BOUNDDEDNESS AND CONVERGENCE OF SINGULAR INTEGRALS ON FRACTAL TYPE SETS

VASILEIOS CHOUSHIONIS

Academic dissertation

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Department of Mathematics and Statistics
Faculty of Science
University of Helsinki
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Vasilis Chousionis
LIST OF INCLUDED ARTICLES

The thesis consists of the following three articles, which will be referred in the sequel by [A], [B] and [C]:

[A] Singular integrals on Sierpinski gaskets.
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To appear in Publ. Mat.

[B] Directed porosity on conformal iterated function systems and weak convergence of singular integrals.
Vasilis Chousionis.

[C] Boundedness and convergence for singular integrals of measures separated by Lipschitz graphs
Vasilis Chousionis and Pertti Mattila.
Reports in Mathematics, Preprint 479, Department of Mathematics and Statistics, University of Helsinki, Helsinki 2008.
1. Introduction

The most frequently encountered singular integral is the Hilbert transform on \( \mathbb{R} \), formally given by

\[
Hf(x) = \int \frac{1}{x-y} f(y) dy.
\]

The works of M. Riesz, Besicovitch, Titchmarsh, and Marcinkiewicz among others, established the Hilbert transform as a central research topic in classic harmonic analysis, during the first half of the 20th century, and led to its rigorous understanding. The modern theory of singular integrals was essentially founded by Calderón and Zygmund in their seminal paper \([CZ]\), where they systematically studied analogues of the Hilbert transform in higher dimensions. Since then, singular integrals proved to be a very fruitful part of analysis; they have been studied in many diverse directions and the accumulated knowledge has been applied in various fields, particularly in partial differential equations. An extensive overview of the subject can be found in \([S2]\).

In this thesis, we consider singular integrals in Euclidean spaces with respect to general measures, and we study how the geometric structure of the measures affects certain analytic properties of the operators. The topic has been studied widely in the last thirty years, see e.g. \([Ch]\), \([M1]\), \([DS2]\), \([MP]\), \([MMV]\), \([T1]\), \([T2]\), \([Hu]\), \([MV]\), \([Pr]\) and \([T4]\). Furthermore, many tools developed in the field, such as the so called “T(b) type theorems”, had been essential in most of the recent developments concerning analytic capacity. Indicatively we refer to the proof of Vitushkin’s conjecture by David, in \([D4]\), and in the proof of the semiadditivity of analytic capacity by Tolsa in \([T3]\).

Our setting in its most general form will consist of a Radon measure \( \mu \) in \( \mathbb{R}^n \) and a \( \mu \)-measurable kernel \( K : \mathbb{R}^n \times \mathbb{R}^n \setminus \{(x, y) : x = y\} \to \mathbb{R} \) that satisfies the antisymmetry condition

\[
K(x, y) = -K(y, x) \quad \text{for} \quad x, y \in \mathbb{R}^n, \quad x \neq y.
\]

Usually these integrals do not exist when \( x \in \text{spt} \mu \), thus one hopes to overcome this obstacle by considering the truncated singular integral operators \( T_{\mu,K}^\infty \), \( \infty > 0 \);

\[
T_{\mu,K}^\infty f(x) = \int_{|x-y| > \varepsilon} K(x, y) f(y) d\mu_y,
\]

and expecting that the principal values,

\[
p.v. T_{\mu,K} f(x) = \lim_{\varepsilon \to 0} T_{\mu,K}^\varepsilon f(x),
\]

would exist \( \mu \) almost everywhere, for \( f \in L^1(\mu) \). This is the case in the classical setting, when \( \mu = L^n \), the Lebesgue measure in \( \mathbb{R}^n \), and \( K \) is a standard Calderón-Zygmund kernel. The strong symmetry properties of \( L^n \) allowing heavy cancelations and the fact that smooth functions are dense in \( L^1(L^n) \) force the principal values to exist almost everywhere for \( L^1 \)-functions, see e.g. \([S1]\). Things are more complicated
when we are considering general measures, since in that case principal values do not exist automatically even for constant functions.

The maximal singular integral operator $T_{\mu,K}^*$,

$$T_{\mu,K}^* f(x) = \sup_{\varepsilon > 0} |T_{\mu,K}^\varepsilon f(x)|,$$

is said to be bounded in $L^2(\mu)$ if there exists some constant $C > 0$ such that for $f \in L^2(\mu)$

$$\int (T_{\mu,K}^* f)^2 d\mu \leq C \int |f|^2 d\mu.$$

Recall that $L^2$ boundedness is a central notion in the theory of singular integral operators. With the previous paragraph in mind, one could naturally ask if the $L^2(\mu)$-boundedness of $T_{\mu,K}^*$ forces the principal values to exist $\mu$ almost everywhere. Surprisingly, even when $\mu$ is an $m$-dimensional Ahlfors-David (AD) regular measure in $\mathbb{R}^n$: $C^{-1}r^m \leq \mu(B(x,r)) \leq C r^m$ for $x \in \text{spt} \mu$, $0 < r < \text{diam}(\text{spt} \mu)$, and $K$ is any of the coordinate Riesz kernels:

$$R^m_i(x,y) = \frac{x_i - y_i}{|x - y|^{m+1}}$$

for $i = 1, \ldots, n$,

the question remains open for $m > 1$. Notice that in the case of $m = 1$, the corresponding Riesz transforms essentially coincide with the Cauchy transform on $\mathbb{C}$,

$$C^* \frac{f(\zeta)}{z - \zeta} d\mu \zeta.$$

By a result of Tolsa, see [T1], the question has positive answer for the Cauchy transform even for more general measures. Previous works of Mattila, Melnikov and Verdera, see [MM] and [MMV], dealt with the affirmative in the case of AD-regular measures. The relations between the Cauchy kernel, $1/\zeta$, and the Menger curvature, discovered by Melnikov, in [Me], play a crucial role in the proofs of the aforementioned results. Farag showed in [F] that the same approach fails for $m > 1$; this is one of the main reasons for the lack of understanding of the Riesz transforms, in this context.

The question we discussed earlier does not always have positive answer. Let $C$ be the 1-dimensional four corners Cantor set $C$ and $\mu$ its natural (1-dimensional Hausdorff) measure. David in [D5] constructed Calderón-Zygmund standard kernels that define operators bounded in $L^2(\mu)$ whose principal values fail to exist $\mu$ almost everywhere. These kernels can be chosen odd or even but they are never homogeneous of degree -1. In [A], our setting consists of classical plane Sierpinski Gaskets $E_d$, of Hausdorff dimension $d$, $0 < d < 1$. For each of these $d$-AD regular sets we construct families of CZ standard, smooth, odd and $d$-homogeneous kernels. These kernels give rise to singular integral operators bounded in $L^2(\mu_d)$, whose principal values diverge $\mu_d$ almost everywhere. Here $\mu_d$ is the restriction of
the $d$-dimensional Hausdorff measure on the sets $E_d$. The proof applies with minor changes to various symmetric self similar sets, e.g. the four corners Cantor sets with Hausdorff dimension less than 1.

In [B], we introduce the notion of directed porosity, and we study its connections with conformal iterated function systems (CIFS) and with singular integrals. The theory of CIFS is a natural way of generalizing self similarity; instead of similitudes the function system consists of uniformly contracting conformal maps, generating a larger variety of limit sets. Dynamic and geometric properties of such limit sets have been actively investigated in the last several years, see e.g. [MU], [MMU], [MayU], [U] and [K]. In [U], Urbański considered porosity in CIFS and gave some interesting applications in number theory. From his work it follows that if limit sets of finite CIFS do not have full Hausdorff dimension they are porous. Under some extra dimensional assumptions, we prove that such limit sets have much stronger porosity properties, extending in a sense Urbański’s result. We then proceed and study the convergence behavior of the operators, $T_{\mu,K}^\varepsilon$ when $\text{spt}\mu$ satisfies different porosity conditions. Among other things we prove that when $E \subset \mathbb{R}^n$ is an $(n-1)$-purely unrectifiable limit set of a given CIFS, and $\mu = \mathcal{H}^{n-1}|E$, the restriction of the $(n-1)$-dimensional Hausdorff measure on $E$, the weak limits

$$\lim_{\varepsilon \to 0} \int T_{\mu,K}^\varepsilon(f)(x)g(x)d\mu$$

exist for $f, g$ in some dense subspaces of $L^2(\mu)$, under very mild assumptions for the kernels.

Recall that a set $E \subset \mathbb{R}^n$ will be called $m$-rectifiable for $m = 1, \ldots, n$, if there exist $m$-dimensional Lipschitz surfaces $M_i$, $i \in \mathbb{N}$, such that

$$\mathcal{H}^m(E \setminus \bigcup_{i=1}^{\infty} M_i) = 0.$$  

Sets intersecting $m$-rectifiable sets in a set of zero $\mathcal{H}^m$ measure are called $m$-purely unrectifiable. More information about rectifiability and related topics can be found in [M2]. Rectifiability is deeply related with singular integrals; if $E$ is an $\mathcal{H}^m$-measurable set with $\mathcal{H}^m(E) < \infty$, and $\mu = \mathcal{H}^m|E$ by the works of Mattila and Preiss [MP], Mattila and Melnikov [MM], Verdera [Ve] and Tolsa [T4] the principal values of the $m$-dimensional Riesz transforms exist $\mu$ almost everywhere if and only if the set E is $m$-rectifiable. This stresses the difference with the weak convergence we establish in [B]; as many operators, including the $(n-1)$-dimensional Riesz transform, converge weakly in that sense, even when the measures are purely unrectifiable. Furthermore, the techniques used to prove the existence of the weak limits there, actually depend on the fractal structure of $\mu$.

The initial motivation for [B], and partially for [C], stems from one recent result of Mattila and Verdera, see [MV]. They prove that, for general measures and kernels $\mu$ and $K$, the $L^2(\mu)$-boundedness of $T_{\mu,K}^\varepsilon$ implies that the operators $T_{\mu,K}^\varepsilon$ converge weakly in $L^2(\mu)$. This means that there exists a bounded linear operator
\[ T : L^2(\mu) \to L^2(\mu) \text{ such that for all } f, g \in L^2(\mu), \]
\[ \lim_{\varepsilon \to 0} \int T_{\mu,K}^\varepsilon(f)(x)g(x)d\mu x = \int T(f)(x)g(x)d\mu x. \quad (1.1) \]

Therefore one can naturally ask whether weak limits like in (1.1) might exist if we remove the strong \( L^2 \)-boundedness assumption, even when the measures are supported in some purely unrectifiable set.

In [C] we consider two measures \( \mu \) and \( \nu \) which live on different sides of some \((n−1)\)-dimensional Lipschitz graph. We shall prove that then \( T_{\nu,K}^* : L^2(\nu) \to L^2(\mu) \) is bounded very generally. We should remark that the case where \( \nu = H^{n−1}\lfloor S \), for a Lipschitz graph \( S \), was proved by David in [D1] and our proof makes use of this result. Furthermore we apply this boundedness theorem to prove that the truncated operators \( T_{\mu,K}^\varepsilon \) converge weakly in some dense subspaces of \( L^2(\mu) \). The difference with the result obtained in [B] is that we have to require much less about the measures while we have to add some extra assumptions for the kernels. Both of these results cannot be extended to \( L^2(\mu) \) because, as it was remarked in [MV], by the Banach-Steinhaus Theorem, the weak convergence in \( L^2(\mu) \) implies that the operators \( T_{\mu,K}^\varepsilon \) are uniformly bounded in \( L^2(\mu) \). This sets a barrier as the Cauchy transform with respect to the 1-dimensional four corners Cantor set, for example, converges weakly in the sense of [B] and [C] but it is not bounded in \( L^2 \).

2. Singular integrals on self similar sets

In this section we make a short description of the results and ideas found in [A]. For \( \lambda \in (0,1/3) \) the three planar similitudes \( s_1^\lambda(x,y) = \lambda(x,y), \ s_2^\lambda(x,y) = \lambda(x,y) + (1-\lambda,0), \) and \( s_3^\lambda(x,y) = \lambda(x,y) + (\frac{1-\lambda}{2},\frac{\sqrt{3}}{2}(1-\lambda)) \) generate the so called \( \lambda \)-Sierpinski gaskets \( E_\lambda \). Let \( I = \{1,2,3\} \). For \( \alpha \in I^n \), say \( \alpha = (i_1,\ldots,i_n) \), define the maps \( s_\alpha^\lambda : \mathbb{R}^2 \to \mathbb{R}^2 \) through iteration
\[ s_\alpha^\lambda = s_{i_1}^\lambda \circ s_{i_2}^\lambda \circ \ldots \circ s_{i_n}^\lambda. \]

Let \( A \) be a suitable equilateral triangle with sidelength 1 and denote \( s_0^\lambda(A) = S_\alpha^\lambda, \ I^0 = \{0\} \) and \( s_0^\lambda = \text{id} \). Then the sets
\[ E_\lambda = \bigcap_{j \geq 0} \bigcup_{\alpha \in I^j} S_\alpha^\lambda \]
are well known self similar sets with Hausdorff dimension
\[ d_\lambda := \dim_H E_\lambda = -\frac{\log 3}{\log \lambda}. \]

Obviously for \( \lambda \in (0,1/3), d_\lambda \in (0,1) \). Notice also that, as a general property of self similar sets, the measures \( \mu_\lambda = \mathcal{H}^{d_\lambda} \lfloor E_\lambda \) are \( d_\lambda \)-AD regular.

Our aim was to find families of Calderón-Zygmund standard kernels on \( E_\lambda \times E_\lambda \setminus \{(x,y) : x = y\} \) that define bounded singular integral operators on \( L^2(\mu_\lambda) \). The
desired kernels, \( K_\lambda : E_\lambda \times E_\lambda \setminus \{(x, y) : x = y\} \to \mathbb{R} \) are defined as

\[
K_\lambda(x, y) = \frac{\Omega_\lambda((x - y)/|x - y|)}{h_\lambda(|x - y|)},
\]

where the functions \( \Omega_\lambda : S^1 \to S^1 \) and \( h_\lambda : (0, \infty) \to \mathbb{R} \) are described rigorously in Section 2 of [A]. Indicatively, both of these should be \( C^\infty \), \( h_\lambda(r) \approx r^{d_\lambda} \) for \( r \in (0, 1] \), and \( K_\lambda(x, y) \) should generate heavy cancelations on \( E_\lambda \). Roughly speaking this means that if \( x \in S^\alpha_\lambda \), \( y \in S^\beta_\lambda \) and \( z \in S^\gamma_\lambda \), where \( \alpha, \beta, \gamma \in I^m, m > 1 \), and they only differ in their last digit then

\[
K_\lambda(x, y) + K_\lambda(z, y) = 0,
\]

see Figure 1. It is rather easy to see that the kernels \( K_\lambda \) are Calderón-Zygmund standard, odd, smooth and \( d_\lambda \) homogeneous. The main theorem of [A] reads as follows.

**Theorem 2.1 ([A], Theorem 3.1).** For all \( \lambda \in (0, 1/3) \) the maximal singular integral operators \( T_\lambda^* \),

\[
T_\lambda^*(f)(x) = \sup_{\varepsilon > 0} \left| \int_{|x-y| > \varepsilon} K_\lambda(x, y) f(y) \, d\mu_\lambda y \right|,
\]

are bounded in \( L^2(\mu_\lambda) \).

We use the \( T(1) \)-theorem of David and Journé, proved in [DJ], applied to our setting. The idea was originally used in [D5]. Consider the sequence of singular integral operators, \( \{T_\lambda^n\}_{n \in \mathbb{N}} \),

\[
T_\lambda^n(f)(x) = \int_{|x-y| > \lambda^n} K_\lambda(x, y) f(y) \, d\mu_\lambda y.
\]

Due to strong symmetry properties of \( E_\lambda \), and the form of the kernels \( K_\lambda \), it turns out that for all \( n \in \mathbb{N}, T_\lambda^n(1) = 0 \). Applying the \( T(1) \) theorem to every member of

\[
E_\lambda
\]

\[
S_\alpha^\lambda \cap S_\beta^\lambda \cap S_\gamma^\lambda
\]

**Figure 1.** The kernels \( K_\lambda \) generate heavy cancelations on the sets \( E_\lambda \). For triplets of points like in the figure, \( K_\lambda(x, y) + K_\lambda(z, y) = 0 \).
the sequence, we derive that the operators $T^n_\lambda$ are bounded in $L^2(\mu_\lambda)$ with bounds not depending on $n$. Hence we can extract some linear $L^2(\mu_\lambda)$-bounded operator $T$ as a weak limit of some subsequence $\{T^n_\lambda\}_{n \in \mathbb{N}}$, and finally making use of a certain variant of Cotlar’s inequality found in [D3] we deduce that $T^n_\lambda$ is bounded in $L^2(\mu_\lambda)$.

In the setting of [A], $L^2$-boundedness does not imply almost everywhere existence of principal values.

**Theorem 2.2** ([A], Theorem 4.1). Let $\lambda \in (0, 1/3)$. For $\mu_\lambda$ almost every point in $E_\lambda$ the principal values of the singular integral operator $T_\lambda$ do not exist.

The proof is based on the fact that for every $x \in E_\lambda$, there exist sequences of annuli $A_n(x, r_n, R_n)$ such that $A_n(x, r_n, R_n) \cap E_\lambda = S^\lambda_{a_n}$ and $\text{diam}(S^\lambda_{a_n}) \approx r_n$. Figure A of [A], illustrates the simplicity of the argument.

### 3. DIRECTED POROSITY ON CIFS AND SINGULAR INTEGRALS

A set $E \subset \mathbb{R}^n$ is called porous if there exists a constant $c > 0$ so that for each $x \in E$ and $0 < r < \text{diam}(E)$ there exists $y \in B(x, r)$ satisfying

$$B(y, cr) \subset B(x, r) \setminus E.$$ 

Porosity related questions arise naturally in fractal geometry and dynamics, see e.g. [JJM], [PRo], [PU] and [U]. This can be understood heuristically since many familiar self similar sets in $\mathbb{R}^n$ are constructed by removing pieces out of some $n$-dimensional set in every step of the iteration process. The theory of conformal iterated function systems, as developed by Mauldin and Urbański in [MU], extends previous results, allowing one to analyze a broad range of limit sets.

We now proceed and briefly describe the setting of CIFS, more information can be found in [MU]. Let $I$ be a finite set with at least two elements and let

$$I^* = \bigcup_{m \geq 1} I^m \text{ and } I^\infty = I^\mathbb{N}.$$ 

If $w = (i_1, i_2, \ldots) \in I^* \cup I^\infty$ and $n \in \mathbb{N}, n \geq 1$, does not exceed $|w|$, the length of $w$, we denote $w|_n = (i_1, \ldots, i_n)$.

Let $\Omega$ be some open, bounded and connected subset of $\mathbb{R}^n$ and consider a family of conformal, injective maps $\{\varphi_i\}_{i \in I}, \varphi_i : \Omega \to \Omega$, such that for every $i \in I$ there exists some $0 < s_i < 1$ such that

$$|\varphi_i(x) - \varphi_i(y)| \leq s_i |x - y|.$$ 

The mappings $\{\varphi_i\}$ are conformal in the sense that $|\varphi'_i|^n = |J\varphi_i|$, where $J$ is the Jacobian and the norm in the left side is the usual “sup-norm” for linear mappings. Assume also that there exists a compact set $X \subset \Omega$ such that $\text{int}(X) \neq \emptyset$ with the property that $\varphi_i(X) \subset X$ for all $i \in I$. The **open set condition** holds for $\{\varphi_i\}_{i \in I}$ if there exists a non-empty open set $U \subset X$ (in the relative $X$-topology) such that $\varphi_i(U) \subset U$ for every $i \in I$ and $\varphi_i(U) \cap \varphi_j(U) \neq \emptyset$ for every pair $i, j \in I$. We will call a family of functions, as described above, a **conformal iterated function**
system (CIFS) if it satisfies the open set condition. For \( w = (i_1, \ldots, i_m) \in I^m \), denote \( \varphi_w = \varphi_{i_1} \circ \cdots \circ \varphi_{i_m} \) and notice that
\[
\text{diam}(\varphi_w(X)) \leq s^m d(X).
\]
As usual the \textit{limit set} of the CIFS is defined as,
\[
E = \bigcup_{w \in I^\infty} \bigcap_{m \geq 1} \varphi_{w|m}(X).
\]
The following two properties of finite CIFS satisfying the open set condition are essential for the proof of Theorem 3.2.

1. \textit{Bounded distortion property:} There exists some \( K \geq 1 \) such that
   \[
   |\varphi'_w(x)| \leq K |\varphi'_w(y)| \text{ for } w \in I^* \text{ and } x, y \in \Omega.
   \]

2. \textit{Finite clustering property:} There exist some positive number \( N \in \mathbb{N} \) and some constant \( C > 0 \) such that for every \( x \in \mathbb{R}^n \) and every \( r > 0 \) there exists some \( I(x, r) \subset I^* \) such that
   \[
   \begin{align*}
   & (a) \text{ card}(I(x, r)) \leq N, \text{ where card}(\cdot) \text{ denotes cardinality,} \\
   & (b) Cr \leq \text{diam}(\varphi_w(E)) \leq r \text{ for } w \in I(x, r), \\
   & (c) E \cap B(x, r) \subset \bigcup_{w \in I(x, r)} \varphi_w(E).
   \end{align*}
   \]

The constants depend only on the initial parameters of the CIFS.

In [U], Urbański gave necessary and sufficient conditions for the limit set of a CIFS on \( \mathbb{R}^n \) to be porous. As a consequence if the CIFS is finite and its limit set has Hausdorff dimension less than \( n \), it is also porous. One of the objectives in [B], is to further investigate porosity in conformal iterated function systems. In this direction we introduce the notion of directed porous sets. For \( m \in \mathbb{N}, 0 < m < n \), we denote by \( G(n, m) \) the set of all \( m \)-dimensional planes in \( \mathbb{R}^n \) crossing the origin.

\textbf{Definition 3.1.} Suppose \( V \in G(n, m) \). A set \( E \subset \mathbb{R}^n \) will be called \( V \)-\textit{directed porous at} \( x \in E \), if there exists a constant \( c_x > 0 \), such that for all \( r > 0 \) we can find \( y \in V + x \) satisfying
\[
B(y, c_x r) \subset B(x, r) \setminus E.
\]
If \( E \) is \( V \)-directed porous at every \( x \in E \), and \( c(V) = \inf\{c_x : x \in E\} > 0 \), it will be called \( V \)-\textit{directed porous}.

The motivation for this definition, stems from observing simple well known self similar sets, like the 1-dimensional Sierpinski gasket in the plane. Although it was already known that such sets are porous, in many cases this does not seem to convey enough information about their geometry. Intuitively one expects that various CIFS’s limit sets satisfy stronger porosity conditions, as they seem to contain holes spread in many directions, see Figure 2. In [B] we show,

\textbf{Theorem 3.2} ([B], Theorem 1.2). \textit{Let} \( E \subset \mathbb{R}^n \) \textit{be the limit set of a given finite CIFS. If} \( E \) \textit{is} \( m \)-\textit{purely unrectifiable then it is} \( V \)-\textit{directed porous for all} \( V \in G(n, m) \).
Combining Theorem 3.2 with Käenmäki’s rigidity result from [K], we obtain the following corollary.

**Corollary 3.3 ([B], Corollary 1.3).** Let $E \subset \mathbb{R}^n$ be the limit set of a given finite CIFS. If $\dim_H E \leq m$ then $E$ is $V$-directed porous at every $x \in E$ for all, except at most one, $V \in G(n, m)$.

In the following theorem we relate directed porosity with weak convergence of singular integral operators. Observe that we essentially assume minimal assumptions for the kernels, not even requiring continuity. By $X_Q(\mathbb{R}^n)$ and $X_B(\mathbb{R}^n)$ we denote respectively the function spaces of all finite linear combinations of characteristic functions of cubes on $\mathbb{R}^n$, with their sides parallel to the axis, and of all finite linear combinations of characteristic functions of balls. Notice that both of these test spaces are dense in $L^2(\mu)$.

**Theorem 3.4 ([B], Theorem 1.4).** Let $\mu$ be a finite Radon measure on $\mathbb{R}^n, n \geq 2$, satisfying

$$\mu(B(x, r)) \leq Cr^{n-1} \text{ for all } x \in \text{spt } \mu \text{ and } r > 0.$$ 

Let $K : \mathbb{R}^n \setminus \{0\} \to \mathbb{R}$ be a $\mu$-measurable antisymmetric kernel, satisfying for all $x \in \mathbb{R}^n \setminus \{0\}$,

$$|K(x)| \leq C_K |x|^{-(n-1)},$$

where $C_K$ is a constant depending on the kernel $K$.

1. If $\text{spt } \mu$ is $V^i$-directed porous for $i = 1, \ldots, n$, where $V^i = \{x \in \mathbb{R}^n : x^i = 0\}$ are the usual coordinate planes of $\mathbb{R}^n$, the truncated singular integral operators $T_{\mu,K}^\varepsilon$ converge weakly in $X_Q(\mathbb{R}^n)$, i.e., the limits

$$\lim_{\varepsilon \to 0} \int T_{\mu,K}^\varepsilon(f)(x)g(x)d\mu$$

exist for $f, g \in X_Q(\mathbb{R}^n)$.
(2) If $\text{spt}\mu$ is $V$-directed porous for all $V \in G(n, n-1)$, the truncated singular integral operators $T_{\mu,K}^\varepsilon$ converge weakly also in $\mathcal{X}_B(\mathbb{R}^n)$.

As usual, $T_{\mu,K}^\varepsilon(f)(x) = \int_{|x-y|>\varepsilon} K(x-y)f(y)d\mu y$. Combining Theorems 3.2 and 3.4 we obtain,

**Corollary 3.5** ([B], Corollary 1.5). Let $E \subset \mathbb{R}^n, n \geq 2$, be a $(n-1)$-purely unrectifiable limit set of a given finite CIFS. If $\mu = \mathcal{H}^{n-1}|E$ and $K : \mathbb{R}^n \setminus \{0\} \rightarrow \mathbb{R}$ is a kernel as in Theorem 3.4, the limits

$$\lim_{\varepsilon \to 0} \int T_{\mu,K}^\varepsilon(f)(x)g(x)d\mu$$

exist for $f, g \in \mathcal{X}_Q(\mathbb{R}^n)$ and $f, g \in \mathcal{X}_B(\mathbb{R}^n)$.

4. **Boundedness of singular integrals of measures separated by Lipschitz maps**

It is well known that even with very nice kernels the boundedness of $T^*_\mu : L^2(\mu) \rightarrow L^2(\mu)$ requires strong regularity properties of the measure $\mu$. In [DS2] David and Semmes introduced a quantitative notion of rectifiability, the so called uniform $m$-rectifiability, for $0 < m \leq n$, in order to investigate for which $m$-dimensional measures on $\mathbb{R}^n$ the natural ($m$-dimensional) Calderón-Zygmund kernels define operators bounded in $L^2(\mu)$. For $m = 1$, an 1-AD regular measure is uniformly rectifiable if its support is contained in an AD curve. The somewhat complicated general definition, as well as many equivalent formulations, can be found in [DS2]. David in [D2] showed that if $\mu$ is uniformly $m$-rectifiable, any antisymmetric $C^\infty$ kernel $K : \mathbb{R}^n \setminus \{0\} \rightarrow \mathbb{R}$ satisfying

$$|\nabla^j K(x)| \leq |x|^{-m-j} \text{ for all } x \in \mathbb{R}^n \setminus \{0\} \text{ and } j = 0, 1, 2, .. \quad (4.1)$$
defines a bounded operator in $L^2(\mu)$. On the other hand, David and Semmes in [DS1] proved that, if $\mu$ is $m$-AD regular and all antisymmetric kernels satisfying (4.1) define bounded operators on $L^2(\mu)$, then $\mu$ has to be uniform $m$-rectifiable.

In contrast with the situation described in the previous paragraph, in [C] we prove that when $\mu$ and $\nu$ are two measures whose supports are separated by a $(n-1)$-dimensional Lipschitz graph the operators $T^*_\nu : L^2(\nu) \rightarrow L^2(\mu)$ are bounded under very mild assumptions for the measures. It is noteworthy that our results hold for a large variety of fractal measures. The setting in [C] is determined by the following two definitions.

**Definition 4.1.** The class $\Delta$ will contain all finite Radon measures $\mu$ on $\mathbb{R}^n$ such that

$$\mu(B(x,r)) \leq C_\mu r^{n-1} \text{ for } x \in \mathbb{R}^n \text{ and } r > 0, \quad (4.2)$$

where $C_\mu$ is some constant depending on $\mu$.

**Definition 4.2.** The class $\mathcal{K}$ will contain all continuously differentiable kernels $K : \mathbb{R}^n \setminus \{0\} \rightarrow \mathbb{R}$ satisfying for all $x \in \mathbb{R}^n \setminus \{0\}$,
(1) $K(-x) = -K(x)$ (Antisymmetry),
(2) $|K(x)| \leq C^K_0 |x|^{-(n-1)}$,
(3) $|\nabla K(x)| \leq C^K_1 |x|^{-n},$

where the constants $C^K_0$ and $C^K_1$ depend on $K$.

The classes $\mathcal{K}$ and $\Delta$ have been studied widely, see e.g. [D3], and they are quite broad, consisting of both regular and irregular cases. The condition (4.2) is not particularly restrictive for the geometry of the measures. In the sense that it is satisfied by measures supported on $(n-1)$-dimensional planes and Lipschitz graphs, as well by many purely $(n-1)$-unrectifiable measures. On the other hand the class $\mathcal{K}$ contains both standard well-known kernels, as the Riesz kernels $|x|^{-n}x_i, x \in \mathbb{R}^n, i = 1, \ldots, n$, and stranger examples like the ones considered in [D5].

Before stating the results of [C] we lay down some basic notation. We denote the graph of a given function $f: \mathbb{R}^{n-1} \to \mathbb{R}$ by
$C_f = \{(x, f(x)) : x \in \mathbb{R}^{n-1}\}$
and the corresponding half spaces by
$H_f^+ = \{(x, y) : x \in \mathbb{R}^{n-1}, y > f(x)\}$ and $H_f^- = \{(x, y) : x \in \mathbb{R}^{n-1}, y < f(x)\}$.

The following theorem is the main result of [C].

**Theorem 4.3** ([C], Theorem 1.5). Let $f: \mathbb{R}^{n-1} \to \mathbb{R}$ be some Lipschitz function and $\mu$ and $\nu$ measures in $\mathbb{R}^n$ such that
(1) $\mu(H_f^-) = \nu(H_f^+) = 0$,
(2) $\mu, \nu \in \Delta$.

There exist constants $C_p, 1 \leq p < \infty$, depending only on $p, n, C_\mu, C_\nu$ and Lip$(f)$ such that for all $g \in L^1(\nu)$,
$$
\int (T_{*K}^* g)^p d\mu \leq C_p \int |g|^p d\nu \quad \text{for } 1 < p < \infty,
$$
and
$$
\mu(\{x \in \mathbb{R}^n : T_{*K}^* g(x) > t\}) \leq \frac{C_1}{t} \int |g| d\nu \quad \text{for } t > 0.
$$
for every $K \in \mathcal{K}$.

We apply Theorem 4.3 to obtain some weak convergence results related with the ones proved in [B]. The first, auxiliary one, reads as follows

**Theorem 4.4** ([C], Theorem 1.7). Let $\mu \in \Delta$ and $K \in \mathcal{K}$. Then for any Lipschitz function $f: \mathbb{R}^{n-1} \to \mathbb{R}$ the limit
$$
\lim_{\varepsilon \to 0} \int_{\mathbb{R}^n \setminus H_f^-} \int_{H_f^-} K(x - y)d\mu(x)d\mu(y)\varepsilon
$$
exists and it is finite.

The previous theorem serves as the main tool in showing that,
Theorem 4.5 ([C], Theorem 1.8). If $\mu \in \Delta$ and $K \in \mathcal{K}$, the finite limit

$$\lim_{\varepsilon \to 0} \int T_{\mu,K}(f)(x)g(x)d\mu x$$

exists for $f, g \in \mathcal{X}_B(\mathbb{R}^n)$ and $f, g \in \mathcal{X}_Q(\mathbb{R}^n)$.

References


