

Polish Notation

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Summary. This article is the first in a series formalizing some results in my joint work with Prof. Joanna Golińska-Pilarek ([12] and [13]) concerning a logic proposed by Prof. Andrzej Grzegorzczak ([14]).

We present some *mathematical folklore* about representing formulas in “Polish notation”, that is, with operators of fixed arity prepended to their arguments. This notation, which was published by Jan Łukasiewicz in [15], eliminates the need for parentheses and is generally well suited for rigorous reasoning about syntactic properties of formulas.

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The notation and terminology used in this paper have been introduced in the following articles: [5], [1], [4], [11], [7], [8], [3], [9], [16], [19], [17], [18], and [10].

1. PRELIMINARIES

From now on k, m, n denote natural numbers, a, b, c, c_1, c_2 denote objects, x, y, z, X, Y, Z denote sets, D denotes a non empty set, p, q, r, s, t, u, v denote finite sequences, $P, Q, R, P_1, P_2, Q_1, Q_2, R_1, R_2$ denote finite sequence-membered sets, and S, T denote non empty, finite sequence-membered sets.

Let D be a non empty set and P, Q be subsets of D^* . The functor $\frown(D, P, Q)$ yielding a subset of D^* is defined by the term

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(Def. 1) $\{p \frown q, \text{ where } p \text{ is a finite sequence of elements of } D, q \text{ is a finite sequence of elements of } D : p \in P \text{ and } q \in Q\}$.

Let us consider P and Q . The functor $P \frown Q$ yielding a finite sequence-membered set is defined by

(Def. 2) for every $a, a \in it$ iff there exists p and there exists q such that $a = p \frown q$ and $p \in P$ and $q \in Q$.

Let β be an empty set. One can check that $\beta \frown P$ is empty and $P \frown \beta$ is empty.

Let us consider S and T . One can check that $S \frown T$ is non empty.

Now we state the propositions:

- (1) If $p \frown q = r \frown s$, then there exists t such that $p \frown t = r$ or $p = r \frown t$.
- (2) $(P \frown Q) \frown R = P \frown (Q \frown R)$.

PROOF: For every $a, a \in (P \frown Q) \frown R$ iff $a \in P \frown (Q \frown R)$ by [4, (32)]. \square

Note that $\{\emptyset\}$ is non empty and finite sequence-membered.

- (3) (i) $P \frown \{\emptyset\} = P$, and
- (ii) $\{\emptyset\} \frown P = P$.

PROOF: For every $a, a \in P \frown \{\emptyset\}$ iff $a \in P$ by [4, (34)]. For every $a, a \in \{\emptyset\} \frown P$ iff $a \in P$ by [4, (34)]. \square

Let us consider P . The functor $P \frown \frown$ yielding a function is defined by

(Def. 3) $\text{dom } it = \mathbb{N}$ and $it(0) = \{\emptyset\}$ and for every n , there exists Q such that $Q = it(n)$ and $it(n + 1) = Q \frown P$.

Let us consider n . The functor $P \frown n$ yielding a finite sequence-membered set is defined by the term

(Def. 4) $(P \frown \frown)(n)$.

Now we state the proposition:

- (4) $\emptyset \in P \frown 0$.

Let us consider P . Let n be a zero natural number. Note that $P \frown n$ is non empty.

Let β be an empty set and n be a non zero natural number. One can verify that $\beta \frown n$ is empty.

Let us consider P . The functor P^* yielding a non empty, finite sequence-membered set is defined by the term

(Def. 5) \bigcup the set of all $P \frown n$ where n is a natural number.

- (5) $a \in P^*$ if and only if there exists n such that $a \in P \frown n$.

Let us consider P .

- (6) (i) $P \frown 0 = \{\emptyset\}$, and
- (ii) for every $n, P \frown (n + 1) = (P \frown n) \frown P$.

(7) $P \frown 1 = P$. The theorem is a consequence of (6) and (3).

(8) $P \frown n \subseteq P^*$.

(9) (i) $\emptyset \in P^*$, and

(ii) $P \subseteq P^*$.

The theorem is a consequence of (4), (5), and (7).

(10) $P \frown (m + n) = (P \frown m) \frown (P \frown n)$.

PROOF: Define $\mathcal{X}[\text{natural number}] \equiv P \frown (m + \$_1) = (P \frown m) \frown (P \frown \$_1)$. $\mathcal{X}[0]$. For every k such that $\mathcal{X}[k]$ holds $\mathcal{X}[k + 1]$. For every k , $\mathcal{X}[k]$ from [2, Sch. 2]. \square

(11) If $p \in P \frown m$ and $q \in P \frown n$, then $p \frown q \in P \frown (m + n)$. The theorem is a consequence of (10).

(12) If $p, q \in P^*$, then $p \frown q \in P^*$. The theorem is a consequence of (5) and (11).

(13) If $P \subseteq R^*$ and $Q \subseteq R^*$, then $P \frown Q \subseteq R^*$. The theorem is a consequence of (12).

(14) If $Q \subseteq P^*$, then $Q \frown n \subseteq P^*$.

PROOF: Define $\mathcal{X}[\text{natural number}] \equiv Q \frown \$_1 \subseteq P^*$. $\mathcal{X}[0]$. For every k such that $\mathcal{X}[k]$ holds $\mathcal{X}[k + 1]$. For every k , $\mathcal{X}[k]$ from [2, Sch. 2]. \square

(15) If $Q \subseteq P^*$, then $Q^* \subseteq P^*$. The theorem is a consequence of (5) and (14).

(16) If $P_1 \subseteq P_2$ and $Q_1 \subseteq Q_2$, then $P_1 \frown Q_1 \subseteq P_2 \frown Q_2$.

(17) If $P \subseteq Q$, then for every n , $P \frown n \subseteq Q \frown n$.

PROOF: Define $\mathcal{S}[\text{natural number}] \equiv P \frown \$_1 \subseteq Q \frown \$_1$. $P \frown 0 = \{\emptyset\}$. For every n such that $\mathcal{S}[n]$ holds $\mathcal{S}[n + 1]$. For every n , $\mathcal{S}[n]$ from [2, Sch. 2]. \square

Let us consider S and n . Let us observe that $S \frown n$ is non empty and finite sequence-membered.

2. THE LANGUAGE

In the sequel α denotes a function from P into \mathbb{N} and U, V, W denote subsets of P^* .

Let us consider P, α , and U . The Polish-expression layer(P, α, U) yielding a subset of P^* is defined by

(Def. 6) for every a , $a \in it$ iff $a \in P^*$ and there exists p and there exists q and there exists n such that $a = p \frown q$ and $p \in P$ and $n = \alpha(p)$ and $q \in U \frown n$.

Now we state the proposition:

(18) Suppose $p \in P$ and $n = \alpha(p)$ and $q \in U \frown n$. Then $p \frown q \in$ the Polish-expression layer(P, α, U). The theorem is a consequence of (14), (9), and (12).

Let us consider P and α . The Polish atoms(P, α) yielding a subset of P^* is defined by

(Def. 7) for every $a, a \in it$ iff $a \in P$ and $\alpha(a) = 0$.

The Polish operations(P, α) yielding a subset of P is defined by the term

(Def. 8) $\{t, \text{ where } t \text{ is an element of } P^* : t \in P \text{ and } \alpha(t) \neq 0\}$.

Now we state the propositions:

(19) The Polish atoms(P, α) \subseteq the Polish-expression layer(P, α, U). The theorem is a consequence of (4) and (18).

(20) Suppose $U \subseteq V$. Then the Polish-expression layer(P, α, U) \subseteq the Polish-expression layer(P, α, V). The theorem is a consequence of (17).

(21) Suppose $u \in$ the Polish-expression layer(P, α, U). Then there exists p and there exists q such that $p \in P$ and $u = p \wedge q$.

Let us consider P and α . The Polish-expression hierarchy(P, α) yielding a function is defined by

(Def. 9) $\text{dom } it = \mathbb{N}$ and $it(0) =$ the Polish atoms(P, α) and for every n , there exists U such that $U = it(n)$ and $it(n + 1) =$ the Polish-expression layer(P, α, U).

Let us consider n . The Polish-expression hierarchy(P, α, n) yielding a subset of P^* is defined by the term

(Def. 10) (the Polish-expression hierarchy(P, α))(n).

Now we state the proposition:

(22) The Polish-expression hierarchy($P, \alpha, 0$) = the Polish atoms(P, α).

Let us consider P, α , and n . Now we state the propositions:

(23) The Polish-expression hierarchy($P, \alpha, n + 1$) = the Polish-expression layer($P, \alpha, \text{ the Polish-expression hierarchy}(P, \alpha, n)$).

(24) The Polish-expression hierarchy(P, α, n) \subseteq the Polish-expression hierarchy($P, \alpha, n + 1$).

PROOF: Define $\mathcal{S}[\text{natural number}] \equiv$ the Polish-expression hierarchy($P, \alpha, \$1$) \subseteq the Polish-expression hierarchy($P, \alpha, \$1 + 1$). $\mathcal{S}[0]$. For every k such that $\mathcal{S}[k]$ holds $\mathcal{S}[k + 1]$. For every $k, \mathcal{S}[k]$ from [2, Sch. 2]. \square

Now we state the proposition:

(25) The Polish-expression hierarchy(P, α, n) \subseteq the Polish-expression hierarchy($P, \alpha, n + m$).

PROOF: Define $\mathcal{S}[\text{natural number}] \equiv$ the Polish-expression hierarchy(P, α, n) \subseteq the Polish-expression hierarchy($P, \alpha, n + \$1$). For every k such that $\mathcal{S}[k]$ holds $\mathcal{S}[k + 1]$. For every $k, \mathcal{S}[k]$ from [2, Sch. 2]. \square

Let us consider P and α . The Polish-expression set(P, α) yielding a subset of P^* is defined by the term

(Def. 11) \cup the set of all the Polish-expression hierarchy(P, α, n) where n is a natural number.

Now we state the propositions:

(26) The Polish-expression hierarchy(P, α, n) \subseteq the Polish-expression set(P, α).

(27) Suppose $q \in$ (the Polish-expression set(P, α)) $\cap n$. Then there exists m such that $q \in$ (the Polish-expression hierarchy(P, α, m)) $\cap n$.

PROOF: Define \mathcal{S} [natural number] \equiv for every q such that $q \in$ (the Polish-expression set(P, α)) $\cap \mathbb{N}_1$ there exists m such that $q \in$ (the Polish-expression hierarchy(P, α, m)) $\cap \mathbb{N}_1$. $\mathcal{S}[0]$. For every k such that $\mathcal{S}[k]$ holds $\mathcal{S}[k+1]$. For every k , $\mathcal{S}[k]$ from [2, Sch. 2]. \square

(28) Suppose $a \in$ the Polish-expression set(P, α). Then there exists n such that $a \in$ the Polish-expression hierarchy($P, \alpha, n+1$). The theorem is a consequence of (24).

Let us consider P and α .

A Polish expression of P and α is an element of the Polish-expression set(P, α). Let us consider n and t . Assume $t \in P$. The Polish operation(P, α, n, t) yielding a function from (the Polish-expression set(P, α)) $\cap n$ into P^* is defined by

(Def. 12) for every q such that $q \in \text{dom } it$ holds $it(q) = t \cap q$.

Let us consider X and Y . Let F be a partial function from X to 2^Y . One can check that F is disjoint valued if and only if the condition (Def. 13) is satisfied.

(Def. 13) for every a and b such that $a, b \in \text{dom } F$ and $a \neq b$ holds $F(a)$ misses $F(b)$.

Let X be a set. One can check that there exists a finite sequence of elements of 2^X which is disjoint valued.

Now we state the proposition:

(29) Let us consider a set X , a disjoint valued finite sequence B of elements of 2^X , a, b , and c . If $a \in B(b)$ and $a \in B(c)$, then $b = c$ and $b \in \text{dom } B$.

Let us consider X . Let B be a disjoint valued finite sequence of elements of 2^X . The arity from list B yielding a function from X into \mathbb{N} is defined by

(Def. 14) for every a such that $a \in X$ holds there exists n such that $a \in B(n)$ and $a \in B(it(a))$ or there exists no n such that $a \in B(n)$ and $it(a) = 0$.

Now we state the propositions:

(30) Let us consider a disjoint valued finite sequence B of elements of 2^X , and a . Suppose $a \in X$. Then (the arity from list B)(a) $\neq 0$ if and only if

there exists n such that $a \in B(n)$. The theorem is a consequence of (29).

(31) Let us consider a disjoint valued finite sequence B of elements of 2^X , a , and n . Suppose $a \in B(n)$. Then (the arity from list B)(a) = n . The theorem is a consequence of (29).

(32) Suppose $r \in$ the Polish-expression set(P, α). Then there exists n and there exists p and there exists q such that $p \in P$ and $n = \alpha(p)$ and $q \in$ (the Polish-expression set(P, α)) $\hat{\ } n$ and $r = p \hat{\ } q$. The theorem is a consequence of (28), (23), (26), and (17).

Let us consider P, α , and Q . We say that Q is α -closed if and only if

(Def. 15) for every p, n , and q such that $p \in P$ and $n = \alpha(p)$ and $q \in Q \hat{\ } n$ holds $p \hat{\ } q \in Q$.

Now we state the propositions:

(33) The Polish-expression set(P, α) is α -closed. The theorem is a consequence of (27), (18), (23), and (26).

(34) If Q is α -closed, then the Polish atoms(P, α) $\subseteq Q$. The theorem is a consequence of (4).

(35) If Q is α -closed, then the Polish-expression hierarchy(P, α, n) $\subseteq Q$.

PROOF: Define \mathcal{X} [natural number] \equiv the Polish-expression hierarchy(P, α, \mathbb{S}_1) $\subseteq Q$. $\mathcal{X}[0]$. For every k such that $\mathcal{X}[k]$ holds $\mathcal{X}[k + 1]$. For every k , $\mathcal{X}[k]$ from [2, Sch. 2]. \square

(36) The Polish atoms(P, α) \subseteq the Polish-expression set(P, α). The theorem is a consequence of (33) and (34).

(37) If Q is α -closed, then the Polish-expression set(P, α) $\subseteq Q$. The theorem is a consequence of (28) and (35).

(38) Suppose $r \in$ the Polish-expression set(P, α). Then there exists n and there exists t and there exists q such that $t \in P$ and $n = \alpha(t)$ and $r =$ (the Polish operation(P, α, n, t))(q). The theorem is a consequence of (28), (23), (26), and (17).

(39) Suppose $p \in P$ and $n = \alpha(p)$ and $q \in$ (the Polish-expression set(P, α)) $\hat{\ } n$. Then (the Polish operation(P, α, n, p))(q) \in the Polish-expression set(P, α). The theorem is a consequence of (33).

The scheme *AInd* deals with a finite sequence-membered set \mathcal{P} and a function α from \mathcal{P} into \mathbb{N} and a unary predicate \mathcal{X} and states that

(Sch. 1) For every a such that $a \in$ the Polish-expression set(\mathcal{P}, α) holds $\mathcal{X}[a]$ provided

- for every p, q , and n such that $p \in \mathcal{P}$ and $n = \alpha(p)$ and $q \in$ (the Polish-expression set(\mathcal{P}, α)) $\hat{\ } n$ holds $\mathcal{X}[p \hat{\ } q]$.

3. PARSING

In the sequel k, l, m, n, i, j denote natural numbers, a, b, c, c_1, c_2 denote objects, x, y, z, X, Y, Z denote sets, D, D_1, D_2 denote non empty sets, p, q, r, s, t, u, v denote finite sequences, and P, Q, R denote finite sequence-membered sets.

Let us consider P . We say that P is antichain-like if and only if

(Def. 16) for every p and q such that $p, p \wedge q \in P$ holds $q = \emptyset$.

Now we state the propositions:

(40) P is antichain-like if and only if for every p and q such that $p, p \wedge q \in P$ holds $p = p \wedge q$.

PROOF: If P is antichain-like, then for every p and q such that $p, p \wedge q \in P$ holds $p = p \wedge q$ by [4, (34)]. \square

(41) If $P \subseteq Q$ and Q is antichain-like, then P is antichain-like.

Note that every finite sequence-membered set which is trivial is also antichain-like.

Now we state the proposition:

(42) If $P = \{a\}$, then P is antichain-like.

Note that there exists a non empty, finite sequence-membered set which is antichain-like and every finite sequence-membered set which is empty is also antichain-like.

An antichain is an antichain-like, finite sequence-membered set. In the sequel B, C denote antichains.

Let us consider B . One can verify that every subset of B is antichain-like and finite sequence-membered.

A Polish-language is a non empty antichain. From now on S, T denote Polish-languages.

Let D be a non empty set and ψ be a subset of D^* . Note that ψ is antichain-like if and only if the condition (Def. 17) is satisfied.

(Def. 17) for every finite sequence g of elements of D and for every finite sequence h of elements of D such that $g, g \wedge h \in \psi$ holds $h = \varepsilon_D$.

Now we state the proposition:

(43) If $p \wedge q = r \wedge s$ and $p, r \in B$, then $p = r$ and $q = s$. The theorem is a consequence of (1) and (40).

Let us consider B and C . Note that $B \wedge C$ is antichain-like.

Now we state the propositions:

(44) If for every p and q such that $p, q \in P$ holds $\text{dom } p = \text{dom } q$, then P is antichain-like.

PROOF: For every p and q such that $p, p \wedge q \in P$ holds $p = p \wedge q$ by [4, (21)]. \square

(45) If for every p such that $p \in P$ holds $\text{dom } p = a$, then P is antichain-like. The theorem is a consequence of (44).

(46) If $\emptyset \in B$, then $B = \{\emptyset\}$.

PROOF: For every a such that $a \in B$ holds $a = \emptyset$ by [4, (34)]. \square

Let us consider B and n . Note that $B \wedge n$ is antichain-like.

Let us consider T . Let us observe that there exists a subset of T^* which is non empty and antichain-like and $T \wedge n$ is non empty.

A Polish-language of T is a non empty, antichain-like subset of T^* .

A Polish arity-function of T is a function from T into \mathbb{N} and is defined by

(Def. 18) there exists a such that $a \in T$ and $it(a) = 0$.

One can verify that every Polish-language of T is non empty, antichain-like, and finite sequence-membered.

In the sequel α denotes a Polish arity-function of T and U, V, W denote Polish-languages of T .

Let us consider T and α . Let t be an element of T . Let us observe that the functor $\alpha(t)$ yields a natural number. Let us consider U . Note that the Polish-expression $\text{layer}(T, \alpha, U)$ is defined by

(Def. 19) for every $a, a \in it$ iff there exists an element t of T and there exists an element u of T^* such that $a = t \wedge u$ and $u \in U \wedge \alpha(t)$.

Let us consider B and p . We say that p is B -headed if and only if

(Def. 20) there exists q and there exists r such that $q \in B$ and $p = q \wedge r$.

Let us consider P . We say that P is B -headed if and only if

(Def. 21) for every p such that $p \in P$ holds p is B -headed.

Now we state the propositions:

(47) If p is B -headed and $B \subseteq C$, then p is C -headed.

(48) If P is B -headed and $B \subseteq C$, then P is C -headed.

Let us consider B and P . Observe that $B \wedge P$ is B -headed.

Now we state the propositions:

(49) If p is $(B \wedge C)$ -headed, then p is B -headed.

(50) B is B -headed. The theorem is a consequence of (3).

Let us consider B . Let us observe that there exists a finite sequence-membered set which is B -headed.

Let P be a B -headed, finite sequence-membered set. Let us note that every subset of P is B -headed.

Let us consider S . Let us observe that there exists a finite sequence-membered set which is non empty and S -headed.

Now we state the proposition:

(51) $S \frown (m + n)$ is $(S \frown m)$ -headed. The theorem is a consequence of (10).

Let us consider S and p . The functor $S\text{-head}(p)$ yielding a finite sequence is defined by

- (Def. 22) (i) $it \in S$ and there exists r such that $p = it \frown r$, **if** p is S -headed,
(ii) $it = \emptyset$, **otherwise**.

The functor $S\text{-tail}(p)$ yielding a finite sequence is defined by

- (Def. 23) $p = (S\text{-head}(p)) \frown it$.

Now we state the propositions:

(52) If $s \in S$, then $S\text{-head}(s \frown t) = s$ and $S\text{-tail}(s \frown t) = t$.

(53) If $s \in S$, then $S\text{-head}(s) = s$ and $S\text{-tail}(s) = \emptyset$. The theorem is a consequence of (52).

Let us consider S , T , and u . Now we state the propositions:

(54) If $u \in S \frown T$, then $S\text{-head}(u) \in S$ and $S\text{-tail}(u) \in T$. The theorem is a consequence of (52).

(55) If $S \subseteq T$ and u is S -headed, then $S\text{-head}(u) = T\text{-head}(u)$ and $S\text{-tail}(u) = T\text{-tail}(u)$. The theorem is a consequence of (52).

Now we state the propositions:

(56) Suppose s is S -headed. Then

- (i) $s \frown t$ is S -headed, and
(ii) $S\text{-head}(s \frown t) = S\text{-head}(s)$, and
(iii) $S\text{-tail}(s \frown t) = (S\text{-tail}(s)) \frown t$.

The theorem is a consequence of (52).

(57) If $m + 1 \leq n$ and $s \in S \frown n$, then s is $(S \frown m)$ -headed and $S \frown m\text{-tail}(s)$ is S -headed. The theorem is a consequence of (51), (10), (54), and (7).

(58) (i) s is $(S \frown 0)$ -headed, and

- (ii) $S \frown 0\text{-head}(s) = \emptyset$, and
(iii) $S \frown 0\text{-tail}(s) = s$.

The theorem is a consequence of (4) and (52).

Let us consider T and α . One can verify that the Polish atoms(T , α) is non empty and antichain-like.

Let us consider U . Let us observe that the Polish-expression layer(T , α , U) is non empty and antichain-like.

One can verify that the Polish-expression layer(T , α , U) yields a Polish-language of T . The Polish operations(T , α) yielding a subset of T is defined by the term

(Def. 24) $\{t, \text{ where } t \text{ is an element of } T : \alpha(t) \neq 0\}$.

Let us consider n . Let us note that the Polish-expression hierarchy (T, α, n) is antichain-like and non empty.

One can check that the Polish-expression hierarchy (T, α, n) yields a Polish-language of T . The functor Polish-WFF-set (T, α) yielding a Polish-language of T is defined by the term

(Def. 25) the Polish-expression set (T, α) .

A Polish WFF of T and α is an element of Polish-WFF-set (T, α) . Let t be an element of T . The Polish operation (T, α, t) yielding a function from Polish-WFF-set $(T, \alpha) \cap \alpha(t)$ into Polish-WFF-set (T, α) is defined by the term

(Def. 26) the Polish operation $(T, \alpha, \alpha(t), t)$.

Assume $\alpha(t) = 1$. The functor Polish-unOp (T, α, t) yielding a unary operation on Polish-WFF-set (T, α) is defined by the term

(Def. 27) the Polish operation (T, α, t) .

Assume $\alpha(t) = 2$. The functor Polish-binOp (T, α, t) yielding a binary operation on Polish-WFF-set (T, α) is defined by

(Def. 28) for every u and v such that $u, v \in \text{Polish-WFF-set}(T, \alpha)$ holds $it(u, v) = (\text{the Polish operation}(T, \alpha, t))(u \cap v)$.

In the sequel φ, ψ denote Polish WFFs of T and α .

Let us consider X and Y . Let F be a partial function from X to 2^Y . We say that F is exhaustive if and only if

(Def. 29) for every a such that $a \in Y$ there exists b such that $b \in \text{dom } F$ and $a \in F(b)$.

Let X be a non empty set. Observe that there exists a finite sequence of elements of 2^X which is non exhaustive and disjoint valued.

Now we state the proposition:

(59) Let us consider a partial function F from X to 2^Y . Then F is not exhaustive if and only if there exists a such that $a \in Y$ and for every b such that $b \in \text{dom } F$ holds $a \notin F(b)$.

Let us consider T . Let B be a non exhaustive, disjoint valued finite sequence of elements of 2^T . The Polish arity from list B yielding a Polish arity-function of T is defined by the term

(Def. 30) the arity from list B .

One can check that there exists an antichain-like, finite sequence-membered set which has non empty elements and there exists a Polish-language which is non trivial and every antichain-like, finite sequence-membered set which is non trivial has also non empty elements.

Let us consider S , n , and m . Let p be an element of $S \frown (n + 1 + m)$. The functor $\text{decomp}(S, n, m, p)$ yielding an element of S is defined by the term

(Def. 31) $S\text{-head}(S \frown n\text{-tail}(p))$.

Let p be an element of $S \frown n$. The functor $\text{decomp}(S, n, p)$ yielding a finite sequence of elements of S is defined by

(Def. 32) $\text{dom } it = \text{Seg } n$ and for every m such that $m \in \text{Seg } n$ there exists k such that $m = k + 1$ and $it(m) = S\text{-head}(S \frown k\text{-tail}(p))$.

Now we state the propositions:

(60) Let us consider an element s of $S \frown n$, and an element t of $T \frown n$. If $S \subseteq T$ and $s = t$, then $\text{decomp}(S, n, s) = \text{decomp}(T, n, t)$.

PROOF: Set $p = \text{decomp}(S, n, s)$. Set $q = \text{decomp}(T, n, t)$. For every a such that $a \in \text{Seg } n$ holds $p(a) = q(a)$ by (17), [4, (1)], (57), (55). \square

(61) Let us consider an element q of $S \frown 0$. Then $\text{decomp}(S, 0, q) = \emptyset$.

(62) Let us consider an element q of $S \frown n$. Then $\text{len } \text{decomp}(S, n, q) = n$.

(63) Let us consider an element q of $S \frown 1$. Then $\text{decomp}(S, 1, q) = \langle q \rangle$. The theorem is a consequence of (7), (58), (53), and (62).

(64) Let us consider elements p, q of S , and an element r of $S \frown 2$. Suppose $r = p \frown q$. Then $\text{decomp}(S, 2, r) = \langle p, q \rangle$. The theorem is a consequence of (58), (52), (7), (53), and (62).

(65) Polish-WFF-set(T, α) is T -headed. The theorem is a consequence of (28), (23), and (21).

(66) The Polish-expression hierarchy(T, α, n) is T -headed. The theorem is a consequence of (26) and (65).

Let us consider T, α , and φ . The functor Polish-WFF-head φ yielding an element of T is defined by the term

(Def. 33) $T\text{-head}(\varphi)$.

Let us consider n . Let φ be an element of the Polish-expression hierarchy(T, α, n). The functor Polish-WFF-head φ yielding an element of T is defined by the term

(Def. 34) $T\text{-head}(\varphi)$.

Let us consider φ . The Polish arity φ yielding a natural number is defined by the term

(Def. 35) $\alpha(\text{Polish-WFF-head } \varphi)$.

Let us consider n . Let φ be an element of the Polish-expression hierarchy(T, α, n). The Polish arity φ yielding a natural number is defined by the term

(Def. 36) $\alpha(\text{Polish-WFF-head } \varphi)$.

Now we state the propositions:

(67) $T\text{-tail}(\varphi) \in \text{Polish-WFF-set}(T, \alpha) \frown (\text{the Polish arity } \varphi)$. The theorem is a consequence of (32) and (52).

(68) Let us consider an element φ of the Polish-expression hierarchy($T, \alpha, n + 1$). Then $T\text{-tail}(\varphi) \in (\text{the Polish-expression hierarchy}(T, \alpha, n)) \frown (\text{the Polish arity } \varphi)$. The theorem is a consequence of (23) and (52).

Let us consider T, α , and φ . The functor $(T, \alpha)\text{-tail } \varphi$ yielding an element of $\text{Polish-WFF-set}(T, \alpha) \frown (\text{the Polish arity } \varphi)$ is defined by the term

(Def. 37) $T\text{-tail}(\varphi)$.

Now we state the proposition:

(69) If $T\text{-head}(\varphi) \in \text{the Polish atoms}(T, \alpha)$, then $\varphi = T\text{-head}(\varphi)$. The theorem is a consequence of (67) and (6).

Let us consider T, α , and n . Let φ be an element of the Polish-expression hierarchy($T, \alpha, n+1$). The functor $(T, \alpha)\text{-tail } \varphi$ yielding an element of $(\text{the Polish-expression hierarchy}(T, \alpha, n)) \frown (\text{the Polish arity } \varphi)$ is defined by the term

(Def. 38) $T\text{-tail}(\varphi)$.

Let us consider φ . The functor $\text{Polish-WFF-args } \varphi$ yielding a finite sequence of elements of $\text{Polish-WFF-set}(T, \alpha)$ is defined by the term

(Def. 39) $\text{decomp}(\text{Polish-WFF-set}(T, \alpha), \text{the Polish arity } \varphi, (T, \alpha)\text{-tail } \varphi)$.

Let us consider n . Let φ be an element of the Polish-expression hierarchy($T, \alpha, n + 1$). The functor $\text{Polish-WFF-args } \varphi$ yielding a finite sequence of elements of the Polish-expression hierarchy(T, α, n) is defined by the term

(Def. 40) $\text{decomp}(\text{the Polish-expression hierarchy}(T, \alpha, n), \text{the Polish arity } \varphi, (T, \alpha)\text{-tail } \varphi)$.

Now we state the propositions:

(70) Let us consider an element t of T , and u .

Suppose $u \in \text{Polish-WFF-set}(T, \alpha) \frown \alpha(t)$.

Then $T\text{-tail}((\text{the Polish operation}(T, \alpha, t))(u)) = u$. The theorem is a consequence of (52).

(71) Suppose $\varphi \in \text{the Polish-expression hierarchy}(T, \alpha, n + 1)$.

Then $\text{rng Polish-WFF-args } \varphi \subseteq \text{the Polish-expression hierarchy}(T, \alpha, n)$. The theorem is a consequence of (60) and (26).

(72) Let us consider a finite sequence p , a function f from Y into D , and a function g from Z into D . Suppose $\text{rng } p \subseteq Y$ and $\text{rng } p \subseteq Z$ and for every a such that $a \in \text{rng } p$ holds $f(a) = g(a)$. Then $f \cdot p = g \cdot p$.

PROOF: Reconsider $p_1 = p$ as a finite sequence of elements of Y . Reconsider $q = f \cdot p_1$ as a finite sequence. Reconsider $p_2 = p$ as a finite sequence of elements of Z . Reconsider $r = g \cdot p_2$ as a finite sequence. $q = r$ by [6, (33)], [4, (1)], [7, (13), (3)]. \square

Let us consider T , α , and D . The Polish recursion-domain(α , D) yielding a subset of $T \times D^*$ is defined by the term

(Def. 41) $\{\langle t, p \rangle, \text{ where } t \text{ is an element of } T, p \text{ is a finite sequence of elements of } D : \text{len } p = \alpha(t)\}$.

A Polish recursion-function of α and D is a function from the Polish recursion-domain(α , D) into D . From now on f denotes a Polish recursion-function of α and D and γ , γ_1 , γ_2 denote functions from Polish-WFF-set(T , α) into D .

Let us consider T , α , D , f , and γ . We say that γ is f -recursive if and only if

(Def. 42) for every φ , $\gamma(\varphi) = f(\langle T\text{-head}(\varphi), \gamma \cdot \text{Polish-WFF-args } \varphi \rangle)$.

Now we state the proposition:

(73) If γ_1 is f -recursive and γ_2 is f -recursive, then $\gamma_1 = \gamma_2$. The theorem is a consequence of (36), (17), (33), (52), (60), (72), and (37).

From now on L denotes a non trivial Polish-language, β denotes a Polish arity-function of L , g denotes a Polish recursion-function of β and D , J , J_1 denote subsets of Polish-WFF-set(L , β), H denotes a function from J into D , H_1 denotes a function from J_1 into D .

Let us consider L , β , D , g , J , and H . We say that H is g -recursive if and only if

(Def. 43) for every Polish WFF φ of L and β such that $\varphi \in J$ and $\text{rng Polish-WFF-args } \varphi \subseteq J$ holds $H(\varphi) = g(\langle L\text{-head}(\varphi), H \cdot \text{Polish-WFF-args } \varphi \rangle)$.

Now we state the propositions:

(74) There exists J and there exists H such that $J =$ the Polish-expression hierarchy(L , β , n) and H is g -recursive.

PROOF: Define $\mathcal{X}[\text{natural number}] \equiv$ there exists J and there exists H such that $J =$ the Polish-expression hierarchy(L , β , $\$1$) and H is g -recursive. For every n , $\mathcal{X}[n]$ from [2, Sch. 2]. \square

(75) There exists a function γ from Polish-WFF-set(L , β) into D such that γ is g -recursive.

PROOF: Set $W = \text{Polish-WFF-set}(L, \beta)$. Define $\mathcal{X}[\text{object, object}] \equiv$ there exists n and there exists J_1 and there exists H_1 such that $J_1 =$ the Polish-expression hierarchy(L , β , n) and H_1 is g -recursive and $\$1 \in J_1$ and $\$2 = H_1(\$1)$. For every a such that $a \in W$ there exists b such that $b \in D$ and $\mathcal{X}[a, b]$ by (28), (74), [8, (5)]. Consider γ being a function from W into D such that for every a such that $a \in W$ holds $\mathcal{X}[a, \gamma(a)]$ from [8, Sch. 1]. \square

(76) Let us consider an element t of L . Then the Polish operation(L , β , t) is one-to-one.

PROOF: Set $f =$ the Polish operation(L, β, t). For every a and b such that $a, b \in \text{dom } f$ and $f(a) = f(b)$ holds $a = b$ by [4, (33)]. \square

(77) Let us consider elements t, u of L . Suppose $\text{rng}(\text{the Polish operation}(L, \beta, t))$ meets $\text{rng}(\text{the Polish operation}(L, \beta, u))$. Then $t = u$. The theorem is a consequence of (43).

(78) Let us consider an element t of L , and a . Suppose $a \in \text{dom}(\text{the Polish operation}(L, \beta, t))$. Then there exists p such that

- (i) $p = (\text{the Polish operation}(L, \beta, t))(a)$, and
- (ii) $L\text{-head}(p) = t$.

The theorem is a consequence of (52).

Let us consider L, β , an element t of L , and a Polish WFF φ of L and β . Now we state the proposition:

(79) Polish-WFF-head $\varphi = t$ if and only if there exists an element u of Polish-WFF-set(L, β) $\cap \beta(t)$ such that $\varphi = (\text{the Polish operation}(L, \beta, t))(u)$. The theorem is a consequence of (52).

Let us assume that $\beta(t) = 1$. Now we state the propositions:

(80) If Polish-WFF-head $\varphi = t$, then there exists a Polish WFF ψ of L and β such that $\varphi = (\text{Polish-unOp}(L, \beta, t))(\psi)$. The theorem is a consequence of (79) and (7).

(81) (i) Polish-WFF-head($(\text{Polish-unOp}(L, \beta, t))(\varphi)$) = t , and
(ii) Polish-WFF-args($(\text{Polish-unOp}(L, \beta, t))(\varphi)$) = $\langle \varphi \rangle$.

The theorem is a consequence of (7), (79), (70), and (63).

Now we state the proposition:

(82) Suppose $\beta(t) = 2$. Then suppose Polish-WFF-head $\varphi = t$. Then there exist Polish WFFs ψ, H of L and β such that $\varphi = (\text{Polish-binOp}(L, \beta, t))(\psi, H)$. The theorem is a consequence of (79), (6), and (7).

Now we state the propositions:

(83) Let us consider an element t of L . Suppose $\beta(t) = 2$. Let us consider Polish WFFs φ, ψ of L and β . Then

- (i) Polish-WFF-head($\text{Polish-binOp}(L, \beta, t))(\varphi, \psi) = t$, and
- (ii) Polish-WFF-args($\text{Polish-binOp}(L, \beta, t))(\varphi, \psi) = \langle \varphi, \psi \rangle$.

The theorem is a consequence of (7), (11), (79), (64), and (70).

(84) Let us consider a Polish WFF φ of L and β . Then $\varphi \in$ the Polish atoms(L, β) if and only if the Polish arity $\varphi = 0$. The theorem is a consequence of (53), (67), and (6).

(85) Let us consider a function γ from Polish-WFF-set(L, β) into D , an element t of L , and a Polish WFF φ of L and β . Suppose γ is g -recursive and $\beta(t) = 1$. Then $\gamma((\text{Polish-unOp}(L, \beta, t))(\varphi)) = g(t, \langle \gamma(\varphi) \rangle)$. The theorem is a consequence of (81).

Let us consider S . Let p be a finite sequence of elements of S . The functor Flat(p) yielding an element of $S \wedge \text{len } p$ is defined by

(Def. 44) $\text{decomp}(S, \text{len } p, it) = p$.

Let us consider L and β .

A substitution of L and β is a partial function from the Polish atoms(L, β) to Polish-WFF-set(L, β). Let s be a substitution of L and β . The functor Subst s yielding a Polish recursion-function of β and Polish-WFF-set(L, β) is defined by

(Def. 45) for every element t of L and for every finite sequence p of elements of Polish-WFF-set(L, β) such that $\text{len } p = \beta(t)$ holds if $t \in \text{dom } s$, then $it(t, p) = s(t)$ and if $t \notin \text{dom } s$, then $it(t, p) = t \wedge \text{Flat}(p)$.

Let φ be a Polish WFF of L and β . The functor $s[\varphi]$ yielding a Polish WFF of L and β is defined by

(Def. 46) there exists a function H from Polish-WFF-set(L, β) into Polish-WFF-set(L, β) such that H is (Subst s)-recursive and $it = H(\varphi)$.

Now we state the proposition:

(86) Let us consider a substitution s of L and β , and a Polish WFF φ of L and β . If $s = \emptyset$, then $s[\varphi] = \varphi$.

PROOF: Set $W = \text{Polish-WFF-set}(L, \beta)$. Set $g = \text{Subst } s$. Set $\gamma = \text{id}_W$. γ is g -recursive by (62), [6, (32), (33)], [7, (3), (17), (13)]. \square

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