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Noncommutative Quantum Field Theories and UV/IR Mixing

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Tiivistelmä — Referat — Abstract

Nykyinen ymmärryksemme alkeishiukkasista ja niiden välisistä perusvuorovaikutuksista perustuu hiukkasfysiikan Standardimalliin, joka nojautuu kvantisoituihin mittakenttäteorioihin. Toisaalta ymmärryksemme aika-avaruuden dynaamisesta luonteesta suurilla etäisyyksillä perustuu Einsteinin yleiseen suhteellisuusteoriaan. Näiden luonnon kahden toisiaan täydentävän kuvauksen, kvanttiteorian ja gravitaation, yhteen sulauttaminen on eräs nykyfysiikan perustutkimuksen suurimmista tavoitteista, jonka saavuttaminen auttaisi ymmärtämään paremmin aika-avaruuden rakennetta lyhyillä etäisyyksillä, ja siten valoittamaan tapahtumia yleisen suhteellisuusteorian singulariteeteissä, kuten mustissa aukoissa ja alkuräjähdyksessä, joissa nykyiset teoriamme eivät päde. Kvanttikenttäteorioiden muotoilu epäkommutatiivisessa aika-avaruudessa on yritys toteuttaa ajatus aika-avaruuden epälokaaliudesta lyhyillä etäisyyksillä, johon nykyinen ymmärryksemme edellä mainituista luonnon perusperiaatteista viittaa, ja siten löytää kokeellisia viitteitä aika-avaruuden piilevästä kvanttiluonteesta.

Näiden epäkommutatiivisten teorioiden muotoilu kohtaa useita ennen näkemättömiä ongelmia, jotka juontuvat niiden erikoisesta epälokaalista luonteesta. Vakavin näistä ongelmista on niin kutsuttu UV/IR sekoittuminen, joka vaikeuttaa kokeellisten ennustusten tekemistä aiheuttamalla epäkommutatiivisiin kvanttikenttäteorioihin uusia vaikeasti korjattavia äärettömyyksiä, joihin perinteiset kvanttikenttäteorioiden renormalisaatiometodit eivät sovellu. Tutkielmassani käyn läpi epäkommutatiivisen aika-avaruuden matemaattisen peruskäsitteistön, epäkommutatiivisten kvanttikenttäteorioiden muotoilun, ja esittelen UV/IR sekoittumisen teoreettisen perustan. Erityisesti osoitan, että myös niin kutsutun Seiberg-Witten kuvauksen avulla muotoiltu epäkommutatiivinen kvanttielektrodynamiikan teoria kärsii UV/IR sekoittumisesta. Tämä tulos on uusi, ja odottaa julkaisua. Lopuksi tarkastelen muutamia lupaavimpia ehdotuksia ongelman korjaamiseksi. Lopullinen ratkaisu säilyy haasteena tulevaisuuteen.

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noncommutative, spacetime, quantum, scalar, gauge, field theory, UV/IR mixing

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Tiivistelmä — Referat — Abstract

Our present-day understanding of fundamental constituents of matter and their interactions is based on the Standard Model of particle physics, which relies on quantum gauge field theories. On the other hand, the large scale dynamical behaviour of spacetime is understood via the general theory of relativity of Einstein. The merging of these two complementary aspects of nature, quantum and gravity, is one of the greatest goals of modern fundamental physics, the achievement of which would help us understand the short-distance structure of spacetime, thus shedding light on the events in the singular states of general relativity, such as black holes and the Big Bang, where our current models of nature break down. The formulation of quantum field theories in noncommutative spacetime is an attempt to realize the idea of nonlocality at short distances, which our present understanding of these different aspects of Nature suggests, and consequently to find testable hints of the underlying quantum behaviour of spacetime.

The formulation of noncommutative theories encounters various unprecedented problems, which derive from their peculiar inherent nonlocality. Arguably the most serious of these is the so-called UV/IR mixing, which makes the derivation of observable predictions especially hard by causing new tedious divergencies, to which our previous well-developed renormalization methods for quantum field theories do not apply. In the thesis I review the basic mathematical concepts of noncommutative spacetime, different formulations of quantum field theories in the context, and the theoretical understanding of UV/IR mixing. In particular, I put forward new results to be published, which show that also the theory of quantum electrodynamics in noncommutative spacetime defined via Seiberg-Witten map suffers from UV/IR mixing. Finally, I review some of the most promising ways to overcome the problem. The final solution remains a challenge for the future.

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Contents

†‡These sections contain original research conducted for the purposes of this thesis.

Chapter 1

Two Revolutions and More

1.1 Classical Theories of Space, Time and Matter

The beginning of 20th century brought about two fundamental revolutions in physics. In 1905 Albert Einstein published his solution, the special theory of relativity [1], to the problem of combining Newton's theory of classical mechanics with Maxwell's theory of electromagnetism, which contradicted each other gravely. Maxwell's theory predicted the speed of light in vacuum to be constant for any observer[1], while in Newton's theory there was no preferred speed what-so-ever, which allowed for the Newtonian notions of absolute space and time. It was precisely the requirement to maintain the invariability of the speed of light for all observers with constant relative speeds, which forced Einstein to abandon the previously unquestioned notions of space and time, and to make them dependent on the observer's state of motion. Also other important consequences followed from the formalism of special relativity, such as the fact that classical information cannot be transmitted faster than light, because this could violate causality, and the most famous equation of all time,

$$E = mc^2 \quad , \tag{1.1}$$

which relates the notions of energy E of a particle and its mass m to each other in a fundamental way via the constant vacuum speed of light $c \approx 2.9979 \times 10^8$m/s.

But this, of course, was not enough for Einstein. Having achieved the formulation of the special theory of relativity, he set out to replace the old theory of gravitation by Newton with a new theory, which in turn would be compatible with the special theory of relativity. The revolutionary result, the general theory of relativity, was

[1]From Maxwell's field equations for electromagnetic fields the vacuum propagation speed of electromagnetic radiation is found to be $c = (\epsilon_0\mu_0)^{-\frac{1}{2}}$, where ϵ_0 and μ_0 are the electric permittivity and the magnetic permeability of vacuum, respectively, which were understood to be constants of vacuum. Also, the famous Michelson-Morley experiment in 1887 [2] contributed to the belief that the speed of light was independent of the state of motion, and that there was no 'luminiferous aether' in which electromagnetic waves propagate.

published by Einstein in 1916 [3]. In it, gravity is not a mere force field in space, but a property, namely, the curvature of spacetime itself. In the general theory of relativity spacetime is described in differential geometrical terms, where the geometry, i.e., the metric of spacetime itself is a dynamical entity, having a close relationship with matter, which affects the geometry of spacetime causing it to curve, whereas the geometry of spacetime, on the other hand, affects the motion of matter. More accurately, the metric $g_{\mu\nu}(x)$ on the spacetime manifold $\mathcal{M}$ gives a notion of diffeomorphism invariant distance, the proper time τ measurable by observers, in the manifold by defining the invariant line element $\mathrm{d}\tau$ as[2]

$$\mathrm{d}\tau^2 = g_{\mu\nu}(x)\mathrm{d}x^\mu \mathrm{d}x^\nu \tag{1.2}$$

at any point $x \in \mathcal{M}$. Matter then, roughly speaking, follows the geodesics, the paths with the longest proper time, given by the metric. The metric, on the other hand, depends on the matter content of spacetime via ten second-order nonlinear partial differential equations in the components of the metric tensor, the Einstein field equations[3]

$$R_{\mu\nu} - \frac{1}{2}R g_{\mu\nu} + \Lambda g_{\mu\nu} = -\frac{8\pi G}{c^4} T_{\mu\nu} \quad , \tag{1.3}$$

where $T_{\mu\nu}(x)$ is the stress-energy tensor of matter and $R_{\mu\nu}(x)$ is the Ricci curvature tensor, while $R := R_{\mu\nu}g^{\mu\nu}$ is the Ricci scalar curvature, Λ the cosmological constant, and $G \approx 6.674 \times 10^{-11} \frac{\mathrm{m}^3}{\mathrm{kg\ s}^2}$ the gravitational constant [4].

One could say that the general theory of relativity gave the final blow for the Newtonian notions of absolute space and time by demanding the invariance of physics under the group of all differentiable transformations of spacetime coordinates. But, what is more, it also gave predictions for a host of new exciting physical phenomena, such as gravitational lensing, gravitational waves and black holes, which have almost all been confirmed thereafter [5].

The most interesting of these, at least for the purposes of this thesis, are black holes, whose existence was theoretically predicted by Karl Schwarzschild [6] only few months after the publication of Einstein's general theory of relativity. Since then, their physical existence has also been largely confirmed, for example, by observations of super-massive black holes at the centers of galaxies. A black hole is, simply put, a region of spacetime, where an extreme energy density of matter causes a gravitational field so strong that the escape velocity from the region exceeds the speed of light, thus preventing any classical information from get-

[2] Throughout this thesis we will use the sign convention $(+---)$ for the metric tensor unless mentioned otherwise, the Einstein summation convention for indices (the repeated indices are summed over), and also adopt the natural units convention $c = \hbar = G \equiv 1$, where it is not explicitly violated.

[3] Actually, the Einstein field equations also contain equations of motion for matter, since by the Bianchi identities and the Einstein equations, we have $\nabla_\mu T^{\mu\nu} \equiv 0$, where ∇_μ are the covariant derivatives. This identity gives the local conservation of the combined four-momentum of matter and gravitational field.

ting out of the region.[4] The boundary of the region, where the escape velocity equals the speed of light, is called the event horizon of the black hole, and its radius for a spherical nonrotating electrically uncharged black hole is given by the Schwarzschild radius [4]

$$r_s = \frac{2Gm}{c^2} \quad , \tag{1.4}$$

where m is the mass of the black hole. We will revive the discussion about black holes in Section (1.3), where we consider the implications of their existence for quantum theories of spacetime.

1.2 Quantum Theories of Matter

Another of the aforementioned major revolutions was the birth of quantum mechanics around 1920's, which was mainly motivated by experimental results, in contrast to the theories of relativity. In order to explain a myriad of experimental results, such as the spectrum of black-body radiation, the photoelectric effect, Compton scattering, the electron diffraction experiments and the stability of atoms, one is forced to assume that the electromagnetic waves have particle-like properties and, vice versa, that matter particles have wave-like properties, the explicit connection between the four-momentum $p_\mu = (E, \vec{p})$ of a particle and its wave four-vector $k_\mu = (\omega, \vec{k})$ being the Planck-deBroglie relation

$$p_\mu = \hbar k_\mu \quad , \tag{1.5}$$

where $2\pi\hbar = h \approx 6.626 \times 10^{-34}$Js is the Planck constant characterizing the magnitude of quantum effects. Not only that, but it was soon realized that quantum mechanics introduces to physics a fundamental probabilistic aspect: Whereas in classical[5] theories the time evolution of a physical system is deterministic, in quantum theory only probabilities and expectation values of physical observables can be predicted. Moreover, whereas classically any physical quantity can be measured at any instant of time with an arbitrary accuracy, quantum theory includes innate restrictions on the possible accuracy of simultaneous measurements of certain observables. These peculiarities follow straight-forwardly from the basic formalism of quantum theory, of which we will shortly review the most relevant aspects for our further considerations.[6]

In quantum theory an observable quantity A of a physical system is described by a corresponding Hermitean operator $\hat{A}$, which operates on elements of the

[4]There are some very subtle unresolved issues here, namely, the "black hole information paradox" [7] concerning the entropy of black holes, the possible violation of unitarity and/or causality of quantum mechanical time-evolution of black holes and the loss of information in the singularity. However, we will not concern ourselves with these issues in this thesis.

[5]Throughout this thesis we always refer by the term 'classical' to the aspects of deterministic pre-quantum theories of physics.

[6]For a reference on basic formalism of quantum theory, see [8, 9].

complex Hilbert space of states $\mathcal{S}$, and whose eigenvalues correspond to the possible outcomes of a measurement of A. Since the operator $\hat{A}$ is Hermitean, its eigenstates $|a\rangle \in \mathcal{S}$, for which $\hat{A}|a\rangle = a|a\rangle$, $a \in \mathbb{R}$, can be chosen so that they span an orthonormal basis in $\mathcal{S}$. Therefore any state $|\Psi\rangle$ of the system can be expressed as a complex linear combination[7]

$$|\Psi\rangle = \sum_a \psi_a |a\rangle \quad , \text{where} \quad \psi_a \in \mathbb{C} \quad . \tag{1.6}$$

When properly normalized, i.e., $\langle\Psi|\Psi\rangle = \sum_a |\psi_a|^2 = 1$, the squares $|\psi_a|^2$ of the coefficients can be interpreted as probabilities of obtaining the value a in a measurement. Accordingly, we get the expectation value of A for the state Ψ as

$$\langle A\rangle_\Psi := \langle\Psi|\hat{A}|\Psi\rangle = \sum_a a|\psi_a|^2 \quad . \tag{1.7}$$

But this is all we can ever say about the value of an observable according to quantum theory.

Furthermore, given two observables A and B, the corresponding operators $\hat{A}$ and $\hat{B}$ may not commute, i.e.,

$$[\hat{A}, \hat{B}] := \hat{A}\hat{B} - \hat{B}\hat{A} \neq 0 \quad . \tag{1.8}$$

Now, with some linear algebra one may derive the uncertainty relation [10]

$$(\Delta A)_\Psi (\Delta B)_\Psi \geq \frac{1}{2}|\langle\Psi|[\hat{A}, \hat{B}]|\Psi\rangle| \tag{1.9}$$

for any state $|\Psi\rangle \in \mathcal{S}$, where $(\Delta A)_\Psi := \sqrt{|\langle\Psi|\hat{A}^2 - \langle A\rangle^2|\Psi\rangle|}$ is the standard deviation of the values of A for the state $|\Psi\rangle$. This implies, as first recognized by Heisenberg [11], that there is a fundamental lower limit for our knowledge of simultaneous values of such noncommutative observables. In particular, in quantum mechanics the canonically conjugate observables, the coordinates $\hat{X}^i$ and the momenta $\hat{P}_i$ of a point particle, satisfy the canonical commutation relation $[\hat{X}^i, \hat{P}_j] = i\hbar\delta^i_j$, where δ^i_j is the Kronecker delta. Accordingly, we get by (1.9) the epitome of quantum mechanics, the Heisenberg uncertainty principle of coordinates and momenta

$$(\Delta x^i)(\Delta p_j) \geq \frac{\hbar}{2}\delta^i_j \quad . \tag{1.10}$$

A further revolution of quantum theory was the formulation of relativistic quantum field theories from the late 1920's onwards, now incorporating the lessons of the special theory of relativity into quantum mechanics, culminating in the Standard Model of elementary particle physics around 1970's, which has stand against experiments with an unprecedented accuracy of prediction ever since. In field theories the physical variables are the values of the fields and their derivatives at

[7] The summation here should be understood as an integral for the possible continuous part of the spectrum of $\hat{A}$.

each point of spacetime, which then are quantized in quantum field theories via canonical commutation relations, whereas the spacetime coordinates remain mere real-valued parameters labelling the field values. Otherwise, however, the basic principles of quantum theories introduced above remain intact. What the Standard Model itself consists of are so-called gauge field theories, which rely on the notion of local internal symmetry of the elementary particle fields with respect to some unitary group of complex linear transformations mixing the fields. In particular, in the Standard Model the gauge groups are $SU(3) \times SU(2) \times U(1)$, which describe the strong and weak nuclear forces and the electromagnetic force, respectively [12].

Despite its amazing experimental success, one could argue, however, that the Standard Model is not an entirely pleasing description, mathematically or aesthetically, of the workings of Nature, because of certain divergencies it harbours within its formalism [13]. In quantum electrodynamics (QED), the $U(1)$ sector of the Standard Model, for example, when the quantum mechanical probability amplitude for some process is calculated by expanding it in the powers of the coupling constant, as it is usually done à la Feynman in order to perform numerical calculations, the second and higher order correction terms of propagators foster divergencies, which stem from the virtual particle loops, and the high energy limit of corresponding momentum integrals in Feynman diagrams. To obtain finite results one must typically use some type of a regularization method to get rid of the divergent parts of the integrals, which however works wonders in the case of QED, and the Standard Model in general. Nevertheless, the divergencies at high energies clearly tell us that the validity of the Standard Model is of limited scope. On the other hand, this is hardly surprising, because the Standard Model does not account for gravitational effects, which are bound to become relevant at extremely high energies. Therefore, among other equally relevant reasons, arises the question of how to incorporate also the gravitational force into the quantum mechanical framework.

1.3 Quantum Theory of Space and Time?

Considering the discussion above, one may come to appreciate the fact that a large portion of the major advances in the physical understanding of Nature during the last century or so were due to reconciliations of two conflicting notions or principles of physical theories. Roughly speaking, Einstein's relativistic theories were the outcome of uniting Newtonian mechanics with the invariance of the speed of light in Maxwell's theory of electromagnetism, and the Standard Model was the outcome of bringing together quantum mechanics and special relativity. What is more, the unification has not been completed yet, and we remain to be in a situation of conflicting concepts also in today's understanding of physics. On one hand we have the theory of the very large, the general theory of relativity, which is a classical theory explaining beautifully the workings of the gravitational force at large distances, while on the other hand we have the quantum theory of the very small, the Standard Model of elementary particles, which describes with extreme accuracy the other three fundamental forces and the building blocks of matter. Yet we do not have, at the moment, a well-established quantum theory for the gravitational force, and constructing one has proven to be a demanding effort. It is widely expected, however, that the complete unification of the current contradictory paradigms will entail a further revolution comparable, at least, to the two previous ones.

Currently, the two most popular head-on attempts to address the problem of quantum gravity are String Theory and Loop Quantum Gravity, but there are many more, and no firm consensus on which of the various approaches is the correct one [15].[8] This is understandable, since due to the weakness of the gravitational force, the energy scale, the Planck mass[9] $m_p \approx 1.221 \times 10^{19}$GeV, at which the quantum gravitational effects are expected to become relevant, is largely beyond anything observable in the relatively peaceful patch of the Universe we happen to accommodate, and therefore there are virtually no experimental results to give guidance in choosing one theory over the others. Complementarily, it is extremely hard to produce observable predictions for any theory of quantum gravitation, which should, of course, reduce to the general theory of relativity at low energy scales, since the quantum gravitational corrections are presumably extremely small, of the order of inverse Planck mass or smaller.

However, there is another way to approach the problem, not straightforwardly head-on, but by considering effective properties the underlying theory of quantum gravity is likely to possess. Indeed, at least by a semi-classical reasoning, it is reasonable to expect that infinite localization of fields should not be allowed in a theory of quantum gravity, since infinite localization implies infinite energy density, which again leads to a formation of a black hole, as discussed in Section (1.1). Using the equations (1.1), (1.4) and (1.5) we may arrive to a rough estimate that a localized particle, whose average wave-length equals twice the diameter of the event horizon created by its mass, i.e., $2\pi\omega^{-1} = 4r_s$, has a mass of the order of the Planck mass, which again corresponds to an average localization of the order of Planck length $l_p \approx 1.616 \times 10^{-35}$m. Therefore, roughly speaking, any attempt to measure a feature of smaller size than l_p creates an event horizon around the interaction, which forbids the outflow of information, and accordingly the measurement is doomed to fail. A more definitive analysis of the situation

[8] There is also some disagreement on whether the general theory of relativity should be quantized at all [14], which again connects to the peculiar connection that gravity and entropy seem to have.

[9] Compare this, for example, with the energy scale 1.4×10^4GeV reached by the Large Hadron Collider at CERN, which is currently expected to be operational in 2010 [16].

can be performed based on quantum field theory and general relativity [17], which gives as a rough estimate the limits

$$\begin{aligned}(\Delta x_0)\,[(\Delta x_1)+(\Delta x_2)+(\Delta x_3)] &\gtrsim l_p^2 \\ (\Delta x_1)(\Delta x_2)+(\Delta x_2)(\Delta x_3)+(\Delta x_3)(\Delta x_1) &\gtrsim l_p^2 \end{aligned} \tag{1.11}$$

for the localization of fields, where x_0 is the temporal coordinate and x_i, $i = 1, 2, 3$, the spatial coordinates of spacetime. Such uncertainty relations of coordinates can be realized by postulating commutation relations of the canonical form

$$[\hat{x}^\mu, \hat{x}^\nu] = i\theta^{\mu\nu} \quad , \tag{1.12}$$

where $\hat{x}^\mu$, $\mu = 0, 1, 2, 3$, are noncommutative coordinates and $\theta^{\mu\nu}$ is a constant real-valued anti-symmetric matrix, whose elements are of the order of l_p^2. By the equation (1.9) we then get the slightly more general limits

$$(\Delta x^\mu)(\Delta x^\nu) \geq \frac{1}{2}|\theta^{\mu\nu}| \sim l_p^2 \tag{1.13}$$

for the localization of fields in noncommutative spacetime endowed with the canonical commutation relations among the coordinates.[10] Accordingly, in noncommutative spacetime the notion of a point effectively looses its meaning, since one cannot speak of features smaller than the limits given by the coordinate uncertainty relations (1.13). Indeed, geometry in noncommutative spaces is often described as "pointless", a phrase coined by John von Neumann.

One of the first people to study noncommutative geometry was Alain Connes [18, 19] in the beginning of the 1990's, and in 1995 Doplicher, Fredenhagen and Roberts published their detailed studies [17] on localization of quantum fields reviewed briefly above. Moreover, the idea caught further momentum in 1999, when it was shown by Seiberg and Witten [20] that a certain low energy limit of String Theory naturally induces noncommutativity of spacetime coordinates, offering confirmation for the relevance of noncommutativity for quantum gravity. A lot of effort has been since put by numerous researchers into constructing quantum field theories in noncommutative spacetime, commonly called *noncommutative quantum field theories*, in hope for observable predictions of quantum gravitational effects — hints of the underlying quantum spacetime. Also, a natural regularization of the divergencies of quantum field theories, mentioned in Section (1.2), was initially hoped for. A lot of progress has certainly been made in understanding various features of noncommutative spacetime and the theories in them. There are still, however, certain generic features, understood to be caused by the nonlocal character of noncommutativity, which give trouble for the formulation and predictivity of all noncommutative quantum field theories. Arguably, the most serious of these is the *UV/IR mixing*, which constitutes the main subject of this thesis.

[10] It should be mentioned that noncommutative spacetimes endowed with other kinds of commutation relations have also been studied actively recently [15].

However, to explain properly what it means and how it comes about, we will first need to review the basic mathematical formalism of noncommutative quantum field theories.

Chapter 2

Quantum Deformation of Spacetime

2.1 Noncommutative Coordinate Algebra

Motivated by the arguments in Section (1.3), let us now postulate the commutation relation

$$[\hat{x}^\mu, \hat{x}^\nu] = i\theta^{\mu\nu} \quad , \tag{2.1}$$

where $\theta^{\mu\nu}$ is a constant real-valued anti-symmetric matrix. Then the coordinates generate an algebra of fields $\hat{\phi}$, which are combinations of the noncommutative coordinates $\hat{x}^\mu$ obtained with multiplication and summation, modulo the commutation relation (2.1). The abstract noncommutative coordinates $\hat{x}^\mu$ then gain their physical meaning through their action on the noncommutative fields.

We may also define partial derivative operators $\hat{\partial}_\mu$ via the commutation relations

$$[\hat{\partial}_\mu, \hat{x}^\nu] = \delta_\mu^\nu \quad \text{and} \quad [\hat{\partial}_\mu, \hat{\partial}_\nu] = 0 \quad , \tag{2.2}$$

thus leaving the partial derivative algebra commutative [21]. Furthermore, a linear trace operation $\mathrm{Tr}_{\hat{x}}$ can be defined for noncommutative fields via the requirement

$$\mathrm{Tr}_{\hat{x}}[e^{ip\cdot\hat{x}}] = (2\pi)^D \delta^D(p) \quad , \tag{2.3}$$

where D is the dimensionality of spacetime, and p_μ are real coefficients to be identified as vector components in the dual momentum space in the following section.

2.2 Noncommutative Fourier Transformation

We may now define a Fourier-like transformation F as

$$\hat{\phi} \overset{F}{\mapsto} F\hat{\phi} = \mathrm{Tr}_{\hat{x}}[e^{-ip\cdot\hat{x}}\hat{\phi}] =: \tilde{\phi}(p) \tag{2.4}$$

for any noncommutative field $\hat{\phi}$, for which $\mathrm{Tr}_{\hat{x}}\left[[e^{-ip\cdot\hat{x}},\hat{\phi}]\right] \equiv 0$ and $\tilde{\phi}(p)$ is a Schwartz function, i.e., all the fields obtained from it by partial derivations decay rapidly at infinity. F maps the noncommutative field $\hat{\phi}$ to a field $\tilde{\phi}(p)$ in a commutative momentum space, and is one-to-one[1] by the existence of the inverse transformation F^{-1} given by

$$\tilde{\phi}(p) \overset{F^{-1}}{\mapsto} F^{-1}\tilde{\phi}(p) := \int \frac{\mathrm{d}^D p}{(2\pi)^D} e^{ip\cdot\hat{x}}\tilde{\phi}(p) = \hat{\phi} \quad . \tag{2.5}$$

Now, using the formula (2.5), we find

$$\mathrm{Tr}_{\hat{x}}[\hat{\phi}] = \tilde{\phi}(0) \quad , \tag{2.6}$$

which shows explicitly the uniqueness of the trace, which however is already implicit in the one-to-one correspondence of the noncommutative Fourier transformation (2.4).

By applying the transformations (2.4) and (2.5) we find that[2]

$$\hat{x}^\mu\hat{\phi} \overset{F}{\mapsto} F(\hat{x}^\mu\hat{\phi}) = (F\hat{x}^\mu F^{-1})(F\hat{\phi}) = \tilde{x}^\mu\tilde{\phi}(p) \quad , \tag{2.7}$$

where

$$\tilde{x}^\mu := i\frac{\partial}{\partial p_\mu} - \frac{1}{2}\theta^{\mu\nu}p_\nu \tag{2.8}$$

are the momentum space representations of the noncommutative coordinate operators. They satisfy the commutation relation

$$[\tilde{x}^\mu, \tilde{x}^\nu] = i\theta^{\mu\nu} \quad , \tag{2.9}$$

as they, of course, should for consistency. Accordingly, one obtains a linear representation of the noncommutative coordinate algebra in the commutative momentum space via the operators (2.8). However, to get a representation of the algebra of functions in noncommutative spacetime in terms of commutative space, typically the Weyl-Moyal correspondence is used, which is introduced in the next section.

2.3 Weyl-Moyal Correspondence

In the view of performing concrete calculations, there is a convenient way to associate a noncommutative field $\hat{\phi}$ to a field $\phi(x)$ in commutative spacetime via the *Weyl-Moyal correspondence* [23]. Namely, assuming that the field $\phi(x)$ satisfies the

[1]It is worth pointing out that the relation need not be bijective, in other words, not all Schwartz fields $\tilde{\phi}(p)$ need to correspond to a noncommutative field.

[2]The following result is due to personal research [22] conducted beside the thesis project.

Schwartz condition, i.e., any field obtained from it by partial derivations decays rapidly at infinity, it has a unique Fourier transform

$$\tilde{\phi}(p) = \int \mathrm{d}^D x \; e^{-ip\cdot x}\phi(x) \quad . \tag{2.10}$$

Now, we may associate to $\phi(x)$ the noncommutative field $\hat{\phi}$, whose noncommutative Fourier transform (2.4) coincides with (2.10). Accordingly, we find the associating relation to be

$$\hat{\mathcal{W}}[\phi] := \int \mathrm{d}^D x \; \hat{\Delta}(x)\phi(x) = \hat{\phi} \quad , \tag{2.11}$$

where

$$\hat{\Delta}(x) := \int \frac{\mathrm{d}^D p}{(2\pi)^D} e^{ip\cdot\hat{x}} e^{-ip\cdot x} \quad , \tag{2.12}$$

and $\hat{\mathcal{W}}[\phi]$ is called the *Weyl symbol* of $\phi(x)$. This association is one-to-one by the virtue of the noncommutative and the commutative Fourier transformations both being one-to-one. The inverse transformation is given by

$$\phi(x) = \mathrm{Tr}_{\hat{x}}[\hat{\Delta}(x)\hat{\phi}] \quad . \tag{2.13}$$

Moreover, by (2.3) we obtain

$$\mathrm{Tr}_{\hat{x}}[\hat{\phi}] = \int \mathrm{d}^D x \; \phi(x) \quad \text{and} \quad \mathrm{Tr}_{\hat{x}}[\hat{\Delta}(x)\hat{\Delta}(y)] = \delta^D(x-y) \quad . \tag{2.14}$$

One can also show that [21]

$$\left[\hat{\partial}_\mu, \hat{\mathcal{W}}[\phi]\right] = \hat{\mathcal{W}}[\partial_\mu \phi] \quad , \tag{2.15}$$

further justifying the choice of the defining commutation relations (2.2) of the partial derivative operators.

However, to obtain a full isomorphism between the commutative and noncommutative algebras of fields, we must also come up with a product, denoted commonly by '$\star$', in the commutative algebra, which satisfies the homomorphism relation $\hat{\mathcal{W}}[\phi]\hat{\mathcal{W}}[\psi] = \hat{\mathcal{W}}[\phi \star \psi]$, where $\phi(x)$ and $\psi(x)$ are two fields in the commutative spacetime. This requirement can be shown to be satisfied by the *Groenewold-Moyal product* [24, 25], referred to from now on as the '$\star$-product', which is given by

$$(\phi \star \psi)(x) := \left[e^{\frac{i}{2}\theta^{\mu\nu}\frac{\partial}{\partial\eta}\frac{\partial}{\partial\zeta}} \phi(x+\eta)\psi(x+\zeta) \right]_{\eta=\zeta=0} \equiv \phi(x) e^{\frac{i}{2}\overleftarrow{\partial_\mu}\theta^{\mu\nu}\overrightarrow{\partial_\nu}} \psi(x) \quad . \tag{2.16}$$

Accordingly, we get an equivalent representation of the noncommutative algebra of fields by considering fields in commutative spacetime, but replacing the ordinary point-wise products with nonlocal $\star$-products. This is readily realized in the Moyal bracket of the commutative spacetime coordinates

$$[x^\mu \stackrel{\star}{,} x^\nu] := x^\mu \star x^\nu - x^\nu \star x^\mu = i\theta^{\mu\nu} \quad . \tag{2.17}$$

If $\theta^{\mu\nu}$ is invertible, we may also express (2.16) as [26]

$$(\phi \star \psi)(x) = \iint \frac{\mathrm{d}^D\xi\ \mathrm{d}^D\zeta}{|\det(\theta)|} \phi(\xi)\psi(\zeta) e^{2i(x-\xi)^\mu \theta^{-1}_{\mu\nu}(x-\zeta)^\nu} \quad , \tag{2.18}$$

which highlights clearly the nonlocality of the $\star$-product.

Important properties of the $\star$-product are that it is associative, i.e., $(\phi \star \psi) \star \chi = \phi \star (\psi \star \chi)$, and that

$$\int \mathrm{d}^D x\ (\phi \star \psi)(x) = \int \mathrm{d}^D x\ \phi(x)\psi(x) \quad , \tag{2.19}$$

when at least one of ϕ and ψ is a Schwartz function [21]. Also a curious property, for which we will find use later on, is that the plane waves $e^{ik\cdot x}$ act effectively as translation operators in the Moyal space, since we have

$$e^{ik\cdot x} \star f(x) \star e^{-ik\cdot x} = f(x + \tilde{k}) \quad , \text{where} \quad \tilde{k}^\mu := \theta^{\mu\nu} k_\nu \quad . \tag{2.20}$$

2.4 Other Noncommutative Transformations†

The new invariant quantities $\theta^{\mu\nu}$ introduced via the coordinate commutation relation (2.1) give rise to interesting new transformations within the Schwartz space of fields in momentum space, and similarly for the commutative coordinate space obtained through the Weyl-Moyal correspondence, given that $\theta^{\mu\nu}$ is invertible[3]. This is highly reminiscent of the way the (reduced) Planck constant $\hbar$ gives rise to the Fourier transformation between fields in coordinate and momentum spaces.

In particular, the fields

$$e_P(p, p') := \sqrt{\frac{|\det(\theta)|}{(4\pi)^D}} e^{\frac{i}{2} p \wedge p'} \quad , \text{where} \quad p \wedge p' := p_\mu \theta^{\mu\nu} p'_\nu \quad , \tag{2.21}$$

form an orthonormal basis in the space of fields in momentum space[4], because they satisfy the relation

$$\int \mathrm{d}^D p\ e^*_P(p, p') e_P(p, p'') = \delta^D(p' - p'') \quad , \tag{2.22}$$

where the superscript '$*$' denotes complex conjugation. Accordingly, one obtains the one-to-one transformation T_P from the Schwartz space of fields in momentum space to itself as

$$\tilde{\phi}(p) \overset{T_P}{\mapsto} (T_P\tilde{\phi})(p) = \int \mathrm{d}^D p'\ e_P(p, p')\tilde{\phi}(p') \quad , \tag{2.23}$$

† The results in this section are due to personal research [22] conducted beside the thesis project.

[3] For $\theta^{\mu\nu}$ to be invertible, we must require the dimensionality D of spacetime to be even, which is, of course, plausible.

[4] Note that this is possible only because $\theta^{\mu\nu}$ has dimensions of length squared, which is required to render the exponent dimensionless. In analogy, one needs $\hbar$ to render the exponent dimensionless in the Fourier case, where we have the product $x \cdot p$ in the exponent.

and the inverse transformation T_P^{-1} as

$$\tilde{\phi}(p) \stackrel{T_P^{-1}}{\mapsto} (T_P^{-1}\tilde{\phi})(p) = \int d^D p' \; e_P^*(p,p')\tilde{\phi}(p') \quad . \tag{2.24}$$

Similarly, for the space of fields in the Weyl-Moyal coordinate space one finds the basis fields

$$e_X(x,x') := \frac{e^{2ix\vee x'}}{\sqrt{\pi^D |\det(\theta)|}} \quad \text{, where} \quad x \vee x' = x^\alpha \theta^{-1}_{\alpha\beta} x'^\beta \quad , \tag{2.25}$$

which, analogously to (2.22), satisfy

$$\int d^D x \; e_X^*(x,x') e_X(x,x'') = \delta^D(x'-x'') \quad , \tag{2.26}$$

and, accordingly, give the one-to-one transformation and inverse transformation

$$\begin{aligned} \phi(x) \;&\stackrel{T_X}{\mapsto}\; (T_X\phi)(x) = \int d^D x' \; e_X(x,x')\phi(x') \\ \phi(x) \;&\stackrel{T_X^{-1}}{\mapsto}\; (T_X^{-1}\phi)(x) = \int d^D x' \; e_X^*(x,x')\phi(x') \quad , \end{aligned} \tag{2.27}$$

respectively.

These new transformations are clearly closely related to the noncommutative structure of spacetime induced by the $\star$-product. In fact, one can even rewrite the formula (2.18) for the $\star$-product as

$$(\phi \star \psi)(x) = \sqrt{\frac{\pi^D}{|\det(\theta)|}} \iint d^D\xi \; d^D\zeta \; \phi(\xi) e_X(x-\xi, x-\zeta)\psi(\zeta) \quad . \tag{2.28}$$

However, the interpretation of the transformations is not well understood at the moment. They may reflect the over-completeness of the Dirac delta function basis, i.e., the spacetime points labelling the field values, in analogy with the commutative Fourier transform reflecting the over-completeness of phase space points. Indeed, we have not one but two one-to-one transformations from the fields in commutative momentum space to the fields in noncommutative coordinate space:

$$\hat{\phi} = F^{-1}\tilde{\phi}_1(p) = (T_P F)^{-1}\tilde{\phi}_2(p) \quad , \tag{2.29}$$

where $\tilde{\phi}_2(p) = T_P\tilde{\phi}_1(p)$, which may imply that we should understand the both fields $\tilde{\phi}_1(p)$ and $\tilde{\phi}_2(p)$ as corresponding to the same physical state. The over-completeness would be understandable, since the noncommutative spacetime does not have well-defined points due to the coordinate commutation relation (2.1), which imposes the uncertainty principle (1.13). Yet the dual representations in commutative spaces with no restrictions on fields are expected to be equivalent to the noncommutative one, even though they possess fields representing well-defined points of spacetime. However, these heuristic arguments require further investigation to gain justification and credibility.

2.5 Twisted Poincaré Symmetry

One of the most important questions related to the physical plausibility of the commutation relation (2.1) is, undoubtedly, to which extent does it violate the usual Poincaré symmetry of commutative spacetime, which is a crucial ingredient in all of modern high energy physics. In particular, the elementary particles are understood to correspond to the fundamental representations of the Poincaré algebra, and are classified according to the Poincaré invariant quantities, namely, mass and spin. One might expect that introducing the constant matrix $\theta^{\mu\nu}$, which is required to be invariant in all frames of reference, would certainly violate Lorentz symmetry, since it does not transform tensorially under Lorentz transformations. However, an elegant answer to this question exists in terms of the quantum group theory [28, 29].[5]

In the commutative case the relativistic symmetries of spacetime, i.e., translations, rotations and Lorentz boosts, are generated by the elements of the Poincaré algebra $\mathcal{P}$, whose linear representation in Minkowski space is given by the differential operators

$$P_\mu = -i\frac{\partial}{\partial x^\mu} \quad \text{and} \quad M_{\mu\nu} = x_\mu P_\nu - x_\nu P_\mu \quad , \tag{2.30}$$

which satisfy the commutation relations

$$\begin{aligned} i[P_\mu, P_\nu] &= 0 \\ i[P_\mu, M_{\rho\sigma}] &= \eta_{\mu\rho}P_\sigma - \eta_{\mu\sigma}P_\rho \\ i[M_{\mu\nu}, M_{\rho\sigma}] &= \eta_{\mu\rho}M_{\sigma\nu} + \eta_{\mu\sigma}M_{\nu\rho} + \eta_{\nu\rho}M_{\mu\sigma} + \eta_{\nu\sigma}M_{\rho\mu} \quad . \end{aligned} \tag{2.31}$$

The universal enveloping algebra $\mathcal{U}(\mathcal{P})$ of the Poincaré algebra $\mathcal{P}$ is then generated by all the symmetrized products of the elements of $\mathcal{P}$. When operating on fields, the elements of the universal enveloping $\mathcal{U}(\mathcal{P})$ satisfy the Leibniz rule of derivation, which in the quantum group terms is encoded in the coproduct $\Delta : \mathcal{U}(\mathcal{P}) \to \mathcal{U}(\mathcal{P}) \otimes \mathcal{U}(\mathcal{P})$. In particular, it tells how the elements of $\mathcal{U}(\mathcal{P})$ operate on products of fields in the following manner. Let $\mathcal{F}$ be the algebra of commutative fields, and $m : \mathcal{F} \otimes \mathcal{F} \to \mathcal{F}$ be the multiplication map giving the point-wise product of fields. Then we have

$$Y(m(\phi \otimes \psi)) = m \circ \Delta(Y)(\phi \otimes \psi) \qquad \forall\, Y \in \mathcal{U}(\mathcal{P}) \quad \text{and} \quad \phi, \psi \in \mathcal{F} \quad , \tag{2.32}$$

where Δ is the Leibniz rule

$$\Delta(Y) = Y \otimes 1 + 1 \otimes Y \quad . \tag{2.33}$$

But now, for the noncommutative spacetime, we have altered the multiplication map m of fields as in (2.16) by *twisting* it by an Abelian twist element

$$\mathcal{T} = \exp\left[\frac{i}{2}\theta^{\mu\nu} P_\mu \otimes P_\nu\right] \tag{2.34}$$

[5] See, for example, [27] for a thorough presentation of quantum groups.

as $m \mapsto m \circ \mathcal{T}^{-1} =: m_t$, and therefore to preserve the generators of symmetries, and accordingly the representations, as those of the Poincaré algebra, we must also alter the coproduct of the quantum group correspondingly, so that the equation (2.32) retains its validity. This is accomplished by twisting the coproduct [28] as well by the Abelian twist element, so that the coproduct becomes $\Delta_t := \mathcal{T} \circ \Delta \circ \mathcal{T}^{-1}$. Indeed, it is easy to see that this cures the problem:

$$\begin{aligned} m_t \circ \Delta_t(Y)(\phi \otimes \psi) &= m \circ \mathcal{T}^{-1} \circ \mathcal{T} \circ \Delta(Y) \circ \mathcal{T}^{-1}(\phi \otimes \psi) \\ &= m \circ \Delta(Y) \circ \mathcal{T}^{-1}(\phi \otimes \psi) \\ &= Y(m \circ \mathcal{T}^{-1}(\phi \otimes \psi)) \\ &= Y(m_t(\phi \otimes \psi)) \quad . \end{aligned} \tag{2.35}$$

The resulting algebra is called the *twisted* Poincaré algebra. Explicitly, the twisted coproducts become

$$\begin{aligned} \Delta_t(P_\mu) &= P_\mu \otimes 1 + 1 \otimes P_\mu \\ \Delta_t(M_{\mu\nu}) &= M_{\mu\nu} \otimes 1 + 1 \otimes M_{\mu\nu} \\ &\quad -\frac{1}{2}\theta^{\alpha\beta}\left[(\eta_{\alpha\mu}P_\nu - \eta_{\alpha\nu}P_\mu) \otimes P_\beta + P_\alpha \otimes (\eta_{\beta\mu}P_\nu - \eta_{\beta\nu}P_\mu)\right] \quad . \end{aligned} \tag{2.36}$$

Thus, by the virtue of the twist of the coproduct, we are able to preserve the representations of the Poincaré algebra under the complementary twist of the multiplication map. Accordingly, the elementary particle content of Poincaré invariant theories survives the quantum deformation (2.1) of spacetime, which is, of course, a highly important piece of knowledge, and also our stepping stone into the following chapters, where we concentrate on theories of elementary particles in noncommutative spacetime.

Chapter 3

Noncommutative Scalar Field Theories

3.1 Formulation and Some Properties

As explained in Section (2.3), we obtain an isomorphic representation of the noncommutative algebra of fields in terms of fields in commutative spacetime by replacing ordinary point-wise products with $\star$-products (2.16). Therefore the most straight-forward way, and the one typically used, to formulate a (quantum) field theory in noncommutative spacetime is simply to modify the Lagrangian of the theory according to the above prescription [26, 30]. It has been shown, however, that this naïve approach to noncommutative scalar field theories retains unitarity and causality in Minkowski spacetime only if $\theta^{0i} = 0$ or $\theta^{\mu\rho}\theta_{\rho}{}^{\nu} = 0$ [31, 32]. The CPT invariance remains generally valid [33, 34]. The preference for the case of mere space-space noncommutativity (i.e., $\theta^{0i} = 0$) can be traced back to the fact that only this case is obtainable as a low energy limit of String Theory [20], which is a well-behaved model itself.

Because of its enlightening nature, let us begin by considering the simplest example: the $\lambda\phi^{4\star}$ scalar field theory in noncommutative Euclidean $D = 4$ spacetime [26, 30]. In this case the action of the noncommutative theory acquires the form

$$\mathcal{S}_{\lambda\phi^{4\star}} = \int \mathrm{d}^4 x \left[\frac{1}{2}\phi(x)(-\partial^2 + m)\phi(x) + \frac{\lambda}{4!}(\phi \star \phi \star \phi \star \phi)(x) \right] \quad . \tag{3.1}$$

Here the first term, the free part of the action, remains the same as that of the corresponding commutative theory because of the property (2.19) of the $\star$-product. This implies that the free propagator of the quantum field theory also remains unchanged, and only the interaction part, i.e., the vertex function of Feynman rules, gains extra contributions from the noncommutativity. In particular, the

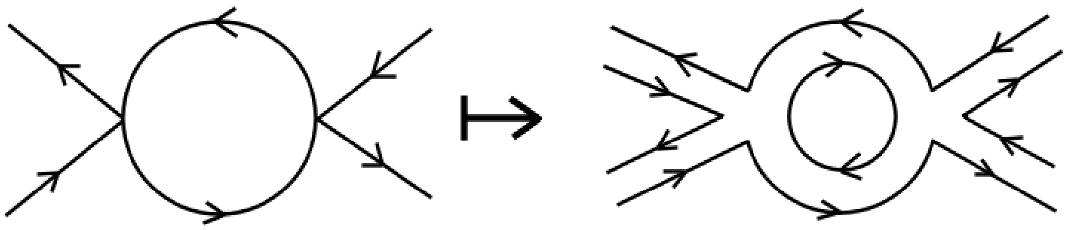

Figure 3.1: An example of the shift to a double-line diagram (the arrows represent momentum flow)

commutative vertex function $i\lambda$ gains an extra phase factor of the form

$$V(k_i) = \sum_{i<j} e^{-\frac{i}{2}k_i \wedge k_j} \quad , \text{ where } \quad p \wedge q = p_\mu \theta^{\mu\nu} q_\nu \quad , \tag{3.2}$$

and k_i are the momenta flowing into the vertex. It is important to note that $V(k_i)$ is not invariant under arbitrary permutations of the momenta but only under cyclic permutations, so one has to keep track of the order in which propagators are connected to the vertices of Feynman diagrams.

Let us call the diagrams, which can be drawn on a plane without intersecting propagators, 'planar'. For planar diagrams there is a neat way to keep track of the momenta [30]. Namely, we may replace every line in a planar Feynman diagram by a double line so that we end up with a diagram, which has only non-intersecting solid lines and loops. (See Fig. (3.1) for a graphical example.) Now, due to the momentum conservation at the vertices and the planarity, we may label the lines of the double line notation by 'momenta' l_i, which correspond to the original momenta via the relation $k_i = l_{i_1} - l_{i_2}$. Accordingly, when k_i, $i = 1, \ldots, 4$, are the incoming momenta for a vertex in cyclic order, the phase factor (3.2) becomes

$$e^{-\frac{i}{2}\sum_{i=1}^{4} l_{i_j} \wedge l_{i_{j+1}}} \quad , \tag{3.3}$$

where each of the expressions $l_{i_j} \wedge l_{i_{j+1}}$ corresponds to one of the incoming propagators. The over-all phase factor for a diagram is then the product of the phase factors corresponding to each of the vertices of the diagram. Therefore, by the expression (3.3), we find that for a planar diagram the factors corresponding to the internal propagators cancel out, since they contribute opposite terms $\pm l_{i_j} \wedge l_{i_{j+1}}$ to the exponent of the over-all phase factor. Consequently, we are left with the over-all phase factor

$$V(p_i) = e^{-\frac{1}{2}\sum_{i<j} p_i \wedge p_j} \tag{3.4}$$

for a planar diagram, where p_i are the momenta associated to the external lines of the original single-line diagram in the cyclic order [30]. It immediately follows that the UV-divergencies of the commutative quantum scalar field theory, which arise from the integrals over the internal momenta of Feynman diagrams, are also

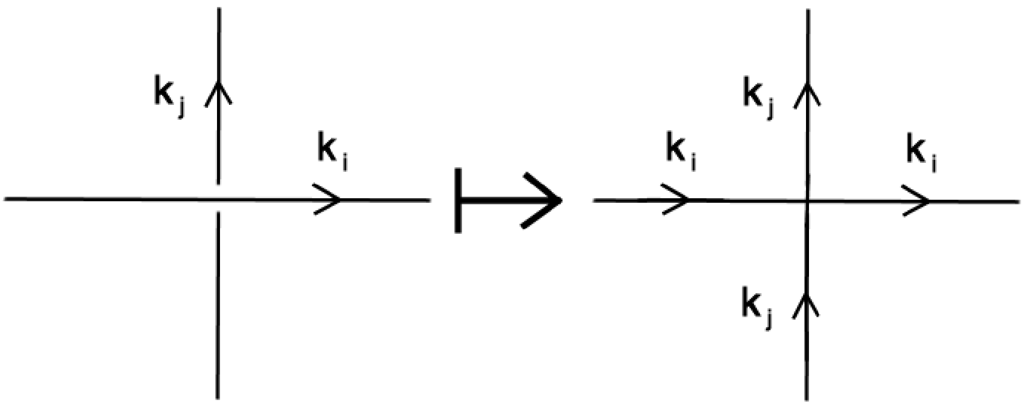

Figure 3.2: Replacing a crossing with a vertex

present in the planar diagrams of the noncommutative theory, and therefore, due to this example, the noncommutativity of spacetime does not seem to help to naturally regularize the divergencies of quantum field theories.

For nonplanar diagrams, however, this result does not hold, but we do receive contributions from the internal momenta. By the following argument we easily find the form of the contributions: Suppose we have a crossing of propagators with momenta k_i and k_j in the diagram in question. Now, we may render the neighbourhood of the crossing planar by replacing the crossing by a vertex. (See Fig. (3.2) for a graphical illustration.) Such a vertex would contribute a phase factor

$$e^{-ik_j \wedge k_i} \tag{3.5}$$

to the over-all phase factor, which would eventually be cancelled by the phase factors due to the other vertices. Therefore, in the *absence* of a vertex, we must have the opposite term

$$e^{ik_j \wedge k_i} \tag{3.6}$$

for each of the crossings of the diagram in addition to the planar phase factor $V(p_i)$ of the corresponding planar diagram. Consequently, we get an over-all phase factor of the form [26]

$$V(p_i)e^{-\frac{i}{2}\sum_{i,j} C_{ij} k_i \wedge k_j} \tag{3.7}$$

where $V(p_i)$ is the planar phase factor (3.4), and C_{ij} is an intersection matrix counting the crossings of propagators so that[1]

$$C_{ij} = \begin{cases} 1 & \text{if } k_i \text{ crosses over } k_j \text{ with } k_j \text{ pointing to the left of } k_i. \\ 0 & \text{if } k_i \text{ and } k_j \text{ does not cross each other.} \\ -1 & \text{if } k_i \text{ crosses over } k_j \text{ with } k_j \text{ pointing to the right of } k_i. \end{cases} \tag{3.8}$$

[1]It is important to note that the intersection matrix (3.8) depends explicitly on the way the diagram is drawn. However, it can be shown that the conservation of momentum at the vertices renders all the different phase factors resulting from such arbitrary choices of drawing the diagram equivalent [26].

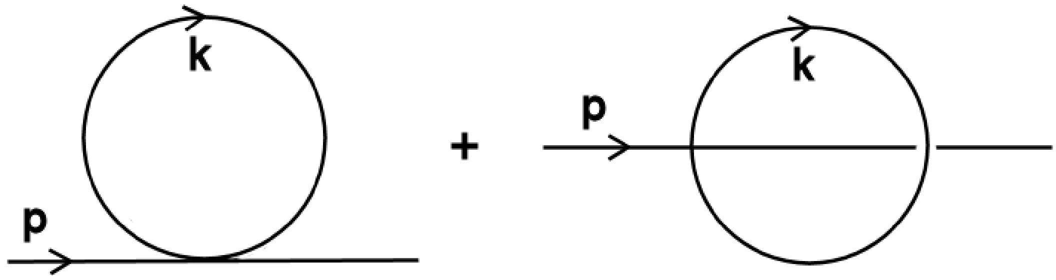

Figure 3.3: Planar and nonplanar loop diagrams, respectively

Accordingly, the internal momenta do appear in the phase factors of nonplanar Feynman diagrams, and therefore for them the UV-divergent momentum loop integrals are, indeed, modified in noncommutative scalar field theory.[2] However, the resulting modification is of a highly nontrivial form, giving rise to a peculiar mixing of high and low energy sectors of the theory described in the next section.

3.2 One-Loop Corrections and UV/IR Mixing

Let us now consider the one-particle-irreducible (1PI) two point function Γ of the noncommutative $\lambda\phi^{4\star}$ scalar field theory. At the zeroth order in the coupling constant λ, the two point function is simply the inverse propagator $\Gamma^{(0)} = p^2 + m^2$ as in the commutative theory, since the free action is not altered. The first order corrections arise from the loop diagrams depicted in Fig. (3.3) [26]. According to the considerations of the previous section, we must now make a distinction between planar and nonplanar diagrams, since they receive different contributions from the noncommutativity, and therefore the corrections are of the form $\Gamma^{(n)} = \Gamma^{(n)}_{\mathrm{p}} + \Gamma^{(n)}_{\mathrm{np}}$, where the first term is the planar contribution and the second term the nonplanar one. In fact, since we have only one external momentum in the case of the two point function, the planar phase factor (3.4) equals unity, and thus the planar diagrams give exactly the same correction as in the commutative case, apart from combinatorial factors. The first order nonplanar correction $\Gamma^{(1)}_{\mathrm{np}}$, however, acquires a phase factor $e^{ik\wedge p}$, where k is the internal momentum in the loop and p the external momentum of the diagram. Accordingly, the first order planar and nonplanar correction terms become

$$\begin{aligned}\Gamma^{(1)}_{\mathrm{p}} &= \frac{\lambda}{3(2\pi)^4}\int\frac{\mathrm{d}^4k}{k^2+m^2}\\ \Gamma^{(1)}_{\mathrm{np}} &= \frac{\lambda}{6(2\pi)^4}\int\frac{\mathrm{d}^4k}{k^2+m^2}e^{ik\wedge p}\quad ,\end{aligned}\tag{3.9}$$

[2]The above arguments are readily generalizable to a scalar field theory with a potential of any order in the scalar field, not just four, and also to Minkowski space. In general, we get exactly the same form for the over-all phase factor (3.7) [30].

respectively. Now, regularizing the momentum integrals at the energy scale Λ, we find [26]

$$\begin{aligned}\Gamma^{(1)}_{\mathrm{p}} &= \frac{\lambda}{48\pi^2}\left[\Lambda^2 - m^2\ln\left(\frac{\Lambda^2}{m^2}\right) + \dots\right]\\ \Gamma^{(1)}_{\mathrm{np}} &= \frac{\lambda}{96\pi^2}\left[\Lambda^2_{\mathrm{eff}} - m^2\ln\left(\frac{\Lambda^2_{\mathrm{eff}}}{m^2}\right) + \dots\right] \quad ,\end{aligned} \tag{3.10}$$

where

$$\Lambda_{\mathrm{eff}} = \frac{1}{1/\Lambda^2 + p\circ p} \quad , \quad p\circ p := -p^\mu\theta^2_{\mu\nu}p^\nu \geq 0\ \forall p \quad , \tag{3.11}$$

and the ellipses "…" correspond to lower order terms in Λ and Λ_{eff}, respectively. As already noted, the planar correction is proportional to the usual expression one gets in the commutative theory, which diverges at the high energy limit $\Lambda \to \infty$. The nonplanar correction, however, has an additional regularization given by the term $p\circ p$, which renders the limit $\Lambda \to \infty$ finite, when $\tilde{p}^\mu := \theta^{\mu\nu}p_\nu \neq 0$. In particular, when we take the UV-limit $\Lambda \to \infty$ of the internal momentum, we find

$$\Gamma^{(1)}_{\mathrm{np}} \stackrel{\Lambda\to\infty}{\longrightarrow} \frac{\lambda}{96\pi^2}\left[\left(\frac{1}{p\circ p}\right)^2 - m^2\ln\left(\frac{1}{m^2(p\circ p)^2}\right) + \dots\right] \quad . \tag{3.12}$$

But now, this expression diverges at the low energy limit $p \to 0$ of the external momentum, or more generally[3] when $\tilde{p}^\mu = \theta^{\mu\nu}p_\nu \to 0$. This is not too surprising, since the nonplanar phase factor $e^{ik\wedge p}$ in (3.9), which dampens the singularity of the momentum integral, approaches unity as $\tilde{p}^\mu \to 0$, and accordingly the dampening is lost.[4] This exotic mixing of the high energy (UV) and low energy (IR) scales in noncommutative theories, which does not have a counter-part in commutative theories, is called *UV/IR mixing*.

The UV/IR mixing causes severe problems for the ordinary renormalization procedure of quantum field theories [26, 35]. Typically, in commutative theories, one is able to renormalize the UV-divergencies by introducing a dependence on the high energy cut-off scale Λ into the free parameters of the theory. With suitable choices of renormalized mass $m(\Lambda)$, coupling constant $\lambda(\Lambda)$ and over-all scaling $Z(\Lambda)$ of the field one obtains the renormalized action

$$\mathcal{S}_{\text{eff-}\lambda\phi^4}(\Lambda) = \int \mathrm{d}^4x \left\{\frac{Z(\Lambda)}{2}\phi(x)\left[-\partial^2 + m^2(\Lambda)\right]\phi(x) + \frac{\lambda(\Lambda)Z^2(\Lambda)}{4!}\phi^4(x)\right\} \quad , \tag{3.13}$$

which gives finite correlation functions at the UV-limit $\Lambda \to \infty$. Moreover, in the case of two point functions, for example, the results should converge uniformly to their limiting values for all values of the external momentum p above, because the renormalization is not allowed to depend on p. Now, while in the noncommutative

[3] If $\theta^{\mu\nu}$ is invertible, these limits are equivalent.

[4] It is also important for our later considerations to note that the dampening by the phase factor is lost, if one performs a finite-order approximation in powers of $\theta^{\mu\nu}$, since this destroys the oscillations at infinity.

case the limits may exist, the two point function Γ does not converge uniformly for all values of the external momentum, since due to the UV/IR mixing the limits $\Lambda \to 0$ and $\tilde{p} \to 0$ do not commute. This causes unprecedented problems for the renormalization of noncommutative quantum scalar field theories.

3.3 Origin of UV/IR Mixing

The appearance of UV/IR mixing was given an enlightening explanation by Minwalla et al. in [26] in terms of the nonlocality of the $\star$-product, which we will review in this section. The peculiar nonlocal character of the $\star$-product is, perhaps, most obvious in the equality [36]

$$(\delta^D \star \delta^D)(x) = \frac{1}{|\det(\theta)|} \quad , \tag{3.14}$$

that is, the $\star$-product of two infinitely narrow fields is spread constantly over the whole spacetime. However, to better quantify our understanding of the nonlocality, let us consider a 2-dimensional Euclidean noncommutative plane with the coordinate commutation relation $[x_i, x_j] = i\theta\epsilon_{ij}$, where ϵ_{ij} is the completely antisymmetric matrix. Now, the equation (2.18) for the $\star$-product reads

$$\begin{aligned}(\phi \star \psi)(\vec{x}) &= \iint \frac{\mathrm{d}^2\xi\ \mathrm{d}^2\zeta}{|\det(\theta)|} \phi(\vec{\xi})\psi(\vec{\zeta}) e^{\frac{2i}{\theta}(x-\xi)^i\epsilon_{ij}(x-\zeta)^j} \\ &= \int \frac{\mathrm{d}^2\xi}{|\det(\theta)|}\phi(\vec{\xi}) \int \mathrm{d}^2\zeta\ \psi(\vec{\zeta}) e^{\frac{2i}{\theta}(x-\xi)^i\epsilon_{ij}(x-\zeta)^j} \quad .\end{aligned} \tag{3.15}$$

Furthermore, suppose the fields ϕ and ψ are slowly varying, and that ψ has average widths $\Delta_{\psi 1}$ and $\Delta_{\psi 2}$ in x_1 and x_2 directions, respectively. Then, the phase factor suppresses the integral

$$\int \mathrm{d}^2\zeta\ \psi(\vec{\zeta}) e^{\frac{2i}{\theta}(x-\xi)^i\epsilon_{ij}(x-\zeta)^j} \tag{3.16}$$

in (3.15), when

$$|x_1 - \xi_1||x_2 - \zeta_2| \gg \theta \quad \text{or} \quad |x_2 - \xi_2||x_1 - \zeta_1| \gg \theta \quad . \tag{3.17}$$

Therefore the integral (3.16) is nonzero, roughly, when

$$\Delta_{\psi 1}|x_2 - \xi_2| \gg \theta \quad \text{and} \quad \Delta_{\psi 2}|x_1 - \xi_1| \gg \theta \quad , \tag{3.18}$$

and accordingly $(\phi\star\psi)(x)$ 'samples' ϕ with accuracy $\delta_{\phi 1} \approx \theta/\Delta_{\psi 2}$ and $\delta_{\phi 2} \approx \theta/\Delta_{\psi 1}$ in x_1 and x_2 directions, respectively [26]. Moreover, we may repeat the same consideration, but in terms of the sampling of ψ instead with similar results, and we obtain the approximate equalities

$$\delta_{\phi 1}\Delta_{\psi 2} \approx \theta \quad , \quad \delta_{\phi 2}\Delta_{\psi 1} \approx \theta \quad , \quad \delta_{\psi 1}\Delta_{\phi 2} \approx \theta \quad \text{and} \quad \delta_{\psi 2}\Delta_{\phi 1} \approx \theta \quad .$$

Now, in the special case $\psi = \phi$, where the widths of ϕ in directions x_1 and x_2 are Δ_1 and Δ_2, we consequently find the limits

$$\delta_1 \approx \max\left\{\Delta_1, \frac{\theta}{\Delta_2}\right\} \quad \text{and} \quad \delta_2 \approx \max\left\{\Delta_2, \frac{\theta}{\Delta_1}\right\} \tag{3.19}$$

for the widths of $(\phi \star \phi)(x)$ in the directions x_1 and x_2, respectively [26].

Now, consider the noncommutative $\lambda\phi^{3\star}$ scalar field theory, for example, in which the classical field equation is [26]

$$(\partial^2 - m^2)\phi(x) = \frac{\lambda}{2}(\phi \star \phi)(x) \quad . \tag{3.20}$$

If ϕ_0 is a solution to the free field equation $(\partial^2 - m^2)\phi(x) = 0$, we may approximate the solution of (3.20) with a perturbative expansion as

$$\phi(x) = \phi_0(x) - \frac{\lambda}{2}\int \mathrm{d}^4y\ G(x-y)(\phi_0 \star \phi_0)(y) + \ldots \quad , \tag{3.21}$$

where $G(x-y)$ is the appropriate Green's function. Since the source term $-\frac{\lambda}{2}(\phi\star\phi)$ contains $\star$-products of the field, due to the property (3.19), narrow high energy wave-packets of width $\sim \Delta$ are spread to a width $\sim \theta/\Delta$ upon interacting (where θ is the characteristic magnitude of $\theta^{\mu\nu}$), and therefore, strangely, the UV-sector contributes to the IR-sector of the theory. This explains why, as observed in the previous section, in quantum theory, where even low energy processes receive contributions from the high energy virtual particles, a UV cut-off at the energy scale Λ also imposes an effective IR cut-off at the scale $(\theta\Lambda)^{-1}$, regulating the IR divergence of nonplanar diagrams [26]. In this way, the UV/IR mixing of noncommutative quantum field theories can be understood to be deeply rooted in the inherent nonlocal character of the $\star$-product. Accordingly, it seems reasonable to expect to encounter it in any quantum field theory formulated via $\star$-products in the above naïve manner.

Chapter 4

Noncommutative Gauge Field Theories

4.1 Formulation and Some Properties

In order to formulate more physical noncommutative models, one must move beyond scalar field theories to consider gauge field theories in noncommutative space-time. Therefore a great portion of effort has been aimed at their consistent construction. The naïve yet straightforward formulation follows, again, simply by replacing the commutative point-wise product of fields by the noncommutative $\star$-product (2.16) of Weyl-Moyal correspondence, whence one obtains the action

$$\mathcal{S}_{\text{NCGT}} = \int \mathrm{d}^4 x \left[\bar{\Psi}(i\partial\!\!\!/ - m)\Psi - g\bar{\Psi} \star A\!\!\!/ \star \Psi - \frac{1}{4}\mathrm{tr}(F_{\mu\nu} \star F^{\mu\nu}) \right] \quad , \tag{4.1}$$

where Ψ is a spinor field, A_μ a gauge field, g a coupling constant and

$$F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu - ig[A_\mu \stackrel{\star}{,} A_\nu] \tag{4.2}$$

the noncommutative field strength. We also employ the Feynman slash notation $A\!\!\!/ := \gamma^\mu A_\mu$, where γ^μ are the Dirac gamma matrices. The action (4.1) is then invariant under noncommutative gauge transformations infinitesimally given by the formulae

$$\begin{aligned} \delta_\Lambda A_\mu &= \partial_\mu \Lambda + ig[\Lambda \stackrel{\star}{,} A_\mu] \\ \delta_\Lambda \Psi &= ig\Lambda \star \Psi \quad , \end{aligned} \tag{4.3}$$

where Λ is the infinitesimal gauge transformation parameter. An interesting detail is that we have to replace the usual commutator of matrices in the field strength by the Moyal bracket (2.17), which renders also the noncommutative QED a non-Abelian theory giving rise to photon-photon interaction. Thus, new physics appears.

The case of gauge field theories, however, is even more subtle than that of scalar field theories, because of the fundamental notion of *local* symmetry their are built on. Indeed, it turns out that severe restrictions on gauge groups and their representations follow from the nonlocality. First of all, the special unitary groups $SU(N)$ are not closed under multiplication by the $\star$–product, and therefore one cannot have the usual $SU(3) \times SU(2) \times U(1)$ composition for a noncommutative Standard Model. However, the unitary groups $U(N)$ equipped with the $\star$-product, denoted commonly by $U_\star(N)$, are closed under multiplication, and therefore the closest one can get to the commutative Standard Model is $U_\star(3) \times U_\star(2) \times U_\star(1)$. Moreover, the values of charges are limited to $0, \pm 1$ for the $U_\star(1)$ sector, since it is a non-Abelian group due to the noncommutative multiplication [37, 38, 39]. Thirdly, due to a no-go theorem [39], the matter fields can transform nontrivially under only two $U_\star(N)$ gauge groups at most. There are, however, ways to overcome some of these restrictions [40, 41, 42]. It turns out that a noncommutative Standard Model based on the gauge groups $U_\star(3) \times U_\star(2) \times U_\star(1)$ can be constructed with the proper particle content, except for two additional particles corresponding to the reduction of the $U(1)$ factors of $U_\star(3)$ and $U_\star(2)$ via a symmetry breaking mechanism called the *Higgsac mechanism* [40], in parallel with the usual Higgs mechanism of the commutative Standard Model. Remarkably, the restrictions imposed by noncommutativity have the potential to explain the values of quark electric charges.

4.2 Observables in Noncommutative Yang-Mills Theories

Let us say a few words about observables in noncommutative Yang-Mills gauge theories, since it is such a subtle subject, which clearly highlights the nonlocality of these theories, and is also deeply related to UV/IR mixing. Indeed, in noncommutative spacetime local operators, such as $\mathrm{tr}F^2(x)$, are not gauge invariant. Instead, one may construct gauge invariant observables using *open Wilson lines*, which are nonlocal operators defined as [43, 44]

$$W(x,\zeta) := P_\star \exp\left[ig\int_0^1 \mathrm{d}\sigma \frac{\mathrm{d}\zeta^\mu}{\mathrm{d}\sigma} A_\mu(x+\zeta(\sigma))\right] \quad , \tag{4.4}$$

where $\zeta^\mu(\sigma)$ is a curve in spacetime parametrized by $0 \leq \sigma \leq 1$, so that $\zeta^\mu(0) = 0$ and $\zeta^\mu(1) = l^\mu =$ constant. $P_\star$ denotes the path ordering of factors with respect to the $\star$-product, so that

$$\begin{aligned} W(x,\zeta) &= \sum_{n=0}^{\infty} (ig)^n \int_0^1 \mathrm{d}\sigma_1 \int_{\sigma_1}^1 \mathrm{d}\sigma_2 \cdots \int_{\sigma_{n-1}}^1 \mathrm{d}\sigma_n \left(\prod_{i=1}^n \frac{\mathrm{d}\zeta^{\mu_i}}{\mathrm{d}\sigma_i}\right) \\ &\quad \times A_\mu(x+\zeta(\sigma_1)) \star A_\mu(x+\zeta(\sigma_2)) \star \cdots \star A_\mu(x+\zeta(\sigma_n)) \quad , \end{aligned} \tag{4.5}$$

where $\zeta^{\mu_i} \equiv \zeta^{\mu_i}(\sigma_i)$. The important property of an open Wilson line, which we will take advantage of, is that it transforms as

$$W(x,\zeta) \mapsto U(x) \star W(x,\zeta) \star U^\dagger(x+l) \tag{4.6}$$

under finite noncommutative gauge transformations [44]. Now, consider the operator

$$\tilde{W}(k,\zeta) := \int \mathrm{d}^4x \ \mathrm{tr}[W(x,\zeta)] \star e^{ik\cdot x} \quad . \tag{4.7}$$

According to (4.6), it transforms under a gauge transformation as

$$\begin{aligned} \tilde{W}(k,\zeta) \ &\mapsto \ \int \mathrm{d}^4x \ \mathrm{tr}[U(x) \star W(x,\zeta) \star U^\dagger(x+l) \star e^{ik\cdot x}] \\ &= \ \int \mathrm{d}^4x \ \mathrm{tr}[U(x) \star W(x,\zeta) \star e^{ik\cdot x} \star U^\dagger(x+l-\tilde{k})] \end{aligned} \tag{4.8}$$

due to the translation property (2.20) of plane waves. Therefore we find that $\tilde{W}(k,\zeta)$ is a gauge invariant operator, if $l = \tilde{k} = \theta^{\mu\nu}k_\nu$, since the trace and the spacetime integral are cyclic in their arguments.

Furthermore, one may construct a gauge invariant operator out of any local operator $O(x)$ gauge invariant in commutative gauge theory by attaching it to one end of a Wilson line with $\zeta^\mu(1) = \tilde{k}^\mu$ as [44]

$$\tilde{O}(k,\zeta) := \int \mathrm{d}^4x \ \mathrm{tr}[O(x) \star W(x,\zeta)] \star e^{ik\cdot x} \quad . \tag{4.9}$$

However, we get an over-complete set of operators by allowing all possible paths $\zeta^\mu(\sigma)$. By making the natural choice of a straight path $\zeta^\mu(\sigma) = \tilde{k}^\mu\sigma$ we may cure the over-completeness, and also make the operator $\tilde{O}(k,\zeta)$ independent of the particular location on the path we choose to attach $O(x)$ to [44]. Accordingly, we get the gauge invariant operator

$$\tilde{O}(k) := \int \mathrm{d}^4x \ \mathrm{tr}[O(x) \star W(x,\tilde{k}\sigma)] \star e^{ik\cdot x} \quad , \tag{4.10}$$

which is local in momentum space, but distributed along a distance $\tilde{k}$ transverse to the momentum k in coordinate space ($k \cdot \tilde{k} \equiv 0$). It also reduces to the Fourier transform of the local commutative operator $O(x)$ at the limit $\theta^{\mu\nu} \to 0$.

This suggests that particles carrying gauge invariant quantities in noncommutative spacetime should not be viewed as point-like but as extended string-like objects of size $\tilde{k}^\mu = \theta^{\mu\nu}k_\nu$. Therefore it is conceivable why it is particularly the nonplanar Feynman diagrams, where the extended particles may 'collide', which obtain momentum dependent regularization, and accordingly UV/IR mixing [36, 44].

4.3 Noncommutative QED and UV/IR Mixing

Despite the aforementioned achievements and growing understanding in formulating noncommutative gauge field theories, the problem of UV/IR mixing still shows

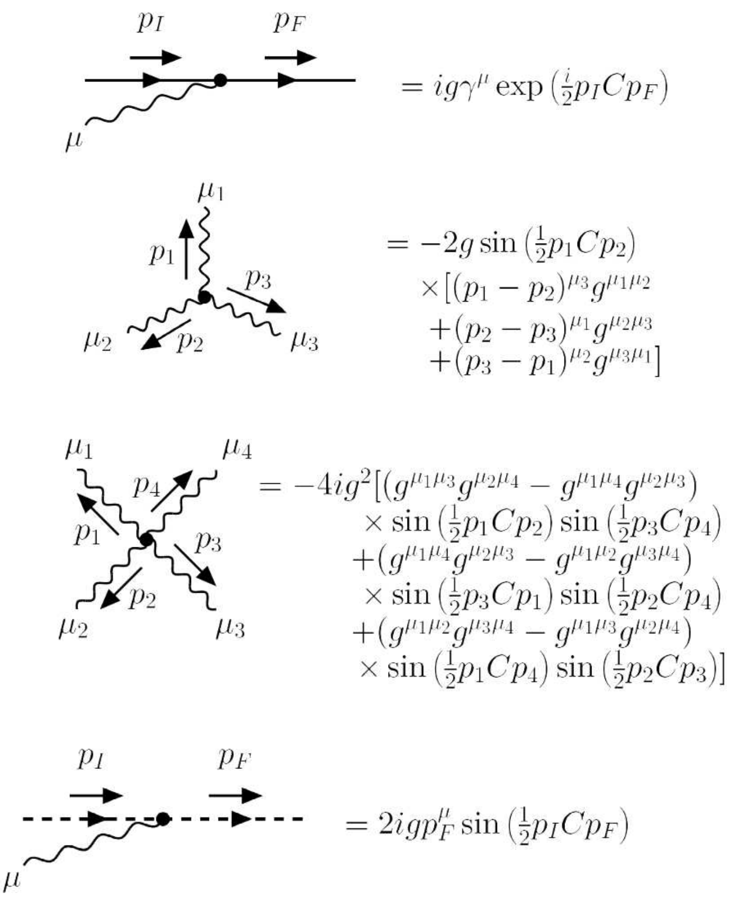

Figure 4.1: Vertex functions for noncommutative QED [38]

up, as expected, when one calculates the higher order diagrams of noncommutative gauge field theories. The propagators of noncommutative gauge theories are again equal to those of their commutative counter-parts, but the vertex functions contain nontrivial phase factors, as in the case of noncommutative scalar field theories, which ultimately lead to the UV/IR mixing. In particular, for noncommutative QED one obtains (using BRST gauge fixing) the vertex functions in Fig. (4.1) for Feynman diagrams [38].[1] For example, for the correction $\Pi_\Psi^{\mu\nu}$ given by a massless fermion loop to the photon propagator one finds [45]

$$i\Pi_\Psi^{\mu\nu}(k) = -4g^2\int\frac{\mathrm{d}^4p}{(2\pi^4)}\frac{\mathrm{tr}[\gamma^\mu(\not{p}-\not{k})\gamma^\nu\not{p}]}{(p-k)^2p^2}\sin^2\left(\frac{1}{2}p\wedge k\right) \quad , \tag{4.11}$$

where the appearing phase factor can now be divided into a sum of planar and nonplanar parts as

$$\sin^2\left(\frac{1}{2}p\wedge k\right) = \frac{1}{2}\left[1-\cos(p\wedge k)\right] \quad , \tag{4.12}$$

respectively. The planar part gives then the usual logarithmically UV-divergent but renormalizable contribution, whereas the nonplanar part with the dampening

[1] In the diagrams of Fig. (4.1) $pCq := p_\mu\theta^{\mu\nu}q_\nu$. In addition, solid straight lines represent the fermion field, wavy lines represent the photon field, and dotted lines represent the Faddeev-Popov ghost field, as usual.

phase factor $\cos(p \wedge k)$ gives the leading order term

$$i\Pi^{\mu\nu}_{\Psi \mathrm{np}}(k) \sim \frac{\tilde{k}^\mu \tilde{k}^\nu}{\tilde{k}^4} \tag{4.13}$$

at the IR-limit of the external momentum, which clearly diverges quadratically as $\tilde{k} \to 0$. Therefore we again encounter the UV/IR mixing, where an IR-divergence of the external momentum arises from the UV-limit of the integral over the internal loop momentum. Similar IR-divergencies arise also from other higher order corrections to propagators and vertices [45].

It should be stressed that this pathological IR-behaviour, as explained in the previous chapter, not only seems radically different from the commutative one, but also causes great trouble for the renormalization of noncommutative quantum field theories, and thus prevents us from making sound quantitative predictions.[2] So far we have seen that in the straightforward formulation of quantum field theories, where one simply replaces the point-wise multiplication of fields with the $\star$-product, UV/IR mixing is a generic property deriving from the nonlocality of interactions. Therefore it seems desirable to seek for other ways to define noncommutative models. Indeed, there are such ways, a particularly interesting one being the so-called Seiberg-Witten map, which we explore in the next section.

4.4 UV/IR Mixing in Noncommutative QED via Seiberg-Witten Map‡

In their seminal paper [20] on the connection between noncommutative geometry and String Theory, Seiberg and Witten introduced a mapping, which relates gauge field theories in noncommutative spacetime to ordinary commutative ones, known as the *Seiberg-Witten map*. This mapping has virtues, since some aspects of gauge theories, such as observables and gauge fixing, are more easily understood and dealt with in the commutative theories. On the other hand, it also has certain uniqueness ambiguities explored in [48, 49]. Moreover, it does not seem to affect at all some problems stemming from the noncommutativity, an example of which is the no-go theorem [39, 50] mentioned above.

In particular, it has been argued, for example in [51][3], that the UV/IR mixing is absent in the Seiberg-Witten formalism. However, we will find that this is presumably due to the expansion in the noncommutativity parameter matrix θ in the

[2] It should be mentioned, however, that noncommutative $U_\star(N)$ gauge theories have been shown to be renormalizable in the planar sector in [46].

‡The results of this section constitute the main research effort for the thesis conducted in collaboration with Doc. Anca Tureanu and Prof. Masud Chaichian [47].

[3] In [51] it was shown that the photon self-energy can be renormalized to any *finite* order in θ by shifting the nonrenormalizable 'mess' to the next order by redefinitions of fields within the limits of the freedom/ambiguity of Seiberg-Witten map.

θ-expanded Seiberg-Witten map. In the θ-exact Seiberg-Witten map for noncommutative QED the UV/IR mixing reappears, as we will demonstrate. This same argument was expressed by Schupp and You in [52], where a noncommutative model with a gauge field coupled with a spinor field in the adjoint representation was considered. The adjoint representation of the gauge group, however, corresponds to a chargeless particle with an electric dipole moment proportional to θ, and therefore in their model the interaction vanishes at the commutative limit $\theta \to 0$. Accordingly, their model does not correspond to a noncommutative theory of electrically charged fermions, which should reduce (classically) to the commutative QED in the commutative limit.

Here we extend the results of [52] to the case of noncommutative QED with charged fermions. We first derive the θ-exact Seiberg-Witten map for a theory with a spinor field in the fundamental representation of the gauge field, corresponding to a charged fermion, and then demonstrate the persistence of UV/IR mixing in the photon self-energy diagram.

4.4.1 θ-exact Seiberg-Witten Map with Charged Fermions

The Seiberg-Witten map is a technique to induce a gauge orbit preserving mapping $(A_\mu, \Lambda) \mapsto (\hat{A}_\mu, \hat{\Lambda})$ between gauge fields and gauge transformation parameters in commutative and noncommutative spacetimes, respectively. The mapping can be realized either as an expansion in the noncommutativity parameters $\theta^{\mu\nu}$ or in the gauge field A_μ. Since an expansion in $\theta^{\mu\nu}$ may hide the possible UV/IR mixing of the noncommutative theory, as indeed is the case with scalar field theories above, we will here follow the latter approach of Seiberg and Witten, but adding also a spinor field into the picture, thus inducing a mapping $(\Psi, A_\mu, \Lambda) \mapsto (\hat{\Psi}, \hat{A}_\mu, \hat{\Lambda})$. A θ-exact Seiberg-Witten map for an Abelian gauge field theory without charged fermions has already been established in [53, 54, 52], in the respective order. However, we will carry out the derivation from scratch in a slightly more direct manner, while expanding the results by including a spinor field in the fundamental representation of the gauge group.

The strategy in deriving the θ-exact Seiberg-Witten map, in a nutshell, is first to relate two gauge field theories in noncommutative spacetimes with infinitesimally differing noncommutativity parameter matrices, say θ and θ', to each other in a gauge orbit preserving way, and then to integrate this relation from the origin $\theta_0 \equiv 0$ to some constant matrix θ_1 along a path in the space of 4×4 real-valued anti-symmetric matrices. Thus, let us consider two noncommutative gauge field theories with spinor fields, denoted by $\mathcal{T}[\theta^{\mu\nu}, A_\mu, \Psi]$ and $\mathcal{T}'[\theta'^{\mu\nu}, A'_\mu, \Psi']$, where the arguments are the noncommutativity parameters, the gauge fields and the spinor fields, respectively. Let us also introduce the notation

$$\begin{aligned} \theta'^{\mu\nu} - \theta^{\mu\nu} &= \delta\theta^{\mu\nu} \\ A'_\mu - A_\mu &= a_\mu \\ \Psi' - \Psi &= \psi \quad . \end{aligned} \tag{4.14}$$

As prescribed, assume that $\delta\theta^{\mu\nu}$ are infinitesimal, and that the fields depend smoothly on the noncommutativity parameters, so that a_μ, ψ and all their partial derivatives are also infinitesimal.

Let us now consider a mapping of the fields from $\mathcal{T}$ to $\mathcal{T}'$. We may think of the fields in $\mathcal{T}'$ as depending on the fields in $\mathcal{T}$ according to this mapping, so that[4]

$$A'_\mu \equiv A'_\mu(A) = A_\mu + a_\mu(A) \quad \text{and} \quad \Psi' \equiv \Psi'(\Psi, A) = \Psi + \psi(\Psi, A) \quad . \tag{4.15}$$

Now, apply a gauge transformation in the theory $\mathcal{T}$ with a gauge transformation parameter Λ. For a noncommutative gauge field theory a gauge transformation is given by the formulae[5]

$$\begin{aligned} \delta_\Lambda A_\mu &= \partial_\mu \Lambda + i[\Lambda \stackrel{\star}{,} A_\mu] \\ \delta_\Lambda \Psi &= i\Lambda \star \Psi \quad . \end{aligned} \tag{4.16}$$

The fundamental requirement for the Seiberg-Witten map is that it should preserve the gauge equivalence classes of the theory, so that the transformation Λ in $\mathcal{T}$ corresponds to a gauge transformation

$$\Lambda' \equiv \Lambda'(\Lambda, A) = \Lambda + \lambda(\Lambda, A) \tag{4.17}$$

in $\mathcal{T}'$, so that

$$A'_\mu(A + \delta_\Lambda A) = A'_\mu(A) + \delta_{\Lambda'} A'_\mu(A) \tag{4.18}$$

$$\Psi'(\Psi + \delta_\Lambda \Psi, A + \delta_\Lambda A) = \Psi'(\Psi, A) + \delta_{\Lambda'} \Psi'(\Psi, A) \quad . \tag{4.19}$$

By substituting the formulae (4.14) and (4.16), and using the relation

$$f \star' g = f \; e^{\frac{i}{2}\overleftarrow{\partial_\mu}(\theta + \delta\theta)^{\mu\nu}\overrightarrow{\partial_\nu}} \; g = f \star g + \frac{i}{2}\delta\theta^{\mu\nu}(\partial_\mu f) \star (\partial_\nu g) \quad , \tag{4.20}$$

we arrive at the equations

$$\begin{aligned} & a_\mu(A + \delta_\Lambda A) - a_\mu(A) - \partial_\mu \lambda(\Lambda, A) - i\,[\lambda(\Lambda, A) \stackrel{\star}{,} A_\mu] - i\,[\Lambda \stackrel{\star}{,} a_\mu(A)] \\ = & -\frac{1}{2}\delta\theta^{\alpha\beta}\,\{\partial_\alpha \Lambda \stackrel{\star}{,} \partial_\beta A_\mu\} \end{aligned} \tag{4.21}$$

[4]Precisely which arguments are needed here depends on, and is revealed by, the solutions found below, but for clarity they are already given here. Moreover, we have dropped out the Lorentz indices of the arguments for simplicity, since it is clear how they are resumed.

[5]We do not worry about gauge fixing here, since it is ultimately performed in the commutative QED. We also set for the coupling constant $g = 1$ hereafter for simplicity.

and

$$\begin{aligned} &\psi(\Psi+\delta_\Lambda\Psi, A+\delta_\Lambda A)-\psi(\Psi,A)-i\Lambda\star\psi(\Psi,A)-i\lambda(\Lambda,A)\star\Psi \\ = &-\frac{1}{2}\delta\theta^{\alpha\beta}(\partial_\alpha\Lambda)\star(\partial_\beta\Psi) \end{aligned} \tag{4.22}$$

for λ, a_μ and ψ. As found out by Seiberg and Witten in [20], the equation (4.21) is solved by

$$\begin{aligned} \lambda &= -\frac{1}{4}\delta\theta^{\alpha\beta}\left\{A_\alpha \stackrel{\star}{,} \partial_\beta\Lambda\right\} \\ a_\mu &= -\frac{1}{4}\delta\theta^{\alpha\beta}\left\{A_\alpha \stackrel{\star}{,} \partial_\beta A_\mu + F_{\beta\mu}\right\} \quad , \end{aligned} \tag{4.23}$$

where $F_{\mu\nu}\equiv\partial_\mu A_\nu-\partial_\nu A_\mu-i\left[A_\mu \stackrel{\star}{,} A_\nu\right]$ is the noncommutative field strength. Using (4.23), we find for the equation (4.22) the solution

$$\psi=-\frac{1}{2}\delta\theta^{\alpha\beta}\left[A_\alpha\star(\partial_\beta\Psi)+\frac{1}{2}(\partial_\beta A_\alpha)\star\Psi\right] \quad . \tag{4.24}$$

As prescribed, the next step in constructing the θ-exact Seiberg-Witten map is to integrate these relations along a path in the space of real-valued anti-symmetric 4×4 matrices to obtain a relation between gauge theories in a commutative spacetime and in a noncommutative one with finite noncommutativity parameters $\theta^{\mu\nu}$. There are certain ambiguities related to choosing a particular path, following from the observation that successive Seiberg-Witten maps do not commute in general, and hence there is an infinite number of free parameters related to the path fixing, some of which, but not all, correspond to gauge transformations and field redefinitions [48, 49]. However, for simplicity, we choose to consider a straight path $\gamma:[0,1]\to\{\theta\in\mathbb{R}^{4\times4}|\theta \text{ anti-symmetric}\}$ such that $\gamma(s)=s\theta_1$, where θ_1 is the constant matrix reached at $s=1$. Let us denote the fields, now considered as dependent on the spacetime coordinates x^μ and the noncommutativity parameters $\theta^{\mu\nu}$, as $A_\mu(x;\theta)$ and $\Psi(x;\theta)$. Integrating the variation (4.21) along the straight path γ by applying integration by parts, we obtain for the gauge field

$$\begin{aligned} A_\mu(x;\theta_1) = \; & A_\mu(x;0)+\lim_{y\to x}\Bigg\{-\frac{\theta_1^{\alpha\beta}}{4}\frac{e^{\frac{i}{2}\theta^{\rho\sigma}\frac{\partial}{\partial x^\rho}\frac{\partial}{\partial y^\sigma}}}{\frac{i}{2}\theta_1^{\gamma\delta}\frac{\partial}{\partial x^\gamma}\frac{\partial}{\partial y^\delta}} \\ & \times\Big[A_\alpha(x;\theta)\big(\partial_\beta A_\mu(y;\theta)+F_{\beta\mu}(y;\theta)\big) \\ & \quad +\big(\partial_\beta A_\mu(x;\theta)+F_{\beta\mu}(x;\theta)\big)A_\alpha(y;\theta)\Big] \\ & +\frac{\theta_1^{\alpha\beta}}{4}\sum_{n=2}^{\infty}(-1)^n\frac{e^{\frac{i}{2}\theta^{\rho\sigma}\frac{\partial}{\partial x^\rho}\frac{\partial}{\partial y^\sigma}}}{\left(\frac{i}{2}\theta_1^{\gamma\delta}\frac{\partial}{\partial x^\gamma}\frac{\partial}{\partial y^\delta}\right)^n}\left(\prod_{k=2}^{n}\theta_1^{\alpha_k\beta_k}\frac{\delta}{\delta\theta^{\alpha_k\beta_k}}\right) \\ & \times\Big[A_\alpha(x;\theta)\big(\partial_\beta A_\mu(y;\theta)+F_{\beta\mu}(y;\theta)\big) \\ & \quad +\big(\partial_\beta A_\mu(x;\theta)+F_{\beta\mu}(x;\theta)\big)A_\alpha(y;\theta)\Big]\Bigg\}_{\theta=0}^{\theta=\theta_1} \quad , \end{aligned} \tag{4.25}$$

and similarly for the spinor field

$$\begin{aligned}\Psi(x;\theta_1) &= \Psi(x;0) + \lim_{y\to x}\Bigg\{ -\frac{\theta_1^{\alpha\beta}}{4}\frac{e^{\frac{i}{2}\theta^{\rho\sigma}\frac{\partial}{\partial x^\rho}\frac{\partial}{\partial y^\sigma}}}{\frac{i}{2}\theta_1^{\gamma\delta}\frac{\partial}{\partial x^\gamma}\frac{\partial}{\partial y^\delta}} \\ &\quad\times\Big[A_\alpha(x;\theta)(\partial_\beta\Psi(y;\theta)) + \frac{1}{2}(\partial_\beta A_\alpha(x;\theta))\Psi(y;\theta)\Big] \\ &\quad+\frac{\theta_1^{\alpha\beta}}{4}\sum_{n=2}^{\infty}(-1)^n\frac{e^{\frac{i}{2}\theta^{\rho\sigma}\frac{\partial}{\partial x^\rho}\frac{\partial}{\partial y^\sigma}}}{\left(\frac{i}{2}\theta_1^{\gamma\delta}\frac{\partial}{\partial x^\gamma}\frac{\partial}{\partial y^\delta}\right)^n}\left(\prod_{k=2}^{n}\theta_1^{\alpha_k\beta_k}\frac{\delta}{\delta\theta^{\alpha_k\beta_k}}\right) \\ &\quad\times\Big[A_\alpha(x;\theta)(\partial_\beta\Psi(y;\theta)) + \frac{1}{2}(\partial_\beta A_\alpha(x;\theta))\Psi(y;\theta)\Big]\Bigg\}_{\theta=0}^{\theta=\theta_1}, \end{aligned} \tag{4.26}$$

which can be calculated iteratively in powers of the gauge field A_μ, since $\frac{\delta}{\delta\theta}A_\mu = \mathcal{O}(A^2)$ and $\frac{\delta}{\delta\theta}\Psi = \mathcal{O}(A)$, so the variations in the sums give terms of ever increasing powers in the gauge field.

4.4.2 Noncommutative QED via Seiberg-Witten Map

We now turn to consider exclusively the gauge group $U_\star(1)$. We want to express the action of noncommutative QED,

$$\mathcal{S}_{\rm NCQED} = \int d^4x\left[\bar{\hat{\Psi}}(i\partial\!\!\!/ - m)\hat{\Psi} - \bar{\hat{\Psi}}\star\hat{A\!\!\!/}\star\hat{\Psi} - \frac{1}{4}\hat{F}_{\mu\nu}\star\hat{F}^{\mu\nu}\right] \quad , \tag{4.27}$$

in terms of the commutative fields up to the first order in A, so that we can calculate the photon propagator correction coming from the one-loop photon self-energy diagram. Denoting the noncommutative fields by hats and dropping the lower index from θ_1, we find for the gauge field via the equation (4.25) the expression

$$\hat{A}_\mu = A_\mu + \mathcal{O}(A^2) \quad , \tag{4.28}$$

and for the spinor field via the equation (4.26) the expression

$$\hat{\Psi} = \Psi - \frac{1}{2}\theta^{\alpha\beta}\left[A_\alpha \star_1 (\partial_\beta\Psi) + \frac{1}{2}(\partial_\beta A_\alpha)\star_1\Psi\right] + \mathcal{O}(A^2) \quad , \tag{4.29}$$

where we use the notation

$$(f\star_1 g)(x) := \left\{\frac{e^{\frac{i}{2}\partial_1\wedge\partial_2} - 1}{\frac{i}{2}\partial_1\wedge\partial_2}f(x_1)g(x_2)\right\}_{x_1=x_2\equiv x} \quad . \tag{4.30}$$

Since $(f\star g)^\dagger = g^\dagger\star f^\dagger$ for any functions (or matrices) f and g, we find that

$$\bar{\hat{\Psi}} \equiv \bar{\hat{\bar{\Psi}}} = \bar{\Psi} - \frac{1}{2}\theta^{\alpha\beta}\left[(\partial_\beta\bar{\Psi})\star_1 A_\alpha + \frac{1}{2}\bar{\Psi}\star_1(\partial_\beta A_\alpha)\right] + \mathcal{O}(A^2) \quad . \tag{4.31}$$

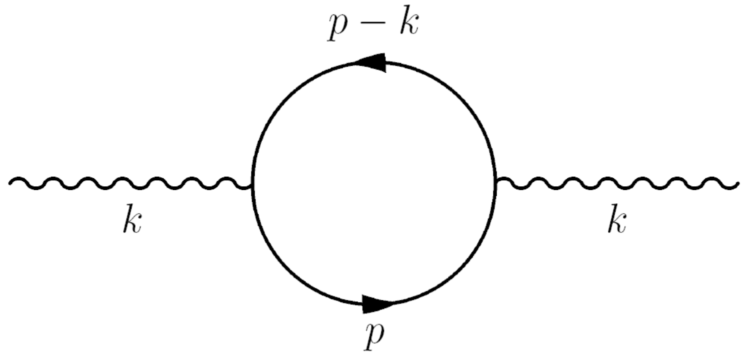

Figure 4.2: One-loop photon self-energy diagram

Substituting (4.28), (4.29) and (4.31) into the action (4.27), we find the fermion-photon interaction term to be, up to first order in A,

$$\begin{aligned}\mathcal{L}^{(1)}_{\Psi A} &= -\bar{\Psi}\star \not{A}\star\Psi \\ &\quad -\frac{1}{2}\theta^{\alpha\beta}\left[(\partial_\beta\bar{\Psi})\star_1 A_\alpha + \frac{1}{2}\bar{\Psi}\star_1(\partial_\beta A_\alpha)\right](i\not{\partial}-m)\Psi \\ &\quad -\frac{1}{2}\theta^{\alpha\beta}\bar{\Psi}(i\not{\partial}-m)\left[A_\alpha\star_1(\partial_\beta\Psi)+\frac{1}{2}(\partial_\beta A_\alpha)\star_1\Psi\right] \quad . \qquad (4.32)\end{aligned}$$

For the corresponding vertex function we get the expression

$$V^\mu(k_1,k_2) = -i\gamma^\mu e^{\frac{i}{2}k_1\wedge k_2} - \frac{i}{2}(\tilde{k}_1-\tilde{k}_2)^\mu(\not{k}_1+\not{k}_2)\frac{e^{\frac{i}{2}k_1\wedge k_2}-1}{k_1\wedge k_2} \quad , \qquad (4.33)$$

where k_1 and k_2 are the incoming momenta of the outgoing and incoming fermions, respectively.

4.4.3 Photon Self-energy and UV/IR Mixing

Now, using the vertex function (4.33), we find the first order fermion loop correction to the photon propagator given by the one-loop photon self-energy diagram in Fig. (4.2) to be

$$\begin{aligned}\Pi^{\mu\nu}_{(1)}(k) &= -4\int\frac{\mathrm{d}^4p}{(2\pi)^4} \\ &\quad \times\Bigg\{T^{\mu\nu} + \frac{i}{2}\frac{\sin(\frac{1}{4}p\wedge k)}{\frac{1}{4}p\wedge k}\Big[(\tilde{p}-\frac{1}{2}\tilde{k})^\mu k_\rho T^{\rho\nu}e^{-\frac{i}{4}p\wedge k} \\ &\qquad -(\tilde{p}-\frac{1}{2}\tilde{k})^\nu k_\rho T^{\rho\mu}e^{\frac{i}{4}p\wedge k}\Big] \\ &\qquad +\frac{1}{4}\frac{\sin^2(\frac{1}{4}p\wedge k)}{(\frac{1}{4}p\wedge k)^2}(\tilde{p}-\frac{1}{2}\tilde{k})^\mu(\tilde{p}-\frac{1}{2}\tilde{k})^\nu k_\rho k_\sigma T^{\rho\sigma}\Bigg\} \quad , \qquad (4.34)\end{aligned}$$

where

$$T^{\mu\nu}(k,p) := \frac{(p-k)^\mu p^\nu + p^\mu (p-k)^\nu + [m^2 - (p-k)\cdot p]\eta^{\mu\nu}}{[(p-k)^2 - m^2][p^2 - m^2]} \quad , \tag{4.35}$$

which is the only term we get in the commutative case. Therefore, the first term in (4.34) is naturally understood to correspond to the planar part of the diagram, and in fact follows straightforwardly from the first terms of the vertex functions (4.33) as the phase factors cancel each other, in the same way as they do for the planar diagrams of a noncommutative scalar field theory. The other terms, on the other hand, clearly correspond to the nonplanar part with nontrivial phase factors that give rise to UV/IR mixing. Indeed, the second term[6] in (4.34) can be shown to yield the leading order contribution

$$i\Pi^{\mu\nu}_{(1)\mathrm{np}}(k) \approx \frac{8}{\pi^2}\frac{\tilde{k}^\mu \tilde{k}^\nu}{\tilde{k}^4} + \frac{4}{\pi^2}\frac{\tilde{\tilde{k}}^\mu k^\nu + k^\mu \tilde{\tilde{k}}^\nu}{\tilde{k}^4} \tag{4.36}$$

at the IR-limit of the external momentum. (See Appendix A for the details of the calculation.) The first term in (4.36) is similar to (4.13) found in the naïve formulation above, whereas the second term is gauge variant and should cancel, when all the second order contributions in the coupling constant are taken into account. Therefore we conclude that also gauge field theories defined via Seiberg-Witten map appear to fail to be renormalizable because of UV/IR mixing, which further shows that this is a generic property of noncommutative theories.

[6]The third term leads also to similar IR-divergent terms.

Chapter 5

Curing the Pathologies

5.1 Scalar Models with Modified Lagrangians

Based on the current evidence it seems that UV/IR mixing is a very generic property of noncommutative quantum field theories. Therefore the question arises whether there are indeed any noncommutative models without such pathological behaviour. The answer turns out to be positive. There are at least a few known models with modified Lagrangians, where the problem does not seem to arise, or it can be dealt with.

The first one of the renormalizable noncommutative models discovered is a $\lambda\phi^{4\star}$ scalar field model in Euclidean spacetime, due to Grosse and Wulkenhaar [55, 56], which introduces a harmonic potential term into the action, giving it the form

$$\mathcal{S}_{\mathrm{GW}} = \int \mathrm{d}^4x \left[\frac{1}{2}(\partial_\mu\phi)(\partial^\mu\phi) + \frac{\Omega^2}{2}(\tilde{x}_\mu\phi)(\tilde{x}^\mu\phi) + \frac{m^2}{2}\phi^2 + \frac{\lambda}{4!}\phi\star\phi\star\phi\star\phi\right] \quad , \tag{5.1}$$

where $\tilde{x}_\mu := 2\theta^{-1}_{\mu\nu}x^\nu$. The harmonic potential dampens the low energy sector of the theory banishing the IR-divergencies related to UV/IR mixing as a consequence, thus making the model renormalizable in all orders of perturbation theory. A serious down-side of such a term is, however, that it explicitly breaks translational invariance, and consequently energy-momentum conservation is violated.

Another renormalizable noncommutative scalar field model, recently put forward by Gurau et al. [57], also adds an extra term to the action, but in their model the translational invariance is preserved. The action becomes

$$\mathcal{S}_{\mathrm{Gur.}} = \int \mathrm{d}^4x \left[\frac{1}{2}(\partial_\mu\phi)(\partial^\mu\phi) - \frac{1}{2}\phi\frac{a^2}{\tilde{\partial}^2}\phi + \frac{m^2}{2}\phi^2 + \frac{\lambda}{4!}\phi\star\phi\star\phi\star\phi\right] \quad , \tag{5.2}$$

where $\tilde{\partial}^\mu := \theta^{\mu\nu}\partial_\nu$. The free propagator in this model receives an extra contribution from the new term, and is given by

$$G(p) = \frac{1}{p^2 + m^2 + \frac{a^2}{\tilde{p}^2}} \quad . \tag{5.3}$$

By the virtue of this contribution the model becomes renormalizable in all orders of perturbation theory, since now one can renormalize the parameter a to counter-act the IR-divergent contributions [58, 59].

These techniques are, however, hard to replicate in gauge theories because of the additional requirement for gauge invariance and different structure of divergencies [35]. Accordingly, the extra terms are required to be quite complicated. Several attempts to this direction have recently been initiated, for example in [60], but so far there are no definite proofs of the renormalization properties of these models.

5.2 Generalizations of the QFT Methods for Nonlocal Interactions

There is yet another noteworthy way to approach the problem of UV/IR mixing. Since locality is a fundamental assumption of commutative quantum field theories, it is reasonable to question the validity of the ordinary methods of quantization in noncommutative case, where nonlocality is inherent. Some attempts have been made in this direction to critically analyze and generalize the quantization methods for nonlocal interactions [61, 62, 63, 64, 65].

In the *interaction-point time-ordered perturbation theory* (IPTOPT) approach [63] an alternative form for the $\star$-product (2.16)

$$(f \star g)(x) = \iint \frac{\mathrm{d}^4 k}{(2\pi)^4} \mathrm{d}^4 y \; f(x - \frac{1}{2}\tilde{k}) g(x+y) \quad , \text{where} \quad \tilde{k}^{\mu\nu} := \theta^{\mu\nu} k_\nu \quad , \tag{5.4}$$

is used, which gives the interaction Lagrangian of $\lambda\phi^{4\star}$ scalar field model the form

$$\begin{aligned} \mathcal{L}_\mathrm{I}(x) &= \frac{\lambda}{4!}(\phi \star \phi \star \phi \star \phi)(x) \\ &= \int \cdots \int \left(\prod_{i=1}^{3} \frac{\mathrm{d}^4 k}{(2\pi)^4} \mathrm{d}^4 y_i \; e^{i y_i \cdot k_i} \right) \phi(x - \frac{1}{2}\tilde{k}_1) \phi(x + y_1 - \frac{1}{2}\tilde{k}_2) \\ &\qquad \times \phi(x + y_1 + y_2 - \frac{1}{2}\tilde{k}_3) \phi(x + y_1 + y_2 + y_3) \quad . \end{aligned} \tag{5.5}$$

Then the time-ordering of interactions in the perturbative expansion is done with respect to the coordinate x, called the interaction-point, before performing the $\star$-product multiplications. This results in smeared nonlocal regions of interaction, where 'micro'causality is violated. Nevertheless, the procedure is understood to be mathematically consistent, and unitarity is conserved even with $\theta^{0i} \neq 0$. There is some evidence to suggest that IPTOPT may also cure the UV/IR mixing problem in the case of scalar field models, but currently a definite proof is absent [66].

Whether or not the persistent problems of noncommutative quantum field theories are solved by these new approaches, it is clear that a careful revision of the usual methods of commutative QFT is certainly necessary to re-evaluate their applicability, when one of the most fundamental assumption of the previous theories, locality, is no longer available.

Chapter 6

Summary & Conclusion

We started off the thesis by motivating the research of noncommutative quantum field theories with an argument invoking principles of quantum theory and general relativity, which together hint at the possible nonlocal character of spacetime. The realization of this new radical aspect is obtained by postulating a canonical commutation relation $[\hat{x}^\mu, \hat{x}^\nu] = i\theta^{\mu\nu}$ for noncommutative spacetime coordinates $\hat{x}^\mu$. We then reviewed the basic mathematical properties of noncommutative spacetime together with introducing the Weyl-Moyal correspondence, which allows us to replicate the algebra of fields in noncommutative spacetime in a corresponding commutative spacetime with a modified nonlocal $\star$-product of fields. Using this to our advantage, we formulated noncommutative scalar and gauge field theories simply by replacing the ordinary point-wise product of fields in Lagrangians with the $\star$-product. We found accodingly that this method of formulation leads to unprecendented problems in renormalization of noncommutative theories due to new momentum-dependent IR-divergencies. This effect, called UV/IR mixing, was explained to be caused by the inherent nonlocal character of the $\star$-product, which mixes high and low energy scales in these theories. In particular, we reviewed the formulation of gauge field theories via θ-exact Seiberg-Witten map, and explained new results, which show that this approach also suffers from the UV/IR mixing. Finally, we presented some proposed solutions to the problem.

The study of noncommutative quantum field theories offers a real opportunity to probe the microscopic structure of spacetime by possessing the potential to produce easily interpretable predictions, which may give hints of new Planck scale physics and the underlying quantum theory of gravity. Great progress has been made towards fulfilling this goal, but the final breakthrough lies still ahead mainly due to the theoretical obstacles reviewed in this thesis. A lot of work is to be done in understanding the fundamental mathematical framework of noncommutative spacetime, which will undoubtedly shed light on several unresolved issues with noncommutative theories. Indeed, there is much hope that the problems will ultimately be resolved, glimpses of which can be already seen in the recent progress, and therefore, it seems, many important discoveries in this field remain to be made.

Appendix A

Momentum integral calculation

In this appendix we review the details of the momentum integration in Section (4.4.3), equation (4.34).

To evaluate the parts of the second term in (4.34) with phase factors we use the trick of Schupp and You [52] by expressing them as

$$\begin{aligned} & -\int \frac{\mathrm{d}^4 p}{(2\pi)^4} \frac{1}{\frac{1}{2} p \wedge k} \left[(\tilde{p} - \frac{1}{2}\tilde{k})^\mu k_\rho T^{\rho\nu} e^{-\frac{i}{2} p\wedge k} + (\tilde{p} - \frac{1}{2}\tilde{k})^\nu k_\rho T^{\rho\mu} e^{\frac{i}{2} p\wedge k} \right] \\ = & \; i \int \mathrm{d}\lambda \; I^{\mu\nu}(k;\lambda) \Big|_{\lambda=-1} + i \int \mathrm{d}\lambda \; I^{\nu\mu}(k;\lambda) \Big|_{\lambda=1} \quad , \end{aligned} \tag{A.1}$$

where

$$I^{\mu\nu}(k;\lambda) = \int \frac{\mathrm{d}^4 p}{(2\pi)^4} \left(\tilde{p} - \frac{1}{2}\tilde{k} \right) k_\rho T^{\rho\nu} e^{\frac{i}{2}\lambda p\wedge k} \quad . \tag{A.2}$$

By performing a Wick rotation $p^\mu = e_i^\mu \bar{p}^i$, where $e_i^\mu = \mathrm{diag}(i,1,1,1)$ and $\bar{p}^i$ is the Euclidean momentum, and using Schwinger parametrization

$$\frac{1}{\bar{p}^2 + m^2} = \int_0^\infty \mathrm{d}\alpha \; e^{-\alpha(\bar{p}^2 + m^2)} \quad , \tag{A.3}$$

we get

$$\begin{aligned} I^{\mu\nu}(k;\lambda) \;=\; & i e_i^\mu e_j^\nu \iint_0^\infty \mathrm{d}\alpha \mathrm{d}\beta \int \frac{\mathrm{d}^4 p}{(2\pi)^2} (\bar{\tilde{p}} - \frac{1}{2}\bar{\tilde{k}})^i \left[(\bar{k}^2 - 2\bar{k}\cdot\bar{p})\bar{p}^j + (\bar{p}^2 + m^2)\bar{k}^j \right] \\ & \times e^{-\alpha[(\bar{p}-\bar{k})^2 + m^2] - \beta[\bar{p}^2 + m^2] + \frac{i}{2}\lambda \bar{p}\cdot\bar{\tilde{k}}} \quad . \end{aligned} \tag{A.4}$$

We may render the exponent Gaussian by applying the change of variables

$$\bar{q} := \bar{p} - \frac{\alpha}{\alpha+\beta}\bar{k} - \frac{i\lambda}{4(\alpha+\beta)}\bar{\tilde{k}} \quad , \tag{A.5}$$

after which we can perform the integration over $\bar{q}$. Further multiplying the integrand by

$$1 = \int_0^\infty \mathrm{d}c \; \delta(c - \alpha - \beta) \quad , \tag{A.6}$$

changing the order of integrations, and applying the change of variables $\alpha = ca$, $\beta = cb$, we get

$$I^{\mu\nu}(k;\lambda) \approx \frac{ie_i^{\mu}e_j^{\nu}\bar{\theta}^{ik}}{(4\pi)^2}\iint_0^1 \mathrm{d}a\mathrm{d}b\;\delta(1-a-b)\int_0^{\infty}\mathrm{d}c\;c^{-3} \times\left[\left(\frac{i\lambda}{2}-\frac{i\lambda^3\bar{\tilde{k}}^2}{64c}\right)\bar{\tilde{k}}_k\bar{k}^j - \frac{i\lambda}{4}\bar{k}_k\bar{\tilde{k}}^j\right] e^{-c(ab\bar{k}^2+m^2)-\frac{\lambda^2}{16c}\bar{\tilde{k}}^2} \quad , \tag{A.7}$$

where the less IR-divergent terms are dropped out.[1] The integral over λ is now straightforward to perform. Moreover, the integral over c can be performed and expressed for small k using the properties of modified Bessel functions $K_r(x,y)$ [67]:

$$\int_0^{\infty}\mathrm{d}c\;c^{-r-1}e^{-xc-y/c} = 2\left(\frac{x}{y}\right)^{\frac{r}{2}} K_r\left[2\sqrt{xy}\right] \quad , \text{where} \quad \mathrm{Re}[x], \mathrm{Re}[y] > 0 \quad ,$$
$$\text{and} \quad K_r(z) \approx \frac{\Gamma(r)}{2}\left(\frac{2}{z}\right)^r \quad , \text{when} \quad 0 < z \ll \sqrt{r+1} \quad . \tag{A.8}$$

The dependence on a and b cancel out, and we get for small $\bar{\tilde{k}}^2 \ll m^{-2}$ accordingly

$$i\Pi^{\mu\nu}_{(1)\mathrm{np}}(k) \approx \frac{8}{\pi^2}\frac{\tilde{k}^{\mu}\tilde{k}^{\nu}}{\tilde{k}^4} + \frac{4}{\pi^2}\frac{\tilde{\tilde{k}}^{\mu}k^{\nu}+k^{\mu}\tilde{\tilde{k}}^{\nu}}{\tilde{k}^4} \quad . \tag{A.9}$$

[1] Here we have to take into account the following integration over c, where $c \sim \bar{\tilde{k}}^2$. The integration over λ does not affect the relative powers of divergence.

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