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S-functions, spectral functions of hyperbolic geometry, and vertex operators with applications to structure for Weyl and orthogonal group invariants

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Dedicated to the memory of our friend and colleague, Petr P. Kulish

Abstract

In this paper we analyze the quantum homological invariants (the Poincaré polynomials of the \mathfrak{sl}_N link homology). In the case when the dimensions of homologies of appropriate topological spaces are precisely known, the procedure of the calculation of the Kovanov–Rozansky type homology, based on the Euler–Poincaré formula can be appreciably simplified. We express the formal character of the irreducible tensor representation of the classical groups in terms of the symmetric and spectral functions of hyperbolic geometry. On the basis of Labastida–Mariño–Ooguri–Vafa conjecture, we derive a representation of the Chern–Simons partition function in the form of an infinite product in terms of the Ruelle spectral functions (the cases of a knot, unknot, and links have been considered). We also derive an infinite-product formula for the orthogonal Chern–Simons partition functions and analyze the singularities and the symmetry properties of the infinite-product structures.

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1. Introduction

Graded Poincaré polynomials; infinite-dimensional algebras. The aim of this paper is to exploit and emphasize the structure of quantum group invariants. Recall some recent activities related to quantum invariants and the homological invariants of the Hopf link.

Note a certain importance in many diverse areas of mathematics and physics a class of infinite-dimensional algebras, in particular (affine) Kac–Moody algebras, which has been introduced in the late 1960s. Unlike the finite-dimensional case (where simple Lie algebras can be realized in terms of a finite number of bosonic/fermionic *modes*), infinite-dimensional Kac–Moody algebras have various vertex operator realizations (in terms of a finite number of bosonic *free fields*, the modes of which generate a Heisenberg algebra). Generally speaking, all simple (twisted and untwisted) Kac–Moody algebras can be embedded in the infinite-dimensional algebra $\mathfrak{gl}(\infty)$ of infinite matrices with a finite number of non-zero entries (see Sect. 3), which has a simple realization in terms of generators of a Clifford algebra.

Virasoro algebra as another type of infinite-dimensional algebra arises in different areas of physics (see Sect. 3 for details); this is the algebra of conformal transformations in two-dimensions. The operator algebra structure of two-dimensional conformally-invariant quantum field theories is determined by the representation theory of the Virasoro algebra. The Ramond and Neveu–Schwarz superalgebras (or the $N = 1$ superconformal algebras) are supersymmetric extensions of the Virasoro algebra.

We also mention the quantum affine algebras, which are q -deformations of Kac–Moody algebras. By analogy with the undeformed case, vertex operator realizations, initially for level one representations and then for arbitrary high level representations, were constructed.

Symmetric functions (or S -functions) with its connection to replicated plethysms, has been involved in the applications to certain infinite-dimensional Lie algebras (in particular the quantum affine algebras $U_q(\mathfrak{gl}_N)$) and generating functions of quantum \mathfrak{gl}_N invariants. The S -functions were first studied by Jacobi and have been generalized in various applications in physics and mathematics. There are numerous generalizations of S -functions, and among them Jack symmetric functions. Jack symmetric functions are just a special limit of a generalized Hall–Littlewood function considered by Macdonald [1]. Macdonald’s polynomials can be formulated as the trace of an interwiner (algebra homomorphism) of modules over the quantum group [2].

Quantum group invariants; finite-dimensional algebras. Another type of algebra, which has had a wide variety of applications in physics, is the so-called quantum group; this may be regarded as a deformation, depending on a parameter q , of the universal enveloping algebra of a semi-simple Lie algebra. Thus they are not finite-dimensional algebras, but are finitely generated. These algebras were first constructed by Kulish and Reshetikhin [3] and as a Hopf algebra by Sklyanin [4]. Their representation theory for q not a root of unity was found to be similar to the corresponding semi-simple Lie algebra, complications arise when q is a root of unity due to the fact that the centre of the algebra becomes larger. Quantum groups are an example of quasi-triangular Hopf algebras and as such, for each quantum group there exists a universal R -matrix which intertwines with the action of the coproduct.

By exploiting calculations for oriented and unoriented links the two-variable HOMFLY and one-variable Kauffman polynomials have been analyzed in [5,6] and [7] respectively. Note that HOMFLY polynomial can be generalized to Alexander and Jones polynomial; the two-variable Kauffman polynomial has been also introduced in [8] by generalization of the Jones polynomial. A polynomial invariant of oriented knots has been discovered in [9,10]. A quantum field theory

interpretation of the Jones polynomial by the Chern–Simons path integral method has been given by Witten [11], who also predicted the existence of 3-manifold quantum invariants.

A construction of three-manifold invariants via a track of quantum universal enveloping algebra (quantum group) $U_q(\mathfrak{sl}_2)$ at roots of unity has been discovered by Reshetikhin and Turaev in [12,13]; it leads to the *colored* version of classical HOMFLY and Kauffman polynomial invariants. *Color* in this context means the representation of quantum groups, and colored HOMFLY polynomial is a special linear quantum group invariants, the quantum group of A_n type, whereas colored Kauffman polynomial invariants is a quantum group invariants of B_n , C_n and D_n type. These achievements actually give a unified understanding of the quantum group invariants of links. Generalization to affine Macdonald’s polynomials by considering homomorphisms of the affine quantum group $U_q(\mathfrak{gl}_N)$ has been considered in [14].

In articles [15–17] a conjectural description of relationship between reformulated invariants of colored HOMFLY link has been proposed; we will refer to this proposal as Labastida–Mariño–Ooguri–Vafa (LMOV) conjecture. Later this conjecture was proved in [18]. The LMOV conjecture can be expressed by using mathematical language, theory of irreducible representation of quantum groups. The physics background can be addressed to works in late 1970s on large N expansion of $U(N)$ gauge field theories. In this connection, the exact result for closed topological string theory on the resolved conifold, dual to the $U(N)$ Chern–Simons theory on S^3 , has been described by Gopakumar and Vafa [19]. The Gromov–Witten theory of the resolved conifold corresponds to the Chern–Simons theory of an unknot, while the LMOV conjecture considers the case when the link or knot is nontrivial and the corresponding Wilson loop expectation values, in fact colored HOMFLY polynomial of the link. Therefore the LMOV conjecture could be viewed as a counterpart of Gopakumar–Vafa conjecture. It is important that the LMOV conjecture predicts an intrinsic symmetry of series in $q - q^{-1}$ about reformulated invariants of the colored HOMFLY polynomial as well as hidden integrality encoded in the colored HOMFLY polynomial.

The orthogonal LMOV conjecture was formulated using the colored Kauffman solely in [20], and the relation between the colored HOMFLY polynomials and the colored Kauffman polynomials in [21] correspondingly. In recent articles [22,23] a new structure of the colored HOMFLY polynomial has been analyzed by means of the Chern–Simons partition function, which appear in the LMOV conjecture as an infinite product, and indicates some potential modularity of this partition function.

The organization of the paper and our key results.

– The quantum homological invariants we consider in Sect. 2. We analyze Poincaré polynomial of the \mathfrak{sl}_N link homology $\overline{\mathcal{P}}_{\mathfrak{sl}_N; \lambda, \mu}(q, t)$. We show that in the case when dimensions of homologies of appropriate topological spaces are precisely known the procedure of the calculation of the Kovanov–Rozansky type homology, based on the Euler–Poincaré formula, can be appreciably simplified.

– In Sect. 3 we exploit the Hopf algebra structure of the ring $\Lambda(X)$ of S -functions of the independent variables (x_1, x_2, \dots) (finite or countably infinite in number), that constitute the alphabet X , mainly following to notation and discussion of important article [24]. After some notational preliminaries we discuss algebraic properties of the ring $\Lambda(X)$ in Sect. 3.1. Then in Sect. 3.2 we define plethysms (Schur function plethysm, scalar and inner (co)products), the notation for the mutually inverse pair and the inner products is: $M(t; X) = \prod_{i \geq 1} (1 - tx_i)^{-1}$, $L(t; X) = \prod_{i \geq 1} (1 - tx_i)$ and $M(XY) = \prod_{i,j} (1 - x_i y_j^{-1})$, $L(XY) = \prod_{i,j} (1 - x_i y_j)$; the Cauchy kernel, $M(XY)$, is a dual version of the Schur–Hall scalar product. The next impor-

tant result realizes more plethysms associated with multipartite generating functions, Sect. 3.3. It is obtained through the use of one of the restricted specializations. In addition, we use the Bell polynomials in multipartite function problem with its connection to the spectral functions of hyperbolic 3-geometry. We conclude with some examples of hierarchy which can be treated as a product of copies, each of them corresponds to a free two-dimensional CFT (Eq. (3.45)). The relevant formulas of this section expressed in terms of so-called Ruelle (Patterson–Selberg) spectral zeta-functions (3.39)–(3.42) are very efficient.

– The characters of the orthogonal and symplectic groups have been found by Schur [25,26] and Weyl [27] respectively. The method that has been used is transcendental (it depends on integration over the group manifold). However the appropriate characters may also be obtained by algebraic methods [28]: “*This algebraic method would seem to offer a better prospect of successful application to other restricted groups than the method of group integration.*” Following [24] we have used algebraic methods. The Hopf algebra can be exploited in the determination of (sub)group branching rules and the decomposition of tensor products. We use this analysis for vertex operator traces in Sect. 3.4.

– The HOMFLY skein and the link invariants from vertex models we analyzed in Sect. 4.1. Then in Sect. 4.3 on the base of LMOV conjecture we derive a new representation of the Chern–Simons partition function in form of an infinite product in terms of Ruelle spectral functions of hyperbolic geometry. In addition, we consider the case of a knot, unknot, and links. We discuss singularities and symmetry properties of these infinite-product structures in Sect. 4.4. Infinite-product formula for orthogonal Chern–Simons partition functions we derive in Sect. 5.

2. Poincaré polynomials of the homological invariants of the Hopf link

Polygraded algebras and polynomial invariants. The relationship between Lie algebras and combinatorial identities (the famous Euler identity, as a particular famous example) was first discovered by Macdonald. A general outline for proving combinatorial identities is based on the Euler–Poincaré formula. Let \mathfrak{g} be a polygraded Lie algebra, $\mathfrak{g} = \bigoplus_{\substack{\lambda_1 \geq 0, \dots, \lambda_k \geq 0 \\ \lambda_1 + \dots + \lambda_k > 0}} \mathfrak{g}_{(\lambda_1, \dots, \lambda_k)}$, satisfying the condition $\dim \mathfrak{g}_{(\lambda_1, \dots, \lambda_k)} < \infty$. For formal power series in q_1, \dots, q_k , one can get the following identity (the Euler–Poincaré formula):

$$\mathcal{P}_{\mathfrak{g}}(q) = \sum_{m, \lambda_1, \dots, \lambda_k} (-1)^m q_1^{\lambda_1} \dots q_k^{\lambda_k} \dim H_m^{(\lambda_1, \dots, \lambda_k)} = \prod_{n_1, \dots, n_k} (1 - q_1^{n_1} \dots q_k^{n_k})^{\dim \mathfrak{g}_{n_1, \dots, n_k}}. \tag{2.1}$$

Interesting combinatorial identities may be obtained by applying (2.1) to graded algebras, for example, to the subalgebras \mathfrak{g}^A of Kac–Moody algebras. From the point of view of the applications, homologies associated with subalgebras $\mathfrak{g} = \mathfrak{sl}_N(\mathbb{C})$ are much more important than the homology of the current algebra themselves, since they constitute the technical basis of the proof of the combinatorial identities of Euler–Gauss–Jacobi–MacDonald.

Poincaré polynomial of the \mathfrak{sl}_N link homology $\overline{\mathcal{P}}_{\mathfrak{sl}_N; \lambda, \mu}(q, t)$. It is known that the invariants of the colored Hopf link are identified as topological open string amplitudes on the deformed conifold T^*S^3 . Indeed, the $U(N)$ Chern–Simons theory is realized by topological strings on T^*S^3 with N topological D-branes wrapping on the base Lagrangian submanifold S^3 . On the other hand the Hopf link in S^3 consisting of two knots \mathcal{K}_1 and \mathcal{K}_2 can be introduced by a pair of new D-branes wrapping on Lagrangian three-cycles L_1 and L_2 such that $S^3 \cap L_i = \mathcal{K}_i$ [17]. For this brane system the topological open string amplitude (rewritten in terms of symmetric

functions and spectral Selberg-type functions) is supposed to give the invariants of the Hopf link. The representation (or coloring) attached to each knot \mathcal{K}_i is related to the boundary states of the open string ending on L_i by the Frobenius relation. Because of geometric transition or the large N duality [19,29,30], this brane configuration is mapped to the resolved conifold $\mathcal{O}(-1) \oplus \mathcal{O}(-1) \rightarrow \mathbb{P}^1$. Eventually, the D-branes wrapping on S^3 disappear, but a pair of Lagrangian D-branes remains as a remnant of the Hopf link. The resulting D-brane system can be described in terms of the toric diagram and the corresponding amplitude computed by the topological vertex method [31,32].

Our interest is the superpolynomial which is a polynomial in $(a, q, t) \in \mathbb{C}^3$, such that a specialization $a = q^N$ leads the Poincaré polynomial of the \mathfrak{sl}_N link homology $\overline{\mathcal{P}}_{\mathfrak{sl}_N; \lambda, \mu}(q, t)$, which is a two parameter (q, t) version of the \mathfrak{sl}_N link invariants.

In the case the coloring is the N dimensional defining representation, it is called the Khovanov–Rozansky homology [33]. In the case $a = q^N$ with $N \in \mathbb{Z}_+$, the superpolynomial of the homological invariants of the colored Hopf link reduces to $\sum_{i, j \in \mathbb{Z}} q^i t^j \dim \mathcal{H}_{i, j}^{\mathfrak{sl}_N; \lambda, \mu}$ with certain doubly graded homology $\mathcal{H}_{i, j}^{\mathfrak{sl}_N; \lambda, \mu}$ [34] (by definition it should be a polynomial in q and t with non-negative integer coefficients). It has been argued [35] that homological link invariants are related to a refinement of the BPS state counting in topological open string theory. The conjecture on homological link invariants of the Hopf link [34] is based on this proposal. For the Hopf link \mathcal{L} this means that there is a doubly graded homology theory $\mathcal{H}_{i, j}^{\mathfrak{sl}_N; \lambda, \mu}(\mathcal{L})$ whose graded Poincaré polynomial is (cf. Eq. (2.1))

$$\overline{\mathcal{P}}_{\mathfrak{sl}_N; \lambda, \mu}(q, t) = \sum_{i, j \in \mathbb{Z}} q^i t^j \dim \mathcal{H}_{i, j}^{\mathfrak{sl}_N; \lambda, \mu}(\mathcal{L}). \tag{2.2}$$

Remark 2.1. It is convenient for a computational reasons to investigate the effect of a relation on the Poincaré series of a graded algebra. Let \mathfrak{g}_1 and \mathfrak{g}_2 be two graded algebras. Suppose a basis for \mathfrak{g}_1 (as a vector space) is $\{x_i\}_{i \in I}$, while a basis for \mathfrak{g}_2 is $\{y_j\}_{j \in J}$. Then a basis for $\mathfrak{g}_1 \otimes \mathfrak{g}_2$ (as a vector space) is $\{x_i \otimes y_j\}_{i \in I, j \in J}$. As a result, $\mathcal{P}_{\mathfrak{g}_1 \otimes \mathfrak{g}_2}(q) = \mathcal{P}_{\mathfrak{g}_1}(q) \mathcal{P}_{\mathfrak{g}_2}(q)$.

Suppose that \mathfrak{g} is a ring. We need the following definition [36]: A sequence of elements $\{\gamma_j\}_{j=1}^r$ in \mathfrak{g} is called a *regular sequence* if γ_1 is not a zero-divisor in \mathfrak{g} and for each $j \geq 2$ the image of γ_j in $\mathfrak{g}/(\gamma_1, \dots, \gamma_{j-1})$ is not a zero-divisor. Let, as before, \mathfrak{g} be a graded algebra and $\gamma_1, \gamma_2, \dots, \gamma_r$ a regular sequence of homogeneous elements of degrees n_1, n_2, \dots, n_r . We have (see for detail [36]) $\mathcal{P}_{\mathfrak{g}/(\gamma_1, \dots, \gamma_r)}(q) = \mathcal{P}_{\mathfrak{g}}(q)(1 - q^{n_1}) \cdots (1 - q^{n_r})$.

For more examples let us proceed to describing the properties of link homologies suggested by the their relation to Hilbert spaces of BPS states [35]. Let $\mathcal{H}_{k, j}^{\mathfrak{sl}_N; R_1, \dots, R_\ell}(\mathcal{L})$ be the doubly-graded homology theory whose graded Euler characteristic is the polynomial invariant $\overline{\mathcal{P}}_{\mathfrak{sl}_N; R_1, \dots, R_\ell}(q)$ (the *bar* means that this invariant is unnormalized invariant; its normalized version obtained by dividing by the invariant of the unknot)

$$\overline{\mathcal{P}}_{\mathfrak{sl}_N; R_1, \dots, R_\ell}(q) = \sum_{k, j \in \mathbb{Z}} (-1)^j q^k \dim \mathcal{H}_{k, j}^{\mathfrak{sl}_N; R_1, \dots, R_\ell}(\mathcal{L}). \tag{2.3}$$

Here \mathcal{L} is an oriented link in S^3 , we consider the Lie algebra $\mathfrak{g} = \mathfrak{sl}_N$ (there is a natural generalization to other classical Lie algebras B, C , and D [35]) and a link colored is given by a collection of representations R_1, \dots, R_ℓ of \mathfrak{sl}_N . The graded Poincaré polynomial has the form

$$\bar{P}_{\mathfrak{sl}_N; R_1, \dots, R_\ell}(q, t) := \sum_{k, j \in \mathbb{Z}} q^k t^j \dim \mathcal{H}_{k, j}^{\mathfrak{sl}_N; R_1, \dots, R_\ell}(\mathcal{L}). \tag{2.4}$$

By definition, it is a polynomial in $q^{\pm 1}$ and $t^{\pm 1}$ with integer non-negative coefficients. In addition, evaluating (2.4) at $t = -1$ gives (2.3). In the case $R_a = \square$ for all $a = 1, \dots, \ell$, the homology $\mathcal{H}_{k, j}^{\mathfrak{sl}_N; \square, \dots, \square}(\mathcal{L})$ is known as the Khovanov–Rozansky homology, $(KR)\bar{H}_{k, j}^N(\mathcal{L})$. The further physical interpretation of homological link invariants via Hilbert spaces of BPS states leads to certain predictions regarding the behavior of link homologies with rank N (for more discussion see [35,37]).

In the case when dimensions of homologies of appropriate topological spaces are precisely known the procedure of the calculation can be appreciably simplified. Such a situation happens, for example, if asymptotic behavior the total dimension of $\mathcal{H}_{*, * }^{\mathfrak{sl}_N; R_1, \dots, R_\ell}(\mathcal{L})$ grows as [34]: $\dim \mathcal{H}_{*, * }^{\mathfrak{sl}_N; R_1, \dots, R_\ell}(\mathcal{L})|_{N \rightarrow \infty} \rightarrow N^d, d = \sum_{j=1}^{\ell} \dim R_j$.

3. Hopf algebraic approach and group theory

Usually the Euler–Poincaré formula applies to chain complexes of finite dimensional Lie algebras. In the infinite dimensional case matters can be fixed up by considering polygraded Lie algebras. Thus the partition functions can indeed be converted into product expressions. The expression on the right-hand side of (2.1) looks like *counting* the states in the Hilbert space of a second quantized theory. Certain formulas for the partition functions or Poincaré polynomials

$$\prod_{n_1, \dots, n_k} (1 - q_1^{n_1} \dots q_k^{n_k})^{\dim \mathfrak{g}_{n_1, \dots, n_k}}, \quad \prod_{n_1, \dots, n_k} (1 - q_1^{n_1} \dots q_k^{n_k})^{\text{rank } \mathfrak{g}_{n_1, \dots, n_k}} \tag{3.1}$$

are associated with dimensions of homologies of appropriate topological spaces and linked to generating functions and elliptic genera. Note that this conclusively explains the sequence of dimensions (*distinguished powers*) of the simple Lie algebras.

Before examine a ring of formal power series (of type (3.1)) we provide a short discussion on the theory of a higher-weight modules over Lie algebra. We start with very well known Lie algebra $\mathfrak{gl}(n, k)$.¹ Results for $\mathfrak{gl}(n, k)$ survive the passage to the limit $n \rightarrow \infty$, if one assumes that $\mathfrak{gl}(\infty, k)$ is the Lie algebra of infinite finitary matrices, which means $\bigcup_n \mathfrak{gl}(n, k)$. In this remark we deal with the Lie algebra $\mathfrak{gl}_{\mathcal{J}}(k)$ of generalized Jacobian matrices.² The algebra $\mathfrak{gl}_{\mathcal{J}}(k)$ can be considered as a nontrivial one-dimensional central extension of the Lie algebra $\mathfrak{gl}_{\mathcal{J}}(k)$ (for details, see [38]). It is obvious that $\mathfrak{gl}_{\mathcal{J}}(k) \supset \mathfrak{gl}_{\mathcal{J}}(\infty, k)$. The importance of this Lie algebra stems from the following facts:

- Many of the classical constructions of the theory of representations of the Lie algebra $\mathfrak{gl}_{\mathcal{J}}(k)$ can be also applied to the algebra $\mathfrak{gl}_{\mathcal{J}}(k)$. This creates a sizable supply of $\mathfrak{gl}_{\mathcal{J}}(k)$ -modules.
- Important infinite-dimensional Lie algebras can be embedded in $\mathfrak{gl}_{\mathcal{J}}(k)$. Thus, the already mentioned representations of $\mathfrak{gl}_{\mathcal{J}}(k)$ become representations of these algebras.

¹ The symbol k denotes the field of real numbers \mathbb{R} or the field \mathbb{C} of complex numbers. In particular, $\mathfrak{gl}(n, \mathbb{C})$ is the Lie algebra of all complex $n \times n$ matrices with the operation $A, B \mapsto [A, B] = AB - BA$.

² The bilateral matrix $\|a_{ij}\|_{i, j \in \mathbb{Z}}$ is called a generalized Jacobian matrix if it has a finite number of nonzero diagonals (that is, if there exists a positive N such that $a_{ij} = 0$ for $|j - i| > N$). It is clear that the set of generalized Jacobian matrices constitutes a Lie algebra, with respect to the usual commutation rule.

- The subalgebra of $\mathfrak{gl}_{\mathcal{J}}(k)$ composed of n -periodic matrices, $\|a_{ij}\|$ with $a_{i+n,j+n} = a_{ij}$, is isomorphic to the algebra of currents [38].³ A non-trivial central extension of \mathfrak{g}^X – a Kac–Moody algebra – is embedded in $\mathfrak{gl}_{\mathcal{J}}(k)$. The Lie algebra $L^{\text{pol}} = \mathbb{C}(\text{Vect } S^1)^{\text{pol}}$ of complex polynomial vector fields on the circle can be embedded in $\mathfrak{gl}_{\mathcal{J}}(k = \mathbb{C})$. Recall that L^{pol} has a basis \mathbf{e}_i and commutators of the form

$$[\mathbf{e}_i, \mathbf{e}_j] = (i - j)\mathbf{e}_{i+j} \quad (j \in \mathbb{Z}), \quad \mathbf{e}_j = -x^{j+1}d/dx \quad \text{on } \mathbb{C} \setminus \{0\}. \tag{3.2}$$

(The cohomologies of the algebra L^{pol} are known; in particular, $H^2(L^{\text{pol}}) = \mathbb{C}$.) The Virasoro algebra is a Lie algebra over \mathbb{C} with basis L_n ($n \in \mathbb{Z}$), c . Because of Eq. (3.2), the Lie Virasoro algebra is a (universal) central extension of the Lie algebra of holomorphic vector fields on the punctured complex plane having finite Laurent series. It is for this reason that the Virasoro algebra plays a key role in conformal field theory.

We briefly note some elements of the representation theory of the Virasoro algebra which are, in fact, very similar to those for Kac–Moody algebras. Let us consider the highest representation of the Virasoro algebra. Let $M(c, h)$ ($c, h \in \mathbb{C}$) be the Verma module over the Virasoro algebra. The conformal central charge c acts on $M(c, h)$ as cId . As $[\mathbf{e}_0, \mathbf{e}_{-j}] = n\mathbf{e}_{-j}$, \mathbf{e}_0 is diagonalizable on $M(c, h)$, with spectrum $h + \mathbb{Z}_+$ and eigenspace decomposition given by: $M(c, h) = \bigoplus_{j \in \mathbb{Z}_+} M(c, h)_{h+j}$, where $M(c, h)_{h+j}$ is spanned by elements of the basis $\{\mathbf{e}_{-j_k}\}_{k=1}^n$ of $M(c, h)$. The number $Z_j = \dim M(c, h)_{h+j}$, is the classical partition function. This means that the Konstant partition function for the Virasoro algebra is the classical partition function. On the other hand, the partition functions can be rewritten in the form (cf. Eq. (3.1))

$$\text{Tr}_{M(c,h)} q^{\mathbf{e}_0} := \sum_{\lambda} \dim M(c, h)_{\lambda} q^{\lambda} = q^h \prod_{j=1}^{\infty} (1 - q^j)^{-1}. \tag{3.3}$$

The series $\text{Tr}_{\mathcal{V}} q^{\mathbf{e}_0}$ is called the formal character of the Virasoro-module \mathcal{V} . (A \mathfrak{g} -module $\mathcal{V} \in C$, where C is a category if: There is an expansion $\mathcal{V} = \bigoplus_{\lambda \in \mathfrak{h}^*} \mathcal{V}_{\lambda}$ (\mathfrak{h} is a Cartan subalgebra of \mathfrak{g}) and $e_{\alpha}^{(i)} \mathcal{V}_{\lambda} \subset \mathcal{V}_{\lambda+\alpha}$, where $e_{\alpha}^{(i)}$ are root vectors correspond to root α ; $\dim \mathcal{V}_{\lambda} < \infty$ for all λ ; $D(\lambda) := \{\lambda \in \mathfrak{h}^* \mid \mathcal{V}_{\lambda} \neq 0\} \subset \bigcup_{i=1}^s D(\lambda_i)$ for some $\lambda_1, \dots, \lambda_s \in \mathfrak{h}^*$.)

3.1. The polynomial ring $\Lambda(X)$

Our aim in this Section is to exploit the Hopf algebra of the ring $\Lambda(X)$ of symmetric functions of the independent variables (x_1, x_2, \dots) , finite or countably infinite in number, that constitute the alphabet X . In our notations and basic statements we shall mainly follow the lines of article [24].

Let $\mathbb{Z}[x_1, \dots, x_n]$ be the polynomial ring, or the ring of formal power series, in n commuting variables x_1, \dots, x_n . The symmetric group S_n acting on n letters acts on this ring by permuting the variables. For $\pi \in S_n$ and $f \in \mathbb{Z}[x_1, \dots, x_n]$ we have $\pi f(x_1, \dots, x_n) = f(x_{\pi(1)}, \dots, x_{\pi(n)})$.

³ Recall that the space of smooth maps $X \rightarrow \mathfrak{g}$, where X is a smooth manifold and \mathfrak{g} is a finite-dimensional Lie algebra, with the C^{∞} -topology and the commutator $[\mathfrak{f}, \mathfrak{g}](x) = [\mathfrak{f}(x), \mathfrak{g}(x)]$, is the (topological) current Lie algebra and is denoted by \mathfrak{g}^X . Together with the algebra \mathfrak{g}^{S^1} ($X = S^1$) one can consider its subalgebra $(\mathfrak{g}^{S^1})^{\text{pol}}$, consisting of maps described by trigonometric polynomials. For any commutative associative algebra A , the tensor product $\mathfrak{g} \otimes A$ is a Lie algebra with respect to the commutators $[\mathfrak{g}_1 \otimes a_1, \mathfrak{g}_2 \otimes a_2] = [\mathfrak{g}_1, \mathfrak{g}_2] \otimes a_1 a_2$; also $(\mathfrak{g}^{S^1})^{\text{pol}} = \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}]$.

We are interested in the subring of functions invariant under this action, $\pi f = f$, that is to say the ring of symmetric polynomials in n variables: $\Lambda(x_1, \dots, x_n) = \mathbb{Z}[x_1, \dots, x_n]^{S_n}$. This ring may be graded by the degree of the polynomials, so that $\Lambda(X) = \bigoplus_n \Lambda^{(n)}(X)$, where $\Lambda^{(n)}(X)$ consists of homogeneous symmetric polynomials in x_1, \dots, x_n of total degree n .

In order to work with an arbitrary number of variables, following Macdonald [1], we define the ring of symmetric functions $\Lambda = \lim_{n \rightarrow \infty} \Lambda(x_1, \dots, x_n)$ in its stable limit ($n \rightarrow \infty$). There exist various bases of $\Lambda(X)$:

(i) A \mathbb{Z} basis of $\Lambda^{(n)}$ is provided by the monomial symmetric functions $\{m_\lambda\}$, where λ is any partitions of n .

(ii) The other (integral and rational) bases for $\Lambda^{(n)}$ are indexed by the partitions λ of n . There are the complete, elementary and power sum symmetric functions bases defined multiplicatively in terms of corresponding one part functions by: $h_\lambda = h_{\lambda_1} h_{\lambda_2} \cdots h_{\lambda_n}$, $e_\lambda = e_{\lambda_1} e_{\lambda_2} \cdots e_{\lambda_n}$ and $p_\lambda = p_{\lambda_1} p_{\lambda_2} \cdots p_{\lambda_n}$ where the one part functions are defined for $\forall n \in \mathbb{Z}_+$ by

$$h_n(X) = \sum_{i_1 \leq i_2 \leq \dots \leq i_n} x_{i_1} x_{i_2} \cdots x_{i_n}, \quad e_n(X) = \sum_{i_1 < i_2 < \dots < i_n} x_{i_1} x_{i_2} \cdots x_{i_n}, \quad p_n(X) = \sum_i x_i^n, \tag{3.4}$$

with the convention $h_0 = e_0 = p_0 = 1$, $h_{-n} = e_{-n} = p_{-n} = 0$. Three of these bases are multiplicative, with $h_\lambda = h_{\lambda_1} h_{\lambda_2} \cdots h_{\lambda_n}$, $e_\lambda = e_{\lambda_1} e_{\lambda_2} \cdots e_{\lambda_n}$ and $p_\lambda = p_{\lambda_1} p_{\lambda_2} \cdots p_{\lambda_n}$. The relationships between the various bases we just mention at this stage by the transitions

$$p_\rho(X) = \sum_{\lambda \vdash n} \chi_\rho^\lambda s_\lambda(X) \quad \text{and} \quad s_\lambda(X) = \sum_{\rho \vdash n} \mathfrak{z}_\rho^{-1} \chi_\rho^\lambda p_\rho(X). \tag{3.5}$$

For each partition λ , the Schur function is defined by

$$s_\lambda(X) \equiv s_\lambda(x_1, x_2, \dots, x_n) = \frac{\sum_{\sigma \in S_n} \text{sgn}(\sigma) X^{\sigma(\lambda+\delta)}}{\prod_{i < j} (x_i - x_j)}, \tag{3.6}$$

where $\delta = (n - 1, n - 2, \dots, 1, 0)$. In fact both h_n and e_n are special Schur functions, $h_n = s_{(n)}$, $e_n = s_{(1^n)}$, and their generating functions are expressed in terms of the power-sum p_n :

$$\sum_{n \geq 0} h_n z^n = \exp\left(\sum_{n=1}^{\infty} (p_n/n) z^n\right), \quad \sum_{n \geq 0} e_n z^n = \exp\left(-\sum_{n=1}^{\infty} (p_n/n) (-z)^n\right). \tag{3.7}$$

The Jacobi–Trudi formula [1] express the Schur functions in terms of h_n or e_n : $s_\lambda = \det(h_{\lambda'_i - i + j}) = \det(e_{\lambda'_i - i + j})$, where λ' is the conjugate of λ . An involution $\omega : \Lambda \rightarrow \Lambda$ can be defined by $\omega(p_n) = (-1)^{n-1} p_n$. Then it follows that $\omega(h_n) = e_n$. Also we have $\omega(s_\lambda) = s_{\lambda'}$. χ_ρ^λ is the character of the irreducible representation of the symmetric groups S_n specified by λ in the conjugacy class specified by ρ . These characters satisfy the orthogonality conditions

$$\sum_{\rho \vdash n} \mathfrak{z}_\rho^{-1} \chi_\rho^\lambda \chi_\rho^\mu = \delta_{\lambda, \mu} \quad \text{and} \quad \sum_{\lambda \vdash n} \mathfrak{z}_\rho^{-1} \chi_\rho^\lambda \chi_\sigma^\lambda = \delta_{\rho, \sigma}. \tag{3.8}$$

The significance of the Schur function basis lies in the fact that with respect to the usual Schur–Hall scalar product $\langle \cdot | \cdot \rangle_{\Lambda(X)}$ on $\Lambda(X)$ we have

$$\langle s_\mu(X) | s_\nu(X) \rangle_{\Lambda(X)} = \delta_{\mu, \nu} \quad \text{and therefore} \quad \langle p_\rho(X) | p_\sigma(X) \rangle_{\Lambda(X)} = \mathfrak{z}_\rho \delta_{\rho, \sigma}, \tag{3.9}$$

where $\mathfrak{z}_\lambda = \prod_i i^{m_i} m_i!$ for $\lambda = (1^{m_1}, 2^{m_2}, \dots)$.

Algebraic properties of $\Lambda(X)$. The ring, $\Lambda(X)$, of symmetric functions over X has a Hopf algebra structure, and two further algebraic and two coalgebraic operations. For notation and basic properties we refer the reader to [39,40] and references therein.

- We indicate outer products on Λ either by m , or with infix notation using juxtaposition.
- Inner products are denoted either by m or as infix by $*$.
- Plethysms (compositions) are denoted by \circ or by means of square brackets $[]$; plethysm coproduct is denoted by ∇ .
- The corresponding coproduct maps are specified by Δ for the outer coproduct.
- Notation δ we use for the inner coproduct.⁴

The coproduct coefficients themselves are obtained from the products by duality using the Schur–Hall scalar product and the self-duality of $\Lambda(X)$. For example, for all $A, B \in \Lambda(X)$:

$$\begin{aligned} m(A \otimes B) &= AB; & \Delta(A) &= A_{(1)} \otimes A_{(2)}; \\ m(A \otimes B) &= A * B; & \delta(A) &= A_{[1]} \otimes A_{[2]}; \\ A \circ B &= A[B]; & \nabla(A) &= A_{(1)} \otimes A_{(2)}. \end{aligned}$$

In terms of the Schur function basis $\{s_\lambda\}_{\lambda \vdash n, n \in \mathbb{Z}_+}$ the product and coproduct maps give rise to the particular sets of coefficients specified as follows:

$$\begin{aligned} s_\mu s_\nu &= \sum_\lambda c_{\mu, \nu}^\lambda s_\lambda; & \Delta(s_\lambda) &= s_{\lambda_{(1)}} \otimes s_{\lambda_{(2)}} = \sum_{\mu, \nu} c_{\mu, \nu}^\lambda s_\mu \otimes s_\nu; \\ s_\mu * s_\nu &= \sum_\lambda g_{\mu, \nu}^\lambda s_\lambda; & \delta(s_\lambda) &= s_{\lambda_{[1]}} \otimes s_{\lambda_{[2]}} = \sum_{\mu, \nu} g_{\mu, \nu}^\lambda s_\mu \otimes s_\nu; \\ s_\mu [s_\nu] &= \sum_\lambda p_{\mu, \nu}^\lambda s_\lambda; & \nabla(s_\lambda) &= s_{\lambda_{(1)}} \otimes s_{\lambda_{(2)}} = \sum_{\mu, \nu} p_{\mu, \nu}^\lambda s_\mu \otimes s_\nu. \end{aligned} \tag{3.10}$$

Here the $c_{\mu, \nu}^\lambda$ are Littlewood–Richardson coefficients, the $g_{\mu, \nu}^\lambda$ are Kronecker coefficients and the $p_{\mu, \nu}^\lambda$ are plethysm coefficients. All these coefficients are non-negative integers. The Littlewood–Richardson coefficients can be obtained, for example, by means of the Littlewood–Richardson rule [41,42] or the hive model [43]. The Kronecker coefficients may be determined directly from characters of the symmetric group or by exploiting the Jacobi–Trudi identity and the Littlewood–Richardson rule, while plethysm coefficients have been the subject of the variety methods of calculation [44,45]. Note that the above sums are finite, since $c_{\mu, \nu}^\lambda \geq 0$ iff $|\lambda| = |\mu| + |\nu|$; $g_{\mu, \nu}^\lambda \geq 0$ iff $|\lambda| = |\mu| = |\nu|$; $p_{\mu, \nu}^\lambda \geq 0$ iff $|\lambda| = |\mu| |\nu|$.

The Schur–Hall scalar product may be used to define skew Schur functions $s_{\lambda/\mu}$ through the identities $c_{\mu, \nu}^\lambda = \langle s_\mu s_\nu | s_\lambda \rangle = \langle s_\nu | s_\mu^\perp(s_\lambda) \rangle = \langle s_\nu | s_{\lambda/\mu} \rangle$, so that $s_{\lambda/\mu} = \sum_\nu c_{\mu, \nu}^\lambda s_\nu$. Within the outer product Hopf algebra we have a unit Id , a counit ε and an antipode \mathbf{S} such that⁵: $\text{Id}(1) = s_0$; $\varepsilon(s_\lambda) = \delta_{\lambda, (0)}$; $\mathbf{S}(s_\lambda) = (-1)^{|\lambda|} s_{\lambda'}$.

In what follows we shall make considerable use of several infinite series of Schur functions. The most important of these are the mutually inverse pair defined by

⁴ In Sweedler notation the action of these coproducts is distinguished by means of different brackets, round, square and angular, around the Sweedler indices – the so-called Brouder–Schmitt convention.

⁵ Macdonald uses the involution ω which differs from the antipode by a sign factor: $\mathbf{S}(s_\lambda) = (-1)^{\ell(\lambda)} \omega(s_\lambda)$. It is, however, convenient to employ the antipode if Hopf algebra structures are in use.

$$M(t; X) = \prod_{i \geq 1} (1 - t x_i)^{-1} = \sum_{m \geq 0} h_m(X) t^m, \tag{3.11}$$

$$L(t; X) = \prod_{i \geq 1} (1 - t x_i) = \sum_{m \geq 0} (-1)^m e_m(X) t^m, \tag{3.12}$$

where as Schur functions $h_m(X) = s_{(m)}(X)$ and $e_m(X) = s_{(1^m)}(X)$.⁶ For convenience, in the case $t = 1$ we write $M(1; X) = M(X)$ and $L(1; X) = L(X)$.

3.2. Plethysms

Plethysms are defined as compositions whereby for any $A, B \in \Lambda(X)$; the plethysm $A[B]$ is A evaluated over an alphabet Y whose letters are the monomials of $B(X)$, with each letter repeated as many times as the multiplicity of the corresponding monomial. For example, the Schur function plethysm is defined by $s_\lambda[s_\mu](X) = s_\lambda(Y)$, where $Y = s_\mu(X)$.

For all $A, B, C \in \Lambda(X)$ we have the following rules, due to Littlewood [42], for manipulating plethysms⁷:

$$\begin{aligned} (A + B)[C] &= A[C] + B[C]; & A[B + C] &= A_{(1)}[B]A_{(2)}[C]; \\ (AB)[C] &= A[C]B[C]; & A[BC] &= A_{[1]}[B]A_{[2]}[C]; \\ A[B[C]] &= (A[B])[C]. \end{aligned} \tag{3.13}$$

These rules enable us to evaluate plethysms not only of outer and inner products but also of outer and inner coproducts.

The Cauchy kernel. It is often convenient to represent an alphabet in an additive manner X , as itself an element of the ring $\Lambda(X)$ in the sense that $x_1 + x_2 + \dots = h_1(X) = e_1(X) = p_1(X) = s_{(1)}(X)$. As elements of $\Lambda(X) \otimes \Lambda(Y)$ we have $X + Y = \sum_{j=1} x_j + \sum_{j=1} y_j$, $XY = \sum_j x_j \sum_j y_j$. With this notation, the outer coproduct gives

$$\begin{aligned} \Delta(M) &= M_{(1)} \otimes M_{(2)} = M \otimes M; & M(X + Y) &= \prod_i (1 - x_i)^{-1} \prod_j (1 - y_j)^{-1}; \\ \Delta(L) &= L_{(1)} \otimes L_{(2)} = L \otimes L; & L(X + Y) &= \prod_i (1 - x_i) \prod_j (1 - y_j), \end{aligned}$$

so that $M(X + Y) = M(X) M(Y)$ and $L(X + Y) = L(X) L(Y)$. For the inner coproduct:

$$\begin{aligned} \delta(M) &= M_{[1]} \otimes M_{[2]}; & M(XY) &= \prod_{i,j} (1 - x_i y_j)^{-1}; \\ \delta(L) &= L_{[1]} \otimes L_{[2]}; & L(XY) &= \prod_{i,j} (1 - x_i y_j). \end{aligned}$$

The expansions of the products on the right hand sides of these expressions is effected remarkably easily by evaluating the inner coproducts on the left:

⁶ It might be noted that in Macdonald’s notation and λ -ring notation $M(t; X) = H(t) = \sigma_t(X)$ and $L(t; X) = E(-t) = \lambda_{-t}(X)$.

⁷ To these we can add, see [40]: $A[-B] = (S(A))[B]$; $A[S(B)] = S(A[B])$; $A[\Delta(B)] = \Delta(A[B])$; $A[\delta(B)] = \delta(A[B])$, and the plethysm of a tensor product: $A[B \otimes C] = A_{[1]}[B] \otimes A_{[2]}[C]$.

$$\delta(M) = \sum_{k \geq 0} \delta(h_k) = \sum_{k \geq 0} \sum_{\lambda \vdash k} s_\lambda \otimes s_\lambda ;$$

$$\delta(L) = \sum_{k \geq 0} (-1)^k \delta(e_k) = \sum_{k \geq 0} (-1)^k \sum_{\lambda \vdash k} s_\lambda \otimes s_{\lambda'} .$$

This gives immediately the well known Cauchy and Cauchy–Binet formulas:

$$M(XY) = \prod_{i,j} (1 - x_i y_j)^{-1} = \sum_{\lambda} s_\lambda(X) s_\lambda(Y) ; \tag{3.14}$$

$$L(XY) = \prod_{i,j} (1 - x_i y_j) = \sum_{\lambda} (-1)^{|\lambda|} s_\lambda(X) s_{\lambda'}(Y) . \tag{3.15}$$

The Cauchy kernel, $M(XY)$, is a dual version of the Schur–Hall scalar product; indeed:

$$s_\mu(X) M(XY) = \sum_{\lambda} \sum_{\nu} c_{\mu,\lambda}^{\nu} s_\nu(X) s_\lambda(Y)$$

$$= \sum_{\nu} s_\nu(X) s_{\nu/\mu}(Y) = s_\mu^\perp(Y) (M(XY)) . \tag{3.16}$$

Generally speaking, for any $F(X) \in \Lambda(X)$ with dual $F^\perp(X)$, by linearly extending the above result we have $F(X) M(XY) = F^\perp(Y) (M(XY))$.

3.3. Multipartite generating functions and symmetric functions

We note here an alternative definition of plethysms, involving a particular substitution process. For $M(z; X) = \prod_{i \geq 1} (1 - z x_i)^{-1}$, $L(z; X) = \prod_{i \geq 1} (1 - z x_i) = M^{-1}(z; X)$ it follows that

$$\log M(z; X) = - \sum_{i \geq 1} \log(1 - z x_i) = \sum_{i \geq 1} \sum_{k \geq 1} \frac{(z x_i)^k}{k} = \sum_{k \geq 1} \frac{z^k}{k} p_k(X) , \tag{3.17}$$

$$\log L(z; X) = - \log M(z; X) = - \sum_{k \geq 1} \frac{z^k}{k} p_k(X) , \tag{3.18}$$

$$M(z; X) = \exp \left(\sum_{k \geq 1} (z^k / k) p_k(X) \right) , L(z; X) = \exp \left(- \sum_{k \geq 1} (z^k / k) p_k(X) \right) . \tag{3.19}$$

Let for all positive integers k , $p_k^\perp(X) = k \frac{\partial}{\partial p_k(X)}$, then $L^\perp(z; X) = \exp(- \sum_{k \geq 1} z^k \frac{\partial}{\partial p_k(X)})$.

Given a symmetric function $F(X) = \sum_{\alpha} C_\alpha X^\alpha$ expressed in terms of monomials $X^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \dots$ define new variables $\{y_1, y_2, \dots\}$ by

$$\prod_i (1 + z y_i) = \prod_{\alpha} (1 + z x^\alpha)^{C_\alpha} . \tag{3.20}$$

This allows one to define the plethysm of two symmetric functions $F, G \in \Lambda(X)$ by $(F \otimes G)(X) = G(y_1, y_2, \dots)$. This operation is right-distributive, but not left-distributive [46,47]⁸:

⁸ An important result in the theory of plethysms is: $s_\lambda \otimes s_\mu = \sum_{\rho} a_{\lambda,\mu}^{\rho} s_\rho$, where $a_{\lambda,\mu}^{\rho}$ are non-negative integers, and the sum is over all partitions of weight $|\rho| = |\lambda| \cdot |\mu|$. There is a standard method for recursively computing plethysms based on the identity [47]: $\sum_{n=0}^{|\lambda|+|\mu|} D_{(n)}(s_\lambda \otimes s_\mu) = (\sum_{m=0}^{|\mu|} s_{\mu/(m)}) \otimes s_\lambda$. Here D_λ is the adjoint (skew) operator $D_\lambda s_\sigma = s_{\sigma/\lambda}$.

$$(F + G) \otimes s_\lambda = \sum_{\rho} (F \otimes s_{\lambda/\rho})(G \otimes s_{\rho}). \tag{3.21}$$

By taking the logarithm of (3.20) as it has been done in (3.17), (3.18), it follows that $p_n(Y) = F(X^n)$. In order to calculate the plethysm $s_\lambda \otimes s_\mu$, one can express $s_\mu(X)$ as a multinomial in the power sums $p_1(x), p_2(x), \dots$, and then makes the substitution $p_j(X) \rightarrow s_\lambda(x^j)$. Thus a knowledge of how to express $s_\lambda(x^j)$ in terms on S -functions with argument X , along with the Littlewood Richardson rule for multiplying the S -functions together, is sufficient to be able to calculate any plethysm.

Let us consider, for any ordered r -tuple of nonnegative integers not all zeros, $(k_1, k_2, \dots, k_r) = \vec{k}$ (referred to as “ r -partite” or *multipartite* numbers), the (multi)partitions, i.e. distinct representations of (k_1, k_2, \dots, k_r) as sums of multipartite numbers. Let us call $\mathcal{C}_-^{(\mathcal{O},r)}(\vec{k}) = \mathcal{C}_-^{(\mathcal{O})}(\mathcal{O}; k_1, k_2, \dots, k_r)$ the number of such multipartitions, and introduce in addition the symbol $\mathcal{C}_+^{(\mathcal{O},r)}(\vec{k}) = \mathcal{C}_+^{(r)}(\mathcal{O}; k_1, k_2, \dots, k_r)$. Their generating functions are defined by

$$\mathcal{F}(\mathcal{O}; X) := \prod_{\vec{k} \geq 0} \left(1 - \mathcal{O}x_1^{k_1}x_2^{k_2}\dots x_r^{k_r}\right)^{-1} = \sum_{\vec{k} \geq 0} \mathcal{C}_-^{(\mathcal{O},r)}(\vec{k})x_1^{k_1}x_2^{k_2}\dots x_r^{k_r}, \tag{3.22}$$

$$\mathcal{G}(\mathcal{O}; X) := \prod_{\vec{k} \geq 0} \left(1 + \mathcal{O}x_1^{k_1}x_2^{k_2}\dots x_r^{k_r}\right) = \sum_{\vec{k} \geq 0} \mathcal{C}_+^{(\mathcal{O},r)}(\vec{k})x_1^{k_1}x_2^{k_2}\dots x_r^{k_r}. \tag{3.23}$$

Therefore,

$$\begin{aligned} \log \mathcal{F}(\mathcal{O}; X) &= - \sum_{\vec{k} \geq 0} \log \left(1 - \mathcal{O}x_1^{k_1}x_2^{k_2}\dots x_r^{k_r}\right) = \sum_{\vec{k} \geq 0} \sum_{m=1}^{\infty} \frac{\mathcal{O}^m}{m} x_1^{mk_1}x_2^{mk_2}\dots x_r^{mk_r} \\ &= \sum_{m=1}^{\infty} \frac{\mathcal{O}^m}{m} (1 - x_1^m)^{-1} (1 - x_2^m)^{-1} \dots (1 - x_r^m)^{-1} \\ &= \sum_{m=1}^{\infty} \frac{\mathcal{O}^m}{m} \prod_{j=1}^r (1 - x_j^m)^{-1}, \end{aligned} \tag{3.24}$$

$$\log \mathcal{G}(-\mathcal{O}; X) = \log \mathcal{F}(\mathcal{O}; X). \tag{3.25}$$

Finally,

$$\mathcal{F}(\mathcal{O}; X) = \sum_{\vec{k} \geq 0} \mathcal{C}_-^{(\mathcal{O},r)}(\vec{k})x_1^{k_1}x_2^{k_2}\dots x_r^{k_r} = \exp \left(\sum_{m=1}^{\infty} \frac{\mathcal{O}^m}{m} \prod_{j=1}^r (1 - x_j^m)^{-1} \right), \tag{3.26}$$

$$\mathcal{G}(\mathcal{O}; X) = \sum_{\vec{k} \geq 0} \mathcal{C}_+^{(\mathcal{O},r)}(\vec{k})x_1^{k_1}x_2^{k_2}\dots x_r^{k_r} = \exp \left(\sum_{m=1}^{\infty} \frac{(-\mathcal{O})^m}{m} \prod_{j=1}^r (1 - x_j^m)^{-1} \right). \tag{3.27}$$

Restricted specialization. Setting $X = (x_1, x_2, \dots, x_r, 0, 0, \dots)$; for finite additive manner let $p_m(X) := \prod_{j=1}^r (1 - x_j^m)^{-1}$, then

$$\mathcal{F}(\mathcal{O}; X) = \sum_{\vec{k} \geq 0} \mathcal{C}_-^{(\mathcal{O},r)}(\vec{k})x_1^{k_1}x_2^{k_2}\dots x_r^{k_r} = \exp \left(\sum_{m=1}^{\infty} \frac{\mathcal{O}^m}{m} \exp \left(\sum_{k \geq 1} \frac{1}{k} p_m(X) \right) \right). \tag{3.28}$$

$$\mathcal{G}(\mathcal{O}; X) = \sum_{\vec{k} \geq 0} \mathcal{C}_+^{(\mathcal{O}, r)}(\vec{k}) x_1^{k_1} x_2^{k_2} \cdots x_r^{k_r} = \exp \left(\sum_{m=1}^{\infty} \frac{(-\mathcal{O})^m}{m} \exp \left(\sum_{k \geq 1}^r \frac{1}{k} p_m(X) \right) \right). \tag{3.29}$$

It is known that the Bell polynomials are very useful in many problems in combinatorics. We would like to note their application in multipartite partition problem [48]. The Bell polynomials technique can be used for the calculation $\mathcal{C}_-^{(r)}(\vec{k})$ and $\mathcal{C}_+^{(r)}(\vec{k})$. Let

$$\mathcal{F}(\mathcal{O}; X) := 1 + \sum_{j=1}^{\infty} \mathcal{P}_j(x_1, x_2, \dots, x_r) \mathcal{O}^j, \quad \mathcal{P}_j = 1 + \sum_{\vec{k} > 0} P(\vec{k}; j) x_1^{n_1} \cdots x_r^{n_r}, \tag{3.30}$$

$$\mathcal{Q}(\mathcal{O}; X) := 1 + \sum_{j=1}^{\infty} \mathcal{Q}_j(x_1, x_2, \dots, x_r) \mathcal{O}^j, \quad \mathcal{Q}_j = 1 + \sum_{\vec{k} > 0} Q(\vec{k}; j) x_1^{n_1} \cdots x_r^{n_r}. \tag{3.31}$$

Useful expressions for the recurrence relation of the Bell polynomial $Y_n(g_1, g_2, \dots, g_n)$ and generating function $\mathcal{B}(\mathcal{O})$ have the forms [48]:

$$Y_{n+1}(g_1, g_2, \dots, g_{n+1}) = \sum_{k=0}^n \binom{n}{k} Y_{n-k}(g_1, g_2, \dots, g_{n-k}) g_{k+1}, \tag{3.32}$$

$\mathcal{B}(\mathcal{O}) = \sum_{n=0}^{\infty} Y_n \mathcal{O}^n / n! \implies \log \mathcal{B}(\mathcal{O}) = \sum_{n=1}^{\infty} g_n \mathcal{O}^n / n!$. To verify the last formula we need to differentiate with respect to \mathcal{O} and observe that a comparison of the coefficients of \mathcal{O}^n in the resulting equation produces an identity equivalent to (3.32). From Eq. (3.32) one can obtain the following explicit formula for the Bell polynomials (it is known as Faa di Bruno’s formula)

$$Y_n(g_1, g_2, \dots, g_n) = \sum_{\mathbf{k} \vdash n} \frac{n!}{k_1! \cdots k_n!} \prod_{j=1}^n \left(\frac{g_j}{j!} \right)^{k_j}. \tag{3.33}$$

The following result holds (see for detail [48]):

$$\mathcal{P}_j = \frac{1}{j!} Y_j(0!p_1(X), 1!p_2(X), \dots, (j-1)!p_j(X)), \tag{3.34}$$

$$\mathcal{Q}_j = \frac{1}{(-1)^j j!} Y_j(-0!p_1(X), -1!p_2(X), \dots, -(j-1)!p_j(X)). \tag{3.35}$$

For some specializations, when $X = q^\rho = (q, q^2, \dots, q^r)$ we get

$$\mathcal{F}(\mathcal{O}; X) = \prod_{\vec{k} \geq 0} \left(1 - \mathcal{O} q^{k_1+k_2+\dots+k_r} \right)^{-1} = \exp \left(- \sum_{m=1}^{\infty} \frac{\mathcal{O}^m}{m} \prod_{\ell=1}^r (1 - q^{\ell m})^{-1} \right), \tag{3.36}$$

$$\mathcal{G}(\mathcal{O}; X) = \prod_{\vec{k} \geq 0} \left(1 + \mathcal{O} q^{k_1+k_2+\dots+k_r} \right) = \exp \left(- \sum_{m=1}^{\infty} \frac{(-\mathcal{O})^m}{m} \prod_{\ell=1}^r (1 - q^{\ell m})^{-1} \right). \tag{3.37}$$

Spectral functions of hyperbolic geometry. Let us begin by explaining the general lore on the characteristic classes and \mathfrak{g} -structure on compact groups.

Statement 3.38. *Suppose that \mathfrak{g} is the Lie algebra of a Lie group G . Let us consider the pair (H, G) of Lie groups, where H is a closed subgroup of G with normalizer subgroup $N_H \subset G$. Then the pair (H, G) with the discrete quotient group N_H/H corresponds to the inclusion $\mathfrak{g} \hookrightarrow W_n$, where W_n is the Lie algebra of formal vector fields in $n = \dim G/H$ variables, while the homogeneous space G/H possesses a canonical \mathfrak{g} -structure ω . Combining this \mathfrak{g} -structure with the inclusion $\mathfrak{g} \hookrightarrow W_n$, one obtains a W_n -structure on the quotient space G/Γ for any discrete subgroup Γ of the Lie group G ; this is precisely the W_n -structure which corresponds to the H -equivariant foliation of G by left cosets of Γ [38]. The homomorphism $\text{char}_\omega : H^\sharp(W_n) \rightarrow H^\sharp(G/\Gamma, \mathbb{R})$ associated with characteristic classes of W_n -structures decomposes into the composition of two homomorphisms $H^\sharp(W_n) \rightarrow H^\sharp(\mathfrak{g})$ and $H^\sharp(\mathfrak{g}) \rightarrow H^\sharp(G/\Gamma, \mathbb{R})$; the first homomorphism is independent of Γ and is induced by the inclusion $\mathfrak{g} \hookrightarrow W_n$, while the second homomorphism is independent of H and corresponds to the canonical homomorphism which determines the characteristic classes of the canonical \mathfrak{g} -structure ω on G/Γ . If the group G is semi-simple, then the Lie algebra \mathfrak{g} is unitary and G contains a discrete subgroup Γ for which G/Γ is compact; for appropriate choice of Γ the kernel of the homomorphism $H^\sharp(W_n) \rightarrow H^\sharp(G/\Gamma, \mathbb{R})$ coincides with the kernel of the homomorphism $H^\sharp(W_n) \rightarrow H^\sharp(\mathfrak{g})$.*

In our applications we shall consider a compact hyperbolic three-manifold G/Γ with $G = SL(2, \mathbb{C})$. By combining the characteristic class representatives of field theory elliptic genera with the homomorphism char_ω , one can compute quantum partition functions in terms of the spectral functions of hyperbolic three-geometry [49].

Let us introduce next the Ruelle spectral function $\mathcal{R}(s)$ associated with hyperbolic three-geometry [50,49]. The function $\mathcal{R}(s)$ is an alternating product of more complicate factors, each of which is so-called Patterson–Selberg zeta-functions $Z_{\Gamma\gamma}$ (see Sect. 4.4 and [51]). Functions $\mathcal{R}(s)$ can be continued meromorphically to the entire complex plane \mathbb{C}

$$\begin{aligned} \prod_{n=\ell}^{\infty} (1 - q^{an+\varepsilon}) &= \prod_{p=0,1} Z_{\Gamma\gamma} (\underbrace{(al + \varepsilon)(1 - i\varrho(\vartheta)) + 1 - a}_s + a(1 + i\varrho(\vartheta))p)^{(-1)^p} \\ &= \mathcal{R}(s = (al + \varepsilon)(1 - i\varrho(\vartheta)) + 1 - a), \end{aligned} \tag{3.39}$$

$$\begin{aligned} \prod_{n=\ell}^{\infty} (1 + q^{an+\varepsilon}) &= \prod_{p=0,1} Z_{\Gamma\gamma} (\underbrace{(al + \varepsilon)(1 - i\varrho(\vartheta)) + 1 - a + i\sigma(\vartheta)}_s + a(1 + i\varrho(\vartheta))p)^{(-1)^p} \\ &= \mathcal{R}(s = (al + \varepsilon)(1 - i\varrho(\vartheta)) + 1 - a + i\sigma(\vartheta)), \end{aligned} \tag{3.40}$$

$$\begin{aligned} \prod_{n=\ell}^{\infty} (1 - q^{an+\varepsilon})^{bn} &= \mathcal{R}(s = (al + \varepsilon)(1 - i\varrho(\vartheta)) + 1 - a)^{b\ell} \\ &\times \prod_{n=\ell+1}^{\infty} \mathcal{R}(s = (an + \varepsilon)(1 - i\varrho(\vartheta)) + 1 - a)^b, \end{aligned} \tag{3.41}$$

$$\begin{aligned} \prod_{n=\ell}^{\infty} (1 + q^{an+\varepsilon})^{bn} &= \mathcal{R}(s = (al + \varepsilon)(1 - i\varrho(\vartheta)) + 1 - a + i\sigma(\vartheta))^{b\ell} \\ &\times \prod_{n=\ell+1}^{\infty} \mathcal{R}(s = (an + \varepsilon)(1 - i\varrho(\vartheta)) + 1 - a + i\sigma(\vartheta))^b, \end{aligned} \tag{3.42}$$

being $q \equiv e^{2\pi i \vartheta}$, $\varrho(\vartheta) = \text{Re } \vartheta / \text{Im } \vartheta$, $\sigma(\vartheta) = (2 \text{Im } \vartheta)^{-1}$, a is a real number, $\varepsilon, b \in \mathbb{C}$, $\ell \in \mathbb{Z}_+$.

Obviously, $\prod_{\ell=1}^r (1 - q^{\ell m}) \equiv \prod_{\ell=1}^{\infty} (1 - q^{\ell m}) \prod_{\ell=r+1}^{\infty} (1 - q^{\ell m})^{-1}$ and

$$\begin{aligned} \mathcal{F}(\mathcal{O}; X) &= \prod_{\vec{k} \geq 0} \left(1 - \mathcal{O} q^{k_1+k_2+\dots+k_r} \right)^{-1} \\ &= \exp \left(- \sum_{m=1}^{\infty} \frac{\mathcal{O}^m}{m} \left(\frac{\mathcal{R}(s = -im\varrho(\vartheta)(r+1) + mr + 1)}{\mathcal{R}(s = -im\varrho(\vartheta) + 1)} \right) \right), \end{aligned} \tag{3.43}$$

$$\begin{aligned} \mathcal{G}(\mathcal{O}; X) &= \prod_{\vec{k} \geq 0} \left(1 + \mathcal{O} q^{k_1+k_2+\dots+k_r} \right) \\ &= \exp \left(- \sum_{m=1}^{\infty} \frac{(-\mathcal{O})^m}{m} \left(\frac{\mathcal{R}(s = -im\varrho(\vartheta)(r+1) + mr + 1)}{\mathcal{R}(s = -im\varrho(\vartheta) + 1)} \right) \right). \end{aligned} \tag{3.44}$$

Hierarchy. Setting $\mathcal{O} q^{k_0+k_1+\dots+k_r} = \mathcal{O}_{\vec{k}} q^{k_0}$ with $\mathcal{O}_{\vec{k}} = \mathcal{O} q^{k_1+\dots+k_r}$ ($\vec{k} = (k_1, \dots, k_r)$) we get

$$Z_2(\mathcal{O}_{\vec{k}}, q) = \prod_{k_0=0}^{\infty} \left[1 - \mathcal{O}_{\vec{k}} q^{k_0} \right]^{-1} = [(1 - \mathcal{O}_{\vec{k}}) \mathcal{R}(s = (k_1 + \dots + k_r)(1 - i\varrho(\tau)))]^{-1}. \tag{3.45}$$

Therefore the infinite products $\prod_{k_r=0}^{\infty} \prod_{k_{r-1}=0}^{\infty} \dots \prod_{k_1=0}^{\infty} \prod_{k_0=0}^{\infty} (1 - q^{k_0+k_1+\dots+k_r})^{-1}$ can be factorized as $\prod_{\vec{k} \geq \vec{0}} Z_2(\mathcal{O}_{\vec{k}}, q)$. We can treat this factorization as a product of r copies, each of them is $Z_2(\mathcal{O}_{\vec{k}}, q)$ and corresponds to a free two-dimensional conformal field theory.

3.4. Characters, branching rules and vertex operator traces

Let us discuss Schur functions interpretation as universal characters [52]. In this section our aim is try to make clear the connection between the Hopf algebraic approach and the group theory. Our basic starting point is very well known Weyl’s character formula

$$\text{ch}(\Lambda) = \frac{\sum_{w \in W} \varepsilon(w) e^{w(\Lambda + \rho)}}{\sum_{w \in W} \varepsilon(w) e^{w(\rho)}}. \tag{3.46}$$

Here Λ is the highest weight vector, ρ is half the sum of the positive roots and W is the appropriate Weyl group with ε the sign of w .

Characters of the classical groups. Recall that the Cartan classification of the simple complex classical Lie groups is given by the series A_n, B_n, C_n and D_n (not to be confused with Schur function series). These series correspond to the complexified versions of the groups $SU(n+1), SO(2n+1), Sp(2n)$ and $SO(2n)$, which can be considered as subgroups of unitary groups $U(N)$ for $N = n+1, 2n+1, 2n$ and $2n$. Let us denote eigenvalues as $x_k = \exp(i\varphi_k)$; one can write the eigenvalues of group elements as follows:

$$\begin{aligned} GL(n) : & \quad x_1, x_2, \dots, x_n \quad \text{and} \quad x_1 x_2 \dots x_n \neq 0. \\ SL(n) : & \quad x_1, x_2, \dots, x_n \quad \text{and} \quad x_1 x_2 \dots x_n = 1. \\ SU(n+1) : & \quad x_1, x_2, \dots, x_n \quad \text{and} \quad x_1 x_2 \dots x_n = 1. \end{aligned}$$

$$\begin{aligned}
 O(2n + 1) \setminus SO(2n + 1) : & \quad x_1, x_2, \dots, x_n, x_1^{-1}, x_2^{-1}, \dots, x_n^{-1}, -1. \\
 O(2n) \setminus SO(2n) : & \quad x_1, x_2, \dots, x_n, x_1^{-1}, x_2^{-1}, \dots, x_{n-1}^{-1}, -1. \\
 SO(2n) : & \quad x_1, x_2, \dots, x_n, x_1^{-1}, x_2^{-1}, \dots, x_n^{-1}. \\
 SO(2n + 1) : & \quad x_1, x_2, \dots, x_n, x_1^{-1}, x_2^{-1}, \dots, x_n^{-1}, 1. \\
 Sp(2n) : & \quad x_1, x_2, \dots, x_n, x_1^{-1}, x_2^{-1}, \dots, x_n^{-1}. \\
 Sp(2n + 1) : & \quad x_1, x_2, \dots, x_n, x_1^{-1}, x_2^{-1}, \dots, x_n^{-1}, 1.
 \end{aligned}$$

For example, in the case of $SO(2n)$ and $Sp(2n)$ we have $p_n(X, X^{-1}) = p_n(X) + p_n(X^{-1})$, while for $SO(2n + 1)$ and $Sp(2n + 1)$, $p_n(X, X^{-1}, 1) = p_n(X) + p_n(X^{-1}) + 1$. The connection to the group characters is obtained by inserting the eigenvalues into the Weyl character formula and interpreting the exponentials as

$$e^\lambda = x_1^{\lambda_1} x_2^{\lambda_2} \dots x_n^{\lambda_n}, \quad \lambda = (\lambda_1, \dots, \lambda_n). \tag{3.47}$$

In the case of $U(n)$, the Weyl group is just the symmetric group (on n letters). Hence the characters are labeled by partitions and the Weyl character formula turns into the defining relation of the Schur functions. Let μ be the conjugacy class of the permutation. One finds

$$\text{ch}_\mu(\lambda) = \frac{\sum_{w \in S_n} \varepsilon(w) e^{w(\lambda + \rho) \cdot \mu}}{\sum_{w \in S_n} \varepsilon(w) e^{w(\rho \cdot \mu)}} \tag{3.48}$$

where $\rho = (n - 1, n - 2, \dots, 1, 0)$. Both numerator and denominator reduce to determinants after inserting the x_i (the denominator being the Vandermonde determinant), and the quotient of the two alternating functions is a standard construction of the Schur function.

Each irreducible tensor representation, $V_{GL(n)}^\lambda$, of $GL(n)$ is specified by a partition λ of length $\ell(\lambda) \leq n$. Let $X \in GL(n)$ have eigenvalues (x_1, x_2, \dots, x_n) and let $\rho = (n - 1, n - 2, \dots, 1, 0)$. Then the character of this irreducible representation is given by [47,1]: $\text{ch } V_{GL(n)}^\lambda = a_{\lambda+\rho}(X)/a_\rho(X) = |x_i^{\lambda_j+n-j}|/|x_i^{n-j}| = s_\lambda(X)$. One can define the corresponding universal character of $GL(n)$ by

$$\text{ch } V_{GL}^\lambda = \{\lambda\}(X) = s_\lambda(X), \tag{3.49}$$

where $X = (x_1, x_2, \dots)$. For each finite n the characters $\text{ch } V_{GL(n)}^\lambda$ are recovered from the universal characters $\text{ch } V_{GL}^\lambda$ merely by setting $X = (x_1, x_2, \dots, x_n, 0, 0, \dots, 0)$. In a similar way, there exists irreducible tensor representation, $V_{O(n)}^\lambda$ and $V_{Sp(n)}^\lambda$, of $O(n)$ and $Sp(n)$, respectively. The corresponding characters $\text{ch } V_{O(n)}^\lambda$ and $\text{ch } V_{Sp(n)}^\lambda$ may each be defined in terms of determinants. More important is that there exist corresponding universal characters [53], denoted by

$$\text{ch } V_O^\lambda = [\lambda](X) \quad \text{and} \quad \text{ch } V_{Sp}^\lambda = \langle \lambda \rangle(X), \tag{3.50}$$

with $X = (x_1, x_2, \dots)$ arbitrary. These are universal in the sense that for any finite n the characters $\text{ch } V_{O(n)}^\lambda$ and $\text{ch } V_{Sp(n)}^\lambda$ are obtained by specializing X to $(x_1, x_2, \dots, x_n, 0, 0, \dots, 0)$ with x_1, x_2, \dots, x_n restricted to the eigenvalues of the appropriate group elements parametrized as above. The universal characters (3.50) are themselves defined by means of the generating functions [47]:

$$\prod_{i,j} (1 - x_i y_j)^{-1} \prod_{i \leq j} (1 - y_i y_j) = \sum_{\lambda} [\lambda](X) \{\lambda\}(Y); \tag{3.51}$$

$$\prod_{i,j} (1 - x_i y_j)^{-1} \prod_{i < j} (1 - y_i y_j) = \sum_{\lambda} \langle \lambda \rangle(X) \{\lambda\}(Y). \tag{3.52}$$

Theorem 3.1. (See Theorem 3.1 in [54].) The universal characters $\text{ch } V_G^\lambda$ of the orthogonal and symplectic groups are given respectively by

$$[\lambda](X) = \{\lambda/\mathbf{C}\}(X) = s_{\lambda/\mathbf{C}}(X), \quad \text{where } \mathbf{C}(X) = \prod_{i \leq j} (1 - x_i x_j); \tag{3.53}$$

$$\langle \lambda \rangle(X) = \{\lambda/\mathbf{A}\}(X) = s_{\lambda/\mathbf{A}}(X), \quad \text{where } \mathbf{A}(X) = \prod_{i < j} (1 - x_i x_j). \tag{3.54}$$

Branching rules. In order to allow the possibility of extending the results to a wider class of subgroups of the general linear group, we consider any subgroup $H(n)$ of $GL(n)$, whose group elements $X \in H(n) \subset GL(n)$ have the eigenvalues $X = (x_1, x_2, \dots, x_n)$. We assume, just as in the case of $O(n)$ and $Sp(n)$, that there exist irreducible representations $V_{H(n)}^\lambda$ of $H(n)$, specified by partitions λ , with characters $\text{ch } V_{H(n)}^\lambda$ that may be determined by specializing from $X = (x_1, x_2, \dots)$ to $X = (x_1, x_2, \dots, x_n, 0, \dots, 0)$, with appropriate x_1, x_2, \dots, x_n . Applying this to $O(n)$ and $Sp(n)$, we have:

Theorem 3.2. (See Theorem 4.1 in [54].) The branching rules for the decomposition of representations of $GL(n)$ under restriction to the subgroups $O(n)$ and $Sp(n)$ take the form:

$$GL(n) \supset O(n) : \quad \{\lambda\} \rightarrow [\lambda/D] \quad \text{with } D = \mathbf{C}^{-1} = \prod_{i \leq j} (1 - x_i x_j)^{-1}; \tag{3.55}$$

$$GL(n) \supset Sp(n) : \quad \{\lambda\} \rightarrow \langle \lambda/B \rangle \quad \text{with } B = \mathbf{A}^{-1} = \prod_{i < j} (1 - x_i x_j)^{-1}. \tag{3.56}$$

One of the most important features of group representations is the modular invariance of their Kač–Weyl character formula, which allows us to derive many new results and to unify many important results in topology. Let us recall that the Weyl character formula in representation theory describes the characters of irreducible representations of compact Lie groups in terms of their highest weights.

Remark 3.1. All group like $\prod_j (1 - F(x_j))^\alpha$, for polynomial F , (based on 1-cocycles) induce trivial branchings, i.e. branchings equivalent to $U(n)$ ([39]). In the special case of the trivial one-dimensional representation the character is 1, so the Weyl character formula becomes the Weyl denominator formula:

$$\sum_{w \in W} \varepsilon(w) e^{w(\rho)} = e^\rho \prod_{\alpha \in \Delta_+} (1 - e^{-\alpha}). \tag{3.57}$$

For special unitary groups this is equivalent to the expression $\sum_{\sigma \in S_n} \text{sgn}(\sigma) x_1^{\sigma(1)-1} \dots x_n^{\sigma(n)-1} = \prod_{1 \leq i < j \leq n} (x_j - x_i)$ i.e. the Vandermonde determinant.

Vertex operator traces. Vertex operators have played a fruitful role in string theory, mathematical constructions of group representations as well as combinatorial constructions. We cite their applications to affine and quantum affine Lie algebras [46,55–57] and sporadic discrete groups [58] (see also [59], Chapter 14). Variations on the theme of symmetric functions are applications, for example, to Q -functions [60,61], Hall–Littlewood functions [62], Macdonald functions [63,14], Jack functions [64], Kerov’s symmetric functions [65] (and a specialization of S -functions introduced by Kerov).

By considering different specializations of Kerov’s symmetric functions the trace calculations in representations of the levels quantum affine algebra $U_q(\mathfrak{gl}_N)$ can be feasible. The extension of this mathematical tools to other (quantum) affine algebras and superalgebras is also practicable and provides the relevant vertex operator realizations of those algebras.

As an approach to generalizing the vertex operators, the observations made in previous sections allow us to write down the expressions for replicated or parameterized vertex operators. In the simplest case, this is exemplified by

$$V_\alpha(z) = M(\alpha z) L^\perp(\alpha z^{-1}) = \exp\left(\alpha \sum_{k \geq 1} \frac{z^k}{k} p_k\right) \exp\left(-\alpha \sum_{k \geq 1} z^{-k} \frac{\partial}{\partial p_k}\right), \tag{3.58}$$

for any α , integer, rational, real or complex. Then we have:

$$\begin{aligned} M(\alpha z; X) &= M(z; X)^\alpha = \prod_{i \geq 1} (1 - z x_i)^{-\alpha} = \sum_{\sigma} s_\sigma(\alpha z) s_\sigma(X) \\ &= \sum_{\sigma} z^{|\sigma|} \dim_{\sigma}(\alpha) s_\sigma(X), \end{aligned} \tag{3.59}$$

$$\begin{aligned} L(\alpha z^{-1}; X) &= L(z^{-1}; X)^\alpha = \prod_{i \geq 1} (1 - z^{-1} x_i)^\alpha = \sum_{\tau} (-1)^{|\tau|} s_\tau(\alpha z^{-1}) s_{\tau'}(X) \\ &= \sum_{\tau} (-z)^{-|\tau|} \dim_{\tau}(\alpha) s_{\tau'}(X), \end{aligned} \tag{3.60}$$

as given first in [66]. The following result holds

$$V_{\alpha_1}(x_1) \cdots V_{\alpha_n}(x_n) = \prod_{i < j} (1 - x_j x_i^{-1})^{\alpha_i \alpha_j} : V_{\alpha_1}(x_1) \cdots V_{\alpha_n}(x_n) : \tag{3.61}$$

Here by $::$ we mean the procedure normal ordering. The Cauchy kernel $M(XZ)$ serves as a generating function for characters of $GL(n)$ in the sense that

$$M(XY) = \prod_{i,j} (1 - x_i y_j) = \sum_{\lambda} s_\lambda(X) s_\lambda(Y), \tag{3.62}$$

where $s_\lambda(X)$ is the character of the irreducible representation $V_{GL(n)}^\lambda$ of highest weight λ evaluated at group elements whose eigenvalues are the element of X . We summarize some useful formulas:

$$\begin{aligned} M(q; XY) &= \prod_{i,j} (1 - q x_i y_j)^{-1} = \sum_{\alpha} q^\alpha s_\alpha(X) s_\alpha(Y), \\ L(q; XY) &= \prod_{i,j} (1 - q x_i y_j) = \sum_{\alpha} (-q)^{|\alpha|} s_\alpha(X) s_{\alpha'}(Y). \end{aligned} \tag{3.63}$$

The standard infinite symmetric function series which used in representation theory of finite dimensional algebras and superalgebras can be given by a Fock space interpretation [67]

$${}_q\langle \mathbf{A} | : V_1(x_1) \cdot V_2(x_2) \cdots : |0\rangle = \prod_{i < j} (1 - qx_i x_j) = \sum_{\alpha \in \mathbf{A}} (-q)^{|\alpha|/2} s_\alpha(X), \tag{3.64}$$

$${}_q\langle \mathbf{B} | : V_1(x_1) \cdot V_2(x_2) \cdots : |0\rangle = \prod_{i < j} (1 - qx_i x_j)^{-1} = \sum_{\beta \in \mathbf{B}} q^{|\beta|/2} s_\beta(X), \tag{3.65}$$

$${}_q\langle \mathbf{C} | : V_1(x_1) \cdot V_2(x_2) \cdots : |0\rangle = \prod_{i \leq j} (1 - qx_i x_j) = \sum_{\gamma \in \mathbf{C}} (-q)^{|\gamma|/2} s_\gamma(X), \tag{3.66}$$

$${}_q\langle \mathbf{D} | : V_1(x_1) \cdot V_2(x_2) \cdots : |0\rangle = \prod_{i \leq j} (1 - qx_i x_j)^{-1} = \sum_{\delta \in \mathbf{D}} q^{|\delta|/2} s_\delta(X). \tag{3.67}$$

To define A, B, C, D is convenient to use the Frobenius notation:

$$\mathfrak{F}_n = \left\{ \begin{pmatrix} a_1 & a_2 & \dots & a_r \\ b_1 & b_2 & \dots & b_r \end{pmatrix} \in \mathfrak{F} \mid \begin{matrix} a_k - b_k = n & \text{for all } k = 1, 2, \dots, r \\ r = 0, 1, 2, \dots \end{matrix} \right\} \tag{3.68}$$

With this notation $\mathbf{A} = \mathfrak{F}_{-1}$, $\mathbf{C} = \mathfrak{F}_1$, \mathbf{D} is the set of partitions all of whose parts are even, and \mathbf{B} is the set of partitions all of whose distinct parts are repeated an even number of times. Let us define modified symmetric functions (and corresponding kets vectors): $S_{\lambda/\mathbf{C}}(X)$, $\langle \lambda/\mathbf{C} |$, $S_{\lambda/\mathbf{A}}(X)$, $\langle \lambda/\mathbf{A} |$, where the notation indicates symmetric function division distributed over all admissible elements of the indicated series. Then matrix elements of vertex operators with the corresponding reservoir states

$$\begin{aligned} {}_q\langle \mathbf{A}/\mathbf{C} | : V_1(x_1) \cdot V_2(x_2) \cdots : |0\rangle &= \prod_{k=1}^{\infty} \prod_{i < j} (1 - q^k x_i x_j) \\ &= \prod_{i < j} \mathcal{R}(s = (1 + \Omega(x_i x_j; \vartheta))(1 - i\varrho(\vartheta))), \end{aligned} \tag{3.69}$$

$$\begin{aligned} {}_q\langle \mathbf{C}/\mathbf{A} | : V_1(x_1) \cdot V_2(x_2) \cdots : |0\rangle &= \prod_{k=1}^{\infty} \prod_{i \leq j} (1 - q^k x_i x_j) \\ &= \prod_{i \leq j} \mathcal{R}(s = (1 + \Omega(x_i x_j; \vartheta))(1 - i\varrho(\vartheta))), \end{aligned} \tag{3.70}$$

where $\Omega(x_i x_j; \vartheta) := \log(x_i x_j) / 2\pi i \vartheta$.

4. The quantum group invariants

4.1. The HOMFLY skein and the quantum group invariants

Preliminaries. Our notations are summarized as follows: denote by \mathcal{Y} the set of all Young diagrams. Let χ_A be the character of irreducible representation of symmetric group, labeled by partition A . Given a partition μ , define $m_j = \text{card}(\mu_k = j; k \geq 1)$. The order of the conjugate class of type μ is given as before by: $z_\mu = \prod_{j \geq 1} j^{m_j} m_j!$. The symmetric power functions of a given set of variables $X = \{x_j\}_{j \geq 1}$ are defined as the direct limit of the Newton polynomials:

$p_n(X) = \sum_{j \geq 1} x_j^n$, $p_\mu(X) = \prod_{i \geq 1} p_{\mu_i}(X)$, and we have the following formulae which determines the Schur function and the orthogonality property of the character

$$s_A(X) = \sum_{\mu} \frac{\chi_A(C_\mu)}{\delta_\mu} p_\mu(X), \quad \sum_{\mu} \frac{\chi_A(C_\mu)\chi_B(C_\mu)}{\delta_\mu} = \delta_{A,B}. \tag{4.1}$$

Given $X = \{x_i\}_{i \geq 1}$, $Y = \{y_j\}_{j \geq 1}$, define $X * Y = \{x_i \cdot y_j\}_{i \geq 1, j \geq 1}$. We also define $X^d = \{x_i^d\}_{i \geq 1}$. The d -th Adam operation of a Schur function is given by $s_A(X^d)$. We use the following convention for the notations:

- Denote by \mathcal{L} a link and by L the number of components in \mathcal{L} .
- The irreducible $U_q(\mathfrak{sl}_N)$ module associated to \mathcal{L} will be labeled by their highest weights, thus by Young diagrams. We usually denote it by a vector form $\vec{A} = (A^1, \dots, A^L)$.
- Let $\vec{X} = (x_1, \dots, x_L)$ is L sets of variables, each of which is associated to a component of \mathcal{L} and $\vec{\mu} = (\mu^1, \dots, \mu^L) \in \mathcal{Y}^L$ be a tuple of L partitions, and

$$[\vec{\mu}] = \prod_{\alpha=1}^L [\mu^\alpha], \quad \delta_{\vec{\mu}} = \prod_{\alpha=1}^L \delta_{\mu^\alpha}, \quad \chi_{\vec{A}}(C_{\vec{\mu}}) = \prod_{\alpha=1}^L \chi_{A^\alpha}(C_{\mu^\alpha}),$$

$$s_{\vec{A}}(\vec{X}) = \prod_{\alpha=1}^L s_{A^\alpha}(x_\alpha), \quad p_\mu(X) = \prod_{i=1}^{\ell(\mu)} p_{\mu_i}(X), \quad p_{\vec{\mu}}(\vec{X}) = \prod_{\alpha=1}^L p_{\mu^\alpha}(x_\alpha).$$

The quantum group invariants can be defined over any semi-simple Lie algebra \mathfrak{g} . In the $SU(N)$ Chern–Simons gauge theory we study the quantum \mathfrak{sl}_N invariants, which can be identified as the colored HOMFLY polynomials.

The framed HOMFLY polynomial of links (an invariant of framed oriented links), denote by $\mathcal{H}(\mathcal{L})$, and can be normalize as: $\mathcal{H}(\bigcirc) = (t^{-\frac{1}{2}} - t^{\frac{1}{2}})/(q^{-\frac{1}{2}} - q^{\frac{1}{2}})$. These invariants can be recursively computed through the HOMFLY skein.

The colored HOMFLY polynomials are defined through *satellite knot*. A satellite of \mathcal{K} is determined by choosing a diagram Q in the annulus. Draw Q on the annular neighborhood of \mathcal{K} determined by the framing to give a satellite knot $\mathcal{K} \star Q$. One can refer to this construction as *decorating \mathcal{K} with the pattern Q* . The HOMFLY polynomial $\mathcal{H}(\mathcal{K} \star Q)$ of the satellite depends on Q only as an element of the skein \mathcal{C} of the annulus. $\{Q_\lambda\}_{\lambda \in \mathcal{Y}}$ form a basis of \mathcal{C} . \mathcal{C} can be regarded as the parameter space for these invariants of \mathcal{K} , and can be called as the HOMFLY *satellite invariants of \mathcal{K}* .

4.2. Link invariants from vertex models

Vertex calculation (see Sect. 3) can be used to associate oriented graphs, provided one decides on a way of numbering the edges which meet at a given vertex. It is known that plan projection of any link is a valent graph and therefore the partition function associates a number to every oriented link. One can choose the weights of vertices (beyond the quantum Yang–Baxter equation) in such a way that the partition function depends only on the equivalent class of the link.

The quantum \mathfrak{sl}_N invariant for the irreducible module V_{A^1}, \dots, V_{A^L} , labeled by the corresponding partitions A^1, \dots, A^L , can be identified as the HOMFLY invariants for the link decorated by Q_{A^1}, \dots, Q_{A^L} . The quantum \mathfrak{sl}_N invariants of the link is given by

$$W_{\vec{A}}(\mathcal{L}; q, t) = \mathcal{H}(\mathcal{L} \star \bigotimes_{\alpha=1}^L Q_{A^\alpha}). \tag{4.2}$$

The colored HOMFLY polynomial of the link \mathcal{L} (4.2) can be defined by [68]

$$W_{\vec{A}} = q^{-\sum_{\alpha=1}^L k_{A^\alpha} \omega(\mathcal{K}_\alpha)} t^{-\sum_{\alpha=1}^L |A^\alpha| \omega(\mathcal{K}_\alpha)} \langle \mathcal{L} \star \otimes_{\alpha=1}^L Q_{A^\alpha} \rangle, \tag{4.3}$$

where $\omega(\mathcal{K}_\alpha)$ is the number of the α -component \mathcal{K}_α of \mathcal{L} and the bracket $\langle \mathcal{L} \star \otimes_{\alpha=1}^L Q_{A^\alpha} \rangle$ denotes the framed HOMFLY polynomial of the satellite link $\mathcal{L} \star \otimes_{\alpha=1}^L Q_{A^\alpha}$. We can define the following invariants:

$$Z_{\vec{\mu}}(\mathcal{L}; q, t) = \sum_{\vec{A}=(A^1, \dots, A^L)} \left(\prod_{\alpha=1}^L \chi_{A^\alpha}(C_{\mu^\alpha}) \right) W_{\vec{A}}(\mathcal{L}; q, t). \tag{4.4}$$

The Chern–Simons partition function $Z_{CS}(\mathcal{L}; q, t)$ and the free energy $F(\mathcal{L}; q, t)$ of the link \mathcal{L} are the following generating series of quantum group invariants weighted by Schur functions and by the invariants $Z_{\vec{\mu}}$:

$$Z_{CS}(\mathcal{L}; q, t) = 1 + \sum_{\vec{A}} W_{\vec{A}}(\mathcal{L}; q, t) s_{\vec{A}}(\vec{X}) = 1 + \sum_{\vec{\mu}} \frac{Z_{\vec{\mu}}(\mathcal{L}; q, t)}{\delta_{\vec{\mu}}} p_{\vec{\mu}}(\vec{X}), \tag{4.5}$$

$$F(\mathcal{L}; q, t) = \log Z_{CS}(\mathcal{L}; q, t) = \sum_{\vec{\mu}} \frac{F_{\vec{\mu}}(\mathcal{L}; q, t)}{\delta_{\vec{\mu}}} p_{\vec{\mu}}(\vec{X}). \tag{4.6}$$

4.3. From summations to infinite products

The Chern–Simons theory has been conjectured to be equivalent to a topological string theory $1/N$ expansion in physics. This duality conjecture builds a fundamental connection in mathematics. On the one hand, Chern–Simons theory leads to the construction of knot invariants; on the other hand, topological string theory gives rise to Gromov–Witten theory in geometry.

The Chern–Simons/topological string duality conjecture identifies the generating function of Gromov–Witten invariants as Chern–Simons knot invariants [17]. Based on these thoughts, the existence of a sequence of integer invariants is conjectured [17,16] in a similar spirit to Gopakumar–Vafa setting [19], which provides an essential evidence of the duality between Chern–Simons theory and topological string theory. This integrality conjecture is called the LMOV conjecture. One important corollary of the LMOV conjecture is to express Chern–Simons partition function as an infinite product derived in this article. The motivation of studying such an infinite-product formula is based on a guess on the modularity property of topological string partition function.

To derive an infinite-product formula, we will state the result for a knot at first, since the notations in the computation for a knot are relatively simpler.

The case of a knot. Based on LMOV conjecture the following infinite product for a knot has been obtained in [22]:

$$Z_{CS}^{SL}(\mathcal{K}; q, t; X) = \prod_{\mu} \prod_{Q \in \mathbb{Z}/2} \prod_{m=1}^{\infty} \prod_{k=-\infty}^{\infty} \langle 1 - q^{k+m} t^Q X^\mu \rangle^{-m n_{\mu; g, Q}}. \tag{4.7}$$

Here $n_{\mu; g, Q}$ are invariants related to the integer invariants in the LMOV conjecture. For a given μ , $n_{\mu; g, Q}$ vanish for sufficiently large $|Q|$ due to the vanishing property of $n_{B; g, Q}$; the products involved with Q and k are finite products for a fixed partition μ . The symmetric product $\langle \cdot \rangle$ defined by the formula

$$\langle 1 - \psi X^\mu \rangle = \prod_{x_{i_1}, \dots, x_{i_{\ell(\mu)}}} \left(1 - \psi x_{i_1}^{\mu_1} \dots x_{i_{\ell(\mu)}}^{\mu_{\ell(\mu)}} \right). \tag{4.8}$$

Remark 4.9. The symmetric product (4.8) can be simplify by using of the Bell polynomials (3.34), (3.35), Sect. 3.3. In that case the final result can be represent in form similar to (3.43) and (3.44).

In terms of Ruelle spectral functions $Z_{CS}^{SL}(\mathcal{K}; q, t; X)$ takes the form

$$\begin{aligned} Z_{CS}^{SL}(\mathcal{K}; q, t; X) &= \prod_{\mu} \prod_{Q \in \mathbb{Z}/2} \prod_{k=-\infty}^{\infty} \prod_{m=1}^{\infty} \prod_{x_{i_1}, \dots, x_{i_{\ell(\mu)}}} \left(1 - q^{k+m} t^Q x_{i_1}^{\mu_1} \dots x_{i_{\ell(\mu)}}^{\mu_{\ell(\mu)}} \right)^{-m n_{\mu}; g, Q} \\ &= \prod_{\mu} \prod_{Q \in \mathbb{Z}/2} \prod_{k=-\infty}^{\infty} \prod_{m=1}^{\infty} \prod_{x_{i_1}, \dots, x_{i_{\ell(\mu)}}} \left(1 - q^{m+\Omega} (q^k t^Q X^\mu; \vartheta) \right)^{-m n_{\mu}; g, Q} \\ &= \prod_{\mu} \prod_{Q \in \mathbb{Z}/2} \prod_{k=-\infty}^{\infty} \prod_{x_{i_1}, \dots, x_{i_{\ell(\mu)}}} \prod_{m=1}^{\infty} \left(\mathcal{R}(s = (m + \Omega(t^Q X^\mu q^k; \vartheta))(1 - i_Q(\vartheta))) \right)^{-n_{\mu}; g, Q}, \end{aligned} \tag{4.10}$$

where as before $\Omega(q^k t^Q X^\mu; \vartheta) \equiv \log(q^k t^Q x_{i_1}^{\mu_1} \dots x_{i_{\ell(\mu)}}^{\mu_{\ell(\mu)}}) / 2\pi i \vartheta$.

The case of links. The generalization of this result for the case of links can be easy derived. Let $\vec{\mu} = (\mu^1, \dots, \mu^L)$ and $\vec{X} = (x_1, \dots, x_L)$. Denote by ℓ_i the length of μ^i . Generalize the symmetric product in Eq. (4.8) to $\vec{\mu}$ and \vec{X} as:

$$\langle 1 - \psi \vec{X}^\mu \rangle = \prod_{\alpha=1}^L \prod_{i_{\alpha,1}, \dots, i_{\alpha, \ell_\alpha}} \left(1 - \psi \prod_{\alpha=1}^L ((x_\alpha)_{i_{\alpha,1}}^{\mu_\alpha^1} \dots (x_\alpha)_{i_{\alpha, \ell_\alpha}}^{\mu_\alpha^{\ell_\alpha}}) \right). \tag{4.11}$$

The infinite-product formula for the Chern–Simons partition function of \mathcal{L} :

$$\begin{aligned} Z_{CS}^{SL}(\mathcal{L}; q, t; \vec{X}) &= \\ &= \prod_{\vec{\mu}} \prod_{Q \in \mathbb{Z}/2} \prod_{k=-\infty}^{\infty} \prod_{m=1}^{\infty} \prod_{\alpha=1}^L \prod_{i_{\alpha,1}, \dots, i_{\alpha, \ell_\alpha}} \left(1 - q^{m+g-2k} \prod_{\alpha=1}^L ((x_\alpha)_{i_{\alpha,1}}^{\mu_\alpha^1} \dots (x_\alpha)_{i_{\alpha, \ell_\alpha}}^{\mu_\alpha^{\ell_\alpha}}) \right)^{-m n_{\vec{\mu}}; g, Q} = \\ &= \prod_{\vec{\mu}} \prod_{Q \in \mathbb{Z}/2} \prod_{k=-\infty}^{\infty} \prod_{\alpha=1}^L \prod_{i_{\alpha,1}, \dots, i_{\alpha, \ell_\alpha}} \prod_{m=1}^{\infty} \mathcal{R}(s = (m + \Omega(q^{g-2k} t^Q \vec{X}^\mu; \vartheta))(1 - i_Q(\vartheta)))^{-n_{\vec{\mu}}; g, Q}. \end{aligned} \tag{4.12}$$

The case of the unknot. The Chern–Simons partition function of the unknot is given by

$$Z_{CS}^{SL}(\bigcirc; q, t) = 1 + \sum_A \dim_q V_A \cdot s_A(X), \tag{4.13}$$

where $\dim_q V_A$ is the quantum dimension of the irreducible $U_q(\mathfrak{sl}_N)$ module V_A . The formula of quantum dimension is well known (see for example [18]):

$$\dim_q V_A = \sum_{\mu} \frac{\chi_A(C_{\mu})}{\delta_{\mu}} \prod_{j=1}^{\ell(\mu)} \frac{t^{-\frac{\mu_j}{2}} - t^{\frac{\mu_j}{2}}}{q^{-\frac{\mu_j}{2}} - q^{\frac{\mu_j}{2}}}. \tag{4.14}$$

Then a similar computation leads to the following formula:

$$\begin{aligned} Z_{CS}^{SL}(\mathbb{O}; q, t; X) &= \prod_{m=1}^{\infty} \prod_i \frac{(1 - q^m t^{1/2} x_i)^m}{(1 - q^m t^{-1/2} x_i)^m} = \prod_{m=1}^{\infty} \prod_i \frac{(1 - q^{m+\Omega(t^{1/2} x_i; \vartheta)})^m}{(1 - q^{m+\Omega(t^{-1/2} x_i; \vartheta)})^m} \\ &= \prod_i \prod_{m=1}^{\infty} \frac{\mathcal{R}(s = (m + \Omega(t^{1/2} x_i; \vartheta))(1 - i\varrho(\vartheta)))}{\mathcal{R}(s = (m + \Omega(t^{-1/2} x_i; \vartheta))(1 - i\varrho(\vartheta)))}. \end{aligned} \tag{4.15}$$

4.4. Singularities and symmetries in infinite-product structure

In this section we discuss a basic symmetric property of infinite-product structure obtained from the LMOV partition function. First we recall some results on the Ruelle (Patterson–Selberg type) spectral functions. For details we refer the reader to [50,49] where spectral functions of hyperbolic three-geometry were considered in connection with three-dimensional Euclidean black holes, pure supergravity, and string amplitudes.

Let $\Gamma^{\gamma} \in G = SL(2, \mathbb{C})$ be the discrete group defined by

$$\begin{aligned} \Gamma^{\gamma} &= \{ \text{diag}(e^{2n\pi(\text{Im } \vartheta + i\text{Re } \vartheta)}, e^{-2n\pi(\text{Im } \vartheta + i\text{Re } \vartheta)}) : n \in \mathbb{Z} \} = \{ \gamma^n : n \in \mathbb{Z} \}, \\ \gamma &= \text{diag}(e^{2\pi(\text{Im } \vartheta + i\text{Re } \vartheta)}, e^{-2\pi(\text{Im } \vartheta + i\text{Re } \vartheta)}). \end{aligned} \tag{4.16}$$

One can construct a zeta function of Selberg-type for the group $\Gamma^{\gamma} \equiv \Gamma_{(\alpha, \beta)}^{\gamma}$ generated by a single hyperbolic element of the form $\gamma_{(\alpha, \beta)} = \text{diag}(e^z, e^{-z})$, where $z = \alpha + i\beta$ for $\alpha, \beta > 0$. Actually $\alpha = 2\pi \text{Im } \vartheta$ and $\beta = 2\pi \text{Re } \vartheta$. The Patterson–Selberg spectral function $Z_{\Gamma^{\gamma}}(s)$ and its logarithm for $\text{Re } s > 0$ can be attached to H^3/Γ^{γ} as follows:

$$Z_{\Gamma^{\gamma}}(s) := \prod_{k_1, k_2 \geq 0} [1 - (e^{i\beta})^{k_1} (e^{-i\beta})^{k_2} e^{-(k_1+k_2+s)\alpha}], \tag{4.17}$$

$$\log Z_{\Gamma^{\gamma}}(s) = -\frac{1}{4} \sum_{n=1}^{\infty} \frac{e^{-n\alpha(s-1)}}{n [\sinh^2(\frac{\alpha n}{2}) + \sin^2(\frac{\beta n}{2})]}. \tag{4.18}$$

The zeros of $Z_{\mathfrak{G}^{\gamma}}(s)$ are precisely the set of complex numbers

$$\zeta_{n, k_1, k_2} = -(k_1 + k_2) + i(k_1 - k_2)\beta/\alpha + 2\pi i n/\alpha, \tag{4.19}$$

with $n \in \mathbb{Z}$. The Ruelle functions $\mathcal{R}(s)$, (3.39)–(3.42), are an alternating product of factors, each of which is a Selberg-type zeta function; $\mathcal{R}(s)$ can be continued meromorphically to the entire complex plane \mathbb{C} . For more information about the analytic properties of this spectral function we refer the reader to the papers [51,49]. The magnitude of the zeta-function is bounded for both $\text{Re } s \geq 0$ and $\text{Re } s \leq 0$, and its growth can be estimated as

$$|Z_{\mathfrak{G}^{\gamma}}(s)| \leq \left(\prod_{k_1+k_2 \leq |s|} e^{|\ell|} \right) \left(\prod_{k_1+k_2 \geq |s|} (1 - e^{-(|s|-k_1-k_2)\ell}) \right) \leq C_1 e^{C_2 |s|^3} \tag{4.20}$$

for suitable constants ℓ, C_1, C_2 . The first product on the right-hand side of (4.20) gives the exponential growth, while the second product is bounded. The spectral function $Z_{\mathfrak{G}^{\gamma}}(s)$ is an entire function of order three and of finite type which can be written as a Hadamard product [49]

$$Z_{\mathfrak{G}^\gamma}(s) = e^{Q(s)} \prod_{\zeta \in \Sigma} \left(1 - \frac{s}{\zeta} \right) \exp \left(\frac{s}{\zeta} + \frac{s^2}{2\zeta^2} + \frac{s^3}{3\zeta^3} \right), \tag{4.21}$$

where Σ is the set of zeroes $\zeta := \zeta_{n,k_1,k_2}$ and $Q(s)$ is a polynomial of degree at most three. (The product formula for entire function (4.21) is also known as Weierstrass formula (1876).)

As a function of $\Omega(\dots; \vartheta)$ the partition function Z_{CS} has infinitely many poles of orders $m \in \mathbb{Z}_+$ (in fact the functions $\mathcal{R}(s)$ are the meromorphic functions; poles of $\mathcal{R}(s)$ correspond to zeros of $Z_{\mathfrak{G}^\gamma}(s)$).

Symmetry property. Let us discuss symmetry properties of the infinite-product formula given in Sect. 4.3. For this reason we can use functional equations for the spectral Ruelle functions (3.39)–(3.42):

$$\begin{aligned} &\mathcal{R}(s = (z + b)(1 - i\rho(\vartheta)) + i\sigma(\vartheta)) \cdot \mathcal{R}(s = -(1 + z + b)(1 - i\rho(\vartheta)) + i\sigma(\vartheta)) \\ &= q^{-zb - b(b+1)/2} \mathcal{R}(s = -z(1 - i\rho(\vartheta)) + i\sigma(\vartheta)) \cdot \mathcal{R}(s = (1 + z)(1 - i\rho(\vartheta)) + i\sigma(\vartheta)) \\ &= q^{-z(b-1) - b(b+1)/2} \mathcal{R}(s = (1 - z)(1 - i\rho(\vartheta)) + i\sigma(\vartheta)) \cdot \mathcal{R}(s = z(1 - i\rho(\vartheta)) + i\sigma(\vartheta)). \end{aligned} \tag{4.22}$$

The first key is Eq. (4.10) (the case of links is exactly similar). The simple case $b = 0$ in Eq. (4.22) leads to the symmetry $\vartheta \rightarrow -\vartheta$, i.e the symmetry $q \rightarrow q^{-1}$.

There is also the following symmetry about μ and Q $n_{\mu; g, -Q} = (-1)^{\ell(\mu)} n_{\mu; g, Q}$, which can be interpreted as the rank-level duality of the $SU(N)_k$ and $SU(k)_N$ Chern–Simons gauge theories [22]. Rank-level duality is essentially a symmetry of quantum group invariants relating a labeling color to its transpose [22]. It can be expressed using symmetry about μ , Q , and modularity properties of Ruelle functions as follows: $W_{A^t}(s^{-1}, -v) = W_A(s, v)$, where $s = q^{1/2}$, $v = t^{1/2}$. The stronger version is [22,68,69]: $W_{A^t}(s^{-1}, v) = (-1)^{|A|} W_A(s, v)$, $W_A(s, -v) = (-1)^{|A|} W_A(s, v)$.

5. Orthogonal group and colored Kauffman polynomials

Quantum Invariants of Links. Recall that for the unknot \bigcirc , $W_A(\bigcirc)$ is the quantum dimension $\dim_q(V_A)$ of the corresponding representation space V_A (see Eq. (4.13)).

– If $\mathfrak{g} = \mathfrak{sl}_N$ and $A^1 = A^2 = \dots = A^L = (1)$, the quantum group invariant of links equal to the HOMFLY polynomial at $t = q^N$ up to a universal factor $(t - t^{-1})/(q - q^{-1})$.

– If $\mathfrak{g} = \mathfrak{so}_{2N+1}$ and $A^1 = A^2 = \dots = A^L = (1)$, quantum group invariant of links equal to Kauffman polynomial at $t = q^{2N}$ up to a universal factor $1 + (t - t^{-1})/(q - q^{-1})$ and some t power of the linking numbers.

The quantum group invariant associated to $\mathfrak{g} = \mathfrak{sl}_N$ and $\mathfrak{g} = \mathfrak{so}_{2N+1}$ are called the colored HOMFLY and the colored Kauffman polynomials respectively. Actually the irreducible representation of the quantum groups of special linear and orthogonal cases can be labeled by the Young Tableau.

For each link \mathcal{L} , the type- A Chern–Simons partition function of \mathcal{L} is defined by (4.5). The original LMOV conjecture describes a very subtle structure of $Z_{CS}^{SL}(\mathcal{L}; q, t; \vec{X})$, which was proved in [18], based on the cabling technique and a careful degree analysis of the cut-join equation. As an application, the LMOV conjecture gives highly non-trivial relations between colored HOMFLY polynomials. The first such relation is the classical Lichorish–Millett theorem [70].

The study of the colored Kauffman polynomials are more difficult. For instance, the definition of the Chern–Simons partition function for the orthogonal quantum groups involves the representations of Brauer centralizer algebras, which admit a more complicated orthogonal relations

[71–73]. The orthogonal quantum group version of LMOV conjecture has been formulated in [20] by using the representation of the Brauer centralizer algebra.

Let $Z_{CS}^{SO}(\mathcal{L}, q, t)$ be the orthogonal Chern–Simons partition function defined by

$$Z_{CS}^{SO}(\mathcal{L}; q, t; \vec{X}) = \sum_{\vec{\mu} \in \mathcal{P}^L} \frac{p_{\vec{\mu}}(\vec{X})}{\delta_{\vec{\mu}}} \sum_{\vec{A} \in \widehat{Br}_{|\vec{\mu}|}} \chi_{\vec{A}}(\gamma_{\vec{\mu}}) W_{\vec{A}}^{SO}(\mathcal{L}; q, t), \tag{5.1}$$

where $\widehat{Br}_{|\vec{\mu}|}$ denotes the set $\widehat{Br}_{d^1} \times \dots \times \widehat{Br}_{d^L}$ (every element is a representation of the Brauer algebra), $\vec{\mu} = (\mu^1, \dots, \mu^L)$ for partitions μ^i of $d^i \in \mathbb{Z}$ and $\chi_{\vec{A}}(\gamma_{\vec{\mu}}) = \prod_{i=1}^L \chi_{A^i}(\gamma_{\mu^i})$ for the character χ_{A^i} of Br_{d^i} labeled by A^i . Expend the free energy

$$F_{CS}^{SO}(\mathcal{L}; q, t; \vec{X}) = \log Z_{CS}^{SO}(\mathcal{L}; q, t; \vec{X}) = \sum_{\vec{\mu} \neq \vec{0}} F_{\vec{\mu}}^{SO}(\mathcal{L}; q, t) p_{\vec{\mu}}(\vec{X}). \tag{5.2}$$

Then the reformulated invariants are defined by

$$g_{\vec{\mu}}(\mathcal{L}; q, t) = \sum_{k|\vec{\mu}} \frac{\mu(k)}{k} F_{\vec{\mu}/k}(\mathcal{L}; q^k, t^k). \tag{5.3}$$

The orthogonal LMOV conjecture assumes that [20]

$$\frac{\delta_{\vec{\mu}} [1]^2}{2[\vec{\mu}]} (g_{\vec{\mu}}(\mathcal{L}; q, t) - g_{\vec{\mu}}(\mathcal{L}; q, -t)) = \sum_{g=0}^{\infty} \sum_{\beta \in \mathbb{Z}} N_{\vec{\mu}, g, \beta} [1]_q^g t^{\beta}, \tag{5.4}$$

where $N_{\vec{\mu}, g, \beta}$ are the integer coefficients and vanish for sufficiently large g and $|\beta|$. This conjecture is a rigorous mathematical formulation of the LMOV type conjecture about the colored Kauffman polynomial; while in [74,75], their conjecture emphasizes on the relationship between colored HOMFLY and colored Kauffman. The integer coefficients $N_{\vec{\mu}, g, \beta}$ are closely related to the BPS numbers. To derive an infinite-product formula, we will state the result for a knot first, since the notations in the computation for a knot are relatively simpler.

The case of a knot. By the orthogonal LMOV conjecture, $N_{B; g, \beta}$ vanish for sufficiently large g and $|\beta|$, thus $n_{B; g, \beta}$ vanish for sufficiently large g and $|\beta|$. Finally the Chern–Simons partition function for orthogonal quantum group invariants can be expressed as the following infinite-product formula

$$\begin{aligned} \frac{Z_{CS}^{SO}(\mathcal{K}; q, t; X)}{Z_{CS}^{SO}(\mathcal{K}; q, -t; X)} &= \prod_{\mu \neq 0} \prod_{g=0}^{\infty} \prod_{\beta \in \mathbb{Z}} \prod_{m=1}^{\infty} \prod_{k=0}^g \left(\frac{(1 + q^{g-2k+2m} t^{\beta} X^{\mu})}{(1 - q^{g-2k+2m} t^{\beta} X^{\mu})} \right)^{\frac{mn_{\mu, g, \beta}}{\delta_{\mu}}} \\ &= \prod_{\mu \neq 0} \prod_{g=0}^{\infty} \prod_{\beta \in \mathbb{Z}} \prod_{k=0}^g \prod_{m=1}^{\infty} \left(\frac{\mathcal{R}(s = (2m + \Omega(q^{g-2k} t^{\beta} X^{\mu}; \vartheta))(1 - i_{\varrho}(\vartheta)) - 1 + i\sigma(\vartheta))}{\mathcal{R}(s = (2m + \Omega(q^{g-2k} t^{\beta} X^{\mu}; \vartheta))(1 - i_{\varrho}(\vartheta)) - 1)} \right)^{\frac{n_{\mu, g, \beta}}{\delta_{\mu}}}. \end{aligned} \tag{5.5}$$

The case of a link. We generalize the symmetric product to the case of link as follows:

$$\langle 1 \pm \psi \vec{X}^{\mu} \rangle = \prod_{i_{1,1}, \dots, i_{1, \ell(\mu^1)}, \dots, i_{L,1}, \dots, i_{L, \ell(\mu^L)}} \left(1 \pm \psi \prod_{\alpha=1}^L \left((x_{i_{\alpha,1}}^{\alpha})^{\mu_1^{\alpha}} \dots (x_{i_{\alpha, \ell(\mu^{\alpha})}}^{\alpha})^{\mu_{\ell(\mu^{\alpha})}^{\alpha}} \right) \right).$$

In a similar way the Chern–Simons partition function for orthogonal quantum group invariants can be expressed as the following infinite-product formula

$$\begin{aligned} \frac{Z_{CS}^{SO}(\mathcal{L}; q, t; \vec{X})}{Z_{CS}^{SO}(\mathcal{L}; q, -t; \vec{X})} &= \prod_{\vec{\mu} \neq \vec{0}} \prod_{g=0}^{\infty} \prod_{\beta \in \mathbb{Z}} \prod_{m=1}^{\infty} \prod_{k=0}^g \left(\frac{(1 + q^{g-2k+2m} t^{\beta} (x^1)^{\mu^1} \dots (x^L)^{\mu^L})}{(1 - q^{g-2k+2m} t^{\beta} (x^1)^{\mu^1} \dots (x^L)^{\mu^L})} \right)^{\frac{mn_{\vec{\mu}, g, \beta}}{3\vec{\mu}}} \\ &= \prod_{\vec{\mu} \neq \vec{0}} \prod_{g=0}^{\infty} \prod_{\beta \in \mathbb{Z}} \prod_{m=1}^{\infty} \prod_{k=0}^g \left(\frac{\mathcal{R}(s = (2m + \Omega(q^{g-2k} t^{\beta} \vec{X}^{\mu}); \vartheta))(1 - i\varrho(\vartheta)) - 1 + i\sigma(\vartheta))}{\mathcal{R}(s = (2m + \Omega(q^{g-2k} t^{\beta} \vec{X}^{\mu}); \vartheta))(1 - i\varrho(\vartheta)) - 1)} \right)^{\frac{n_{\vec{\mu}, g, \beta}}{3\vec{\mu}}} \end{aligned} \tag{5.6}$$

The case of the unknot.

$$\begin{aligned} \frac{Z_{CS}^{SO}(\bigcirc; q, t; X)}{Z_{CS}^{SO}(\bigcirc; q, -t; X)} &= \prod_{m=1}^{\infty} \prod_{i=-\infty}^{\infty} \left(\frac{(1 + q^{2m} t x_i)(1 - q^{2m} t^{-1} x_i)}{(1 - q^{2m} t z_i)(1 + q^{2m} t^{-1} x_i)} \right)^m \\ &= \prod_{i=-\infty}^{\infty} \prod_{m=1}^{\infty} \left(\frac{\mathcal{R}(s = (2m + \Omega(t x_i; \vartheta)) - 1 + i\sigma(\vartheta))}{\mathcal{R}(s = (2m + \Omega(t^{-1} x_i; \vartheta)) - 1 + i\sigma(\vartheta))} \right) \\ &\times \left(\frac{\mathcal{R}(s = (2m + \Omega(t^{-1} x_i; \vartheta)) - 1)}{\mathcal{R}(s = (2m + \Omega(t x_i; \vartheta)) - 1)} \right) \end{aligned} \tag{5.7}$$

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