The Hurewicz theorem by CW approximation

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In this thesis we prove the Hurewicz theorem which states that the $n$-th homology and homotopy groups are isomorphic for an $(n-1)$-connected topological space. There exists proofs of the Hurewicz theorem in which one constructs a concrete isomorphism between the spaces, but in this thesis we avoid the construction by transferring the problem to the realm of CW complexes and cellular structures by a technique known as cellular approximation. Combined with the cellular homology groups and related results this technique allows us to analyse the space on a cell-by-cell basis. This reduces the problem significantly and gives rise to many methods not applicable otherwise.

To prove the theorem we lay out the foundations of homotopy theory and homology theory. The singular homology theory is introduced, which in turn is used together with the concept of degree to define the cellular homology groups suitable for the analysis of CW complexes. Since CW complexes are built out of homeomorphic copies of the open unit disk extending to its boundary, it became crucial to prove various properties of these subspaces in both homotopy and homology. Fibrations, fiber bundles, and the Freudenthal suspension theorem were introduced for the homotopical viewpoint, while long exact sequences and contractibility played a great role in the homological considerations. CW approximation then made it possible to apply all this machinery to the topological space in question. Finally, the boundary homomorphisms from the long exact sequence in both homotopy and cellular homology theory turn out to be the same which made it possible to show the existence of an isomorphism between the groups.
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Introduction

Algebraic topology is often concerned with topological invariants of the algebraic kind which allow mathematicians to transfer topological properties to more easily formulated algebraic properties. The development of homotopy and homology theories that give insight into the structure of a topological space from an algebraic standpoint has been a major milestone of algebraic topology. Homotopy theory formalises the idea of encapsulating subspaces in loops of varying dimensions and analysing the distinct ways this can be done to classify the space in question. Homology theory can in a way be viewed in a similar manner, but fundamentally it is more concerned with the ways a space can be deconstructed into its building blocks and how these building blocks affect each other and the surrounding space.

On an intuitive level one might imagine that there must be a deeper connection between homotopy and homology theories; after all, they give similar results in a great many cases and can often be used to aid in the calculations in the other theory. The reality has proven to be quite different, since the homotopy groups exhibit very different behavior on a general level compared to homology groups. By restricting the space in question it is still possible to construct bridges and connections between these two theories.

The purpose of this thesis is to prove the famous Hurewicz theorem by using the technique of CW approximation. The Hurewicz theorem states that the homotopy and homology groups agree for a suitably connected space at the first dimension with non-trivial groups. This implies that both homology and homotopy groups agree on the most fundamental structural level that the two theories are able to distinguish. We begin by introducing some preliminary concepts, namely cell structures, homotopy and homology theory, fibrations and fiber bundles, and CW approximation. After that we develop the cellular homology theory using degree theory, culminating finally in the development of the needed lemmas to prove the main result.

This thesis will largely follow Hatcher in \[H\] in the structure and proofs of the needed results. In addition to Hatcher we make use of the axiomatisation of homology theories by Eilenberg and Steenrod in \[ES\], the theory of fibrations and fiber bundles in the work of Switzer in \[Sw\], and some brief references to Väisälä in \[Va\] and tom Dieck in \[Di\].
Preliminary concepts

2.1 CW complexes

To be able to use the later techniques of homology and homotopy efficiently we must define some way in which to give structure and rigidity to a space. It turns out that many spaces can be constructed from homeomorphic copies of the open unit disk $D^n \setminus S^{n-1}$ for dimensions $n \geq 0$. Here we adopt the custom of defining the 0-dimensional copy of the open unit disk as a point by the facts that $D^0 = \{0\}$ and $S^{-1} = \emptyset$. By attaching these copies of the open unit disks in varying dimensions to each other we can construct a great many of the most common spaces. As a generalisation of the common polyhedra this way of looking at spaces gives the much needed stability and structure to the analysis of spaces.

**Definition 2.1.** Let $J$ be an indexing set. The homeomorphic copy of $D^n \setminus S^{n-1}$ is called an $n$-cell and is denoted by $e^n_j$ for some indexing $j \in J$, sometimes just $e^n$ if the dimension and indexing are not relevant.

**Definition 2.2.** Let $E$ be a collection of cells and let $X$ be a topological space that is a disjoint union of these cells. The $k$-skeleton of a space $X$, denoted by $X^k$, is the collection of cells with dimension less than or equal to $k$, or in other words

$$X^k = \bigcup \{ e \in E \mid \dim(e) \leq k \}$$

The skeletons are subsets of each other and

$$X = \bigcup_{k \geq 0} X^k.$$ 

By taking a set of 0-cells or points and attaching by boundary 1-cells, which are homeomorphic copies of the open unit interval, we end up with a network of closed paths ending and/or starting at some 0-cell. Let us now attach 2-cells, which are homeomorphic copies of the unit disk, to this construction by again attaching the boundary to the lower dimensional cells. If we keep going like this, we are constructing what is known as a CW complex.

**Definition 2.3.** A CW complex is a topological space which is a disjoint union of cells, where the boundaries of cells $e^n_j$ are attached to the $X^{n-1}$ skeleton by attaching maps $\varphi_j: S^{n-1} \to X^{n-1}$. This attaching map extends to the characteristic map of the cell $e^n_j$, namely $\Phi_j: D^n_j \to X$ which is a homeomorphism from the open unit disk onto $e^n_j$. If a subspace $A \subset X$ contains the closures of all the cells that make up $A$, then $A$ is called a CW subcomplex of $X$. 

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The space \( X \) is called finite-dimensional if there exists an \( n \), called the dimension of \( X \), such that \( X = X^n \). Otherwise,
\[
X = \bigcup_{n \geq 0} X^n.
\]
The standard example of a cell structure is the canonical structure given to the \( n \)-sphere \( S^n \) which consists of one \( n \)-cell with its boundary attached to one 0-cell.

A map \( f: X \to Y \) between CW complexes is called cellular if \( f(X^n) \subseteq Y^n \) for each \( n \geq 0 \). A CW complex is given the weak topology with respect to its cells by defining that a set \( A \subseteq X \) is open if and only if the space \( A \cap e \) is open in \( e \) for every cell \( e \). The letters C and W in the name CW complex stem from the fact that it has the properties of closure finiteness and weak topology with respect to its cells. Closure finiteness means that the closure of each cell meets only finitely many other cells. Since the closure of a cell is compact, this property follows from the following proposition.

**Proposition 2.4.** Let \( X \) be a CW complex and let \( A \subseteq X \) be any compact subspace. Then the subspace \( A \) is contained in a finite CW subcomplex of \( X \).

**Proof.** See [Ha, Proposition A.1.] for a proof.

A powerful result in the study of CW complexes is the result that every continuous map between CW complexes is homotopic to a cellular map. We will define homotopy in the next section. The proof of the theorem is beyond the scope of this thesis but it will be stated here for later use.

**Theorem 2.5** (Cellular approximation theorem). Let \( X \) and \( Y \) be CW complexes and let \( f: X \to Y \) be a continuous map between said complexes. Then \( f \) is homotopic to a cellular map.

**Proof.** See [Ha, Theorem 4.8.] for a proof.

### 2.2 Homotopy and homology groups

When analysing topological spaces one is often concerned of the extent to which these spaces are path-connected, connected, and have various other topological properties. The homotopy and homology groups of a space measure how loops and cycles are allowed to behave, and thus provide more information about the structure of the space. Homotopy groups deal with loops that are continuous images of \( n \)-spheres, while homology groups describe cycles on a more general level utilising the boundaries of these cycles.

#### 2.2.1 Homotopy groups

We begin by defining the concept of homotopy between continuous maps, that is the property which says when two mappings can be transformed into one another continuously.
**Definition 2.6.** Let $X$ and $Y$ be topological spaces and let $f, g : X \to Y$ be continuous maps. A *homotopy* between these maps is a continuous map $H : X \times [0, 1] \to Y$ that satisfies $H(x, 0) = f(x)$ and $H(x, 1) = g(x)$ for all $x \in X$ and $t \in [0, 1]$. The homotopy is thus defined by a collection of maps $\{ H_t : X \to Y \mid t \in I \}$ where $H_t(x) = H(x, t)$. We denote $H : f \sim g$ or just $f \sim g$ for homotopic maps $f$ and $g$. If the homotopy keeps points of a subset $A \subset X$ stationary for all $t \in I$, then the homotopy is called relative and is denoted by $H : f \sim g \text{ rel } A$.

Maps homotopic to a constant map $c_{x_0} : X \to Y$ with $c_{x_0}(x) = x_0$ for all $x \in X$ are called *nullhomotopic* and can thus be deformed continuously into a single point. It is easy to see that $\sim$ is an equivalence relation on the set of continuous maps, and thus homotopic maps form equivalence classes that partition the space of continuous mappings. So an equivalence class $[f]$ with representative $f$ is the set of all continuous maps which are homotopic to $f$ and can thus be continuously transformed into this map. Sometimes we wish to form a homotopy class where the homotopies are relative to some subspace, often a chosen basepoint. We denote by $[f]_0$ the equivalence class of maps homotopic to the map $f$ where the homotopy is relative to the basepoint in the domain of $f$.

Homotopic maps enable us to define a way of classifying spaces as being the same that is not as strong and rigid as homeomorphisms, namely *homotopy equivalences*:

**Definition 2.7.** A continuous map $f : X \to Y$ between topological spaces is a *homotopy equivalence* if there exists a continuous map $g : Y \to X$ such that $f \circ g \sim \text{id}_Y$ and $g \circ f \sim \text{id}_X$. Topological spaces $X$ and $Y$ are called *homotopically equivalent* if such a map exists, and this is denoted by $X \simeq Y$. A homotopy equivalence of pairs $(X, A) \simeq (Y, B)$ is defined similarly on each space for maps $f : (X, A) \to (Y, B)$.

As a reminder, continuous maps $f : (X, A) \to (Y, B)$ are continuous maps $f : X \to Y$ such that $f(A) \subset B$. The most usual case for us is when $A$ is a singleton $\{x_0\}$, which defines a basepoint $x_0$ in the space $X$ and a corresponding basepoint $f(x_0)$ in the space $Y$.

A *retraction* is a continuous map $r : X \to A$ with $A \subset X$ such that the restriction $r|_A$ is the identity on $A$. A *deformation retraction* of $X$ to a subspace $A$ is a homotopy $H : \text{id}_X \sim i \circ r$ where $r$ is a retraction between $X$ and $A$ and $i : A \hookrightarrow X$ is the inclusion. If the points in $A$ are kept stationary during the homotopy, then the homotopy is called a *strong deformation retraction*. A deformation retraction implies that the retraction is a homotopy equivalence. The standard example is the deformation retraction of the unit disc $D^n$ to the origin, which in turn implies that the unit disk is homotopy equivalent to a point. In general, a space $X$ which deformation retracts onto a point is called *contractible*.

As a quick reminder, the disjoint union $\coprod_i X_i$ of sets $X_i$ is defined as the union of the indexed sets with additional indexing that distinguishes from which set a particular element originates from. The *cone* $CX$ of a space $X$ is defined as the quotient

$$CX = (X \times I)/(X \times \{1\})$$

where the base of the cone $X \times \{0\}$ is identified with $X$. The *suspension* $SX$ of a space $X$ is the quotient space of $X \times I$ where the points in $X \times \{0\}$ are identified with some point.
$x_0 \in X \times \{0\}$, and the points in $X \times \{1\}$ are identified with some point $x_1 \in X \times \{1\}$. Note that $SS^n = S^{n+1}$ since the space $SX$ can be thought of as two cones $CX$ attached to each other by their common intersection.

An important property of nullhomotopic maps from the $(n-1)$-sphere which will be useful later on in cell extensions is the following result:

**Proposition 2.8.** Let $X$ be a topological space and let $f: S^{n-1} \to X$ be any continuous map. The map $f$ is nullhomotopic if and only if it has a continuous extension $g: D^n \to X$.

**Proof.** The nullhomotopy gives a direct extension of $f$ to the cone $CS^{n-1}$ and it is easily checked that the cone of $S^{n-1}$ is homeomorphic to the closed disk $\bar{D}^n$. Thus $f$ extends to $D^n$ if and only if $f$ extends to the cone of $S^{n-1}$. The other direction follows from the fact that the $n$-disk is a convex set, and thus a nullhomotopy of $f$ to a point can be constructed as a linear homotopy through the disk $D^n$.

Let us now define the $n$-th homotopy group, first as a set without group structure.

**Definition 2.9.** The $n$-th homotopy group (as a set) with basepoint $x_0$ is denoted by $\pi_n(X, x_0)$ and is the set of homotopy classes of maps $f: (S^n, x_0) \to (X, f(x_0))$ where the homotopies are relative to the basepoint $x_0$. In other words

$$\pi_n(X, x_0) = \{ [f]_0 \mid f: (S^n, x_0) \to (X, f(x_0)) \text{ is continuous} \}.$$ 

This set can be given a group structure for $n > 0$ by first identifying the $n$-sphere $S^n$ with the quotient space $I^n/\partial I^n$ where the unit $n$-cube has its boundary identified to a single point. Thus we end up with mappings of the form $f: (I^n, \partial I^n) \to (X, x_0)$ for which the sum operation can be defined as follows:

$$(f + g)(x_1, x_2, \ldots, x_n) = \begin{cases} f(2x_1, x_2, \ldots, x_n), & x_1 \in [0, 1/2] \\ g(2x_1 - 1, x_2, \ldots, x_n), & x_1 \in [1/2, 1] \end{cases}$$

It is straightforward to verify that the $n$-th homotopy group equipped with this operation is a group with neutral element the class of the constant loop at the basepoint $x_0$. Each continuous map $f: (X, x_0) \to (Y, f(x_0))$ induces a well-defined homomorphism

$$f_*: \pi_n(X, x_0) \to \pi_n(Y, f(x_0))$$

by setting $f_*([g]) = [f \circ g]$ for each loop $g: (S^n, x_0) \to (X, g(x_0))$.

There exists homotopically equivalent spaces which are not homeomorphic, although all homeomorphic spaces are also homotopically equivalent. The homomorphism

$$f_*: \pi_n(X, x_0) \to \pi_n(Y, f(x_0))$$

induced by a homotopy equivalence $f: X \to Y$ is an isomorphism for $n \geq 1$ and for every basepoint $x_0 \in X$, see [Va] Theorem 23.16] for a proof for the case $n = 1$ which generalises
The concept of homotopically equivalent spaces will be revisited in the section on CW approximation.

The special case $\pi_0(X,x_0)$ is generally not a group but gives important information about the path-connectedness of the space $X$. Given maps $f: (S^0, x_0) \to (X, f(x_0))$ which map two points to the space $X$, one into the basepoint $f(x_0)$, we note that every homotopy class of these maps defines uniquely a path-component of the space $X$. A trivial homotopy group $\pi_0(X,x_0)$ implies thus that $X$ is path-connected.

Example 2.10. The $k$-th homotopy group $\pi_k(S^n)$ of an $n$-sphere is trivial for all $k < n$. This follows from the fact that when the $n$-sphere is given the canonical cell structure, a homotopy class is represented by a map $f: S^k \to S^n$ which is homotopic to a cellular map by Theorem 2.5. The only way this is possible for $k < n$ is if the image of the map $f$ is contained in the 0-cell, and thus $f$ has to be nullhomotopic.

Example 2.11. An important special case of the homotopy groups of $n$-spheres is the so-called fundamental group of the 1-sphere, namely $\pi_1(S^1)$. It turns out that $\pi_1(S^1) \cong \mathbb{Z}$ and that it is generated by the identity map $\text{id}: S^1 \to S^1$. The result is usually proven using the theory of covering spaces; one such proof can be found in [Va, Theorem 25.2]. This result will be used in the calculation of $\pi_n(S^n)$ later on.

It is sometimes useful to trivialise loops in a space $X$ which are contained in some subspace $A$. This leads to the concept of relative homotopy groups which are especially useful in computations of homotopy groups since they fit into a long exact sequence of homotopy groups.

Definition 2.12. The $n$-th relative homotopy group $\pi_n(X,A,x_0)$ of a pair $(X,A)$ with basepoint $x_0 \in A$ is the set of homotopy classes of maps $f: (D^n, S^{n-1}, x_0) \to (X,A,f(x_0))$ where the homotopies are relative to the basepoint $x_0$, or in other words

$$\pi_n(X,A,x_0) = \{ [f]_0 \mid f: (D^n, S^{n-1}, x_0) \to (X,A,f(x_0)) \text{ is continuous} \}.$$ 

In addition, each pair $(X,A)$ has associated to it a boundary homomorphism

$$\partial: \pi_n(X,A,x_0) \to \pi_{n-1}(A,x_0)$$

for each $n \geq 0$ defined as the restriction $\partial([f]) = [f|_{S^{n-1}}]$ for each class representative $f: (D^n, S^{n-1}, x_0) \to (X,A,f(x_0))$.

We digress for a bit to define exact sequences:

Definition 2.13. Let $\{C_k\}_{k \in \mathbb{Z}}$ be a sequence of groups with associated homomorphisms $\partial_n: C_n \to C_{n-1}$ in the following fashion:

$$\ldots \to C_{n+1} \xrightarrow{\partial_{n+1}} C_n \xrightarrow{\partial_n} C_{n-1} \to \ldots$$

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The sequence is called exact if $\ker \partial_n = \operatorname{im} \partial_{n+1}$ for each $n$. If $A, B$ and $C$ are groups with homomorphisms $\alpha : A \to B$ and $\beta : B \to C$ such that they form an exact sequence

$$0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0$$

then this sequence is called a short exact sequence.

Exact sequences can be used to great effect in both homotopy and homology theory. The following theorem in homotopy theory is especially useful.

**Theorem 2.14.** The sequence

$$\ldots \xrightarrow{\partial} \pi_n(A,x_0) \xrightarrow{i_*} \pi_n(X,x_0) \xrightarrow{j_*} \pi_n(X,A,x_0) \xrightarrow{\partial} \pi_{n-1}(A,x_0) \xrightarrow{i_*} \ldots \to \pi_0(X,x_0)$$

is exact when $i_*$ and $j_*$ are induced by the inclusions $i : (A,x_0) \hookrightarrow (X,x_0)$ and $j : (X,x_0,x_0) \hookrightarrow (X,A,x_0)$, while the boundary homomorphism $\partial$ is as detailed above. This exact sequence is called the long exact sequence of homotopy groups.

**Proof.** See [Ha, Thm. 4.3] for a proof.

Lastly in this section we will consider results needed in the calculation of the $n$-th homotopy group of the $n$-sphere $S^n$ in Section 3.1. For this we will state the so called homotopy excision property for CW complexes and prove the Freudenthal suspension theorem using the excision property. The excision property is central to homology theory as will be seen in the next section, and it essentially says that the homology groups of pairs do not change if we remove some suitably contained subspace. This idea is not transferred completely to homotopy theory, but it does exist in a slightly different manner depending on the connectedness of the space in question. Then we have to take a small detour to the theory of fibrations to calculate $\pi_2(S^2)$.

We start by defining the concepts of simply-connected and $n$-connected spaces, which will also be central in the formulation of the Hurewicz theorem.

**Definition 2.15.** A space $X$ with basepoint $x_0$ is called simply connected if $\pi_1(X,x_0) = 0$ and $X$ is path-connected (so even $\pi_0(X,x_0) = 0$). The space is called $n$-connected if $\pi_i(X,x_0) = 0$ for all $i \leq n$. Similarly, a pair $(X,A)$ is $n$-connected if $\pi_i(X,A,x_0) = 0$ for all $i \leq n$.

The homotopy excision property for CW complexes is stated here without proof since it extends beyond the scope of this thesis.

**Theorem 2.16** (Homotopy excision property). Let $X$ be a CW complex constructed as the union of two subcomplexes $A$ and $B$ such that they have a non-empty connected intersection $C = A \cap B$. Furthermore, let $(A,C)$ be $r$-connected and let $(B,C)$ be $s$-connected for $r, s \geq 0$. Then the inclusion $i : (A,C) \hookrightarrow (X,B)$ induces a homomorphism $i_* : \pi_k(A,C) \to \pi_k(X,B)$ which is an isomorphism for $k < r + s$ and a surjection for $k = r + s$. 

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The Freudenthal suspension theorem for CW complexes states that there is a very specific relationship between the homotopy groups of an \((n - 1)-\)connected CW complex and its suspension in dimensions less than \(2n - 1\). It follows quite naturally from the homotopy excision property:

**Corollary 2.17** (Freudenthal suspension theorem). Let \(X\) be an \((n - 1)-\)connected CW complex. Then the homomorphism \(\gamma: \pi_i(X) \rightarrow \pi_{i+1}(SX)\) is an isomorphism for \(i < 2n - 1\) and a surjection for \(i = 2n - 1\).

**Proof.** Let the suspension \(SX\) be considered as the union of two cones \(C_0X\) and \(C_1X\) with a common intersection of the CW complex \(X\). Now consider the long exact sequences of the pairs \((C_1X,X)\) and \((SX,C_0X)\):

\[\ldots \xrightarrow{i} \pi_{i+1}(C_1X) \xrightarrow{j} \pi_{i+1}(C_1X,X) \xrightarrow{\partial} \pi_i(X) \xrightarrow{i} \pi_i(C_1X) \xrightarrow{j} \ldots\]

and

\[\ldots \xrightarrow{\partial} \pi_{i+1}(C_0X) \xrightarrow{i} \pi_{i+1}(SX) \xrightarrow{j} \pi_{i+1}(SX,C_0X) \xrightarrow{\partial} \pi_i(C_0X) \xrightarrow{i} \ldots\]

Because the cones \(C_0X\) and \(C_1X\) are contractible it follows that their homotopy groups are trivial, which means that the long exact sequences provide isomorphisms \(\partial: \pi_{i+1}(C_1X,X) \rightarrow \pi_i(X)\) and \(j_*: \pi_{i+1}(SX) \rightarrow \pi_{i+1}(SX,C_0X)\). In addition to this, the first exact sequence implies that the pair \((C_1X,X)\) is \(n\)-connected because \(X\) is \((n-1)\)-connected by hypothesis. Similarly we may conclude that also \((C_0X,X)\) is \(n\)-connected.

This observation makes \(SX = C_0X \cup C_1X\) with \(X = C_0X \cap C_1X\) satisfy the requirements of Theorem 2.16. Thus the inclusion \(k: (C_0X, X) \rightarrow (SX, C_1X)\) induces an homomorphism \(k_*: \pi_{i+1}(C_1X,X) \rightarrow \pi_{i+1}(SX,C_0X)\) which is an isomorphism for \(i + 1 < n + n\) and hence for \(i < 2n - 1\) and a surjection for \(i + 1 = n + n\) and hence for \(i = 2n - 1\). The homomorphism \(\gamma\) can now be expressed as a composition of \(\partial^{-1}\), \(k_*\) and \(j_*^{-1}\) as in the commutative diagram:

\[
\begin{array}{ccc}
\pi_{i+1}(C_1X, ) & \xrightarrow{k_*} & \pi_{i+1}(SX, C_0X) \\
\downarrow{\approx} & & \downarrow{\approx} \\
\pi_i(X) & \xrightarrow{\gamma} & \pi_{i+1}(SX)
\end{array}
\]

This implies that \(\gamma\) is an isomorphism for \(i < 2n - 1\) and a surjection for \(i = 2n - 1\). \(\square\)

Now to complete the puzzle of calculating \(\pi_n(S^n)\) we need a way to calculate \(\pi_2(S^2)\). The theory of fibrations combined with a long exact sequence of homotopy groups that fits the Hopf fibration \(S^3 \rightarrow S^2\) gives us this result. The details for these results can be found in [Sw] Chapter 4.
Definition 2.18. Let $E$ and $B$ be any spaces and let $p: E \to B$ be a map. Furthermore, let $f: I^n \to E$ be any map and let $h: I^n \times I \to B$ be any homotopy, $n \geq 0$, for which $h_0 = p \circ f$ holds. If there exists a homotopy $H: I^n \times I \to E$ such that $H_0 = f$ and $p \circ H = h$, then the map $p$ is a Serre fibration.

In other words, Serre fibrations have the property of lifting homotopies from $B$ to $E$. A Serre fibration fits into a long exact sequence of homotopy groups in the following manner:

Proposition 2.19. Let $p: E \to B$ be a Serre fibration and let $b_0 \in B$ be any point. Furthermore, define $F := p^{-1}(b_0)$ and let $e_0 \in F$ be a suitable basepoint. Then there exists a boundary homomorphism

$$\partial: \pi_n(B, b_0) \to \pi_n(F, e_0)$$

for all $n \geq 1$ which together with the homomorphisms $i_*: \pi_n(F, e_0) \to \pi_n(E, e_0)$ and $p_*: \pi_n(E, e_0) \to \pi_n(B, b_0)$ induced by the inclusion $i$ and the Serre fibration $p$ fit into the following long exact sequence of homotopy groups:

$$\ldots \xrightarrow{\partial} \pi_n(F, e_0) \xrightarrow{i_*} \pi_n(E, e_0) \xrightarrow{p_*} \pi_n(B, b_0) \xrightarrow{\partial} \pi_{n-1}(F, e_0) \xrightarrow{i_*} \ldots$$

Proof. See [Sw, Theorem 4.6] with following remarks for a proof.

Serre fibrations work well in so called fiber bundles which can be analysed well using the above long exact sequence:

Definition 2.20. A fiber bundle is a quadruplet $(B, p, E, F)$ where $B$, $E$ and $F$ are spaces and $p: E \to B$ is any continuous surjection which satisfy the following property: There exists an open covering $\{A_i\}_{i \in I}$ of $B$ and a homeomorphism $\varphi_i: A_i \times F \to p^{-1}(A_i)$ for each $i \in I$ such that $p \circ \varphi_i = \text{pr}_1$ where $\text{pr}_1$ is the first projection of the product $A_i \times F$.

It turns out that the projection of a fiber bundle is always a Serre fibration:

Proposition 2.21. Let $(B, p, E, F)$ be a fiber bundle. Then the projection $p: E \to B$ is a Serre fibration.

Proof. See [Sw, Proposition 4.10] for a proof.

Finally we can calculate $\pi_2(S^2)$ using the long exact sequence of homotopy groups formed by a very special fiber bundle known as the Hopf bundle or the Hopf fibration:

Proposition 2.22. There exists a fiber bundle $(S^2, p, S^3, S^1)$ known as the Hopf bundle. This implies that $\pi_2(S^2) \cong \mathbb{Z}$.

Proof. The construction of the Hopf bundle $(S^2, p, S^3, S^1)$ can be found in [Sw, Example 4.14] and [Sw, Proposition 6.27]. By the previous proposition it follows that $p$ is a Serre fibration and thus induces the following part of a long exact sequence of homotopy groups:

$$\ldots \xrightarrow{\partial} \pi_2(S^1) \xrightarrow{i_*} \pi_2(S^3) \xrightarrow{p_*} \pi_2(S^2) \xrightarrow{\partial} \pi_1(S^1) \xrightarrow{i_*} \pi_1(S^3) \xrightarrow{p_*} \ldots$$
From Example 2.10 we know that \( \pi_2(S^3) = \pi_1(S^3) = 0 \), and thus \( \partial \) must be an isomorphism by exactness. From Example 2.11 we know that \( \pi_1(S^1) \cong \mathbb{Z} \) and hence \( \pi_2(S^2) \cong \mathbb{Z} \) as well.

### 2.2.2 Homology groups

Compared to homotopy groups the homology groups of a space are much easier to calculate but require a higher level of abstraction for their definition. In essence, while homotopy groups deal with equivalence classes of loops that are images of \( n \)-spheres, the homology groups consist of chains or sequences of cycles. These cycles can be defined in varying ways which each produce different, although equivalent, homology theories. The axiomatic homology theory developed by Eilenberg and Steenrod in [ES, p. 10–12], the singular homology theory satisfying these axioms and the cellular homology theory built on the singular homology theory will be sufficient for the purposes of this thesis, and the theory will be presented here without proofs.

**Definition 2.23** (Axiomatic homology theory). A homology theory associates with each (topological) pair \((X,A)\), with \(A \subset X\) closed, and for each \(n \geq 0\) an abelian group denoted by \(H_n(X,A)\) which is the \(n\)-th homology group of \((X,A)\). The homology theory also associates to each continuous map \(f : (X,A) \to (Y,B)\) an induced homomorphism \(f_* : H_n(X,A) \to H_n(Y,B)\) for each \(n \geq 0\). In addition to this the homology theory defines a boundary homomorphism

\[
\partial : H_n(X,A) \to H_{n-1}(A)
\]

for each \(n \geq 1\). These groups and induced maps satisfy the following seven axioms of homology:

(i) **Identity.** If \(\text{id} : (X,A) \to (X,A)\) is the identity map on \((X,A)\), then

\[
\text{id}_* = \text{id} : H_n(X,A) \to H_n(X,A)
\]

is the identity on homology.

(ii) **Composition.** For maps \(f : (X,A) \to (Y,B)\) and \(g : (Y,B) \to (Z,C)\) it holds that

\[
(g \circ f)_* = g_* \circ f_*
\]

(iii) **Commutativity.** The following square of the boundary map \(\partial\) and any induced map \(f_* : H_n(X,A) \to H_n(Y,B)\) commutes, which means that \(f_* \circ \partial = \partial(f|_A)_*\):

\[
\begin{array}{ccc}
H_n(X,A) & \xrightarrow{f_*} & H_n(Y,B) \\
\downarrow\partial & & \downarrow\partial \\
H_{n-1}(A) & \xrightarrow{(f|_A)_*} & H_{n-1}(B)
\end{array}
\]
(iv) **Homotopy.** If continuous maps $f$ and $g$ are homotopic then the induced maps $f_*$ and $g_*$ are the same maps on homology.

(v) **Exactness.** There exists a sequence of homology groups

$$
\ldots \xrightarrow{\partial} H_n(A) \xrightarrow{i_*} H_n(X) \xrightarrow{j_*} H_n(X, A) \xrightarrow{\partial} H_{n-1}(A) \xrightarrow{i_*} \ldots
$$

with inclusions $i: A \to X$ and $j: X \to (X, A)$ such that the kernel of each homomorphism is equal to the image of the previous group. In other words, the inclusions and the boundary map induces a *long exact sequence* in homology.

(vi) **Excision.** Let $(X, A)$ be a pair and $U \subset A$ a subspace such that the closure of $U$ is contained in some open set $V \subset A$. Then the inclusion map $i: (X \setminus U, A \setminus U) \to (X, A)$ induces an isomorphism $i_*: H_n(X \setminus U, A \setminus U) \cong H_n(X, A)$ in homology.

(vi) **Dimension.** The homology groups of a point $x_0 \in X$ are trivial for $n \geq 1$, i.e. $H_n(x_0) = 0$.

Now we will briefly describe the construction of *singular homology theory* which will be later used to define cellular homology. This homology theory satisfies the axioms outlined above. We begin by defining $n$-simplices which serve as the basis for singular $n$-chains, singular $n$-simplices and singular chain complexes.

An *$n$-simplex*, denoted by $[x_0, \ldots, x_n]$, is defined as the smallest convex set containing $n + 1$ points $x_0, \ldots, x_n \in \mathbb{R}^{n+1}$ so that no lower dimensional hyperplane goes through all these points. It is thus the generalisation of a triangle in two dimensions. The *face* of an $n$-simplex, denoted $[x_0, \ldots, \hat{x}_i, \ldots, x_n]$, is the $(n - 1)$-simplex we are left with when we remove one vertex $x_i$. The *standard $n$-simplex* is denoted by $\Delta^n$ and is defined as

$$
\Delta^n = \{ (t_0, \ldots, t_n) \in \mathbb{R}^{n+1} \mid \sum_i t_i = 1 \text{ and } t_i \geq 0 \text{ for all } i \}.
$$

A *singular $n$-simplex* in a space $X$ is a continuous map $\sigma: \Delta^n \to X$ from the standard $n$-simplex to the space $X$. A *singular $n$-chain* in $X$ is a formal sum $\sum_i n_i \sigma_i$, or linear combination, of singular $n$-simplices with $n_i \in \mathbb{Z}$. The set of singular $n$-chains is thus generated by the singular $n$-simplices. A singular $n$-chain can be interpreted as a path through the images of standard $n$-simplices that make up the space $X$ in the same way as the loops of homotopy groups form different paths in the space. Homology theory is concerned more with the concept of boundary and how the space built of building blocks (singular $n$-simplices) can be reduced to the information in their boundaries. Thus we require a map which defines the boundary of a given singular $n$-simplex.

Let $C_n(X)$ denote the free abelian group with basis the singular $n$-simplices of $X$, or in other words, the group of singular $n$-chains in $X$. We can define a *boundary homomorphism* $\partial_n: C_n(X) \to C_{n-1}(X)$ which takes singular $n$-simplices to their boundaries which are singular $(n - 1)$-simplices in themselves:

$$
\partial_n(\sigma) = \sum_i (-1)^i \sigma|[x_0, \ldots, \hat{x}_i, \ldots, x_n]
$$
By linearity this can be extended to the free abelian group of $n$-chains. The groups $C_n(X)$ now fit into the following sequence, called a chain complex:

$$\cdots \xrightarrow{\partial_{n+1}} C_n(X) \xrightarrow{\partial_n} C_{n-1}(X) \xrightarrow{\partial_{n-1}} \cdots \xrightarrow{\partial_1} C_0(X) \xrightarrow{\partial_0} 0$$

It is a straightforward, albeit a bit cumbersome, calculation to verify that $\partial_n \circ \partial_{n+1} = 0$ for all $n \geq 0$, which means that the boundaries of boundaries are trivial. This also implies that the set of boundaries of singular $(n+1)$-chains, that is $\text{Im} \partial_{n+1}$, is contained in the set of singular $n$-chains which have zero boundary, that is in $\text{Ker} \partial_n$. In other words, $\text{Im} \partial_{n+1} \subset \text{Ker} \partial_n$, which means it is relevant to talk about the quotient $\text{Ker} \partial_n / \text{Im} \partial_{n+1}$. Since the boundary of a singular $n$-simplex can be contracted to a single point on the boundary by passing through the interior of the simplex we wish to trivialise them. Thus we end up with the following definition:

**Definition 2.24.** The $n$-th singular homology group of a space $X$ is the quotient space $H_n(X) = \text{Ker} \partial_n / \text{Im} \partial_{n+1}$ with the boundary homomorphism $\partial_n$ defined as above. The elements of $\text{Ker} \partial_n$ are called *cycles* and the elements of $\text{Im} \partial_{n+1}$ are called *boundaries*.

If we augment the chain complex for singular homology with an additional group $\mathbb{Z}$ and a homomorphism $\varepsilon: C_0(X) \to \mathbb{Z}$ defined by $\varepsilon(\sum_i n_i \sigma_i) = \sum_i n_i$ we end up with a version of singular homology where points have trivial homology groups, even for dimension zero. The homology groups of the chain complex

$$\cdots \xrightarrow{\partial_{n+1}} C_n(X) \xrightarrow{\partial_n} C_{n-1}(X) \xrightarrow{\partial_{n-1}} \cdots \xrightarrow{\partial_1} C_0(X) \xrightarrow{\varepsilon} \mathbb{Z} \longrightarrow 0.$$ 

are called the *reduced singular homology groups* $\tilde{H}_n(X)$. These coincide with the singular homology groups for $n > 0$ with the addition that $H_0(X) \cong \tilde{H}_0(X) \oplus \mathbb{Z}$.

The *relative singular homology groups* $H_n(X, A)$ are constructed by trivialising singular $n$-chains in $C_n(A)$ by building a chain complex of groups $C_n(X, A) = C_n(X)/C_n(A)$ and calculating the homology groups of that complex. This is made possible by the fact that the standard boundary map takes singular $n$-chains of $A$ to singular $(n-1)$-chains of $A$, which means that it induces a quotient boundary map. These relative homology groups fit into a long exact sequence as outlined in the axiomatisation of homology theories.

A *good pair* is a pair of spaces $(X, A)$ where $A$ is a nonempty closed subspace such that there exists a neighbourhood $V \subset X$ of $A$ and a strong deformation retraction of $V$ to $A$. The following proposition will be useful later:

**Proposition 2.25.** Let $(X, A)$ be a good pair and let $p: (X, A) \to (X/A, A/A)$ be the quotient map which collapses $A$ to a point. Then the quotient map induces an isomorphism $p_*: H_n(X, A) \cong H_n(X/A, A/A)$.

Since $A/A = \{x_0\}$ for some basepoint $x_0 \in X$ we especially have that $H_n(X, A) \cong \tilde{H}_n(X/A)$.

**Proof.** See [Ha, Proposition 2.22.] for a proof.
One last property of homology groups can be found from the homotopy section, namely that homotopy equivalences induce isomorphisms on homology groups:

**Proposition 2.26.** Let \( f : X \to Y \) be a homotopy equivalence. Then \( f_* : H_n(X) \to H_n(Y) \) is an isomorphism for all \( n \geq 0 \).

**Proof.** Let \( g : Y \to X \) be the homotopy inverse of \( f \). Then \( f \circ g \sim \text{id}_Y \) which in light of the axiom on homotopy implies that \( (f \circ g)_* = (\text{id}_Y)_* \). By the identity and associativity axioms it follows that

\[
    f_* \circ g_* = (f \circ g)_* = (\text{id}_Y)_* = \text{id}_{H_n(Y)}
\]

and similarly for \( g_* \circ f_* = \text{id}_{H_n(X)} \) for all \( n \geq 0 \). \( \square \)

Finally we present a few examples that will be useful later on.

**Example 2.27.** The homology group \( H_0 \) can be expressed as a direct sum of the homology groups of its path-components. Additionally, the homology group \( H_0(x_0) \) of a point space is isomorphic to \( \mathbb{Z} \). Thus \( H_0(S^0) \cong \mathbb{Z} \oplus \mathbb{Z} \). The general result can be found in [Ha, Proposition 2.6] and [Ha, Proposition 2.8].

**Example 2.28.** Since the unit disc \( D^n \) is homotopy equivalent to a point, the dimension axiom and the previous proposition gives that \( H_k(D^n) = 0 \) for all \( n \geq 0 \) and \( k \geq 1 \). Using reduced homology this can be expressed as \( \tilde{H}_k(D^n) = 0 \) for \( n,k \geq 0 \).

**Example 2.29.** The long exact sequence of the pair \( (D^n, S^{n-1}) \) reduces by the previous example to short exact sequences

\[
    0 \to H_k(D^n, S^{n-1}) \xrightarrow{\partial} H_{k-1}(S^{n-1}) \to 0
\]

which by exactness implies that \( \partial : H_k(D^n, S^{n-1}) \to H_{k-1}(S^{n-1}) \) is an isomorphism for all \( k > 1 \). Furthermore, since \( H_k(D^n, S^{n-1}) \cong \tilde{H}_k(D^n/S^{n-1}) = \tilde{H}_k(S^n) \) by Proposition 2.25 we get an isomorphism \( \tilde{H}_k(S^n) \cong H_{k-1}(S^{n-1}) \). For reduced groups this holds even for \( k = 1 \) which combined with \( H_0(S^0) \cong \mathbb{Z} \oplus \mathbb{Z} \) from Example 2.27 gives inductively that \( \tilde{H}_n(S^n) \cong \mathbb{Z} \) for all \( n \geq 0 \).

**Example 2.30.** The long exact sequence of the pair \( (X, x_0) \) with \( x_0 \in X \) reduces by the dimension axiom to short exact sequences

\[
    0 \to H_k(X) \xrightarrow{j_*} H_k(X, x_0) \to 0
\]

which implies that \( j_* : H_k(X) \to H_k(X, x_0) \) is an isomorphism for all \( k \geq 0 \). Since \( H_k(X) \cong \tilde{H}_k(X) \) for all \( k > 0 \) we get the isomorphism \( H_k(X, x_0) \cong \tilde{H}_k(X) \) for \( k > 0 \) identifying the reduced homology groups with homology groups equipped with a basepoint.
2.3 CW approximation

The cellular homology groups (defined in Section 3.2) are a powerful tool when dealing with CW complexes. To be able to utilise this machinery in the proof of Hurewicz theorem for general topological spaces that don’t necessarily have a natural cell structure, we need to introduce the theory of cellular approximation. From the section on homotopy groups we know that a homotopy equivalence between path-connected spaces induces isomorphisms on homotopy groups. Taking this as a criterion we get the following definition:

**Definition 2.31.** A map \( f: X \to Y \) is a weak homotopy equivalence if it induces isomorphisms \( f_*: \pi_n(X,x_0) \to \pi_n(Y,f(x_0)) \) for all \( n \geq 0 \) and all basepoints \( x_0 \in X \).

In this sense a weak homotopy equivalence contains all the information about the structure of the homotopy groups. The goal of this section is to show that for every space \( X \) there exists a CW complex \( Z \) with a weak homotopy equivalence \( f: Z \to X \), which we will call a CW approximation of \( X \), we first need the following lemma:

**Lemma 2.32.** Let \( X \) and \( A \) be CW complexes with a map \( f: A \to X \) and one 0-cell \( a_j = e_j^0 \) fixed in each connected component of \( A \). Then there exists a CW complex \( B \), constructed by attaching \( k \)-cells to \( A \) for some \( k \geq 0 \), with a map \( f': B \to X \) extending \( f \), so that the induced map

\[
f'_*: \pi_i(B,a_j) \to \pi_i(X,f'(a_j))
\]

is injective for \( i = k - 1 \) and surjective for \( i = k \) for all \( a_j \).

**Proof.** We begin by constructing an injection in the dimension \( k - 1 \). Consider the kernel \( \text{Ker} \ f_* \) of the induced map \( f_*: \pi_{k-1}(A,a_j) \to \pi_{k-1}(X,f(a_j)) \) for each basepoint \( a_j \). The kernel consists of the loops in dimension \( k - 1 \) which \( f \) maps to trivial loops in \( X \). Let the maps \( \varphi_i: (S^{k-1}, x_0) \to (A,a_j) \) generate the kernel of \( f_* \). These maps may be assumed to be cellular by cellular approximation (Theorem 2.5) so that they do not change the cell structure of \( A \) in dimensions larger than \( k \). By using the maps \( \varphi_i \) as attaching maps for the \( k \)-cells \( e_i^k \) we attach these to \( A \) to produce a new CW complex \( A' \).

We know that \( f_*([\alpha]) = 0 \) for all classes of loops \( [\alpha] \) in the kernel of \( f_* \), which in turns implies that \( f \circ \alpha \sim c_{a_j} \). The fact that the maps \( \varphi_i \) generate the kernel of \( f_* \) is equivalent to saying that \( f \circ \varphi_i \sim c_{a_j} \) for each \( \varphi_i \). By Proposition 2.8 the composition \( f \circ \varphi_i: (S^{n-1}, x_0) \to (X,f(a_j)) \) can be extended to a map \( (D^n, x_0) \to (X,f(a_j)) \). This in turn gives us a map \( g: (A',a_j) \to (X,g(a_j)) \) extending \( f \).

Let us now extend this map further into a surjection in dimension \( k \). Let the maps \( f_l: S^k \to X \) generate the groups \( \pi_k(X,f(a_j)) \), so every element in \( \pi_k(X,f(a_j)) \) can be represented as a formal sum of these maps. For each generator \( f_l \) attach a \( k \)-cell \( e_l^k \) to \( A' \).
Theorem 2.33. Let X be any space. Then there exists a CW complex Z and a weak homotopy equivalence \( f: Z \to X \). Thus every space X has a CW approximation.

Proof. Let \( A_0 = \{ a_j \mid j \in J \} \) be a set consisting of one point for every path-component of X. Let \( a_k \in A_0 \) and let \( f^0: (A_0, a_k) \to (X, f^0(a_k)) \) be a map which takes each of the points \( a_j \in A_0 \) to a point in the corresponding path-component \( f_0(a_k) \). Since the homotopy classes in \( \pi_0 \) correspond to the path-components of the space in question, the induced map \( f_0^*: \pi_0(A_0, a_k) \to \pi_0(X, f^0(a_k)) \) is a bijection.
By Lemma 2.32 this map $f^0$ can be extended to a map $f^1: (A_1, a_k) \to (X, f^1(a_k))$ with $A_0 \subseteq A_1$ so that the induced map $f^1_*: \pi_1(A_1, a_k) \to \pi_1(X, f^1(a_k))$ is a surjection. Lemma 2.32 can be applied once more to extend $f^1$ into $f^2$ such that the induced map is surjective on $\pi_2$ and injective on $\pi_1$, which was already surjective and thus an isomorphism. By repeating this procedure inductively we end up with a CW complex $Z := A_l$ for some $l \geq 0$ such that $f := f^l_*: \pi_n(Z, a_k) \to \pi_n(X, f^l(a_k))$ is an isomorphism for $n \geq 0$. 

Note that $n$-cells of a space $X$ only affect the homotopy groups of dimension $n - 1$ and higher since maps from $(n - 2)$-spheres are trivial on spheres of dimension $n - 1$ and higher. This means that attaching $n$-cells cannot alter the injectivity or surjectivity of previously constructed isomorphisms.

Constructing a CW approximation of a pair $(X, A)$ can be done based on the above construction by first constructing a CW approximation $f': B \to A$. This approximation can through the composition $B \to A \to X$ be used as a starting point for the construction above to produce a CW approximation $f: Z \to X$ that extends $f'$. Using the well-known Five Lemma it can be shown that $f: (Z, B) \to (X, A)$ is a weak homotopy equivalence (see for example [Di, Lemma 11.1.4.] for a direct proof).

The following theorem by Whitehead is a famous tool in the analysis of CW complexes in algebraic topology and will be used briefly in the next proof.

**Theorem 2.34** (Whitehead’s theorem). Let $X$ and $Y$ be CW complexes and let $f: X \to Y$ be a weak homotopy equivalence. Then $f$ is also a homotopy equivalence.

**Proof.** See [Ha, Theorem 4.5.] for proof. 

As a corollary to CW approximation and Whitehead’s theorem we find that we can trivialise the $(n - 1)$-skeleton for $n$-connected CW complexes:

**Corollary 2.35.** Let $X$ be an $n$-connected CW complex. Then there exists a homotopy equivalent CW complex $Z$ which consists of cells of dimension $n$ and higher with a single 0-cell as a basepoint.

**Proof.** Since the homotopy groups of $X$ are trivial for all $i \leq n$ the construction in Lemma 2.32 applies when starting from stage $n$ in the inductive attaching process where the $n$-cells are attached to a single 0-cell. The weak homotopy equivalence constructed is by Whitehead’s theorem a homotopy equivalence. 

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Homology of CW complexes

3.1 Degree of a map

When discussing loops and cycles in topological spaces, the natural question to ask is how many times these loops and cycles circle the said space or point. When calculating the fundamental group $\pi_1(S^1) \cong \mathbb{Z}$ we identified the group with homotopy classes consisting of loops that rotate the same amount of times either clockwise or counter-clockwise. Thus the homotopy class is uniquely identified by this integer. The intuition behind the homology group of $S^1$ works in a similar fashion.

Let $f : S^n \rightarrow S^n$ be any continuous map between $n$-spheres. If we consider the induced homomorphism $f_* : H_n(S^n) \rightarrow H_n(S^n)$ we note that it reduces to a homomorphism $f_* : \mathbb{Z} \rightarrow \mathbb{Z}$ for $n > 0$ by Example 2.29. The homomorphism $f_*$ must thus be of the form

$$f_*(\alpha) = d\alpha$$

for all cycles $\alpha \in H_n(S^n)$ and some integer $d \in \mathbb{Z}$. The integer is unique to $f$ and is called the degree of $f$. In the intuitive manner of the earlier paragraph this can be thought of as the extent to which the map $f$ increases the amount of rotations that the cycles do on the $n$-sphere. We denote the degree of $f$ by $\text{deg}(f)$.

The following theorem presents some elementary properties of the degree of a continuous map between $n$-spheres:

**Proposition 3.1.** The degree of a continuous map $f : S^n \rightarrow S^n$ satisfies the following properties:

(i) The degree of the identity map is $\text{deg}(\text{id}_{S^n}) = 1$.

(ii) If $f$ is not surjective, then $\text{deg}(f) = 0$.

(iii) For a composition of maps it holds that $\text{deg}(f \circ g) = \text{deg}(f) \text{deg}(g)$ for maps $f, g : S^n \rightarrow S^n$.

(iv) If $f$ is a reflection respective to some subsphere $S^{n-1} \subset S^n$, then $\text{deg}(f) = -1$.

(v) If $f \sim g$ then $\text{deg}(f) = \text{deg}(g)$.

**Proof.** Property (i) follows directly from the identity axiom on homology. Property (ii) is a consequence of $S^n \setminus \{x_0\}$ being contractible and thus having trivial homology group. A point $x_0 \in S^n \setminus f(S^n)$ can be factored out from $f$ resulting in a factoring where the middle factor is trivial, and thus $f_*$ must consequently be the trivial map. Property (iii) follows from the homology axiom that states that $(f \circ g)_* = f_* \circ g_*$. Property (iv) follows from simplicial homology theory (see [Ha] p. 134 for a quick proof). Finally, property (v) follows directly from the homotopy axiom of homology which states that $f_* = g_*$ if $f \sim g$. □
The calculation of the degree of a continuous map can often be a cumbersome task, but luckily the so called global degree can be calculated using local degrees that measure the behaviour of the map in a neighbourhood of a point. We begin by constructing some preliminary isomorphisms.

Let \( f: S^n \to S^n \) be a continuous map and suppose that \( f^{-1}(y) \) is finite for some \( y \in S^n \). Then \( f^{-1}(y) = \{x_1, \ldots, x_k\} \) which means that we may for each point \( x_i \) choose a neighbourhood \( U_i \) disjoint from the others such that all the neighbourhoods map into some neighbourhood \( V \) of \( y \). If we consider the long exact sequences of the pairs \((S^n, S^n \setminus \{y\})\) and \((S^n, S^n \setminus \{x_i\})\) we get separate isomorphisms \( H_n(S^n) \cong H_n(S^n, S^n \setminus \{y\}) \) and \( H_n(S^n) \cong H_n(S^n, S^n \setminus \{x_i\}) \) since \( S^n \) with a point removed is contractible.

Furthermore, both \( S^n \setminus \{x_i\} \) and \( S^n \setminus \{y\} \) are open, and thus the excision axiom allows us to remove spaces \( S^n \setminus U_i \) and \( S^n \setminus V \) from the spaces \((S^n, S^n \setminus \{x_i\})\) and \((S^n, S^n \setminus \{y\})\) respectively to get isomorphisms \( H_n(U_i, U_i \setminus \{x_i\}) \cong H_n(S^n, S^n \setminus \{x_i\}) \) and \( H_n(V, V \setminus \{y\}) \cong H_n(S^n, S^n \setminus \{y\}) \). Using these isomorphisms we end up with the following definition:

**Definition 3.2.** Let \( f: S^n \to S^n \) be a continuous map. Using the theory above, the local degree of \( f \) at point \( x_0 \in f^{-1}(y) \) with \( y \in S^n \) is the degree of the restriction \( f: (U_0, U_0 \setminus \{x_0\}) \to (V, V \setminus \{y\}) \) where the homology groups of both the domain and codomain are identified with \( H_n(S^n) \cong \mathbb{Z} \). The local degree of \( f \) at point \( x_0 \) is denoted by \( \deg(f)|_{x_0} \).

The reason why local degrees are very useful in the calculations of degree, is that the degree of a map \( f \) is equal to the sum of the local degrees at all points in the preimage of any point \( y \in S^n \). The following proposition makes this clear.

**Proposition 3.3.** Let \( f: S^n \to S^n \) be a continuous map and let \( y \in S^n \) be any point with finite preimage. If we denote \( X := f^{-1}(y) \), then

\[
\deg(f) = \sum_{x \in X} \deg(f)|_{x_0}.
\]

**Proof.** Let \( j_*: H_n(S^n) \to H_n(S^n, S^n \setminus f^{-1}(y)) \) be the homomorphism from the long exact sequence of the pair \((S^n, S^n \setminus f^{-1}(y))\) and let \( k'_*: H_n(U_i, U_i \setminus \{x_i\}) \to H_n(S^n \setminus f^{-1}(y)) \) and \( p'_*: H_n(S^n \setminus f^{-1}(y)) \to H_n(S^n, S^n \setminus \{x_i\}) \) be the homomorphisms induced by the inclusions \( k_*' \) and \( p'_* \). Now combined with the isomorphisms introduced in the definition of local degree we get the following commutative diagram

\[
\begin{array}{ccc}
H_n(U_i, U_i \setminus \{x_i\}) & \xrightarrow{j'_*} & H_n(V, V \setminus \{y\}) \\
\downarrow{k'_*} & & \downarrow{k_*} \\
H_n(S^n, S^n \setminus \{x_i\}) & \xrightarrow{p'_*} & H_n(S^n, S^n \setminus \{y\}) \\
\downarrow{j_*} & & \downarrow{j_*} \\
H_n(S^n) & \xrightarrow{f_*} & H_n(S^n)
\end{array}
\]
The diagram commutes because the isomorphisms are essentially relativisations of the induced inclusion maps in every triangle and square. Note that $f''_*$ and $f'_*$ are just renamings of $f_*$ to preserve clarity. Since $H_n(U_i, U_i \setminus \{x_i\}) \cong \mathbb{Z}$ it means that $H_n(S^n, S^n \setminus f^{-1}(y))$ can be interpreted as a direct sum of infinite cyclic groups, one for each point in the preimage of $y$. In this way the homomorphisms $k^*_i$ and $p^*_i$ can be interpreted as the inclusions and projections of the $i$-th summand.

By commutativity of the lower left triangle we see that the composition $p^*_i \circ j^*_i$ takes $1$ to $1$. Since $p^*_i$ is the projection of the $i$-th summand this implies that $j^*_i(1) = (1, \ldots, 1)$. Because $k^*_i$ is the inclusion, this again implies that $j^*_i(1) = \sum_i k^*_i(1)$. By the isomorphism in the upper right square and the fact that the square commutes, we note that $f''_*(k^*_i(1)) = f''_*(1) = \deg(f)|_{x_i}$. Thus

$$f'_i(j^*_i(1)) = f'_i(\sum_i k^*_i(1)) = \sum_i \deg(f)|_{x_i}$$

and by the isomorphism in the lower right square and the fact that the square commutes we end up with the formula

$$\deg(f) = f_*(1) = f'_i(j^*_i(1)) = \sum_i \deg(f)|_{x_i}.$$ 

One useful property of degree is that the suspension is degree-invariant:

**Proposition 3.4.** Let $f : S^n \to S^n$ be a map and let $Sf : SS^n \to SS^n$ be the suspension of the map $f$. Then $\deg(f) = \deg(Sf)$.

**Proof.** We begin by considering the cone $CS^n$ of the $n$-sphere with $S^n$ embedded as the base of the cone. This is a convenient representation, since we can now collapse $S^n$ to a point and end up with the suspension $CS^n/S^n \cong SS^n \cong S^{n+1}$. Let us now look at the long exact sequence of the pair $(CS^n, S^n)$:

$$\cdots \to H_{n+2}(CS^n) \to H_{n+1}(CS^n, S^n) \to H_n(S^n) \to H_{n+1}(CS^n) \to \cdots$$

Since $CS^n$ is contractible and has thus trivial homology groups in all dimensions the long exact sequence implies that the boundary map $\partial : H_{n+1}(CS^n, S^n) \to H_n(S^n)$ is an isomorphism. The map $f$ extends continuously to a cone map $Cf : (CS^n, S^n) \to (CS^n, S^n)$ whose quotient is $Sf$. Now consider the following diagram

$$\begin{array}{ccc}
H_{n+1}(SS^n) & \xrightarrow{p_*} & H_{n+1}(CS^n, S^n) \\
\downarrow{Sf_*} & & \downarrow{Cf_*} \\
H_{n+1}(SS^n) & \xrightarrow{p_*} & H_{n+1}(CS^n, S^n)
\end{array}$$

$$\begin{array}{ccc}
\xrightarrow{\partial} & H_n(S^n) \\
\downarrow{f_*} & & \downarrow{f_*} \\
\xrightarrow{\partial} & H_n(S^n)
\end{array}$$
where the isomorphism $p_\ast: H_{n+1}(CS^n, S^n) \rightarrow H_{n+1}(SS^n)$ is from Proposition 2.25. Note that the reduced homology group in Proposition 2.25 is not needed in the previous isomorphism so it was replaced by the unreduced group. The right square in the diagram commutes by the commutativity axiom in homology, as does the left square by the definition of $Cf$ and $Sf$. Thus the entire diagram commutes, which implies that if $f_\ast$ is a multiplication by $d$ then also $Sf_\ast$ must be such.

Using the results from the end of the section on homotopy groups we can finally calculate the $n$-th homotopy group of the $n$-sphere $S^n$:

**Theorem 3.5.** The $n$-th homotopy group of the $n$-sphere is $\pi_n(S^n) \cong \mathbb{Z}$. Furthermore, if we consider the map $\deg: \pi_n(S^n) \rightarrow \mathbb{Z}$ which maps each homotopy class to the degree of its representative, then $\deg$ is an isomorphism.

**Proof.** Recall from Proposition 2.22 that there exists an isomorphism $\alpha: \pi_1(S^1) \rightarrow \pi_2(S^2)$. Since $S^k$ is a $(k-1)$-connected CW complex by cellular approximation (see Example 2.10) we may apply the Freudenthal suspension theorem (Corollary 2.17) to construct isomorphisms $\gamma_k: \pi_i(S^k) \rightarrow \pi_{i+1}(S^{k+1})$ for all $0 < i < 2k-1$ for each $k > 1$. Thus we can construct a sequence of isomorphisms

$$
\pi_1(S^1) \xrightarrow{\alpha} \pi_2(S^2) \xrightarrow{\gamma_2} \pi_3(S^3) \xrightarrow{\gamma_3} \pi_4(S^4) \xrightarrow{\gamma_4} \ldots
$$

which implies that $\pi_n(S^n) \cong \mathbb{Z}$ since $\pi_1(S^1) \cong \mathbb{Z}$.

The group $\pi_1(S^1)$ can be represented by the winding maps $f_k: S^1 \rightarrow S^1$ for $k \in \mathbb{Z}$ defined by $f_k(z) = z^k$. For $k = 0$ this is just a constant map and has thus zero degree by property (ii) from Proposition 3.1. When $k < 0$ we may compose the map with the map $g(z) = z^{-1}$ which is a reflection and has thus degree $-1$ by property (iv). By property (iii) we see that the case $k < 0$ reduces to the case $k > 0$.

When $k > 0$ we note that $f^{-1}(y)$ consists of $k$ distinct points for any $y \in S^n$, and that $f_k$ is locally just a rotation with a scaling factor $k$ that scales neighbourhoods of $x_i \in f^{-1}(y)$ into neighbourhoods of $y$. Locally this scaling can be removed by deforming $f_k$ to a simple rotation on $S^1$. This deformation doesn’t change the local degree since the unscaled neighbourhood of $x_i$ is still mapped inside the original neighbourhood of $y$. The local degree of $f_k$ is thus the same as for a rotation. A rotation on $S^1$ is a homeomorphism and has thus degree 1 by properties (i), (iii) and (v) in Proposition 3.1. Because the preimage of $y$ consists of $k$ distinct points, it must hold that $\deg(f) = \sum_i \deg(f)|_{x_i} = k$.

Since suspension preserves degree it must follow that each class in $\pi_n(S^n)$ can be represented by the degree of the repeated suspensions of the maps $f_k$. Thus the map $\deg$ is an isomorphism.

### 3.2 Cellular homology

Cellular homology theory is an exceptionally useful tool in calculating the homology groups of CW complexes. The cellular homology groups are isomorphic to the singular homology...
groups but are much more straightforward to calculate due to the additional techniques associated to CW complexes. The boundary homomorphism of singular homology can at times be difficult to define for more complex spaces, but the boundary homomorphism of cellular homology can be directly calculated using the degree theory outlined in the previous section.

For this construction we need a few properties concerning the singular homology groups of the $n$-skeletons of a CW complex $X$:

**Theorem 3.6.** Let $X$ be a CW complex. Then $H_k(X^n, X^{n-1}) = 0$ for $k \neq n$. For $k = n$ it follows that $H_k(X^n, X^{n-1})$ is free abelian, and that there exists a bijection between its basis and the $n$-cells of $X$. Furthermore, $H_k(X^n) = 0$ for $k > n$.

**Proof.** See [Ha, Theorem 3.1.1] for a proof. □

This implies that the group $H_n(X^n, X^{n-1})$, similarly as with the the singular chain groups, can be interpreted as a linear combination of the $n$-cells of the CW complex $X$. This makes it possible to build a homology theory based on a sequence of these groups. By the previous theorem we may combine the long exact sequences of the pairs $(X^{n-1}, X^{n-2})$, $(X^n, X^{n-1})$ and $(X^{n+1}, X^n)$ with boundary homomorphisms $\partial$ and inclusions $i_*$ and $j_*$ (dimensions omitted for clarity) into the following diagram:

$$
\begin{array}{ccccccccc}
0 & \rightarrow & H_n(X^{n+1}) & \xrightarrow{\cong} & H_n(X) & \xrightarrow{} & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \rightarrow & H_n(X^n) & \xrightarrow{\partial} & H_n(X^{n-1}) & \xrightarrow{j_*} & H_n(X^n, X^{n-1}) & \xrightarrow{d_n} & H_{n-1}(X^{n-1}, X^{n-2}) & \xrightarrow{} & \ldots \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \rightarrow & H_n(X^{n-1}) & \xrightarrow{\partial} & H_{n-1}(X^{n-1}) & \xrightarrow{j_*} & H_{n-1}(X^{n-1}, X^{n-2}) & \xrightarrow{} & \ldots \\
\end{array}
$$

The middle row forms a chain complex called the **cellular chain complex** with boundary homomorphisms $d_n := j_* \circ \partial$ (note that $d_n \circ d_{n+1} = 0$ since the same holds for $\partial$). Thus similarly as for singular homology groups we can formulate the following definition:

**Definition 3.7.** Let $X$ be a CW complex with a cellular chain complex as outlined above. The **$n$-th cellular homology group** $H_n(X)$ is defined as

$$H_n(X) = \text{Ker } d_n / \text{Im } d_{n+1}.$$
Note that we use the same notation $H_n(X)$ for the cellular homology groups and the singular homology groups. This is because the groups are isomorphic for all $n \geq 0$ (see [Ha, Theorem 2.35.] for a simple proof). From now on we will exclusively mean the cellular homology groups when using this notation, if nothing else is specified. One strength of the cellular homology groups is that there exists quite a straightforward characterisation of the cellular boundary homomorphism which simplifies calculation and will illuminate the proof of the Hurewicz theorem later on.

**Theorem 3.8.** Let $n > 1$ and let $X$ be a CW complex. Let $p: X^{n-1} \to X^{n-1}/X^{n-2}$ be the quotient map (or quotient projection) which collapses the $(n - 2)$-skeleton into a point and let $p_j: X^{n-1}/X^{n-2} \to S_j^{n-1}$ be some map which collapses the complement of the cell $e_j^{n-1}$ to a point. The cellular boundary homomorphism $d_n: H_n(X^n, X^{n-1}) \to H_{n-1}(X^{n-1}, X^{n-2})$ can be calculated on the basis elements $e_i^n$ by the formula

$$d_n(e_i^n) = \sum_j d_{ij} e_j^{n-1}$$

where the coefficients are $d_{ij} := \deg(\omega_{ij})$ where $\omega_{ij} = p_j \circ p \circ \varphi_i$. Here $\varphi_i: S_i^{n-1} \to X^{n-1}$ is the attaching map of the cell $e_i^n$ and $p$ and $p_j$ are as defined above.

The cells $e_i^n$ are the cells in the $(n - 1)$-skeleton to which the attaching map $\varphi_i$ attaches the boundary of the cell $e_i^n$. Note that the sum in the formula above is finite since the image $\varphi_i(e_i^n)$ is compact and can thus meet only finitely many of the $(n - 1)$-cells in the $(n - 1)$-skeleton $X^{n-1}$.

**Proof.** Consider the following diagram:

$$
\begin{array}{cccc}
H_n(D_i^n, S_i^{n-1}) & \xrightarrow{\partial_n} & H_{n-1}(S_i^{n-1}) & \xrightarrow{\omega_{ij}^*} & H_{n-1}(S_j^{n-1}) \\
\downarrow{\Phi_i} & & \downarrow{\varphi_i} & & \uparrow{p_j} \\
H_n(X^n, X^{n-1}) & \xrightarrow{d_n} & H_{n-1}(X^{n-1}) & \xrightarrow{p_*} & H_{n-1}(X^{n-1}/X^{n-2}) \\
\downarrow{j_*} & & \downarrow{p_*} & & \cong \downarrow{\cong} \\
H_{n-1}(X^{n-1}, X^{n-2}) & \cong & H_{n-1}(X^{n-1}/X^{n-2}, X^{n-2}/X^{n-2})
\end{array}
$$

The maps in the diagram are as outlined in the theorem and in the definition of cellular homology, with the addition of the characteristic map $\Phi_i: (D_i^n, \partial S_i^{n-1}) \to (X^n, X^{n-1})$ of the cell $e_i^n$ and the isomorphisms in the lower right square. The isomorphism in the upper left square is given in Example 2.29 and the isomorphisms in the lower right square are given in Proposition 2.25 and in Example 2.30 (since $X^{n-2}/X^{n-2} = \{x_0\}$ for some $x_0 \in X$).

The upper left square commutes since the attaching map is a restriction of the characteristic map and thus commutes with the boundary. The triangle commutes by the
definition of $d_n$, as does the upper right square by the definition of $\omega_{ij}$. The lower left square commutes since the right side isomorphism is exactly the inclusion $j_*$ from the long exact sequence on relative homology and the lower isomorphism collapses the 2-skeleton just as $p_*$ does. Thus the entire diagram commutes.

Since $H_n(X^n, X^{n-1})$ is generated by the $n$-cells of $X$ the characteristic map $\Phi_{is}$ takes some chosen generator $[D^n_i] \in H_n(D^n_i, S^{n-1})$ to one such $n$-cell $e^n_i$, so $\Phi_{is}([D^n_i]) = e^n_i$. Thus by the commutativity of the upper left square and the triangle we have that

$$d_n(e^n_i) = (d_n \circ \Phi_{is})([D^n_i]) = (j_* \circ \varphi_{is} \circ \partial_n)([D^n_i]).$$

By the commutativity of the lower right square we may exchange $j_*$ for $p_*$ and end up with the formula

$$d_n(e^n_i) = (p_* \circ \varphi_{is} \circ \partial_n)([D^n_i]).$$

Note that we have omitted any isomorphisms for clarity. The map $p_{js}$ collapses all the $(n-1)$-cells except for $e^{n-1}_j$. Since $H_{n-1}(S^{n-1}_j)$ can be interpreted as a subgroup of $H_{n-1}(X^{n-1}, X^{n-2})$, specifically as one of its summands corresponds to $e^{n-1}_j$, we may through the isomorphisms in the lower right square express $d_n$ as a sum over these $(n-1)$-cells $e^{n-1}_j$ through the map $p_{js}$:

$$d_n(e^n_i) = \sum_j (p_{js} \circ p_* \circ \varphi_{is} \circ \partial_n)([D^n_i]).$$

By the commutativity of the upper right square we can deduce that

$$d_n(e^n_i) = \sum_j (\omega_{ij} \circ \partial_n)([D^n_i]).$$

Finally, since $\omega_{ij}$ is multiplication by $d_{ij} = \deg(\omega_{ij})$ and $\partial_n([D^n_i]) = e^{n-1}_i$ we can conclude the formula to be proved:

$$d_n(e^n_i) = \sum_j \omega_{ij} e^{n-1}_j = \sum_j d_{ij} e^{n-1}_j.$$
Hurewicz theorem

The Hurewicz theorem describes a connection between homotopy and homology groups of an \( n \)-connected space \( X \), namely that the first dimension where the homotopy and homology groups are not trivial gives an isomorphism between the groups. This relationship can be put to great use especially in homotopy theory since homology groups are often easier to calculate compared to homotopy groups. The main components in the proof of the theorem are CW approximation, cellular homology with degree-based boundary maps, and a method of calculating homotopy groups for certain types of spaces using the boundary map from the long exact sequence. We begin this last section by stating the main theorem of this thesis:

**Theorem 4.1 (Hurewicz theorem).** Let \( X \) be an \( (n-1) \)-connected topological space with \( n \geq 2 \). Then \( \pi_n(X) \cong H_n(X) \) and \( H_i(X) = 0 \) for \( i < n \). Furthermore, if \( (X,A) \) is \( (n-1) \)-connected for some \( n \geq 2 \) and \( A \) is nonempty and simply connected, then \( H_i(X,A) = 0 \) for \( i < n \) and \( \pi_n(X,A) \cong H_n(X,A) \).

To prove the Hurewicz theorem we need some additional results that build upon the theory explored in the previous chapters. The first step is to use CW approximation to replace the space \( X \) with a more suitable CW complex, in particular one that can be expressed in terms of wedge sums:

**Definition 4.2.** Let \( \{(X_i, x_i)\}_{i \in J} \) be a collection of topological spaces \( X_i \) with basepoints \( x_i \) for some index set \( J \). The wedge sum \( \bigvee_i X_i \) is the quotient space formed of the disjoint union \( \bigsqcup_i X_i \) where all the basepoints \( x_i \in X_i \) are identified with one single point \( x_0 \).

The most useful property of wedge sums is how they relate to CW complexes. Namely if \( X \) is a CW complex and we collapse the \( (n-1) \)-skeleton to a single point, then the \( n \)-skeleton can be identified with a wedge sum of \( n \)-spheres \( \bigvee_i S^n_i \). Additionally, there is a one-to-one correspondence between these \( n \)-spheres and the \( n \)-cells of \( X \). This is easily visualised in lower dimensions, since for each \( n \)-cell that is left untouched by the identification we take its boundary and collapse it to a single point. This equates to the familiar quotient space \( D^n/S^{n-1} \cong S^n \).

The \( n \)-th homotopy group of a wedge sum of \( n \)-spheres behaves nicely and can be shown to be generated by the inclusions of the \( n \)-spheres into the wedge sum.

**Lemma 4.3.** Let \( X = \bigvee_i S^n_i \) be a wedge sum of \( n \)-spheres for some \( n \geq 2 \). Then the group \( \pi_n(X) \) is free abelian with basis the homotopy classes of the inclusions \( i: S^n_i \hookrightarrow X \).

**Proof.** See [Ha, Example 4.26] for proof.

The next lemma will be very useful in reducing the proof of the Hurewicz theorem to the absolute case to avoid dealing with the relative homotopy groups.
Lemma 4.4. Let $X$ be a CW complex and let $A \subset X$ be a CW subcomplex of $X$. If the pair $(X, A)$ is $r$-connected and $A$ is $s$-connected for some $r, s \geq 0$, then the homomorphism $q_*: \pi_n(X, A) \to \pi_n(X/A)$ induced by the quotient map $q: X \to X/A$ is an isomorphism for $n \leq r + s$ and a surjection for $n = r + s + 1$.

Proof. Let $n \geq 0$ and let $q: X \to X/A$ be the quotient map. Additionally, let $CA$ be the cone of $A$ and attach it to $X$ in such a way that $A \times \{0\}$ in the cone is attached to $A \subset X$. This yields a CW complex $X \cup CA$. The cone of a space is contractible and CW pairs have the homotopy extension property which means that the quotient map $p: X \cup CA \to (X \cup CA)/CA = X/A$ is a homotopy equivalence (see [Ha 0.16 and 0.17] for straightforward proofs). This homotopy equivalence gives us an isomorphism $p_*: \pi_n(X \cup CA) \to \pi_n((X \cup CA)/CA) = \pi_n(X/A)$.

Now consider the long exact sequence of the pairs $(CA, A)$ and $(X \cup CA, CA)$:

$$
\cdots \to \pi_n(A) \xrightarrow{i_*} \pi_n(CA) \xrightarrow{j_*} \pi_n(CA, A) \to \pi_{n-1}(A) \xrightarrow{i_*} \cdots
$$

and

$$
\cdots \to \pi_n(CA) \xrightarrow{i_*} \pi_n(X \cup CA) \xrightarrow{j_*} \pi_n(X \cup CA, CA) \to \pi_{n-1}(CA) \xrightarrow{i_*} \cdots
$$

Since $CA$ is contractible and has thus trivial homotopy groups, the first exact sequence gives an isomorphism $\partial: \pi_n(CA, A) \to \pi_{n-1}(A)$ which implies that $(CA, A)$ is $(s + 1)$-connected since $A$ is $s$-connected. The homotopy excision property now applies so that the inclusion $k: (X, A) \hookrightarrow (X \cup CA, CA)$ induces a homomorphism $k_*: \pi_n(X, A) \to \pi_n(X \cup CA, CA)$ which is an isomorphism for $n \leq r + s$ and a surjection for $n = r + s + 1$.

The second exact sequences gives that the inclusion $j$ induces an isomorphism $j_*: \pi_n(X \cup CA) \to \pi_n(X \cup CA, CA)$, again because the homotopy groups of $CA$ are trivial. Thus we can assemble the three homomorphisms into the following commutative diagram:

$$
\begin{array}{c}
\pi_n(X, A) \xrightarrow{k_*} \pi_n(X \cup CA, CA) \xrightarrow{\cong} \pi_n((X \cup CA)/CA) \xrightarrow{p_*} \pi_n(X/A) \\
\downarrow \scriptstyle{j_*} \cong \scriptstyle{\cong} \downarrow \scriptstyle{p_*} \\
\pi_n(X \cup CA) 
\end{array}
$$

The map $q$ can be expressed as $q = p \circ j^{-1} \circ k$ which implies that $q_* = p_* \circ j_*^{-1} \circ k_*$ is an isomorphism when $n \leq r + s$. The homomorphism $k_*$ is surjective when $n = r + s + 1$ which implies that also $q_*$ is surjective when $n = r + s + 1$. \hfill \Box

A key component of the proof of the Hurewicz theorem is to find a concrete connection between homotopy and homology groups. It turns out that one such connection can be found in the boundary homomorphisms for respective theories. The following lemma exemplifies this.
Lemma 4.5. Let X be a CW complex of the form \((\bigvee_i S^0_i) \cup_j e_j^{n+1}\) with the attaching maps of the cells \(e_j^n\) being constant and with basepoint-preserving attaching maps \(\varphi_j\) of the cells \(e_j^{n+1}\). Then the boundary homomorphism \(\partial_{n+1} : \pi_{n+1}(X,X^n) \to \pi_n(X^n)\) from the long exact sequence of the pair \((X,X^n)\) and the boundary homomorphism \(d_{n+1} : H_{n+1}(X^{n+1},X^n) \to H_n(X^n,X^{n-1})\) from the cellular chain complex of cellular homology are the same.

Proof. We must first show that the domains of the two homomorphisms coincide. The basis of the homology group \(H_{n+1}(X^{n+1},X^n)\) can be by Theorem 3.6 be interpreted as the \((n+1)\)-cells of the space \(X\). The same can be shown for the homotopy group \(\pi_{n+1}(X,X^n)\) as follows.

Lemma 4.4 applies to the pair \((X,X^n)\) so that especially \(\pi_n(X,X^n)\) is isomorphic to \(\pi_n(X/X^n)\). Now remember that if we collapse the boundaries of \(n\)-cells to a single point we are left with a wedge of the boundaries of \((n+1)\)-cells, namely \(\bigvee_i S_i^{n+1}\). Thus we can express \(\pi_n(X,X^n) = \pi_n(X,\bigvee_i S_i^n)\) as the homotopy group of the wedge of \((n+1)\)-spheres \(\pi_n(\bigvee_i S_i^{n+1})\). Lemma 4.3 now applies to express \(\pi_n(X,X^n)\) as a free abelian group with basis the homotopy classes of the inclusions \(S_i^{n+1} \to \bigvee_j S_j^n\). These inclusions are equivalently interpreted as the characteristic maps \(\Phi_j : D_j^{n+1} \to X\) of the cells \(e_j^{n+1}\) which map the boundary of \(D_j^{n+1}\) to the basepoint of the wedge sum. Denote by \(\alpha : H_{n+1}(X^{n+1},X^n) \to \pi_{n+1}(X,X^n)\) the isomorphism which takes each \((n+1)\)-cell \(e_j^{n+1}\) to its characteristic map \(\Phi_j\).

The boundary map \(d_{n+1}\) takes these classes of characteristic maps to the classes of the corresponding attaching maps \(\varphi_j : S_j^n \to X^n\). It now suffices to show that there exists an isomorphism that takes \([\varphi_j]\) to \(d_{n+1}(e_j^{n+1})\). From Theorem 3.5 we know that the degree map \(\deg : \pi_n(S^n) \to \mathbb{Z}\) is an isomorphism. Furthermore, Theorem 3.8 gives that \(d(e_j^{n+1}) = \sum_i d_{ij} e_i^n\) with \(d_{ij} = \deg(p_i \circ \varphi_j)\) where \(p_i\) collapses the complement of the cell \(e_i^n\) to a point. Note that the quotient map \(p\) in Theorem 3.8 is not included since it is equal to the identity map because the \((n-1)\)-skeleton is already a point set by construction.

The basis elements \([\varphi_i]\) can again be identified with the inclusions \(i : S_i^n \to X^n\) which are in one-to-one correspondence with the \(n\)-cells \(e_i^n\). The isomorphism from \([\varphi_i]\) to \(\sum_i d_{ij} e_i^n\) is thus obtained on each of these cells through the isomorphic degree map, combined together as a direct sum over the wedge sum. Thus it must follow that \(d_{n+1} = \partial_{n+1}\). \(\square\)

It turns out that the homotopy groups of a CW complex constructed from a wedge sum of \(n\)-spheres and some additional \((n+1)\)-cells can be calculated in a straightforward fashion as the cokernel of the boundary map of the corresponding long exact sequence. This result combined with the previous lemma will finish off the proof of the Hurewicz theorem.

Lemma 4.6. Let \(X\) be an \((n+1)\)-dimensional CW complex for \(n \geq 2\), which has been constructed by attaching \((n+1)\)-cells \(e_i^{n+1}\) to a wedge of spheres \(\bigvee_i S_i^n\) via basepoint-preserving attaching maps \(\varphi_j : S^n \to \bigvee_i S_i^n\). Then \(\pi_n(X) \cong \text{Coker} \partial_{n+1}\) where \(\text{Coker} \partial_{n+1}\) is the cokernel of the boundary map \(\partial_{n+1} : \pi_{n+1}(X,X^n) \to \pi_n(X^n)\) in the long exact sequence of the pair \((X,X^n)\).
Proof. Let us first consider the long exact sequence of the pair \((X, X^n) = (X, \bigvee_i S^n_i)\):

\[
... \xrightarrow{j^*} \pi_{n+1}(X, \bigvee_i S^n_i) \xrightarrow{\partial} \pi_n(\bigvee_i S^n_i) \xrightarrow{i_*} \pi_n(X) \xrightarrow{j_*} \pi_n(X, \bigvee_i S^n_i) \xrightarrow{\partial} ... 
\]

Similarly as in Example 2.10, both the space \(X\) and the space \(\bigvee_i S^n_i\) are \((n-1)\)-connected by cellular approximation. Thus \(\pi_i(X) = \pi_i(\bigvee_i S^n_i) = 0\) for all \(i < n\). Since the homotopy groups of both \(X\) and the space \(\bigvee_i S^n_i\) are trivial in dimensions less than \(n\), the long exact sequence takes the following form:

\[
... \xrightarrow{j^*} \pi_{n+1}(X, \bigvee_i S^n_i) \xrightarrow{\partial} \pi_n(\bigvee_i S^n_i) \xrightarrow{i_*} \pi_n(X) \longrightarrow 0.
\]

Recall from elementary algebra that the image \(\text{Im} f\) of a group homomorphism \(f: G \to H\) is isomorphic to the quotient \(G/\text{Ker} f\). Hence we see that \(\text{Im} i_* \cong \pi_n(\bigvee_i S^n_i)/\text{Ker} i_*\). By exactness we know that \(i_*\) is surjective and that \(\text{Ker} i_* = \text{Im} \partial\), and thus we conclude that

\[
\pi_n(X) = \text{Im} i_* \cong \pi_n(\bigvee_i S^n_i)/\text{Ker} i_* = \pi_n(\bigvee_i S^n_i)/\text{Im} \partial = \text{Coker} \partial.
\]

Finally we are ready to prove the Hurewicz theorem which we will restate here:

**Theorem 4.7** (Hurewicz theorem). Let \(X\) be an \((n-1)\)-connected topological space with \(n \geq 2\). Then \(\pi_n(X) \cong H_n(X)\) and \(H_i(X) = 0\) for \(i < n\). Furthermore, if \((X, A)\) is \((n-1)\)-connected for some \(n \geq 2\) and \(A\) is nonempty and simply connected, then \(H_i(X, A) = 0\) for \(i < n\) and \(\pi_n(X, A) \cong H_n(X, A)\).

**Proof.** Let us first replace \((X, A)\) by its CW approximation, the existence of which is guaranteed by Theorem 2.33. By Lemma 4.4 we have an isomorphism \(\pi_i(X, A) \cong \pi_i(X/A)\) for \(i < n + 1\) which reduces the question for homotopy groups to the absolute case. Note that it is crucial here that \(A\) is simply connected, for otherwise the isomorphism would not necessarily hold for \(i = n\). Similarly Proposition 2.25 gives an isomorphism \(H_i(X, A) \cong H_i(X/A)\) for all \(i \geq 0\) because CW pairs are good pairs. Thus the proof of the theorem reduces entirely to the absolute case.

We begin by replacing the space \(X\) with its homotopy equivalent CW approximation. The resulting CW complex can again be replaced by Corollary 2.35 with a homotopy equivalent CW complex consisting of cells of dimension \(n\) and higher with a single 0-cell as a basepoint. This implies that \(H_i(X^i, X^{i-1}) = 0\) for all \(i < n\) by Theorem 3.6 and thus \(H_i(X) = 0\) for \(i < n\).

Similarly we may remove any cells of dimension greater than \((n+1)\) from the space \(X\). This is because the group \(H_n(X)\) only depends on the groups \(H_{n+1}(X^{n+1}, X^n)\) and \(H_{n-1}(X^{n-1}, X^{n-2})\) which are by Theorem 3.6 generated by \((n+1)\)-cells and \((n-1)\)-cells respectively. The groups \(\pi_n(X)\) remain unaffected since all class representatives

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\( f: S^n \to X \) are homotopic to cellular maps by cellular approximation (Theorem 2.33), and thus affect only the \( n \)-skeleton of \( X \). The above construction leaves us with a CW complex \( X = (\bigvee_i S^n_i) \bigcup \bigcup_j e_j^{n+1} \) where the attaching maps \( \varphi_j \) of the cells \( e_j^{n+1} \) are basepoint-preserving.

Lemma 4.6 applies here to calculate \( \pi_n(X) \) as the Coker \( \partial_{n+1} \) which by Lemma 4.3 is the same as Coker \( d_{n+1} \) where \( d_{n+1}: H_{n+1}(X^{n+1}, X^n) \to H_n(X^n, X^{n-1}) \) is the boundary homomorphism from the cellular chain complex of cellular homology. The space \( X \) contains no \((n-1)\)-cells by construction which implies that \( H_{n-1}(X^{n-1}, X^{n-2}) = 0 \). This in turn implies that \( \text{Ker} \ d_n = H_n(X^n, X^{n-1}) \). Combining this with the previous observations we can conclude that

\[
H_n(X) = \text{Ker} \ d_n / \text{Im} \ d_{n+1} = H_n(X^n, X^{n-1}) / \text{Im} \ d_{n+1} = \text{Coker} \ d_{n+1} \cong \pi_n(X).
\]
Bibliography


