

HOMOGENEOUS MODEL THEORY OF METRIC STRUCTURES

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Academic dissertation

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1. INTRODUCTION

In this Ph.D. thesis we study homogeneous classes of complete metric structures. Today there are two important directions for generalizing model theory, abstract elementary classes (AEC) and metric model theory. We take a step in the direction of combining these two by introducing an AEC-like setting for studying metric structures. We then work in a homogeneous context, thus extending the usual compact approach to metric model theory, and prove a categoricity transfer theorem. We also present a way to treat generalized isomorphisms and develop a notion of independence 'up to perturbation'.

The thesis consists of the following two independent papers, both of which are joint work with Tapani Hyttinen:

- I Å. Hirvonen and T. Hyttinen, *Categoricity in homogeneous complete metric spaces*, *Archive for Mathematical Logic*, 48 (2009), pp. 269–322.
- II Å. Hirvonen and T. Hyttinen, *Metric abstract elementary classes with perturbations*. Submitted.

The metric structures we study are many-sorted structures, each sort of which is a complete metric space, and with one sort an isomorphic copy of the reals \mathbb{R} . The metrics are then part of the vocabulary as functions with values in \mathbb{R} . There may also be other functions, relations and constants in the structures. Note that we do not need to assume the spaces to be bounded.

We say that $A' \subseteq A$ is *dense* in A if for every $\varepsilon > 0$ and $a \in A$ there is some $a' \in A'$ with $d(a', a) < \varepsilon$, where d is the metric of the sort of a (and a'). The *density character* of a set A is defined as the smallest cardinality of a dense subset of A . Since we consider only complete metric spaces, density character is a more natural notion of size than cardinality. Hence we use the notation $|A|$ to denote the density character of A . Also categoricity is measured in terms of densities. Hence a class is λ -categorical if all its structures of density character λ are isomorphic. Separably categorical means ω -categorical.

By isomorphisms we mean isometric isomorphisms. Adding generalized isomorphisms in Paper II allows us to treat also the functional analytic isomorphisms, which are linear homeomorphisms. To bring out the difference between these two notions of isomorphism we will call the functional analytic ones linear isomorphisms.

2. METRIC MODEL THEORY

Several approaches to the model theory of metric structures have been developed over the past decades. In the 1970s Henson studied Banach spaces and their nonstandard hulls using first-order languages [13, 14] and developed the approach of *positive bounded formulas* and approximative semantics [15]. The framework is designed for the study of normed spaces and structures derived from these. The atomic formulas are of the form

$$t \leq r, \quad t \geq r,$$

where t is a term and r a rational number. The positive bounded formulas are then built from these by positive Boolean combinations and bounded quantification

$$\exists x(\|x\| \leq r \wedge \varphi), \quad \forall x(\|x\| \leq r \rightarrow \varphi).$$

Approximations of the formulas are constructed by relaxing the bounds both in the atomic formulas and the quantifications. Finally a formula is said to be *approximately satisfied* if all its approximations are satisfied. The framework was extensively investigated by Henson and Iovino in [16].

With the positive bounded framework as a working ground Iovino developed stability theory for Banach space structures [18], defining stability based on the density character of the type space, developing a notion of forking and proving a stability spectrum result. The general definition of stability allows for any uniform structure on the type space satisfying some requirements, but Iovino studied mainly two: the infimum metric \mathbf{d} defined as the infimum of distances of realizations of the types, and the Banach-Mazur metric based on the linear homeomorphisms which in functional analysis are called isomorphisms.

Another approach to the model theory of metric structures is Ben Yaacov's *compact abstract theories* or *CATs* [2]. The general case was developed to study type spaces which are not totally disconnected, hence with the set of basic formulas being closed only under *positive* boolean combinations. Ben Yaacov defines CATs in three ways: as positive Robinson theories, as compact type-space functors, and as compact elementary categories.

The approach Ben Yaacov uses for metric structures is that of *Hausdorff CATs*, i.e. CATs with Hausdorff type spaces, with countable language. As an abstract approach this avoids the question of 'correct language' and thus of metric formula twisting, although Ben Yaacov does have formulas in his approach. Also, Ben Yaacov [3] proves that Hausdorff CATs admit definable metrics which are unique up to uniform equivalence. Hausdorff CATs closely resemble Shelah's Kind II together with Assumption III in

[20]. It also turned out that the frameworks of positive bounded formulas and that of Hausdorff CATs are equivalent.

In [22] Shelah and Usvyatsov use the approach of *monster metric spaces*. Here they use a set of formulas closed under conjunction, existential quantification and subformulas, and satisfying assumptions concerning the interplay between formulas and the metric and giving a local approximation of negation. The general framework considers a large homogeneous monster structure, but the main results of [22] are carried out using compactness. The approach of compact monster metric spaces is equivalent to Ben Yaacov's (non-Hausdorff) CATs.

The latest and at the moment most widespread approach to metric model theory is continuous logic. It is based on Chang's and Keisler's work from the 1960s [10] and was introduced by Ben Yaacov and Usvyatsov [7] and studied extensively in [6]. It is a variant of Chang's and Keisler's logic, restricting the truth values to the interval $[0, 1]$ instead of any compact Hausdorff space, and the quantifiers to sup and inf. This gives a very natural generalization of classical first order logic which at the same time aims at being easily readable for analysts. In [7] it is also shown that continuous formulas give rise to a good analogue of first order local stability theory. Finally, the approach turns out to be equivalent to positive bounded theories and Hausdorff CATs.

Berenstein and Buechler [9] have investigated homogeneous metric structures. However, their work only considers expansions of Hilbert spaces and not general metric spaces. They characterize independence in Hilbert spaces expanded by, on the one hand, pointwise multiplication, on the other hand, self-adjoint commuting operators.

In our approach we have solved the question of correct language by omitting formulas altogether and working in a setting resembling Shelah's *abstract elementary classes (AEC)*, introduced in [21]. They stand for the most general approach for the study of non-elementary classes and (equipped with various additional assumptions) are the subject of vivid research. Our version of these classes are the *metric abstract elementary classes (MAEC)* where the differences arise from the studied structures being complete metric spaces instead of discrete ones. Therefore, completions need to be taken when taking unions of chains and the size of sets are measured by density characters instead of cardinalities.

Definition 1. We call a class $(\mathbb{K}, \preceq_{\mathbb{K}})$ of τ -structures for some fixed vocabulary τ a *metric abstract elementary class, MAEC*, if the following hold:

- (1) Both \mathbb{K} and the binary relation $\preceq_{\mathbb{K}}$ are closed under isomorphism.
- (2) If $\mathcal{A} \preceq_{\mathbb{K}} \mathcal{B}$ then \mathcal{A} is a substructure of \mathcal{B} (i.e. each sort of \mathcal{A} is a substructure of the corresponding sort of \mathcal{B}).
- (3) $\preceq_{\mathbb{K}}$ is a partial order on \mathbb{K} .
- (4) If $(\mathcal{A}_i)_{i < \delta}$ is a $\preceq_{\mathbb{K}}$ -increasing chain then there is a model $\overline{\bigcup_{i < \delta} \mathcal{A}_i}$, unique up to the choice of completion such that
 - (a) $\overline{\bigcup_{i < \delta} \mathcal{A}_i} \in \mathbb{K}$,

(b) for each $j < \delta$, $\mathcal{A}_j \preceq_{\mathbb{K}} \overline{\bigcup_{i < \delta} \mathcal{A}_i}$.

In addition if each $\mathcal{A}_i \preceq_{\mathbb{K}} \mathcal{B} \in \mathbb{K}$ and the completion is the metric closure in \mathcal{B} then $\overline{\bigcup_{i < \delta} \mathcal{A}_i} \preceq_{\mathbb{K}} \mathcal{B}$.

(5) If $\mathcal{A}, \mathcal{B}, \mathcal{C} \in \mathbb{K}$, $\mathcal{A} \preceq_{\mathbb{K}} \mathcal{C}$, $\mathcal{B} \preceq_{\mathbb{K}} \mathcal{C}$ and $\mathcal{A} \subseteq \mathcal{B}$ then $\mathcal{A} \preceq_{\mathbb{K}} \mathcal{B}$.

(6) There exists a Löwenheim-Skolem number $\text{LS}^d(\mathbb{K})$ such that if $\mathcal{A} \in \mathbb{K}$ and $A \subset \mathcal{A}$ then there is $B \supseteq A$ such that $|B| = |A| + \text{LS}^d(\mathbb{K})$ and $\overline{B} \preceq_{\mathbb{K}} \mathcal{A}$.

In applying an AEC-like setting we are not aiming at the full generality of AECs but do assume amalgamation, joint embedding and homogeneity. We *are* however working in a more general setting than the ones described above, since we do not assume compactness.

We have two motivations for working in a syntax-free environment. One is convenience. When the work on Paper I was started, continuous logic was not yet established as the main working ground for metric structures, and any choice of syntax would have felt somewhat arbitrary. By working without a syntax we ensured that the results would be applicable as widely as possible. A more important reason still is the handling of generalized isomorphisms. In Paper II we add to the MAECs classes of functions which we consider isomorphisms. Working in an abstract setting enables adding various classes of functions and treating them as isomorphisms in a way no syntactic approach seems to allow.

3. GENERALIZING ISOMORPHISMS

In model theory an isomorphism is a bijection preserving the structure of the model. Hence, everything that can be said (within the used vocabulary) in one structure is true in its isomorphic image. Since the norm is part of the vocabulary in a Banach space structure this clashes with what analysts call isomorphisms. To a logician an isomorphism should preserve the norm, but functional analysts think of linear homeomorphisms when they talk about isomorphisms.

The functional analytic notion of isomorphism was studied by Heinrich and Henson [12] in the 1980s when they characterized the existence of linearly isomorphic ultrapowers of two Banach spaces in terms of a variant of elementary equivalence closely tied to the approximations of positive bounded formulas. However, no stability theory was developed using these linear isomorphisms.

A recent development in metric model theory is the introduction of *perturbations*. These were introduced by Ben Yaacov [1, 4] as a means of handling small changes in parts of the structures. The need for these arose from a result by Chatzidakis and Pillay [11] showing that if a first order theory T is superstable and $T \cup \{\sigma \text{ is an automorphism}\}$ has a model companion T_A , then every completion of T_A is supersimple. It turned out [5] that with the metric versions of stability and superstability the result was not true for continuous first order theories i.e. theories in continuous logic:

Although the continuous theory APr of probability algebras is \aleph_0 -stable and $APr \cup \{\sigma \text{ is an automorphism}\}$ admits a model companion APr_A , Ben Yaacov showed [5] that probability algebras with a generic automorphism are not superstable and hence not supersimple as a theory is superstable if and only if it is stable and supersimple. However, allowing for arbitrarily small changes of the automorphism makes APr_A \aleph_0 -stable, i.e. APr_A is \aleph_0 -stable *up to perturbations*.

The addition of perturbations also allows for a Ryll-Nardzewski style characterization of continuous theories separably categorical up to perturbations [1], generalizing a result by Henson on pure Banach spaces separably categorical up to almost isometry.

Perturbations are defined via perturbation metrics on spaces of types as functions mapping types within a given distance. Thus, they need not be elementary, but are required to respect equality, existential quantification and be invariant under permutations. This is often combined with the standard metric on types allowing for perturbations to both perturb structure and move realizations.

In [4] Ben Yaacov gives an abstract treatment of spaces equipped with both a topology and a metric. This generalizes the treatment of the type spaces of metric structures with, on the one hand, the logic topology, on the other hand, the metric given by the infimum distance \mathbf{d} of realizations of types in a monster model. Ben Yaacov also proves a list of properties of perturbations [4, Theorem 4.4].

We have chosen an abstract approach to generalized isomorphisms of metric structures, adding as part of the class a class of isomorphisms satisfying axioms resembling Ben Yaacov's list in [4, Theorem 4.4]. This gives a natural treatment of perturbations as built into the framework and enables us to define a notion of independence 'up to perturbation'.

4. CATEGORICITY

Morley's theorem from 1965, answering a conjecture by Łos, is often regarded as the beginning of modern model theory. Morley [19] proved that a countable first order theory is categorical in some uncountable power if and only if it is categorical in all uncountable cardinalities. The theorem has been generalized in various directions, both to uncountable languages and to various non-elementary classes. Within the research on abstract elementary classes many results concern what assumptions are needed for various categoricity transfer results. Generalizing the theorem is a good testbed for new model theoretic approaches.

In metric model theory, categoricity has to be considered with respect to density characters to make sense, and correspondingly only complete structures are considered. In the metric case, Ben Yaacov [3] has proved that a countable Hausdorff CAT is categorical in some uncountable cardinal if and only if it is categorical in every uncountable cardinal. Since the framework of Hausdorff CATs is equivalent to that of continuous first order

logic, Ben Yaacov's result covers the metric version of the countable first order case.

Shelah and Usvyatsov [22] prove Morley's theorem for compact monster metric spaces. In fact, they start with an even weaker assumption, assuming, instead of uncountable categoricity, *weak uncountable categoricity*, which states that for each $\varepsilon > 0$ there exists a cardinal λ such that for any models $\mathcal{M}_1, \mathcal{M}_2$ of density λ and all $\delta > \varepsilon$ there is a bijection $f : M_1 \rightarrow M_2$ preserving a sort of δ -approximations of formulas.

5. I CATEGORICITY IN HOMOGENEOUS COMPLETE METRIC SPACES

In the first paper we introduce the concept of *metric abstract elementary class* (see Definition 1) as a framework for studying metric space structures. With the abstract approach we do not need to bound the structures as is usually done in continuous logic [6], but can directly regard unbounded metric structures. We consider as a metric structure a many-sorted structure where the sorts are complete metric spaces, one of them being an isometric copy of the ordered field of real numbers \mathbb{R} . The metrics of the spaces are part of the signature as well as possibly other constant, relation and function symbols. We assume the signature and the number of sorts to be at most countable.

We assume that the class we are investigating forms a MAEC (Definition 1) with countable Löwenheim-Skolem number (where size is measured by density character). We also assume joint embedding and amalgamation, which gives us a well-defined notion of type, the Galois type. Further, we assume the existence of arbitrarily large models, homogeneity and a property which we call the perturbation property. This states that the limit element of a convergent sequence of elements of the same type has the same type as the elements of the sequence. The purpose of it is to replace the Perturbation Lemma of [16] stating that if a tuple a satisfies a positive bounded formula then tuples close enough to a will satisfy given approximations of the formula.

We use the standard infimum metric on the spaces of types. As usual in metric model theory, stability is then defined by measuring the density of the space of types over sets of a given size. Using a modification of the standard Ehrenfeucht-Mostowski technique we prove that κ -categoricity in some cardinality $\kappa = \kappa^{\aleph_0} > \aleph_1$ (with respect to densities) implies \aleph_0 -stability. Working in the \aleph_0 -stable case we develop a notion of independence based on ε -splitting and prove that over \aleph_1 -saturated models this notion coincides with the independence notion from homogeneous model theory developed by Hyttinen and Shelah [17]. Thus we can use results from [17]. We also develop a notion of ε -isolation, build primary models and prove dominance for these. Our main result in this homogeneous setting is (Theorem 8.6. of Paper I):

Theorem 2. *Assume \mathbb{K} is κ -categorical for some $\kappa = \kappa^{\aleph_0} > \aleph_1$. Then there is $\xi < \beth_{(2^{\aleph_0})^+}$ such that \mathbb{K} is categorical in all λ satisfying*

- (1) $\lambda \geq \min\{\xi, \kappa\}$,
- (2) $\lambda^{\aleph_0} = \lambda$,
- (3) for all $\zeta < \lambda$, $\zeta^{\aleph_0} < \lambda$.

The conditions (ii) and (iii) as well as the additional demands on κ arise from our difficulties to prove stability transfer. Without further assumptions we only manage to prove that \aleph_0 -stable classes are stable in all λ which satisfy $\lambda^{\aleph_0} = \lambda$. In order to improve the result we introduce one more assumption, that of *metric homogeneity*:

Definition 3. \mathbb{K} is *metricly homogeneous* if for all $\varepsilon > 0$ there is some $\delta > 0$ such that for all finite $a, b \in \mathfrak{M}$ and all sets A , if

$$\mathbf{d}(t^g(a/A), t^g(b/A)) > \varepsilon$$

then there is a finite $A' \subset A$ such that

$$\mathbf{d}(t^g(a/A'), t^g(b/A')) > \delta.$$

With this addition we prove (Corollary 10.32 of Paper I):

Theorem 4. *Suppose \mathbb{K} is metricly homogeneous and κ -categorical (with respect to densities) for some uncountable κ . Assume further that either $\kappa > \aleph_1$ or separable $F_\omega^{\mathfrak{M}}$ -saturated models exist. Then there exists some $\xi < \beth_{c^+}$ such that \mathbb{K} is categorical in all $\lambda \geq \min\{\kappa, \xi\}$.*

A natural example of a MAEC is the class of all Banach spaces. Paper I also contains an example class which is categorical in all infinite cardinalities. Within 'elementary' metric model theory (i.e. investigating classes of models of complete *continuous* first order theories) there are no functional analytic examples of categorical classes which are not based on Hilbert spaces. However, with an abstract approach we have more freedom to define classes. Our example consists of Banach spaces isometrically isomorphic to some $\ell^p(I)$ with I a well-order, for $1 \leq p < \infty$, $p \neq 2$. This class is clearly categorical in all cardinalities and satisfies metric homogeneity with $\delta = \varepsilon$.

6. II METRIC ABSTRACT ELEMENTARY CLASSES WITH PERTURBATIONS

In the second paper we widen the use of metric abstract elementary classes by allowing for generalized isomorphisms. This is done by adding to the MAEC classes \mathbb{F}_ε of so called ε -isomorphisms. We demand that the classes \mathbb{F}_ε are collections of bijective mappings between members of the MAEC \mathbb{K} satisfying:

- (a) $\mathbb{F}_\delta \subseteq \mathbb{F}_\varepsilon$ for $\delta < \varepsilon$, $\mathbb{F}_0 = \bigcap_{\varepsilon > 0} \mathbb{F}_\varepsilon$ and \mathbb{F}_0 is the collection of genuine isomorphisms between models in \mathbb{K} ,
- (b) if $f : \mathcal{A} \rightarrow \mathcal{B}$ and $f \in \mathbb{F}_\varepsilon$, then f is a e^ε -bi-Lipschitz mapping with respect to the metric, i.e. $e^{-\varepsilon}d(x, y) \leq d(f(x), f(y)) \leq e^\varepsilon d(x, y)$ for all $x, y \in \mathcal{A}$,
- (c) if $f \in \mathbb{F}_\varepsilon$ then $f^{-1} \in \mathbb{F}_\varepsilon$,
- (d) if $f \in \mathbb{F}_\varepsilon$, $g \in \mathbb{F}_\delta$ and $\text{dom}(g) = \text{rng}(f)$ then $g \circ f \in \mathbb{F}_{\varepsilon+\delta}$.

- (e) if $(f_i)_{i < \alpha}$ is an increasing chain of ε -isomorphisms, i.e. $f_i \in \mathbb{F}_\varepsilon$, $f_i : \mathcal{A}_i \rightarrow \mathcal{B}_i$, $\mathcal{A}_i \preceq \mathcal{A}_{i+1}$, $\mathcal{B}_i \preceq \mathcal{B}_{i+1}$ and $f_i \subseteq f_{i+1}$ for all $i < \alpha$, then there is an ε -isomorphism $f : \bigcup_{i < \alpha} \mathcal{A}_i \rightarrow \bigcup_{i < \alpha} \mathcal{B}_i$ such that $f \upharpoonright \mathcal{A}_i = f_i$ for all $i < \alpha$.

As in Paper I we assume countable Löwenheim-Skolem number, the existence of arbitrarily large models, joint embedding, amalgamation, homogeneity and a perturbation property, but both the amalgamation and perturbation properties assumed here are stronger than the ones in Paper I.

Amalgamation is not only assumed for ordinary isomorphisms (in this context called *0-isomorphisms*) but also for ε -isomorphisms:

Definition 5 (Amalgamation property). \mathbb{K} is said to have the *amalgamation property* if whenever $\mathcal{A}, \mathcal{B}, \mathcal{C} \in \mathbb{K}$, $\mathcal{A} \preceq \mathcal{B}$ and $f : \mathcal{A} \rightarrow \mathcal{C}$ is an ε -embedding then there are $\mathcal{B}', \mathcal{C}' \in \mathbb{K}$ with $\mathcal{B}' \succcurlyeq \mathcal{B}$, $\mathcal{C}' \succcurlyeq \mathcal{C}$ and an ε -embedding $g : \mathcal{B}' \rightarrow \mathcal{C}'$ with $g \supseteq f$.

Based on a construction by Kislyakov, it is shown in Paper I that the class of Banach spaces together with linear homeomorphisms satisfy an even stronger version of this amalgamation: If \mathcal{A}, \mathcal{B} and \mathcal{C} are Banach spaces, \mathcal{A} a closed subspace of \mathcal{C} and $f : \mathcal{A} \rightarrow \mathcal{B}$ a \mathcal{C} -embedding, i.e. a linear homeomorphism onto a closed subspace of \mathcal{B} with $\|f\| \leq C$, $\|f^{-1}\| \leq C$ then there are $\mathcal{B}' \succcurlyeq \mathcal{B}$ and a \mathcal{C} -embedding $g : \mathcal{A} \rightarrow \mathcal{B}'$ with $g \supseteq f$. However, this turned out to be too strong an assumption when dealing with perturbations of automorphisms on Hilbert spaces, thus giving rise to Definition 5. As a consequence, for $\varepsilon > 0$, ε -isomorphisms need not preserve submodels, i.e. if $f : \mathcal{A} \rightarrow \mathcal{B}$ is such that $f \in \mathbb{F}_\varepsilon$ and $\mathcal{A}' \preceq \mathcal{A}$ then it need not be the case that $f(\mathcal{A}') \preceq \mathcal{B}$.

Our notion of type is the standard Galois-type (w.r.t. 0-isomorphisms). The role of ε -mappings shows in the topology we define on the type space. Instead of using the standard infimum distance we define a *quasimetric* \mathbf{d}^p on pairs of Galois-types of finite tuples over the empty set.

Definition 6. For $a, b \in \mathfrak{M}$ and $\varepsilon > 0$ we write $\mathbf{d}^p(t^g(a/\emptyset), t^g(b/\emptyset)) \leq \varepsilon$ if there are ε -automorphisms f and g of \mathfrak{M} such that $d(f(a), b) \leq \varepsilon$ and $d(g(b), a) \leq \varepsilon$.

Our new perturbation property, stronger than the one used in Paper I, says that whenever a and b are finite tuples of the monster model \mathfrak{M} such that $\mathbf{d}^p(t^g(a/\emptyset), t^g(b/\emptyset)) = 0$, then $t^g(a/\emptyset) = t^g(b/\emptyset)$.

The properties mentioned above are assumed throughout Paper II. For constructions requiring better limit properties we often assume one further property, namely that of *complete type-spaces*. By the perturbation property and truncating \mathbf{d}^p at a given distance, \mathbf{d}^p becomes a quasimetric. Thus it makes sense to talk about \mathbf{d}^p -Cauchy sequences and we can define:

Definition 7. We say that \mathbb{K} *has complete type-spaces* if for all finite sets A \mathbf{d}^p -Cauchy sequences of types over A have a limit, i.e. if $(a_i)_{i < \omega}$ is a sequence with the property that for all $\varepsilon > 0$ there is $n_0 < \omega$ such that for

all $m, n \geq n_0$

$$\mathbf{d}^p(\mathfrak{t}^g(a_m A/\emptyset), \mathfrak{t}^g(a_n A/\emptyset)) < \varepsilon,$$

then there exists some a with the property that for all $\varepsilon > 0$ there is $n_0 < \omega$ such that for all $n > n_0$

$$\mathbf{d}^p(\mathfrak{t}^g(aA/\emptyset), \mathfrak{t}^g(a_n A/\emptyset)) < \varepsilon.$$

In some constructions the assumption of complete type-spaces can be replaced by the assumption of models being *good* (Definition 2.19 in Paper II). This roughly tells that limits of δ -embeddings are ε -embeddings.

We show how the example of Hilbert spaces with an automorphism investigated in [8] and [5] fit into this framework.

We develop a notion of independence based on splitting. Since perturbations are built into our framework this gives a notion of independence up to perturbation. We show (Theorem 6.6 in Paper II) that this notion of independence satisfies the usual axioms for an independence notion. As we work in a metric setting, the axiom of finite character is replaced by countable character.

Theorem 8. *If \mathbb{K} is ω - \mathbf{d}^p -stable and has complete type spaces then independence satisfies the following axioms:*

- (1) **Isomorphism invariance** *If $a \downarrow_A^s B$ and f is a 0-automorphism of \mathfrak{M} then $f(a) \downarrow_{f(A)}^s f(B)$.*
- (2) **Monotonicity** *If $A \subset B \subset C \subset D$ and $a \downarrow_A^s D$ then $a \downarrow_B^s C$.*
- (3) **Countable character of non-freeness** *If A is ω - \mathbf{d}^p -saturated and $a \not\downarrow_A^s B$ then there is a countable $A' \subset A$ and a finite $B' \subset B$ such that if $\mathfrak{t}^g(b/A'B') = \mathfrak{t}^g(a/A'B')$ then $b \not\downarrow_{A'}^s B'$.*
- (4) **Local character** *$a \downarrow_A^s A$ for all a and A , i.e. for every a , A and $\varepsilon > 0$ there is some finite $E \subset A$ such that $a \downarrow_E^\varepsilon A$.*
- (5) **Extension** *If $a \downarrow_A^s B$, B is ω - \mathbf{d}^p -saturated and $A \subseteq B \subset C$ then there is b with $\mathfrak{t}^g(b/B) = \mathfrak{t}^g(a/B)$ satisfying $b \downarrow_A^s C$.*
- (6) **Stationarity** *If A is ω - \mathbf{d}^p -saturated, $A \subset B$, $\mathfrak{t}^g(a/A) = \mathfrak{t}^g(b/A)$, $a \downarrow_A^s B$ and $b \downarrow_A^s B$, then $\mathfrak{t}^g(a/B) = \mathfrak{t}^g(b/B)$.*
- (7) **Transitivity** *If $A \subset B \subset C$, B is ω - \mathbf{d}^p -saturated, $a \downarrow_A^s B$ and $a \downarrow_B^s C$ then $a \downarrow_A^s C$.*
- (8) **Symmetry** *If A is ω - \mathbf{d}^p -saturated then $a \downarrow_A^s b$ if and only if $b \downarrow_A^s a$.*
- (9) **Reflexivity** *If A is ω - \mathbf{d}^p -saturated and metricly closed and $a \notin A$ then $a \not\downarrow_A^s a$.*

We also show (Theorem 6.12) that our notion of independence coincides with the independence notion developed by Hyttinen and Shelah in [17] over saturated enough models.

Theorem 9. *Assume \mathbb{K} is ω - \mathbf{d}^p -stable and has complete type-spaces. Then if $\mathcal{A} \subseteq B$, $a \in \mathfrak{M}$ and \mathcal{A} is 2^{\aleph_0} -saturated (i.e. realizes all Galois types over parameter sets of size $< 2^{\aleph_0}$) then*

$$a \downarrow_{\mathcal{A}}^s B \text{ if and only if } a \downarrow_{\mathcal{A}} B,$$

i.e. our notion of independence agrees with that in classical homogeneous model theory over 2^{\aleph_0} -saturated models.

We also develop an isolation notion and prove dominance for it (Theorem 7.9):

Theorem 10. *Assume \mathbb{K} is ω - \mathbf{d}^p -stable and has complete type-spaces. If A is ω - \mathbf{d}^p -saturated, $B \supseteq A$ and B^* is constructible over B then a $\downarrow_A^s B$ implies a $\downarrow_A^s B^*$.*

As an example of the use of dominance we can prove the following result (Lemma 7.11) which implies that the class of Hilbert spaces with an automorphism has the maximal number of models in density character \aleph_1 . The proof resembles the proof in first order logic showing that non-unidimensional theories are not categorical.

Proposition 11. *Suppose \mathbb{K} is ω - \mathbf{d}^p -stable, has complete type spaces and every complete ω - \mathbf{d}^p -saturated set is a model. Assume that there are ω - \mathbf{d}^p -saturated \mathcal{A} and $a_i \notin \mathcal{A}$, $i < \omega_1$, such that $a_i \downarrow_{\emptyset}^s \mathcal{A}$ and for $i < j$, $\text{t}^g(a_i/\emptyset)$ is orthogonal to $\text{t}^g(a_j/\emptyset)$. Then the number of models in density character ω_1 is 2^{ω_1} .*

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