Dirichlet problem at infinity for $p$-harmonic functions on negatively curved spaces

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List of included articles

This thesis consists of an introductory part and the following three articles:

[A] A. Vähäkangas:
   *Dirichlet problem at infinity for A-harmonic functions.*
   Potential Anal. 27 (2007), 27–44.

[B] I. Holopainen and A. Vähäkangas:
   *Asymptotic Dirichlet problem on negatively curved spaces.*
   Proceedings of the International Conference on Geometric Function Theory,
   Special Functions and Applications (ICGFT) (R. W. Barnard and S. Ponnusamy, eds.),

[C] I. Holopainen, U. Lang, and A. Vähäkangas:
   *Dirichlet problem at infinity on Gromov hyperbolic metric measure spaces.*

In the introductory part these articles are referred to as [A],[B],[C] and other references will be numbered as [1],[2],...
1 Overview

The object of this thesis is to study globally defined bounded $p$-harmonic functions on negatively curved manifolds and metric spaces. We construct such functions by solving the so called Dirichlet problem at infinity.

We give a short overview of the articles included in this dissertation. In [A] we consider the Dirichlet problem at infinity for $\mathcal{A}$-harmonic functions by developing further the proof method used by Cheng in [13]. In [B] we solve the Dirichlet problem at infinity for $p$-harmonic functions under very general curvature bounds. In [C] we consider the Dirichlet problem at infinity for $p$-harmonic functions on Gromov hyperbolic metric measure spaces and prove existence and uniqueness results in this context.

The rest of the summary is organized as follows. In Section 2 we define $\mathcal{A}$-harmonic functions and the Dirichlet problem at infinity. In Section 3 we present history of the Dirichlet problem at infinity on Cartan-Hadamard manifolds. Section 4 contains results from [A] and [B] and we discuss their relation to previously known results. In Section 5 we present our results from [C] and study the problem in the more general setting of metric measure spaces.

2 Global aspects in potential theory

In this section we define the key concepts related to this thesis. We suppose that $M$ is a connected Riemannian $n$-manifold without boundary and that $U \subset M$ is an open subset. We fix an exponent $p \in (1, \infty)$.

A function $u \in W_{\text{loc}}^{1,p}(U)$ is said to be $\mathcal{A}$-harmonic if it is a continuous weak solution to the quasilinear elliptic equation

\begin{equation}
- \text{div} \mathcal{A}(\nabla u) = 0
\end{equation}

in $U$. Here the operator $\mathcal{A}$ satisfies $\langle \mathcal{A}(v), v \rangle \approx |v|^p$ and is said to be of type $p$. See [A] for precise requirements of $\mathcal{A}$. A lower semicontinuous function $u : U \to \mathbb{R} \cup \{\infty\}$ is $\mathcal{A}$-superharmonic if for every domain $D \subset\subset U$ and each $\mathcal{A}$-harmonic $h \in C(\bar{D})$, $h \leq u$ on $\partial D$ implies $h \leq u$ in $D$.

The equation (1) is modelled after the $p$-Laplace equation

\[-\text{div}(|\nabla u|^{p-2}\nabla u) = 0\]

and in this case the terminology $p$-harmonic and $p$-superharmonic is used. The case $p = 2$ reduces to the Laplace equation and classical harmonic and superharmonic functions.

Properties of these $\mathcal{A}$-harmonic and $\mathcal{A}$-superharmonic functions have been extensively studied, especially in the Euclidean setting. This has given rise to nonlinear potential theory where $\mathcal{A}$-superharmonic functions play a role similar to that of superharmonic functions in the classical potential theory. Many results from the linear theory have natural generalizations in this nonlinear setting although their proofs often require new methods. The standard reference for nonlinear potential theory on Euclidean spaces is [19]. In the Riemannian setting one should see [20].

In the Riemannian setting studying the local properties of $\mathcal{A}$-harmonic functions can be reduced to studying such properties in the Euclidean setting via bilipschitz coordinate
charts. Namely, if \( u : U \rightarrow \mathbb{R} \) is an \( \mathcal{A} \)-harmonic function and \( \varphi : U \rightarrow \varphi U \) is a bilipschitz coordinate chart, then \( u \circ \varphi^{-1} : \varphi U \rightarrow \mathbb{R} \) is \( \mathcal{A}' \)-harmonic function on \( \varphi U \subset \mathbb{R}^n \) for a certain \( \mathcal{A}' \). However, the global properties of \( \mathcal{A} \)-harmonic and \( \mathcal{A} \)-superharmonic functions on Riemannian manifolds depend nontrivially on the geometry of the manifold and this is what we consider next.

2.1 Classification scheme

Riemannian manifolds can be classified according to whether they carry globally defined \( \mathcal{A} \)-harmonic functions of a given type. Here are four properties one can consider.

2 Definition. (i) We say that \( M \) is \( p \)-parabolic if every positive \( \mathcal{A} \)-superharmonic function on \( M \) is constant for all \( \mathcal{A} \) of type \( p \).

(ii) We say that \( M \) has strong \( p \)-Liouville property if every positive \( \mathcal{A} \)-harmonic function on \( M \) is constant for all \( \mathcal{A} \) of type \( p \).

(iii) We say that \( M \) has \( p \)-Liouville property if every bounded \( \mathcal{A} \)-harmonic function on \( M \) is constant for all \( \mathcal{A} \) of type \( p \).

(iv) We say that \( M \) has \( D_p \)-Liouville property if every bounded \( \mathcal{A} \)-harmonic function \( u \) on \( M \) with \( \int_M |\nabla u|^p < \infty \) is constant for all \( \mathcal{A} \) of type \( p \).

It is clear that these definitions obey

\[
(i) \implies (ii) \implies (iii) \implies (iv).
\]

The following gives us alternative definitions for \( p \)-parabolicity.

3 Theorem (cf. [20, Theorem 5.2]). The following are equivalent:

(a) \( M \) is \( p \)-parabolic.

(b) there does not exist a positive Green’s function on \( M \) for any \( \mathcal{A} \) of type \( p \).

(c) \( \text{cap}_p(K, M) = 0 \) for every compact set \( K \subset M \).

By the equivalence of (a) and (c), we can interpret \( p \)-parabolicity as certain kind of smallness of the manifold. For example, every compact manifold is \( p \)-parabolic.

In the classical setting related to the Laplacian, such Liouville properties have been extensively studied. We refer to the survey articles by Grigor’yan [18] and Li [26]. In the nonlinear case see [15], [21], and [22].

2.2 Cartan-Hadamard manifolds and Dirichlet problem at infinity

A Cartan-Hadamard manifold \( M \) is a simply connected complete Riemannian manifold with nonpositive sectional curvature. Such manifolds include the Euclidean space \( \mathbb{R}^n \) and the hyperbolic space \( \mathbb{H}^n \).

Cartan-Hadamard theorem states that the exponential map is a diffeomorphism at each point on \( M \). Hence the manifold has a very simple structure as a differentiable manifold. However, its geometry depends on the curvature in a nontrivial way. This makes studying various aspects of analysis interesting on Cartan-Hadamard manifolds. For example, \( \mathbb{R}^n \) is (strong) \( p \)-Liouville for every \( p \in (1, \infty) \) [19, Corollary 6.11] but \( \mathbb{H}^n \) does not have the \( p \)-Liouville property for any \( p \in (1, \infty) \) [A]. Is there a way to tell whether or not a
given Cartan-Hadamard manifold has the $p$-Liouville property by looking at the curvature tensor?

Greene and Wu [17] laid the foundations for the study of harmonic functions on Cartan-Hadamard manifolds. Motivated by their search for a higher dimensional counterpart for the uniformization theorem of Riemann surfaces they conjectured in [17] that if the sectional curvature on a Cartan-Hadamard manifold $M$ satisfies

$$K_M \leq -\frac{C}{\rho^2},$$

outside a compact set, where $C > 0$ is a positive constant and $\rho$ is distance from a fixed point, then there exists a non-constant bounded harmonic function on $M$. Despite numerous partial results, the conjecture is still open in its general form in dimensions three and above [7].

A natural framework to discuss the existence of globally defined bounded harmonic functions is the Dirichlet problem at infinity. In order to define this we first describe a natural compactification of the manifold.

A Cartan-Hadamard manifold $M$ has a natural geometric boundary, the sphere at infinity (also called the ideal boundary) $M(\infty)$, such that $\bar{M} := M \cup M(\infty)$ equipped with so called cone topology is homeomorphic to the ball $B(0,1) \subset \mathbb{R}^n$ with $M$ corresponding to $B(0,1)$ and $M(\infty)$ to $S^n$ in this mapping, cf. [10]. In this topology using polar coordinates $(r,v)$ with respect to a fixed point, a sequence of points $(r_j, v_j)$ in $M$ converges to a point in $M(\infty)$ corresponding to $v \in S^n$ if and only if $r_j \to \infty$ and $v_j \to v$ as $j \to \infty$.

This allows us to define the central concept of our thesis:

4 Problem. (Dirichlet problem at infinity) Given $\theta \in C(M(\infty))$, does there exist a unique function $u \in C(\bar{M})$ such that $u|_M$ is $A$-harmonic and $u|_{M(\infty)} = \theta$?

If such a function $u$ exists for every $\theta \in C(M(\infty))$, we say that the Dirichlet problem at infinity is solvable. It turns out that the existence of such function $u$ depends heavily on the manifold $M$ and also on $p$. However, uniqueness is always guaranteed in this setting (cf. proof of [C, Theorem 7.1]). This problem is also called the asymptotic Dirichlet problem.

3 History of the Dirichlet problem at infinity

In this section we give an overview of the history of the Dirichlet problem at infinity. We exclude [A], [B], and [C] from this consideration. From now on $M$ is a Cartan-Hadamard manifold and $o \in M$ is fixed. We denote $\rho = d(\cdot,o)$ for the distance to $o$.

Choi [14] was the first to solve the Dirichlet problem at infinity under the assumption that the sectional curvature satisfies $K_M \leq -a^2 < 0$ and any two points on $M(\infty)$ can be separated by convex neighborhoods. Such neighborhoods were constructed by Anderson [5] in 1983 and he proved the following.

5 Theorem. Suppose that the sectional curvature of $M$ satisfies the pinched curvature condition

$$-b^2 \leq K_M \leq -a^2,$$

where $0 < a \leq b$ are arbitrary positive constants. Then the Dirichlet problem at infinity is uniquely solvable.
At the same time Sullivan [31] solved the Dirichlet problem at infinity on a manifold of pinched sectional curvature by analyzing the behavior of the Brownian motion on the manifold. Anderson and Schoen [6] presented a simple solution to the Dirichlet problem and showed that the Martin boundary of $M$ is homeomorphic to the ideal boundary in the case of pinched sectional curvature.

Ballmann [8] solved the Dirichlet problem at infinity for irreducible, rank one manifolds of non-positive curvature admitting a compact quotient. He later generalized this result in a joint paper with Ledrappier [9] by proving a Poisson representation formula for harmonic functions on the manifold.

Ancona studied the Dirichlet problem at infinity on Gromov hyperbolic graphs [1] and Gromov hyperbolic Riemannian manifolds [2]. In [3] he was also able to solve the Dirichlet problem at infinity on Cartan-Hadamard manifolds with sectional curvature upper bound $K_M \leq -a^2$ and a bounded geometry assumption that each ball up to a fixed radius is $L$-bi-Lipschitz equivalent to an open set in $\mathbb{R}^n$ for some fixed $L \geq 1$.

In 1993 Cheng [13] proved the following result.

\textbf{7 Theorem.} Assume that the bottom of the spectrum $\lambda_1(M)$ for the Laplacian on $M$ is positive. Suppose that there exists a point $o \in M$ and a constant $C$ such that the sectional curvature satisfies

$$|K_M(P)| \leq C|K_M(P')|$$

for 2-plane sections $P$ and $P'$ at $x$ containing the tangent vector of the geodesic joining $o$ to $x$. Then the Dirichlet problem at infinity is solvable for the Laplacian.

This result is interesting since unlike many other results it does not assume a fixed curvature upper bound or a fixed curvature lower bound. It allows the possibility that the manifold has points outside any compact set where the curvature vanishes or has paths going to infinity where the curvature grows arbitrarily fast.

\textbf{3.1 Relaxing the curvature assumptions}

Numerous results have been obtained to relax the curvature assumption (6) in order to solve the Dirichlet problem at infinity. Hsu and March [25] solved the Dirichlet problem under the curvature assumption

$$-b^2 \leq K_M \leq -C/r^2$$

for some constants $b > 0$ and $C > 2$. Borbély [11] was able to solve the Dirichlet problem at infinity with a weaker curvature lower bound assumption

$$-be^{\lambda r} \leq K_M \leq -a$$

for some constants $b \geq a > 0$ and $\lambda < 1/3$.

The most recent result of this kind is by Hsu [24] in 2003. Excluding [B] his results below are the most general known curvature growth conditions that imply the solvability of the Dirichlet problem at infinity.
8 Theorem ([24, Theorem 1.2]). Suppose that there exist positive constants $r_0$, $\alpha > 2$, and $\beta < \alpha - 2$ such that

$$-r^{2\beta} \leq \text{Ric}_x \quad \text{and} \quad \text{Sect}_x \leq -\alpha(\alpha - 1)/r^2$$

for all $r = r(x) \geq r_0$. Then the Dirichlet problem at infinity is solvable on $M$.

9 Theorem ([24, Theorem 1.1]). Suppose that there exists a positive constant $a$ and a positive and nonincreasing function $h$ with $\int_0^\infty rh(r)\,dr < \infty$ such that

$$-h(r)^2e^{2ar} \leq \text{Ric}_x \quad \text{and} \quad \text{Sect}_x \leq -a^2.$$

Then the Dirichlet problem at infinity is solvable on $M$.

3.2 Rotationally symmetric manifolds

When $\mathbb{R}^n$ is given a smooth Riemannian metric which is written in polar coordinates as $dr^2 + f(r)^2d\theta^2$, where $f : [0, \infty) \to [0, \infty)$, it becomes a rotationally symmetric manifold. In this case we denote the resulting Riemannian manifold as $M_f = (\mathbb{R}^n, dr^2 + f(r)^2d\theta^2)$. These manifolds provide simple concrete examples of Cartan-Hadamard manifolds (provided that $f'' \geq 0$). They are also often used as comparison manifolds in association with various comparison theorems in Riemannian geometry. The Dirichlet problem at infinity has a natural definition on all of these manifolds and this definition coincides with the one that we have for Cartan-Hadamard manifolds if $M_f$ is such, see [14] for this definition.

In the case of Riemann surfaces ($n = 2$) Milnor [28] showed that $M_f$ is parabolic if and only if $\int_1^\infty dr/f(r) = \infty$. Choi [14] proved that if the radial curvature is bounded from above by $-C/r^2 \log r$ for some $A > 1$ outside a compact set, then the Dirichlet problem at infinity is solvable. In 1986 using probabilistic methods March [27] characterized the existence of bounded non-constant harmonic functions on $M_f$ with the integral condition

$$\int_1^\infty f(r)^n - 3 \int_1^\infty f(t)^{1-n}dt\,dr < \infty.$$  

Murata [29] later showed that (10) is equivalent to either of the following conditions:

(i) $M_f$ does not have strong Liouville property,

(ii) the Dirichlet problem at infinity is solvable on $M_f$.

It would be very interesting to know how the existence of bounded non-constant $p$-harmonic functions could be characterized in terms of $f$, $p$, and $n$ in the case $p \in (1, \infty)$, $p \neq 2$. This, however, is unknown to us except in the conformal case $p = n$.

3.3 Nonsolvability results

The necessity of some kind of curvature upper bound in (6) is clear in Theorem 5 as is seen from the example $M = \mathbb{R}^n$. However, if one only considers rotationally symmetric manifolds then there is no need for curvature lower bound as can be deduced from the results stated above. Also, if one only considers two dimensional manifolds then there is no need for curvature lower bound as proven by Choi [14]. This raises the question: is
the curvature lower bound needed in the general case? The answer is yes by a result of Ancona [4] from 1994. Ancona constructed a Cartan-Hadamard 3-manifold with sectional curvature $K_M \leq -1$ so that every bounded harmonic function extending continuously to the ideal boundary is constant. Borbély [12] gave another construction with the same property and also proved the existence of bounded non-constant harmonic functions on his manifold. These manifolds have been further studied and generalized by Ulsamer [32] and Arnaudon, Thalmaier, and Ulsamer [7] by using probabilistic methods.

3.4 Nonlinear setting

Excluding [A], [B], and [C], relatively little is known about the behavior of $p$-harmonic functions on Cartan-Hadamard manifolds if $p \neq 2$. Pansu [30] proved the existence of non-constant bounded $p$-harmonic functions with finite $p$-energy on Cartan-Hadamard manifolds $M$ satisfying (6) with $p > (n - 1)b/a$ and the non-existence of such functions if $p \leq (n - 1)a/b$. The Dirichlet problem at infinity for $p$-harmonic functions was solved in 2001 by Holopainen [23] under the pinched curvature condition (6). To do this he generalized the proof of [6, Theorem 3.1] by Anderson and Schoen.

4 Summary of articles [A] and [B]

In this section we summarize the results and methods used in [A] and [B]. We fix our notation so that $M$ is a Cartan-Hadamard manifold, $o \in M$ is a basepoint, and $\rho = d(o, \cdot)$. Exponent $p \in (1, \infty)$ and operator $\mathcal{A}$, $\langle \mathcal{A}(v), v \rangle \approx |v|^p$, are also fixed.

In order to break down the Dirichlet problem at infinity and to better understand it, we approach it by using Perron’s method, which is available in the nonlinear setting, see [19].

11 Definition. A function $u : M \to (-\infty, \infty]$ belongs to the upper class $\mathcal{U}_{\theta}$ of $\theta : M(\infty) \to [-\infty, \infty]$ if

(i) $u$ is $\mathcal{A}$-superharmonic in $M$,

(ii) $u$ is bounded below, and

(iii) $\liminf_{x \to x_0} u(x) \geq \theta(x_0)$ for all $x_0 \in M(\infty)$.

The function

$$\overline{\Pi}_{\theta} = \inf\{u : u \in \mathcal{U}_{\theta}\}$$

is called the upper Perron solution.

12 Definition. A point $x_0 \in M(\infty)$ is $\mathcal{A}$-regular (or $p$-regular if $\mathcal{A}$ is the $p$-Laplacian), if

$$\lim_{x \to x_0} \overline{\Pi}_{\theta}(x) = \theta(x_0)$$

for every continuous function $\theta : M(\infty) \to \mathbb{R}$.

The concept of regularity is related to the Dirichlet problem in the way that the Dirichlet problem at infinity is solvable for $\mathcal{A}$-harmonic functions if and only if every point at infinity is $\mathcal{A}$-regular.
4.1 Article [B]

In [B] we generalize the proofs by Holopainen [23] and Anderson and Schoen [6, Theorem 3.1] to obtain generous curvature bounds that allow us to find $p$-regular points at infinity.

The rough idea behind our argument is the following. We are given $x_0 \in M(\infty)$ and we want to prove that $x_0$ is $p$-regular. We take a continuous function $h$ defined on the sphere at infinity, and radially extend it to the whole $\bar{M} \setminus \{o\}$. We then smoothen the extended function by using a convolution type procedure. It is then shown that by making a small perturbation to this smoothened function we obtain a $p$-superharmonic function $w$. By choosing $h$ suitably, the resulting function $w$ behaves like a barrier function at $x_0$. This is used to prove that $x_0$ is $p$-regular.

One important difference between our argument and that of Holopainen and also Anderson and Schoen is that our smoothening procedure depends on the given curvature lower bound whereas theirs do not. This results in technical difficulties and tedious computations but allows the more general results.

Our results in [B] are the following.

13 Corollary ([B, Corollary 3.22]). Let $\phi > 1$ and $\varepsilon > 0$. Let $x_0 \in M(\infty)$ and let $U$ be a neighborhood of $x_0$ in the cone topology. Suppose that

$$-\rho(x)^{2\phi-4-\varepsilon} \leq K_M(P) \leq -\frac{\phi(\phi-1)}{\rho(x)^2}$$

for every $x \in U \cap M$ and every 2-dimensional subspace $P \subset T_x M$. Then $x_0$ is a $p$-regular point at infinity for every $p \in (1, 1 + (n-1)\phi)$.

We do not know if the curvature lower bound given here is close to optimal or not. We do know, however, that the curvature upper bound is optimal in the sense that the claim does not always hold if we allow the border line case $p = 1 + (n-1)\phi$. This is proven with an example in [A, Example 2].

We note that Corollary 13 corresponds closely to Theorem 8 by Hsu in the case $p = 2$ except that he restricts $\alpha > 2$ whereas we allow any $\phi > 1$. Hence Theorem 8 follows from Corollary 13 as a special case.

14 Corollary ([B, Corollary 3.23]). Let $k > 0$ and $\varepsilon > 0$. Let $x_0 \in M(\infty)$ and let $U$ be a neighborhood of $x_0$ in the cone topology. Suppose that

$$-\rho(x)^{-2-\varepsilon} e^{2k\rho(x)} \leq K_M(P) \leq -k^2$$

for every $x \in U \cap M$ and every 2-dimensional subspace $P \subset T_x M$. Then $x_0$ is a $p$-regular point at infinity for every $p \in (1, \infty)$.

Again we do not know whether the curvature lower bound is close to optimal or not. Corollary 14 corresponds closely to Theorem 9 by Hsu in the case $p = 2$. However, they are slightly different and neither follows directly from the other in the case $p = 2$.

4.2 Article [A]

In Article [A] we solve the Dirichlet problem at infinity for $A$-harmonic functions. The method used in [B] does not generalize readily to this setting since in general we cannot
test $\mathcal{A}$-superharmonicity pointwise. Luckily the technique used by Cheng in [13] to prove Theorem 7 generalizes to cover $\mathcal{A}$-harmonic functions as well. In addition to considering a larger class of operators, we make two modifications to the original method: we localize the argument to prove $\mathcal{A}$-regularity of a point at infinity and instead of assuming that the bottom of the spectrum satisfies $\lambda_1(M) > 0$, we assume a weak local upper bound for the sectional curvature. We obtain the following result.

15 Theorem ([A, Theorem 4]). Let $x_0 \in M(\infty)$ and $\phi > 1$. Suppose that $x_0$ has a neighborhood $U$ (in the cone topology) such that

$$K_M(P) \leq -\phi(\phi - 1)/\rho(x)^2$$

for every $x \in U \cap M$ and every 2-dimensional subspace $P \subset T_xM$ that contains the radial vector $\nabla \rho(x)$. Suppose also that there exists a constant $C < \infty$ such that

(16) $$|K_M(P)| \leq C|K_M(P')|$$

whenever $x \in U \cap M$ and $P, P' \subset T_xM$ are 2-dimensional subspaces containing $\nabla \rho(x)$. Suppose that

$$1 < p < \frac{\alpha}{\beta}(1 + (n - 1)\phi),$$

where $\alpha$ and $\beta$ are the structure constants of $\mathcal{A}$. Then $x_0$ is $\mathcal{A}$-regular.

Although Condition (16) is very restrictive in general, it is trivially satisfied in three important special cases: when the manifold satisfies the pinched curvature condition (6), when $n = 2$, and when $M$ is rotationally symmetric at $o$. In the two-dimensional case we obtain the following satisfactory result.

17 Corollary. Suppose that $n = 2$ and that there exists a constant $\phi > 1$ such that

$$K_M \leq -\phi(\phi - 1)/\rho^2$$

outside a compact set. Then the Dirichlet problem at infinity is solvable for $p$-harmonic functions on $M$ if $p \in (1, 1 + \phi)$.

For rotationally symmetric manifolds we obtain the following.

18 Corollary. Suppose that the Cartan-Hadamard manifold $M$ is rotationally symmetric at $o$ and that there exists a constant $\phi > 1$ such that

$$K_M \leq -\phi(\phi - 1)/\rho^2$$

outside a compact set. Then the Dirichlet problem at infinity is solvable for $p$-harmonic functions on $M$ if $p \in (1, 1 + (n - 1)\phi)$.

Corollaries 17 and 18 are close to sharp in light of the following.

19 Proposition. (a) Suppose that there exists a constant $\phi > 1$ and a compact set $K \subset M$ such that

$$K_M(P) \geq -\phi(\phi - 1)/\rho(x)^2$$

8
for every $x \in M \setminus K$ and every 2-dimensional subspace $P \subset T_x M$ that contains the radial vector $\nabla \rho(x)$. Then $M$ is $p$-parabolic, and hence has the strong $p$-Liouville property, if $p \geq 1 + (n - 1)\phi$.

(b) Suppose that $b : [0, \infty) \to [0, \infty)$ is a function such that the sectional curvature satisfies $K_M(P) \geq -(b \circ \rho)(x)^2$ for every $x \in M$ and 2-dimensional subspace $P \subset T_x M$. Suppose also that

$$\int_0^\infty tb(t)^2 \, dt < \infty.$$ 

Then $\exp_o$ is a bilipschitz diffeomorphism. In particular, $M$ has the strong $p$-Liouville property.

The proof of (a) is given in [A, Proposition 1] and (b) in [17, Theorem C].

### 4.3 Cheng’s proof adapts to many situations

After writing [A] we have found that the proof method used by Cheng in [13] and subsequently by us in [A] for solving the Dirichlet problem at infinity can be adapted successfully in many settings. The method relies on a global (or semi-global) Sobolev type inequality and a Caccioppoli type estimate. These are tools available in many settings making this proof method suitable for generalization.

Recently we have generalized Theorem 15 to include curvature growth conditions similar to those given in Corollaries 13 and 14. This yields a much simpler proof for results like those that we present in [B]. A paper detailing this is in preparation.

We have also applied Cheng’s proof method to characterize rotationally symmetric manifolds $M_f$ that satisfy the $D_p$-Liouville property. This result was presented in the authors Licentiate thesis [33].

Finally, Cheng’s approach is used in the setting of Gromov hyperbolic metric spaces in Article [C] that we present next.

### 5 Summary of Article [C]

The observation that the proof given by Cheng in [13] used essentially only Caccioppoli inequality and various Sobolev inequalities led the authors of [C] to study whether one could solve the Dirichlet problem at infinity on Gromov hyperbolic metric measure spaces. In this section we outline the content of Article [C], where we show how a variation of Cheng’s proof method can be applied in such spaces. In order to do this we first broaden our concept of Dirichlet problem at infinity appropriately.

Suppose that $(X, d, \mu)$ is a metric measure space with some “boundary at infinity” $\partial X$ such that $X \cup \partial X$ is a topological space. The Dirichlet problem at infinity (or asymptotic Dirichlet problem) is to find for given continuous boundary data $\theta : \partial X \to \mathbb{R}$ a continuous function $u : X \cup \partial X \to \mathbb{R}$ such that $u|_X$ is $p$-harmonic. The problem is said to be solvable if such a function $u$ exists for every continuous function $\theta$. One obtains variations of this problem by choosing the boundary at infinity or the condition for $u|_X$ in different ways.

A metric space $X$ is (Gromov) $\delta$-hyperbolic with $\delta \geq 0$, if

$$(x|y)_{\partial} \geq \min\{(x|y)_{\partial}, (y|z)_{\partial}\} - \delta$$
for all $x, y, z, o \in X$. Here
\[
(x|y)_o = \frac{(|x-o| + |y-o| - |x-y|)}{2}
\]
is the **Gromov product** of $x$ and $y$ with respect to $o$. In the case that $X$ is geodesic this is equivalent to the **slim triangles condition**; there exists $\delta_0 > 0$ such that for every geodesic triangle $\Delta$ each side $\tau$ of $\Delta$ is contained in $B(|\Delta| \setminus \tau, \delta_0)$ [16, p. 41]. For our purposes the most important examples of Gromov hyperbolic metric space are connected complete Riemannian manifolds with sectional curvature bounded from above by a negative constant.

A sequence $(x_i)$ of points in $X$ is called a **Gromov sequence** if
\[
\lim_{i,j \to \infty} (x_i|x_j)_o = \infty,
\]
where the basepoint $o$ is fixed. Two Gromov sequences $\bar{x} = (x_i)$ and $\bar{y} = (y_i)$ are **equivalent**, $\bar{x} \sim \bar{y}$, if $(x_i|y_i)_o \to \infty$ as $i \to \infty$. This defines an equivalence relation and the set of all equivalence classes is called the **Gromov boundary** of $X$,
\[
\partial_G X = \{[\bar{x}] : \bar{x} \text{ is a Gromov sequence in } X\}.
\]
The set
\[
X^* = X \cup \partial_G X
\]
is called the **Gromov closure** of $X$ and it is equipped with a topology that is natural in the sense that if $X$ is Cartan-Hadamard with $K_X \leq -a^2$ for some $a > 0$, then the topology agrees with the cone topology.

Our main result in [C] concerns the solvability of the Dirichlet problem at infinity when the boundary at infinity is the Gromov boundary.

**20 Theorem ([C, Theorem 1.1]).** Suppose that $X$ is a connected, locally compact, Gromov hyperbolic metric measure space equipped with a Borel regular measure $\mu$. We assume that $(X, d, \mu)$ has local bounded geometry in the sense that $\mu$ is locally doubling, the measures of balls with sufficiently small radius have a uniform positive lower bound, and that $X$ supports a local Poincaré inequality. Suppose also that $X$ has at most exponential volume growth and that a global Sobolev inequality holds for compactly supported functions.

Under these assumptions the Dirichlet problem at infinity for $p$-harmonic functions is solvable. In other words, for every continuous bounded function $f : \partial_G X \to \mathbb{R}$, there exists a continuous function $u : X^* \to \mathbb{R}$ so that $u|X$ is $p$-harmonic and that $u|\partial_G X = f$.

In this context $p$-harmonic functions are defined as continuous minimizers of $p$-energy among the functions in Newtonian space with the same boundary values. Now uniqueness of a solution is not always guaranteed unlike in the case of Cartan-Hadamard manifolds. The most simple example of a space that satisfies the assumptions of Theorem 20 but where a solution is not unique is the punctured hyperbolic space $\mathbb{H}^n \setminus \{0\}$. However, we prove that solutions are unique if $X^*$ is sequentially compact [C, Theorem 7.1].

The assumptions in Theorem 20 are modelled after a curvature upper bound $K \leq -a^2$, $a > 0$, and a bounded geometry assumption as used by Ancona [3]. In particular, if $M$ is a Cartan-Hadamard manifold that satisfies the pinched curvature condition, then it satisfies the assumptions of Theorem 20. More generally we have the following.
21 Corollary ([C, Theorem 8.3]). Let $N$ be a Cartan-Hadamard manifold of bounded geometry whose sectional curvatures satisfy $K_N \leq -a^2 < 0$. Suppose that $M$ is a complete Riemannian manifold of bounded geometry that is quasi-isometric to $N$. Then $M$ is Gromov hyperbolic and for every continuous function $h$ on $\partial_G M$ there exists a unique function $u \in C(M^*)$ that is $p$-harmonic in $M$ and satisfies $u|_{\partial_G M} = h$.

As a final remark we note that the context of Gromov hyperbolic metric spaces has the drawback that there are many Cartan-Hadamard manifolds where the Dirichlet problem at infinity is solvable but that are not Gromov hyperbolic. Examples are given by Cartan-Hadamard manifolds with sectional curvature that tend to zero when approaching infinity but slow enough that one can apply [B, Corollary 3.22].

References


