V^*$ is of the form $\{(\tau^*_j); (z^*_j)\}$ where the $\tau^*_j$ are optimal stopping times to be determined subject to $(z^*_j) = (1, 0, ..., 1, 0, 1, ...)$. For distinguishing between the operating modes, we below define the participating functions in $(x, z)$-space. We also define a switching operator $\mathcal{M}$ as

$$\mathcal{M}f(x, z) = f(x, 1 - z).$$
Suppressing the arguments, the quasi-variational system for $V^* = V^*(x, z)$ may then be written as

\begin{align*}
rV^* & \geq \mathcal{A}^* V^* + \pi x + z(\lambda \Delta^* - c) \quad (18a) \\
V^* & \geq \mathcal{M}V^* \quad (18b) \\
(\mathcal{A}^* V^* - rV^* + \pi x + z(\lambda \Delta^* - c))(V^* - \mathcal{M}V^*) &= 0. \quad (18c)
\end{align*}

The complementary equation (18c) imposes necessary continuation and switching conditions as follows. When continuing in the current operating mode is optimal ($z^* = z^*(x) = z$), (18a) holds with equality, as a Bellman equation. When switching out of the current operating mode is optimal ($z^* = 1 - z$), (18b) holds with equality instead. Then also

$$\mathcal{A}^*(\mathcal{M}V^*) - r\mathcal{M}V^* + (1 - z)(\lambda \mathcal{M}\Delta^* - c) = 0.$$  

The suspension region for the shock is defined as

$$C_0 = \{x \geq 0 : z^*(x) = 0\},$$

and the R&D region by

$$C_1 = \{x \geq 0 : z^*(x) = 1\}.$$  

Hence, the continuation region is defined in $(x, z)$-space by

$$C = \{(x, z) : x \in C_0 \text{ and } z^*(x) = 0 \text{ or } x \in C_1 \text{ and } z^*(x) = 1\}.$$  

The fact that outside $C$ (18a) obtains concomitantly with $V^* = \mathcal{M}V^*$ and (19) implies

$$z(\lambda \mathcal{M}\Delta^* - c) \leq (1 - z)(\lambda \mathcal{M}\Delta^* - c).$$

Hence,

$$(x, z) = (x, 0) \notin C \Rightarrow \lambda \mathcal{M}\Delta^* \geq c,$$  

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and

\[(x, z) = (x, 1) \notin \mathcal{C} \Rightarrow \lambda \mathcal{M} \Delta^* \leq c. \tag{23}\]

The economic intuition for these conditions is simply as follows. For a switch out of the suspension mode to be optimal, it is by (22) necessary that the value contribution of R&D be non-negative: the additional rate of capital gain \(\lambda \mathcal{M} \Delta^*\) created by the possibility of leapfrogging the leader in the imminent future must be at least as large as the flow cost of R&D. For a switch out of the R&D mode to be optimal, it is by (23) necessary that the value contribution be non-positive. More to the point, suppose that \((x, 1)\) is outside a given \(\mathcal{C}\) and we find that \(\lambda \mathcal{M} \Delta \geq c\). Since by \(V^* = \mathcal{M} V^*\) also \(\lambda \Delta^* \geq c\), we arrive at a contradiction indicating that \(\mathcal{C}\) and the values do not conform to the optimum. Most crucially, in the absence of hysteresis the sufficiency implications are operative as well, so (22) and (23) can be extended to equivalences. The indifference/optimality condition brought about in Remark 1 then collapses to

\[\Delta^*(x) = \frac{c}{\lambda}. \tag{24}\]

This is associated with

\[\mathcal{C}_1 = \left\{x \geq 0 : \Delta^*(x) \geq \frac{c}{\lambda}\right\},\]

and \(\mathcal{C}_0 = R_+ - \mathcal{C}_1\).

Working inside \(\mathcal{C}\) now breaks (16) and (17) into two systems of Bellman equations. First,

\[\mathcal{A}_x V^* + \pi x = r V^*, \ (x, z) = (x, 0) \in \mathcal{C} \tag{25}\]

and

\[\mathcal{A}_x \tilde{V}^* + \hat{\pi} x = r \tilde{V}^*, \ (x, z) = (x, 0) \in \mathcal{C}. \tag{26}\]

Second,

\[\mathcal{A}_x V^* + \pi x + (\lambda \Delta^* - c) = r V^*, \ (x, z) = (x, 1) \in \mathcal{C} \tag{27}\]

and

\[\mathcal{A}_x \tilde{V}^* + \hat{\pi} x - \lambda \Delta^* = r \tilde{V}^*, \ (x, z) = (x, 1) \in \mathcal{C}. \tag{28}\]
Representing absence of arbitrage conditions, the Bellman equations state that the total rate of return on a firm must equal the risk free rate under risk neutrality. It is also worth noting that the system (27)-(28) is recursive through $\Delta^\ast$. As the two systems and the $C_z$ provide only a general characterisation of the solution, additional structure in the form of optimality and regularity conditions on the values must be devised in order to close the model. This is done in what is to follow.

3.2 The solution

The flow of product market profits represents a real underlying asset, as opposed to a financial asset such as a share of stock. In the well established taxonomy of Trigeorgis (1995, pp. 3-4) then, the switching options are (compound American-style) real options to alter the operating scale. Correspondingly, the yet unknown $V^\ast$ and $\hat{V}^\ast$ represent real options values. Classical contributions in the same realm include the temporary suspension model of McDonald and Siegel (1985), the natural resource investment model of Brennan and Schwartz (1985), the capacity choice model of Pindyck (1988) and the time-to-build model of Majd and Pindyck (1987).

The general insight of real options analysis is that the discounted cash flow rules of investment tend to misallocate capital by underestimating or totally ignoring the value of operational flexibility. Pindyck (1991) provides a prototype example by studying the optimal timing of an irreversible investment under idiosyncratic payoff uncertainty. The key finding is that the net present value (NPV) rule leads to a suboptimally early investment by ignoring the value of a delaying option as an additional opportunity cost. Along the same lines, the NPV rule will now be suboptimal since it ignores the switching options. While the shock diffuses inside $C_z$ the follower keeps the switching option alive in the fashion of a delaying option. The moment the shock diffuses from $C_1$ into $C_{1-z}$ the follower exercises this option and activates an option to make the opposite switch the moment the shock diffuses back from $C_{1-z}$ into $C_z$. Furthermore, we draw on the investment lag model of Bar-Ilan and Strange (1996) to argue that the optimal R&D rule also accounts for an opportunity cost of delaying specific to the role of the suspension option in the presence of technological uncertainty, as manifested by the (expected) lag between engaging in R&D and leapfrogging the leader. Namely, the suspension option creates an asymmetry between the upside and downside risk associated with R&D: while undertaking R&D the follower (i) taps into the upside risk on the incremental product market profits accrued over the period of technological leadership (referred to as the leapfrogging rent) while (ii) at the same time being protected from the downside by the suspension option. Put differently, the follower holds operational flexibility to abandon the current R&D undertaking by switching to the suspension mode in the event that the expected return on the R&D investments becomes too small relative the expected R&D cost. Running counter to the
value of delaying, the asymmetry contributes an inducement for R&D in the form of an additional opportunity cost of delaying.

We develop the analysis in three stages. As the point of departure, the firm NPVs and the NPV rule are found in the first stage. The NPVs are employed in the second stage to find the optimal R&D rule in the absence of product market uncertainty, with $\sigma = 0$. In the third stage we finally provide the real options solution under product market uncertainty. Having done that, we generalise the analysis by allowing for R&D cost uncertainty also.

**NPV rule.** Let $v_z$ denote the follower NPV and $\hat{v}_z$ denote the leader NPV in operating mode $z$. The particular solutions of the Bellman systems (25)-(26) and (27)-(28) then yield

$$v_z(x) = \frac{r - \mu + z\lambda \pi + z\lambda \hat{\pi}}{r - \mu + 2z\lambda} \frac{x}{r - \mu} - \frac{r + z\lambda c}{r + 2z\lambda r},$$

and

$$\hat{v}_z(x) = \frac{r - \mu + z\lambda \pi + z\lambda \pi}{r - \mu + 2z\lambda} \frac{x}{r - \mu} - \frac{z\lambda c}{r + 2z\lambda r} > v_z(x).$$

Now let $x_0$ denote the critical shock threshold at which the $v_z$ are equal. Then

$$\frac{\lambda}{r - \mu + 2\lambda} \frac{\Delta x_0}{r - \mu} - \frac{r + \lambda c}{r + 2\lambda r} = 0,$$

which implies

$$x_0 = \frac{r + \lambda}{r + 2\lambda} \frac{(r - \mu + 2\lambda) r - \mu c}{\Delta r}.$$  \hfill (29)

The NPV rule calls for undertaking R&D when the shock lies in $(x_0, \infty]$ and for suspending R&D when the shock lies in $(0, x_0)$.

**Optimal management without uncertainty.** Let us assume $\sigma = 0$ so that

$$dx_t = \mu x_t dt,$$

and $A_x = \mu x$. Since the shock then grows exponentially at rate $\mu > 0$, it is natural to assume there exists a unique threshold $x^*_0$ distinct from $x_0$ such that a switch from the suspension mode to the R&D mode is a one shot occurrence optimally taking place at the deterministic first passage time.
\[ \tau_0^* = \inf \{ t \geq 0 : x_t \geq x_0^* \}. \]

It is a standard result that

\[
\tau_0^* = \begin{cases} \ln \left( \frac{x_0^*}{x_t^*} \right)^{\frac{1}{\lambda}}, & x < x_0^* \\ 0, & x \geq x_0^* \end{cases}
\]

(30)

Hence, the follower value in the suspension mode is given by

\[
\int_{0}^{\tau_0^*} e^{(\mu-r)t} \pi x dt + e^{-r \tau^*(x)} v_1(x^*) =
\]

\[
\left( \int_{0}^{\tau_0^*} e^{(\mu-r)t} dt - \int_{\ln(x/x_0^*)}^{\infty} \frac{e^{(\mu-r)t}}{\pi} dt \right) \pi x +
\]

\[
e^{-\ln \left( \frac{x_0^*}{x_t^*} \right)} \left( \frac{(r-\mu+\lambda)\pi+\lambda \pi}{r-\mu+2\lambda} \frac{x_0^*}{r-\mu} - \frac{r+\lambda c}{r+2\lambda r} \right),
\]

where \( \beta = \frac{r}{\pi} > 1 \). The integral terms account for the profits up until switching to the R&D mode. With

\[
e^{\beta \ln \left( \frac{x_0^*}{x_t^*} \right)} = \left( \frac{x}{x_0^*} \right)^{\beta} < 1
\]

as a deterministic riskfree discount factor, the remaining term accounts for the follower value upon the switch. Since the operational flexibility to switch back to the suspension mode is redundant, this value is given by the R&D mode NPV \( v_1(x^*) \). Since also

\[
\int_{0}^{\infty} e^{(\mu-r)t} dt - \int_{-\frac{c}{\pi \ln(x/x_0^*)}}^{\infty} e^{(\mu-r)s} ds = \frac{1}{r-\mu} - \left( \frac{x}{x_0^*} \right)^{\beta-1} \frac{1}{r-\mu},
\]

we find that the follower value is given by
Similarly, the leader value is given by

\[ V^*(x, z) = \begin{cases} \\
\frac{\pi x}{r - \mu} + \left( \frac{x}{z^0} \right) \beta \left( \frac{\pi x^0}{r - \mu} + \left( \frac{r - \mu + \lambda}{r - \mu + 2\lambda} \right) \frac{x^0}{r - \mu} - \frac{\lambda + \lambda \tilde{c}}{r + 2\lambda} \right), & x < x_0^* + z = 0 \\
\mathcal{M}V^*(x, z) & x < x_0^*, z = 1 \\
\frac{(r - \mu + \lambda) + \lambda \pi}{r - \mu + 2\lambda} - \frac{x}{r - \mu} = \frac{\lambda + \lambda \tilde{c}}{r + 2\lambda}, & x \geq x_0^*, z = 1
\end{cases} \]

By construction, both \( V^* \) and \( \tilde{V}^* \) are continuous around \( x^* \) so there are no abrupt value changes upon switching. Subtracting yields

\[ \Delta^*(x, z) = \begin{cases} \\
\frac{\Delta x}{r - \mu} + \left( \frac{x}{z^0} \right) \beta \left( \frac{-\Delta x^0}{r - \mu} + \frac{\Delta x^0}{r - \mu + 2\lambda} + \frac{c}{r + 2\lambda} \right), & x < x_0^* + z = 0 \\
\mathcal{M}\Delta(x, z) & x < x_0^*, z = 1 \\
\frac{\Delta x}{r - \mu + 2\lambda} + \frac{\mu}{r + 2\lambda}, & x \geq x_0^*, z = 1
\end{cases} \]

Hence, the indifference/optimality condition (24) becomes

\[ \frac{\Delta x^0}{r - \mu + 2\lambda} + \frac{c}{r + 2\lambda} = \frac{c}{\lambda} \]

which implies

\[ x_0^* = \frac{r + \lambda}{\lambda} \frac{r - \mu + 2\lambda}{\Delta} \]

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By use of $x_0^*$ we can write

$$V^*(x, z) = \begin{cases} \frac{\dot{x}}{r - \mu} + V^+(x), & x < x_0^*, z = 0 \\ MV^*(x, z) & x < x_0^*, z = 1 \\ \frac{(r - \mu + \lambda \pi + \lambda \hat{\pi}) x}{r - \mu} - \frac{r + \lambda c}{r + 2 \lambda r}, & x \geq x_0^*, z = 1 \end{cases} \quad (36)$$

and

$$\hat{V}^*(x, z) = \begin{cases} \frac{\dot{x}}{x_0^*} + \hat{V}^-(x), & x < x_0^*, z = 0 \\ MV^{\hat*}(x, z) & x < x_0^*, z = 1 \\ \frac{(r - \mu + \lambda \pi + \lambda \hat{\pi}) x}{r - \mu} - \frac{r + \lambda c}{r + 2 \lambda r}, & x \geq x_0^*, z = 1 \end{cases} \quad (37)$$

where

$$V^+(x) = \left(\frac{x}{x_0^*}\right)^\beta \frac{1}{\beta} \frac{\lambda}{r - \mu + 2 \lambda r - \mu} \Delta x_0^* \geq 0,$$

and

$$\hat{V}^-(x) = -\left(\frac{x}{x_0^*}\right)^\beta \left(\frac{\Delta x_0^*}{r - \mu} - \frac{c}{\lambda}\right) + V^+(x) \leq 0.$$ 

Moreover, it is straightforward to verify that (34) equivalent to an explicit first-order condition.

**Remark 2** Define a threshold $\bar{x}$ and for $x < \bar{x}$ the function

$$G(x, \bar{x}) = \left(\int_0^\infty e^{(\mu - r)t} dt - \int_{\frac{1}{\mu} \ln(\frac{1}{\nu})}^\infty e^{(\mu - r)t} dt \right) \pi x +$$

$$\left(\frac{x}{\bar{x}}\right)^\beta \left(\frac{(r - \mu + \lambda \pi + \lambda \hat{\pi}) \bar{x}}{r - \mu + 2 \lambda r - \mu} - \frac{r + \lambda c}{r + 2 \lambda r}\right).$$

By definition then
\[ V^*(x) = \sup_{\bar{x}} G(x, \bar{x}). \]

The first order condition \( \frac{dV}{dx} = 0 \) (for all \( x < \bar{x} \)) yields

\[ \bar{x}^* = \frac{\beta}{\beta - 1} x_0 = \frac{\beta}{\beta - 1} \frac{r + \lambda}{r + 2\lambda} (r - \mu + 2\lambda) \frac{r - \mu - c}{\Delta}, \tag{38} \]

which satisfies the second order condition \( \frac{d^2V}{dx^2}(\bar{x}^*) \leq 0 \) as well. Since

\[ \frac{\beta}{\beta - 1} = \frac{r}{r - \mu}, \]

we also have \( x_0^* = \bar{x}^* \) which shows that (34) is equivalent to explicitly maximising the follower value with respect to the switching trigger. The maximisation (and hence (34)) is also equivalent to the high contact condition of dynamic programming that the follower value be smooth around the switching trigger.

From (38) we notice that \( x_0^* \) is increased above the NPV threshold by a markup in \( \beta \). This markup accounts for \( V^+ \), the option premium (value of delaying) associated with the optimal one shot switch at time \( \tau_0^* \). The presence of \( V^+ \) renders the follower value convex in the shock for \( x < x_0^* \). Ordinary differentiation further verifies that the follower value is smooth around \( x_0^* \) as required by dynamic programming optimality considerations. The shape exhibited by the leader value emerges frequently in dynamic oligopoly models for firms operating under the threat that a one shot investment made a competitor triggers a discontinuous drop in the expected future profits. Such models include, for example, Smets (1991) and Weeds (2002). Firstly, for \( x < x_0^* \) the leader value is rendered concave by \( V^+ \), the value loss caused by the threat of switching. More specifically, the leader value increases less than linearly with the shock since the positive effect on the NPV is counteracted by the fact that switching and the associated discontinuous drop in the NPV become more imminent as the shock draws closer to \( x_0^* \). Secondly, the leader value is kinked around \( x_0^* \) since under one shot switching \( x_0^* \) acts as a first hitting boundary instead of a transitional boundary in the sense of Dixit (1993, pp. 212-213). The two values are illustrated by Figure 1.

Optimal management under uncertainty. Let us next extend the analysis by allowing for product market uncertainty through the assumption \( 0 < \sigma < \sigma_{\text{max}} \). Extending from the above, we facilitate the analysis with the following basic notions:
1. Based on the absence of hysteresis and the positive persistence of uncertainty exhibited by the shock (e.g. Dixit and Pindyck 1994, pp. 128-129), we assume/guess there exists a unique switching trigger $x^*$ distinct from $x_0^*$ that defines $C_0$ and $C_1$ as non-overlapping connected intervals $C_0 = [0, x^*)$ and $C_1 = [x^*, \infty)$. Moreover, $x^*$ is of the multiplicative form

$$x^* = \omega(\lambda) \frac{e}{\Delta}.$$ 

By use of an indicator function the optimal R&D rule may be defined as $z^*(x) = 1_{x \geq x^*}(x)$.

2. In both operating modes the follower value $V^*$ is increased above the NPV by a convex option premium. We denote the premium in mode $z$ by $V_z^+$. The interaction between the two premia is accounted for by $\omega(\lambda)$. The relationship between $\omega(\lambda)$ and the multiplier $\frac{r+\lambda}{r+2\lambda}(r - \mu + 2\lambda)$ in $x_0^*$ is a priori unclear. Particularly, it is possible that the irreversible investment prediction of uncertainty delaying investment is turned over by option interaction, or the interaction between the value of delaying and the opportunity cost of delaying.

3. In the suspension mode the leader value $\bar{V}^*$ is decreased below the NPV by a value loss caused be the threat of the follower switching to the R&D mode. Conversely, in the R&D mode $\bar{V}^*$ is increased above the NPV by a value gain created by the possibility of the follower switching to the suspension mode.

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1. That is, a higher current shock value shifts the cumulative probability distribution of future values (and thereby the distribution of future leapfrogging rents) uniformly to the right. This property is also referred to as first order stochastic dominance (e.g. Sjodal 1997).
4. Since the shock diffuses stochastically, $x^*$ acts as a transitional boundary (reversible switching point) instead of a first hitting boundary.

We attack the problem of finding the triplet $V^*, \hat{V}^*, x^*$ by applying an indirect solution strategy as follows. First, letting

$$\Sigma^*(x) = V^*(x) + \hat{V}^*(x),$$

and noting that

$$V^*(x) = \frac{\Sigma^*(x) - \Delta^*(x)}{2}, \quad (39)$$

and

$$\hat{V}^*(x) = \frac{\Sigma^*(x) + \Delta^*(x)}{2}, \quad (40)$$

we replace the additional optimality and regularity conditions on $V^*$ and $\hat{V}^*$ with their equivalents on $\Delta^*$ and $\Sigma^*$. With the triplet $\Delta^*, \Sigma^*, x^*$ then found, we solve for $V^*$ and $\hat{V}^*$ by direct substitution. This strategy is instrumental as it gets rid off mathematical clutter and enables us to employ the indifference/optimality condition $\Delta^*(x^*) = \frac{1}{x}$ as an integral part of the analysis.

From the Bellman system (25)-(28), we find that $\Sigma^*$ is of the form

$$\Sigma^*(x, z) = \begin{cases} 
\Sigma_0^*(x), & z = 0, x < x^* \\
M \Sigma^*(x, z), & z = 1, x < x^* \text{ or } z = 0, x \geq x^* \\
\Sigma_1^*(x), & z = 1, x \geq x^*, 
\end{cases} \quad (41)$$

where the $\Sigma_2^*$ satisfy

$$A_2 \Sigma_2^* - r \Sigma_2^* + \Sigma x - ze = 0. \quad (42)$$

Hence,

$$\Sigma_0^*(x) = a_1 x^{\beta^+} + a_2 x^{\beta^-} + \frac{\Sigma x}{r - \mu}, \quad (43)$$

and
\[ \Sigma^*_i(x) = b_i x^{\beta^+} + b_2 x^{\beta^-} + \frac{\Sigma x}{r - \mu} - \frac{c}{r}, \tag{44} \]

where \( \beta^+ \) is the positive root and \( \beta^- \) is the negative root of the characteristic polynomial \( \sigma^2 \beta (\beta - 1) + 2 \mu \beta - 2 r = 0 \):

\[ \beta^\pm = \left( \frac{1}{2} - \frac{\mu}{\sigma^2} \right) \pm \sqrt{\left( \frac{1}{2} - \frac{\mu}{\sigma^2} \right)^2 + 2 \frac{r}{\sigma^2}}. \]

Similarly, \( \Delta^* \) is of the form

\[ \Delta^*(x, z) = \begin{cases} 
\Delta_0^*(x), & z = 0, x < x^* \\
\mathcal{M} \Delta^*(x, z), & z = 1, x < x^* \text{ or } z = 0, x \geq x^* \\
\Delta_1^*(x), & z = 1, x \geq x^*,
\end{cases} \tag{45} \]

where the \( \Delta_0^* \) satisfy the Bellman equations

\[ A_x \Delta_0^* - (r + 2 z \lambda) \Delta_0^* + \Delta x + z c = 0. \tag{46} \]

Hence,

\[ \Delta_0^*(x) = a_3 x^{\alpha^+(\lambda)} + a_4 x^{\alpha^-(\lambda)} + \frac{\Delta x}{r - \mu}, \tag{47} \]

and

\[ \Delta_1^*(x) = b_3 x^{\alpha^+(\lambda)} + b_4 x^{\alpha^-(\lambda)} + \frac{\Delta x}{r - \mu + 2 \lambda} + \frac{c}{r + 2 \lambda}, \tag{48} \]

where \( \alpha^+(\lambda) \) is the positive root and \( \alpha^-(\lambda) \) is the negative root of the characteristic polynomial \( \sigma^2 \alpha (\alpha - 1) + 2 \mu \alpha - 2 (r + 2 \lambda) = 0 \):

\[ \alpha^\pm(\lambda) = \left( \frac{1}{2} - \frac{\mu}{\sigma^2} \right) \pm \sqrt{\left( \frac{1}{2} - \frac{\mu}{\sigma^2} \right)^2 + 2 \frac{r + 2 \lambda}{\sigma^2}}. \]

By direct inspection, \( \alpha^+(\lambda) > \beta^+ \) and \( \alpha^-(\lambda) < \beta^- \) for all \( \lambda > 0 \).

We now state the additional optimality and regularity conditions as follows:
1. The smoothness of values: both $V^*$ and $\tilde{V}^*$ must be continuously differentiable around $x^*$. Since this is equivalent to $\Sigma^*$ and $\Delta^*$ being continuously differentiable around $x^*$, we have the pair value matching conditions

$$\Sigma^*_z(x^*) = \Sigma^*_{1-z}(x^*), \quad (49)$$

and

$$\Delta^*_z(x^*) = \Delta^*_{1-z}(x^*), \quad (50)$$

in conjunction with the pair of high contact conditions

$$\frac{d\Sigma^*_z}{dx}(x^*) = \frac{d\Sigma^*_{1-z}}{dx}(x^*), \quad (51)$$

and

$$\frac{d\Delta^*_z}{dx}(x^*) = \frac{d\Delta^*_{1-z}}{dx}(x^*). \quad (52)$$

2. The values must vanish with the shock: $V^* \to 0$ and $\tilde{V}^* \to 0$ as $x \to 0$. From (43) and (47), this is equivalent to the regularity conditions

$$a_1 x^{\beta^+} + a_2 x^{\beta^-} \to 0, \quad (53)$$

and

$$a_3 x^{\beta^+} + a_4 x^{\beta^-} \to 0, \quad (54)$$

as $x \to 0$. In addition, both $V^*$ and $\tilde{V}^*$ must satisfy a no bubbles condition ruling out explosive growth as $x \to +\infty$. From (44) and (48), this is equivalent to the regularity conditions

$$b_1 x^{\alpha^+} + b_2 x^{\alpha^-} \to 0, \quad (55)$$

and

$$b_3 x^{\alpha^+}(\lambda) + b_4 x^{\alpha^-}(\lambda) \to 0, \quad (56)$$

as $x \to \infty$. 

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3. Bang-bang conditions: $V^*$ and $\tilde{V}^*$ must satisfy the indifference/optimality condition $\Delta^*(x^*) = \frac{c}{\lambda}$ in conjunction with

$$\Delta^*(x) > \frac{c}{\lambda},$$  \hspace{1cm} (57)

for all $x > x^*$ and

$$\Delta^*(x) < \frac{c}{\lambda},$$  \hspace{1cm} (58)

for all $x < x^*$.

The bang-bang conditions (57) and (58) cannot be a priori imposed, nor are they implied by the other conditions. Hence, with $\Delta^*, \Sigma^*, x^*$ found, we must verify whether $\Delta^*$ is well behaving such that (57) and (58) are indeed satisfied. Should $\Delta^*$ violate (57) by dropping below $\frac{c}{\lambda}$ somewhere above $x^*$ or violate (58) by raising above $\frac{c}{\lambda}$ somewhere below $x^*$, the solution hand at hand makes no economic sense whatsoever and we conclude that no solution for the problem exists. The smoothness of the follower value around $x^*$ is due to the condition that an optimal performance function be stochastically $C^2$ in the continuation domain with respect to the driving Itô diffusion (eg. Vollert, 2003, pp. 67-68). Basically, an optimal performance function on an Itô diffusion must be $C^2$ inside the continuation domain (now in the interiors of the $C_x$) and smooth on the boundary of this domain, a set of zero measure (now the point $x^*$). The economic intuition for the smoothness can be derived from heuristic arbitrage considerations or from optimality considerations ruling out value function kinks around optimal investment thresholds, as demonstrated by Dixit and Pindyck (1994, pp. 130-132). The smoothness of the leader value, on the other hand, is due to the condition that value functions be smooth around transitional boundaries on which the parameters of the driving process and/or the flow payoff function experience a change (Dixit, 1993, pp. 30-31). This is easiest verified by inspecting the Bellman equations for $\Sigma^*$, as condensed by (42). Since the (net) flow payoff switching between $\Sigma x$ and $\Sigma x - c$ is the only change experienced by these equations at $x^*$, it is immediate that $\Sigma^*$ is smooth around $x^*$. Since also $\tilde{V}^* = \Sigma^* - V^*$ and $V^*$ is smooth around $x^*$, we have that $\tilde{V}^*$ is smooth around $x^*$ as well. The regularity conditions (53) and (54) are due to the absorbing barrier property of $x = 0$: should the shock ever hit zero, it will stay there forever, thereby rendering the product market and the firms worthless. The no bubble conditions (55) and (55) imply that the suspension option premium $V_{1}^{+}$ and the value gain $\tilde{V}_{1}^{+}$ vanish as suspension becomes an extremely remote possibility:

$$V_{1}^{+}(x) \to 0,$$
and

$$V_1^+(x) \to 0,$$

as \( x \to +\infty. \)

By (53)-(56), the coefficients \( a_2, a_4, b_1 \) and \( b_3 \) all equal zero. An application of (50) together with \( \Delta^+(x^*) = \frac{c}{\Delta} \) then yields

$$\Delta_0^+(x) = \frac{\Delta x}{r - \mu} + \left( \frac{x}{x^*} \right)^{\beta^+} \left( -\frac{\Delta x^*}{r - \mu} + c \right),$$

and

$$\Delta_1^+(x) = \frac{\Delta x}{r - \mu + 2\lambda} + \frac{c}{r + 2\lambda} + \left( \frac{x}{x^*} \right)^{\alpha^{-}(\lambda)} \left( -\frac{\Delta x^*}{r - \mu + 2\lambda} - \frac{c}{r + 2\lambda} + \frac{c}{\lambda} \right).$$

Similarly, an application of (49) yields

$$\Sigma_0^+(x) = \frac{\Sigma x}{r - \mu} + \left( \frac{x}{x^*} \right)^{\beta^+} \left( -\frac{\Sigma x^*}{r - \mu} + \Sigma^{**} \right),$$

and

$$\Sigma_1^+(x) = \frac{\Sigma x}{r - \mu} - \frac{c}{\beta^+} + \left( \frac{x}{x^*} \right)^{\beta^-} \left( -\frac{\Sigma x^*}{r - \mu} + \frac{c}{\beta^+} + \Sigma^{**} \right),$$

where \( \Sigma^{**} \) is the total value upon switching, \( \Sigma^+(x^*) \). Solving (51) for it yields

$$\Sigma^{**} = \frac{\Sigma x^*}{r - \mu} + \frac{\beta^- c}{\beta^+ - \beta^-} \frac{c}{r}.$$

Solving (52) for \( x^* \) finally yields the proposed multiplicative form

$$x^* = \omega(\lambda) \frac{c}{\Delta}, \quad (59)$$

with
\[ \omega(\lambda) = \frac{\alpha(\lambda) \frac{r+\lambda}{r+2\lambda} - \beta}{(\alpha(\lambda) - 1) \frac{r-\mu}{r-\mu+2\lambda} - (\beta - 1)} \frac{r-\mu}{\lambda}, \]

where we have abbreviated \( \alpha(\lambda) = \alpha^- (\lambda) \) and \( \beta = \beta^+. \) From \( \beta > 1 \) it follows that \( \omega(\lambda) > 0 \) for all parameter configurations. Since also \( \alpha(\lambda) \to -\infty \) and \( \beta \to \frac{\lambda}{\mu} \) as \( \sigma \to 0 \), an application of the l'Hôpital rule yields

\[ \omega(\lambda) \to \frac{r+\lambda}{r+2\lambda} \frac{r-\mu+2\lambda}{\lambda}, \]

as \( \sigma \to 0 \). This verifies that \( x^*_0 \) arises as a limiting case of \( x^* \).

Substituting the obtained \( \Sigma^*_x \) and \( \Delta^*_x \) into (41) and (45) yields

\[
\Sigma^*(x, z) = \begin{cases} 
\frac{\Sigma_x}{r-\mu} + \left( \frac{\Sigma_x}{r} \right)^\beta \left( -\frac{\Sigma_x}{r-\mu} + \frac{\beta^+}{\beta^-} \frac{x}{r} \right), & z = 0, x < x^* \\
\mathcal{M} \Sigma^*(x, z), & z = 1, x < x^* \\
\frac{\Sigma_x}{r-\mu} - \frac{x}{r} + \left( \frac{\Sigma_x}{r} \right)^\beta \left( -\frac{\Sigma_x}{r-\mu} + \frac{\beta^+}{\beta^-} \frac{x}{r} \right), & z = 0, x \geq x^* \\
\mathcal{M} \Sigma^*(x, z), & z = 1, x \geq x^* 
\end{cases}
\]  

and

\[
\Delta^*(x, z) = \begin{cases} 
\frac{\Delta_x}{r-\mu} + \left( \frac{\Delta_x}{r} \right)^\beta \left( -\frac{\Delta_x}{r-\mu} + \frac{x}{r} \right), & z = 0, x < x^* \\
\mathcal{M} \Delta^*(x, z), & z = 1, x < x^* \\
\frac{\Delta_x}{r-\mu+2\lambda} + \frac{x}{r+2\lambda} + \left( \frac{\Delta_x}{r} \right)^\beta \left( -\frac{\Delta_x}{r-\mu+2\lambda} + \frac{\beta^+}{\beta^-} \frac{x}{r+2\lambda} \right), & z = 1, x \geq x^*. 
\end{cases}
\]  

Let us now turn to the bang bang conditions (57) and (58). We find from (61) that the properties of \( \Delta \) allow a lower boundary restriction for \( \omega(\lambda) \) under which these conditions are concomitantly satisfied. This result is stated by the following lemma.

**Lemma 1** For \( \Delta^* \) to be well behaving such that the bang bang conditions (57) and (58) are concomitantly satisfied, it is sufficient and necessary that \( \Delta^* \) has a non-negative slope around \( x^* \). The condition for the non-negative slope is

\[ \omega(\lambda) \geq \omega_{\min}(\lambda), \]
where
\[ \omega_{\text{min}}(\lambda) = \frac{\alpha(\lambda)}{\alpha(\lambda) - 1} \frac{r + \lambda}{r + 2\lambda}(r + \mu + 2\lambda). \]

**Proof.** Differentiating \( \Delta^* = \Delta^+_\lambda \) with respect to \( x \) yields
\[
\frac{d\Delta^*}{dx} = \frac{\Delta}{r - \mu + 2\lambda} + \alpha(\lambda) \left( \frac{x}{x^*} \right)^{\alpha(\lambda)} \left( -\frac{\Delta}{r - \mu + 2\lambda} \frac{x^*}{x} + \frac{r + \lambda}{r + 2\lambda} \frac{c}{x} \right). \tag{62}
\]
Evaluating this at \( x^* \) we find that
\[
\frac{d\Delta^*}{dx}(x^*) \geq 0 \iff \omega(\lambda) \geq \omega_{\text{min}}(\lambda),
\]
which is the proposed non-negative slope condition. Assume that this condition holds. Then
\[
-\frac{\Delta x^*}{r - \mu + 2\lambda} + \frac{r + \lambda}{r + 2\lambda} \frac{c}{x} \geq -\frac{1}{\alpha(\lambda)} \frac{\Delta x^*}{r - \mu + 2\lambda},
\]
which on substitution into (62) yields
\[
\frac{d\Delta^*}{dx} \geq \frac{\Delta}{r - \mu + 2\lambda} - \left( \frac{x}{x^*} \right)^{\alpha(\lambda) - 1} \frac{\Delta}{r - \mu + 2\lambda} > 0,
\]
for all \( x > x^* \). We have shown that a non-negative slope in the right neighbourhood of \( x^* \) implies a positive slope for all \( x > x^* \). This proves the sufficiency claim for (57). In order to prove the sufficiency claim for (58), we note that by smoothness \( \omega(\lambda) \geq \omega_{\text{min}}(\lambda) \) concomitantly implies a non-negative slope in the left neighbourhood of \( x^* \) and that this (from differentiating \( \Delta^+_\lambda \)) implies a positive slope for all \( x < x^* \). The necessity claims being trivial, we are done. □

Lemma 1 establishes \( \omega(\lambda) \geq \omega_{\text{min}}(\lambda) \) as a condition under which the obtained triplet \( \Delta^*, \Sigma^*, x^* \) constitutes a unique solution to the problem. We denote the implied part of the \( (\sigma, r, \mu, \lambda) \)-space by \( \mathcal{W} \). Outside \( \mathcal{W} \) it is not possible to enforce all the valuation conditions associated with the assumption/guess that the \( \mathcal{C}_\zeta \) are non-overlapping connected intervals separated by a unique switching trigger. Naturally, the question arises whether alternative solutions can be found outside \( \mathcal{W} \) by departing from a different assumption. Overlapping is ruled out by the absence of hysteresis. The possibility of the \( \mathcal{C}_\zeta \) being formed by disconnected
intervals is ruled out by the positive persistence of uncertainty exhibited by the shock. Specifically, since the expected return on the R&D investments increases monotonously with the current shock value, at optimum it cannot be the case that the follower starts R&D for some shock value and then goes ahead to suspend R&D for a larger value. All in all, we conclude that outside $W$ there exists no solutions and particularly that the only solution is the one obtained, inside $W$. In this light, Lemma 1 can be seen as an 'existence result' in the presence of product market uncertainty. Figures 2 and 3 illustrate $\Delta^*$ inside and outside $W$.

Noting from the respective characteristic polynomials that

$$\alpha(\lambda) - 1 = \frac{(r + 2\lambda) - \mu \alpha(\lambda)}{\frac{1}{2} \sigma^2 \alpha(\lambda)},$$

and

$$\beta - 1 = \frac{r - \mu \beta}{\frac{1}{2} \sigma^2 \beta},$$

the definition of $W$ expands to read

$$W = \left\{ (\sigma, r, \mu, \lambda) : \alpha^2(\lambda) \frac{(r + \lambda) - \mu \alpha(\lambda)}{r + 2\lambda} \frac{r - \mu + 2\lambda}{r - \mu \alpha(\lambda) + 2\lambda} \leq \beta^2 \frac{r - \mu}{r - \mu \beta} \right\}. \quad (63)$$

The $\beta$-term is positive since $\beta < \frac{r}{\mu}$ for all $\sigma > 0$. Moreover, since we can show that the $\alpha(\lambda)$-term increases monotonously in $\lambda$, the definition can be condensed as

$$\lambda \leq \lambda_{\text{max}},$$

with $\lambda_{\text{max}}$ implicitly solving

$$\alpha^2(\lambda_{\text{max}}) \frac{r + \lambda_{\text{max}}}{r + 2\lambda_{\text{max}}} \frac{r - \mu + 2\lambda_{\text{max}}}{r - \mu \alpha(\lambda_{\text{max}}) + 2\lambda_{\text{max}}} = \beta^2 \frac{r - \mu}{r - \mu \beta}. \quad (64)$$

Hence, an optimal R&D rule exists under product market uncertainty provided that the expected leapfrogging frequency in the R&D mode is sufficiently low. In valuation terms this means that there must be sufficient distinction between the leader and the follower. Since $\beta = \frac{r}{\mu}$ and $\alpha(\lambda) = \frac{r + 2\lambda}{\mu}$ for $\sigma = 0$, we also verify from (64) that a solution then exists for all $\lambda$. To get an idea of its magnitude, Table 2 displays $\lambda_{\text{max}}$ for selected levels of $\sigma$ when $(r, \mu) = (0.05, 0.03)$. 

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Equipped with \( W \), we finally summarise the main result of the analysis in the following proposition.

**Proposition 1** Assume that \((\sigma, r, \mu, \lambda) \in W\). The optimal R&D rule for a firm in the follower position is then

\[
W \Rightarrow \lambda^* = \lambda_{\text{max}}
\]
where the switching trigger $x^*$ is given by

$$x^* = \frac{\alpha(\lambda) \frac{r+\lambda}{r+2\lambda} - \beta}{(\alpha(\lambda) - 1) \frac{r+\mu}{r-\mu+2\lambda} - (\beta - 1) \frac{r-\mu}{2\lambda}} r - \mu c$$

Outside $W$ there does not exist an optimal rule.

Substituting (60) and (61) into (39) and (40), we find that the associated follower and leader values are given by

$$V^*(x, z) = \begin{cases} \frac{sx}{r-\mu} + \frac{1}{2} \left( \frac{x}{c} \right)^2 \left( \frac{r-\mu}{r-\mu+2\lambda} - \frac{s+x}{r-\mu+2\lambda} \right) + \frac{sx}{r-\mu} + \frac{1}{2} \left( \frac{x}{c} \right)^2 \left( \frac{r-\mu}{r-\mu+2\lambda} - \frac{s+x}{r-\mu+2\lambda} \right), & z = 0, x < x^* \\ \frac{(r-\mu+\lambda)x + \lambda x}{r-\mu+2\lambda} - \frac{sx}{r-\mu} - \frac{1}{2} \left( \frac{x}{c} \right)^2 \frac{r-\mu}{r-\mu+2\lambda} - \frac{s+x}{r-\mu+2\lambda} + \frac{1}{2} \left( \frac{x}{c} \right)^2 \frac{r-\mu}{r-\mu+2\lambda} - \frac{s+x}{r-\mu+2\lambda} & z = 1, x < x^* \text{ or } z = 0, x \geq x^* \end{cases}$$

and

$$\hat{V}^*(x, z) = \begin{cases} \frac{sx}{r-\mu} + \frac{1}{2} \left( \frac{x}{c} \right)^2 \left( \frac{r-\mu}{r-\mu+2\lambda} - \frac{s+x}{r-\mu+2\lambda} \right) + \frac{sx}{r-\mu} + \frac{1}{2} \left( \frac{x}{c} \right)^2 \left( \frac{r-\mu}{r-\mu+2\lambda} - \frac{s+x}{r-\mu+2\lambda} \right), & z = 0, x < x^* \\ \frac{(r-\mu+\lambda)x + \lambda x}{r-\mu+2\lambda} - \frac{sx}{r-\mu} - \frac{1}{2} \left( \frac{x}{c} \right)^2 \frac{r-\mu}{r-\mu+2\lambda} - \frac{s+x}{r-\mu+2\lambda} + \frac{1}{2} \left( \frac{x}{c} \right)^2 \frac{r-\mu}{r-\mu+2\lambda} - \frac{s+x}{r-\mu+2\lambda} & z = 1, x < x^* \text{ or } z = 0, x \geq x^* \end{cases}$$

The nonlinear components in $V^*$ represent the option premia $V_2^+$. The two values exhibit the same shapes (in $x$-space) as in Figure 1, with the one exception that $V^*$ is smooth by the transitional boundary argument. Since $x^*$ is linear in $c$, the values are also linear homogenous in $(x, c)$. Adopting the subscript notation in the fashion of $\Delta^*_2$ and $\Sigma^*_2$, we can hence write

$$V_z^*(x, c) = cv_z^* \left( \frac{x}{c} \right),$$

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and

$$V^*_x(x, c) = c \hat{\epsilon}^*_x \left( \frac{x}{c} \right),$$

where the $v^*_x$ and $\hat{\epsilon}^*_x$ are the functions to be determined. As suggested by the analysis of McDonald and Siegel (1986), linear homogeneity enables an extension towards R&D cost uncertainty (two dimensional uncertainty) such that a closed form solution for the optimal rule can still be obtained through a reduction of dimensionality (see also Dixit and Pindyck 1994, pp. 207-212).

**Extension: R&D cost uncertainty.** Let us suppose the flow R&D cost is driven by a geometric Brownian motion

$$dc_t = \tilde{\mu} c_t dt + \tilde{\sigma} c_t d\tilde{w}_t, \quad c_0 = c,$$

where $d\tilde{w}_t$ is the increment of another standard Brownian motion, the diffusion coefficient $\tilde{\sigma}$ measures the degree of volatility in the R&D input market $a_n$ and $\tilde{\mu}$ is the difference between riskfree rate of return and the equilibrium rate of return shortfall on the cost. Potential correlation between product market and R&D input market fluctuations is allowed for by a covariance coefficient $\rho \in [-1, 1]$. The expectation of $dw_t d\tilde{w}_t$ is correspondingly given by $\rho dt$.

The R&D and suspension regions are separated in $(c, x)$-space by a trigger curve along which the follower is indifferent between the two operating modes. By linear homogeneity, a free boundary problem is averted as this curve is defined by the ray

$$x^*(c) = y^* c,$$

where the slope $y^*$ is a positive scalar constituting a trigger level for the ratio process

$$y_t = \frac{x_t}{c_t}.$$  

When the shock is sufficiently large or/and the flow cost is sufficiently small such that the ratio exceeds $y^*$, it is optimal to continue in the R&D more or switch immediately out of the suspension mode. Formally, the R&D region reduces to one dimension as

$$C_1 = \{ y \geq y^* \},$$

and the optimal rule as $z^*(y) = 1_{y \geq y^*}(y)$. 

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Since the solution strategy for $y^*$ basically replicates that for $x^*$, we only highlight the main steps. First noticing that any twice differentiable function

$$G(x, c) = cg(y)$$

satisfies

$$\frac{dG}{dx} = \frac{dg}{dy} \frac{dG}{dc} = g - y \frac{dg}{dy},$$

and

$$\frac{d^2G}{dx^2} = \frac{1}{c} \frac{d^2g}{dy^2} \frac{dG}{dc} = \frac{y^2 d^2g}{c dy^2},$$

we can show that the counterpart of the Bellman system (25)-(26) reads as

$$\frac{1}{2} b(\rho) y^2 \frac{d^2v^*}{dy^2} + (\mu - \bar{\mu}) y \frac{dv^*}{dy} - (r - \bar{\mu}) v^* + \lambda (\dot{v}^* - v^*) + \pi y - 1 = 0, \quad (65)$$

and

$$\frac{1}{2} b(\rho) y^2 \frac{d^2\dot{v}^*}{dy^2} + (\mu - \bar{\mu}) y \frac{d\dot{v}^*}{dy} - (r - \bar{\mu}) \dot{v}^* - \lambda (\dot{v}^* - v^*) + \dot{\pi} y = 0, \quad (66)$$

where $b(\rho) = \sigma^2 - 2\rho \sigma \bar{\sigma} + \bar{\sigma}^2 > 0$. Similarly, the counterpart of (27)-(28) reads as

$$\frac{1}{2} b(\rho) y^2 \frac{d^2v^*}{dy^2} + (\mu - \bar{\mu}) y \frac{dv^*}{dy} - (r - \bar{\mu}) v^* + \pi y = 0, \quad (67)$$

and

$$\frac{1}{2} b(\rho) y^2 \frac{d^2\dot{v}^*}{dy^2} + (\mu - \bar{\mu}) y \frac{d\dot{v}^*}{dy} - (r - \bar{\mu}) \dot{v}^* + \dot{\pi} y = 0. \quad (68)$$

Now reusing notation as $\Delta^*(y) = \dot{v}^*(y) - v^*(y)$ and $\Sigma^*(y) = \dot{v}^*(y) + v^*(y)$, the indirect value matching conditions along $x^*(c)$ become

$$\Delta_z^*(y^*) = \Delta_{1-z}^*(y^*),$$

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and
\[ \Sigma^*_\delta(y^*) = \Sigma^*_{1-x}(y^*). \]

The high contact conditions with respect to \( x \) become
\[ \frac{d\Delta^*_\delta}{dy}(y^*) = \frac{d\Delta^*_{1-x}}{dy}(y^*), \]
and
\[ \frac{d\Sigma^*_\delta}{dy}(y^*) = \frac{d\Sigma^*_{1-x}}{dy}(y^*). \]

The additional high contact conditions with respect to \( c \) become
\[ \Delta^*_\delta(y^*) - y^* \frac{d\Delta^*_\delta}{dy}(y^*) = \Delta^*_{1-x}(y^*) - y^* \frac{d\Delta^*_{1-x}}{dy}(y^*), \]
and
\[ \Sigma^*_\delta(y^*) - y^* \frac{d\Sigma^*_\delta}{dy}(y^*) = \Sigma^*_{1-x}(y^*) - y^* \frac{d\Sigma^*_{1-x}}{dy}(y^*). \]

For both \( \Delta^* \) and \( \Sigma^* \) any of the three conditions can be dropped as an implication of the other two. Adding the bang bang condition \( \lambda \Delta^*(y^*) = 1 \), this leaves us with five conditions. Imposing also the four regularity conditions pertaining to the limiting cases \( y \to 0 \) and \( y \to \infty \) (so the number of unknowns is matched), we find that
\[ y^* = \frac{\zeta(\lambda) \frac{r-\mu+\lambda}{b(\rho)} - \eta}{(\zeta(\lambda) - 1) \frac{r-\mu}{b(\rho)} - (\eta - 1) \frac{1}{\lambda \Delta}}, \]
where
\[ \zeta(\lambda) = \left( \frac{1}{2} - \frac{\mu - \bar{\mu}}{b(\rho)} \right) - \sqrt{\left( \frac{1}{2} - \frac{\mu - \bar{\mu}}{b(\rho)} \right)^2 + \frac{2(r - \bar{\mu} + 2\lambda)}{b(\rho)}} < 0, \]
and
\[ \eta = \left( \frac{1}{2} - \frac{\mu - \bar{\mu}}{b(\rho)} \right) + \sqrt{\left( \frac{1}{2} - \frac{\mu - \bar{\mu}}{b(\rho)} \right)^2 + \frac{2(r - \bar{\mu})}{b(\rho)}} > 1, \]
as implied by the respective characteristic polynomials.
4 Comparative statics

We now gain further insight into the optimal R&D rule by investigating the comparative statics properties of \( x^* \) and \( y^* \). By direct inspection, it is clear that these triggers display the same properties with respect to the shared parameters \( \sigma, r, \mu \) and \( \lambda \). This noted, we focus primarily on the simpler \( x^* \).

**Product market uncertainty.** Given the asymmetry of the shock distribution, we isolate the effect of product market uncertainty on \( x^* \) by way of two distribution spreads that preserve distinct measures of central tendency: mean preserving spreads (in the spirit of Rothschild and Stiglitz, 1970) and median preserving spreads. Since \( E_x x_t = x e^{\mu t} \), a mean preserving spread is obtained by increasing \( \sigma \). Since the median of \( x_t \) on the other hand is \( x e^{\mu t} \) with \( \nu = \mu - \frac{1}{2} \sigma^2 \), a median preserving spread is obtained by increasing \( \mu \) (the rate of return shortfall \( \delta \)) concomitantly with \( \sigma \) according to

\[
\frac{d\mu}{d\sigma} = \sigma. \tag{69}
\]

Let us start from a mean preserving spread. The respective characteristic polynomials imply

\[
\frac{d\nu}{d\sigma} = \sigma \hat{g}(\alpha; \sigma), \tag{70}
\]

and

\[
\frac{d\beta}{d\sigma} = \sigma \hat{g}(\beta; \sigma), \tag{71}
\]

where

\[
g(u; \sigma) = -\frac{u(u - 1)}{\sigma^2 u + \nu}. \tag{72}
\]

These differentials account for two opposite effects on the optimal rule. On one hand, there is a positive effect on \( x^* \) since the value of delaying increases when the return on the R&D investments becomes more variable. Technically, since the premium on the option to switch to the R&D mode is convex in the shock, it increases as an implication of Jensen’s Inequality. On the other hand, there is a negative effect since also the opportunity cost of delaying increases. While the increase in the value of delaying reflects the well known bad news principle of Bernanke (1983), the increase in the opportunity cost reflects what could be called ’a good news principle’: an increase in \( \sigma \) renders R&D investments more
attractive since under the downside protection (an asymmetry between upside and downside risk on the leapfrogging rent) the expected return on these investments goes up together with the likelihood of extreme shock values. Overall, we find that \( x^* \) is a non-monotonous function of \( \sigma \). Moreover, a comparison to the certainty trigger \( x_0^* \) in (35) reveals that the absence of product market uncertainty always acts as a deterrent for R&D investments. These results are stated by the following proposition.

**Proposition 2** There exists a unique threshold volatility \( \sigma_0 \in (0, \sigma_{\text{max}}) \) implicitly solving

\[
\frac{g(\beta; \sigma_0)}{k(\beta)} = \frac{g(\alpha; \sigma_0)}{k(\alpha) - 2\lambda}, \tag{73}
\]

with

\[ k(u) = (r + \mu + 2\lambda)u - 2(r + \lambda), \]

such that

\[
\frac{dx^*}{d\sigma} \begin{cases}
< 0 & \text{for } \alpha < \sigma_0, \\
> 0 & \text{for } \alpha > \sigma_0.
\end{cases}
\]

Moreover, in the presence of product market uncertainty R&D investments are always more attractive than in the absence of product market volatility:

\[ x^* < x_0^*, \]

for all \( \sigma \leq \sigma_{\text{max}} \).

**Proof.** By ordinary differentiation, (73) follows from \( \frac{dx^*}{d\sigma} = \frac{d\omega}{d\sigma} \) and

\[
\frac{d\omega}{d\sigma} = \frac{\sigma}{(r - \mu + 2\lambda)(r + 2\lambda)} \frac{(k(\alpha) - 2\lambda)g(\beta, \sigma) - k(\beta)g(\alpha, \sigma)}{(\alpha(\lambda) - 1) \frac{r - \mu}{r - \mu + 2\lambda} - (\beta - 1)} (r - \mu).
\]

The LHS of (73) increases monotonously for all \( \sigma \leq \sigma_{\text{max}} \) and goes to zero as \( \sigma \to 0 \). The RHS, on other hand, decreases monotonously and goes to \( +\infty \) as \( \sigma \to 0 \). In addition, the LHS is larger than the RHS for \( \sigma = \sigma_{\text{max}} \). Hence, \( \sigma_0 \in (0, \sigma_{\text{max}}) \) exists and is unique. For showing \( x^* < x_0^* \), it now suffices to verify that \( x^* < x_0^* \) for \( \sigma = \sigma_{\text{max}} \), or that
\[
\frac{\alpha(\lambda) \frac{r+\lambda}{r+2\lambda} - \beta}{(\alpha(\lambda) - 1) \frac{r - \mu}{r - \mu + 2\lambda} - (\beta - 1) (r - \mu) < \frac{r + \lambda}{r + 2\lambda} (r - \mu + 2\lambda),}
\]

(74)

for \( \sigma = \sigma_{\text{max}} \). Evaluating \( \alpha(\lambda) \) and \( \beta \) for \( \sigma = \sigma_{\text{max}} \) yields

\[
\beta = \sqrt{\frac{r}{\mu}},
\]

and

\[
\alpha(\lambda) = -\sqrt{\frac{r + 2\lambda}{\mu}}.
\]

Substituting these and simplifying, we find that (74) is satisfied if

\[
\sqrt{\frac{\mu}{r}} < \frac{1}{2} + \frac{\mu}{r}.
\]

But this holds true trivially for all \( \frac{\mu}{r} \geq 0 \), so we are done. \( \blacksquare \)

Proposition 2 establishes a U-shaped relationship between the attractiveness of R&D investments and mean preserving increases in product market uncertainty. This is illustrated by Figure 4. When \( \sigma = \sigma_0 \) the effects on the value of delaying and on the opportunity cost of delaying cancel each other out. On the downward sloping part \((0, \sigma_0)\) the effect on the opportunity cost dominates whereas on the upward sloping part \((\sigma_0, \sigma_{\text{max}})\) there is sufficient uncertainty for the effect on the value delaying to take over. At this juncture, it is worth pointing out that the implications of uncertainty for the short run R&D activity and technological progress are not directly inferable from the U-shaped relationship since the speed at which the shock tends towards (or away from) a given level is inversely related to \( \sigma \) through \( \nu \). For elaborating on this point, let us define the first passage time

\[
\tau^* = \inf\{t \geq 0 : x_t \geq x^*\}.
\]

It is a standard result (eg. Øksendal 2003, p. 125) that

\[
\mathbb{E}_x \tau^* = \begin{cases} 
\frac{1}{\nu} \ln \left( \frac{x^*}{x} \right), & x < x^* \\
0, & x \geq x^*.
\end{cases}
\]
Letting $\sigma = 0$ yields the $\tau_0^*$ in (30) as a special case. The coefficient $\frac{1}{\nu}$ accounts for the speed at which the shock tends towards $x^*$. When $\sigma$ lies in $(\sigma_0, \sigma_{\text{max}})$ the positive effect on $x^*$ is compounded by the speed effect so higher uncertainty is detrimental to the expected short run R&D activity. When $\sigma$ lies in $(0, \sigma_0)$ the speed effect and the effect on $x^*$ work to the opposite directions so the relationship between uncertainty and the expected short run R&D activity is ambiguous. Specifically, the speed effect takes over when the switch out of the suspension mode is a sufficiently remote possibility ($x$ is small relative to $x^*$).

Next turning a median preserving spread by use of (69), we find that the U-shaped relationship does not carry over as $x^*$ becomes a monotonously decreasing function of $\sigma$. This result is stated by the following proposition.

**Proposition 3** The optimal switching trigger is a monotonously decreasing function of median preserving increases in product market uncertainty: conditional on (69),

$$\frac{dx^*}{d\sigma} < 0,$$

for all $\sigma \leq \sigma_{\text{max}}$.

**Proof.** Conditional on (69), ordinary differentiation yields

$$\frac{d\omega}{d\sigma} = \frac{\sigma}{(r - \mu + 2\lambda)(r + 2\lambda)} \frac{(k(\alpha) - 2\lambda)g(\beta; \sigma) - k(\beta)g(\alpha; \sigma) + g_0(r - \mu)}{(\alpha(\lambda) - 1) \frac{r - \mu}{r - \mu + 2\lambda} - (\beta - 1))^2},$$

where $g$ has simplified from (72) to
\[ g(u; \sigma) = -\frac{u^2}{\sigma^2 u + \nu}, \]

and

\[ g_0 = \frac{(r + 2\lambda)\beta - (r + \lambda)\alpha(\lambda)}{\lambda} \left[ \frac{r - \mu}{r - \mu + 2\lambda} (\alpha(\lambda) - 1) - \frac{r - \mu + 2\lambda}{r - \mu} (\beta - 1) \right]. \]

The problem is to show that the denominator

\[ (k(\alpha) - 2\lambda)g(\beta; \sigma) - k(\beta)g(\alpha; \sigma) + g_0 < 0. \] (75)

As the procedure is quite elaborate, also invoking the definition of \( W \), it is presented in Appendix A.

Proposition 3 establishes an important qualitative distinction between a mean preserving increase and a median preserving increase in product market uncertainty, stating the latter always renders R&D investments more attractive. Technically, there does not exist a counterpart for \( \sigma_0 \). This is illustrated by Figure 5. The economic intuition for the monotonous relationship derives from (69) as follows. Relative to a mean preserving increase, a median preserving increase entails a more pronounced opportunity cost of delaying since the expected return on the R&D investments goes up together with \( \mu \) (is inversely related to \( \delta \)). What is more, the effect on the opportunity cost now dominates the effect on the value of delaying across all \( \sigma \), so the U-shape is abolished. Since the speed effect is eliminated, we go on to conclude that a median preserving increase tends to boost R&D activity and technological progress in the short run.

Unfortunately, the shock properties do not allow an economically interesting long run analysis (in the absence of reflecting barriers). Specifically, for all \( \sigma < \sigma_{\text{max}} \) the expected relative occupation time of the shock above \( x^* \) approaches unity as the time horizon approaches infinity. For related discussion on long run stationary distributions, we refer to Dixit (1993, pp. 58-69) and Dixit and Pindyck (1994, pp. 83-84).

R&D cost uncertainty. The effect of mean preserving increases in R&D cost uncertainty is accounted for by \( \zeta(\lambda) \) and \( \eta \) in \( y^* \). It is clear from \( b(\rho) \) that these increases work to the same direction as those in product market uncertainty\(^2\).

\(^{2}\)It is worth noting from

\[ \frac{dy}{y} = (\delta^2 + \mu - \mu)dt + (\sigma dw_1 - \delta d\hat{w}_t - \sigma \delta dw_1 d\hat{w}_t) \]

that mean preserving increases in \( \sigma \) and \( \delta \) do not preserve the mean of the ratio.
The economic intuition goes along the same lines also. On one hand, the value of delaying increases with \( \bar{\sigma} \) since the net return on the R&D investments becomes more variable. On the other hand, the opportunity cost of delaying increases as well under the downside protection. Specifically, while undertaking R&D the follower taps into the upside risk on the shock-cost ratio (upside risk on the shock and downside risk on the cost) while at the same time being protected from the downside by the suspension option. Overall, we conclude that the relationship between \( g^* \) and mean preserving increases in \( \bar{\sigma} \) replicates the U-shape of Figure 4. Finally, we also see from \( b(\rho) \) that a positive \( \rho \) dampens the effect of \( \sigma \) and \( \bar{\sigma} \). This reflects the fact that higher covariance between two stochastic processes implies lower variability over their ratio.

**Technological uncertainty.** The effect of technological uncertainty manifests itself through the expected leapfrogging lag

\[
L = \frac{1}{\lambda}.
\]

This noted, we rewrite the switching trigger as

\[
x^* = \frac{\phi(L)(r+\frac{1}{L}) - \beta}{(\phi(L) - 1)(\frac{r-\mu}{\sigma}) - (\beta - 1)}L \frac{c}{\Delta},
\]

with

\[
\phi(L) = \alpha \left( \frac{1}{L} \right) = \left( \frac{1}{2} - \frac{\mu}{\sigma^2} \right) - \sqrt{\left( \frac{1}{2} - \frac{\mu}{\sigma^2} \right)^2 + 2 \frac{rL + 2}{\sigma^2 L}} < 0.
\]
The characteristic polynomial \( \sigma^2 L \phi (\phi - 1) + 2 \mu L \phi - 2 (rL + 2) = 0 \) implies

\[
\frac{d \phi}{dL} = -\frac{2}{L} \left( \frac{\phi}{\frac{1}{2} (\sigma \phi)^2 + r} \right) L + 2 > 0. \tag{78}
\]

We also notice that \( \lambda < \lambda_{\text{max}} \) is replaced with \( L > L_{\text{min}} = \frac{1}{\lambda_{\text{max}}} \).

An increase in \( L \) (‘time to build’) has four effects. Firstly, there is the obvious positive effect on \( x^* \) due to the fact that the expected R&D costs until leapfrogging go up. Secondly, since the (expected) variance over the return on the R&D investments is jointly increased by \( L \) and \( \sigma \), as suggested by the nominator \( \sigma^2 L \) inside the square root (77), there is another positive effect through the value of delaying (assuming \( \sigma > 0 \)). This in essence reproduces the reasoning of Majd and Pindyck (1987) why longer time to build makes delaying more attractive. Thirdly, there is a negative effect through the opportunity cost of delaying since the expected return on the investments is higher when the competitor tends to recapture the technological leadership with a longer lag. Finally, under the downside protection there is another negative effect through the opportunity cost since the risk of observing an extreme shock value midstream, while still undertaking R&D, increases with \( L \). Overall, we find that \( x^* \) is a monotonous function of \( L \). This result is stated by the following proposition.

**Proposition 4** The optimal switching trigger increases monotonously in the expected leapfrogging lag \( L \):

\[
\frac{dx^*}{dL} > 0,
\]

for all \( L > L_{\text{min}} \).

**Proof.** Define

\[
g_1 = \phi \frac{rL + 1}{rL + 2} - \beta,
\]

and

\[
g_2 = (\phi - 1) \frac{(r - \mu)L}{(r - \mu)L + 2} - (\beta - 1).
\]

Then \( x^* = \frac{g_1}{g_2} L (r - \mu) \) and the problem is to show that \( \frac{g_1}{g_2} L \) increases monotonously in \( L \). By ordinary differentiation,

\[
\frac{d}{dL} \frac{g_1}{g_2} L > 0 \iff -\frac{dg_1}{dL} \frac{L}{g_1} + \frac{dg_2}{dL} \frac{L}{g_2} < 1.
\]
Employing (78) yields

\[
\begin{align*}
\frac{dg_1}{dL} g_1 L &= L \frac{\frac{\phi}{\phi_1^2} (rL + 1) + \frac{r}{r_{L+2}} \phi}{(rL + 1)\phi - (rL + 2)\beta} = L \frac{\frac{\phi}{\phi_1^2} (rL + 1) - \frac{rL}{r_{L+2}}}{(rL + 1)\phi - (rL + 2)\beta} \\
&= \frac{\phi}{(\frac{1}{\phi} + \frac{r}{r_{L+2}}) L + 2} < 0,
\end{align*}
\]

and

\[
\begin{align*}
\frac{dg_2}{dL} g_2 L &= L \frac{\frac{\phi}{\phi_1^2} (rL + 1) + \frac{r}{r_{L+2}} \phi}{(rL + 1)\phi - (rL + 2)\beta} = L \frac{\frac{\phi}{\phi_1^2} (rL + 1) + \frac{rL}{r_{L+2}}}{(rL + 1)\phi - (rL + 2)\beta} \\
&= \frac{\phi}{(\frac{1}{\phi} + \frac{r}{r_{L+2}}) L + 2} < 0.
\end{align*}
\]

Noting that \( \frac{\phi L}{(\frac{1}{\phi} + \frac{r}{r_{L+2}}) L + 2} < 0 \) in \( \frac{dg_2}{dL} g_2 L \) and rearranging, we show that

\[
\begin{align*}
- \frac{\phi}{(\frac{1}{\phi} + \frac{r}{r_{L+2}}) L + 2} < 0.
\end{align*}
\]

Since \( \frac{dg_1}{dL} g_1 L + \frac{dg_2}{dL} g_2 L \) decreases monotonously in \( \sigma < \sigma_{\text{max}} \) for all parameter configurations (proof omitted), it suffices that

\[
\begin{align*}
- \frac{\phi}{(\frac{1}{\phi} + \frac{r}{r_{L+2}}) L + 2} < 0.
\end{align*}
\]

as \( \sigma \to 0 \). Since \( \phi \to -\infty \) and \( \sigma \phi \to -\infty \) as \( \sigma \to 0 \), the limit is given by
\[ K = - \frac{rL}{(rL + 2)(rL + 1)} + \frac{2}{(r - \mu)L + 2}. \]

Obviously \( K < 1 \) for all \( L > 0 \), so we are done. \( \blacksquare \)

Proposition 4 establishes that the effects on the expected R&D costs and on the value of delaying dominate the two opportunity cost effects with the implication that R&D investments become less attractive as the expected leapfrogging lag gets larger. The relationship between \( x^* \) and \( L \) is illustrated by Figure 6. Table 3 goes on to compare \( x^* \) with the certainty trigger \( x_0^* \) for different values of \( L \) when the other parameters are fixed at \((\sigma, r, \mu) = (0.2, 0.05, 0.03)\). We notice that the percentage markup of \( x_0^* \) over \( x^* \) goes down from around 29 percent to around 6.3 percent when \( L \) goes up from 15 > \( L_{\text{min}} = 14.46 \) to 100. Consistently with Bar-Ilan and Strange (1996), this finding gives evidence that the optimal R&D rule is less sensitive to product market uncertainty (in the mean preserving sense) when leapfrogging tends to be more lagged\(^3\). The economic intuition stems from the joint variance effect of \( L \) and \( \sigma \). Namely, since for large \( L \) the return on the R&D investments is highly variable to begin with (assuming \( \sigma > 0 \)), small increases in \( \sigma \) will not call for drastic revisions on the optimal rule, in relative terms.

\[
\begin{array}{|c|c|c|c|}
\hline
L & x^* & x_0^* & \frac{x_0^*-x^*}{x^*} \\
\hline
15 & 45.40 & 58.55 & 0.290 \\
20 & 52.69 & 64.00 & 0.215 \\
25 & 59.25 & 69.23 & 0.169 \\
30 & 65.30 & 73.29 & 0.138 \\
35 & 70.99 & 79.20 & 0.116 \\
100 & 132.65 & 137.14 & 0.063 \\
\hline
\end{array}
\]

Table 3: effect of lag on \( x^* \) and \( x_0^* \)

\textit{Competition intensity.} The effect of product market competition intensity, as measured by the conjectural variations parameter \( \theta \), is captured by \( \Delta = \Delta(\theta, \nu) \) where we recall that \( \nu < 1 \) is the cost down-scaling factor associated with innovations. Since in the strong leapfrogging case \( 1 < \frac{\Delta}{\nu} \leq \frac{1}{\nu} \) the product market is characterised by a persistent monopoly as in the Scumpeterian model of innovation (endogenous growth), the Boone reallocation criterion is not operative and \( x^* \) is independent of \( \theta \). In the weak leapfrogging case \( 0 \leq \frac{\Delta}{\nu} \leq 1 \) the criterion is manifested by \( \frac{dx^*}{d\theta} > 0 \), so we have

\[
\frac{dx^*}{d\theta} < 0.
\]

\(^3\)Here \( \sigma > \sigma_0 = 0.152 \). The same sensitivity result holds true also for \( \sigma < \sigma_0 \).
This finding reproduces in a real options context what is in the modern innovation literature known as the selection effect of competition, originating with Vickers (1995). The selection effect in general terms pertains to industries where multiple firms coexist and are earning profits: since intensifying competition increases the incremental profits earned by the technological leader relative to the competitors (payoff from innovating), also the incentives for R&D get higher. For further discussion, we refer to Aghion et al. (1997), Boone (1999), Aghion et al. (2001) and Encaoua et al. (2004).

Since under pure Cournot competition

$$\Delta = \frac{1 - v}{1 + v},$$

and under perfect competition

$$\Delta = 2 \frac{1 - v}{1 + v},$$

we have as a special case

$$x^* = \frac{\alpha(\lambda) \frac{v + \lambda}{r + 2 \lambda} - \beta}{(\alpha(\lambda) - 1) \frac{r - \mu}{r - \mu + 2 \lambda} - (\beta - 1)} \frac{r - \mu}{\lambda} M c,$$

where $M \in \left\{ \frac{1 + v}{1 - v}, \frac{1 + v}{2(1 - v)} \right\}$, respectively.
5 Concluding remarks

In this paper we analysed the optimal management of R&D investments in a continuous time duopoly in which innovations yield competitive advantage in the product market by enhancing the cost efficiency of production, there is technological uncertainty through a Poisson innovation process, product market uncertainty through a demand shock driven by a geometric Brownian motion, and R&D cost uncertainty through a flow cost driven by another geometric Brownian motion. For ease of argumentation, the analysis was constructed in a way that rules out genuine strategic interaction between the duopoly firms. Specifically, neck-and-neck competition in the product market and the escape competition motive for R&D were ruled out by combining a leapfrogging dynamics with a one innovation limit on the technological gap. Furthermore, by assuming both the R&D cost and the hazard rate of innovations are linear functions of the R&D intensity, we took the real options perspective by stating that the technological follower holds an infinite sequence of options to switch between an R&D mode and a suspension mode. Starting from the one dimensional case where R&D cost uncertainty is excluded, we showed by standard methods of stochastic control and dynamic programming that, in the absence of switching costs (hysteresis), these options are associated with a unique switching trigger that defines the optimal R&D rule for the technological follower under a particular restriction of the parameter space.

The comparative statics of the switching trigger produced a number of important results. First and foremost, we established a U-shaped relationship between the attractiveness of R&D investments and mean preserving increases in product market uncertainty, by showing there exists a shock volatility threshold at which the effects on the value of delaying and on the opportunity cost of delaying cancel each other out. Moreover, a quick comparison to the no-uncertainty trigger revealed that the absence of product market uncertainty always acts as a deterrent for R&D investments. By formal analogy, mean preserving increases in R&D cost uncertainty were argued to have the same effect as those in product market uncertainty. Consequently, short run technological progress in the duopoly is at its slowest in expected terms when there is technological uncertainty only. The asymmetry of the shock distribution led us to consider median preserving increases in product market uncertainty as well. Interestingly, it turned out that the U-shape does not carry over: a median preserving increase always renders R&D investments more attractive. The reason, we argued, is that a median preserving increase entails an amplification in the opportunity cost of delaying such that the effect on the value of delaying is dominated also for higher volatility levels. The implications of technological uncertainty were captured by inspecting the relationship between the switching trigger and the expected leapfrogging lag (‘time to build’). We showed that a larger lag always renders R&D investments less attractive, arguing that the effects on the value of delaying and on the expected R&D costs are dominated by the effects on the opportunity cost of delaying and on the expected return...
on the investments. Finally, deriving from the Boone reallocation criterion, we reproduced a central result of modern innovation literature, the selection effect of competition. That is, since the flow profit spread between the technological leader and the technological follower increases in the intensity of competition, the converse holds true for the switching trigger.

Being highly simplified to enable a closed form solution, the model can be extended towards various directions. The first obvious extension would be to relax the assumption that the firms are identical in terms of R&D, by allowing variation in the expected leapfrogging lag (efficiency of R&D) or/and in the R&D costs. A second extension would be to allow neck-and-neck competition and the escape competition motive by employing a step-by-step dynamics where the technological gap can exceed one innovation. The ensuing strategic interaction and increasing number of state value functions would presumably lead to numerical analysis. A third extension would be to introduce hysteresis by assuming positive switching costs. A fourth extension would be to allow spillovers and cumulative effects from past R&D by use of a more elaborate innovation process. Finally, in order to study the implications of uncertainty for long run technological progress, the shock should be replaced with a process that has a long run stationary distribution, such as a mean reverting Ornstein-Uhlenbeck process.
Appendix A. Proof of Proposition 3

In this appendix we show that $x^*$ is a monotonously decreasing function of median-preserving spreads in $\sigma$. Let $F$ denote the inverse of the LHS of (75):

$$F(\sigma; \lambda) = -((r + \mu + 2\lambda)\beta - 2(r + \lambda))\frac{\alpha^2(\lambda)}{\sigma^2\alpha(\lambda) + \nu} +$$

$$((r + \mu + 2\lambda)\alpha(\lambda) - 2(r + 2\lambda))\frac{\beta^2}{\sigma^2\beta + \nu} +$$

$$\frac{(r + 2\lambda)\beta - (r + \lambda)\alpha(\lambda)}{\lambda} \left[\frac{r - \mu + 2\lambda}{r - \mu}(\beta - 1) - \frac{r - \mu}{r - \mu + 2\lambda}(\alpha(\lambda) - 1)\right].$$

The problem then is to show that $F > 0$ inside the feasible parameter space $\mathcal{W}$ defined in terms innovation hazard rate as $\lambda \leq \lambda_{\text{max}}$, with $\lambda_{\text{max}}$ solving

$$\frac{\alpha^2(\lambda_{\text{max}})}{(r + 2\lambda_{\text{max}}) - \mu\alpha(\lambda_{\text{max}})} \frac{r + \lambda_{\text{max}}}{r + 2\lambda_{\text{max}}} (r - \mu + 2\lambda_{\text{max}}) \leq \frac{\beta^2}{r + \mu\beta}(r - \mu). \quad (79)$$

By ordinary differentiation, $F$ satisfies

$$\frac{dF}{d\sigma} < 0,$$

and

$$\frac{dF}{d\lambda} < 0. \quad (80)$$

Since also

$$\frac{d\lambda_{\text{max}}}{d\sigma} > 0,$$

it suffices to show that

$$F(\sigma; \lambda) > 0,$$

for $\sigma = \sigma_{\text{max}}$ and $\lambda_{\text{max}} = \lambda_{\text{max}}(r, \mu; \sigma_{\text{max}})$. From (79), it can be shown that
\[ \lambda_{\text{max}}(r, \mu; \sigma_{\text{max}}) < (1 + 2\beta) \mu = \left( 1 + 2 \sqrt{\frac{r}{\mu}} \right) \mu. \]  

(81)

We also recall from the proof of Proposition 2 that

\[ \beta = \sqrt{\frac{r}{\mu}}, \]  

(82)

and

\[ \alpha(\lambda) = -\sqrt{\frac{r + 2\lambda}{\mu}}, \]  

(83)

for \( \sigma = \sigma_{\text{max}} \).

Evaluating \( F \) for \( \sigma = \sigma_{\text{max}} \) yields

\[
F(\sigma_{\text{max}}; \lambda) = \frac{1}{2\mu} \left[ -((r + \mu + 2\lambda)\beta - 2(r + \lambda))\alpha(\lambda) + ((r + \mu + 2\lambda)\alpha(\lambda) - 2(r + 2\lambda))\beta \right] + \frac{(r + 2\lambda)\beta - (r + \lambda)\alpha(\lambda)}{\lambda} \left( \frac{r - \mu + \lambda}{r - \mu} (\beta - 1) - \frac{r - \mu}{r - \mu + 2\lambda} (\alpha(\lambda) - 1) \right) \\
= \underbrace{(r + 2\lambda)\beta - (r + \lambda)\alpha(\lambda)}_{\lambda > 0} \underbrace{G(\sigma_{\text{max}}; \lambda)}_{\lambda > 0},
\]

where

\[ G(\sigma_{\text{max}}; \lambda) = \frac{r - \mu + 2\lambda}{r - \mu} (\beta - 1) - \frac{r - \mu}{r - \mu + 2\lambda} (\alpha(\lambda) - 1) - \frac{\lambda}{\mu}. \]

Since \( \frac{d}{d\lambda} \frac{\beta(r + 2\lambda) - \alpha(\lambda)(r + \lambda)}{\lambda} > 0 \), it is by (80) necessary that \( \frac{dG}{d\lambda} < 0 \). Making use of this together with (81)-(83) yields for all \( \lambda < \lambda_{\text{max}} \) that
\(G(\sigma; \lambda) > G(\sigma; \lambda)\)

\[\frac{r + \mu + 4\mu\sqrt{\frac{r}{\mu}}}{r - \mu} (\sqrt{\frac{r}{\mu}} - 1) + \frac{r - \mu}{r + \mu + 4\mu\sqrt{\frac{r}{\mu}}} \left( \sqrt{\frac{r}{\mu} + 2 + 4\sqrt{\frac{r}{\mu}}} + 1 \right) - \left( 1 + 2\sqrt{\frac{r}{\mu}} \right) \]

\[\frac{r + \mu + 4\mu\sqrt{\frac{r}{\mu}}}{r - \mu} (\sqrt{\frac{r}{\mu}} - 1) + \frac{r - \mu}{r + \mu + 4\mu\sqrt{\frac{r}{\mu}}} \left( \sqrt{\frac{r}{\mu} + 2 + 4\sqrt{\frac{r}{\mu}}} + 1 \right) - \left( 1 + 2\sqrt{\frac{r}{\mu}} \right) \]

\[= (r - \mu) \frac{2 + \sqrt{\frac{r}{\mu}}}{r + \mu + 4\mu\sqrt{\frac{r}{\mu}}} - \frac{\sqrt{\frac{r}{\mu} + 4\mu\sqrt{\frac{r}{\mu}}} + 2r}{r - \mu} \]

\[= 2 \frac{\mu^2}{r - \mu} 1 + \frac{r}{\mu} \left( 2\sqrt{\frac{r}{\mu} - 3} \right) \]

For all \(\frac{r}{\mu} \geq 0\) (particularly, for all \(\frac{r}{\mu} \geq 1\)) the denominator

\[1 + \frac{r}{\mu} \left( 2\sqrt{\frac{r}{\mu} - 3} \right) \geq 0.\]

Hence, \(G(\max; \lambda) > 0\) for all \(\lambda < \lambda_{\max}\). Since this implies that \(F > 0\) inside \(W\), we are done.
References


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