LOCALITY AND ORDER-INVARIANT LOGICS

HANNU NIEMISTÖ

Academic dissertation

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Contents

1. Introduction

A typical problem in finite model theory is to determine if two logics have the same expressive power. To prove that one of the logics is more expressive, by definition, it has to be shown that there is a query that one of the logics can express which the other cannot.

Showing that a query is expressible is usually straightforward, one has to give only a sentence and prove that it expresses the query, although sometimes finding the sentence may be non-trivial. Much of the machinery of finite model theory has been developed for proving the opposite, that the query is not expressible.

The most direct approach to prove non-expressivity is to construct a sequence of pairs of structures, such that the query considered separates the structures in every pair, but any sentence of the logic fails to separate the structures in some pair. Instead of considering single sentences, some kind of notion of complexity of sentence is usually defined, which gives an equivalence relation "no sentence of certain complexity can separate the structures".

The standard notion of complexity for extensions of first-order logic with generalized quantifiers is the quantifier-rank of the sentence. The equivalence relation given by this notion can be characterized by two-player games parameterized by pairs of structures, where one of the players has a winning strategy if the structures are equivalent. The moves of the game correspond to the choice of the interpretations of the variables bound by quantifiers. For many important logics, the generic game characterization can be simplified.

While the game characterizations give a general framework for non-expressivity proofs, finding and describing a winning strategy of the game is often laborious even in the case of first-order logic. Therefore there has been a motivation to find tools that would simplify the proofs. Locality can be seen as such a tool. From the point of view of logical games, the idea is that elements of the structure that are in some sense near each other are dependent on each other and we have to consider their relationship, while the game strategy near the elements with a great distance separating them may be designed independently. If the logic admits this kind of strategy, i.e., it is local, then we can usually give a sufficient condition for the equivalence of the structures by using only local information about the structures.

The idea of locality first occurred in different forms in Hanf's [Han65] and Gaifman's theorems [Gai82], both of which can be applied also in the context of infinite model theory. The first gives a sufficient condition for first-order equivalence of structures up to given quantifier rank in terms of neighborhood types occurring in the structures. The second theorem gives a normal form for first-order formulas such that quantifications in the subformulas are bounded to neighborhoods of fixed elements. These theorems were generalized as abstract properties by Hella, Libkin and Nurmonen [HLN99] and used for example to prove inexpressibility results of certain database query languages [HLNW01].

Locality is usually considered as a property of a single query. From this point of view, it seems fragile: for example a local query on graphs may become non-local, if we complement every graph in the query, although intuitively complementation should not change the complexity of the query.

In this thesis, locality is studied as a property of a logic satisfying various regularity properties, most important case being the extensions of first-order logic with generalized quantifiers. We begin in Section 3 with the question, when a quantifier gives rise to a local logic. Clearly, the query corresponding to the quantifier has to stay local under quantifier-free interpretations. This is however not enough. We give in the section two sufficient criteria: either locality has to be preserved under infinitary quantifier-free interpretations or the query has to admit a decomposition resembling the normal form of Gaifman's theorem. At the end of Section 3 we show how infinite counting logic relates to the concepts.

Section 4 considers the relation of uniform reduction to the criteria developed in the previous section and we see also that there are logics that are local without satisfying the first condition. This means in particular that there does not exist a greatest local extension of first-order logic with generalized quantifiers.

An order-invariant logic is a logic whose sentences may use a built-in linear order on structures but the truth of the sentence must not depend on the linear order chosen. The definition is motivated by the fact that order-invariant extensions of logics that capture complexity classes only on ordered structures, capture them strongly, i.e, on all structures. We show in Section 6 that the logics can also be defined quite naturally as maximal extensions of a certain logic such that the extension does not increase the expressive power of the logic on a given class of structures, in the case of order-invariant logics, on the class of ordered structures.

The starting point for the material in Section 5 and the rest of the thesis was the question by Luc Segoufin about the expressive power of order-invariant first-order logic on finite trees. The question was answered independently in [Nie05] and two subsequent papers [BS05a] and [BS05b]. In both proofs, first-order logic (possibly with counting modulo quantifiers) is characterized on trees using a set of necessary and sufficient conditions.

An important intermediate result of the former proof, which will be presented in Section 7, was that Gaifman-locality implies Hanf-locality on trees assuming that the logic is sufficiently regular. This leads to the question, how generally this happens.

The main result of Section 5 is that first-order logic can be extended with generalized quantifiers so that it remains Gaifman-local but loses Hanf-locality. The example given works on graphs with bounded genus and degree. Before giving the example, Gaifman-locality is characterized by a definition that resembles Hanf-locality. This allows also to define a natural hierarchy of locality notions between Gaifman- and Hanf-locality such that every level of the hierarchy contains an extension of first-order logic.

Order-invariant logics are the subject of Section 6. After recalling some properties of order-invariant logics, we show that order-invariant first-order logic is not definable on any level of the quantifier hierarchy. We turn then our attention to locality of order-invariant logics.

By the result of Grohe and Schwentick [GS00], order-invariant first-order logic is Gaifman-local. We show first that no order-invariant proper extension of first-order logic with relativized unary quantifiers is Gaifman-local, however order-invariant extensions with counting modulo quantifiers satisfy weaker versions of Gaifman-locality. As pointed out in [BS05b], Gaifman-locality is not a necessary condition for the expressive power of a logic to collapse to first-order logic on trees, although the logic clearly has to be Gaifman-local on trees. We show that a weak version of Gaifman-locality is enough to give Hanf-locality on trees which implies that order-invariant extension of first-order logic with counting modulo k quantifier $\mathrm{FO}_<(D_k)$ collapses to the corresponding logic $\mathrm{FO}(D_k)$ without order, if k is odd. If k is even, we show that the order-invariant logic has greater expressive power on trees.

2. Foundations

2.1. **General notations.** A sequence $(a_0, \dots, a_{k-1})$ is abbreviated as $\overline{a}$ when its length is clear from the context. We drop often commas and parenthesis when speaking about sequencens and may also write $a_0 \dots a_{k-1}$. If $\overline{a}$ is a k-sequence, $[\overline{a}] = \{a_0, \dots, a_{k-1}\}$. If f is a function $A \to B$ and $\overline{a} \in A^k$, then $(f(a_0), \dots, f(a_{k-1}))$ is abbreviated as $f(\overline{a})$, and if $X \subseteq A^k$, then $f(X) = \{f(\overline{a}) \mid \overline{a} \in X\}$. The concatenation of two sequences $\overline{a}$ and $\overline{b}$ is denoted by $\overline{a}\overline{b}$. Given a set X, we use the convention $X^0 = 1 = \{0\} = \{\emptyset\}$.

We use quite extensively the convention that the free variables at the right side of the set comprehension $\{\dots \mid \dots\}$ are quantified existentially.

2.2. **Structures.** *A vocabulary* τ is a set of relation and constant symbols. Denote the set of all relation symbols of τ by $\mathrm{Rel}(\tau)$ and the set of all constant symbols of τ by $\mathrm{Con}(\tau)$. We assume the following convention throughout the thesis.

Convention 2.1. *The set* $\mathrm{Con}(\tau)$ *is always finite.* $\mathrm{Rel}(\tau)$ *may be infinite.*

Because we allow infinitely many unary relation symbols, the convention only prohibits constructions where we would need formulas with infinitely many free variables. These are seldom useful in finite model theory.

Relation symbols are usually written in upper-case letters and constant symbols in lower case. We associate each relation symbol R with a natural number $\mathrm{ar}(R)$ called *the arity* of the symbol. Also 0-ary relation symbols are allowed and called *proposition symbols*.

A τ-structure $\mathfrak{A}$ is a pair (A, ρ), where A is a non-empty set called the *domain* $\mathrm{Dom}(\mathfrak{A})$ of the structure and ρ is a function mapping each symbol R of τ to its *interpretation* $R^{\mathfrak{A}} = \rho(R)$. If R is a relation symbol, its interpretation in $\mathfrak{A}$ is a relation $R^{\mathfrak{A}} \subseteq \mathrm{Dom}(\mathfrak{A})^{\mathrm{ar}(R)}$ and if c is a constant symbol, its interpretation in $\mathfrak{A}$ is a constant $c^{\mathfrak{A}} \in \mathrm{Dom}(\mathfrak{A})$. Note, that if R is a proposition symbol, by the convention we have, $R^{\mathfrak{A}} \in \mathcal{P}(\mathrm{Dom}(\mathfrak{A})^0) = \{0, 1\}$. *A partial τ-structure* is defined like a structure, but all constant symbols need not have an interpretation.

A structure is finite if its domain is finite. We denote the class of all finite τ-structures by $\mathrm{Mod}(\tau)$. All structures in the thesis will be finite without further mention, although some definitions and results may be valid also on infinite structures.

Two partial τ-structures $\mathfrak{A}$ and $\mathfrak{B}$ are *isomorphic*, $\mathfrak{A} \cong \mathfrak{B}$, if there exists a bijection $\alpha\colon \mathrm{Dom}(\mathfrak{A}) \to \mathrm{Dom}(\mathfrak{B})$ such that for all relation symbols $R \in \tau$, $R^{\mathfrak{B}} = \alpha(R^{\mathfrak{A}})$, for all proposition symbols $S \in \tau$, $S^{\mathfrak{B}} = S^{\mathfrak{A}}$ and for all constant symbols $c \in \tau$, either $c^{\mathfrak{B}} = \alpha(c^{\mathfrak{A}})$ or both $c^{\mathfrak{B}}$ and $c^{\mathfrak{A}}$ are undefined. Although $\mathrm{Mod}(\tau)$ is a proper class, $\mathrm{Mod}(\tau)/\cong$ is countable, when τ is finite, and of size $2^{|\tau|}$ if τ is infinite.

If $\alpha\colon \tau \to \tau'$ is a function preserving types and arities of the symbols, α induces a function $I_\alpha\colon \mathrm{Mod}(\tau') \to \mathrm{Mod}(\tau)$ such that for each $\mathfrak{A} \in \mathrm{Mod}(\tau')$, $\mathrm{Dom}(I_\alpha(\mathfrak{A})) = \mathrm{Dom}(\mathfrak{A})$ and for each $R \in \tau$, $R^{I_\alpha(\mathfrak{A})} = \alpha(R)^{\mathfrak{A}}$. In the special case, where α is an inclusion $\tau \subset \tau'$, we denote $I_\alpha(\mathfrak{A})$ by $\mathfrak{A}\restriction\tau$.

Assume τ and $\tau' = \{R_0, \ldots, R_{n-1}\}$ are disjoint vocabularies. If $\mathfrak{A}$ is a partial τ-structure, $(\mathfrak{A}, E_0/R_0, \ldots, E_{n-1}/R_{n-1})$ denotes the partial $\tau \cup \tau'$-structure $\mathfrak{B}$, whose domain is $\mathrm{Dom}(\mathfrak{A})$, $\mathfrak{B} \restriction \tau = \mathfrak{A}$ and $R_i^{\mathfrak{B}} = E_i$. We call the new structure *an expansion* of $\mathfrak{A}$. If $\overline{x} = x_0 \ldots x_{n-1}$ is a sequence of constant symbols, and $\overline{a} \in \mathrm{Dom}(\mathfrak{A})^n$, we write $(\mathfrak{A}, \overline{a}/\overline{x})$ to abbreviate $(\mathfrak{A}, a_0/x_0, \ldots, a_{n-1}/x_{n-1})$. If X is a set, $(X, E_0/R_0, \ldots, E_{n-1}/R_{n-1})$ denotes the structure we get by expanding a $\emptyset$-structure of domain X.

If $\mathfrak{A}$ is a partial τ-structure and $X \subseteq \mathrm{Dom}(\mathfrak{A})$, *the substructure of* $\mathfrak{A}$ *with domain* X, $\langle X\rangle^{\mathfrak{A}}$, is a partial τ-structure such that $\mathrm{Dom}(\langle X\rangle^{\mathfrak{A}}) = X$, and for relation symbols $R \in \tau$, $R^{\langle X\rangle^{\mathfrak{A}}} = R^{\mathfrak{A}} \cap X^{\mathrm{ar}(R)}$. If $c \in \tau$ is a constant symbol and $c^{\mathfrak{A}} \in X$, then $c^{\langle X\rangle^{\mathfrak{A}}} = c^{\mathfrak{A}}$, otherwise $c^{\langle X\rangle^{\mathfrak{A}}}$ is not defined.

Suppose τ and τ' are vocabularies such that $\tau \cap \tau'$ contains only relation symbols of arity at least one. If $\mathfrak{A}$ is a partial τ-structure and $\mathfrak{B}$ a partial τ'-structure, then their *disjoint union* $\mathfrak{A} \sqcup \mathfrak{B}$ is a partial $\tau \cup \tau'$-structure with domain $(\{0\} \times \mathrm{Dom}(\mathfrak{A})) \cup (\{1\} \times \mathrm{Dom}(\mathfrak{B}))$. Let $\iota_i(a) = (i, a)$. Then for each relation symbol $R \in \tau \cap \tau'$, $R^{\mathfrak{A}\sqcup\mathfrak{B}} = \iota_0(R^{\mathfrak{A}}) \cup \iota_1(R^{\mathfrak{B}})$, for each symbol $R \in \tau \setminus \tau'$, $R^{\mathfrak{A}\sqcup\mathfrak{B}} = \iota_0(R^{\mathfrak{A}})$

and for each symbol $R \in \tau' \setminus \tau$, $R^{\mathfrak{A} \sqcup \mathfrak{B}} = \iota_1(R^{\mathfrak{B}})$. If $\mathfrak{A}$ and $\mathfrak{B}$ are structures, then $\mathfrak{A} \sqcup \mathfrak{B}$ is also a structure.

2.3. **Logics.** *A τ-query* q is a subclass of $\mathrm{Mod}(\tau)$ closed under isomorphism. We denote $\mathfrak{A} \in q$ by $\mathfrak{A} \models q$. *A k-ary τ-query* is a pair $q = (q', \overline{x})$, where $\overline{x}$ is a sequence of k constant symbols not in τ and q' is a $\tau \cup [\overline{x}]$-query. We write $\mathfrak{A} \models q(\overline{a})$ for $(\mathfrak{A}, \overline{a}/\overline{x}) \in q'$ and $q(\mathfrak{A}) = \{\overline{a} \in \mathrm{Dom}(\mathfrak{A})^k \mid \mathfrak{A} \models q(\overline{a})\}$.

A logic is a pair $(\mathcal{L}, \models_{\mathcal{L}})$, where $\mathcal{L}$ is a function mapping each vocabulary τ to a set $\mathcal{L}[\tau]$ of *τ-sentences* and $\models_{\mathcal{L}}$ is a relation between models and $\mathcal{L}$-sentences giving the semantics to the logic so that the following conditions are satisfied:

a) Every sentence $\varphi \in \mathcal{L}[\tau]$ *defines* a τ-query $q_\varphi = \{\mathfrak{A} \in \mathrm{Mod}(\tau) \mid \mathfrak{A} \models_{\mathcal{L}} \varphi\}$.
b) If $\alpha \colon \tau \to \tau'$ is a function preserving types and arities of the symbols, it induces a bijection $S_\alpha \colon \mathcal{L}[\tau] \to \mathcal{L}[\tau']$ such that for all $\varphi \in \mathcal{L}[\tau]$ and $\mathfrak{A} \in \mathrm{Mod}(\tau)$, $I_\alpha(\mathfrak{A}) \models_{\mathcal{L}} \varphi \iff \mathfrak{A} \models_{\mathcal{L}} S_\alpha(\varphi)$.
c) We can associate every $\mathcal{L}$-sentence φ with a vocabulary $\tau(\varphi)$ such that $\varphi \in \mathcal{L}[\tau] \iff \tau(\varphi) \subseteq \tau$ and for all τ-structures $\mathfrak{A}$, $\mathfrak{A} \models_{\mathcal{L}} \varphi$ if and only if $\tau \supseteq \tau(\varphi)$ and $\mathfrak{A} \restriction \tau(\varphi) \models_{\mathcal{L}} \varphi$.

It is usually clear from the context which $\models_{\mathcal{L}}$-relation we are using and so the subscript is mostly dropped. Denote $S_\alpha(\varphi)$ by $\varphi[\overline{x'}/\overline{x}, \overline{R}'/\overline{R}]$, when α is a function mapping $\overline{x}$ to $\overline{x}'$, $\overline{R}$ to $\overline{R'}$ and fixing all other symbols in the vocabulary of φ.

Definition 2.2. A logic $\mathcal{L}$ is *infinitary* if for some $\varphi \in \mathcal{L}[\tau]$, $\tau(\varphi)$ is infinite. If a logic is not infinitary, it is *finitary.* A logic is finite (countable), if it contains only finitely (countably) many different sentences up to renaming relation and constant symbols.

A k-ary τ-formula of a logic $\mathcal{L}$ is a pair $\varphi = (\varphi', \overline{x})$, where $\overline{x}$ is a sequence of k constant symbols enumerating $\mathrm{Con}(\tau(\varphi')) \setminus \tau$ and $\varphi' \in \mathcal{L}[\tau \cup [\overline{x}]]$. We denote the set of all k-ary τ-formulas of $\mathcal{L}$ by $\mathcal{L}_k[\tau]$. Given another sequence $\overline{y}$ of constant symbols let $\varphi(\overline{y}) = (\varphi'[\overline{y}/\overline{x}], \overline{y})$. Every k-ary formula $\varphi = (\varphi', \overline{x})$ defines a k-ary query $q_\varphi = (q_{\varphi'}, \overline{x})$. Hence we can define the notations $\mathfrak{A} \models \varphi(\overline{a}) \iff \mathfrak{A} \models q_\varphi(\overline{a})$ and $\varphi(\mathfrak{A}) = q_\varphi(\mathfrak{A})$. We will sometimes refer to the parameter sequence $\overline{x}$ of the formula φ as $\overline{x}^\varphi$.

Most of the concepts and operations we define for sentences are also valid and will be used for formulas, we just apply them to the underlying sentences. Therefore we usually omit defining them separately for formulas. We also identify queries and 0-ary queries as well as sentences and 0-ary formulas.

Two formulas φ and φ' are *equivalent*, $\varphi \equiv \varphi'$, if they define the same query: $q_\varphi = q_{\varphi'}$. The expressive power of a logic $\mathcal{L}$ is greater than or equal to the expressive power of $\mathcal{L}'$, if every query definable in $\mathcal{L}'$ is also definable in $\mathcal{L}$. This is denoted by $\mathcal{L} \geq \mathcal{L}'$. If the logics define the same queries, we say that they have the same expressive power and write $\mathcal{L} \equiv \mathcal{L}'$. Finally, $\mathcal{L} < \mathcal{L}'$, if $\mathcal{L} \leq \mathcal{L}'$, but not $\mathcal{L} \geq \mathcal{L}'$.

If L is a set of logics, we define minimal, maximal, the least and the greatest logics of the set as usual using $\leq$ as a partial order.

It sometimes makes sense to examine the expressive power of logics relativized to some class of structures. Let M be any class of structures, possibly with different vocabularies, closed under isomorphism. A k-ary τ-query q is definable in $\mathcal{L}$ modulo M, if there exists $\varphi \in \mathcal{L}_k[\tau]$ such that $q(\mathfrak{A}) = \varphi(\mathfrak{A})$ for all $\mathfrak{A} \in M$. We write $\mathcal{L} \geq \mathcal{L}'$ (mod M), if every query definable in $\mathcal{L}'$ modulo M is also definable in $\mathcal{L}$ modulo M. If $\mathcal{L} \geq \mathcal{L}'$ (mod M) and $\mathcal{L} \leq \mathcal{L}'$ (mod M), we write $\mathcal{L} \equiv \mathcal{L}'$ (mod M).

2.4. **Composition of formulas and logics.** Let τ and τ' be vocabularies such that $\mathrm{Con}(\tau') \subseteq \mathrm{Con}(\tau)$. *An $\mathcal{L}$-interpretation of τ' in τ* is a function $\psi\colon \mathrm{Rel}(\tau') \to \bigcup_{k\in\mathbb{N}} \mathcal{L}_k[\tau]$, $R \mapsto \psi_R$, such that for all $R \in \mathrm{Rel}(\tau')$, ψ_R is an $\mathrm{ar}(R)$-ary $\mathcal{L}$-formula. We denote the set of all such interpretations by $I(\mathcal{L}, \tau, \tau')$.

An interpretation $\psi \in I(\mathcal{L}, \tau, \tau')$ induces a function $\psi^*\colon \mathrm{Mod}(\tau) \to \mathrm{Mod}(\tau')$ such that for all $\mathfrak{A} \in \mathrm{Mod}(\tau)$, $\mathrm{Dom}(\psi^*(\mathfrak{A})) = \mathrm{Dom}(\mathfrak{A})$, for all relation symbols $R \in \tau'$, $R^{\psi^*(\mathfrak{A})} = \psi_R(\mathfrak{A})$ and for all constant symbols $c \in \tau'$, $c^{\psi^*(\mathfrak{A})} = c^{\mathfrak{A}}$.

If $\mathcal{L}$ and $\mathcal{L}'$ are logics, we define a new logic $\mathcal{L} \circ \mathcal{L}'$ such that

$$(\mathcal{L} \circ \mathcal{L}')[\tau] = \{\varphi \circ \psi \mid \tau' \text{ is a vocabulary}, \varphi \in \mathcal{L}[\tau'], \psi \in I(\mathcal{L}', \tau, \tau')\}.$$

The logic has the following semantics:

$$\mathfrak{A} \models_{\mathcal{L}\circ\mathcal{L}'} \varphi \circ \psi \iff \psi^*(\mathfrak{A}) \models_{\mathcal{L}} \varphi.$$

We can set $\tau(\varphi \circ \psi) = \mathrm{Con}(\tau(\varphi)) \cup \bigcup_{R\in\mathrm{Rel}(\tau(\varphi))} \tau(\psi_R)$.

If $\psi \in I(\mathcal{L}', \tau', \tau'')$ and $\theta \in I(\mathcal{L}'', \tau, \tau')$, we define $\psi \circ \theta \in I(\mathcal{L}' \circ \mathcal{L}'', \tau, \tau'')$ such that for all $R \in \mathrm{Rel}(\tau'')$, $(\psi \circ \theta)_R = \psi_R \circ \theta$. This definition satisfies the equation $(\psi \circ \theta)^* = \psi^* \circ \theta^*$. Now, if $\varphi \in \mathcal{L}[\tau'']$, then $(\varphi \circ \psi) \circ \theta \equiv \varphi \circ (\psi \circ \theta)$ and we conclude $(\mathcal{L} \circ \mathcal{L}') \circ \mathcal{L}'' \leq \mathcal{L} \circ (\mathcal{L}' \circ \mathcal{L}'')$. The converse does not generally hold, because there may exist an $\mathcal{L}' \circ \mathcal{L}''$-interpretation that is not a composition of $\mathcal{L}'$- and $\mathcal{L}''$-interpretations. It is however hard to give a natural example of this.

We can define the composition also in a more restricted way in order to acquire associativity. Let $I^w(\mathcal{L}, \tau, \tau')$ be a subset of $I(\mathcal{L}, \tau, \tau')$ containing only interpretations ψ such that for all $R \in \mathrm{Rel}(\tau')$, $\mathrm{Con}(\tau(\psi_R)) = [\overline{x}^{\psi_R}]$. We say that the elements of $I^w(\mathcal{L}, \tau, \tau')$ are *interpretations without parameters.* Define $\mathcal{L} \circ^w \mathcal{L}'$ as $\mathcal{L} \circ \mathcal{L}'$, but I replaced by I^w.

Lemma 2.3. *Every interpretation $\gamma \in I(\mathcal{L} \circ^w \mathcal{L}', \tau, \tau')$ is equivalent to an interpretation $\psi \circ \theta$, for some $\psi \in I(\mathcal{L}, \tau'', \tau')$ and $\theta \in I^w(\mathcal{L}', \tau, \tau'')$. If γ is an intepretation without parameters, also ψ can be chosen to be an interpretation without parameters.*

Proof. For every $R \in \mathrm{Rel}(\tau')$, $\gamma_R \equiv (\psi_R \circ \theta^R, \overline{x}_R)$, where $\psi_R \in \mathcal{L}[\tau_R]$ and $\theta^R \in I^w(\mathcal{L}', \tau, \tau_R)$. We may assume without loss of generality that the sets $\mathrm{Rel}(\tau_R)$ are disjoint for different relation symbols R. Since $\theta^R \in I^w(\mathcal{L}', \tau, \tau_R)$, we may also assume $[\overline{x}_R] \cap \mathrm{Con}(\tau(\theta^R_S)) = \emptyset$ for all $S \in \tau_R$. Let $\tau'' = \bigcup_{R\in\mathrm{Rel}(\tau')} \tau_R$ and $\theta = \bigcup_{R\in\mathrm{Rel}(\tau')} \theta_R \in I^w(\mathcal{L}', \tau, \tau'')$. We may consider formulas $(\psi_R, \overline{x}_R)$ together as an interpretation $\psi \in I(\mathcal{L}, \tau'', \tau')$ and so $\gamma \equiv \psi \circ \theta$. □

Corollary 2.4. *For all logics $\mathcal{L}$, $\mathcal{L}'$ and $\mathcal{L}''$, we have $\mathcal{L} \circ (\mathcal{L}' \circ^w \mathcal{L}'') \equiv (\mathcal{L} \circ \mathcal{L}') \circ^w \mathcal{L}''$.*

We define next a sufficient condition when the composition with and without parameters are equivalent and the associativity of the composition holds.

Let τ be a vocabulary and $\bar{c}$ a sequence of constants. Let τ' be a vocabulary, that contains for every relation symbol $R \in \tau$, an $(\mathrm{ar}(R) + |\bar{c}|)$-ary relation symbol R' and contains the same constant symbols as τ. If $\mathfrak{A}'$ is a $\tau' \cup [\bar{c}]$-structure, let $\mathfrak{A} = \mathrm{pr}_{\bar{c}}(\mathfrak{A}')$ be a τ-structure such that $\mathrm{Dom}(\mathfrak{A}) = \mathrm{Dom}(\mathfrak{A}')$ and for every relation symbol $R \in \tau$, $R^{\mathfrak{A}} = \{\bar{a} \in \mathrm{Dom}(\mathfrak{A})^{\mathrm{ar}(R)} \mid \overline{ac}^{\mathfrak{A}'} \in R^{\mathfrak{A}'}\}$ and all constant symbols of τ have the same interpretation as in $\mathfrak{A}'$. If q is a τ-query, we say that a τ'-query q' is its *$\bar{c}$-decoration*, if

$$\mathrm{pr}_{\bar{c}}(\mathfrak{A}') \models q \iff \mathfrak{A}' \models q'.$$

A logic is *closed under decorations* if for every query definable in the logic all its decorations are also definable.

Lemma 2.5. *If $\mathcal{L}$ is closed under decorations and $\mathcal{L}'$ is an arbitrary logic, $\mathcal{L} \circ \mathcal{L}' \equiv \mathcal{L} \circ^w \mathcal{L}'$.*

Proof. Clearly $\mathcal{L} \circ^w \mathcal{L}' \leq \mathcal{L} \circ \mathcal{L}'$. Suppose $\mathcal{L}$ is closed under decorations and $\varphi \circ \psi \in (\mathcal{L} \circ \mathcal{L}')[\tau]$. Let $\bar{c}$ be a sequence of constant symbols such that $[\bar{c}] = \bigcup_{R \in \tau(\varphi)} \mathrm{Con}(\tau(\psi_R)) \setminus [\bar{x}^{\psi_R}]$. Let φ' be an $\mathcal{L}$-sentence expressing the $\bar{c}$-decoration of the query that φ defines and let ψ' be the interpretation such that $\psi'_R = \psi_R(\bar{x}^{\psi_R}\bar{c})$ Now $\varphi' \circ \psi'$ is in $\mathcal{L} \circ^w \mathcal{L}'$ and equivalent to $\varphi \circ \psi$. □

Lemma 2.6. *If $\mathcal{L}'$ is closed under decorations, $\mathcal{L} \circ \mathcal{L}'$ is also closed under decorations.*

Proof. Let $\varphi \circ \psi \in (\mathcal{L} \circ \mathcal{L}')[\tau]$ and suppose $\bar{c}$ is a sequence of constant symbols. For all $R \in \mathrm{Rel}(\tau(\varphi))$, let ψ'_R be a formula expressing the $\bar{c}$-decoration of the query defined by ψ_R. Then ψ' is an interpretation in the vocabulary $\tau \cup [\bar{c}]$ such that $\psi^*(\mathrm{pr}_{\bar{c}}(\mathfrak{A})) = (\psi')^*(\mathfrak{A})$. Now $\varphi \circ \psi'$ expresses the $\bar{c}$-decoration of the query defined by $\varphi \circ \psi$ which proves the lemma. □

Corollary 2.7. *If $\mathcal{L}'$ is closed under decorations, $\mathcal{L} \circ (\mathcal{L}' \circ \mathcal{L}'') \equiv (\mathcal{L} \circ \mathcal{L}') \circ \mathcal{L}''$.*

Proof. By the lemmas just proven,

$$\mathcal{L} \circ (\mathcal{L}' \circ \mathcal{L}'') \equiv \mathcal{L} \circ (\mathcal{L}' \circ^w \mathcal{L}'') \equiv (\mathcal{L} \circ \mathcal{L}') \circ^w \mathcal{L}'' \equiv (\mathcal{L} \circ \mathcal{L}') \circ \mathcal{L}''.$$

□

If $(\mathcal{L}^i)_{i \in I}$ is a sequence of logics, we can define a new logic as their union. $\mathcal{L} = \bigcup_{i \in I} \mathcal{L}^i$ is a logic such that $\mathcal{L}[\tau] = \{(i, \varphi) \mid i \in I, \varphi \in \mathcal{L}^i[\tau]\}$ and $\mathfrak{A} \models_{\mathcal{L}} (i, \varphi) \iff \mathfrak{A} \models_{\mathcal{L}^i} \varphi$.

A logic $\mathcal{L}$ is closed under substitution, if $\mathcal{L} \circ \mathcal{L} \leq \mathcal{L}$ holds. Let $\langle \mathcal{L} \rangle^1 = \mathcal{L}$, $\langle \mathcal{L} \rangle^{k+1} = \mathcal{L} \cup (\mathcal{L} \circ \langle \mathcal{L} \rangle^k)$ and $\langle \mathcal{L} \rangle = \bigcup_{k \in \mathbb{Z}_+} \langle \mathcal{L} \rangle^k$. Then we have

Lemma 2.8. *If $\mathcal{L}$ is finitary, $\langle \mathcal{L} \rangle$ is the least extension of $\mathcal{L}$ closed under substitution.*

Proof. If $\mathcal{L}'$ is a logic such that $\mathcal{L} \leq \mathcal{L}'$ and $\mathcal{L}'$ is closed under substitution, we can show by induction that $\langle \mathcal{L} \rangle^k \leq \mathcal{L}'$. Hence, if $\langle \mathcal{L} \rangle$ is closed under substitution, it must be the minimal such logic.

Now, suppose $\mathcal{L}$ is finitary. Let $\varphi \circ \theta \in \langle \mathcal{L} \rangle \circ \langle \mathcal{L} \rangle$. Then for some $k \in \mathbb{N}$, $\varphi \in \langle \mathcal{L} \rangle^k$. Because $\langle \mathcal{L} \rangle^k$ is finitary, $\tau(\varphi)$ is finite and there exists $l \in \mathbb{N}$ such that $\theta \restriction \tau(\varphi) \in \langle \mathcal{L} \rangle^l$. Thus $\varphi \circ \theta$ is equivalent to an $\langle \mathcal{L} \rangle^k \circ \langle \mathcal{L} \rangle^l$-sentence. Because $\langle \mathcal{L} \rangle^k \circ \langle \mathcal{L} \rangle^l \leq \langle \mathcal{L} \rangle^{k+l} \leq \langle \mathcal{L} \rangle$, we have $\langle \mathcal{L} \rangle \circ \langle \mathcal{L} \rangle \leq \langle \mathcal{L} \rangle$. □

2.5. **Specific logics.** Let FO be ordinary first-order logic and QF its quantifier-free fragment. Let QF_∞ be a logic of all infinitary quantifier free sentences, i.e., $\mathrm{QF}_\infty[\tau]$ contains all atomic sentences $R\overline{c}$, for $R, \overline{c} \in \tau$ and negations, infinite conjunctions and disjunctions of the sentences already in $\mathrm{QF}_\infty[\tau]$. Our convention to consider only vocabularies with finitely many constant symbols affects the nature of this logic. All three logics are closed under decorations.

If Q is a τ-query, we may define the least logic $\mathcal{L}_\mathrm{Q}$ expressing Q. The conventional way to define the logic if $\tau = \{R_0, \ldots, R_{n-1}\}$ is to use a syntactic construction called *generalized quantifier*: the sentences of $\mathcal{L}_\mathrm{Q}[\tau']$ are of the form

$$\varphi \equiv \mathrm{Q}\overline{x}_0, \ldots, \overline{x}_{n-1}(S_0\overline{y}_0, \ldots, S_{n-1}\overline{y}_{n-1}),$$

where each S_i is a relation symbol in the vocabulary τ', $|\overline{x}_i| = \mathrm{ar}(R_i)$ and each sequence $\overline{y}_i$ contains variables in $\overline{x}_i$ and constant symbols in the vocabulary τ'. We have $\mathfrak{A} \models \varphi$, if and only if $(\mathrm{Dom}(\mathfrak{A}), \psi_0(\mathfrak{A})/R_0, \ldots, \psi_{n-1}(\mathfrak{A})/R_{n-1}) \in \mathrm{Q}$, where $\psi_i(\overline{x}_i) = S_i\overline{y}_i$.

Using the notations defined so far, we can write $\mathrm{FO} \equiv \langle \mathrm{QF} \cup \mathcal{L}_\exists \rangle$ and $\mathrm{FO}(\mathrm{Q}) \equiv \langle \mathrm{QF} \cup \mathcal{L}_\exists \cup \mathcal{L}_\mathrm{Q} \rangle$, where $\exists$ is a $\{U\}$-query such that $\mathfrak{A} \in \exists$ if and only if $U^\mathfrak{A} \neq \emptyset$. If Q is a set of quantifiers, then we have to take union of the logics $\mathcal{L}_\mathrm{Q}$, $\mathrm{Q} \in Q$, and $\mathrm{QF} \cup \mathcal{L}_\exists$ and close it by $\langle \cdot \rangle$ to form $\mathrm{FO}(Q)$.

Let UL_k be a logic that defines all τ-queries q such that τ is finite, relational and for all $R \in \tau$, $\mathrm{ar}(R) \leq k$.

Given a logic $\mathcal{L}$, let $\mathcal{L} \restriction k$ be a fragment of $\mathcal{L}$ containing only sentences φ such that $|\mathrm{Con}(\tau(\varphi))| \leq k$.

2.6. **Equivalence relations and types.** Every set of sentences $\Phi \subseteq \mathcal{L}[\tau]$ induces an equivalence relation $\equiv_\Phi$ on $\mathrm{Mod}(\tau)$ defined as

$$\mathfrak{A} \equiv_\Phi \mathfrak{B} \iff \{\varphi \in \Phi \mid \mathfrak{A} \models \varphi\} = \{\varphi \in \Phi \mid \mathfrak{B} \models \varphi\}.$$

When $\Phi = \{\varphi\}$ we write $\equiv_\Phi$ as $\equiv_\varphi$. If Φ is finite $\equiv_\Phi$ is *a finite equivalence relation*, i.e., it has finitely many equivalence classes.

We say that a τ-query q *preserves* an equivalence relation $\equiv$ on $\mathrm{Mod}(\tau)$, if $\mathfrak{A} \equiv \mathfrak{B}$ implies $\mathfrak{A} \models q \iff \mathfrak{B} \models q$.

Lemma 2.9. *A logic $\mathcal{L}$ can express all queries preserving $\equiv_\mathcal{L}$ if and only if* $\mathrm{QF}_\infty \circ \mathcal{L} \equiv \mathcal{L}$.

Proof. Assume first that $\mathrm{QF}_\infty \circ \mathcal{L} \equiv \mathcal{L}$. For every τ-structure $\mathfrak{A}$, define

$$\psi_{\mathfrak{A}} \equiv \bigwedge_{\substack{\varphi \in \mathcal{L}[\tau] \\ \mathfrak{A} \models \varphi}} \varphi \wedge \bigwedge_{\substack{\varphi \in \mathcal{L}[\tau] \\ \mathfrak{A} \not\models \varphi}} \neg\varphi.$$

Then $\mathfrak{A} \equiv_{\mathcal{L}} \mathfrak{B}$ if and only if $\mathfrak{B} \models \psi_{\mathfrak{A}}$. Now, if q is a τ-query preserving $\equiv_{\mathcal{L}}$, we can express it as $\bigvee_{\mathfrak{A} \in q} \psi_{\mathfrak{A}}$. The sentence is in $\mathrm{QF}_\infty \circ \mathcal{L}$ and so the query is also expressible in $\mathcal{L}$.

Assume then that $\mathcal{L}$ can express all queries preserving $\equiv_{\mathcal{L}}$. Every boolean combination of the sentences preserving $\equiv_{\mathcal{L}}$ preserves $\equiv_{\mathcal{L}}$ and so $\mathrm{QF}_\infty \circ \mathcal{L} \leq \mathcal{L}$. $\square$

Types are conceptually equivalence classes of $\equiv_\Phi$. For notational reasons, we however define Φ-type $t(\overline{x})$ as a subset of Φ such that for some structure $\mathfrak{A}$ and $\overline{a} \in \mathrm{Dom}(\mathfrak{A})$, $t(\overline{x}) = \mathrm{tp}^{\mathfrak{A}}_{\Phi}(\overline{a}) = \{\varphi \in \Phi \mid (\mathfrak{A}, \overline{a}/\overline{x}) \models \varphi\}$. Atomic types are types with $\Phi = \mathrm{QF}$. We denote $\mathrm{tp}^{\mathfrak{A}}_{\mathrm{QF}}(\overline{a})$ by $\mathrm{atp}^{\mathfrak{A}}(\overline{a})$.

If $t(\overline{x})$ is a type and $\varphi \in \Phi$, we sometimes denote $\varphi \in t(\overline{x})$ by $t(\overline{x}) \models \varphi(\overline{x})$. The notation $t(\overline{x}) \restriction \overline{y}$ means the maximal subset of $t(\overline{x})$ containing only sentences φ with $\tau(\varphi) \cap ([\overline{x}] \setminus [\overline{y}]) = \emptyset$.

2.7. **Characterizing equivalence relations with games.** We assume that the reader is familiar with the Ehrenfeucht-Fraisse game for FO. We describe in this section a game characterization for $\langle \mathrm{QF} \cup \mathrm{UL}_k \rangle$. The characterization, defined in [Hel89], is particularly elegant and it is called *k-bijective game.*

Definition 2.10. Let $\mathfrak{A}$ and $\mathfrak{B}$ be τ-structures. $\mathrm{BG}^k_n(\mathfrak{A}, \mathfrak{B})$ is a game between two players I and II. If $n = 0$, Player II wins the game if and only if $\mathfrak{A} \equiv_{\mathrm{QF}} \mathfrak{B}$. Otherwise, Player II chooses a bijection $\alpha\colon \mathrm{Dom}(\mathfrak{A}) \to \mathrm{Dom}(\mathfrak{B})$ and after that Player I chooses k elements $\overline{a} \in \mathrm{Dom}(\mathfrak{A})^k$. The game continues as $\mathrm{BG}^k_{n-1}((\mathfrak{A}, \overline{a}/\overline{x}), (\mathfrak{B}, \alpha(\overline{a})/\overline{x}))$, where $\overline{x}$ is a sequence of constants not in τ.

Lemma 2.11. $\mathfrak{A} \equiv_{\mathrm{UL}_k \circ \mathcal{L}} \mathfrak{B}$ *if and only if there exists a bijection* $\alpha\colon \mathrm{Dom}(\mathfrak{A}) \to \mathrm{Dom}(\mathfrak{B})$ *such that for all at most k-ary $\mathcal{L}$-formulas $\psi(\overline{x})$, $\alpha(\psi(\mathfrak{A})) = \psi(\mathfrak{B})$.*

Proof. If $\mathfrak{A}$ and $\mathfrak{B}$ have different cardinality, there exist $\varphi \in \mathrm{UL}_k[\emptyset]$ and $\psi \in I(\mathcal{L}, \tau, \emptyset)$ such that $\mathfrak{A} \not\equiv_{\varphi \circ \psi} \mathfrak{B}$, so we may assume that $\mathfrak{A}$ and $\mathfrak{B}$ have the same cardinality. Assume that a bijection described in the lemma does not exist. Then for every bijection $\alpha\colon \mathrm{Dom}(\mathfrak{A}) \to \mathrm{Dom}(\mathfrak{B})$ there exists an $\mathcal{L}$-formula θ_α with arity at most k, such that $\alpha(\theta_\alpha(\mathfrak{A})) \neq \theta_\alpha(\mathfrak{B})$. Because there are only finitely many bijections between finite structures, we may form an interpretation ψ such that every formula θ_α interprets some relation symbol. No bijection α can be an isomorphism from $\psi^*(\mathfrak{A})$ to $\psi^*(\mathfrak{B})$ and so $\psi^*(\mathfrak{A}) \not\cong \psi^*(\mathfrak{B})$. Because UL_k can express all queries on finite vocabularies with arity at most k, there exists $\varphi \in \mathrm{UL}_k$ such that $\mathfrak{A} \not\equiv_{\varphi \circ \psi} \mathfrak{B}$.

On the other hand, if such a bijection $\alpha\colon \mathrm{Dom}(\mathfrak{A}) \to \mathrm{Dom}(\mathfrak{B})$ exists, then for any $\mathcal{L}$-interpretation ψ such that the arity of all formulas in ψ is at most k, we have $\alpha\colon \psi^*(\mathfrak{A}) \cong \psi^*(\mathfrak{B})$ and no UL_k-sentence can separate the structures. $\square$

Theorem 2.12. *Let* $\mathcal{L}_k^0 = \mathrm{QF}$, $\mathcal{L}_k'^0 = \mathrm{QF}_\infty$, $\mathcal{L}_k^{n+1} = \mathrm{UL}_k \circ \mathcal{L}_k^n$ *and* $\mathcal{L}_k'^{n+1} = \mathrm{UL}_k \circ \mathrm{QF}_\infty \circ \mathcal{L}_k'^n$*. The following are equivalent for all* $n \geq 0$*:*

a) Player II wins $\mathrm{BG}_n^k(\mathfrak{A}, \mathfrak{B})$

b) $\mathfrak{A} \equiv_{\mathcal{L}_k^n} \mathfrak{B}$

c) $\mathfrak{A} \equiv_{\mathcal{L}_k'^n} \mathfrak{B}$*.*

Proof. The claim is clear for $n = 0$. Assume that the theorem holds for n. We prove it for $n + 1$.

By definition, Player II wins $\mathrm{BG}_{n+1}^k(\mathfrak{A}, \mathfrak{B})$ if and only if there exists a bijection $\alpha\colon \mathrm{Dom}(\mathfrak{A}) \to \mathrm{Dom}(\mathfrak{B})$ such that for any sequence $\overline{a} \in \mathrm{Dom}(\mathfrak{A})^k$, Player II wins $\mathrm{BG}_n^k((\mathfrak{A}, \overline{a}/\overline{x}), (\mathfrak{B}, \alpha(\overline{a})/\overline{x}))$. By induction hypothesis, this is equivalent to $(\mathfrak{A}, \overline{a}/\overline{x}) \equiv_{\mathcal{L}_k^n} (\mathfrak{B}, \alpha(\overline{a})/\overline{x})$ which is equivalent to $\alpha(\psi(\mathfrak{A})) = \psi(\mathfrak{B})$ for all at most k-ary $\mathcal{L}_k^n$-formulas ψ. Thus Player II wins the game by Lemma 2.11 if and only if $\mathfrak{A} \equiv_{\mathrm{UL}_k \circ \mathcal{L}_k^n} \mathfrak{B}$. This proves the equivalence of (a) and (b). The equivalence of (a) and (c) is proved in the same way using the fact that $\equiv_{\mathcal{L}_k'^n}$ and $\equiv_{\mathrm{QF}_\infty \circ \mathcal{L}_k'^n}$ are the same equivalence relation. □

2.8. **Regularity.** We call a logic *semi-regular*, if it is closed under substitution and contains FO. We will be mainly interested in logics of the form $\mathrm{FO}(Q)$, where Q is a set of generalized quantifiers. These logics are finitary and semi-regular and every finitary semi-regular logic is also equivalent to some $\mathrm{FO}(Q)$.

Given a τ-query q and a unary relation symbol U not in τ, the *relativization* of q is a $\tau \cup \{U\}$-query q^U such that $\mathfrak{A} \in q^U$ if and only if $\langle U^{\mathfrak{A}} \rangle^{\mathfrak{A} \restriction \tau} \in q$.

We say that a logic $\mathcal{L}$ is *regular*, if it is semi-regular and additionally *closed under relativization*, i.e., if $\varphi \in \mathcal{L}[\tau]$ and U is a unary relation symbol not in τ, then there exists $\varphi^U \in \mathcal{L}[\tau \cup \{U\}]$ that defines the relativization of the query defined by φ.

Although regularity is defined in [Ebb85], there does not seem to be an established name for semi-regularity.

Proposition 2.13. FO(Q) *is regular if and only if the relativization of* Q *is definable in* FO(Q). □

In particular, if Q is a relativization of some query, FO(Q) is regular.

Let $k \in \mathbb{Z}_+$ and let τ be a vocabulary. Let $\tau^{(k)} = \{R^{(k)} \mid R \in \tau\} \cup \{c_0, \ldots, c_{k-1} \mid c \in \tau\}$ be a vocabulary, where $R^(k)$ is a new relation symbol with $\mathrm{ar}(R^{(k)}) = k\,\mathrm{ar}(R)$ and $c_0, \ldots, c_{k-1}$ are new constant symbols. If $\mathfrak{A}$ is a $\tau^{(k)}$-structure, its k-vectorization, $\mathfrak{A}^{(k)}$, is a τ-structure with universe $\mathrm{Dom}(\mathfrak{A})^k$ and if $R \in \tau$ and $r = \mathrm{ar}(R)$,

$$R^{\mathfrak{A}^{(k)}} = \{(\overline{a}^0, \ldots, \overline{a}^{r-1}) \in (\mathrm{Dom}(\mathfrak{A})^k)^r \mid \overline{a}^0 \ldots \overline{a}^{r-1} \in (R^{(k)})^{\mathfrak{A}}\},$$

and if $c \in \tau$ is a constant symbol

$$c^{\mathfrak{A}^{(k)}} = (c_0^{\mathfrak{A}}, \ldots, c_{k-1}^{\mathfrak{A}})$$

A logic $\mathcal{L}$ is *closed under vectorization* if for all $k \in \mathbb{Z}_+$, τ, and $\varphi \in \mathcal{L}[\tau]$, there is $\varphi^{(k)} \in \mathcal{L}[\tau^{(k)}]$ such that for all $\mathfrak{A} \in \mathrm{Mod}(\tau^{(k)})$,

$$\mathfrak{A} \models \varphi^{(k)} \iff \mathfrak{A}^{(k)} \models \varphi.$$

A logic is *vectorized regular* if it is regular and closed under vectorization.

3. Locality

3.1. **Gaifman graph.** Locality of a logic means that the logic can only speak about small neighborhoods of some elements at a time. In order to define different forms of locality formally, we have to introduce the basic concepts of neighborhood, distance and components in structures. These concepts are defined for graphs and thus we can define them for all kind of structures by defining first a graph that represents our intuition of nearness. The basic idea of locality can be applied also using some other notion of neighborhood as we show in Section 3.4.

Given a structure $\mathfrak{A}$, its *Gaifman graph* is $\mathcal{G}(\mathfrak{A}) = (\mathrm{Dom}(\mathfrak{A}), E)$, where

$$E = \{(a, b) \in \mathrm{Dom}(\mathfrak{A})^2 \mid a \neq b, \{a, b\} \subseteq [\bar{c}], \bar{c} \in R^{\mathfrak{A}}, R \in \tau\}.$$

Given $a, b \in \mathrm{Dom}(\mathfrak{A})$, we denote the length of a shortest path between a and b on $\mathcal{G}(\mathfrak{A})$ by $d^{\mathfrak{A}}(a, b)$. If there is no path between a and b, we put $d^{\mathfrak{A}}(a, b) = \infty$. The definition generalizes in the standard way to the distance between a set and an element and for the distance between two sets.

We define an *r-neighborhood* of $\bar{a} \in \mathrm{Dom}(\mathfrak{A})^{<\omega}$ as

$$N_r^{\mathfrak{A}}(\bar{a}) = \{b \in \mathrm{Dom}(\mathfrak{A}) \mid d^{\mathfrak{A}}([\bar{a}], b) \leq r\}.$$

Neighborhoods can also be thought as (partial) substructures of $\mathfrak{A}$. Besides the substructure, neighborhood structure also specifies the center of the neighborhood:

$$\mathfrak{N}_r^{\mathfrak{A}}(\bar{a}) = \bigcup_{i=0}^{|\bar{a}|-1} \langle N_r^{\mathfrak{A}}(a_i)\rangle^{(\mathfrak{A}, \bar{a}/\bar{v})},$$

where $\bar{v}$ is a sequence of constants we reserve for the purpose of defining the center. Note, that the neighborhood given by this definition is not always isomorphic to $\langle N_r^{\mathfrak{A}}(\bar{a})\rangle^{(\mathfrak{A}, \bar{a}/\bar{v})}$, for example if $\bar{a} = a_0a_1$ and $d^{\mathfrak{A}}(a_0, a_1) = 2r + 1$. The reason for choosing the former definition is that it satisfies the equivalence: $\mathfrak{N}_r^{\mathfrak{A}}(ab)$ is connected if and only if $d^{\mathfrak{A}}(a, b) \leq 2r$.

3.2. **Strong Gaifman-locality.** We develop in this section the basic tools for handling compositions of local logics.

Definition 3.1. A sentence φ is *k-bounded*, if for all $a, b \in \mathrm{Con}(\tau(\varphi))$, $\mathfrak{A} \models \varphi$ implies $d^{\mathfrak{A}}(a^{\mathfrak{A}}, b^{\mathfrak{A}}) \leq k$. We call an interpretation k-bounded, if it consists of k-bounded formulas.

The following lemma motivates the definition.

Lemma 3.2. *If an interpretation ψ from τ to τ' is k-bounded then for all τ-structures $\mathfrak{A}$ and all $a, b \in \mathrm{Dom}(\mathfrak{A})$, $d^{\mathfrak{A}}(a, b) \leq kd^{\psi^*(\mathfrak{A})}(a, b)$. If ψ is an interpretation without parameters and for all $\mathfrak{A}$ and $a, b \in \mathrm{Dom}(\mathfrak{A})$, $d^{\mathfrak{A}}(a, b) \leq kd^{\psi^*(\mathfrak{A})}(a, b)$, then ψ is k-bounded.*

Proof. Suppose that ψ is k-bounded. If $d^{\psi^*(\mathfrak{A})}(a,b) = 1$, then there exists a relation symbol $R \in \tau'$ and a sequence $\bar{c} \in R^{\psi^*(\mathfrak{A})}$ such that $a, b \in [\bar{c}]$. Because $\mathfrak{A} \models \psi_R(\bar{c})$ and ψ_R is k-bounded, $d^{\mathfrak{A}}(a,b) \leq k$. Note, that we need here the assumption that if $\psi_R(\bar{x})$ is a formula, $[\bar{x}] \subseteq \tau(\psi_R)$.

Now, if $d^{\psi^*(\mathfrak{A})}(a,b) = r$, there exists a path of length r from a to b, and using triangle inequality and the observation above, we get $d^{\mathfrak{A}}(a,b) \leq rk$, proving one direction of the lemma.

The other direction is easy: If $\mathfrak{A} \models \psi_R(\bar{c})$ and $a, b \in [\bar{c}]$, then $d^{\psi^*(\mathfrak{A})}(a,b) \leq 1$. Then if the inequality holds, $d^{\mathfrak{A}}(a,b) \leq k$, and ψ_R is k-bounded. The assumption that ψ is interpretation without parameters is necessary, because otherwise we could only bound the distance between constants that occur as parameters of ψ_R. □

The following definitions are introduced in [HLN99].

Definition 3.3. Let $[\bar{c}] = \mathrm{Con}(\tau)$, and let q be a τ-query. The query q is *strongly r-Gaifman-local*, if for all τ-structures $\mathfrak{A}$ and $\mathfrak{B}$, $\mathfrak{N}_r^{\mathfrak{A}}(\bar{c}^{\mathfrak{A}}) \cong \mathfrak{N}_r^{\mathfrak{B}}(\bar{c}^{\mathfrak{B}})$ implies $\mathfrak{A} \equiv_q \mathfrak{B}$. The query is *$r$-Gaifman-local*, if for all τ-structures $\mathfrak{A}$ and $\mathfrak{B}$, $\mathfrak{N}_r^{\mathfrak{A}}(\bar{c}^{\mathfrak{A}}) \cong \mathfrak{N}_r^{\mathfrak{B}}(\bar{c}^{\mathfrak{B}})$ and $\mathfrak{A} \restriction \mathrm{Rel}(\tau) \cong \mathfrak{B} \restriction \mathrm{Rel}(\tau)$ implies $\mathfrak{A} \equiv_q \mathfrak{B}$. A logic is Gaifman-local, if all its sentences define Gaifman-local queries.

The definition differs slightly but is equivalent to the usual one.

Let SGL^r be a logic expressing all strongly r-Gaifman-local queries and let SGL^r_k be its fragment containing only the sentences that define k-bounded queries.

Lemma 3.4. *Let $a, b \in \mathrm{Con}(\tau)$. There exists $\theta_{2r}^{\tau,a,b} \in \mathrm{SGL}^r_{2r}[\tau]$ defining the query $q_{2r}^{\tau,a,b} = \{\mathfrak{A} \in \mathrm{Mod}(\tau) \mid d^{\mathfrak{A}\restriction\tau}(a^{\mathfrak{A}}, b^{\mathfrak{B}}) \leq 2r\}$.*

Proof. A sentence defining $q_{2r}^{\tau,a,b}$ is clearly $2r$-bounded. It can be defined in $\mathrm{SGL}^r[\tau]$ because, if $\mathfrak{N}_r^{\mathfrak{A}\restriction\tau}(a^{\mathfrak{A}}b^{\mathfrak{A}}) \cong \mathfrak{N}_r^{\mathfrak{B}\restriction\tau}(a^{\mathfrak{B}}b^{\mathfrak{B}})$ then $d^{\mathfrak{A}\restriction\tau}(a^{\mathfrak{A}}, b^{\mathfrak{A}}) \leq 2r \iff d^{\mathfrak{B}\restriction\tau}(a^{\mathfrak{B}}, b^{\mathfrak{B}}) \leq 2r$. Note, that $\tau(\theta_{2r}^{\tau,a,b}) = \mathrm{Rel}(\tau) \cup \{a, b\}$. □

Lemma 3.5. $\mathrm{SGL}^r \restriction k \equiv (\mathrm{QF}_\infty \circ \mathrm{SGL}^r_{2(k-1)r}) \restriction k$. *In particular,* $\mathrm{SGL}^r \equiv \bigcup_{k\in\mathbb{N}} \mathrm{QF}_\infty \circ \mathrm{SGL}^r_k$.

Proof. Assume that τ is a vocabulary with $k \in \mathbb{N}$ constant symbols and let $\mathfrak{A}$ be a τ-structure. Define a graph $\mathcal{G} = (\mathrm{Con}(\tau), E^{\mathcal{G}})$ such that $(a,b) \in E^{\mathcal{G}}$ if and only if $a \neq b$ and $d^{\mathfrak{A}}(a^{\mathfrak{A}}, b^{\mathfrak{A}}) \leq 2r$.

Let $C \subseteq \mathrm{Con}(\tau)$ be a component of $\mathcal{G}$. Then there exists a sentence $\gamma_{\mathfrak{A}}^C \in \mathrm{SGL}^r_{2dr}$, where d is the diameter of C in $\mathcal{G}$, such that $\mathfrak{B} \models \gamma_{\mathfrak{A}}^C \iff \mathfrak{N}_r^{\mathfrak{A}}(C^{\mathfrak{A}}) \cong \mathfrak{N}_r^{\mathfrak{B}}(C^{\mathfrak{B}})$. Because $\mathfrak{N}_r^{\mathfrak{A}}(C^{\mathfrak{A}})$ is connected, $\mathfrak{B} \models \gamma_{\mathfrak{A}}^C$ implies that $\mathfrak{N}_r^{\mathfrak{B}}(C^{\mathfrak{B}})$ also is connected, and so $\gamma_{\mathfrak{A}}^C$ is $2dr$-bounded. Because $d \leq k-1$, it is also $2(k-1)r$-bounded.

Let $\mathcal{C}$ be the set of all components of $\mathcal{G}$. We can now write a sentence

$$\psi_{\mathfrak{A}} \equiv \bigwedge_{C\in\mathcal{C}} \gamma_{\mathfrak{A}}^C \wedge \bigwedge_{\substack{(a,b)\in\mathrm{Con}(\tau)^2\setminus E^{\mathcal{G}}\\ a\neq b}} \neg\theta_{2r}^{\tau,a,b}$$

in $\mathrm{QF}_\infty \circ \mathrm{SGL}^r_{2(k-1)r}$. The sentence is true on $\mathfrak{B}$ if and only if $\mathfrak{N}_r^{\mathfrak{A}}(\mathrm{Con}(\tau)^{\mathfrak{A}}) \cong \mathfrak{N}_r^{\mathfrak{B}}(\mathrm{Con}(\tau)^{\mathfrak{B}})$.

Now any $\varphi \in (\mathrm{SGL}^r \restriction k)[\tau]$ is equivalent to the sentence

$$\bigvee_{\substack{\mathfrak{A} \in \mathrm{Mod}(\tau) \\ \mathfrak{A} \models \varphi}} \psi_{\mathfrak{A}}.$$

This shows $\mathrm{SGL}^r \restriction k \leq \mathrm{QF}_\infty \circ \mathrm{SGL}^r_{2(k-1)r}$. The other direction is easy since $\mathrm{QF}_\infty \circ \mathrm{SGL}^r \equiv \mathrm{SGL}^r$ and $\mathrm{SGL}^r_{2(k-1)r}$ is a fragment of SGL^r. □

Lemma 3.6. *Let* $\psi \in I(\mathrm{SGL}^{r_1}_{k_1}, \tau, \tau')$ *and suppose* $\mathrm{Con}(\tau) = \mathrm{Con}(\tau') = [\overline{c}]$. *If* $\mathfrak{N}^{\mathfrak{A}}_{r_0k_1+r_1}(\overline{c}^{\mathfrak{A}}) \cong \mathfrak{N}^{\mathfrak{B}}_{r_0k_1+r_1}(\overline{c}^{\mathfrak{B}})$ *then* $\mathfrak{N}^{\psi^*(\mathfrak{A})}_{r_0}(\overline{c}^{\mathfrak{A}}) \cong \mathfrak{N}^{\psi^*(\mathfrak{B})}_{r_0}(\overline{c}^{\mathfrak{B}})$.

Proof. Let α be an isomorphism $\alpha\colon \mathfrak{N}^{\mathfrak{A}}_{r_0k_1+r_1}(\overline{c}^{\mathfrak{A}}) \cong \mathfrak{N}^{\mathfrak{B}}_{r_0k_1+r_1}(\overline{c}^{\mathfrak{B}})$. If $\overline{a} \in N^{\mathfrak{B}}_{r_0k_1}(\overline{c}^{\mathfrak{B}})$, then a restriction of α gives $\mathfrak{N}^{\mathfrak{A}}_{r_1}(\overline{a}\overline{c}^{\mathfrak{A}}) \cong \mathfrak{N}^{\mathfrak{B}}_{r_1}(\alpha(\overline{a})\overline{c}^{\mathfrak{B}})$. This means that for any $R \in \tau'$, $\mathfrak{A} \models \psi_R(\overline{a}) \iff \mathfrak{B} \models \psi_R(\alpha(\overline{a}))$. Thus $\langle N^{\mathfrak{A}}_{r_0k_1}(\overline{c}^{\mathfrak{A}})\rangle^{\psi^*(\mathfrak{A})} \cong \langle N^{\mathfrak{B}}_{r_0k_1}(\overline{c}^{\mathfrak{B}})\rangle^{\psi^*(\mathfrak{B})}$.

By Lemma 3.2, $N^{\psi^*(\mathfrak{A})}_{r_0}(\overline{c}^{\mathfrak{A}}) \subseteq N^{\mathfrak{A}}_{r_0k_1}(\overline{c}^{\mathfrak{A}})$ and $N^{\psi^*(\mathfrak{B})}_{r_0}(\overline{c}^{\mathfrak{B}}) \subseteq N^{\mathfrak{B}}_{r_0k_1}(\overline{c}^{\mathfrak{B}})$. Thus a restriction of α gives us $\mathfrak{N}^{\psi^*(\mathfrak{A})}_{r_0}(\overline{c}^{\mathfrak{A}}) \cong \mathfrak{N}^{\psi^*(\mathfrak{B})}_{r_0}(\overline{c}^{\mathfrak{B}})$. □

Corollary 3.7. $\mathrm{SGL}^{r_0} \circ \mathrm{SGL}^{r_1}_{k_1} \leq \mathrm{SGL}^{r_0k_1+r_1}$. □

Lemma 3.8. *Suppose that* $\mathcal{L}$ *contains only* k_0*-bounded sentences and* $\mathcal{L}'$ *only* k_1*-bounded sentences. Then all sentences in* $\mathcal{L} \circ^w \mathcal{L}'$ *are* k_0k_1*-bounded.*

Proof. We prove that all $\mathcal{L}\circ^w\mathcal{L}'$-interpretations without parameters are k_0k_1-bounded. Any such interpretation can be written by Lemma 2.3 as $\psi \circ \theta$, where ψ is a $\mathcal{L}$-interpretation without parameters and θ a $\mathcal{L}'$-interpretation without parameters. For any structure $\mathfrak{A}$ and $a, b \in \mathrm{Dom}(\mathfrak{A})$, we have $d^{\mathfrak{A}}(a,b) \leq k_1 d^{\theta^*(\mathfrak{A})}(a,b) \leq k_0k_1 d^{(\psi^*\circ\theta^*)(\mathfrak{A})}(a,b) \leq k_0k_1 d^{(\psi\circ\theta)^*(\mathfrak{A})}(a,b)$ by Lemma 3.2 and so, by the same lemma, $\psi \circ \theta$ is k_0k_1-bounded. □

Remark. The lemma does not hold for the ordinary composition of logics. As an example, let $\tau = \{a, b\}$ and $\tau' = \{U, a, b\}$ where U is unary. Let ψ be an interpretation from τ to τ' such that $\psi_U(x) \equiv x = a$. Let $\varphi \equiv \neg Ub$.

Now, $\tau(\psi_U) = \{x, a\}$ and ψ_U is 0-bounded, because if ψ_U holds distance between x and a is zero. Also φ is 0-bounded, because $\tau(\varphi)$ contains only one constant symbol. However, $\varphi \circ \psi$ is not k-bounded for any k, since $\mathfrak{A} \models \varphi \circ \psi$ always when $a^{\mathfrak{A}} \neq b^{\mathfrak{A}}$.

Corollary 3.9. $\mathrm{SGL}^{r_0}_{k_0} \circ^w \mathrm{SGL}^{r_1}_{k_1} \leq \mathrm{SGL}^{r_0k_1+r_1}_{k_0k_1}$. □

3.3. **Hanf-like locality.** The purpose of this section is to give some sufficient conditions for FO(Q) to be Hanf-local. We will later give a definition for Gaifman-locality that resembles Hanf-locality and therefore we prove the results in a general setting so that they can also be applied to Gaifman-locality.

Suppose we are given a sequence $(\equiv^r_\tau)_{r\in\mathbb{N},\tau}$ of equivalence relations on all finite τ-structures for all vocabularies τ. We assume that the following conditions hold

a) If $\mathfrak{A} \cong \mathfrak{B}$ then $\mathfrak{A} \equiv^r_\tau \mathfrak{B}$.
b) For all $r \leq r'$ and $\tau \subseteq \tau'$, $\mathfrak{A} \equiv^{r'}_{\tau'} \mathfrak{B}$ implies $\mathfrak{A}\restriction\tau \equiv^r_\tau \mathfrak{B}\restriction\tau$.

c) If $\mathfrak{A} \equiv_\tau^{r_0 k_1 + r_1} \mathfrak{B}$ and $\psi \in I(\mathrm{SGL}_{k_1}^{r_1}, \tau, \tau')$, then $\psi^*(\mathfrak{A}) \equiv_{\tau'}^{r_0} \psi^*(\mathfrak{B})$.
d) There exist functions $s_0, s_1 : \mathbb{N} \to \mathbb{N}$ such that if $\overline{x}$ is a sequence of constant symbols, $\mathfrak{A} \equiv_\tau^{s_0(|\overline{x}|)r} \mathfrak{B}$ and $\mathfrak{N}_{s_1(|\overline{x}|)r}^{\mathfrak{A}}(\overline{a}) \cong \mathfrak{N}_{s_1(|\overline{x}|)r}^{\mathfrak{B}}(\overline{b})$, then $(\mathfrak{A}, \overline{a}/\overline{x}) \equiv_{\tau \cup [\overline{x}]}^r (\mathfrak{B}, \overline{b}/\overline{x})$.

We will usually drop the subscript because it is determined by the structures.

The functions s_0 and s_1 appearing in the conditions are constants in the case of Hanf-locality, but we need to allow them to increase later when we consider Gaifman-locality.

The following definition is again from [HLN99].

Definition 3.10. We write $\alpha\colon \mathfrak{A} \leftrightarrows_r \mathfrak{B}$, if α is a bijection $\mathrm{Dom}(\mathfrak{A}) \to \mathrm{Dom}(\mathfrak{B})$ such that for all $a \in \mathrm{Dom}(\mathfrak{A})$, $\mathfrak{N}_r^{\mathfrak{A}}(a) \cong \mathfrak{N}_r^{\mathfrak{B}}(\alpha(a))$. We denote the existence of such bijection by $\mathfrak{A} \leftrightarrows_r \mathfrak{B}$ and call $\mathfrak{A}$ and $\mathfrak{B}$ *r-Hanf-equivalent*. A query or a sentence φ is *r-Hanf-local*, if for all $\mathfrak{A} \leftrightarrows_r \mathfrak{B}$, we have $\mathfrak{A} \equiv_\varphi \mathfrak{B}$. A logic is Hanf-local, if all its sentences are r-Hanf-local for some $r \in \mathbb{N}$.

We verify that Hanf-equivalence $(\leftrightarrows_r)_{r\in\mathbb{N}}$ satisfies the conditions given above. Conditions (a) and (b) are clear. In order to verify condition (c), let $\alpha\colon \mathfrak{A} \leftrightarrows_{r_0 k_1 + r_1} \mathfrak{B}$, $\psi \in I(\mathrm{SGL}_{k_1}^{r_1}, \tau, \tau')$ and $a \in \mathrm{Dom}(\mathfrak{A})$. For any $\varphi \in \mathrm{SGL}^{r_0}[\tau' \cup \{x\}]$, $(\mathfrak{A}, a/x) \equiv_{\varphi\circ\psi} (\mathfrak{B}, \alpha(a)/x)$ by Corollary 3.7, and so $(\psi^*(\mathfrak{A}), a/x) \equiv_\varphi (\psi^*(\mathfrak{B}), \alpha(a)/x)$. This means $\mathfrak{N}_{r_0}^{\psi^*(\mathfrak{A})}(a) \cong \mathfrak{N}_{r_0}^{\psi^*(\mathfrak{B})}(\alpha(a))$. Because a was arbitrary, $\alpha\colon \psi^*(\mathfrak{A}) \leftrightarrows_{r_0} \psi^*(\mathfrak{B})$. Condition (d) holds with $s_0(k) = 1$ and $s_1(k) = 2$ by the following lemma.

Lemma 3.11. *If* $\mathfrak{A} \leftrightarrows_r \mathfrak{B}$ *and* $\mathfrak{N}_{2r}^{\mathfrak{A}}(\overline{a}) \cong \mathfrak{N}_{2r}^{\mathfrak{B}}(\overline{b})$*, then* $(\mathfrak{A}, \overline{a}/\overline{x}) \leftrightarrows_r (\mathfrak{B}, \overline{b}/\overline{x})$.

Proof. If $\mathfrak{A} \leftrightarrows_r \mathfrak{B}$, then any partial injection $\alpha'\colon \mathrm{Dom}(\mathfrak{A}) \to \mathrm{Dom}(\mathfrak{B})$ such that for all $c \in \mathrm{dom}(\alpha')$, $\mathfrak{N}_r^{\mathfrak{A}}(c) \cong \mathfrak{N}_r^{\mathfrak{B}}(\alpha'(c))$, can be extended to $\alpha \supseteq \alpha'$ such that $\alpha\colon \mathfrak{A} \leftrightarrows_r \mathfrak{B}$.

Assume $\beta\colon \mathfrak{N}_{2r}^{\mathfrak{A}}(\overline{a}) \cong \mathfrak{N}_{2r}^{\mathfrak{B}}(\overline{b})$. Then $\alpha' = \beta \restriction N_r^{\mathfrak{A}}(\overline{a})$ satisfies the condition above and can be extended to α. But then $\alpha\colon (\mathfrak{A}, \overline{a}/\overline{x}) \cong (\mathfrak{B}, \overline{b}/\overline{x})$: If $c \in N_r^{\mathfrak{A}}(\overline{a})$, then $\beta \restriction N_r^{\mathfrak{A}}(c)\colon \mathfrak{N}_r^{\mathfrak{A}}(c) \cong \mathfrak{N}_r^{\mathfrak{A}}(\alpha(c))$. If $c \in \mathrm{Dom}(\mathfrak{A}) \subseteq N_r^{\mathfrak{A}}(\overline{a})$, then no constant symbols in $\overline{x}$ are defined in $\mathfrak{N}^{(\mathfrak{A},\overline{a}/\overline{x})}(c)$ or $\mathfrak{N}^{(\mathfrak{B},\overline{b}/\overline{x})}(\alpha(c))$ and so $\mathfrak{N}^{(\mathfrak{A},\overline{a}/\overline{x})}(c) \cong \mathfrak{N}^{(\mathfrak{B},\overline{b}/\overline{x})}(\alpha(c))$. ☐

Let $\mathrm{HL}^r = \mathrm{HL}(\equiv^r)$ be a logic that defines all queries preserving $\equiv^r$ and no others, and define $\mathrm{HL} = \bigcup_{r\in\mathbb{N}} \mathrm{HL}^r$.

Lemma 3.12. $\mathrm{HL}^{r_0} \circ \mathrm{SGL}_{k_1}^{r_1} \leq \mathrm{HL}^{r_0 k_1 + r_1}$.

Proof. This is the condition (c) rephrased. ☐

Corollary 3.13. $\mathrm{HL}^{r_0} \circ (\mathrm{HL}^{r_0 k_1 + r_1} \restriction 0 \cup \mathrm{SGL}_{k_1}^{r_1}) \leq \mathrm{HL}^{r_0 k_1 + r_1}$.

Proof. If $\varphi \circ \psi \in \mathrm{HL}^{r_0} \circ (\mathrm{HL}^{r_0 k_1 + r_1} \restriction 0 \cup \mathrm{SGL}_{k_1}^{r_1})$ and $\mathfrak{A} \equiv^{r_0 k_1 + r_1} \mathfrak{B}$, then all $\mathrm{HL}^{r_0 k_1 + r_1}$-sentences agree on $\mathfrak{A}$ and $\mathfrak{B}$ and can be eliminated from the sentence. Then the previous lemma gives the result. ☐

Lemma 3.14. $\mathrm{HL}^r \restriction k \leq \mathrm{QF}_\infty \circ (\mathrm{HL}^{s_0(k)r} \restriction 0 \cup \mathrm{SGL}^{s_1(k)r} \restriction k)$.

Proof. This is a direct consequence of the condition (d). ☐

If we combine this with Lemma 3.5, we get

Corollary 3.15. $\mathrm{HL}^r \restriction k \leq \mathrm{QF}_\infty \circ (\mathrm{HL}^{s_0(k)r} \restriction 0 \cup \mathrm{SGL}^{s_1(k)r}_{2r(k-1)} \restriction k)$.

Given a logic $\mathcal{L}$ let $\mathcal{L}^{\mathrm{fin}}$ be its finitary fragment, i.e. a logic containing all finitary sentences of $\mathcal{L}$. Now, denote the logic $\bigcup_{r,r',k\in\mathbb{N}} \mathrm{QF} \circ (\mathrm{HL}^r \restriction 0 \cup \mathrm{SGL}^{r'}_k)^{\mathrm{fin}}$ by $\mathrm{HL}^* = \mathrm{HL}^*(\equiv_r)_{r\in\mathbb{N}}$.

We can now prove the main result of this section.

Theorem 3.16. *If one of the following conditions hold, then* $\langle\mathcal{L}\rangle \leq \mathrm{HL}$.
1) $\mathcal{L}$ *is finitary and* $\mathcal{L} \circ \mathrm{QF}_\infty \leq \mathrm{HL}$.
2) $\mathcal{L}\circ\mathrm{QF}_\infty \leq \mathrm{HL}^r$ *for some* $r \in \mathbb{N}$ *and for all* $k \in \mathbb{N}$, $\sup\{\mathrm{ar}(R) \mid R \in \mathrm{Rel}(\tau(\varphi)), \varphi \in \mathcal{L}\restriction k\} < \infty$.
3) $\mathcal{L}$ *is finitary and* $\mathcal{L} \circ \mathrm{QF} \leq \mathrm{HL}^*$.

We prove each condition in Lemmas 3.17–3.20.

Lemma 3.17. *If* $\mathcal{L}$ *is finitary and* $\mathcal{L} \circ \mathrm{QF}_\infty \leq \mathrm{HL}$, *then* $\mathcal{L} \circ \mathrm{HL} \leq \mathrm{HL}$.

Proof. Because $\mathcal{L}$ is finitary, $\varphi \circ \psi \in \mathcal{L} \circ \mathrm{HL}$ implies that for some $r, k \in \mathbb{N}$, $\varphi \circ \psi \in \mathcal{L} \circ (\mathrm{HL}^r \restriction k)$. We have

$$\begin{aligned}
\mathcal{L} \circ (\mathrm{HL}^r \restriction k) &\leq \mathcal{L} \circ (\mathrm{QF}_\infty \circ (\mathrm{HL}^{s_0(k)r} \restriction 0 \cup \mathrm{SGL}^{s_1(k)r}_{2r(k-1)} \restriction k)) \\
&\equiv (\mathcal{L} \circ \mathrm{QF}_\infty) \circ (\mathrm{HL}^{s_0(k)r} \restriction 0 \cup \mathrm{SGL}^{s_1(k)r}_{2r(k-1)} \restriction k) \\
&\leq \mathrm{HL} \circ (\mathrm{HL}^{s_0(k)r} \restriction 0 \cup \mathrm{SGL}^{s_1(k)r}_{2r(k-1)} \restriction k) \\
&\leq \mathrm{HL},
\end{aligned}$$

where we have applied Corollary 3.15, Corollary 2.7, the assumptions of the lemma and Corollary 3.13. □

This implies the sufficiency of the condition (1) by induction.

We can drop the assumption of finitarity, if we can bound the locality rank of the logic $\mathcal{L} \circ \mathrm{QF}_\infty$ uniformly.

Lemma 3.18. *If* $\mathcal{L} \circ \mathrm{QF}_\infty \leq \mathrm{HL}^{r_0}$, *then* $\mathcal{L} \circ (\mathrm{HL}^{r_1} \restriction k) \leq \mathrm{HL}^{m(r_0,r_1,k)}$, *where* $m(r_0, r_1, k) = \max\{s_0(k)r_0, 2r_0r_1(k-1) + s_1(k)r_1\}$.

Proof. We have

$$\begin{aligned}
\mathcal{L} \circ (\mathrm{HL}^{r_1} \restriction k) &\leq \mathcal{L} \circ (\mathrm{QF}_\infty \circ (\mathrm{HL}^{s_0(k)r_1} \restriction 0 \cup \mathrm{SGL}^{s_1(k)r_1}_{2r_1(k-1)} \restriction k)) \\
&\equiv (\mathcal{L} \circ \mathrm{QF}_\infty) \circ (\mathrm{HL}^{s_0(k)r_1} \restriction 0 \cup \mathrm{SGL}^{s_1(k)r_1}_{2r(k-1)} \restriction k) \\
&\leq \mathrm{HL}^{r_0} \circ (\mathrm{HL}^{s_0(k)r_1} \restriction 0 \cup \mathrm{SGL}^{s_1(k)r_1}_{2r_1(k-1)} \restriction k) \\
&\leq \mathrm{HL}^{m(r_0,r_1,k)}.
\end{aligned}$$

□

Now, if condition (2) holds, and $m_k = \sup\{\mathrm{ar}(R) \mid R \in \mathrm{Rel}(\tau(\varphi)), \varphi \in \mathcal{L} \restriction k\}$, then for all $\varphi \circ \psi \in (\mathcal{L} \circ \mathrm{HL}^{r_1}) \restriction k$, $\psi \in I(\mathrm{HL}^{r_1} \restriction (k + m_k), \tau, \tau')$, since τ has at most k constant symbols and arities of the relation symbols in τ' are bounded by m_k. Thus $(\mathcal{L} \circ \mathrm{HL}^{r_1}) \restriction k \equiv (\mathcal{L} \circ (\mathrm{HL}^{r_1} \restriction (k + m_k))) \restriction k \leq \mathrm{HL}^r \restriction k$ for some r by the previous lemma. Thus we can prove for all i that $\langle \mathcal{L} \rangle^i \restriction k \leq \mathrm{HL}$, which implies $\langle \mathcal{L} \rangle \leq \mathrm{HL}$.

Lemma 3.19. $(\mathrm{QF} \circ (\mathrm{HL}^{r_0} \restriction 0 \cup \mathrm{SGL}^{r'_0}_{k_0})^{\mathrm{fin}}) \circ^w (\mathrm{HL}^{r_1} \restriction 0 \cup \mathrm{SGL}^{r'_1}_{k_1})^{\mathrm{fin}} \leq \mathrm{QF} \circ (\mathrm{HL}^{r_0 k_1 + r'_1} \restriction 0 \cup \mathrm{HL}^{r_1} \restriction 0 \cup \mathrm{SGL}^{r'_0 k_1 + r'_1}_{k_0 k_1})^{\mathrm{fin}}$. *In particular,* $\mathrm{HL}^* \circ^w (\mathrm{HL}^{r_1} \restriction 0 \cup \mathrm{SGL}^{r'_1}_{k_1})^{\mathrm{fin}} \leq \mathrm{HL}^*$. □

Proof. We can write

$$(\mathrm{QF} \circ (\mathrm{HL}^{r_0} \restriction 0 \cup \mathrm{SGL}^{r'_0}_{k_0})^{\mathrm{fin}}) \circ^w (\mathrm{HL}^{r_1} \restriction 0 \cup \mathrm{SGL}^{r'_1}_{k_1})^{\mathrm{fin}}$$

as

$$\mathrm{QF} \circ ((\mathrm{HL}^{r_1} \restriction 0)^{\mathrm{fin}} \cup (\mathrm{HL}^{r_0} \restriction 0 \cup \mathrm{SGL}^{r'_0}_{k_0})^{\mathrm{fin}} \circ^w (\mathrm{SGL}^{r'_1}_{k_1})^{\mathrm{fin}})$$

which is equivalent to

$$\mathrm{QF} \circ (\mathrm{HL}^{r_1} \restriction 0 \cup (\mathrm{HL}^{r_0} \restriction 0 \circ^w \mathrm{SGL}^{r'_1}_{k_1}) \cup (\mathrm{SGL}^{r'_0}_{k_0} \circ^w \mathrm{SGL}^{r'_1}_{k_1}))^{\mathrm{fin}}$$

and that can be written as in the lemma. □

Lemma 3.20. *If* $\mathcal{L}$ *is finitary and* $\mathcal{L} \circ \mathrm{QF} \leq \mathrm{HL}^*$ *then* $\mathcal{L} \circ \mathrm{HL}^* \leq \mathrm{HL}^*$.

Proof. Because $\mathcal{L}$ is finitary, $\varphi \in \mathcal{L} \circ \mathrm{HL}^*$ implies that for some $r_1, r'_1, k_1 \in \mathbb{N}$,

$$\begin{aligned}
\varphi &\in \mathcal{L} \circ (\mathrm{QF} \circ (\mathrm{HL}^{r_1} \restriction 0 \cup \mathrm{SGL}^{r'_1}_{k_1})^{\mathrm{fin}}) \\
&\equiv (\mathcal{L} \circ \mathrm{QF}) \circ^w (\mathrm{HL}^{r_1} \restriction 0 \cup \mathrm{SGL}^{r'_1}_{k_1})^{\mathrm{fin}} \\
&\leq \mathrm{HL}^* \circ^w (\mathrm{HL}^{r_1} \restriction 0 \cup \mathrm{SGL}^{r'_1}_{k_1}) \\
&\leq \mathrm{HL}^*.
\end{aligned}$$

□

This implies the sufficiency of the condition (3) in Theorem 3.16.

The main purpose of Theorem 3.16 was to give sufficient conditions when FO(Q) is Hanf-local. Because $\mathrm{FO(Q)} \equiv \langle \mathrm{QF} \cup \mathcal{L}_\exists \cup \mathcal{L}_\mathrm{Q} \rangle$, we need to show that QF and $\mathcal{L}_\exists$ satisfy the conditions.

Strictly speaking, QF does not satisfy the second part of the condition (2). However, if $\varphi \circ \psi \in (\mathrm{QF} \restriction k) \circ \mathcal{L}$ and $R \in \tau(\varphi)$ with $\mathrm{ar}(R) > k$, then some constant has to be repeated at every occurrence of R. Assume that $Rx_0x_0 \ldots x_{n-1}$ is one of the occurrences. Let φ' be a QF-sentence identical to φ except $Rx_0x_0 \ldots x_{n-1}$ replaced by $R'x_0 \ldots x_{n-1}$, where $\mathrm{ar}(R') = \mathrm{ar}(R) - 1$. Let ψ' be a new $\mathcal{L}$-interpretation, where we have added a formula $\psi_{R'}$ to ψ and $\psi_{R'}$ is get from $\psi_R(\overline{y})$ by identifying the parameters y_0 and y_1. This is possible by the requirements we assumed every logic to satisfy. Then $\varphi \circ \psi \equiv \varphi' \circ \psi'$. In this way, we can eliminate all occurrences of the relation symbols with arity greater than k from φ. So for the purposes of Theorem 3.16, we may assume that QF satisfies the condition (2). The same argument applies also for QF_∞.

Proposition 3.21. *The logics* QF *and* UL_1 *satisfy (essentially) the conditions (1) and (2) of Theorem 3.16 and* QF_∞ *satisfies (essentially) the condition (2).*

Proof. Because 1-neighborhoods of the constants of a structure determine the atomic type of the constants, $\mathrm{QF}_\infty \circ \mathrm{QF}_\infty \equiv \mathrm{QF}_\infty$ is 1-Hanf-local.

For the same reason, if we know the isomorphism types of all 1-neighborhoods of the elements in a τ-structure $\mathfrak{A}$, we can determine $\psi^*(\mathfrak{A})$, where $\psi \in I(\mathrm{QF}_\infty, \tau, \tau')$ and $\mathrm{ar}(R) \leq 1$ for all $R \in \tau'$. Thus $\mathrm{UL}_1 \circ \mathrm{QF}_\infty$ is 1-Hanf-local. □

Because $\mathcal{L}_\exists \leq \mathrm{UL}_1$, this shows also that $\mathcal{L}_\exists$ satisfies the conditions (1) and (2). We show that it satisfies the condition (3) later in Proposition 4.14.

Now, by condition (2), we get the result proved in [HLN99].

Corollary 3.22. *The logic* $\langle \mathrm{QF}_\infty \cup \mathrm{UL}_1 \rangle$ *is Hanf-local.*

This implies also that the finitary fragment of $\langle \mathrm{QF}_\infty \cup \mathrm{UL}_1 \rangle$ satisfies the condition (1) of Theorem 3.16.

3.4. **Neighborhoods and bijective game.** The logic $\langle \mathrm{QF}_\infty \cup \mathrm{UL}_1 \rangle$ captures Hanf-locality on classes of structures with bounded degree. In [Lib01] the logic was extended so that it captures Hanf-locality on all finite structures. The extended logic however is not closed under substitution. By Theorem 3.16, there exists a greatest Hanf-local extension of the finitary fragment of $\langle \mathrm{QF}_\infty \cup \mathrm{UL}_1 \rangle$ that is closed under substitution. The main point of this section is to show that the extension is proper, because there are other notions of neighborhoods and $\langle \mathrm{QF}_\infty \cup \mathrm{UL}_1 \rangle$ is Hanf-local also with respect to these notions.

Let τ be a relational vocabulary, M a class of τ-structures that is closed under isomorphism and I a nonempty set. Suppose that we have been given for every $r \in I$ and $\mathfrak{A} \in M$ a function $\mathrm{cl}_r^{\mathfrak{A}} \colon \mathcal{P}(\mathrm{Dom}(\mathfrak{A})) \to \mathcal{P}(\mathrm{Dom}(\mathfrak{A}))$. We denote $cl_r^{\mathfrak{A}}([\overline{a}])$ by $cl_r^{\mathfrak{A}}(\overline{a})$ and the structure $(\langle \mathrm{cl}_r^{\mathfrak{A}}(\overline{a}) \rangle^{\mathfrak{A}}, \overline{a})$ by $\mathfrak{Cl}_r^{\mathfrak{A}}(\overline{a})$.

Definition 3.23. We say that (M, I, cl) is *a notion of neighborhood* if the following conditions are satisfied, where all structures are from M:

a) $X \subseteq \mathrm{cl}_r^{\mathfrak{A}}(X)$
b) $X \subseteq Y \Rightarrow \mathrm{cl}_r^{\mathfrak{A}}(X) \subseteq \mathrm{cl}_r^{\mathfrak{A}}(Y)$
c) If $p \colon \langle \mathrm{cl}_r^{\mathfrak{A}}(X) \rangle^{\mathfrak{A}} \to \mathfrak{B}$ is an embedding, then $\mathrm{rng}\, p \subseteq \mathrm{cl}_r^{\mathfrak{B}}(p(X))$.
d) There exists a function $\gamma \colon I \times I \to I$ such that for all $p, q \in I$, $\mathfrak{A}$ and $X \subseteq \mathrm{Dom}(\mathfrak{A})$, $\mathrm{cl}_p^{\mathfrak{A}}(\mathrm{cl}_q^{\mathfrak{A}}(X)) \subseteq \mathrm{cl}_{\gamma(p,q)}^{\mathfrak{A}}(X)$.
e) There exists a function $\beta \colon I \to I$ such that for all $r \in I$, $\overline{a} \in \mathrm{Dom}(\mathfrak{A})^{<\omega}$, $\overline{b} \in \mathrm{Dom}(\mathfrak{B})^{<\omega}$, $a' \in \mathrm{Dom}(\mathfrak{A}) \setminus \mathrm{cl}_{\beta(r)}^{\mathfrak{A}}(\overline{a})$ and $b' \in \mathrm{Dom}(\mathfrak{B}) \setminus \mathrm{cl}_{\beta(r)}^{\mathfrak{B}}(\overline{b})$, if $\mathfrak{Cl}_r^{\mathfrak{A}}(\overline{a}) \cong \mathfrak{Cl}_r^{\mathfrak{B}}(\overline{b})$ and $\mathfrak{Cl}_r^{\mathfrak{A}}(a') \cong \mathfrak{Cl}_r^{\mathfrak{B}}(b')$, then $\mathfrak{Cl}_r^{\mathfrak{A}}(\overline{a}a') \cong \mathfrak{Cl}_r^{\mathfrak{B}}(\overline{b}b')$.

The conditions (a), (b) and (d) are just the definition of closure, where the usual $\mathrm{cl}(\mathrm{cl}(X)) = \mathrm{cl}(X)$ is replaced by (d). The condition (c) ensures that extensions of the structure do not make the neighborhood smaller. The condition (e) says that the elements not in the closure of a tuple are in some sense independent from the tuple.

We write $\alpha\colon \mathfrak{A} \sim_r \mathfrak{B}$ if α is a bijection $\mathrm{Dom}(\mathfrak{A}) \to \mathrm{Dom}(\mathfrak{B})$ such that for all $a \in \mathrm{Dom}(\mathfrak{A})$, $\mathfrak{Cl}^{\mathfrak{A}}_r(a) \cong \mathfrak{Cl}^{\mathfrak{B}}_r(b)$. Notation $\mathfrak{A} \sim_r \mathfrak{B}$ means that such a bijection exists.

Lemma 3.24. *If* $p\colon \mathfrak{Cl}^{\mathfrak{A}}_{\gamma(r,r')}(X) \cong \mathfrak{Cl}^{\mathfrak{B}}_{\gamma(r,r')}(p(X))$ *and* $Y \subseteq \mathrm{cl}^{\mathfrak{A}}_{r'}(X)$*, then*

$$p \restriction \mathrm{cl}^{\mathfrak{A}}_r(Y)\colon \mathfrak{Cl}^{\mathfrak{A}}_r(Y) \cong \mathfrak{Cl}^{\mathfrak{B}}_r(p(Y)).$$

Proof. By the conditions (d), (a) and (b), $\mathrm{cl}^{\mathfrak{A}}_r(Y) \subseteq \mathrm{cl}^{\mathfrak{A}}_{\gamma(r,r')}(X)$. By the condition (c), $p(\mathrm{cl}^{\mathfrak{A}}_r(Y)) \subseteq \mathrm{cl}^{\mathfrak{B}}_r(p(Y))$. The condition (c) implies also that $p(\mathrm{cl}^{\mathfrak{A}}_{r'}(X)) \subseteq \mathrm{cl}^{\mathfrak{A}}_{r'}(p(X))$ and so $\mathrm{cl}^{\mathfrak{B}}_r(p(Y)) \subseteq \mathrm{cl}^{\mathfrak{B}}_{\gamma(r,r')}(p(X))$. Thus $p^{-1}(\mathrm{cl}^{\mathfrak{B}}_r(p(Y)) \subseteq \mathrm{cl}^{\mathfrak{A}}_r(Y)$ and we get $p(\mathrm{cl}^{\mathfrak{A}}_r(Y)) = \mathrm{cl}^{\mathfrak{B}}_r(p(Y))$. Because a restriction of an isomorphism is an isomorphism, the lemma follows. □

Theorem 3.25. *Assume* (M, I, cl) *is a notion of neighborhood. For every* $n \in \mathbb{N}$*, there exists* $r \in I$ *such that, if* $\mathfrak{A}, \mathfrak{B} \in M$ *and* $\mathfrak{A} \sim_r \mathfrak{B}$*, then Player II has a winning strategy in* $\mathrm{BG}^1_n(\mathfrak{A}, \mathfrak{B})$*.*

Proof. Choose $r_0 \in I$ arbitrarily and define inductively $r_{i+1} = \gamma(r_i, \beta(r_i))$. Note that $\mathrm{cl}^{\mathfrak{A}}_{r_i}(X) \subseteq \mathrm{cl}^{\mathfrak{A}}_{r_i}(\mathrm{cl}^{\mathfrak{A}}_{\beta(r_i)}(X)) \subseteq \mathrm{cl}^{\mathfrak{A}}_{r_{i+1}}(X)$.

We prove inductively the following claim that implies the lemma: If $\mathfrak{A} \sim_{r_{n-1}} \mathfrak{B}$ and the partial function $p_n = \{(a_i, b_i) \mid i < m\}$ can be extended into an isomorphism $p'_n\colon \mathfrak{Cl}^{\mathfrak{A}}_{r_n}(\mathrm{dom}(p_n)) \cong \mathfrak{Cl}^{\mathfrak{B}}_{r_n}(\mathrm{rng}(p_n))$, then Player II has a winning strategy in the game $\mathrm{BG}^1_n((\mathfrak{A}, \overline{a}/\overline{x}), (\mathfrak{B}, \overline{b}/\overline{x}))$. If $n = 0$, we do not require the second condition.

When $n = 0$, the claim is clear. Assume that the claim has been proven for n. We prove the claim for $n+1$. Let $\alpha\colon \mathfrak{A} \sim_{r_n} \mathfrak{B}$ such that $p'_{n+1} \restriction \mathrm{cl}^{\mathfrak{A}}_{\beta(r_n)}(\mathrm{dom}(p_{n+1})) = \alpha \restriction \mathrm{cl}^{\mathfrak{A}}_{\beta(r_n)}(\mathrm{dom}(p_{n+1}))$. This is possible, because by Lemma 3.24, for all $c \in \mathrm{cl}^{\mathfrak{A}}_{\beta(r_n)}(\mathrm{dom}(p_{n+1}))$, $\mathfrak{Cl}^{\mathfrak{A}}_{r_n}(c) \cong \mathfrak{Cl}^{\mathfrak{B}}_{r_n}(p_{n+1}(c))$.

We prove that playing α leads to winning strategy for Player II. For that we have to show that for all $c \in \mathrm{Dom}(\mathfrak{A})$, Player II has a winning strategy in the game $\mathrm{BG}^1_n((\mathfrak{A}, \overline{a}/\overline{x}, c/y), (\mathfrak{B}, \overline{b}/\overline{x}, \alpha(c)/y))$. By the induction hypothesis, it suffices to find $p'_n\colon \mathfrak{Cl}^{\mathfrak{A}}_{r_n}(\mathrm{dom}(p_n)) \cong \mathfrak{Cl}^{\mathfrak{B}}_{r_n}(\mathrm{rng}(p_n))$ extending $p_n = p_{n+1} \cup \{(c, \alpha(c))\}$.

If $c \in \mathrm{cl}^{\mathfrak{A}}_{\beta(r_n)}$, we may put $p'_n = p'_{n+1} \restriction \mathrm{cl}^{\mathfrak{A}}_{r_n}(\mathrm{dom}(p_{n+1}) \cup \{c\})$ by Lemma 3.24. Otherwise, $c \notin \mathrm{cl}^{\mathfrak{A}}_{\beta(r_n)}(\mathrm{dom}(p_{n+1}))$ and we have also $\alpha(c) \notin \mathrm{cl}^{\mathfrak{B}}_{\beta(r_n)}(\mathrm{rng}(p_{n+1}))$. Because $\alpha\colon \mathfrak{A} \sim_{r_n} \mathfrak{B}$, $\mathfrak{Cl}^{\mathfrak{A}}_{r_n}(c) \cong \mathfrak{Cl}^{\mathfrak{B}}_{r_n}(\alpha(c))$ and so by condition (e) $\mathfrak{Cl}^{\mathfrak{A}}_{r_n}(\overline{a}c) \cong \mathfrak{Cl}^{\mathfrak{B}}_{r_n}(\overline{b}\alpha(c))$. We let p'_n be this partial isomorphism. □

We give now an application of Theorem 3.25. Let $\tau = \{R\}$, where R is a ternary relation symbol. Let M be the class of all finite abelian groups, where R encodes the group operation. We say that $\mathfrak{A} \in M$ is k-divisible, $k \in \mathbb{Z}_+$, if $\forall x(kx = 0 \leftrightarrow x = 0)$ holds in $\mathfrak{A}$. If $\mathfrak{A}$ is k-divisible, the equation $kx = b$ has a unique solution for all $b \in \mathfrak{A}$.

Let $I = \mathbb{Z}_+$. If $r \in I$, $\mathfrak{A} \in M$ is k-divisible for all $1 \leq k \leq r$ and $X \subseteq \mathrm{Dom}(\mathfrak{A})$, we define

$$\mathrm{cl}_r^{\mathfrak{A}}(X) = \left\{ a \in \mathrm{Dom}(\mathfrak{A}) \mid qa = \sum_{i<r} b_i, 1 \leq q \leq r, b_0, \dots, b_{r-1} \in X \cup (-X) \cup \{0\} \right\}.$$

If $\mathfrak{A}$ is not k-divisible for some $1 \leq k \leq r$, we put $\mathrm{cl}_r^{\mathfrak{A}}(X) = \mathrm{Dom}(\mathfrak{A})$.

The triple (M, I, cl) is a notion of neighborhood. The condition (a) follows because for every $x \in X$ we can show that $x \in \mathrm{cl}_r^{\mathfrak{A}}(X)$ by choosing $q = 1$, $b_0 = x$ and $b_i = 0$ for all $i > 0$. The conditions (b) and (c) are clear from the definition.

In order to establish (d), we define $\gamma(r, s) = s^r$. If $a \in \mathrm{cl}_r^{\mathfrak{A}}(\mathrm{cl}_s^{\mathfrak{A}}(X))$, there exists a $1 \leq q \leq r$ and $b_0, \dots, b_{r-1} \in \mathrm{cl}_s^{\mathfrak{A}}(X)$ such that $qa = \sum_{i<r} b_i$ (note that a closure is always closed under opposite and contains 0). Then there exists also elements $1 \leq p_i \leq s$ and $c_{i,j} \in X \cup (-X) \cup \{0\}$, where $i < r$ and $j < s$ such that $p_i b_i = \sum_{j<s} c_{i,j}$. Let $m = \prod_{i<r} p_i$. Then a satisfies the equation

$$ma = \sum_{i<r} \frac{m}{p_i} \sum_{j<s} c_{i,j}.$$

Because $m \leq s^r$ and the right side of the equation can be written as a sum of at most s^r elements, $a \in \mathrm{cl}_{\gamma(r,s)}^{\mathfrak{A}}(X)$.

Finally, we put $\beta(r) = 2r^3$. Assume $|\overline{a}| = |\overline{b}| = m$, $\alpha_0 \colon \mathfrak{Cl}_r^{\mathfrak{A}}(\overline{a}) \cong \mathfrak{Cl}_r^{\mathfrak{B}}(\overline{b})$, $\alpha_1 \colon \mathfrak{Cl}_r^{\mathfrak{A}}(a') \cong \mathfrak{Cl}_r^{\mathfrak{B}}(b')$, $a' \notin \mathrm{cl}_{\beta(r)}^{\mathfrak{A}}(\overline{a})$ and $b' \notin \mathrm{cl}_{\beta(r)}^{\mathfrak{B}}(\overline{b})$. These assumptions imply that $\mathfrak{A}$ and $\mathfrak{B}$ are k-divisible for all $k \leq \beta(r)$.

Let $x \in \mathrm{cl}_r^{\mathfrak{A}}(\overline{a}a')$. Then for some $1 \leq q \leq r$ and $s_0, \dots, s_{m-1}, s' \in \mathbb{Z}$ such that $|s'| + \sum_{i<m} |s_i| \leq r$, we have $qx = s'a' + \sum_{i<m} s_i a_i$. Because $\mathfrak{A}$ is q-divisible, we find x_0 and x_1 such that $x = x_0 + x_1$, $qx_0 = \sum_{i<m} s_i a_i$ and $qx_1 = s'a'$. Clearly $x_0 \in \mathrm{cl}_r^{\mathfrak{A}}(\overline{a})$ and $x_1 \in \mathrm{cl}_r^{\mathfrak{A}}(a')$.

Suppose now that for some $y_0 \in \mathrm{cl}_r^{\mathfrak{A}}(\overline{a}) \setminus \{x_0\}$ and $y_1 \in \mathrm{cl}_r^{\mathfrak{A}}(a') \setminus \{x_1\}$, we have $x = y_0 + y_1$. By the definition of the closure, for some $1 \leq p' \leq r$ and $-r \leq t' \leq r$, we have $p'y_1 = t'a'$ and for some $1 \leq p \leq r$ and $t_0, \dots, t_{m-1} \in \mathbb{Z}$ such that $\sum_{i<m} |t_i| \leq r$, we have $py_0 = \sum_{i<m} t_i a_i$. From this we can conclude

$$\begin{aligned} pq(y_0 - x_0) &= \sum_{i<m} (qt_i - ps_i) a_i \\ p'q(x_1 - y_1) &= (p's' - qt') a' \end{aligned}$$

Because $y_0 - x_0 = x_1 - y_1$, we have

$$p(p's' - qt')a' = \sum_{i<m} p'(qt_i - ps_i) a_i.$$

Because $p'q \leq \beta(r)$, $\mathfrak{A}$ is $p'q$-divisible. We assumed that $x_0 \neq y_0$ and so $p's' - qt'$ cannot be zero. On the other hand $|p's' - qt'| \leq 2r^2 \leq \beta(r)$. We have also $\sum_{i<m} |p'(qt_i - ps_i)| \leq p'q \sum_{i<m} |t_i| + p'p \sum_{i<m} |s_i| \leq 2r^3 \leq \beta(r)$. Thus $a' \in \mathrm{cl}_{\beta(r)}^{\mathfrak{A}}(\overline{a})$.

This contradicts our assumptions and we conclude that the representation of x as a sum of elements in $\mathrm{cl}_r^{\mathfrak{A}}(a')$ and $\mathrm{cl}_r^{\mathfrak{A}}(\overline{a})$ is unique.

The resolution above holds also on $\mathfrak{B}$ and so

$$\alpha(x_0 + x_1) = \alpha_0(x_0) + \alpha_1(x_1)$$

defines a function with an inverse

$$\alpha^{-1}(y_0 + y_1) = \alpha_0^{-1}(y_0) + \alpha_1^{-1}(y_1).$$

This is an isomorphism $\alpha\colon \mathfrak{Cl}_r^{\mathfrak{A}}(\overline{a}a') \cong \mathfrak{Cl}_r^{\mathfrak{B}}(\overline{b}b')$.

Now we can show:

Proposition 3.26. *For all $n \in \mathbb{N}$ there exists $r \in \mathbb{Z}_+$ such that if $\mathfrak{A}$ and $\mathfrak{B}$ are Abelian groups of the same cardinality and they are k-divisible for all $k \leq r$, then Player II has a winning strategy in $\mathrm{BG}_n^1(\mathfrak{A}, \mathfrak{B})$.*

Proof. Let r be as in Theorem 3.25 for the notion of neighborhood defined above. If $\mathfrak{A}$ is divisible for all $k \leq r$, then $\mathrm{cl}_r^{\mathfrak{A}}(0_{\mathfrak{A}}) = \{0_{\mathfrak{A}}\}$. We show that if $a \in \mathrm{Dom}(\mathfrak{A}) \setminus \{0_{\mathfrak{A}}\}$, then $\mathfrak{Cl}_r^{\mathfrak{A}}(a) \cong \mathfrak{Cl}_r^{\mathbb{Q}}(1)$, where we interpret $\mathbb{Q}$ as an abelian group. The isomorphism is defined so that we map $x \in \mathrm{cl}_r^{\mathfrak{A}}(a)$, where $qx = sa$, to s/q. By divisibility, the map is well defined and clearly a partial isomorphism from $\mathfrak{A}$ to $\mathbb{Q}$. Thus any bijection from $\mathrm{Dom}(\mathfrak{A}) \to \mathrm{Dom}(\mathfrak{B})$ mapping $0^{\mathfrak{A}}$ to $0^{\mathfrak{B}}$ shows $\mathfrak{A} \sim_r \mathfrak{B}$ and by Theorem 3.25, Player II has a winning strategy in $\mathrm{BG}_n^1(\mathfrak{A}, \mathfrak{B})$. □

Lemma 3.27. *If $\psi \in I(\mathrm{QF}_\infty, \tau', \tau)$, $\psi^*(\mathfrak{A})$ and $\psi^*(\mathfrak{B})$ are finite groups (with one ternary relation R in the vocabulary) and $\mathfrak{A} \leftrightarrows_{16} \mathfrak{B}$, then $\mathfrak{A} \cong \mathfrak{B}$.*

Proof. Denote the product of a and b in $\psi^*(\mathfrak{A})$ by ab, the neutral element by 1 and the inverse of a by a^{-1}. By group axioms, for all $a, b \in \mathrm{Dom}(\mathfrak{A})$, $\psi_R(a, b, ab)$, $\psi_R(a, 1, a)$ and $\psi_R(a, a^{-1}, 1)$.

If all elements of $\mathfrak{A}$ have different atomic types, the lemma clearly holds. Assume therefore that there exist $a, b \in \mathrm{Dom}(\mathfrak{A})$ such that $a \neq b$, but $\mathrm{atp}^{\mathfrak{A}}(a) = \mathrm{atp}^{\mathfrak{A}}(b)$. For all $c \in \mathrm{Dom}(\mathfrak{A})$, $\mathfrak{A} \models \psi_R(a, a^{-1}c, c) \wedge \neg\psi_R(b, a^{-1}c, c)$. This means that $\mathrm{atp}^{\mathfrak{A}}(a, a^{-1}c, c) \neq \mathrm{atp}^{\mathfrak{A}}(b, a^{-1}c, c)$, and so either $a^{-1}c$ or c belongs to $N_1^{\mathfrak{A}}(ab)$. Because this holds for all $c \in \mathrm{Dom}(\mathfrak{A})$ and $c \mapsto a^{-1}c$ is a bijection, we conclude that $|N_1^{\mathfrak{A}}(ab)| \geq |\mathfrak{A}|/2$.

If there exist $c, d \in \mathrm{Dom}(\mathfrak{A}) \setminus N_2^{\mathfrak{A}}(ab)$ such that $c \neq d$ but $\mathrm{atp}^{\mathfrak{A}}(c) = \mathrm{atp}^{\mathfrak{A}}(d)$, then we have $|N_1^{\mathfrak{A}}(cd)| \geq |\mathfrak{A}|/2$, but $N_1^{\mathfrak{A}}(ab) \cap N_1^{\mathfrak{A}}(cd) = \emptyset$. Necessarily, $\mathrm{Dom}(\mathfrak{A}) = N_1^{\mathfrak{A}}(abcd)$, which implies that every component of $\mathfrak{A}$ has radius at most 6, the worst case being the one where a, b, c and d are in the same component. Thus, in this case, $\mathfrak{A} \leftrightarrows_7 \mathfrak{B}$ implies $\mathfrak{A} \cong \mathfrak{B}$.

Assume then that such elements do not exist. Then every pair of different elements in $S = \mathrm{Dom}(\mathfrak{A}) \setminus N_2^{\mathfrak{A}}(ab)$ have different atomic types. Let

$$T = \{x \in \mathrm{Dom}(\mathfrak{A}) \mid (\forall y \in N_2^{\mathfrak{A}}(x))(\mathrm{atp}^{\mathfrak{A}}(y) \neq \mathrm{atp}^{\mathfrak{A}}(a))\}.$$

Clearly $T \subseteq S$, because all elements in $N_2^{\mathfrak{A}}(ab)$ have a or b in their 2-neighborhoods. Every element x in T has a 2-neighborhood that can be distinguished from any other

2-neighborhood in $\mathfrak{A}$ because if the other element is in T its atomic type differs from $\mathrm{atp}^{\mathfrak{A}}(x)$ and if it is in $\mathrm{Dom}(\mathfrak{A}) \setminus T$ it has an element with atomic type $\mathrm{atp}^{\mathfrak{A}}(a)$ in its neighborhood.

The set $S \setminus T$ is either contained in $N_4^{\mathfrak{A}}(ab)$ or in $N_4^{\mathfrak{A}}(abc)$, where $c \in S \setminus T$ is unique element with $\mathrm{atp}^{\mathfrak{A}}(c) = \mathrm{atp}^{\mathfrak{A}}(a)$. So we can choose for every component of the Gaifman graph of $\mathfrak{A}$ relativized to $\mathrm{Dom}(\mathfrak{A}) \setminus T$ an element such that its 13-neighborhood contains all elements of the component. Because all elements at distance 14 from these elements have unique 2-neighborhoods, knowing all 16-neighborhoods suffices for reconstructing the structure. □

By combining Proposition 3.26 and Lemma 3.27 we get an example showing that there are logics satisfying the condition (1) of Theorem 3.16 that are not in the extensions of first-order logic with unary quantifiers.

Proposition 3.28. *There are finitary logics $\mathcal{L}$ satisfying $\mathcal{L} \circ \mathrm{QF}_\infty \leq \mathrm{HL}^k$ for some $k \in \mathbb{Z}_+$, such that $\mathcal{L} \not\leq \langle \mathrm{QF}_\infty \cup \mathrm{UL}_1 \rangle$.*

Proof. Let Q be a query that is true, if $\mathfrak{A} \cong \mathbb{Z}_{p^2}$, where p is a prime. Then $\mathcal{L}_Q \circ \mathrm{QF} \leq \mathrm{HL}^{16}$ by Lemma 3.27. For any $n \in \mathbb{Z}_+$, let r be as in Proposition 3.26 and let p be a prime greater than r. Then the groups $\mathbb{Z}_{p^2}$ and $\mathbb{Z}_p^2$ are both divisible for all $k \leq r$ and have the same cardinality. So by Proposition 3.26, $\mathbb{Z}_{p^2} \equiv_{\langle \mathrm{QF}_\infty \cup \mathrm{UL}_1 \rangle^n} \mathbb{Z}_p^2$. However, Q separates the groups. Thus $\langle \mathrm{QF}_\infty \cup \mathrm{UL}_1 \rangle$ cannot define Q. □

4. Locality and uniform reduction

4.1. **Uniform reduction.** We call two vocabularies τ_0 and τ_1 *compatible*, if $\tau_0 \cap \tau_1$ contains only relation symbols of arity at least 1. If τ_0 and τ_1 are compatible, $\mathfrak{A} \in \mathrm{Mod}(\tau_0)$ and $\mathfrak{A}' \in \mathrm{Mod}(\tau_1)$, then $\mathfrak{A} \sqcup \mathfrak{A}'$ is a $\tau_0 \cup \tau_1$ -structure.

Definition 4.1. A logic $\mathcal{L}$ has *weak uniform reduction*, if for all compatible τ_0 and τ_1 and every finite $\Phi \subseteq \mathcal{L}[\tau_0 \cup \tau_1]$ there exists a finite equivalence relation $\sim_\Phi$ on $\mathrm{Mod}(\tau_0)$ such that for all structures $\mathfrak{A}, \mathfrak{A}' \in \mathrm{Mod}(\tau_0)$ and $\mathfrak{A}'' \in \mathrm{Mod}(\tau_1)$, equivalence $\mathfrak{A} \sim_\Phi \mathfrak{A}'$ implies $\mathfrak{A} \sqcup \mathfrak{A}'' \equiv_\Phi \mathfrak{A}' \sqcup \mathfrak{A}''$.

A logic has *uniform reduction*, if $\sim_\Phi$ can be chosen to be $\equiv_{\Psi_\Phi}$, where $\Psi_\Phi \subseteq \mathcal{L}[\tau_0]$ is finite.

The following fact is well known:

Proposition 4.2. *First-order logic has uniform reduction.*

Our goal in the rest of this subsection is to show what kind of relationship uniform reduction and its weak variant have.

Definition 4.3. A logic $\mathcal{L}$ is *closed under model extensions* if for all compatible vocabularies τ_0 and τ_1, for every $\varphi \in \mathcal{L}[\tau_0 \cup \tau_1]$ and $\mathfrak{C} \in \mathrm{Mod}(\tau_1)$ there exists $\varphi^{\mathfrak{C}} \in \mathcal{L}[\tau_0]$ such that $\mathfrak{A} \models \varphi^{\mathfrak{C}} \iff \mathfrak{A} \sqcup \mathfrak{C} \models \varphi$.

Lemma 4.4. *Suppose* $\mathrm{QF} \circ \mathcal{L} \leq \mathcal{L}$. *Then $\mathcal{L}$ has uniform reduction, if and only if it has weak uniform reduction and it is closed under model extensions.*

Proof. Let $\mathcal{L}$ be a logic with uniform reduction. By our definition, it is clear that it also has weak uniform reduction. In order to prove that it is closed under model extensions, let $\varphi \in \mathcal{L}[\tau_0 \cup \tau_1]$ and $\mathfrak{C} \in \mathrm{Mod}(\tau_1)$ be arbitrary. Then we put

$$\varphi^{\mathfrak{C}} \equiv \bigvee \Big\{ \bigwedge_{\substack{\psi \in \Psi_{\{\varphi\}} \\ \mathfrak{A} \models \psi}} \psi \wedge \bigwedge_{\substack{\psi \in \Psi_{\{\varphi\}} \\ \mathfrak{A} \not\models \psi}} \neg\psi \mid \mathfrak{A} \in \mathrm{Mod}(\tau_0), \mathfrak{A} \sqcup \mathfrak{C} \models \varphi \Big\}.$$

If $\mathfrak{A} \sqcup \mathfrak{C} \models \varphi$, then clearly $\mathfrak{A} \models \varphi^{\mathfrak{C}}$. If $\mathfrak{A} \models \varphi^{\mathfrak{C}}$, then there exists a structure $\mathfrak{A}' \in \mathrm{Mod}(\tau_0)$ such that $\mathfrak{A}' \sqcup \mathfrak{C} \models \varphi$ and

$$\mathfrak{A} \models \bigwedge_{\substack{\psi \in \Psi_{\{\varphi\}} \\ \mathfrak{A}' \models \psi}} \psi \wedge \bigwedge_{\substack{\psi \in \Psi_{\{\varphi\}} \\ \mathfrak{A}' \not\models \psi}} \neg\psi,$$

but the latter means that $\mathfrak{A} \equiv_{\Psi_{\{\varphi\}}} \mathfrak{A}'$ and so $\mathfrak{A} \sqcup \mathfrak{C} \models \varphi$.

Suppose then $\mathcal{L}$ is closed under model extensions and has weak uniform reduction. Let $\Phi \subseteq \mathcal{L}[\tau_0 \cup \tau_1]$ be a finite set. We put $\Psi'_\Phi = \{\varphi^{\mathfrak{C}} \mid \varphi \in \Phi, \mathfrak{C} \in \mathrm{Mod}(\tau_1)\}$. If $\mathfrak{A} \equiv_{\Psi'_\Phi} \mathfrak{A}'$, then for every $\varphi \in \Phi$,

$$\mathfrak{A} \sqcup \mathfrak{C} \models \varphi \iff \mathfrak{A} \models \varphi^{\mathfrak{C}} \iff \mathfrak{A}' \models \varphi^{\mathfrak{C}} \iff \mathfrak{A}' \sqcup \mathfrak{C} \models \varphi,$$

i.e. $\mathfrak{A} \sqcup \mathfrak{C} \equiv_\Phi \mathfrak{A}' \sqcup \mathfrak{C}$. However, the set Ψ'_Φ is not necessarily finite. Let $\sim_\Phi$ be a finite equivalence relation on $\mathrm{Mod}(\tau_1)$ given by weak uniform reduction for $\tau_1 \cup \tau_0$. If $\mathfrak{C}, \mathfrak{C}' \in \mathrm{Mod}(\tau_1)$ and $\mathfrak{C} \sim_\Phi \mathfrak{C}'$, then

$$\mathfrak{A} \models \varphi^{\mathfrak{C}} \iff \mathfrak{A} \sqcup \mathfrak{C} \models \varphi \iff \mathfrak{A} \sqcup \mathfrak{C}' \models \varphi \iff \mathfrak{A} \models \varphi^{\mathfrak{C}'}.$$

Hence we can choose a finite $\Psi_\Phi \subseteq \Psi'_\Phi$ such that every sentence in Ψ'_Φ is equivalent to some sentence in Ψ_Φ. This shows $\mathcal{L}$ has uniform reduction. □

Lemma 4.5. *If $\mathcal{L} \circ \mathrm{QF}$ and $\mathcal{L}'$ have weak uniform reduction, then $\mathcal{L} \circ \mathcal{L}'$ has weak uniform reduction.*

Proof. Suppose $\mathcal{L} \circ \mathrm{QF}$ and $\mathcal{L}'$ have weak uniform reduction and let $\varphi \circ \psi \in (\mathcal{L} \circ \mathcal{L}')[\tau_0 \cup \tau_1]$. We may assume $\psi \in I(\mathcal{L}', \tau_0 \cup \tau_1, \tau(\varphi))$. For every $\psi_R(\overline{x})$ and $\overline{y} \subseteq \overline{x}$, let $\sim_{\psi_R,\overline{y}}$ be an equivalence relation witnessing weak uniform reduction of ψ_R for $\tau_0 \cup [\overline{y}]$ and $\tau_1 \cup ([\overline{x}] \setminus [\overline{y}])$. Let $C_{\psi_R,\overline{y}}$ be the set of all equivalence classes of $\sim_{\psi_R,\overline{y}}$.

Given a structure $\mathfrak{A} \in \mathrm{Mod}(\tau_0)$, let $Y_0(\mathfrak{A})$ be a structure with the same domain and the same constants as $\mathfrak{A}$ such that for every $\psi_R(\overline{x})$, $\overline{y} \subseteq \overline{x}$ and if $t \in C_{\psi_R,\overline{y}}$ the structure has a relation

$$H_t^{Y_0(\mathfrak{A})} = \{\overline{a} \in \mathrm{Dom}(\mathfrak{A})^{|\overline{y}|} \mid (\mathfrak{A}, \overline{a}/\overline{y}) \in t\}.$$

Let τ'_0 be the vocabulary of this structure.

Define $Y_1(\mathfrak{A})$ for all structures $\mathfrak{A} \in \mathrm{Mod}(\tau_1)$ in a similar way, but with τ_0 and τ_1 reversed. Let τ'_1 be the vocabulary of these structures and suppose $\tau'_0 \cap \tau'_1 = \emptyset$. We find now for every $R \in \tau(\varphi)$ a formula $\psi'_R \in \mathrm{QF}[\tau'_0 \cup \tau'_1]$ such that

$$\mathfrak{A} \sqcup \mathfrak{A}' \models \psi_R(\overline{a}) \iff Y_0(\mathfrak{A}) \sqcup Y_1(\mathfrak{A}') \models \psi'_R(\overline{a}).$$

These formulas form an interpretation $\psi' \in I(\mathrm{QF}, \tau'_0 \cup \tau'_1, \tau(\varphi))$.

By the assumption of the lemma, there exists an equivalence relation $\sim_{\varphi\circ\psi'}$ witnessing weak uniform reduction of $\varphi \circ \psi'$. Let $\sim'$ be the equivalence relation

$$\mathfrak{A} \sim' \mathfrak{A}' \iff Y_0(\mathfrak{A}) \sim_{\varphi\circ\psi'} Y_0(\mathfrak{A}').$$

Clearly, $\mathfrak{A} \sim' \mathfrak{A}'$ implies $\mathfrak{A} \sqcup \mathfrak{A}'' \equiv_{\varphi\circ\psi} \mathfrak{A}' \sqcup \mathfrak{A}''$ for all $\mathfrak{A}'' \in \mathrm{Mod}(\tau_1)$. The equivalence relation $\sim'$ is finite, because it is an inverse image of a finite equivalence relation and thus it witnesses weak uniform reduction of $\varphi \circ \psi$. □

Given a logic $\mathcal{L}$, let $E(\mathcal{L})$ be a logic containing for every $\varphi \in \mathcal{L}[\tau_0 \cup \tau_1]$ and $\mathfrak{C} \in \mathrm{Mod}(\tau_1)$ a sentence $\varphi^{\mathfrak{C}} \in E(\mathcal{L})[\tau_0]$ such that $\mathfrak{A} \models \varphi^{\mathfrak{C}} \iff \mathfrak{A} \sqcup \mathfrak{C} \models \varphi$. Because we may choose $\mathfrak{C}$ to be an empty structure, $E(\mathcal{L}) \geq \mathcal{L}$.

Lemma 4.6. *If $\mathcal{L} \circ \mathrm{QF}$ has weak uniform reduction, then $E(\mathcal{L}) \circ \mathrm{QF}$ has weak uniform reduction.*

Proof. Let U be a new unary relation symbol and $F\colon \mathrm{Mod}(\tau) \to \mathrm{Mod}(\tau \cup \{U\})$, $\mathfrak{A} \mapsto (\mathfrak{A}, \mathrm{Dom}(\mathfrak{A})/U)$. Let $\varphi^{\mathfrak{C}} \circ \psi \in (E(\mathcal{L}) \circ \mathrm{QF})[\tau]$. There is $\psi' \in I(\mathrm{QF}, \tau \cup \{U\}, \tau(\varphi^{\mathfrak{C}}))$ such that $\psi^*(\mathfrak{A}) \sqcup \mathfrak{C} = (\psi')^*(\mathfrak{A} \sqcup F(\mathfrak{C}))$. Now,

$$\begin{aligned} \mathfrak{A} \sqcup \mathfrak{A}'' \models \varphi^{\mathfrak{C}} \circ \psi &\iff \psi^*(\mathfrak{A} \sqcup \mathfrak{A}'') \sqcup \mathfrak{C} \models \varphi \\ &\iff (\psi')^*(\mathfrak{A} \sqcup \mathfrak{A}'' \sqcup F(\mathfrak{C})) \models \varphi \\ &\iff \mathfrak{A} \sqcup (\mathfrak{A}'' \sqcup F(\mathfrak{C})) \models \varphi \circ \psi'. \end{aligned}$$

So, if $\mathfrak{A} \sim_{\varphi\circ\psi'} \mathfrak{A}'$, then $\mathfrak{A} \sqcup \mathfrak{A}'' \equiv_{\varphi^{\mathfrak{C}}\circ\psi} \mathfrak{A}' \sqcup \mathfrak{A}''$. This shows $E(\mathcal{L}) \circ \mathrm{QF}$ has weak uniform reduction. □

Combining the lemmas we get the following theorem showing in particular that in the class of the extensions of FO with generalized quantifiers, every logic with weak uniform reduction can be extended to a logic with uniform reduction.

Theorem 4.7. *If $\mathcal{L} \circ \mathrm{QF}$ has weak uniform reduction, there exists a logic $\mathcal{L}' \geq \mathcal{L} \cup \mathrm{QF}$ that is closed under substitution, model extensions and has uniform reduction.*

Proof. If $\mathcal{L} \circ \mathrm{QF}$ has weak uniform reduction, then by Lemma 4.5, $\mathcal{L} \circ (\mathcal{L} \circ \mathrm{QF})$ has weak uniform reduction as well as $(\mathcal{L} \circ \mathcal{L}) \circ \mathrm{QF} \leq \mathcal{L} \circ (\mathcal{L} \circ \mathrm{QF})$. By Lemma 4.6, $E(\mathcal{L}) \circ \mathrm{QF}$ has weak uniform reduction.

Define a sequence $(\mathcal{L}_i)_i$ of logics where i is an ordinal. Put $\mathcal{L}_0 = \mathcal{L} \cup \mathrm{QF}$, $\mathcal{L}_{i+1} = E((\mathcal{L}_i \circ \mathcal{L}_i) \cup \mathcal{L}_i)$. and $\mathcal{L}_i = \bigcup_{j<i} \mathcal{L}_j$, if i is a limit ordinal. Every logic in the sequence has weak uniform reduction.

Let $\mathcal{L}'$ be the union of the logics $\mathcal{L}_i$. The logic has weak uniform reduction. It is closed under model extensions, because if $\varphi \in \mathcal{L}'$ is equivalent to a sentence in $\mathcal{L}_i$, then $\varphi^{\mathfrak{C}}$ is equivalent to a sentence in $\mathcal{L}_{i+1}$. It is also closed under substitution, because if φ is equivalent to a sentence in $\mathcal{L}_i$ and ψ is an interpretation such that for every formula ψ_R there exists an equivalent $\mathcal{L}_{j_R}$-formula, then $\varphi \circ \psi$ is equivalent to some $\mathcal{L}_{\sup\{i, j_R \mid R \in \tau(\varphi)\}+1}$-sentence. □

4.2. **A characterization of weak uniform reduction.** We characterize in this section the expressive power of the logics without weak uniform reduction.

We call $Y \subseteq X$ *free from* $S \subseteq \mathcal{P}(X)$, if there exists $a \in Y$ and $b \in X \setminus Y$ such that for every $Y' \in S$, either $\{a, b\} \subseteq Y'$ or $\{a, b\} \subseteq X \setminus Y'$. A set S is *free*, if every $Y \in S$ is free from $S \setminus \{Y\}$.

Lemma 4.8. *If $S \subseteq \mathcal{P}(X)$ is infinite, it has a free countably infinite subset.*

Proof. We construct a countable subset $\{Y_0, Y_1, \ldots,\}$ by induction. Let $S_0 = S$. Our induction hypothesis is that S_i is infinite and for all $j < i$, Y_j is free from S_i.

Let U_i be a non-principal ultrafilter on S_i. It exists because S_i is infinite. Let $Z_i = \{a \in X \mid \{Y \in S_i \mid a \in Y\} \in U_i\}$. Because S_i is infinite, there exists some $Y_i \in S_i$ such that $Y_i \neq Z_i$. Choose $a \in Y_i$ and $b \in X \setminus Y_i$ so that either $\{a, b\} \subseteq Z_i$ or $\{a, b\} \subseteq X \setminus Z_i$. Now, let $S_{i+1} = \{Y \in S_i \mid \{a, b\} \subseteq Y \text{ or } \{a, b\} \subseteq X \setminus Y\}$.

Y_i is clearly free from S_{i+1}. If $\{a, b\} \subseteq Z_i$, then $S_{i+1} \supseteq \{Y \in S_i \mid a \in Y\} \cap \{Y \in S_i \mid b \in Y\} \in U_i$ and if $\{a, b\} \subseteq X \setminus Z_i$, $S_{i+1} \supseteq \{Y \in S_i \mid a \notin Y\} \cap \{Y \in S_i \mid b \notin Y\} \in U_i$. Thus $S_{i+1} \in U_i$ and S_{i+1} is infinite, because U_i is non-principal. □

Lemma 4.9. *Let $G = (V \cup V', E)$ be a bipartite graph, where $E \subseteq V \times V'$. For all $v \in V$, denote the set $\{v' \in V' \mid (v, v') \in E\}$ by $N^G(v)$. Assume that V is infinite and for all $v_0, v_1 \in V$, if $N^G(v_0) = N^G(v_1)$, then $v_0 = v_1$. Then, for some relation R among $=$, $\neq$, $\leq$ or $\geq$, we can embed $(\mathbb{N} \times \{0, 1\}, \{((i, 0), (j, 1)) \mid i, j \in \mathbb{N}, iRj\})$ in G.*

Proof. Let $S = \{N^G(v) \mid v \in V\}$. Because the graph G is infinite and every pair of vertices in V have different neighborhoods, the set S is infinite. Therefore, by the previous lemma, we find a countable infinite set $H \subseteq V$ such that $\{N^G(v) \mid v \in H\}$ is free. For every $v \in H$, choose $a_v \in N^G(v)$ and $b_v \in V' \setminus N^G(v)$ such that for every $v' \in H \setminus \{v\}$, either $\{a_v, b_v\} \subseteq N^G(v')$ or $\{a_v, b_v\} \subseteq V' \setminus N^G(v')$.

Now, define a countable ordered graph $G' = (H, \{(v, v') \in H^2 \mid v \neq v', a_v \in N^G(v')\}, <)$, where the ordering can be arbitrary well order. By Ramsey's theorem, it has a countable homogeneous subset $H' \subseteq H$. We can assume, if necessary restricting the set H', that the order type of $<$ restricted to H' is ω

Let $E' = \{(v, v') \in (H')^2 \mid v \neq v', a_v \in N^G(v')\}$. Then we have four possibilities, either $E' = (<) \cap (H')^2$, $E' = (>) \cap (H')^2$, $E' = \emptyset$ or $E' = (H')^2 \setminus \{(v, v) \mid v \in H'\}$. In the first three cases let $K = \{a_v \mid v \in H'\}$. Then the subgraph of G with domain $H' \cup K$ is isomorphic to a desired graph. In the fourth case, let $K = \{b_v \mid v \in H'\}$. Then subgraph of G with domain $H' \cup K$ is isomorphic to $(\mathbb{N} \times \{0, 1\}, \{((i, 0), (j, 1)) \mid i, j \in \mathbb{N}, i \neq j\})$. □

Theorem 4.10. *$\mathcal{L}$ does not have weak uniform reduction, if and only if there exists a sentence $\varphi \in \mathcal{L}[\tau]$ and two sequences of structures $(\mathfrak{A}_i)_{i \in \mathbb{N}}$ and $(\mathfrak{A}'_i)_{i \in \mathbb{N}}$ such that $\mathfrak{A}_i \sqcup \mathfrak{A}'_j \models \varphi \iff iRj$, where R is one of the relations $=$, $\neq$, $\leq$ or $\geq$.*

Proof. If weak uniform reduction fails in $\mathcal{L}$, there exists a sentence $\varphi \in \mathcal{L}[\tau_0 \cup \tau_1]$ and a countable infinite set of structures $H \subseteq \mathrm{Mod}(\tau_1)$ such that for all $\mathfrak{A}'_0, \mathfrak{A}'_1 \in H$, if $K(\mathfrak{A}'_0) = K(\mathfrak{A}'_1)$, then $\mathfrak{A}'_0 = \mathfrak{A}'_1$, where $K(\mathfrak{A}') = \{\mathfrak{A} \in \mathrm{Mod}(\tau_0) \mid \mathfrak{A} \sqcup \mathfrak{A}' \models \varphi\}$.

We find the sequences $(\mathfrak{A}_i)_{i\in\mathbb{N}}$ and $(\mathfrak{A}'_i)_{i\in\mathbb{N}}$ by applying the previous lemma to the graph $(H \cup \mathrm{Mod}(\tau_0), \bigcup_{\mathfrak{A}'\in H}\{\mathfrak{A}'\} \times K(\mathfrak{A}'))$.

If there exists a sentence φ and sequences $(\mathfrak{A}_i)_{i\in\mathbb{N}}$ and $(\mathfrak{A}'_i)_{i\in\mathbb{N}}$ as described in the theorem, $\mathcal{L}$ cannot have weak uniform reduction, because for every finite equivalence relation $\sim$ on $\mathrm{Mod}(\tau_0)$ there are $\mathfrak{A}_i$ and $\mathfrak{A}_j$ in the same $\sim$-equivalence class, but for some $\mathfrak{A}'_k$, $\mathfrak{A}_i \sqcup \mathfrak{A}'_k \not\equiv_\varphi \mathfrak{A}_j \sqcup \mathfrak{A}'_k$. ☐

Corollary 4.11. *If* $\mathrm{QF} \circ \mathcal{L} \leq \mathcal{L}$*,* $\mathcal{L}$ *is closed under relativization and does not have weak uniform reduction, then there exists a sentence* $\varphi \in \mathcal{L}[\tau]$ *and two sequences of structures* $(\mathfrak{A}_i)_{i\in\mathbb{N}}$ *and* $(\mathfrak{A}'_i)_{i\in\mathbb{N}}$ *such that* $\mathfrak{A}_i \sqcup \mathfrak{A}'_j \models \varphi \iff i = j$*.*

Proof. Let φ be a sentence and $(\mathfrak{A}_i)_{i\in\mathbb{N}}$ and $(\mathfrak{A}'_i)_{i\in\mathbb{N}}$ sequences given by the previous theorem. If $R = (=)$, we are fine. If $R = (\neq)$, we can use the same sequences and the sentence $\neg\varphi$. If $R = (\leq)$ or $R = (\geq)$, let U and V be new unary relation symbols. Then the sequences $(\mathfrak{A}_i \sqcup \mathfrak{A}'_i, \mathrm{Dom}(\mathfrak{A}_i)/U, \mathrm{Dom}(\mathfrak{A}'_i)/V)_{i\in\mathbb{N}}$ and $(\mathfrak{A}_i \sqcup \mathfrak{A}'_i, \mathrm{Dom}(\mathfrak{A}'_i)/U, \mathrm{Dom}(\mathfrak{A}_i)/V)_{i\in\mathbb{N}}$ and the sentence $\varphi^U \wedge \varphi^V$ satisfy the corollary. ☐

4.3. **Hanf-locality and uniform reduction.** With weak uniform reduction, we can strengthen Lemma 3.5.

Lemma 4.12. *If* $\mathcal{L} \leq \mathrm{SGL}^r \restriction k$ *has weak uniform reduction then* $\mathcal{L} \leq \mathrm{QF} \circ \mathrm{SGL}^r_{2(k-1)r}$*.*

Proof. Let φ be a $\mathcal{L}[\tau]$-sentence, where $|\mathrm{Con}(\tau)| \leq k$. For all $C \subseteq \mathrm{Con}(\tau)$, let $\sim^C$ be a finite equivalence relation witnessing uniform reduction of φ for $\mathrm{Rel}(\tau) \cup C$ and $\tau \setminus C$. Let T^C be the set of equivalence classes of $\sim^C$.

For every $C \subseteq \mathrm{Con}(\tau)$ and $t \in T^C$, there is a $\mathrm{SGL}^r_{2(k-1)r}[\mathrm{Rel}(\tau) \cup C]$-sentence γ^C_t such that $\mathfrak{A} \models \gamma^C_t$ if and only if $\mathfrak{N}^{\mathfrak{A}}_r(C^{\mathfrak{A}})$ is connected and in t. Let sentences $\theta^{a,b}_{2r} \in \mathrm{SGL}^r_{2r}[\mathrm{Rel}(\tau) \cup \{a, b\}]$ be as in the proof of Lemma 3.5 and let Φ be a finite set of sentences containing $\theta^{a,b}_{2r}$ for all $a, b \in \mathrm{Con}(\tau)$ and γ^C_t for all $C \subseteq \mathrm{Con}(\tau)$ and $t \in T^C$.

Assume $\mathfrak{A} \equiv_\Phi \mathfrak{B}$. We claim then $\mathfrak{A} \equiv_\varphi \mathfrak{B}$, which implies the lemma. Because Φ contains sentences $\theta^{a,b}_{2r}$, for all $a, b \in \mathrm{Con}(\tau)$, $d^{\mathfrak{A}}(a^{\mathfrak{A}}, b^{\mathfrak{A}}) \leq 2r \iff d^{\mathfrak{B}}(a^{\mathfrak{B}}, b^{\mathfrak{B}})$. Let $\mathcal{G} = (\mathrm{Con}(\tau), E)$ be a graph defined as in the proof of Lemma 3.5 and let $\mathcal{C}$ be the set of its components.

Now,

$$\mathfrak{N}^{\mathfrak{A}}_r(\mathrm{Con}(\tau)^{\mathfrak{A}}) \cong \bigsqcup_{C\in\mathcal{C}} \mathfrak{N}^{\mathfrak{A}}_r(C^{\mathfrak{A}}),$$
$$\mathfrak{N}^{\mathfrak{B}}_r(\mathrm{Con}(\tau)^{\mathfrak{B}}) \cong \bigsqcup_{C\in\mathcal{C}} \mathfrak{N}^{\mathfrak{B}}_r(C^{\mathfrak{B}})$$

and for all $C \in \mathcal{C}$, the neighborhoods $\mathfrak{N}^{\mathfrak{A}}_r(C^{\mathfrak{A}})$ and $\mathfrak{N}^{\mathfrak{B}}_r(C^{\mathfrak{B}})$ are connected. Because $\mathcal{L} \leq \mathrm{SGL}^r$, we have $\mathfrak{A} \equiv_\varphi \bigsqcup_{C\in\mathcal{C}} \mathfrak{N}^{\mathfrak{A}}_r(C^{\mathfrak{A}})$ and $\mathfrak{B} \equiv_\varphi \bigsqcup_{C\in\mathcal{C}} \mathfrak{N}^{\mathfrak{B}}_r(C^{\mathfrak{B}})$. The equivalence $\mathfrak{A} \equiv_\Phi \mathfrak{B}$ implies for all $C \in \mathcal{C}$, $\mathfrak{N}^{\mathfrak{A}}_r(C^{\mathfrak{A}}) \sim^C \mathfrak{N}^{\mathfrak{B}}_r(C^{\mathfrak{B}})$. Thus, by applying weak uniform reduction iteratively, $\bigsqcup_{C\in\mathcal{C}} \mathfrak{N}^{\mathfrak{A}}_r(C^{\mathfrak{A}}) \equiv_\varphi \bigsqcup_{C\in\mathcal{C}} \mathfrak{N}^{\mathfrak{B}}_r(C^{\mathfrak{B}})$. ☐

We need the following concept in the formulation of the Theorem 4.13 and in the proof of Proposition 4.14. A first-order formula $\varphi(\overline{x})$ is *r-local*, if all quantifications in φ are of the form $\exists y(d(y,\overline{x}) \leq r \wedge \psi(y))$, where $d(y,\overline{x}) \leq r$ is a FO-formula such that $\mathfrak{A} \models d(a,\overline{b}) \leq r \iff d^{\mathfrak{A}}(a,[\overline{b}]) \leq r$.

Theorem 4.13. (Gaifman's theorem) *Every first-order formula is equivalent to a Boolean combination of local first-order formulas and sentences of the form*

$$\exists x_1 \dots x_n (\bigwedge_{1\leq i\leq n} \varphi(x_i) \wedge \bigwedge_{1\leq i<j\leq n} d(x_i,x_j) > 2r)$$

where φ is r-local first-order formula for some r.

Proof. Originally proved in [Gai82]. The proof is presented also in [EF99] and [Lib04]. □

Proposition 4.14. *First-order logic satisfies the condition (3) of Theorem 3.16.*

Proof. Given a FO-sentence, let $\varphi(\overline{x})$ be a formula such that every constant symbol of the sentence occurs as a parameter of the formula. By Gaifman's theorem, this can be written as a Boolean combination of certain sentences and r-local formulas for some r. Sentences do not contain free variables, and so they are in $\mathrm{HL}^{r'} \restriction 0$ for some r'. The r-local formulas are equivalent to SGL^r-formulas. By Lemma, 4.12 they are equivalent with $\mathrm{QF} \circ \mathrm{SGL}^r_{2r}$-formulas and so the original sentence is equivalent to a $\mathrm{QF} \circ (\mathrm{HL}^{r'} \restriction 0 \cup \mathrm{SGL}^r_{2r})$-sentence. Because FO is finitary, the condition (3) of Theorem 3.16 is satisfied. □

The next lemma shows that if the logic does not have weak uniform reduction and it is closed under relativization, we can encode structures with infinite vocabulary using a finite vocabulary so that the encoding does not break Hanf-equivalence, but we can recover a fixed formula of the logic. The lemma is non-trivial only if the vocabulary τ is infinite.

Lemma 4.15. *Let $\mathcal{L}$ be a logic that is closed under relativization, but does not have weak uniform reduction. Assume also that $\mathrm{FO} \circ \mathcal{L} \leq \mathcal{L}$. Let τ be a relational vocabulary. Let $\psi(\overline{x})$ be a $\mathrm{QF}_\infty[\tau]$-formula. Then there exists a finite vocabulary τ' with $U \in \tau'$, $\mathcal{L}[\tau']$-formula $\psi'(\overline{x})$ and a function $F\colon \mathrm{Mod}(\tau) \to \mathrm{Mod}(\tau')$ such that for all $\mathfrak{A} \in \mathrm{Mod}(\tau)$, $\psi(\mathfrak{A}) = \psi'(F(\mathfrak{A})) \cap (\mathrm{Dom}(\mathfrak{A}))^{|\overline{x}|}$ and for all $\mathfrak{A}, \mathfrak{A}' \in \mathrm{Mod}(\tau)$ and $r \geq 1$, $\mathfrak{A} \leftrightarrows_r \mathfrak{A}'$ if and only if $F(\mathfrak{A}) \leftrightarrows_r F(\mathfrak{A}')$.*

Proof. Let $\theta \in \mathcal{L}[\tau'']$ be a sentence and $(\mathfrak{C}_i)_{i\in\mathbb{N}}$ and $(\mathfrak{C}'_i)_{i\in\mathbb{N}}$ sequences of structures such that $\mathfrak{C}_i \sqcup \mathfrak{C}'_j \models \theta$ if and only if $i = j$. The sequence exists by Corollary 4.11.

Let $m = |\overline{x}|$ and $\tau' = \tau'' \cup \{U, V, \sim, S_0, \dots, S_m, Y, X_{\overline{y}} \mid \overline{y} \in [\overline{x}]^{\leq m}\}$, where U, V, Y and $X_{\overline{y}}$ are unary relation symbols, $\sim$ is a binary relation symbol and for all $0 \leq i \leq m$, S_i is an $i+1$-ary relation symbol.

Let T be the set of all atomic types $t(\overline{y})$ on τ such that $|\overline{y}| \leq m$ and for some $R \in \tau$, and permutation ρ of $|\overline{y}|$, $Ry_{\rho(0)} \dots y_{\rho(|\overline{y}|-1)} \in t$. Let $<$ be any linear-order on T.

For all $\mathfrak{A} \in \mathrm{Mod}(\tau)$, let $T^{\mathfrak{A}} \subset T$ be the set of all types in T that are realized in $\mathfrak{A}$. Let $n^{\mathfrak{A}} = |T^{\mathfrak{A}}| \in \mathbb{N}$. We define an enumeration $(t_i^{\mathfrak{A}})_{i<n^{\mathfrak{A}}}$ of $T^{\mathfrak{A}}$ such that for all $i < j$, $t_i^{\mathfrak{A}} < t_j^{\mathfrak{A}}$. Defined in this way, we have that if $T^{\mathfrak{A}} = T^{\mathfrak{A}'}$, then for all $i < n^{\mathfrak{A}}$, $t_i^{\mathfrak{A}} = t_i^{\mathfrak{A}'}$.

Given $\mathfrak{A} \in \mathrm{Mod}(\tau)$ and a type $t(\overline{y}) \in T^{\mathfrak{A}}$, we denote the set of its realizations, $\{\overline{a} \in \mathfrak{A}^{|\overline{y}|} \mid \mathfrak{A} \models t(\overline{a})\}$, by $t(\mathfrak{A})$.

We define next the function F. We have $F(\mathfrak{A}) = F_0(\mathfrak{A}) \sqcup F_1(\mathfrak{A})$. The first part, $F_0(\mathfrak{A})$, encodes the atomic types of $\mathfrak{A}$ and depends only on $\mathfrak{A}$ and m, but not on ψ. The second part, $F_1(\mathfrak{A})$, encodes a fragment of ψ and depends only on ψ and $T^{\mathfrak{A}}$. The relations $U^{F(\mathfrak{A}}$, $V^{F(\mathfrak{A})}$ and $S_i^{F(\mathfrak{A})}$ are contained in the first part and the relations $X_{\overline{y}}^{F(\mathfrak{A})}$,$Y$ and $\sim$ in the second part.

We let

$$\mathrm{Dom}(F_0(\mathfrak{A})) = \mathrm{Dom}(\mathfrak{A}) \cup \bigcup_{i<n^{\mathfrak{A}}} t_i^{\mathfrak{A}}(\mathfrak{A}) \times \mathrm{Dom}(\mathfrak{C}_i).$$

The non-empty relations are defined in the following way:

$$\begin{aligned} U^{F(\mathfrak{A})} &= \mathrm{Dom}(\mathfrak{A}) \\ V^{F(\mathfrak{A})} &= \mathrm{Dom}(F_0(\mathfrak{A})) \setminus \mathrm{Dom}(\mathfrak{A}) \\ S_k^{F(\mathfrak{A})} &= \{(a_0, \ldots, a_{k-1}, \overline{a}b) \mid i < n^{\mathfrak{A}}, \overline{a} \in t_i^{\mathfrak{A}}(\mathfrak{A}), |\overline{a}| = k, b \in \mathrm{Dom}(\mathfrak{C}_i)\} \end{aligned}$$

and for all $R \in \tau''$, where $\mathrm{ar}(R) = k$,

$$R^{F_0(\mathfrak{A})} = \{(\overline{a}b_0, \ldots, \overline{a}b_{k-1}) \mid i < n^{\mathfrak{A}}, \overline{a} \in t_i^{\mathfrak{A}}(\mathfrak{A}), \overline{b} \in R^{\mathfrak{C}_i}\}.$$

Let $S^{\mathfrak{A}}$ be the set of all atomic m-types $t(\overline{x})$ such that $t(\overline{x}) \models \psi(\overline{x})$ and if $\overline{y} \in [\overline{x}]^{\leq m}$ and for some $R \in \tau$, $R\overline{y} \in t$, then $t \restriction \overline{y} \in T^{\mathfrak{A}}$. Clearly $S^{\mathfrak{A}}$ contains all atomic m-types realized in $\mathfrak{A}$ and satisfying ψ, but it depends only on $T^{\mathfrak{A}}$. If $t \in S^{\mathfrak{A}}$ and we know $t(\overline{x}) \restriction \overline{y}$ for all $[\overline{y}] \subseteq [\overline{x}]$ such that $t(\overline{x}) \restriction \overline{y} \in T^{\mathfrak{A}}$, we can determine t. This implies that $|S^{\mathfrak{A}}| \leq |T^{\mathfrak{A}}|^{m^m}$.

We define now $F_1(\mathfrak{A})$. We let

$$\mathrm{Dom}(F_1(\mathfrak{A})) = \{(t, \overline{y}, b) \mid t \in S^{\mathfrak{A}}, \overline{y} \in [\overline{x}]^{\leq m}, i < n^{\mathfrak{A}}, t \restriction \overline{y} = t_i^{\mathfrak{A}}, b \in \mathrm{Dom}(\mathfrak{C}'_i)\}.$$

The non-empty relations are defined in the following way:

$$\begin{aligned} Y^{F(\mathfrak{A})} &= \mathrm{Dom}(F_1(\mathfrak{A})) \\ \sim^{F(\mathfrak{A})} &= \{((t, \overline{y}, b), (t', \overline{y}', b')) \in (\mathrm{Dom}(F_1(\mathfrak{A})))^2 \mid t = t'\} \\ X_{\overline{y}}^{F(\mathfrak{A})} &= \{(t, \overline{y}', b) \in \mathrm{Dom}(F_1(\mathfrak{A})) \mid \overline{y} = \overline{y}'\} \end{aligned}$$

and for all $R \in \tau''$, where $\mathrm{ar}(R) = k$,

$$\begin{aligned} R^{F_1(\mathfrak{A})} = \{&((t_i^{\mathfrak{A}}, \overline{y}, b_0), \ldots, (t_i^{\mathfrak{A}}, \overline{y}, b_{k-1})) \in (\mathrm{Dom}(F_1(\mathfrak{A})))^k \\ &\mid i < n^{\mathfrak{A}}, \overline{y} \in [\overline{x}]^{\leq m}, \overline{b} \in R^{\mathfrak{C}'_i}\}. \end{aligned}$$

The formula $\psi'(\overline{x})$ is defined in the following way

$$\psi'(\overline{x}) \equiv \bigwedge_{i<m} Ux_i \wedge \exists z \left(Yz \wedge \bigwedge_{\overline{y}\in[\overline{x}]^{\le m}} \left(\bigvee_{R\in\tau} R\overline{y} \right) \to \theta^{\gamma_{\overline{y},z}} \right),$$

where $\theta^{\gamma_{\overline{y},z}}$ is the relativization of θ to the set defined by the formula

$$\gamma_{\overline{y},z}(w) \equiv S_{|\overline{y}|}\overline{y}w \vee (w \sim z \wedge X_{\overline{y}}w).$$

Assume $\overline{a} \in \mathrm{Dom}(\mathfrak{A})^{|\overline{x}|}$, $\overline{y} \in [\overline{x}]^{\le m}$ and $\overline{a'}$ is the interpretation of $\overline{y}$, if the interpretation of $\overline{x}$ is $\overline{a}$. Let $c = (t', \overline{y'}, b') \in Y^{F(\mathfrak{A})}$. If $(F(\mathfrak{A}), \overline{a}/\overline{x}, c/z) \models \gamma_{\overline{y}}(c)$, then c is either in $\{\overline{a'}\} \times \mathrm{Dom}(\mathfrak{C}_i)$, where $i < n^{\mathfrak{A}}$ such that $t_i^{\mathfrak{A}} = \mathrm{atp}^{\mathfrak{A}}(\overline{a'})$, or in $(t', \overline{y}) \times \mathrm{Dom}(\mathfrak{C}'_j)$, where $t_j^{\mathfrak{A}} = t' \restriction \overline{y}$. This means that $F(\mathfrak{A}) \models \psi'(\overline{a})$, if and only if there exists $t' \in S^{\mathfrak{A}}$ such that for all $\overline{y} \in [\overline{x}]^{\le m}$, if $t'' = \mathrm{atp}^{\mathfrak{A}}(\overline{a}/\overline{x}) \restriction \overline{y} \in T^{\mathfrak{A}}$, then $t'' = t'(\overline{x}) \restriction \overline{y}$. As mentioned above, this is equivalent with $\mathrm{atp}(\overline{a}) = t'$ and thus $F(\mathfrak{A}) \models \psi'(\overline{a})$ if and only if $\mathfrak{A} \models \psi(\overline{a})$.

Finally, we have to prove that $\mathfrak{A} \leftrightarrows_r \mathfrak{A}'$ implies $F(\mathfrak{A}) \leftrightarrows_r F(\mathfrak{A}')$, where $r \ge 1$. Assume $\alpha\colon \mathfrak{A} \leftrightarrows_r \mathfrak{A}'$. This implies $T^{\mathfrak{A}} = T^{\mathfrak{A}'}$. Thus $F_1(\mathfrak{A}) = F_1(\mathfrak{A}')$ and because $F_0(\mathfrak{A})$ and $F_1(\mathfrak{A})$ are disjoint parts of $F(\mathfrak{A})$, it suffices to show $F_0(\mathfrak{A}) \leftrightarrows_r F_0(\mathfrak{A}')$.

For every $a \in \mathrm{Dom}(\mathfrak{A})$, let $\beta_a\colon \mathfrak{N}_r^{\mathfrak{A}}(a) \cong \mathfrak{N}_r^{\mathfrak{A}'}(\alpha(a))$. We define $\mu\colon \mathrm{Dom}(F_0(\mathfrak{A})) \to \mathrm{Dom}(F_0(\mathfrak{A}'))$ such that $\mu \restriction \mathrm{Dom}(\mathfrak{A}) = \alpha$ and for all $\overline{a}b \in V^{F(\mathfrak{A})} = V^{F_0(\mathfrak{A})} = \bigcup_{i<n^{\mathfrak{A}}} t_i^{\mathfrak{A}}(\mathfrak{A}) \times \mathrm{Dom}(\mathfrak{C}_i)$, $\mu(\overline{a}b) = \beta_{a_0}(\overline{a})b$. Because β_{a_0} preserves the atomic type of $\overline{a}$, $\mathrm{rng}(\mu) \subseteq \mathrm{Dom}(F_0(\mathfrak{A}'))$. If $\mu(\overline{a}b) = \mu(\overline{a'}b)$, then $\beta_{a_0}(a_0) = \alpha(a_0) = \alpha(a'_0) = \beta_{a'_0}(a'_0)$ implying $a_0 = a'_0$ and because β_{a_0} is a bijection, $\overline{a} = \overline{a'}$. Thus μ is bijective.

Let $c \in \mathrm{Dom}(\mathfrak{A})$. We have $N_r^{F_0(\mathfrak{A})}(c) \subseteq N_r^{\mathfrak{A}}(c) \cup \{\overline{a}b \in V^{\mathfrak{A}} \mid \overline{a} \in N_r^{\mathfrak{A}}(c)\}$. Thus we find an isomorphism $\mathfrak{N}_r^{F_0(\mathfrak{A})}(c) \cong \mathfrak{N}_r^{F_0(\mathfrak{A}')}(\mu(c))$ by mapping all elements $a \in N_r^{F_0(\mathfrak{A})}(c) \cap U^{\mathfrak{A}}$ to $\beta_c(a)$ and all elements $\overline{a}b \in N_r^{F_0(\mathfrak{A})}(c) \cap V^{\mathfrak{A}}$ to $\beta_c(\overline{a})b$. If $c = \overline{a}b \in V^{\mathfrak{A}}$, we get the isomorphism $\mathfrak{N}_r^{F_0(\mathfrak{A})}(c) \cong \mathfrak{N}_r^{F_0(\mathfrak{A}')}(\mu(c))$ by restricting the isomorphism of the neighborhood with center a_0. □

Remark. Lemma 4.15 generalizes quite easily to the case where ψ is an interpretation of a finite vocabulary τ. We need only to add the second part of the structure, $F_1(\mathfrak{A})$, that depends on the formula, for every formula ψ_R, with $R \in \tau$.

Theorem 4.16. *Suppose $\mathcal{L}$ and $\mathcal{L}'$ are closed under relativization, $\mathcal{L}$ is finitary and $\mathrm{FO} \circ \mathcal{L}' \le \mathcal{L}'$. If $\mathcal{L} \circ \mathcal{L}'$ is Hanf-local, but $\mathcal{L} \circ \mathrm{QF}_\infty$ is not, then $\mathcal{L}'$ has weak uniform reduction.*

Proof. Let $\varphi \circ \psi \in \mathcal{L} \circ \mathrm{QF}_\infty[\tau]$ be a sentence that is not Hanf-local. Then there are sequences of structures $(\mathfrak{A}_r)_{r\in\mathbb{N}}$ and $(\mathfrak{A}'_r)_{r\in\mathbb{N}}$ such that for all $r \in \mathbb{N}$, $\psi(\mathfrak{A}_r) \models \varphi$ and $\psi(\mathfrak{A}'_i) \not\models \varphi$ but $\mathfrak{A}_r \leftrightarrows_r \mathfrak{A}'_r$.

Now, if $\mathcal{L}'$ did not have weak uniform reduction, by Lemma 4.15, there would be an $\mathcal{L}'$-interpretation ψ' and a function F such that $\langle U^{F(\mathfrak{A})}\rangle^{(\psi')^*(F(\mathfrak{A}))} = \psi^*(\mathfrak{A})$, where U is a unary relation symbol as in Lemma 4.15, and if $\mathfrak{A} \leftrightarrows_r \mathfrak{A}'$, then $F(\mathfrak{A}) \leftrightarrows_r F(\mathfrak{A}')$. Thus the sequences $(F(\mathfrak{A}_r))_{r\in\mathbb{N}}$ and $(F(\mathfrak{A}'_r))_{r\in\mathbb{N}}$ would witness that the sentence $\varphi^U \circ \psi' \in \mathcal{L} \circ \mathcal{L}'$ is not Hanf-local. □

Corollary 4.17. *A finitary and regular Hanf-local logic $\mathcal{L}$ has weak uniform reduction or satisfies the condition $\mathcal{L} \circ \mathrm{QF}_\infty \leq \mathrm{HL}$.*

Proof. Assume that $\mathcal{L}$ does not satisfy the condition $\mathcal{L} \circ \mathrm{QF}_\infty \leq \mathrm{HL}$. By choosing $\mathcal{L}' = \mathcal{L}$, all conditions of Theorem 4.16 are satisfied and so $\mathcal{L}$ has weak uniform reduction. □

4.4. **A regular Hanf-local logic $\mathcal{L}$ such that $\mathcal{L} \circ \mathrm{QF}_\infty$ is not Gaifman-local.** Corollary 4.17 raises a question, if there exists a logic that does not satisfy the second disjunct, i.e., it is Hanf-local, but $\mathcal{L} \circ \mathrm{QF}_\infty$ is not Hanf-local. The example we use to show that this is possible, is a modification of Gurevich's and Shelah's multipede -structure [GS96]. We present the construction in a little more general form than is necessary here, because we shall reuse it later.

Let $\tau_k = \{U, P, \epsilon, \sim, E\}$, where U and P are unary relation symbols, ϵ and $\sim$ are binary relation symbols and E is a $2k$-ary relation symbol. We define for each $X \subseteq n$ a τ_k-structure $\mathfrak{A}_{k,n,X}$. The universe of $\mathfrak{A}_{k,n,X}$ is $(\mathbb{Z}_n \times k) \cup \mathcal{P}(n)$. The relation symbols U, P, ϵ and $\sim$ have the following interpretations that do not depend on X:

- $U^{\mathfrak{A}_{k,n,X}} = \mathbb{Z}_n \times k$,
- $P^{\mathfrak{A}_{k,n,X}} = \mathcal{P}(\mathbb{Z}_n)$,
- $\epsilon^{\mathfrak{A}_{k,n,X}} = \{\langle (i,j), V\rangle \in U^{\mathfrak{A}_{k,n,X}} \times P^{\mathfrak{A}_{k,n,X}} \mid i \in V\}$ and
- $\sim^{\mathfrak{A}_{k,n,X}} = \{\langle (i,j), (i',j')\rangle \in (U^{\mathfrak{A}_{k,n,X}})^2 \mid i = i'\}$.

Let S_k be the set of all permutations $k \to k$, $A_k \subseteq S_k$ the set of all even permutations and $A'_k = S_k \setminus A_k$ the set of all odd permutations $k \to k$. For all $i \in \mathbb{Z}_n$, let

$$E_i = \{(i, f(0)) \dots (i, f(k-1))(i+1, g(f(0))) \dots (i+1, g(f(k-1))) \mid f \in S_k, g \in A_k\}$$

and

$$E'_i = \{(i, f(0)) \dots (i, f(k-1))(i+1, g(f(0))) \dots (i+1, g(f(k-1))) \mid f \in S_k, g \in A'_k\},$$

where $i+1$ is computed in the cyclic group $\mathbb{Z}_n$.

Now, we define

$$E^{\mathfrak{A}_{k,n,X}} = \bigcup_{i \in X} E'_i \cup \bigcup_{i \in \mathbb{Z}_n \setminus X} E_i.$$

Lemma 4.18. $\mathfrak{A}_{k,n,X} \cong \mathfrak{A}_{k,n,Y}$ *if and only if* $|X| \equiv |Y| \pmod 2$.

Proof. Assign every sequence of permutations $p = (p_i)_{i \in \mathbb{Z}_n} \in S_k^{\mathbb{Z}_n}$ with a permutation α_p on $\mathbb{Z}_n \times k \cup \mathcal{P}(n)$ that fixes $\mathcal{P}(n)$ and maps (i,j) to $(i, p_i(j))$. For all $p \in A_k^{\mathbb{Z}_n}$ and $X \subseteq n$, α_p is an automorphism of $\mathfrak{A}_{k,n,X}$. If $p_i \in A_k$ for all $i \in \mathbb{Z}_k \setminus H$ and $p_i \in A'_k$ for all $i \in H$, then $\alpha_p \colon \mathfrak{A}_{k,n,X} \cong \mathfrak{A}_{k,n,Y}$ for all $X, Y \subseteq \mathbb{Z}_n$ such that $X \Delta Y = H \Delta \{i-1 \mid i \in H\}$. Because for every X with even cardinality, there exists H such that $X = H \Delta \{i-1 \mid i \in H\}$, we conclude that $\mathfrak{A}_{k,n,X} \cong \mathfrak{A}_{k,n,Y}$, if $|X| \equiv |Y| \pmod 2$.

If we choose for each $i \in \mathbb{Z}_x$ a bijection $q_i \colon \{i\} \times k \to \{i+1\} \times k$ such that $\langle (i,0), \dots, (i,k-1), q_i(i,0), \dots, q_i(i,k-1)\rangle \in E^{\mathfrak{A}_{k,n,X}}$, then $q = q_{i-1} \circ q_{i-2} \circ \dots \circ q_0 \circ$

$q_{n-1} \circ \cdots \circ q_{i+1} \circ q_i$ is a bijection $\{i\} \times k \to \{i\} \times k$. Independently of the choice of q_i, the permutation is even, if $|X|$ is even and odd, if $|X|$ is odd. Thus $\mathfrak{A}_{k,n,X} \not\cong \mathfrak{A}_{k,n,Y}$, if $|X| \not\equiv |Y| \pmod 2$. □

Let $M_{k,i} = \{\mathfrak{A}_{k,n,X} \mid n \in \mathbb{Z}_+, X \subseteq Z_n, |X| \equiv i \pmod 2\}$ and $M_k = M_{k,0} \cup M_{k,1}$.

Lemma 4.19. *The class of structures isomorphic to a structure in M_k is* FO-*definable.*

Proof. We can say with the following FO-expressible sentences that a structure $\mathfrak{A}$ is isomorphic to a structure in M_k:

1) $\mathrm{Dom}(\mathfrak{A}) = U^{\mathfrak{A}} \cup P^{\mathfrak{A}}$ and $U^{\mathfrak{A}} \cap P^{\mathfrak{A}} = \emptyset$.
2) $\sim^{\mathfrak{A}}$ is an equivalence relation on $U^{\mathfrak{A}}$ and the cardinality of all its equivalence classes is k.
3) $\epsilon^{\mathfrak{A}} \subseteq U^{\mathfrak{A}} \times P^{\mathfrak{A}}$ and for each $p \in P^{\mathfrak{A}}$, the set $N(p) = \{u \in U^{\mathfrak{A}} \mid (u,p) \in \epsilon^{\mathfrak{A}}\}$ is a union of $\sim^{\mathfrak{A}}$-equivalence classes.
4) If $p, p' \in P^{\mathfrak{A}}$ and $p \neq p'$, then $N(p) \neq N(p')$.
5) $P^{\mathfrak{A}}$ is non-empty and for each $p \in P^{\mathfrak{A}}$ and $u \in U^{\mathfrak{A}}$, there exists $p' \in P^{\mathfrak{A}}$ such that $N(p') = N(p) \Delta [u]_{\sim^{\mathfrak{A}}}$.
6) Let $V = U^{\mathfrak{A}}/{\sim^{\mathfrak{A}}}$ and $D = \{(a,b) \in V \mid E^{\mathfrak{A}} \cap a^k \times b^k \neq \emptyset\}$. Then the directed graph (V, D) is a cycle.
7) For all $(a,b) \in D$, the relation $E^{\mathfrak{A}} \cap a^k \times b^k$ is isomorphic to E_i for any i.

The sentence (6) is first-order expressible, because we know from the sentences (3)-(5) that the elements of $P^{\mathfrak{A}}$ encode all unions of $\sim^{\mathfrak{A}}$-equivalence classes and thus we can say that (V, D) is connected and in-degree and out-degree of every vertex is 1. The sentence (7) is first-order expressible, because we know from (2) that $a \cup b$ has fixed size $2k$. □

Let Q_k be the query containing all structures isomorphic to a structure in $M_{k,0}$ and no others.

Proposition 4.20. *For all $k \geq 2$, $\mathcal{L}_{Q_k} \circ \mathrm{QF}_\infty$ is not Gaifman-local.*

Proof. Let $\tau' = \{U, P, \sim, E\} \cup \{U_i \mid i \in \mathbb{N}\} \cup \{P_X \mid X \subseteq \mathbb{N}, |X| < \omega\}$, where every U_i and P_X is unary, other relation symbols are as before and $\overline{a}$ and $\overline{b}$ are sequences of constant symbols of length k. Let $r \in \mathbb{Z}_+$ and $n = 2r+1$. Define a τ'-structure $\mathfrak{A}'_{k,n}$ so that it has the same universe and the same interpretations of the symbols U, P and $\sim$ as in $\mathfrak{A}_{k,n,\emptyset}$. Define the other relations as

$$
\begin{aligned}
U_i^{\mathfrak{A}'_{k,n}} &= \begin{cases} \{i\} \times k & \text{if } i < n, \\ \emptyset & \text{otherwise} \end{cases} \\
P_X^{\mathfrak{A}'_{k,n}} &= \begin{cases} \{X\} & \text{if } \max X < n, \\ \emptyset & \text{otherwise} \end{cases} \\
E^{\mathfrak{A}'_{k,n}} &= E^{\mathfrak{A}_{k,n,\emptyset}} \setminus (\{n-1\} \times k) \times (\{1\} \times k).
\end{aligned}
$$

We have $d^{\mathfrak{A}'_{k,n}}(\{0\} \times k, \{n-1\} \times k) = n-1$ in this structure.

Define k-sequences $\overline{a}$, $\overline{a}'$ and $\overline{b}$ so that for all $i < k$, $a_i = (0, i)$ and $b_i = (n-1, i)$, for all $i < k-2$, $a'_i = (0, i)$, $a'_{k-2} = (0, k-1)$ and $a'_{k-1} = (0, k-2)$. These sequences satisfy $\mathfrak{N}_r^{\mathfrak{A}'_{k,n}}(\overline{a}\overline{b}) \cong \mathfrak{N}_r^{\mathfrak{A}'_{k,n}}(\overline{a}'\overline{b})$.

Let $\psi \in I(\mathrm{QF}_\infty, \tau' \cup [\overline{z}] \cup [\overline{w}], \tau)$ such that the interpretation does not change the relations U, P and $\sim$,

$$\psi_\epsilon(xy) = \bigvee_{\substack{X \subseteq \mathbb{N} \\ |X| < \omega}} \bigvee_{i \in X} U_i x \wedge P_X y.$$

and

$$\psi_E(\overline{xy}) = E\overline{xy} \vee \bigvee_{\substack{f \in S_k \\ g \in A_k}} \bigwedge_{i<k} (x_i = w_{f(i)} \wedge y_i = z_{g(f(i))}).$$

With this interpretation $\psi^*(\mathfrak{A}'_{k,n}, \overline{a}/\overline{z}, \overline{b}/\overline{w}) = \mathfrak{A}_{k,n,\emptyset} \in Q_k$ and $\psi^*(\mathfrak{A}'_{k,n}, \overline{a}'/\overline{z}, \overline{b}/\overline{w}) = \mathfrak{A}_{k,n,\{n-1\}} \notin Q_k$ and so $\mathcal{L}_{Q_k} \circ \mathrm{QF}_\infty$ is not Gaifman-local. □

Lemma 4.21. *Let τ be a finite vocabulary, $r, n \in \mathbb{N}$, $\psi(x)$ a* QF*-formula and Q a τ-query. Define a τ-query Q' so that $\mathfrak{A} \in Q'$, if and only if there exists $c \in \mathrm{Dom}(\mathfrak{A})$ such that $|\psi(\mathfrak{A}) \setminus N_r^{\mathfrak{A}}(c)| \leq n$ and $\langle \psi(\mathfrak{A}) \rangle^{\mathfrak{A}} \in Q$. Then Q' is expressible in* HL*.*

Proof. For each partial τ-structure $\mathfrak{A}$ and $k \geq r$, let $Q_{\mathfrak{A},k} = \{\mathfrak{A}' \in \mathrm{Mod}(\tau \cup \{c\}) \mid \mathfrak{A} \sqcup \langle \psi(\mathfrak{A}') \cap N_k^{\mathfrak{A}'}(c^{\mathfrak{A}'}) \rangle^{\mathfrak{A}'} \restriction \tau \in Q, |\mathfrak{A}| + |\psi(\mathfrak{A}') \cap N_k^{\mathfrak{A}'}(c^{\mathfrak{A}'}) \setminus N_r^{\mathfrak{A}'}(c^{\mathfrak{A}'})| \leq n\}$. The query can be expressed in $\mathrm{QF} \circ SGL_{2k+2}^{k+1}$ (we have to consider also the constants at the distance $k+1$ from $c^{\mathfrak{A}'}$, because they may modify $\psi(\mathfrak{A}')$). Let $\varphi_{\mathfrak{A},k}$ be a sentence expressing the query. Let $\theta_{\mathfrak{A},k}$ be a FO-sentence expressing the query $\{\mathfrak{A}' \in \mathrm{Mod}(\tau \cup \{c\}) \mid \langle \psi(\mathfrak{A}') \setminus N_k^{\mathfrak{A}'}(c) \rangle^{\mathfrak{A}'} \cong \mathfrak{A}\}$. Let M be the set of all partial τ-structures with at most n elements. Because τ is finite, also M is finite up to isomorphism.

Now we can express Q' as

$$\exists c \bigvee_{r<k\leq r+n+1} \left(\neg \exists x(\psi(x) \wedge d(x,c) = k) \wedge \bigvee_{\mathfrak{A} \in M} (\psi_{\mathfrak{A},k} \wedge \varphi_{\mathfrak{A},k-1}) \right)$$

If the sentence holds on $\mathfrak{A}'$, then there is $c \in \mathrm{Dom}(\mathfrak{A})$ and $k \in \mathbb{N}$ such that $r \leq k \leq r+n+1$ and $\langle \psi(\mathfrak{A}') \rangle^{\mathfrak{A}'} \cong \langle \psi(\mathfrak{A}') \cap N_{k-1}^{\mathfrak{A}}(c) \rangle^{\mathfrak{A}'} \sqcup \langle \psi(\mathfrak{A}') \setminus N_k^{\mathfrak{A}}(c) \rangle^{\mathfrak{A}'}$. Additionally $\mathfrak{A} = \langle \psi(\mathfrak{A}') \setminus N_k^{\mathfrak{A}}(c) \rangle^{\mathfrak{A}'} \in M$ and $\langle \psi(\mathfrak{A}') \cap N_{k-1}^{\mathfrak{A}}(c) \rangle^{\mathfrak{A}'}$ satisfies $Q_{\mathfrak{A},k-1}$, which means that $\langle \psi(\mathfrak{A}') \rangle^{\mathfrak{A}'} \in Q$. Also the condition $|\psi(\mathfrak{A}') \setminus N_r^{\mathfrak{A}'}| \leq r$ is enforced.

The sentence expressing Q' is in FO ∘ HL* and so the query is expressible also in HL* by Lemma 3.20. □

Let Q'_k be a relativization of Q_k, i.e., a τ-query such that $\mathfrak{A} \in Q'_k$ if and only if $\langle U^{\mathfrak{A}} \cup P^{\mathfrak{A}} \rangle^{\mathfrak{A}} \in Q_k$.

Proposition 4.22. $\mathcal{L}_{Q'_k} \circ \mathrm{QF} \leq \mathrm{HL}^*$.

Proof. Let τ' be a finite vocabulary and $\psi \in I(\mathrm{QF}, \tau', \tau)$. We need to show that the query $K_0 = \{\mathfrak{A}' \in \mathrm{Mod}(\tau') \mid \psi(\mathfrak{A}') \in Q'_k\}$ is expressible in HL*.

Let

$$
\begin{aligned}
K_1 &= \{\mathfrak{A}' \in \mathrm{Mod}(\tau') \mid \langle U^{\psi(\mathfrak{A}')} \cup P^{\psi(\mathfrak{A}')} \rangle^{\psi(\mathfrak{A}')} \in M_k\}, \\
K_2 &= \{\langle U^{\psi(\mathfrak{A}')} \cup \mathrm{Con}(\tau')^{\mathfrak{A}'} \rangle^{\mathfrak{A}'} \mid \mathfrak{A}' \in K_0\}, \text{ and} \\
K_3 &= \{\mathfrak{A}' \in \mathrm{Mod}(\tau') \mid \langle U^{\psi(\mathfrak{A}')} \cup \mathrm{Con}(\tau')^{\mathfrak{A}'} \rangle^{\mathfrak{A}'} \in K_2\}.
\end{aligned}
$$

Clearly for all $\mathfrak{A}' \in K_0$, $\mathfrak{A}' \in K_1 \cap K_3$. On the other hand, if $\mathfrak{A}' \in K_1 \cap K_3$, then $\mathfrak{A}' \in K_0$, because the isomorphism type of $\langle U^{\psi(\mathfrak{A}')} \cup \mathrm{Con}(\tau')^{\mathfrak{A}'} \rangle^{\mathfrak{A}'}$ determines the isomorphism type of $E^{\psi(\mathfrak{A}')} \cap (U^{\psi(\mathfrak{A}')})^2$ and so tells if $\mathfrak{A}' \in K_0$ or $\mathfrak{A}' \in K_1 \setminus K_0$. Thus $K_0 = K_1 \cap K_3$.

By Lemma 4.19, the query K_1 is FO-definable. Let m be the number of different unary atomic types realizable on τ'-structures. Let $r = 4$, $n = k(m+2)$,

$$
\begin{aligned}
K_4 &= \{\mathfrak{A}' \in \mathrm{Mod}(\tau') \mid \exists c \in \mathrm{Dom}(\mathfrak{A}')(|U^{\psi(\mathfrak{A}')} \setminus N_r^{\mathfrak{A}'}(c)| \le n)\}, \text{ and} \\
K_5 &= K_3 \cap K_4.
\end{aligned}
$$

By Lemma 4.21, K_5 and so also $K_1 \cap K_5$ is expressible in HL^*. (the unary formula in the lemma is in this case $\psi_U(x) \vee \bigvee_{c \in \mathrm{Con}(\tau')} x = c^{\mathfrak{A}}$). Thus it is enough to show that $K_1 \subseteq K_4$, which implies $K_1 \cap K_5 = K_1 \cap K_3$

Let $\mathfrak{A}'$ be a τ'-structure and $\mathfrak{A} = \psi(\mathfrak{A}')$ such that such that $\mathfrak{A} \in K_1$. The relation $\sim^{\mathfrak{A}}$ is an equivalence relation on $U^{\mathfrak{A}}$. Denote its equivalence classes by $[a] = \{b \in U^{\mathfrak{A}} \mid a \sim^{\mathfrak{A}} b\}$.

Let $a, b, c, d \in U^{\mathfrak{A}}$ be elements from different $\sim^{\mathfrak{A}}$-equivalence classes such that $\mathrm{atp}^{\mathfrak{A}'}(a) = \mathrm{atp}^{\mathfrak{A}'}(b)$ and $\mathrm{atp}^{\mathfrak{A}'}(c) = \mathrm{atp}^{\mathfrak{A}'}(d)$. There exists an element $e \in P^{\mathfrak{A}}$ such that $(a,e), (c,e) \in \epsilon^{\mathfrak{A}}$ and $(b,e), (d,e) \notin \epsilon^{\mathfrak{A}}$. This means $\mathrm{atp}^{\mathfrak{A}'}(ae) \neq \mathrm{atp}^{\mathfrak{A}'}(be)$ and since $\mathrm{atp}^{\mathfrak{A}'}(a) = \mathrm{atp}^{\mathfrak{A}'}(b)$, necessarily a or b is in $N_1^{\mathfrak{A}'}(e)$. The same thing holds for the elements c and d and we conclude $\{c, d\} \cap N_2^{\mathfrak{A}}(ab) \neq \emptyset$.

Assume that $|U^{\mathfrak{A}}| > 3km$. Then there exists $a, b, c, d \in U^{\mathfrak{A}}$ from different $\sim^{\mathfrak{A}}$-equivalence classes having the same atomic type. Then, by the conclusion above, for some pair of the elements, say for (a, b), $d^{\mathfrak{A}}(a, b) \le 2$.

If $X \subseteq U^{\mathfrak{A}} \subseteq ([a] \cup [b])$ is a set of elements with the same atomic type, then we necessarily have $|X \setminus N_2^{\mathfrak{A}}(ab)| \le k$. Otherwise, there were $c, d \in X \setminus N_2^{\mathfrak{A}}(ab)$ from different $\sim^{\mathfrak{A}}$-equivalence classes. This implies $|U^{\mathfrak{A}} \setminus N_2^{\mathfrak{A}}(ab)| \le k(m+2)$. Because $d^{\mathfrak{A}}(a, b) \le 2$, we have $|U^{\mathfrak{A}} \setminus N_4^{\mathfrak{A}}(a)| \le k(m+2)$. □

Corollary 4.23. *If $\mathcal{L}'$ is regular and $\mathcal{L}' \le \mathrm{HL}^*$, then $\mathcal{L}'$ has weak uniform reduction.*

Proof. The claim follows from Theorem 4.16 if we let $\mathcal{L}$ in Theorem to be $\mathcal{L}_{Q'_k}$. The composition $\mathcal{L}_{Q'_k} \circ \mathcal{L}'$ is Hanf-local by Lemma 3.20 and other conditions are clear. □

5. Gaifman-locality

5.1. **A characterization of Gaifman-locality.** If C is a subset of the universe of $\mathfrak{A}$, let

$$N_r^{\mathfrak{A}}(C) = \{a \in A \mid \exists b \in C(d^{\mathfrak{A}}(a,b) \leq r)\} = \bigcup_{b \in C} N_r^{\mathfrak{A}}(b).$$

As with neighborhoods of constants, we define $\mathfrak{N}_r^{\mathfrak{A}}(C) = \langle N_r^{\mathfrak{A}}(C)\rangle^{(\mathfrak{A},C)}$, i.e., C is encoded in $\mathfrak{N}_r^{\mathfrak{A}}(C)$ as a unary relation.

We give in this section a new family of notions of locality, that satisfy the conditions of Section 3.3 and with some regularity assumptions has Gaifman-locality as a special case.

Definition 5.1. We write $\mathfrak{A} \leftrightsquigarrow_{n,r} \mathfrak{B}$, if and only if there exist partitions $\mathrm{Dom}(\mathfrak{A}) = A_0 \cup \cdots \cup A_{n-1}$ and $\mathrm{Dom}(\mathfrak{B}) = B_0 \cup \cdots \cup B_{n-1}$ such that for all $i < n$,

$$\langle N_r^{\mathfrak{A}}(A_i)\rangle^{(\mathfrak{A},A_0/X_0,\ldots,A_{n-1}/X_{n-1})} \cong \langle N_r^{\mathfrak{B}}(B_i)\rangle^{(\mathfrak{A},B_0/X_0,\ldots,B_{n-1}/X_{n-1})}.$$

Definition 5.2. We write $\mathfrak{A} \leftrightsquigarrow_{n,r}^{w} \mathfrak{B}$, if and only if there exist partitions $\mathrm{Dom}(\mathfrak{A}) = A_0 \cup \cdots \cup A_{n-1}$ and $\mathrm{Dom}(\mathfrak{B}) = B_0 \cup \cdots \cup B_{n-1}$ such that for all $i < n$, $\mathfrak{N}_r^{\mathfrak{A}}(A_i) \cong \mathfrak{N}_r^{\mathfrak{B}}(B_i)$.

If $n = 2$, the definitions coincide: $\mathfrak{A} \leftrightsquigarrow_{2,r} \mathfrak{B} \iff \mathfrak{A} \leftrightsquigarrow_{2,r}^{w} \mathfrak{B}$. For $n > 2$, we have $\mathfrak{A} \leftrightsquigarrow_{n,r} \mathfrak{B} \Rightarrow \mathfrak{A} \leftrightsquigarrow_{n,r}^{w} \mathfrak{B}$. If $n < n'$, we have also $\mathfrak{A} \leftrightsquigarrow_{n,r} \mathfrak{B} \Rightarrow \mathfrak{A} \leftrightsquigarrow_{n',r} \mathfrak{B}$ and $\mathfrak{A} \leftrightsquigarrow_{n,r}^{w} \mathfrak{B} \Rightarrow \mathfrak{A} \leftrightsquigarrow_{n',r}^{w} \mathfrak{B}$. For all $n, r \in \mathbb{Z}_+$, $\mathfrak{A} \leftrightsquigarrow_{n,r}^{w} \mathfrak{B} \Rightarrow \mathfrak{A} \leftrightarrows_r \mathfrak{B}$, since if $\mathrm{Dom}(\mathfrak{A}) = A_0 \cup \cdots \cup A_{n-1}$ and $\mathrm{Dom}(\mathfrak{B}) = B_0 \cup \cdots \cup B_{n-1}$ such that for all $i < n$, $\alpha_i \colon \mathfrak{N}_r^{\mathfrak{A}}(A_i) \cong \mathfrak{N}_r^{\mathfrak{B}}(B_i)$, then $\bigcup_{i<n} \alpha_i \restriction A_i \colon \mathfrak{A} \leftrightarrows_r \mathfrak{B}$.

Most of the things we say hold both for $\leftrightsquigarrow_{n,r}$ and $\leftrightsquigarrow_{n,r}^{w}$, however we can prove only that the former gives a notion of locality that satisfies the conditions in Section 3.3.

Lemma 5.3. *Suppose $\mathcal{L} \circ \mathrm{QF} \leq \mathcal{L}$ and for every $\mathcal{L}$-formula $\varphi(x)$ there exists $r \in \mathbb{N}$ such that $\mathfrak{N}_r^{\mathfrak{A}}(a) \cong \mathfrak{N}_r^{\mathfrak{A}}(b)$ implies $(\mathfrak{A}, a/x) \equiv_{\varphi(x)} (\mathfrak{A}, b/x)$. Then for all $\mathcal{L}$-sentences φ there is $r \in \mathbb{N}$ such that $\mathfrak{A} \leftrightsquigarrow_{2,r} \mathfrak{A}'$ implies $\mathfrak{A} \equiv_{\varphi} \mathfrak{A}'$.*

Proof. In order to simplify notations, let us assume that τ contains only one relation symbol R. Suppose $\mathfrak{A} \leftrightsquigarrow_{2,r+1} \mathfrak{A}'$ (we choose r later) which means that there are partitions $A = A_1 \cup A_2$ and $A' = A_1' \cup A_2'$ and partial isomorphisms $\alpha_i \colon N_{r+1}^{\mathfrak{A}}(A_i) \to N_{r+1}^{\mathfrak{A}'}(A_i')$ of $\mathfrak{A}$ such that $\alpha_i(A_i) = A_i'$.

Let $D = N_1^{\mathfrak{A}}(A_1) \cap N_1^{\mathfrak{A}}(A_2)$ and $\beta \colon N_r^{\mathfrak{A}}(D) \to N_r^{\mathfrak{A}}(D)$,

$$\beta(x) = \begin{cases} x & \text{if } x \in A_1, \\ \alpha_2^{-1}(\alpha_1(x)) & \text{if } x \in A_2. \end{cases}$$

Clearly β is bijective and $\beta(D) = D$. For every $i \in \mathbb{N}$, let $R_i = \beta^i(R^{\mathfrak{A}} \cap D^{\mathrm{ar}(R)})$ and let $R' = R^{\mathfrak{A}} \setminus R_0$. Note, that $\beta(R' \cap (N_r(D))^{\mathrm{ar}(R)}) = R' \cap (N_r(D))^{\mathrm{ar}(R)}$. Now $\mathfrak{A} = (\mathrm{Dom}(\mathfrak{A}), (R' \cup R_0)/R)$ and $(\mathrm{Dom}(\mathfrak{A}), (R' \cup R_1)/R) \cong \mathfrak{A}'$, where the latter isomorphism is $\alpha_1 \restriction A_1 \cup \alpha_2 \restriction A_2$.

We say that a permutation $\gamma\colon D \to D$ has a characteristic orbit if there is $x \in D$ such that $\gamma^s(x) = x$ implies $\gamma^s = \mathrm{id}_D$.

Claim 1. If $\beta^k \restriction D$ has a characteristic orbit and r is big enough (depending only on φ), $(\mathrm{Dom}(\mathfrak{A}), R' \cup R_i) \equiv_\varphi (\mathrm{Dom}(\mathfrak{A}), R' \cup R_{i+k})$.

Let $\gamma = \beta^k \restriction D$ and let $c \in D$ be an element such that $\gamma^s(c) = c$ implies $\gamma^s = \mathrm{id}_D$. Let $S = \{(\gamma^s(\overline{y}), \gamma^s(c)) \in D^{\mathrm{ar}(R)+1} \mid \overline{y} \in R^{\mathfrak{A}} \text{ and } s \in \mathbb{Z}\}$ and let $\mathfrak{C} = (A, R', S)$.

If $\psi(\overline{y}) = R\overline{y} \vee S\overline{y}c$ then

$$(\mathrm{Dom}(\mathfrak{A}), \psi(\mathfrak{C}, c)) = (\mathrm{Dom}(\mathfrak{A}), R' \cup R_i)$$

and

$$(\mathrm{Dom}(\mathfrak{A}), \psi(\mathfrak{C}, \gamma(c))) \cong (\mathrm{Dom}(\mathfrak{A}), R' \cup R_{i+k}).$$

This needs the assumption that γ has a characteristic orbit.

Note that $\gamma\colon \mathfrak{N}_r^{\mathfrak{C}}(c) \cong \mathfrak{N}_r^{\mathfrak{C}}(\gamma(c))$. Now, if we let $\varphi'(c) = \varphi \circ \{(R, \psi)\}$ and choose r to be the locality rank of φ' the claim follows.

Claim 2. Let o be the smallest positive integer such that $\beta^o \restriction D = \mathrm{id}_D$. Let $o = \prod_p p^{l_p}$ be the prime factorization of o. Then for every prime p, $\beta^{o/p^{l_p}}$ has a characteristic orbit.

Let $\gamma = \beta^{o/p^{l_p}} \restriction D$. Because $\gamma^{p^{l_p}} = \mathrm{id}_D$, every element $x \in D$ has order that divides p^{l_p}. If no $x \in D$ has order p^{l_p}, $\gamma^{p^{l_p - 1}} = \mathrm{id}_D$ contradicting the choice of o. This proves the claim.

Now, if I is the set of indices i such that $0 \le i < o$ and $(\mathrm{Dom}(\mathfrak{A}), R' \cup R_0) \equiv_\varphi (\mathrm{Dom}(\mathfrak{A}), R' \cup R_i)$, the last two claims combined gives that if $p^l | o$ then $p^l \,|\, |I|$. Because $0 \in I$, $|I| = o$ and in particular $1 \in I$ i.e. $\mathfrak{A} \cong (\mathrm{Dom}(\mathfrak{A}), R', R_0) \equiv_\varphi (\mathrm{Dom}(\mathfrak{A}), R', R_1) \cong \mathfrak{A}'$. □

Since relation $\leftrightsquigarrow_{n,r}$ is probably not an equivalence relation, we define its transitive closure:

Definition 5.4. We write $\mathfrak{A} \leftrightsquigarrow^*_{n,r} \mathfrak{A}'$, if there exists a sequence of structures $\mathfrak{A}_0, \ldots, \mathfrak{A}_k$ such that $\mathfrak{A}_0 = \mathfrak{A}$, $\mathfrak{A}_k = \mathfrak{A}'$ and for all $i < k$, $\mathfrak{A}_i \leftrightsquigarrow_{n,r} \mathfrak{A}_{i+1}$. In a similar way, let $\leftrightsquigarrow^{w,*}_{n,r}$ be the transitive closure of $\leftrightsquigarrow^{w}_{n,r}$.

Lemma 5.5. *Let $s_0 = 0$, and inductively define $s_{k+1} = 6 + 2ks_k$ for $k \ge 0$. Let $\mathfrak{A}$ be an arbitrary structure and $r, k \in \mathbb{N}$. If $\overline{a}, \overline{b} \in \mathrm{Dom}(\mathfrak{A})^k$ such that $\mathfrak{N}^{\mathfrak{A}}_{s_k r}(\overline{a}) \cong \mathfrak{N}^{\mathfrak{A}}_{s_k r}(\overline{b})$, then $(\mathfrak{A}, \overline{a}/\overline{x}) \leftrightsquigarrow^*_{2,r} (\mathfrak{A}, \overline{b}/\overline{x})$.*

Proof. We prove the lemma by induction on k. The claim is trivial for $k = 0$. Suppose, the lemma is true for all $k' < k$ and let $\mathfrak{A}$ be a structure and $\overline{a}, \overline{b} \in \mathrm{Dom}(\mathfrak{A})^k$ such that $\alpha\colon \mathfrak{N}^{\mathfrak{A}}_{s_k r}(\overline{a}) \cong \mathfrak{N}^{\mathfrak{A}}_{s_k r}(\overline{b})$. We prove the claim now by cases.

Case 1. $[\overline{a}] \cap N^{\mathfrak{A}}_{4r}(\overline{b}) = \emptyset$.

Let $A_0 = N^{\mathfrak{A}}_r(\overline{a}) \cup N^{\mathfrak{A}}_r(\overline{b})$ and $A_1 = \mathrm{Dom}(\mathfrak{A}) \setminus A_0$. Because $N^{\mathfrak{A}}_{2r}(\overline{a}) \cap N^{\mathfrak{A}}_{2r}(\overline{b}) = \emptyset$, the function $\alpha \restriction N^{\mathfrak{A}}_{2r}(\overline{a}) \cup \alpha^{-1} \restriction N^{\mathfrak{A}}_{2r}(\overline{b})$ is an automorphism of $\mathfrak{N}^{\mathfrak{A}}_r(A_0)$ and thus

$\mathfrak{N}_r^{(\mathfrak{A},\overline{a}/\overline{x})}(A_0) \cong \mathfrak{N}_r^{(\mathfrak{A},\overline{b}/\overline{x})}(A_0)$. Because $([\overline{a}] \cup [\overline{b}]) \cap N_r^{\mathfrak{A}}(A_1) = \emptyset$, $\mathfrak{N}_r^{(\mathfrak{A},\overline{a}/\overline{x})}(A_1) = \mathfrak{N}_r^{\mathfrak{A}}(A_1) = \mathfrak{N}_r^{(\mathfrak{A},\overline{b}/\overline{x})}(A_1)$ and thus $(\mathfrak{A}, \overline{a}/\overline{x}) \leftrightsquigarrow_{2,r} (\mathfrak{A}, \overline{b}/\overline{x})$.

Case 2. $[\overline{a}] \cap N_{4r}^{\mathfrak{A}}(\overline{b}) \neq \emptyset$ and $[\overline{a}] \not\subseteq N_{4r+(k-1)s_{k-1}r}^{\mathfrak{A}}(\overline{b})$.

For all $i < k$, let $d_i = d^{\mathfrak{A}}(\overline{b}, a_i)$. We may assume without loss of generality that $d_0 \leq \cdots \leq d_{k-1}$. Because $d_0 \leq 4r$ and $d_{k-1} > 4r + (k-1)s_{k-1}r$, there has to exist $i < k-1$ such that $d_{i+1} - d_i > s_{k-1}r$.

Let $\overline{a}' = a_0 \dots a_i$, $\overline{a}'' = a_{i+1} \dots a_{k-1}$, $\overline{b}' = b_0 \dots b_i$ and $\overline{b}'' = b_{i+1} \dots b_{k-1}$. By the choice of i, $d^{\mathfrak{A}}(\overline{a}', \overline{a}'') > s_{k-1}r$. Because α preserves distances up to $2s_k r$ between elements of $\overline{a}$, also $d^{\mathfrak{A}}(\overline{b}', \overline{b}'') > s_{k-1}r$. In addition, we trivially have $d^{\mathfrak{A}}(\overline{b}', \overline{a}'') > s_{k-1}r$. This implies $\mathfrak{N}_{s_{k-1}r}^{(\mathfrak{A},\overline{a}''/\overline{y})}(\overline{a}') \cong \mathfrak{N}_{s_{k-1}r}^{(\mathfrak{A},\overline{a}''/\overline{y})}(\overline{b}')$ and $\mathfrak{N}_{s_{k-1}r}^{(\mathfrak{A},\overline{b}'/\overline{x})}(\overline{a}'') \cong \mathfrak{N}_{s_{k-1}r}^{(\mathfrak{A},\overline{b}'/\overline{x})}(\overline{b}'')$.

Applying the induction hypothesis, we get

$$(\mathfrak{A}, \overline{a}'/\overline{x}, \overline{a}''/\overline{y}) \leftrightsquigarrow_{2,r}^{*} (\mathfrak{A}, \overline{b}'/\overline{x}, \overline{a}''/\overline{y}) \leftrightsquigarrow_{2,r}^{*} (\mathfrak{A}, \overline{b}'/\overline{x}, \overline{b}''/\overline{y}).$$

Case 3. $[\overline{a}] \subseteq N_{4r+(k-1)s_{k-1}r}^{\mathfrak{A}}(\overline{b})$.

Denote $\overline{a}_i = \alpha^i(\overline{a})$. Note, that for some i, $[\overline{a}_i]$ possibly is not anymore in $N_{s_k r}^{\mathfrak{A}}(\overline{a})$ and so $\overline{a}_{i+1}$ is not defined. However at least $\overline{a}_2$ is defined, because $\overline{a}_1 = \overline{b}$ and $[\overline{b}] \subseteq N_{4r+(k-1)s_{k-1}r}^{\mathfrak{A}}(\overline{a})$. We have now two subcases:

Case 3.a. For some i, $[\overline{a}_i] \not\subseteq N_{4r+2(k-1)s_{k-1}r}^{\mathfrak{A}}(\overline{a})$.

This implies that $[\overline{a}_i] \not\subseteq N_{4r+(k-1)s_{k-1}r}^{\mathfrak{A}}(\overline{a}) \cup N_{4r+(k-1)s_{k-1}r}^{\mathfrak{A}}(\overline{b})$ and thus by the cases 1 and 2, we have $(\mathfrak{A}, \overline{a}/\overline{x}) \leftrightsquigarrow_{2,r}^{*} (\mathfrak{A}, \overline{a}_i/\overline{x}) \leftrightsquigarrow_{2,r}^{*} (\mathfrak{A}, \overline{b}/\overline{x})$.

Case 3.b. The sequence $\overline{a}_i$ is defined for all $i \in \mathbb{N}$ and $[\overline{a}_i] \subseteq N_{4r+2(k-1)s_{k-1}r}^{\mathfrak{A}}(\overline{a})$.

Let $A_0 = \bigcup_{i\in\mathbb{N}} N_r^{\mathfrak{A}_i}(\overline{a}_i)$ and $A_1 = \mathrm{Dom}(\mathfrak{A}) \setminus A_0$. Since $N_r^{\mathfrak{A}}(A_0) \subseteq N_{6r+2(k-1)s_{k-1}r}^{\mathfrak{A}}(\overline{a})$, $\alpha \restriction N_r^{\mathfrak{A}}(A_0)$ is an automorphism of $\mathfrak{N}_r^{\mathfrak{A}}(A_0)$ showing that $\mathfrak{N}_r^{(\mathfrak{A},\overline{a}/\overline{x})}(A_0) \cong \mathfrak{N}_r^{(\mathfrak{A},\overline{b}/\overline{x})}(A_0)$. We have also $\mathfrak{N}_r^{(\mathfrak{A},\overline{a}/\overline{x})}(A_1) = \mathfrak{N}_r^{(\mathfrak{A},\overline{b}/\overline{x})}(A_1)$ and so $(\mathfrak{A}, \overline{a}/\overline{x}) \leftrightsquigarrow_{2,r} (\mathfrak{A}, \overline{b}/\overline{x})$. □

Combining these two lemmas we conclude:

Theorem 5.6. *If $\mathcal{L} \circ \mathrm{QF} \leq \mathcal{L}$, then the following are equivalent:*

a) $\mathcal{L}$ is Gaifman-local.
b) For all $\mathcal{L}$-sentences φ, there is r such that $\mathfrak{A} \leftrightsquigarrow_{2,r} \mathfrak{A}'$ implies $\mathfrak{A} \equiv_{\varphi} \mathfrak{A}'$,
c) $\mathcal{L}$ is Gaifman-local with formulas containing only one variable.

□

This gives an alternative proof for the theorem from [HLN99]:

Theorem 5.7. *Hanf-local logics with $\mathcal{L} \circ \mathrm{QF} \leq \mathcal{L}$ are Gaifman-local.*

Proof. This follows from the theorem, because $\mathfrak{A} \leftrightsquigarrow_{2,r} \mathfrak{A}'$ implies $\mathfrak{A} \leftrightarrows_r \mathfrak{A}'$. □

5.2. **Composition properties of Gaifman-locality.** We prove in this section that for all $n \geq 2$, $(\leftrightsquigarrow^*_{n,r})_{r\in\mathbb{N}}$ satisfies the conditions of Section 3.3. The conditions (a) and (b) are clearly satisfied. The following lemma shows that the condition (c) holds.

Lemma 5.8. *If* $\mathfrak{A} \leftrightsquigarrow_{n,r_0k_1+r_1} \mathfrak{B}$ *and* $\psi \in I(\mathrm{SGL}^{r_1}_{k_1}, \tau, \tau')$, *then* $\psi^*(\mathfrak{A}) \leftrightsquigarrow_{n,r_0} \psi^*(\mathfrak{B})$.

Proof. We may add the partition witnessing $\mathfrak{A} \leftrightsquigarrow_{n,r_0k_1+r_1} \mathfrak{B}$ to the structures as unary relations: $\mathrm{Dom}(\mathfrak{A}) = X_0^{\mathfrak{A}} \cup \cdots \cup X_{n-1}^{\mathfrak{A}}$ and $\mathrm{Dom}(\mathfrak{B}) = X_0^{\mathfrak{B}} \cup \cdots \cup X_{n-1}^{\mathfrak{B}}$. We may also assume that $\{X_0, \ldots, X_{n-1}\} \subseteq \tau'$ and $\psi_{X_i}(x) = X_i x$.

For each $i < n$, let $\alpha_i \colon \mathfrak{N}^{\mathfrak{A}}_{r_0k_1+r_1}(X_i^{\mathfrak{A}}) \cong \mathfrak{N}^{\mathfrak{B}}_{r_0k_1+r_1}(X_i^{\mathfrak{B}})$. Choose some enumeration $\overline{a}$ of $X_i^{\mathfrak{A}}$ and let $\overline{b} = \alpha_i(\overline{a})$. Clearly $[\overline{b}] = X_i^{\mathfrak{B}}$. Now $\mathfrak{N}^{\mathfrak{A}}_{r_0k_1+r_1}(\overline{a}) \cong \mathfrak{N}^{\mathfrak{A}}_{r_0k_1+r_1}(\overline{b})$ and Lemma 3.6 implies $\mathfrak{N}^{\psi^*(\mathfrak{A})}_{r_0}(\overline{a}) \cong \mathfrak{N}^{\psi^*(\mathfrak{B})}_{r_0}(\overline{b})$. This shows $\langle \mathfrak{N}^{\psi^*(\mathfrak{A})}_{r_0}(X_i^{\mathfrak{A}}) \cong \mathfrak{N}^{\mathfrak{B}}_{r_0}(X_i^{\psi^*(\mathfrak{B})})$. Because this holds for all $i < n$, we have $\psi^*(\mathfrak{A}) \leftrightsquigarrow_{n,r_0} \psi^*(\mathfrak{B})$. ☐

Remark. The equivalence relations $\leftrightsquigarrow^w_{n,r}$ also satisfy conditions (a)-(c).

We show next that the condition (d) holds.

Lemma 5.9. *If* $\mathfrak{A} \leftrightsquigarrow_{n,(4k+1)r} \mathfrak{B}$ *and* $\overline{a} \in (\mathrm{Dom}(\mathfrak{A}))^k$, *then there exists a sequence* $\overline{b} \in (\mathrm{Dom}(\mathfrak{B}))^k$ *such that* $\mathfrak{N}^{\mathfrak{A}}_r(\overline{a}) \cong \mathfrak{N}^{\mathfrak{B}}_r(\overline{b})$ *and* $(\mathfrak{A}, \overline{a}/\overline{x}) \leftrightsquigarrow_{n,r} (\mathfrak{B}, \overline{b}/\overline{x})$.

Proof. Let $\mathrm{Dom}(\mathfrak{A}) = A_0 \cup \ldots \cup A_{n-1}$ and $\mathrm{Dom}(\mathfrak{B}) = B_0 \cup \ldots \cup B_{n-1}$ be partitions witnessing $\mathfrak{A} \leftrightsquigarrow_{n,(4k+1)r} \mathfrak{B}$. For all $i < n$, let

$$\alpha_i \colon \langle N^{\mathfrak{A}}_{(4k+1)r}(A_i)\rangle^{(\mathfrak{A}, A_0/X_0, \ldots, A_{n-1}/X_{n-1})} \cong \langle N^{\mathfrak{B}}_{(4k+1)r}(B_i)\rangle^{(\mathfrak{B}, B_0/X_0, \ldots, B_{n-1}/X_{n-1})}.$$

Let $K_i = \{d^{\mathfrak{A}}(A_i, a_j) \mid j < k\}$. Because $|K_i| \leq k$, we can choose for all $i < n$, $s_i \in [0, 2kr]$ such that $[s_i - r + 1, s_i + r] \cap K_i = \emptyset$.

For all $i < n$, let $A'_i = N^{\mathfrak{A}}_{s_i}(A_i) \setminus \bigcup_{j<i} A'_j$ and $B'_i = N^{\mathfrak{A}}_{s_i}(B_i) \setminus \bigcup_{j<i} B'_j$. The sequences $(A'_i)_{i<n}$ and $(B'_i)_{i<n}$ form partitions of $\mathrm{Dom}(\mathfrak{A})$ and $\mathrm{Dom}(\mathfrak{B})$.

If $j < k$ and $i < n$ such that $a_j \in A'_i$, then for all $i' < i$, $d^{\mathfrak{A}}(A_{i'}, a_j) > s_{i'}$ and $d^{\mathfrak{A}}(A_i, a_j) \leq s_i$. By the choice of s_i's, for all $i' < i$, $d^{\mathfrak{A}}(A_{i'}, a_j) > s_{i'} + r$ and $d^{\mathfrak{A}}(A_i, a_j) \leq s_i - r$. This implies $N^{\mathfrak{A}}_r(a_j) \subseteq A_i$.

We have clearly $A'_i \subseteq N^{\mathfrak{A}}_{2kr}(A_i) \subseteq \mathrm{dom}(\alpha_i)$. We claim now that for all $i' < n$, $\alpha_i(A'_{i'} \cap N^{\mathfrak{A}}_{(2k+1)r}(A_i)) = B'_{i'} \cap N^{\mathfrak{B}}_{(2k+1)r}(B_i)$. This, in particular, implies $\alpha_i(A'_i) = B'_i$. If $c \in N^{\mathfrak{A}}_{s_{i'}}(A'_{i'}) \cap N^{\mathfrak{A}}_{(2k+1)r}(A_i)$, then there is $d \in A'_{i'}$ such that $d^{\mathfrak{A}}(c, d) \leq s_{i'}$. But now $d^{\mathfrak{A}}(A_i, d) \leq (2k+1)r + s_{i'} \leq (4k+1)r$ and so $d \in N^{\mathfrak{A}}_{(4k+1)r}(A_i)$ and $d^{\mathfrak{A}}(\alpha_i(c), \alpha_i(d)) \leq s_{i'}$. Because α_i maps $A_{i'} \cap \mathrm{dom}(\alpha_i)$ to $B_{i'} \cap \mathrm{rng}(\alpha_i)$, we have $\alpha_i(c) \in N^{\mathfrak{B}}_{s_{i'}}(B_{i'})$. In a similar way, we can show that if $c \in N^{\mathfrak{B}}_{s_{i'}}(B'_{i'}) \cap N^{\mathfrak{B}}_{(2k+1)r}(B_i)$, then $\alpha^{-1}(c) \in N^{\mathfrak{A}}_{s_{i'}}(A_{i'})$. Because the sets A'_i and B'_i are boolean combinations of the neighborhoods $N^{\mathfrak{A}}_{s_{i'}}(A'_{i'})$ and $N^{\mathfrak{B}}_{s_{i'}}(B'_{i'})$, this proves the claim.

Now, define a sequence $\overline{b}$ so that $b_j = \alpha_i(a_j)$, where i is chosen such that $a_j \in A'_i$. Because each a_j belongs to exactly one neighborhood $N^{\mathfrak{A}}_r(A'_i)$, for all $i < n$, $\alpha_i \upharpoonright N^{\mathfrak{A}}_r(A'_i) \colon \langle N^{\mathfrak{A}}_r(A'_i)\rangle^{(\mathfrak{A}, \overline{a}/\overline{x}, A'_0/X_0, \ldots, A'_{n-1}/X_{n-1})} \cong \langle N^{\mathfrak{B}}_r(B'_i)\rangle^{(\mathfrak{B}, \overline{b}/\overline{x}, B'_0/X_0, \ldots, B'_{n-1}/X_{n-1})}$. We have $\mathfrak{N}_r^{\mathfrak{A}}(\overline{a}) \cong \mathfrak{N}_r^{\mathfrak{B}}(\overline{b})$, because every component of $\mathfrak{N}_r^{\mathfrak{A}}(\overline{a})$ is completely included in some of the sets A_i'. □

Lemma 5.10. *If* $\mathfrak{A} \leftrightsquigarrow^*_{n,(4k+1)s_k r} \mathfrak{B}$, $\overline{a} \in \mathrm{Dom}(\mathfrak{A})^k$, $\overline{b} \in \mathrm{Dom}(\mathfrak{B})^k$ *and* $\mathfrak{N}_{s_k r}^{\mathfrak{A}}(\overline{a}) \cong \mathfrak{N}_{s_k r}^{\mathfrak{B}}(\overline{b})$, *then* $(\mathfrak{A}, \overline{a}/\overline{x}) \leftrightsquigarrow^*_{n,r} (\mathfrak{B}, \overline{b}/\overline{x})$.

Proof. By the previous lemma, there exists $\overline{b}' \in \mathrm{Dom}(\mathfrak{B})^k$ such that $\mathfrak{N}_{s_k r}^{\mathfrak{A}}(\overline{a}) \cong \mathfrak{N}_{s_k r}^{\mathfrak{B}}(\overline{b}')$ and $(\mathfrak{A}, \overline{a}/\overline{x}) \leftrightsquigarrow_{n,s_k r} (\mathfrak{B}, \overline{b}'/\overline{x})$. By Lemma 5.5, $(\mathfrak{B}, \overline{b}'/\overline{x}) \leftrightsquigarrow_{2,r} (\mathfrak{B}, \overline{b}/\overline{x})$. □

5.3. **Grids.** Our goal in Sections 5.3–5.5 is to construct examples showing that the hierarchy of logics $(\mathrm{HL}(\leftrightsquigarrow_{n,r})_{r\in\mathbb{N}})_{n\geq 2}$ is strict. We describe in this section the structures we use in the construction and prove some of their basic properties.

Let $\tau = \{E_i \mid i < n\}$, where every E_i is a binary relation symbol. We call a τ-structure $\mathfrak{A}$ an *n-dimensional grid*, if the following conditions are satisfied:

a) For all $0 \leq i < n$, $E_i^{\mathfrak{A}}$ is the graph of a partial injection $f_i^{\mathfrak{A}}\colon \mathrm{Dom}(\mathfrak{A}) \to \mathrm{Dom}(\mathfrak{A})$.
b) For all $0 \leq i < j < n$, $f_i^{\mathfrak{A}} \circ f_j^{\mathfrak{A}} = f_j^{\mathfrak{A}} \circ f_i^{\mathfrak{A}}$, in particular $\mathrm{dom}(f_i^{\mathfrak{A}} \circ f_j^{\mathfrak{A}}) = \mathrm{dom}(f_j^{\mathfrak{A}} \circ f_i^{\mathfrak{A}})$.
c) For all $k \leq n$, $0 \leq i_0 < i_1 < \cdots < i_{k-1} < n$, and $\epsilon_0, \ldots, \epsilon_{k-1} \in \{-1, 1\}$, $\mathrm{dom}((f_{i_0}^{\mathfrak{A}})^{\epsilon_0} \circ \cdots \circ (f_{i_{k-1}}^{\mathfrak{A}})^{\epsilon_{k-1}}) = \mathrm{dom}((f_{i_0}^{\mathfrak{A}})^{\epsilon_0}) \cap \cdots \cap \mathrm{dom}((f_{i_{k-1}}^{\mathfrak{A}})^{\epsilon_{k-1}})$.

Note, that because $f_i^{\mathfrak{A}}$ is a partial injection, $(f_i^{\mathfrak{A}})^{-1}$ also is a well-defined partial injection. The conditions (b) and (c) imply $(f_i^{\mathfrak{A}})^{\epsilon_0} \circ (f_j^{\mathfrak{A}})^{\epsilon_1} = (f_j^{\mathfrak{A}})^{\epsilon_1} \circ (f_i^{\mathfrak{A}})^{\epsilon_0}$ for all $\epsilon_0, \epsilon_1 \in \{1, -1\}$.

Informally, grids can be thought as discretizations of n-dimensional manifolds by rectangular elements. The condition (c) ensures that the boundaries of the grids cannot be jagged and they must meet at common corners.

Let us define some grids. Let $\delta_j \in \mathbb{Z}^n$ be such that $(\delta_j)_j = 1$ and $(\delta_j)_i = 0$ if $i \neq j$. Given $m_0, \ldots, m_{n-1} \in \mathbb{Z}_+$, an *$m_0, \ldots, m_{n-1}$-rectangle*, $\mathfrak{R}_{m_0,\ldots,m_{n-1}}$ is a τ-structure with the universe $\mathrm{Dom}(\mathfrak{R}_{m_0,\ldots,m_{n-1}}) = m_0 \times \cdots \times m_{n-1}$ and

$$E_j^{\mathfrak{R}_{m_0,\ldots,m_{n-1}}} = \{(\overline{i}, \overline{i} + \delta_j) \mid \overline{i} \in m_0 \times \cdots \times m_{j-1} \times (m_j - 1) \times m_{j+1} \times \cdots \times m_{n-1}\}.$$

Given a subgroup N of $\mathbb{Z}^n$, *an N-torus*, $\mathfrak{T}_N$ is a τ-structure with the universe $\mathrm{Dom}(\mathfrak{T}_N) = \mathbb{Z}^n/N$ and

$$E_j^{\mathfrak{T}_N} = \{(\overline{i} + N, \overline{i} + \delta_j + N) \mid \overline{i} \in \mathbb{Z}^n\}.$$

All these structures are n-dimensional grids.

Given n_0- and n_1-dimensional grids $\mathfrak{A}_0$ and $\mathfrak{A}_1$ we define a new n_0+n_1-dimensional grid $\mathfrak{A}_0 \otimes \mathfrak{A}_1$, which we call *the product* of $\mathfrak{A}_0$ and $\mathfrak{A}_1$. We put $\mathrm{Dom}(\mathfrak{A}_0 \otimes \mathfrak{A}_1) = \mathrm{Dom}(\mathfrak{A}_0) \times \mathrm{Dom}(\mathfrak{A}_1)$, for all $0 \leq j < n_0$, let

$$E_j^{\mathfrak{A}_0 \otimes \mathfrak{A}_1} = \{((a, b), (a', b)) \mid (a, a') \in E_j^{\mathfrak{A}_0}, b \in \mathrm{Dom}(\mathfrak{A}_1)\}$$

and for all $0 \leq j < n_1$, let

$$E_{n_0+j}^{\mathfrak{A}_0 \otimes \mathfrak{A}_1} = \{((a, b), (a, b')) \mid a \in \mathrm{Dom}(\mathfrak{A}_0), (b, b') \in E_j^{\mathfrak{A}_1}\}.$$

If $\mathfrak{A}$ is an n-dimensional grid and σ is a permutation of n, we define a new n-dimensional grid $\mathfrak{A}^\sigma$ such that $\mathrm{Dom}(\mathfrak{A}^\sigma) = \mathrm{Dom}(\mathfrak{A})$ and for all $0 \leq j < n$, $E_j^{\mathfrak{A}^\sigma} = E_{\sigma(j)}^{\mathfrak{A}}$.

We can now give a complete characterization of n-dimensional grids in terms of the grids and the operations we have defined so far.

Proposition 5.11. *Every finite n-dimensional grid is isomorphic to a disjoint union of the grids $(\mathfrak{T}_N \otimes \mathfrak{R}_{m_0,\dots,m_{n_1-1}})^\sigma$, where N is a subgroup of $\mathbb{Z}^{n_0}$ and $n = n_0 + n_1$.*

Proof. Suppose $\mathfrak{A}$ is a finite connected n-dimensional grid and $a \in \mathrm{Dom}(\mathfrak{A})$. Define a partial function $g_a^{\mathfrak{A}}\colon \mathbb{Z}^n \to \mathrm{Dom}(\mathfrak{A})$ such that

$$g_a^{\mathfrak{A}}(\bar{i}) = ((f_0^{\mathfrak{A}})^{i_0} \circ \cdots \circ (f_{n-1}^{\mathfrak{A}})^{i_{n-1}})(a).$$

Because the functions $f_0^{\mathfrak{A}}, \dots, f_{n-1}^{\mathfrak{A}}$ commute, for all $0 \leq j < n$, $f_j^{\mathfrak{A}}(g_a^{\mathfrak{A}}(\bar{i})) = g_a^{\mathfrak{A}}(\bar{i} + \delta_j)$ and $(f_j^{\mathfrak{A}})^{-1}(g_a^{\mathfrak{A}}(\bar{i})) = g_a^{\mathfrak{A}}(\bar{i} - \delta_j)$. This and the connectedness of $\mathfrak{A}$ implies $\mathrm{rng}(g_a^{\mathfrak{A}}) = \mathrm{Dom}(\mathfrak{A})$.

If $g_a^{\mathfrak{A}}(\bar{i} + \epsilon_j \delta_j)$ is defined for all $0 \leq j < n$, where $\epsilon_j \in \{-1, 0, 1\}$, by the condition (c) of grids, $g_a^{\mathfrak{A}}(\bar{i} + (\epsilon_0, \dots, \epsilon_{n-1}))$ is defined. This implies that $\mathrm{dom}(g_a^{\mathfrak{A}}) = I_0 \times \cdots \times I_{n-1}$, where every I_j is a possible infinite interval of integers.

Suppose $\min I_j$ exists. If $c = g_a^{\mathfrak{A}}(\bar{i})$ is defined, then $(f_j^{\mathfrak{A}})^{\min I_j - i_j}(c)$ is defined, but $(f_j^{\mathfrak{A}})^{\min I_j - i_j - 1}(c)$ is not. Therefore $g_a^{\mathfrak{A}}(\bar{i}) = g_a^{\mathfrak{A}}(\bar{i}')$ implies $i_j = i'_j$. Now, if I_j were infinite, then the structure $\mathfrak{A}$ would be infinite contrary to our assumption. The same holds, if $\max I_j$ exists. We conclude that every interval I_j is either finite or $\mathbb{Z}$.

We may assume, if necessary by permuting the relations of $\mathfrak{A}$, that there exists $0 \leq n_0 \leq n$ such that for all $0 \leq j < n_0$, $I_j = \mathbb{Z}$ and for all $n_0 \leq j < n$, I_j is finite. Let $n_1 = n - n_0$. For all $j < n_1$, let $m_j = I_{n_0+j}$.

Define $N \subseteq \mathbb{Z}^{n_0}$ as

$$N = \{\bar{i} \in \mathbb{Z}^{n_0} \mid g_a^{\mathfrak{A}}(\overline{i0}) = a\},$$

where $\bar{0}$ is a sequence of $n - n_0$ zeroes and $\overline{i0}$ denotes the concatenation of the sequences. If $\bar{i}, \bar{i}' \in N$, then by commutativity of the functions f_j, $g_a^{\mathfrak{A}}(\overline{i0} + \bar{i}'\bar{0}) = g^{\mathfrak{A}}_{g_a^{\mathfrak{A}}(\overline{i0})}(\bar{i}'\bar{0}) = g_a^{\mathfrak{A}}(\bar{i}'\bar{0}) = a$ and $g_a^{\mathfrak{A}}(-\overline{i0}) = g^{\mathfrak{A}}_{g_a^{\mathfrak{A}}(\overline{i0})}(-\overline{i0}) = g_a^{\mathfrak{A}}(\overline{i0} - \overline{i0}) = a$. Thus $\bar{i} + \bar{i}', -\bar{i} \in N$ and N is a subgroup of $\mathbb{Z}^{n_0}$.

Suppose $g_a^{\mathfrak{A}}(\overline{ij}) = g_a^{\mathfrak{A}}(\bar{i}'\bar{j}')$. We have already seen that $\bar{j} = \bar{j}'$. Now, $g_a^{\mathfrak{A}}(\overline{ij} - \bar{i}'\bar{j}') = g^{\mathfrak{A}}_{g_a^{\mathfrak{A}}(\overline{ij})}(-\bar{i}'\bar{j}') = g^{\mathfrak{A}}_{g_a^{\mathfrak{A}}(\bar{i}'\bar{j}')}(-\bar{i}'\bar{j}') = g_a^{\mathfrak{A}}(\bar{i}'\bar{j}' - \bar{i}'\bar{j}') = a$, i.e, $\bar{i} - \bar{i}' \in N$. On the other hand, if $\bar{i} - \bar{i}' \in N$, $g_a^{\mathfrak{A}}(\overline{ij}) = g^{\mathfrak{A}}_{g_a^{\mathfrak{A}}(\bar{i}'\bar{0} - \overline{i0})}(\overline{ij}) = g_a^{\mathfrak{A}}(\bar{i}'\bar{0} - \overline{i0} + \overline{ij}) = g_a^{\mathfrak{A}}(\bar{i}'\bar{j})$. Thus $\alpha\colon (\mathbb{Z}/N) \times m_0 \times \cdots m_{n_1-1} \to \mathrm{Dom}(\mathfrak{A})$, $(\bar{i} + N, j_0, \dots j_{n-1}) \mapsto g_a^{\mathfrak{A}}(\bar{i}, j_0 + \min I_{n_0}, \dots, j_{n_1-1} + \min I_{n-1})$ is a well-defined bijection.

It is now quite easy to see, that α is in fact an isomorphism $\mathfrak{T}_N \otimes \mathfrak{R}_{m_0,\dots,m_{n_1-1}} \cong \mathfrak{A}$. Because we now have a characterization of connected n-dimensional grids the proposition follows. $\square$

We add now some additional structure into the grids. Let $\tau = \{E_i \mid i < n\} \cup \{U_{i,j} \mid 0 \leq i < j < n\}$, where each $U_{i,j}$ is a unary relation symbol. We call a τ-structure $\mathfrak{A}$ *a rigidified n-dimensional grid*, if $\mathfrak{A} \restriction \{E_0, \ldots, E_{n-1}\}$ is an n-dimensional grid and the following conditions hold for all $0 \leq i < j < n$ and $\epsilon \in \{-1, 1\}$:

a) If $a \in U^{\mathfrak{A}}_{i,j}$ and $b = (f^{\mathfrak{A}}_i \circ f^{\mathfrak{A}}_j)^\epsilon(a)$ is defined, $b \in U^{\mathfrak{A}}_{i,j}$.
b) If $a \in U^{\mathfrak{A}}_{i,j}$, then $(f^{\mathfrak{A}}_i)^\epsilon(a)$ is defined if and only if $(f^{\mathfrak{A}}_j)^\epsilon(a)$ is defined.
c) If $(f^{\mathfrak{A}}_i)^\epsilon(a)$ and $(f^{\mathfrak{A}}_j)^\epsilon(a)$ are undefined, $a \in U^{\mathfrak{A}}_{i,j}$.

Proposition 5.12. *The class of all finite rigidified n-dimensional grids is first-order definable and closed under $\leftrightarrows_n$.* □

We omit the proof, because first-order definability is easy and already implies that the class is closed under $\leftrightarrows_r$ for some r. This is all we need.

Proposition 5.13. *Let $\mathfrak{A}$ be a connected rigidified n-dimensional grid and suppose $\mathfrak{A} \restriction \{E_0, \ldots, E_{n-1}\} = \mathfrak{T}_N \otimes \mathfrak{R}_{m_0,\ldots,m_{n_1-1}}$, where N is a subgroup of $\mathbb{Z}^{n_0}$.*
1) For all $0 \leq i < j < n_1$, $m_i = m_j$ and

$$U^{\mathfrak{A}}_{n_0+i,n_0+j} = \{(a, \overline{k}) \in \mathrm{Dom}(\mathfrak{T}_N) \times \mathrm{Dom}(\mathfrak{R}_{m_0,\ldots,m_{n_1-1}}) \mid k_i = k_j\}.$$

2) For all $0 \leq i < j < n$, if $U_{i,j} \neq \emptyset$, then $0 \leq i < j < n_0$ or $n_0 \leq i < j < n$.

Proof. Let $0 \leq i < j < n_1$ and suppose $(a, \overline{k}) \in \mathrm{Dom}(\mathfrak{T}_N) \times \mathrm{Dom}(\mathfrak{R}_{m_0,\ldots,m_{n_1-1}})$. If $k_i = k_j$, consider an element $b = (a, \overline{k} - k_i(\delta_i + \delta_j))$. This is in $U^{\mathfrak{A}}_{n_0+i,n_0+j}$ by the condition (c), because both $(f^{\mathfrak{A}}_i)^{-1}(b)$ and $(f^{\mathfrak{A}}_j)^{-1}(b)$ are undefined. Applying the condition (a) inductively, we conclude that $(a, \overline{k} - (k_i - s)(\delta_i + \delta_j)) \in U^{\mathfrak{A}}_{n_0+i,n_0+j}$ for all $0 \leq s \leq k_i$. Therefore $b \in U^{\mathfrak{A}}_{n_0+i,n_0+j}$. If $k_i \neq k_j$, we first notice that $(a, \overline{k} - \min\{k_i, k_j\}(\delta_i + \delta_j)) \notin U^{\mathfrak{A}}_{n_0+i,n_0+j}$ by the condition (b) and by induction, $(a, \overline{k})$ is not in $U^{\mathfrak{A}}_{n_0+i,n_0+j}$. Thus

$$U^{\mathfrak{A}}_{n_0+i,n_0+j} = \{(a, \overline{k}) \in \mathrm{Dom}(\mathfrak{T}_N) \times \mathrm{Dom}(\mathfrak{R}_{m_0,\ldots,m_{n_1-1}}) \mid k_i = k_j\}.$$

By the condition (c), every element $(a, \overline{k})$ such that $k_i = m_i - 1$ and $k_j = m_j - 1$ has to be in $U^{\mathfrak{A}}_{n_0+i,n_0+j}$. Hence $m_i = m_j$. We have now shown the first part of the proposition.

Suppose $0 \leq i < n_0 \leq j < n$ and $j' = j - n_0$. The second part of the proposition is equivalent to the claim $U^{\mathfrak{A}}_{i,j} = \emptyset$. Suppose $(a, \overline{k}) \in U^{\mathfrak{A}}_{i,j}$. By the condition (a), $b = (a - k_{j'}\delta_i, \overline{k} - k_{j'}\delta_{j'}) \in U^{\mathfrak{A}}_{i,j}$. However, this is a contradiction, because $(f^{\mathfrak{A}}_i)^{-1}(b)$ is defined, but $(f^{\mathfrak{A}}_j)^{-1}(b)$ is not. □

5.4. **Gaifman locality on grids.** Denote the interval $\{x \in \mathbb{R} \mid a \leq x \leq b\}$ by $[a, b]$ in this section. Let $F^n_{i,0} = \{\overline{x} \in [0, 1]^n \mid x_i = 0\}$ and $F^n_{i,1} = \{\overline{x} \in [0, 1]^n \mid x_i = 1\}$.

Theorem 5.14. (Lebesgue's covering theorem) *Suppose A is a finite family of closed sets covering the unit cube, $[0, 1]^n = \bigcup A$, and none of the sets intersects two opposite sides of the cube, i.e, for all $X \in A$ and $0 \leq i < n$, either $X \cap F^n_{i,0} = \emptyset$ or $X \cap F^n_{i,1} = \emptyset$. Then there exists $A' \subseteq A$, $|A'| = n + 1$ such that $\bigcap A' \neq \emptyset$.*

Proof. The theorem was conjectured by Lebesgue and first proven by Brouwer [Bro24]. The proof in English can be found for example in [HW41]. $\square$

Let $M = m_0 \times \cdots \times m_{n-1}$, where for all $i < n$, $m_i \geq 1$. Given $\overline{a}, \overline{b} \in M$, let $d(\overline{a}, \overline{b}) = \max\{|b_0 - a_0|, \ldots, |b_{n-1} - a_{n-1}|\}$. We say that $X_0, X_1 \subseteq M$ are *separated*, if for all $a \in X_0$ and $b \in X_1$, $d(a, b) > 1$. The set $X \subseteq M$ is *connected*, if there does not exist a partition $X_0 \cup X_1 = X$ such that X_0 and X_1 are separated. $X \subseteq Y$ is *a component* of Y, if X is connected and X and $Y \setminus X$ are separated.

Let $S_{i,0} = \{\overline{a} \in M \mid a_i = 0\}$ and $S_{i,1} = \{\overline{a} \in M \mid a_i = m_i - 1\}$.

Lemma 5.15. *Suppose $X_0 \cup \cdots \cup X_{n-1} = M$. Then for some $i, j \in n$, X_i has a component C such that $C \cap S_{j,0} \neq \emptyset$ and $C \cap S_{j,1} \neq \emptyset$.*

Proof. Let B_i be the family of all components of X_i and let $B = \bigcup_{i<n} B_i$. Suppose that the lemma were not true, i.e., for all $C \in B$ and $j < n$, either $C \cap S_{j,0} = \emptyset$ or $C \cap S_{j,1} = \emptyset$.

For all $\overline{a} \in M$, let $c(\overline{a}) = [a_0, a_0 + 1] \times \cdots \times [a_{n-1}, a_{n-1} + 1]$ and for all $X \subseteq M$, $c(X) = \bigcup_{\overline{a} \in X} c(\overline{a})$. Because finite unions of closed sets are closed, $c(X)$ is closed for all $X \subseteq M$.

Let $A = \{c(C) \mid C \in B\}$. Then $\bigcup A = [0, m_0 + 1] \times \cdots \times [0, m_{n-1} + 1]$ and because for all $C \in B$ and $j < n$, $C \cap S_{j,0} = \emptyset$ or $C \cap S_{j,1} = \emptyset$, none of the sets in A intersects two opposite faces of the cuboid $\bigcup A$. By the Lebesgue's covering theorem, for some $A' \subseteq A$, $|A'| = n + 1$, $\bigcap A' \neq \emptyset$.

Let $B' = \{C \in B \mid c(C) \in A'\}$. The pigeonhole principle and $|B'| = n + 1$ implies that for some $i < n$, $|B_i \cap B'| \geq 2$. Let $C, C' \in B_i \cap B'$, where $C \neq C'$. Because $c(C) \cap c(C') \neq \emptyset$, there exist $\overline{a} \in C$ and $\overline{b} \in C'$ such that $c(\overline{a}) \cap c(\overline{b}) \neq \emptyset$. But then $d(\overline{a}, \overline{b}) \leq 1$, and C and C' are not separated. This is a contradiction, because C and C' were supposed to be different components of X_i. $\square$

Remark. Lemma 5.15 is in the case $n = 2$ closely related to the board game named Hex, where two players try to connect opposite sides of the board by their pieces. Because one color connects the sides on every completely colored board, the game cannot end in a tie. This result is credited to John Nash [Wik07].

Let Q be a query containing all rigidified n-dimensional grids $\mathfrak{A}$ that are isomorphic to $\mathfrak{R}_{m,\ldots,m}$ for some $m \in \mathbb{Z}_+$. We want to show that $\mathrm{FO}(\mathrm{Q}) \leq \mathrm{HL}(\leftrightsquigarrow_{n,r})_{r \in \mathbb{N}}$.

Let $\tau' = \tau \cup \{D_{\overline{\epsilon}}, F_{\overline{\epsilon}}, V_{i,j} \mid \overline{\epsilon} \in \{-1, 0, 1\}^n, 0 \leq i < j < n\}$, where $D_{\overline{\epsilon}}$ and $V_{i,j}$ are unary relation symbols and $F_{\overline{\epsilon}}$ is a binary relation symbol. Let $\theta \in I(\mathrm{FO}, \tau, \tau')$ be such that $\theta^*(\mathfrak{A})$ expands $\mathfrak{A}$, $(a, b) \in F_{\overline{\epsilon}}^{\theta^*(\mathfrak{A})}$ if and only if $b = h_{\overline{\epsilon}}^{\mathfrak{A}}(a) = ((f_0^{\mathfrak{A}})^{\epsilon_0} \circ \cdots \circ (f_{n-1}^{\mathfrak{A}})^{\epsilon_{n-1}})(a)$, $D_{\overline{\epsilon}}^{\theta^*(\mathfrak{A})} = \mathrm{dom}(h_{\overline{\epsilon}}^{\mathfrak{A}})$ and $V_{i,j}^{\theta^*(\mathfrak{A})} = N_1^{\mathfrak{A}}(U_{i,j}^{\mathfrak{A}})$. Let $\mathrm{Q}' = \{\theta^*(\mathfrak{A}) \mid \mathfrak{A} \in \mathrm{Q}\}$. Clearly $\mathrm{FO}(\mathrm{Q}) \equiv \mathrm{FO}(\mathrm{Q}')$.

Lemma 5.16. *Let $\psi \in I(\mathrm{QF}_\infty, \tau'', \tau')$. There is $r \in \mathbb{Z}_+$ such that if $\psi^*(\mathfrak{A}) \in \mathrm{Q}'$ and and $\mathfrak{A} \leftrightsquigarrow_{n,r} \mathfrak{B}$, then $\psi^*(\mathfrak{A}) \cong \psi^*(\mathfrak{B})$.*

Proof. Let φ be a first-order sentence axiomatizing the class of all rigidified n-dimensional grids expanded with θ^*. The sentence $\varphi \circ \psi$ is Hanf-local, let r be its locality rank or $n+1$ depending on which is bigger.

Assume that $\psi^*(\mathfrak{A}) \in \mathrm{Q}'$ and $\mathfrak{A} \leftrightsquigarrow_{n,r} \mathfrak{B}$. We may assume also without loss of generality that $\mathrm{Dom}(\mathfrak{A}) = m \times \cdots \times m$, for some $m \in \mathbb{Z}_+$. Because $\mathfrak{A} \models \varphi \circ \psi$, also $\mathfrak{B} \models \varphi \circ \psi$ and so $\psi^*(\mathfrak{B})$ is a rigidified n-dimensional grid expanded with θ^*.

Let $\mathrm{Dom}(\mathfrak{A}) = A_0 \cup \cdots \cup A_{n-1}$ and $\mathrm{Dom}(\mathfrak{B}) = B_0 \cup \cdots \cup B_{n-1}$ be partitions witnessing $\mathfrak{A} \leftrightsquigarrow_{n,r} \mathfrak{B}$ and for all $i < n$, let $\alpha_i \colon \mathfrak{N}_n^{\mathfrak{A}}(A_i) \cong \mathfrak{N}_n^{\mathfrak{B}}(B_i)$. Because $\mathrm{Dom}(\mathfrak{A}) = m \times \cdots \times m$, Lemma 5.15 gives us a component $C \subseteq A_i$ such that for some $j < n$, C contains elements $\overline{c}$ and $\overline{c}'$ such that $c_j = 0$ and $c'_j = m-1$.

For all $\overline{a} \in C^{<\omega}$, $\mathrm{atp}^{\mathfrak{A}}(\overline{a}) = \mathrm{atp}^{\mathfrak{B}}(\alpha_i(\overline{a}))$ and so for all relation $R \in \tau'$, $\alpha_i(R^{\psi^*(\mathfrak{A})} \cap C^{\mathrm{ar}(R)}) = R^{\psi^*(\mathfrak{B})} \cap \alpha_i(C)^{\mathrm{ar}(R)}$. Because we have added the relations $F_{\overline{\epsilon}}$ into the structures $\psi^*(\mathfrak{A})$ and $\psi^*(\mathfrak{B})$, this implies that $\alpha_i(C)$ is connected in $\psi^*(\mathfrak{B})$.

Choose $j' \in n \setminus \{j\}$ arbitrarily. Define a function $h \colon m \times \cdots \times m \to \mathbb{Z}$, $\overline{a} \mapsto a_{j'} - a_j$. We have $h(\overline{c}) = c_{j'} - c_j = c_{j'} \geq 0$ and $h(\overline{c}') = c'_{j'} - c'_j = m - 1 - c'_j \leq 0$. If $d(\overline{a}, \overline{b}) \leq 1$, then $|h(\overline{a}) - h(\overline{b})| \leq 2$. Therefore the connectivity of C implies that there exists $\overline{c}'' \in C$ such that $|h(\overline{c}'')| \leq 1$. Because $U_{j,j'}^{\psi^*(\mathfrak{A})} = h^{-1}\{0\}$ and $V_{j,j'}^{\psi^*(\mathfrak{A})} = h^{-1}\{-1,0,1\}$, $V_{j,j'}^{\psi^*(\mathfrak{A})} \cap C \neq \emptyset$. Hence $V_{j,j'}^{\psi^*(\mathfrak{B})} \cap \alpha_i(C) \neq \emptyset$ and so $U_{j,j'}^{\psi^*(\mathfrak{B})} \cap N_1^{\psi^*(\mathfrak{B})}(\alpha_i(C)) \neq \emptyset$. for all $j' \in n \setminus \{j\}$.

Let D be a component of $\mathfrak{B}$ such that $\alpha_i(C) \subseteq D$. By Proposition 5.11, $\langle D \rangle^{\mathfrak{B}} \restriction \{E_0, \ldots, E_{n-1}\} \cong (\mathfrak{T}_N \otimes \mathfrak{R}_{m_0,\ldots,m_{n_1-1}})^{\sigma}$, where $N \subseteq \mathbb{N}^{n_0}$. Without loss of generality, we may assume $\sigma = \mathrm{id}_n$. Because $(f_j^{\mathfrak{A}})^{-1}(\overline{c})$ is not defined and the unary relations $D_{\overline{\epsilon}}$ encode the domains of the functions $f_{j'}^{\mathfrak{A}}$, also $(f_j^{\mathfrak{B}})^{-1}(\alpha_i(\overline{c}))$ is undefined. This means $n_0 \leq j < n$. Because $U_{j,j'}^{\langle D \rangle^{\mathfrak{B}}}$ is non-empty for all j', $j' \neq j$, by Proposition 5.13, $n_0 = 0$ and $m_0 = \cdots = m_{n-1} = m'$. We have now $\langle D \rangle^{\mathfrak{B}} \restriction \{E_0, \ldots, E_{n-1}\} = \mathfrak{R}_{m',\ldots,m'}$.

Now, consider a function $u \colon C \to m'$, $u(\overline{a}) = (\alpha_i(\overline{a}))_j$. Let $C' = \{\overline{a} \in C \mid u(\overline{a}) = a_j\}$. Because $u(\overline{c}) = 0 = c_j$, $\overline{c} \in C'$. If $\overline{a}, \overline{b} \in C$ and $d^{\mathfrak{A}}(\overline{a}, \overline{b}) = 1$, then $\overline{a} \in C' \iff \overline{b} \in C'$. Because C is connected, $C' = C$. In particular $u(\overline{c}') = m - 1$, which means $m' = m$. Because $\mathfrak{A}$ and $\mathfrak{B}$ have the same number of elements, they have to be isomorphic. □

Lemma 5.16 and the condition (1) of Theorem 3.16 implies the following theorem.

Theorem 5.17. $\mathrm{FO(Q)} \leq \mathrm{HL}(\leftrightsquigarrow_{n,r})_{r \in \mathbb{N}}$.

5.5. **Non-locality of Q on grids.** Fix $n, r \in \mathbb{Z}_+$ and $d \geq (2n+2)r + 2$. For all $S \subseteq n$ and $m \in \mathbb{Z}_+$, let $\mathfrak{A}_{S,m}$ be a rigidified n-dimensional grid with $\mathfrak{A}_{S,m} \restriction \{E_0, \ldots, E_{n-1}\} = \mathfrak{A}_{S,m}^0 \otimes \cdots \otimes \mathfrak{A}_{S,m}^{n-1}$, where $\mathfrak{A}_{S,m}^i = \mathfrak{R}_m$, if $i \in S$ and $\mathfrak{A}_{S,m}^i = \mathfrak{T}_{d\mathbb{Z}}$, if $i \notin S$. If $i, j \in S$ or $i, j \in n \setminus S$, then $U_{i,j}^{\mathfrak{A}_{S,m}} = \{\overline{a} \in \mathrm{Dom}(\mathfrak{A}_{S,m}) \mid a_i = a_j\}$. Otherwise $U_{i,j}^{\mathfrak{A}_{S,m}} = \emptyset$. Note that $\mathfrak{A}_{n,m} \cong \mathfrak{R}_{m,\ldots,m}$.

For all $m \in \mathbb{Z}_+$, let $\mathfrak{B}_m = \bigsqcup_{S \subseteq n} \mathfrak{A}_{S,m}$. Our goal in this section is to prove the following lemma:

Lemma 5.18. *For all* $k \geq 2$, $\mathfrak{R}_{(k+1)d+1,\dots,(k+1)d+1} \cong \mathfrak{A}_{n,(k+1)d+1} \leftrightsquigarrow_{n+1,r} \mathfrak{B}_{kd+1}$.

We need to define some partition witnessing the lemma. Fix $S \subseteq n$ and $k \in \mathbb{Z}_+$. Let

$$I^{S,k} = \{(T,a) \mid T \subseteq n, a \in (k+1)^S, \text{for all } i \in S \setminus T,\ a(i) < k\}.$$

For all $(T,a) \in I^{S,k}$, let $X'^{S,k}_{T,a} = {}_0X'^{S,k}_{T,a} \times \cdots \times {}_{n-1}X'^{S,k}_{T,a}$, where

$${}_iX'^{S,k}_{T,a} = \begin{cases} [a(i)d - |T|r, a(i)d + |T|r] \cap [0,kd] & \text{if } i \in S \cap T \\ [a(i)d + (|T|+1)r + 1, & \\ \qquad (a(i)+1)d - (|T|+1)r - 1] \cap [0,kd] & \text{if } i \in S \setminus T \\ [-|T|r, |T|r] + d\mathbb{Z} & \text{if } i \in T \setminus S \\ [(|T|+1)r + 1, d - (|T|+1)r - 1] + d\mathbb{Z} & \text{if } i \notin T \cup S. \end{cases}$$

The set $X'^{S,k}_{T,a}$ is a subset of $\mathrm{Dom}(\mathfrak{A}_{S,kd+1})$. Define,

$$X^{S,k}_{T,a} = X'^{S,k}_{T,a} \setminus \bigcup\{X'^{S,k}_{T',a'} \mid (T',a') \in I^{S,k}, |T'| > |T|\}.$$

We shall show that the sets $X^{S,k}_{T,a}$ form a partition of $\mathrm{Dom}(\mathfrak{A}_{S,kd+1})$.

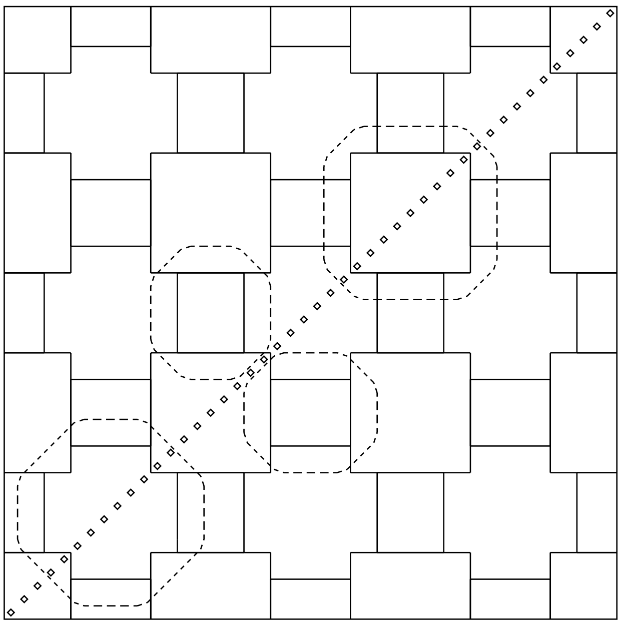


FIGURE 1. The structure $\mathfrak{A}_{S,kd+1}$, with $n = 2$, $r = 2$, $d = 9$ and $k = 3$. The partition $(X^{S,k}_{T,a})$ and r-neighborhoods of some parts are shown.

Lemma 5.19. $\bigcup_{(T,a) \in I^{S,k}} X'^{S,k}_{T,a} = \mathrm{Dom}(\mathfrak{A}_{S,kd+1})$.

Proof. Consider an arbitrary $\overline{b} \in \mathrm{Dom}(\mathfrak{A}_{S,kd+1})$. Define $\overline{h} \in \mathbb{N}^n$ as follows: If $i \in S$, let $c = \lfloor b_i/d + 1/2 \rfloor$ and $h_i = |b_i - dc|$. Then for all $c' \in \mathbb{Z}$, $|b_i - dc'| \leq h_i$. If $i \notin S$, then $b_i \in \mathbb{Z}/d\mathbb{Z}$. Let h_i be the unique element $0 \leq h_i \leq \lfloor d/2 \rfloor$ such that $b_i = h_i + d\mathbb{Z}$ or $b_i = -h_i + d\mathbb{Z}$.

We may assume without loss of generality that for all $i < n-1$, $h_i \leq h_{i+1}$. Let $T = \min\{i < n \mid h_i > (i+1)r\} \cup \{n\}$. Then, for all $i \in T$, $h_i \leq (i+1)r \leq |T|r$ and for all $i \in n \setminus T$, $h_i > (|T|+1)r$.

Now, choose $a \in \mathbb{Z}^S$ so that $a(i) = \lfloor b_i/d + 1/2 \rfloor$, if $i \in S \cap T$, and $a(i) = \lfloor b_i/d \rfloor$, if $i \in S \setminus T$. Then $i \in S \cap T$ implies $0 \leq a(i) \leq k$ and $i \in S \setminus T$ implies $0 \leq a(i) < k$, and so $(T, a) \in I^{S,k}$.

If $i \in S \cap T$, then $|b_i - da(i)| = h_i \leq |T|r$, if $i \in S \setminus T$, then $da(i) + (|T|+1)r < b_i < d(a(i)+1) - (|T|+1)r$, if $i \in T \setminus S$, then $b_i \in [-|T|r, |T|r] + d\mathbb{Z}$ and if $i \notin S \cup T$, then $b_i \in [(|T|+1)r + 1, d - (|T|+1)r - 1] + d\mathbb{Z}$. This shows $b_i \in {}_iX'^{S,k}_{T,a}$ and so $\overline{b} \in X'^{S,k}_{T,a}$. ☐

The lemma implies that also $\bigcup_{(T,a) \in I^{S,k}} X^{S,k}_{T,a}$ covers $\mathrm{Dom}(\mathfrak{A}_{S,kd+1})$. This is because for all $\overline{b} \in \mathrm{Dom}(\mathfrak{A}_{S,kd+1})$, if $(T, a) \in I^{S,k}$ is an element with $\overline{b} \in X'^{S,k}_{T,a}$ maximizing $|T|$, then $\overline{b} \in X^{S,k}_{T,a}$. We show next that the sets $X^{S,k}_{T,a}$ are disjoint. For that and some following considerations we prove:

Lemma 5.20. *If* $(T, a), (T', a') \in I^{S,k}$, $|T| = |T'|$ *and* $(T, a) \neq (T', a')$, *then*

$$d^{\mathfrak{A}_{S,kd+1}}(X'^{S,k}_{T,a}, X'^{S,k}_{T',a'}) \geq 2r + 2.$$

Proof. Let $\overline{b} \in X'^{S,k}_{T,a}$ and $\overline{b}' \in X'^{S,k}_{T,a}$ be arbitrary elements of the sets. We show that $d^{\mathfrak{A}_{S,kd+1}}(\overline{b}, \overline{b}') \geq 2r + 2$.

Consider first the case $T = T'$. Then we have necessarily $a \neq a'$. Suppose first, there exists $i \in T \cap S$ such that $a(i) < a'(i)$. Then $b_i \leq a(i)d + |T|r$ and $b'_i \geq a'(i)d - |T'|r$. Thus $b'_i - b_i \geq a'(i)d - |T|r - a(i)d - |T'|r \geq d - 2nr \geq 2r + 2$ and the claim holds.

Suppose then that for some $i \in S \setminus T$, $a(i) < a'(i)$. In this case, $b_i \leq (a(i)+1)d - (|T|+1)r - 1$ and $b'_i \geq a'(i)d + (|T'|+1)r + 1$. We have now $b'_i - b_i \geq a'(i)d + (|T'|+1)r + 1 - (a(i)+1)d + (|T|+1)r + 1 \geq (a'(i) - a(i) - 1)d + (|T| + |T'| + 2)r + 2 \geq 2r + 2$, again implying the lemma.

Assume next that $T \neq T'$. Since $|T| = |T'|$, there exists $i \in T \setminus T'$ and $j \in T' \setminus T$. If $i \notin S$, then $b_i \in [-|T|r, |T|r] + d\mathbb{Z}$ and $b'_i \in (\mathbb{Z} \setminus [-(|T|+1)r, (|T|+1)r]) + d\mathbb{Z}$. If $i \in S$, the same holds for $b_i + d\mathbb{Z}$ and $b'_i + d\mathbb{Z}$ and we therefore have $|b'_i - b_i| \geq r + 1$. The same holds also for b_j and b'_j, thus $d^{\mathfrak{A}_{S,kd+1}}(\overline{b}, \overline{b}') \geq 2r + 2$. ☐

Lemma 5.20 shows in particular, that if $(T, a) \neq (T', a')$, but $|T| = |T'|$, then $X'^{S,k}_{T,a} \cap X'^{S,k}_{T',a'} = \emptyset$. This implies that the sets $X^{S,k}_{T,a}$ form a partition of $\mathrm{Dom}(\mathfrak{A}_{S,kd+1})$.

For all $(T,a) \in I^{S,k}$, let $\beta^{S,k}_{T,a}$ be a function $T \to \{-1,0,1\}$

$$\beta^{S,k}_{T,a}(i) = \begin{cases} -1 & \text{if } i \in T \cap S \text{ and } a(i) = 0 \\ 0 & \text{if } i \in T \setminus S \text{ or } 0 < a(i) < k \\ 1 & \text{if } i \in T \cap S \text{ and } a(i) = k. \end{cases}$$

Let $\gamma^{S,k}_{T,a} \subseteq n^2$ be such that

$$\begin{aligned} (i,j) \in \gamma^{S,k}_{T,a} \quad \iff \quad & (i \in S \leftrightarrow j \in S) \wedge \\ & (i \in T \leftrightarrow j \in T) \wedge (i,j \in S \to a(i) = a(j)) \end{aligned}$$

Note that $\gamma^{S,k}_{T,a}$ is always an equivalence relation. Let $\mathrm{Inv}^{S,k}_{T,a} = (T, \beta^{S,k}_{T,a}, \gamma^{S,k}_{T,a})$.

Lemma 5.21. *If* $\mathrm{Inv}^{S,k}_{T,a} = \mathrm{Inv}^{S',k'}_{T',a'}$, *there exists an isomorphism* $\beta\colon \mathfrak{N}^{\mathfrak{A}_{S,k}}_r(X^{S,k}_{T,a}) \cong \mathfrak{N}^{\mathfrak{A}_{S',k'}}_r(X^{S',k'}_{T',a'})$. *Additionally, if* $x \in \mathrm{dom}(\beta) \cap X^{S,k}_{T'',b}$, *then for some* b', $\beta(x) \in X^{S',k'}_{T'',b'}$.

Proof. Given $k \in \mathbb{Z}_+$, $S \subseteq n$ and $a\colon S \to \mathbb{Z}$, let $\alpha^{S,k}_a\colon \mathbb{Z}^n \to \mathrm{Dom}(\mathfrak{A}_{S,k})$ be a partial function such that if $i \in S$ and $0 \le b_i + a(i)d \le kd$, then $(\alpha_{S,a}(\bar{b}))_i = b_i + a(i)d$ and if $i \notin S$, then $(\alpha^{S,k}_a(\bar{b}))_i = b_i + d\mathbb{Z}$.

Suppose $\mathrm{Inv}^{S,k}_{T,a} = \mathrm{Inv}^{S',k'}_{T',a'} = (T,\beta,\gamma)$. If $D \subseteq \mathbb{Z}^n$ is such that $\alpha^{S,k}_a \restriction D$ and $\alpha^{S',k'}_{a'} \restriction D$ are total injections on D, there exists a unique bijection $\beta_D\colon \alpha^{S,k}_a(D) \to \alpha^{S',k'}_{a'}(D)$ such that $\alpha^{S',k'}_{a'} = \beta_D \circ \alpha^{S,k}_a$.

Let $D' = D^0 \times \cdots \times D^{n-1}$, where

$$D^i = \begin{cases} [0, (|T|+1)r] & \text{if } i \in T \text{ and } \beta(i) = -1 \\ [-(|T|+1)r, (|T|+1)r] & \text{if } i \in T \text{ and } \beta(i) = 0 \\ [-(|T|+1)r, 0] & \text{if } i \in T \text{ and } \beta(i) = 1 \\ [|T|r+1, d-|T|r-1] & \text{if } i \notin T. \end{cases}$$

and

$$\begin{aligned} D \quad = \quad \Big\{ \bar{b} \in D' \mid & \sum_{i \in T} \max\{|b_i| - |T|r, 0\} + \\ & \sum_{i \in n \setminus T} \max\{0, (|T|+1)r + 1 - b_i, b_i - d + (|T|+1)r + 1\} \le r \Big\}. \end{aligned}$$

We check first that $D \subseteq \mathrm{dom}(\alpha^{S,k}_a)$. This means that for all $\bar{b} \in D$ and $i \in S$, we must show $0 \le b_i + a(i)d \le kd$. If $a(i) = 0$, then $\beta(i) = -1$ or $i \notin T$ and so for all $\bar{b} \in D$, $b_i \ge 0$. If $a(i) > 0$, we have $b_i \ge -(|T|+1)r \ge -(n+1)r \ge -d$ and so $b_i + a(i)d \ge 0$. The upper bound holds for similar reasons.

We show next that $\alpha^{S,k}_a \restriction D$ is an injection. Assume $\bar{b}, \bar{b}' \in D$ and $\alpha^{S,k}_a(\bar{b}) = \alpha^{S,k}_a(\bar{b}')$. If $i \in S$, then $b_i + a(i)d = b'_i + a(i)d$ implying $b_i = b'_i$. If $i \notin S$, then $b_i + d\mathbb{Z} = b'_i + d\mathbb{Z}$ implying $d | b'_i - b_i$. Because D^i is an interval of length at most $\max\{2(|T|+2)+1, d-2|T|r-2\} < d$, we must have $b_i = b'_i$.

These properties hold symmetrically for $\alpha_{a'}^{S',k'} \restriction D$ and so we get a bijection β_D. We show next that this bijection is an isomorphism $\langle\alpha_a^{S,k}(D)\rangle^{\mathfrak{A}_{S,k}} \cong \langle\alpha_{a'}^{S',k'}(D)\rangle^{\mathfrak{A}_{S',k'}}$. Remember that the vocabulary of $\mathfrak{A}_{S,k}$ is $\{E_i \mid i < n\} \cup \{U_{i,j} \mid 0 \le i < j < n\}$. For all $i < n$,

$$\begin{aligned}(\alpha_a^{S,k}(\bar{b}), \alpha_a^{S,k}(\bar{b}')) \in E_i^{\mathfrak{A}_{S,k}} \quad &\Longleftrightarrow \quad (\alpha_a^{S,k}(\bar{b}'))_i = (\alpha_a^{S,k}(\bar{b}))_i + 1 \\ &\qquad \text{and for all } j \neq i, (\alpha_a^{S,k}(\bar{b}'))_j = (\alpha_a^{S,k}(\bar{b}))_j \\ &\Longleftrightarrow \quad b'_i = b_i + 1 \text{ and for all } j \neq i, b'_j = b_j \\ &\Longleftrightarrow \quad (\alpha_{a'}^{S',k'}(\bar{b}), \alpha_{a'}^{S',k'}(\bar{b}')) \in E_i^{\mathfrak{A}_{S',k'}}.\end{aligned}$$

Let $0 \le i < j < n$. If $(i,j) \notin \gamma = \gamma_{T,a}^{S,k}$, there are three possibilities:
(1) $i \in S \not\leftrightarrow j \in S$,
(2) $i \in T \not\leftrightarrow j \in T$, or
(3) $i, j \in S$ and $a(i) \neq a(j)$.
We show that in all cases $U_{i,j}^{\mathfrak{A}_{S,k}} \cap \alpha_a^{S,k}(D) = \emptyset$. In the first case, $U_{i,j}^{\mathfrak{A}_{S,k}} = \emptyset$.

In the second case, we may assume by symmetry that $i \in T$ and $j \notin T$. Then for all $\bar{b} \in D$,

$$\max\{|b_i| - |T|r, 0\} + \max\{0, (|T|+1)r + 1 - b_j, b_j - d + (|T|+1)r + 1\} \le r.$$

In particular,

$$|b_i| - |T|r + (|T|+1)r + 1 - b_j = |b_i| - b_j + r + 1 \le r$$

and

$$|b_i| - |T|r + b_j - d + (|T|+1)r + 1 = |b_i| + b_j - d + r + 1 \le r.$$

Combining the inequalities, we get

$$|b_i| + 1 \le b_j \le d - 1 - |b_i|.$$

This implies

$$1 < |b_i| - b_i + 1 \le b_j - b_i \le d - 1 - |b_i| - b_i < d$$

and so $b_i + a(i)d \neq b_j + a(j)d$, i.e. $\alpha_a^{S,k}(\bar{b}) \notin U_{i,j}^{\mathfrak{A}_{S,k}}$.

In the third case, we may assume $i \in T \leftrightarrow j \in T$ and by symmetry $a(i) < a(j)$. If $i, j \in T$, then for all $\bar{b} \in D$,

$$b_i + a(i)d \le (|T|+1)r + a(i)d < -(|T|+1)r + a(j)d \le b_j + a(j)d.$$

If $i, j \notin T$, then

$$b_i + a(i)d \le d - |T|r - 1 + a(i)d < |T|r + 1 + a(j)d \le b_j + a(j)d.$$

In the similar way, we may show that $U_{i,j}^{\mathfrak{A}_{S',k'}} \cap \alpha_{a'}^{S',k'}(D) = \emptyset$ and so $U_{i,j}$ satisfies the isomorphism condition.

Assume then that $(i,j) \in \gamma$. Now $U_{i,j}^{\mathfrak{A}_{S,k}} \cap \alpha_a^{S,k}(D) \neq \emptyset$ and we have to show that β_D maps it to $U_{i,j}^{\mathfrak{A}_{S',k'}} \cap \alpha_{a'}^{S',k'}(D)$. If $i, j \in S$, then $a(i) = a(j)$ and $a'(i) = a'(j)$. This means $(\alpha_a^{S,k})^{-1}(U_{i,j}^{\mathfrak{A}_{S,k}}) = \{\bar{b} \in \mathrm{dom}(\alpha_a^{S,k}) \mid b_i = b_j\}$ and $(\alpha_{a'}^{S',k'})^{-1}(U_{i,j}^{\mathfrak{A}_{S',k'}}) = \{\bar{b} \in$

$\mathrm{dom}(\alpha_{a'}^{S',k'}) \mid b_i = b_j\}$, Because $D \subseteq \mathrm{dom}(\alpha_a^{S,k}) \cap \mathrm{dom}(\alpha_{a'}^{S',k'})$, this case is clear. If $i, j \notin S$, then $(\alpha_a^{S,k})^{-1}(U_{i,j}^{\mathfrak{A}_{S,k}}) = \{\bar{b} \in \mathrm{dom}(\alpha_a^{S,k}) \mid d|b_j - b_i\}$ and $(\alpha_{a'}^{S',k'})^{-1}(U_{i,j}^{\mathfrak{A}_{S',k'}}) = \{\bar{b} \in \mathrm{dom}(\alpha_{a'}^{S',k'}) \mid d|b_j - b_i\}$ and also this case is clear.

We have now constructed an isomorphism $\beta_D \colon \langle \alpha_a^{S,k}(D) \rangle^{\mathfrak{A}_{S,k}} \to \langle \alpha_{a'}^{S',k'}(D) \rangle^{\mathfrak{A}_{S',k'}}$. It maps elements of the set $X_{T'',b}^{S,k}$ to the set $X_{T'',b-a+a'}^{S',k'}$ and thus satisfies the additional condition of the lemma. Because $N^{\mathfrak{A}_{S,k}}(X_{T,a}^{S,k}) \subseteq \alpha_a^{S,k}(D)$ and $N^{\mathfrak{A}_{S',k'}}(X_{T',a'}^{S',k'}) \subseteq \alpha_{a'}^{S',k'}(D)$, we get an isomorphism satisfying the lemma restricting β_D to $N^{\mathfrak{A}_{S,k}}(X_{T,a}^{S,k})$. □

Suppose $k \in \mathbb{Z}_+$, $S, T \subseteq n$, $\beta \colon T \to \{-1, 0, 1\}$ and γ is an equivalence relation on n. Let

$$N(S, k, T, \beta, \gamma) = |\{a \mid (T, a) \in I^{S,k}, \mathrm{Inv}_{T,a}^{S,k} = (T, \beta, \gamma)\}|.$$

Lemma 5.22. *If $\beta^{-1}\{-1, 1\} \subseteq S \cap T$ and $(i, j) \in \gamma$ implies $i \in S \leftrightarrow j \in S$, $i \in T \leftrightarrow j \in T$ and $\beta(i) = \beta(j)$, then*

$$N(S, k, T, \beta, \gamma) = (k-1)^{|(\beta^{-1}\{0\} \cap S \cap T)/\gamma|} k^{|(S \setminus T)/\gamma|}.$$

Otherwise, $N(S, k, T, \beta, \gamma) = 0$.

Proof. By the definition, $(i, j) \in \gamma_{T,a}^{S,k}$ implies $i \in S \leftrightarrow j \in S$ and $i \in T \leftrightarrow j \in T$. If $i, j \in S$, it also implies $a(i) = a(j)$, which gives us $\beta_{T,a}^{S,k}(i) = \beta_{T,a}^{S,k}(j)$. If $\beta_{T,a}^{S,k}(i) \neq 0$, then $i \in S \cap T$. Thus if (T, β, γ) does not satisfy the conditions, then it differs from $\mathrm{Inv}_{T,a}^{S,k}$ for all a and so $N(S, k, T, \beta, \gamma) = 0$.

Suppose the conditions are satisfied. We may then partition S into three parts: $J_0 = \beta^{-1}\{-1, 1\}$, $J_1 = \beta^{-1}\{0\} \cap S \cap T$ and $J_2 = S \setminus T$. If $i \in J_0$, for all $a \in N(S, k, T, \beta, \gamma)$, $a(i)$ is either 0 or k depending on whether $\beta(i)$ is -1 or 1. If $i \in J_1$, we have $0 < a(i) < k$ and if $i \in J_2$, then $0 \leq a(i) < k$. If $(i, j) \in \gamma$, then $a(i) = a(j)$, but this is only restriction to values of a between different parameters. Because J_1 and J_2 are unions of the equivalence classes of γ, we conclude $N(S, k, T, \beta, \gamma) = (k-1)^{|J_1/\gamma|} k^{|J_2/\gamma|}$. □

Lemma 5.23. *For all $T \subseteq n$, $\beta \colon T \to \{-1, 0, 1\}$ and γ,*

$$N(n, k+1, T, \beta, \gamma) = \sum_{S \subseteq n} N(S, k, T, \beta, \gamma).$$

Proof. We may assume that $\beta^{-1}\{-1, 1\} \subseteq S \cap T$, γ is an equivalence relation, and $(i, j) \in \gamma$ implies $i \in T \leftrightarrow j \in T$ and $\beta(i) = \beta(j)$, because otherwise $N(S, k, T, \beta, \gamma) = 0$ for all S and k and the lemma holds trivially.

Let

$$\mathcal{S} = \{S \subseteq n \mid \beta^{-1}\{-1, 1\} \subseteq S, \forall (i, j) \in \gamma (i \in S \leftrightarrow j \in S).$$

If $S \in \mathcal{S}$, then

$$N(S, k, T, \beta, \gamma) = (k-1)^{|(\beta^{-1}\{0\} \cap S \cap T)/\gamma|} k^{|(S \setminus T)/\gamma|}$$

otherwise $N(S,k,T,\beta,\gamma) = 0$. Clearly $n \in \mathcal{S}$. Let $P_0 = (T \cap \beta^{-1}\{0\})/\gamma$ and $P_1 = (n \setminus T)/\gamma$. We have

$$\mathcal{S} = \left\{\beta^{-1}\{-1,1\} \cup \bigcup S_0 \cup S_1 \mid S_0 \subseteq P_0, S_1 \subseteq P_1\right\}.$$

For all $S_0 \subseteq P_0$ and $S_1 \subseteq P_1$,

$$N\left(\beta^{-1}\{-1,1\} \cup \bigcup S_0 \cup S_1, k, T, \beta, \gamma\right) = (k-1)^{|S_0|}k^{|S_1|}.$$

Thus

$$\begin{aligned}
\sum_{S \subseteq n} N(S,k,T,\beta,\gamma) &= \sum_{S_0 \subseteq P_0}\sum_{S_1 \subseteq P_1}(k-1)^{|S_0|}k^{|S_1|} \\
&= \left(\sum_{S_0 \subseteq P_0}(k-1)^{|S_0|}\right)\left(\sum_{S_1 \subseteq P_1}k^{|S_1|}\right) \\
&= k^{|P_0|}(k+1)^{|P_1|} \\
&= N(n,k+1,T,\beta,\gamma).
\end{aligned}$$

□

Proof of Lemma 5.18. Let $I' = \{(S,T,a) \mid S \subseteq n, (T,a) \in I^{S,k}\}$. Let $I_i^{S,k} = \{(T,a) \in I^{S,k} \mid |T| = i\}$ and $I'_i = \{(S,T,a) \in I' \mid |T| = i\}$. Define

$$A_i = \bigcup_{(T,a) \in I_i^{n,k+1}} X_{T,a}^{n,k+1}$$

and

$$B_i = \bigcup_{(S,T,a) \in I'_i} X_{T,a}^{S,k}.$$

We have already seen that $(A_i)_{i \leq n}$ is a partition of $\mathrm{Dom}(\mathfrak{A}_{n,(k+1)d+1})$ and $(B_i)_{i \leq n}$ is a partition of $\mathfrak{B}_{kd+1}$. We show now that these partitions witness $\mathfrak{A}_{n,(k+1)d+1} \leftrightsquigarrow_{n+1,r} \mathfrak{B}_{kd+1}$.

Fix $0 \leq i \leq n$. We have to show

$$\mathfrak{N}_r^{(\mathfrak{A}_{n,(k+1)d+1},A_0/U_0,\ldots,A_n/U_n)}(A_i) \cong \mathfrak{N}_r^{(\mathfrak{B}_{kd+1},B_0/U_0,\ldots,A_n/U_n)}(B_i).$$

First, fix a bijection $\alpha'_i\colon I_i^{n,k+1} \to I'_i$ such that for every $(T,a) \in I_i^{n,k+1}$ and $(S',T',a') = \alpha(T,a)$, we have $\mathrm{Inv}_{T,a}^{n,k+1} = \mathrm{Inv}_{T',a'}^{S',k}$. This is possible by Lemma 5.23. Then choose for each pair $(T,a) \in I^{n,k+1}$ an isomorphism $\alpha^{T,a}\colon \mathfrak{N}_r^{\mathfrak{A}_{n,k+1}}(X_{T,a}^{n,k+1}) \cong \mathfrak{N}_r^{\mathfrak{A}_{S',k}}(X_{T',a'}^{S',k})$, that exists by Lemma 5.21. Let $\alpha_i = \bigcup_{(T,a) \in I^{n,k+1}} \alpha^{T,a}$. The domains of the isomorphisms $\alpha^{T,a}$ are separated by Lemma 5.20 and so α_i is an isomorphism $\mathfrak{N}_r^{\mathfrak{A}_{n,(k+1)d+1}}(A_i) \cong \mathfrak{N}_r^{\mathfrak{B}_{kd+1}}(B_i)$. The same bijection is also an isomorphism $\mathfrak{N}_r^{(\mathfrak{A}_{n,(k+1)d+1},A_0/U_0,\ldots,A_n/U_n)}(A_i) \cong \mathfrak{N}_r^{(\mathfrak{B}_{kd+1},B_0/U_0,\ldots,A_n/U_n)}(B_i)$, because isomorphisms $\alpha^{T,a}$ maps elements of A_j to B_j by Lemma 5.21 for all $0 \leq j \leq n$. □

Lemma 5.18 implies that for arbitrarily large values of r, there exist $\leftrightsquigarrow_{n+1,r}$-equivalent structures that Q separates. This gives:

Theorem 5.24. $\mathcal{L}_{\mathrm{Q}} \not\leq \mathrm{HL}(\leftrightsquigarrow_{n+1,r})_{r\in\mathbb{N}}$.

Combining Theorems 5.17 and 5.24, we get:

Corollary 5.25. *The logics* $(\mathrm{HL}(\leftrightsquigarrow_{n,r})_{r\in\mathbb{N}}))_{n\geq 2}$ *form a strict hierarchy, i.e for all* $n \geq 2$*,* $\mathrm{HL}(\leftrightsquigarrow_{n,r})_{r\in\mathbb{N}} < \mathrm{HL}(\leftrightsquigarrow_{n+1,r})_{r\in\mathbb{N}}$*. Consecutive levels of the hierarchy can be separated using a logic of the form* FO(Q)*.*

6. Order-invariant logics

6.1. **Invariant logics.** Our main purpose in this section is to define order-invariant logics, but for the amusement of the reader, we first show how the concept of invariant logics can be defined in a quite general way.

Let $\mathcal{L} \geq \mathrm{QF}$ be a finitary logic and M a class of structures, not necessarily from the same vocabulary. Assume that if $\mathfrak{A} \in M$, $\overline{x}$ is a sequence of constant symbols and $\overline{a} \in \mathrm{Dom}(\mathfrak{A})^{|\overline{x}|}$, then $(\mathfrak{A}, \overline{a}/\overline{x}) \in M$. This ensures that if $\mathcal{L} \leq \mathcal{L}'$ (mod M) then $\mathcal{L}'' \circ \mathcal{L} \leq \mathcal{L}'' \circ \mathcal{L}'$ (mod M).

Definition 6.1. Let $\mathcal{R}(\mathcal{L}, M)$ be the greatest finitary logic $\mathcal{L}'$ such that $\mathcal{L}' \circ \mathcal{L} \leq \mathcal{L}$ (mod M).

The greatest logic exists, because we can define it by comprehension from the universal logic capable to express all possible queries.

The following proposition gives a more intuitive definition of the logic.

Proposition 6.2. *If* $\mathcal{L}$ *is closed under substitution, then* $\mathcal{R}(\mathcal{L}, M)$ *is the greatest finitary extension of* $\mathcal{L}$ *satisfying* $\mathcal{R}(\mathcal{L}, M) \leq \mathcal{L}$ (mod M) *and being closed under substitution.*

Proof. Because we assumed $\mathrm{QF} \leq \mathcal{L}$, we have $\mathcal{R}(\mathcal{L}, M) \leq \mathcal{L}$ (mod M). The logic is closed under substitution, since $(\mathcal{R}(\mathcal{L}, M) \circ \mathcal{R}(\mathcal{L}, M)) \circ \mathcal{L} \leq \mathcal{R}(\mathcal{L}, M) \circ (\mathcal{R}(\mathcal{L}, M) \circ \mathcal{L}) \leq \mathcal{R}(\mathcal{L}, M) \circ \mathcal{L} \leq \mathcal{L}$ (mod M). If $\mathcal{L}$ is closed under substitution, then $\mathcal{L} \circ \mathcal{L} \leq \mathcal{L}$, which implies $\mathcal{L} \leq \mathcal{R}(\mathcal{L}, M)$. Finally, if $\mathcal{L}'$ is a finitary extension of $\mathcal{L}$, closed under substitution and $\mathcal{L}' \leq \mathcal{L}$ (mod M), then $\mathcal{L}' \circ \mathcal{L} \leq \mathcal{L}' \circ \mathcal{L}' \leq \mathcal{L}' \leq \mathcal{L}$ (mod M) implying $\mathcal{L}' \leq \mathcal{R}(\mathcal{L}, M)$. □

The size of the class M affects considerably the expressive power of $\mathcal{R}(\mathcal{L}, M)$. We show next that under mild assumptions on $\mathcal{L}$ and M, there is a following dichotomy: either $\mathcal{R}(\mathcal{L}, M)$ is a fragment of second-order logic over $\mathcal{L}$ or it contains uncountably many sentences.

Definition 6.3. Given a logic $\mathcal{L}$, $\Sigma^1_1(\mathcal{L})$ is the logic with

$$\Sigma^1_1(\mathcal{L})[\tau] = \{(\varphi, \tau, \tau') \mid \tau \subseteq \tau', |\tau' \setminus \tau| < \omega, \varphi \in (\mathrm{FO} \circ \mathcal{L})[\tau']\}$$

and $\mathfrak{A} \models_{\Sigma^1_1(\mathcal{L})} (\varphi, \tau, \tau')$ if and only if there exists $\mathfrak{A}' \in \mathrm{Mod}(\tau')$ expanding $\mathfrak{A}$ and $\mathfrak{A}' \models \varphi$. The logic $\Pi^1_1(\mathcal{L})$ has the same sentences as $\Sigma^1_1(\mathcal{L})$, but the semantics:

$\mathfrak{A} \models_{\Pi^1_1(\mathcal{L})} (\varphi, \tau, \tau')$ if and only if for all $\mathfrak{A}' \in \mathrm{Mod}(\tau')$ expanding $\mathfrak{A}$, $\mathfrak{A}' \models \varphi$. The logic $\Delta^1_1(\mathcal{L})$ is a fragment of $\Sigma^1_1(\mathcal{L})$ containing only sentences that are equivalent to some $\Pi^1_1(\mathcal{L})$ sentences.

Definition 6.4. Let M be an arbitrary class of finite structures and let $M' \subseteq \mathrm{Mod}(\tau)$. We say that M *$\mathcal{L}$-covers* M', if there exists a finite sequence of vocabularies $\tau_0, \dots, \tau_{k-1}$ and interpretations $\psi_0, \dots, \psi_{k-1}$, such that $\psi_i \in I(\mathcal{L}, \tau_i, \tau)$ and $M' \subseteq \bigcup_{i<k} \psi_i^*(M \cap \mathrm{Mod}(\tau_i))$.

Proposition 6.5. *Suppose $M \cap \mathrm{Mod}(\tau)$ is $\Delta^1_1(\mathcal{L})$-definable for all finite vocabularies τ and $\mathcal{L}$ is finitary. If for every finite vocabulary τ, the class M $\mathcal{L}$-covers* $\mathrm{Mod}(\tau)$, *then $\mathcal{R}(\mathcal{L}, M) \leq \Delta^1_1(\mathcal{L})$.*

Proof. Let τ be a finite vocabulary, $\varphi \in \mathcal{R}(\mathcal{L}, M)[\tau]$ and suppose that the vocabularies τ_i and ψ_i, $0 \leq i < k$ witness that M $\mathcal{L}$-covers $\mathrm{Mod}(\tau)$. We may assume that every τ_i is finite, because τ is finite and $\mathcal{L}$ is finitary.

By the definition of $\mathcal{R}(\mathcal{L}, M)$, there exist sentences $\varphi_i \in \mathcal{L}[\tau]$ such that $\varphi_i \equiv \varphi \circ \psi_i \pmod{M}$. Then

$$\begin{aligned} \mathfrak{A} \models \varphi \quad &\iff \quad \bigvee_{0 \leq i < k} (\exists \mathfrak{A}' \in M \cap \mathrm{Mod}(\tau_i))(\mathfrak{A} = \psi_i^*(\mathfrak{A}') \wedge \mathfrak{A}' \models \varphi_i) \\ &\iff \quad \bigwedge_{0 \leq i < k} (\forall \mathfrak{A}' \in M \cap \mathrm{Mod}(\tau_i))(\mathfrak{A} = \psi_i^*(\mathfrak{A}') \rightarrow \mathfrak{A}' \models \varphi_i). \end{aligned}$$

We prove the first equivalence. If $\mathfrak{A} \models \varphi$, then by the $\mathcal{L}$-covering, for some $i < k$, there exists $\mathfrak{A}' \in M \cap \mathrm{Mod}(\tau_i)$ such that $\mathfrak{A} = \psi_i^*(\mathfrak{A}')$. Then $\mathfrak{A}' \models \varphi \circ \psi_i$ implying $\mathfrak{A}' \models \varphi_i$. On the other hand, if the right side holds, then for some $i < k$ and $\mathfrak{A}' \in M \cap \mathrm{Mod}(\tau_i)$, we have $\mathfrak{A} = \psi_i^*(\mathfrak{A}')$ and $\mathfrak{A}' \models \varphi_i$. But then $\mathfrak{A}' \models \varphi \circ \psi_i$ which implies $\mathfrak{A} \models \varphi$. The second equivalence is proved similarly.

Because the sets $M \cap \mathrm{Mod}(\tau_i)$ are $\Delta^1_1(\mathcal{L})$-definable and vocabularies τ_i finite, this can be expressed by a $\Delta^1_1(\mathcal{L})$-sentence. □

Proposition 6.6. *Assume $\mathcal{L}$ is countable,* $\mathrm{FO} \circ \mathcal{L} \leq \mathcal{L}$ *and there is a finite vocabulary τ such that M does not $\mathcal{L}$-cover* $\mathrm{Mod}(\tau)$. *Then there exists an infinite set $M' \subseteq \mathrm{Mod}(\tau)$ such that $\mathcal{R}(\mathcal{L}, M)$ can define any query on M'. In particular, $\mathcal{R}(\mathcal{L}, M)$ is uncountable and cannot be a fragment of $\Sigma^1_1(\mathcal{L})$.*

Proof. Because $\mathcal{L}$ is countable, there exist only countably many interpretations in $\bigcup_{\tau' \text{ finite}} I(\mathcal{L}, \tau', \tau)$ up to renaming the symbols in the vocabularies. Let $(\psi_i)_{i \in \mathbb{N}}$ be an enumeration of them and let τ_i be a vocabulary such that $\psi_i \in I(\mathcal{L}, \tau_i, \tau)$. Let $M_i = \psi_i^*(M \cap \mathrm{Mod}(\tau_i))$. Since M does not $\mathcal{L}$-cover $\mathrm{Mod}(\tau)$, the set $\mathrm{Mod}(\tau) \setminus \bigcup_{i<k} M_i$ is nonempty for all $k \in \mathbb{N}$ and we can choose a structure $\mathfrak{A}_k$ from each of them. Let $M' = \{\mathfrak{A}_k \mid k \in \mathbb{N}\}$.

Suppose $Q \subseteq M'$ and $q = \{\mathfrak{A}' \in \mathrm{Mod}(\tau) \mid \mathfrak{A} \in Q, \mathfrak{A} \cong \mathfrak{A}'\}$. In order to show that there exists an $\mathcal{R}(\mathcal{L}, M)$-sentence φ expressing q, we have to show that for all $\mathcal{L}$-interpretations ψ_i, $q \circ \psi_i$ is equivalent to a $\mathcal{L}$-sentence modulo M. Now for all

$\mathfrak{A} \in M \cap \mathrm{Mod}(\tau_i)$,

$$\mathfrak{A} \models \varphi \circ \psi_i \iff \psi_i^*(\mathfrak{A}) \in q \iff \exists j < i(\psi_i^*(\mathfrak{A}) \cong \mathfrak{A}_j \wedge \mathfrak{A}_j \in q).$$

By the assumption $\mathrm{FO} \circ \mathcal{L} \leq \mathcal{L}$, this is expressible in $\mathcal{L}$. Note that the isomorphism is expressible in FO because $\mathfrak{A}_j$ is a fixed finite structure. □

Definition 6.7. An $\mathcal{L}$-bi-interpretation on $M \subseteq \mathrm{Mod}(\tau)$ is a pair (θ, ρ), where $\theta \in I(\mathcal{L}, \tau, \tau')$ and $\rho \in I(\mathcal{L}, \tau', \tau)$, such that for all $\mathfrak{A} \in M$, $(\rho \circ \theta)^*(\mathfrak{A}) \cong \mathfrak{A}$.

Proposition 6.8. *If $\mathcal{L}$ is closed under substitution and for every finite vocabulary λ there exists a finite vocabulary τ_λ and an $\mathcal{L}$-bi-interpretation $(\theta_\lambda, \rho_\lambda)$ on $M \cap \mathrm{Mod}(\lambda)$ such that $\theta_\lambda^*(M \cap \mathrm{Mod}(\lambda)) \subseteq M' \cap \mathrm{Mod}(\tau_\lambda)$, then $\mathcal{R}(\mathcal{L}, M') \leq \mathcal{R}(\mathcal{L}, M)$.*

Proof. It suffices to show that $\mathcal{R}(\mathcal{L}, M') \leq \mathcal{L}$ (mod M). Let $\varphi \in \mathcal{R}(\mathcal{L}, M')[\lambda]$ be an arbitrary sentence. For some $\varphi' \in \mathcal{L}[\tau_\lambda]$, $\varphi' \equiv \varphi \circ \rho_\lambda$ (mod M'). Now $\varphi' \circ \theta_\lambda \equiv \varphi \circ \rho_\lambda \circ \theta_\lambda \equiv \varphi$ (mod M) and $\varphi' \circ \theta_\lambda \in \mathcal{L}[\lambda]$. □

Let $\mathrm{Ord}(\tau) = \{\mathfrak{A} \in \mathrm{Mod}(\tau \cup \{<\}) \mid <^{\mathfrak{A}}$ is a linear order on $\mathrm{Dom}(\mathfrak{A})\}$ and $\mathrm{Ord} = \bigcup_{\tau \text{ finite}} \mathrm{Ord}(\tau)$.

We call a sentence $\varphi \in \mathcal{L}[\tau \cup \{<\}]$, where $(<) \notin \tau$, order-invariant, if for all $\mathfrak{A}, \mathfrak{A}' \in \mathrm{Ord}(\tau)$ such that $\mathfrak{A} \restriction \tau \cong \mathfrak{A}' \restriction \tau$, we have $\mathfrak{A} \equiv_\varphi \mathfrak{A}'$. Let $\mathcal{L}_<$ be a logic with $\mathcal{L}_<[\tau] = \{\varphi \in \mathcal{L}[\tau \cup \{<\}] \mid \varphi \text{ is order-invariant}\}$ (if $(<) \in \tau$, we may rename it when considering the order-invariance of φ) and with semantics $\mathfrak{A} \models_{\mathcal{L}_<} \varphi$, if there exists $\mathfrak{A}' \in \mathrm{Ord}(\tau)$ such that $\mathfrak{A} = \mathfrak{A}' \restriction \tau$ and $\mathfrak{A}' \models_{\mathcal{L}} \varphi$.

Proposition 6.9. *If $\mathcal{L}$ is finitary and closed under substitution, then we have $\mathcal{L}_< \equiv \mathcal{R}(\mathcal{L}, \mathrm{Ord})$.*

Proof. Let $\varphi \in \mathcal{R}(\mathcal{L}, \mathrm{Ord})[\tau]$ and let $\psi \in I(\mathcal{L}, \tau \cup \{<\}, \tau)$ such that $\psi^*(\mathfrak{A}) = \mathfrak{A} \restriction \tau$. Then there exists $\varphi' \in \mathcal{L}[\tau \cup \{<\}]$ such that $\varphi' \equiv \varphi \circ \psi$ (mod Ord). The sentence φ' is order-invariant, because for all $\mathfrak{A}, \mathfrak{A}' \in \mathrm{Ord}(\tau)$, if $\mathfrak{A} \restriction \tau \cong \mathfrak{A}' \restriction \tau$, then $\psi^*(\mathfrak{A}) \cong \psi^*(\mathfrak{A}')$ implying $\mathfrak{A} \equiv_{\varphi'} \mathfrak{A}'$. Thus $\varphi' \in \mathcal{L}_<[\tau]$ and φ' in $\mathcal{L}_<$-semantics is equivalent to φ. This proves $\mathcal{R}(\mathcal{L}, \mathrm{Ord}) \leq \mathcal{L}_<$.

Let $\varphi \in \mathcal{L}_<[\tau]$ and $\psi \in I(\mathcal{L}, \tau', \tau)$ be arbitrary. We show that there is $\varphi' \in \mathcal{L}[\tau']$ such that $\varphi' \equiv \varphi \circ \psi$ (mod Ord). This is trivial, if $(<) \notin \tau'$. If $(<) \in \tau'$, let $\psi' \in I(\mathcal{L}, \tau', \tau \cup \{<\})$ such that $(\psi')^*(\mathfrak{A}) \restriction \tau = \psi^*(\mathfrak{A})$ and $(<^{(\psi')^*(\mathfrak{A})}) = (<^{\mathfrak{A}})$. Then $\varphi \circ \psi' \equiv \varphi \circ \psi$ (mod Ord), where φ occurs first as a $\mathcal{L}[\tau \cup \{<\}]$-sentence and then as a $\mathcal{L}_<[\tau]$-sentence. This shows $\mathcal{L}_< \leq \mathcal{R}(\mathcal{L}, \mathrm{Ord})$. □

The order-invariant logic preserves some basic properties of the base logic.

Proposition 6.10. *If $\mathcal{L}$ is regular, then $\mathcal{L}_<$ is regular.*

Proof. We have already seen that if $\mathcal{L}$ is finitary and closed under substitution, then $\mathcal{R}(\mathcal{L}, \mathrm{Ord}) \equiv \mathcal{L}_<$ have these properties and $\mathcal{L} \leq \mathcal{L}_<$. Thus, if $\mathrm{FO} \leq \mathcal{L}$ then we have also $\mathrm{FO} \leq \mathcal{L}_<$. We have to prove now that $\mathcal{L}_<$ is closed under relativization, whenever $\mathcal{L}$ is.

If $\varphi \in \mathcal{L}[\tau \cup \{<\}]$ is order-invariant, and U is a unary relation symbol not in τ, let φ^U be the relativization of φ to U. If $\mathfrak{A}, \mathfrak{A}' \in \mathrm{Ord}(\tau \cup \{U\})$ and $\mathfrak{A} \restriction \tau \cup \{U\} \cong \mathfrak{A}' \restriction$

$\tau \cup \{U\}$, then $\langle U^{\mathfrak{A}}\rangle^{\mathfrak{A}} \restriction \tau \cong \langle U^{\mathfrak{A}'}\rangle^{\mathfrak{A}'} \restriction \tau$ and $\langle U^{\mathfrak{A}}\rangle^{\mathfrak{A}} \restriction \tau \cup \{<\}, \langle U^{\mathfrak{A}'}\rangle^{\mathfrak{A}'} \restriction \tau \cup \{<\} \in \mathrm{Ord}(\tau)$, which implies $\langle U^{\mathfrak{A}}\rangle^{\mathfrak{A}} \restriction \tau \cup \{<\} \equiv_{\varphi} \langle U^{\mathfrak{A}'}\rangle^{\mathfrak{A}'} \restriction \tau \cup \{<\}$. Thus $\mathfrak{A} \equiv_{\varphi^U} \mathfrak{A}'$, showing that φ^U is order-invariant. It is easy to see that φ^U is also a relativization of φ in $\mathcal{L}_<$-semantics. □

Proposition 6.11. *Let M be a class of structures and $\mathcal{L}$ a logic and assume that for each pair (τ_0, τ_1) of compatible vocabularies, there exist compatible vocabularies τ_0' and τ_1', an interpretation $\psi \in I(\mathcal{L}, \tau_0' \cup \tau_1', \tau_0 \cup \tau_1)$ and functions $f_0 \colon \mathrm{Mod}(\tau_0) \to \mathrm{Mod}(\tau_0')$, and $f_1 \colon \mathrm{Mod}(\tau_1) \to \mathrm{Mod}(\tau_1')$ such that for all $\mathfrak{A}_0 \in \mathrm{Mod}(\tau_0)$ and $\mathfrak{A}_1 \in \mathrm{Mod}(\tau_1)$, $f_0(\mathfrak{A}_0) \sqcup f_1(\mathfrak{A}_1) \in M$ and $\mathfrak{A}_0 \sqcup \mathfrak{A}_1 = \psi^*(f_0(\mathfrak{A}_0) \sqcup f_1(\mathfrak{A}_1))$. If $\mathcal{L}$ has weak uniform reduction, also $\mathcal{R}(\mathcal{L}, M)$ has weak uniform reduction.*

Proof. Assume first that $\mathcal{L}$ has weak uniform reduction. Let τ_0 and τ_1 be compatible vocabularies and φ an arbitrary $\mathcal{R}(\mathcal{L}, M)[\tau_0 \cup \tau_1]$-sentence. Let τ_0', τ_1', f_0, f_1 and ψ be as in the proposition Then there exists $\varphi' \in \mathcal{L}[\tau_0' \cup \tau_1']$ such that $\varphi' \equiv \varphi \circ \psi \pmod{M}$. Let $\sim$ be a finite equivalence relation witnessing weak uniform reduction for the vocabularies τ_0' and τ_1'. Then for all $\mathfrak{A}_0, \mathfrak{A}_0' \in \mathrm{Mod}(\tau_0)$ and $\mathfrak{A}_1 \in \mathrm{Mod}(\tau_1)$, such that $f_0(\mathfrak{A}_0) \sim f_0(\mathfrak{A}_0')$,

$$\begin{aligned}
\mathfrak{A}_0 \sqcup \mathfrak{A}_1 \models \varphi \quad &\Longleftrightarrow \quad \psi^*(f_0(\mathfrak{A}_0) \sqcup f_1(\mathfrak{A}_1)) \models \varphi \\
&\Longleftrightarrow \quad f_0(\mathfrak{A}_0) \sqcup f_1(\mathfrak{A}_1) \models \varphi' \\
&\Longleftrightarrow \quad f_0(\mathfrak{A}_0') \sqcup f_1(\mathfrak{A}_1) \models \varphi' \\
&\Longleftrightarrow \quad \psi^*(f_0(\mathfrak{A}_0') \sqcup f_1(\mathfrak{A}_1)) \models \varphi \\
&\Longleftrightarrow \quad \mathfrak{A}_0' \sqcup \mathfrak{A}_1 \models \varphi.
\end{aligned}$$

Thus $f_0^{-1}(\sim)$ is a finite equivalence relation witnessing weak uniform reduction of $\mathcal{R}(\mathcal{L}, M)$. □

Corollary 6.12. *If $\mathcal{L}$ is finitary, closed under substitution, has (weak) uniform reduction and $\mathcal{L} \geq \mathrm{QF}$, then $\mathcal{L}_<$ also has (weak) uniform reduction.*

Proof. Assume first that $\mathcal{L}$ has weak uniform reduction. Let M be a class of structures that have relations $<'$, U and V in their vocabulary such that the interpretations of U and V form a partition of the universe and $<'$ is a disjoint union of two linear orders one on the set U and one on the set V. We claim that $\mathcal{R}(\mathcal{L}, \mathrm{Ord}) \leq \mathcal{R}(\mathcal{L}, M)$. This is an implication of Proposition 6.8, because we can define a linear order on any structure in M by the QF-formula $\theta(x, y) \equiv x <' y \vee (Ux \wedge Vy)$ and we can so define a bi-interpretation where one direction just adds interpretation to the symbol $<$ and other direction removes the interpretation.

For each pair (τ_0, τ_1) of compatible vocabularies that do not contain relation symbols U, V and $<'$, let $\tau_0' = \tau_0 \cup \{U, <'\}$ define f_0 so that $f_0(\mathfrak{A}) \restriction \tau_0 = \mathfrak{A}$, $U^{f_0(\mathfrak{A})} = \mathrm{Dom}(\mathfrak{A})$ and $(<')^{f_0(\mathfrak{A})}$ is a linear order on $\mathrm{Dom}(\mathfrak{A})$. Define τ_1' and f_1 similarly but using the relation symbol V instead of U. Let $\psi \in I(\mathrm{QF}, \tau_0' \cup \tau_1', \tau_0 \cup \tau_1)$ such that $\psi^*(\mathfrak{A}) = \mathfrak{A} \restriction (\tau_0 \cup \tau_1)$. These definitions show that M and $\mathcal{L}$ satisfy the conditions of Proposition 6.11. Since $\mathcal{R}(\mathcal{L}, \mathrm{Ord}) \leq \mathcal{R}(\mathcal{L}, M)$, if $\mathcal{L}$ has weak uniform reduction, also $\mathcal{L}_<$ has it.

In order to show that $\mathcal{L}_<$ inherits also uniform reduction from $\mathcal{L}$, we have to show that if $\mathcal{L}$ is closed under model extensions, also $\mathcal{L}_<$ is closed under model extensions (Lemma 4.4). Fix compatible vocabularies τ_0 and τ_1, a structure $\mathfrak{C} \in \mathrm{Mod}(\tau_1)$ and a sentence $\varphi \in \mathcal{L}_<[\tau_0 \cup \tau_1]$. We have to show that $\varphi^{\mathfrak{C}}$ defined so that $\mathfrak{A} \models \varphi^{\mathfrak{C}}$ if and only if $\mathfrak{A} \sqcup \mathfrak{C} \models \varphi$ is in $\mathcal{L}_<[\tau_0]$.

Denote the sentence φ by φ' when it is considered in $\mathcal{L}[\tau_0 \cup \tau_1 \cup \{<\}]$-semantics. Let θ be as above and $\mathfrak{C}' = (\mathfrak{C}, \mathrm{Dom}(\mathfrak{C})/V, O/<')$, where O is any linear order on $\mathrm{Dom}(\mathfrak{C})$. Let $\gamma \in I(\mathrm{QF}, \tau_0 \cup \{<\}, \tau_0 \cup \{U, <'\})$ that adds interpretation of U to the structure such that $U^{\gamma^*(\mathfrak{A})} = \mathrm{Dom}(\mathfrak{A})$ and renames $<$ as $<'$. Then for all $\mathfrak{A} \in \mathrm{Ord}(\tau_0)$, $\theta^*(g(\mathfrak{A}) \sqcup \mathfrak{C}') \in \mathrm{Ord}(\tau_0 \cup \tau_1)$. We have now for all $\mathfrak{A} \in \mathrm{Mod}(\tau_0)$,

$$\begin{aligned}
(\mathfrak{A}, O/<) \models (\varphi' \circ \theta)^{\mathfrak{C}'} \circ \gamma \quad &\iff \quad (\mathfrak{A}, \mathrm{Dom}(\mathfrak{A})/U, O/<') \models (\varphi' \circ \theta)^{\mathfrak{C}'} \\
&\iff \quad (\mathfrak{A}, \mathrm{Dom}(\mathfrak{A})/U, O/<') \sqcup \mathfrak{C}' \models \varphi' \circ \theta \\
&\iff \quad (\mathfrak{A} \sqcup \mathfrak{C}, O'/<) \models \varphi'
\end{aligned}$$

Where O is an arbitrary linear order and O' a concatenation of O and linear order in $\mathfrak{C}'$. Because φ' is order-invariant, also $(\varphi' \circ \theta)^{\mathfrak{C}'} \circ \gamma$ is order-invariant and equivalent to $\varphi^{\mathfrak{C}}$. Thus $\varphi^{\mathfrak{C}}$ is in $\mathcal{L}_<$. □

6.2. **Order-invariant first-order logic is hierarchical.** Although $\mathrm{FO}_<$ seems weak — it is even hard to find a query that is expressible in $\mathrm{FO}_<$ but not in FO — the logic is not contained at any level of quantifier hierarchy.

Let Q_k be the query defined in Section 4.4.

Lemma 6.13. *The query Q_k is definable in* $\mathrm{FO}_<$.

Proof. We have already shown in Lemma 4.19 that the class of structures isomorphic to a structure in M_k is FO-definable. We have to show now that $\mathrm{FO}_<$ can separate $M_{k,0}$ and $M_{k,1}$.

Let τ_k be the vocabulary of Q_k and define the following $\mathrm{FO}[\tau_k \cup \{<\}]$-formula:

$$\begin{aligned}
\psi(z) \quad \equiv \quad \exists x_0 \ldots x_{k-1} y_0 \ldots y_{k-1} \Big(& \bigwedge_{i<k} x_i \sim z \\
& \wedge \bigwedge_{i<k-1} (x_i < x_{i+1} \wedge y_i < y_{i+1}) \wedge E x_0 \ldots x_{k-1} y_0 \ldots y_{k-1} \Big).
\end{aligned}$$

Consider a structure $\mathfrak{A} = (\mathfrak{A}_{k,n,X}, <^{\mathfrak{A}}/<)$ in M_k expanded with a linear order. Remember that the domain of $\mathfrak{A}$ is $(\mathbb{Z}_n \times k) \cup \mathcal{P}(n)$. For all $i \in \mathbb{Z}_n$, let $f_i \in S_k$ such that $(i, f_i(0)) <^{\mathfrak{A}} \cdots <^{\mathfrak{A}} (i, f_i(k-1))$. Then if $\mathfrak{A} \models \psi((i,j))$, we must have

$$\langle (i, (f_i(0)), \ldots, (i, (f_i(k-1)), (i+1, (f_{i+1}(0)), \ldots, (i+1, (f_{i+1}(k-1)) \rangle \in E^{\mathfrak{A}}.$$

By the definition of $E^{\mathfrak{A}}$, if $i \in X$ this happens if and only if there exists $g \in A'_k$ such that $f_{i+1} = g \circ f_i$, i.e., $f_{i+1} \circ f_i^{-1} \in A'_k$. If $i \notin X$, we must have $f_{i+1} \circ f_i^{-1} \in A_k$. Hence

$$\psi(\mathfrak{A}) = (X \Delta \{j \in \mathbb{Z}_n \mid f_{i+1} \circ f_i^{-1} \in A_k\}) \times k.$$

Because $(f_0 \circ f_{n-1}^{-1}) \circ (f_{n-1} \circ f_{n-2}^{-1}) \circ \cdots \circ (f_1 \circ f_0^{-1}) = \mathrm{id}_k$ is an even permutation, we must have $|\{j \in \mathbb{Z}_n \mid f_{i+1} \circ f_i^{-1} \in A_k\}| \equiv n \pmod 2$. Thus $|X| = |(U^{\mathfrak{A}} \setminus \psi(\mathfrak{A}))/{\sim^{\mathfrak{A}}}| \pmod 2$. This parity can be expressed in the same way as in the Gurevich's well-known example that separates FO and FO$_<$: We start by ordering the equivalence classes:

$$x \prec y \equiv \exists x'(x \sim x' \wedge \forall y'(y \sim y' \rightarrow x' < y')).$$

Let $\rho(x) \equiv Ux \wedge \neg\psi(x)$ and define a successor relation on $(\rho(\mathfrak{A})/\sim^{\mathfrak{A}}, \prec^{\mathfrak{A}})$ as

$$S(x, y) \equiv x \prec y \wedge \neg\exists z(\rho(z) \wedge x \prec z \prec y).$$

Now the sentence

$$\exists z(Pz \wedge \forall xy((\rho(x) \wedge \rho(y) \wedge S(x, y)) \rightarrow (\epsilon xz \leftrightarrow \neg\epsilon yz)) \wedge \\ \forall x(\rho(x) \wedge (\neg\exists y(\rho(y) \wedge y \prec x) \vee \neg\exists y(\rho(y) \wedge x \prec y)) \rightarrow \epsilon xz))$$

is true on $\mathfrak{A}$ if and only if $|X|$ is odd. Because it does not depend on the order $<^{\mathfrak{A}}$ chosen, it is order-invariant and in $\mathcal{L}_<[\tau_k]$. □

The logics UL$_k$ used in the next lemma were defined in Subsection 2.5.

Lemma 6.14. *The query Q_k is not definable in* FO(UL$_{2k-3}$) .

Proof. We prove the lemma by showing that for all $r \in \mathbb{N}$, if $n > 2(2k)^r$, then Player II has a winning strategy in the game $\mathrm{BG}_r^{2k-3}(\mathfrak{A}_{k,n,\emptyset}, \mathfrak{A}_{k,n,\{-1\}})$.

Given a sequence $\overline{p} \in S_k^{\mathbb{Z}_n}$ of permutations, let $\alpha_{\overline{p}}\colon \mathrm{Dom}(\mathfrak{A}_{k,n,\emptyset}) \rightarrow \mathrm{Dom}(\mathfrak{A}_{k,n,\{-1\}})$ be a bijection defined so that for all $a \in \mathcal{P}(n)$, $\alpha_{\overline{p}}(a) = a$ and $\alpha_{\overline{p}}((i, j)) = (i, p_i(j))$. All bijections Player II will choose in our winning strategy will be of the form $\alpha_{\overline{p}}$.

For all relations R in τ_k excluding E and for all sequences $\overline{p} \in S_k^{\mathbb{Z}_n}$, we have $\alpha_{\overline{p}}(R^{\mathfrak{A}_{k,n,\emptyset}}) = R^{\mathfrak{A}_{k,n,\{-1\}}}$. Let $A(\overline{p}) = \{i \in \mathbb{Z}_n \mid p_i \in A'_k\}$ and $D(\overline{p}) = A(\overline{p})\Delta\{i-1 \mid i \in A(\overline{p})\}\Delta\{-1\}$. The function $\alpha_{\overline{p}} \restriction (\mathrm{Dom}(\mathfrak{A}_{k,n,\emptyset}) \setminus D(\overline{p}) \times k)$ is a partial isomorphism from $\mathfrak{A}_{k,n,\emptyset}$ to $\mathfrak{A}_{k,n,\{-1\}}$.

Given sequences $\overline{a}$ and $\overline{b}$ on $\mathrm{Dom}(\mathfrak{A}_{k,n,\emptyset})$ and $\mathrm{Dom}(\mathfrak{A}_{k,n,\{-1\}})$, let $B(\overline{a}, \overline{b}) = \{\overline{p} \in S_k^{\mathbb{Z}_n} \mid \alpha_{\overline{p}}(\overline{a}) = \overline{b}\}$. This is the set of possible moves in the game situation $(\overline{a}, \overline{b})$.

Let $m(\overline{a}, i) = |[\overline{a}] \cap (i \times k)|$. The pair $(\overline{a}, \overline{b})$ fixes whether $\overline{p} \in B(\overline{a}, \overline{b})$ has $p_i \in A_k$ or $p_i \in A_k$ if and only if $m(\overline{a}, i) \geq k - 1$. Let $M(\overline{a}) = \{i \in \mathbb{Z}_n \mid m(\overline{a}, i) \geq k - 1\}$. Then for all $\overline{p} \in B(\overline{a}, \overline{b})$ and $Z \subseteq \mathbb{Z}_n \setminus M(\overline{a})$, there exists $\overline{p}' \in B(\overline{a}, \overline{b})$ such that $A(\overline{p}') = A(\overline{p})\Delta Z$.

We claim now that Player II has a winning strategy in the game

$$\mathrm{BG}_r^{2k-3}((\mathfrak{A}_{k,n,\emptyset}, \overline{a}/\overline{x}), (\mathfrak{A}_{k,n,\{-1\}}, \overline{b}/\overline{x})),$$

if there exists $\overline{p} \in B(\overline{a}, \overline{b})$ such that $D(\overline{p}) = \{i\}$ and for all $j \in [i - (2k)^r, i + (2k)^r]$, $m(\overline{a}, j) = 0$. The claim implies the existence of a winning strategy.

If $r = 0$, all elements of $\overline{a}$ lie in $\mathrm{Dom}(\mathfrak{A}_{k,n,\emptyset}) \setminus D(\overline{p}) \times k$ for some $\overline{p} \in B(\overline{a}, \overline{b})$ and since $\alpha_{\overline{p}} \restriction [\overline{a}]$ is a partial isomorphism, $\mathrm{atp}^{\mathfrak{A}_{k,n,\emptyset}}(\overline{a}) = \mathrm{atp}^{\mathfrak{A}_{k,n,\{-1\}}}(\overline{b})$ and Player II has won the game.

Assume then that the claim holds for r and there is $\overline{p} \in B(\overline{a}, \overline{b})$ such that $D(\overline{p}) = \{i\}$ and for all $j \in [i-(2k)^{r+1}, i+(2k)^{r+1}]$, $m(\overline{a}, j) = 0$. Player II plays now the bijection $\alpha_{\overline{p}}$.

Assume that Player I plays pairs $(a'_0, b'_0), \ldots, (a'_{2k-4}, b'_{2k-4}) \in \alpha_{\overline{p}}$. If $m(\overline{a}', j) < k-1$ for all $j \in [i, i+(2k)^{r+1}]$, then for all $j \in [i+1, i+(2k)^{r+1}]$, there is $\overline{p}' \in B(\overline{aa}', \overline{bb}')$ such that $A(\overline{p}') = A(\overline{p}) \Delta [i+1, j]$, which means $D(\overline{p}') = j$. Similarly, if $m(\overline{a}', j) < k-1$ for all $j \in [i-(2k)^{r+1}, i]$, then for all $j \in [i-(2k)^{r+1}, i]$ there exists $\overline{p'} \in B(\overline{aa}', \overline{bb}')$ such that $D(\overline{p}') = j$. Because $|\overline{a'}| = 2k-3$, there can be only one j with $m(\overline{a}', j) \geq k-1$ and we find j and $\overline{p}' \in B(\overline{aa}', \overline{bb}')$ such that $D(\overline{p}) = \{i\}$ and for all $j \in [i-(2k)^r, i+(2k)^r]$, $m(\overline{a}, j) = 0$. □

The lemmas together imply the following theorem.

Theorem 6.15. *For all $k \in \mathbb{Z}_+$, $\mathrm{FO}_< \not\leq \mathrm{FO}(\mathrm{UL}_k)$. In particular, $\mathrm{FO}_<$ cannot be defined by extending* FO *with finitely many generalized quantifiers.*

6.3. **Counting.** One of the basic properties of first-order logic is its inability to count. This manifests in many ways. At its simplest, FO cannot separate two sets of different but large enough cardinality. We formalize next these properties.

Let $\mathrm{cut}_{n,k} \colon \mathbb{N} \to n+k$ be the unique function such that $\mathrm{cut}_{n,k} \restriction n = \mathrm{id}_n$ and for all $i \geq n$, $n \leq \mathrm{cut}_{n,k}(i) < n+k$ and $\mathrm{cut}_{n,k}(i) \equiv i \pmod{k}$. We say that a set $S \subseteq \mathbb{N}$ is *k-periodic*, if for some $n \in \mathbb{N}$, $S = \mathrm{cut}_{n,k}^{-1}(S)$. If I is a set and $S \subseteq \mathbb{N} \times I$, we say that S is *uniformly k-periodic*, if for some $n \in \mathbb{N}$, $S = \{(j,i) \mid (\mathrm{cut}_{n,k}(j), i) \in S\}$.

If τ is a vocabulary, $f \colon \mathbb{N} \to \mathrm{Mod}(\tau)$ a function, $K \subseteq \mathbb{N}$ and $\mathcal{L}$ is a logic, we say that $\mathcal{L}$ is *K, f-periodic* if for every $\varphi \in \mathcal{L}[\tau]$, there exists $k \in K$ such that $\{i \in \mathbb{N} \mid f(i) \models \varphi\}$ is k-periodic. If $g \colon \mathbb{N} \times I \to \mathrm{Mod}(\tau)$, we say that $\mathcal{L}$ is *uniformly K, g-periodic* if for every $\varphi \in \mathcal{L}[\tau]$, there exists $k \in K$ such that $\{(j,i) \in \mathbb{N} \times I \mid g(j,i) \models \varphi\}$ is uniformly k-periodic.

Let $f_\emptyset \colon \mathbb{N} \to \mathrm{Mod}(\emptyset)$ and $f_< \colon \mathbb{N} \to \mathrm{Ord}(\emptyset)$ be such that $|f_\emptyset(i)| = |f_<(i)| = i$. Define $g_\tau \colon \mathbb{N} \times \mathrm{Mod}(\tau) \to \mathrm{Mod}(\tau)$ such that $g_\tau(i, \mathfrak{A})$ is isomorphic to the disjoint union of i copies of $\mathfrak{A}$. Define $g_{\tau,<} \colon \mathbb{N} \times \mathrm{Ord}(\tau) \to \mathrm{Ord}(\tau)$ so that $g_{\tau,<}(i, \mathfrak{A}) \restriction \tau = g_\tau(i, \mathfrak{A} \restriction \tau)$ and $<^{g_{\tau,<}(i,\mathfrak{A})}$ is a concatenation of the linear orders on the copies of $\mathfrak{A}$.

Because $K, f_<$-periodicity and uniform $K, g_{\tau,<}$-periodicity depends only on the expressive power of the logic on linearly ordered structures, these properties transform from $\mathcal{L}$ to $\mathcal{L}_<$.

Proposition 6.16. *If $\mathcal{L} \circ \mathrm{QF}$ has weak uniform reduction, then $\mathcal{L}$ is uniformly $\mathbb{N}, g_{\tau,<}$-periodic for all vocabularies τ.*

Proof. Let M be the class of $\tau \cup \{<', U, V\}$-structures defined in the proof of Corollary 6.12, i.e., U and V form a partition of the universe of every structure in the class and $<'$ is a linear order on U and V. Let $\psi \in I(\mathrm{QF}, \tau \cup \{<', U, V\}, \tau \cup \{<\})$ such that for all $R \in \tau$, $R^{\psi^*(\mathfrak{A})} = R^{\mathfrak{A}}$ and $\psi_<(x,y) = x <' y \vee (Ux \wedge Vy)$. For all $\mathfrak{A} \in M$, $\psi^*(\mathfrak{A}) \in \mathrm{Ord}(\tau)$.

Consider an arbitrary $\varphi \in \mathcal{L}[\tau \cup \{<\}]$. Let $\sim$ be a finite equivalence relation witnessing weak uniform reduction of $\varphi \circ \psi$ and let l be the number of equivalence classes of $\sim$. Define $k = \mathrm{lcm}\{1, \ldots, l\}$ and $n = l$.

Given $\mathfrak{A} \in \mathrm{Ord}(\mathfrak{A})$, there exists $i < j \leq l$ such that

$$(g_{\tau,<}(i, \mathfrak{A}), \mathrm{Dom}(g_{\tau,<}(i, \mathfrak{A}))/U) \sim (g_{\tau,<}(j, \mathfrak{A}), \mathrm{Dom}(g_{\tau,<}(j, \mathfrak{A}))/U).$$

Then

$$\begin{aligned} g_{\tau,<}(m+j) &\cong \psi^*((g_{\tau,<}(j), \mathrm{Dom}(g_{\tau,<}(j))/U) \sqcup (g_{\tau,<}(m), \mathrm{Dom}(g_{\tau,<}(m))/V)) \\ &\equiv_\varphi \psi^*((g_{\tau,<}(i), \mathrm{Dom}(g_{\tau,<}(i))/U) \sqcup (g_{\tau,<}(m), \mathrm{Dom}(g_{\tau,<}(m))/V)) \\ &\cong g_{\tau,<}(m+i). \end{aligned}$$

Using this equivalence repeatedly, we can show that for all $\mathfrak{A} \in \mathrm{Ord}(\mathfrak{A})$, $g_{\tau,<}(m) \equiv_\varphi g_{\tau,<}(\mathrm{cut}_{n,k}(m))$, i.e., $\mathcal{L}$ is uniformly $\mathbb{N}, g_{\tau,<}$-periodic. □

If the logic contains only sentences on unary vocabularies, we can say even more.

A query q on unary vocabulary τ is *periodic*, if there is $n, k \in \mathbb{Z}_+$ such that $\mathfrak{A} \equiv_q \mathfrak{B}$ if for all atomic 1-types t on τ, $\mathrm{cut}_{n,k}|\{a \in \mathrm{Dom}(\mathfrak{A}) \mid \mathfrak{A} \models t(a)\}| = \mathrm{cut}_{n,k}|\{b \in \mathrm{Dom}(\mathfrak{B}) \mid \mathfrak{B} \models t(b)\}|$.

Proposition 6.17. *If $\varphi \in \mathcal{L}[\tau]$, τ is a finite unary vocabulary and $\mathcal{L}$ has weak uniform reduction, then φ defines a periodic query.*

Proof. Let $\varphi \in \mathcal{L}[\tau]$, and let $\sim$ be a finite equivalence relation witnessing weak uniform reduction of φ. Let S be the set of all atomic 1-types on τ. For all $t \in S$, there exist structures $\mathfrak{A}_t$ and $\mathfrak{B}_t$, both having only elements realizing t, such that $|\mathfrak{A}_t| < |\mathfrak{B}_t|$ and $\mathfrak{A}_t \sim \mathfrak{B}_t$.

Choose

$$n = \max\{|\mathfrak{B}_t| \mid t \text{ is an atomic 1-type}\}$$

and

$$k = \mathrm{lcm}\{|\mathfrak{B}_t| - |\mathfrak{A}_t| \mid t \text{ is an atomic 1-type.}\}.$$

Let $\mathfrak{A}$ be a τ-structure. If $t \in S$ and $\mathfrak{A}$ has $m_t > |\mathfrak{B}_t|$ realizations of t, then there exists $\mathfrak{A}'$ such that $\mathfrak{A} \cong \mathfrak{A}' \sqcup \mathfrak{B}_t \equiv_\varphi \mathfrak{A}' \sqcup \mathfrak{A}_t$. The structure $\mathfrak{A}' \sqcup \mathfrak{A}_t$ has $m_t - (|\mathfrak{B}_t| - |\mathfrak{A} - t|)$ realizations of t. If $m > n + k$ we can find a structure with $m_t - k$ realizations of t iterating the equivalence. Repeating this we eventually find a structure φ-equivalent to $\mathfrak{A}$ and having $\mathrm{cut}_{n,k}(m_t)$ realizations of each type $t \in S$. Thus φ defines a periodic query.

Note that it is necessary to assume that τ is finite. Otherwise, the least common multiple k would not generally exist. □

Let $\tau_U = \{U\}$, where U is a unary relation symbol. Let $D_{n,k,i}$ be a τ_U-query such that $\mathfrak{A} \in D_{n,k,i}$ if and only if $\mathrm{cut}_{n,k}(|U^{\mathfrak{A}}|) = i$. Let $D_k = D_{0,k,0}$. It is easy to see that $D_{n,k,i}$ is definable in $\mathrm{FO}(D_k)$ for all $n, i \in \mathbb{N}$.

Proposition 6.18. *Let τ be an arbitrary finite unary vocabulary. If* Q *is a periodic τ-query, then* $\mathrm{FO}(\mathrm{Q}) \leq \mathrm{FO}(D_k)$ *for some k. If* $\mathrm{FO}(\mathrm{Q})$ *is additionally regular, k can be chosen such that* $\mathrm{FO}(D_k) \equiv \mathrm{FO}(\mathrm{Q})$*.*

Proof. Let k be the least positive integer such that for some $n \in \mathbb{Z}_+$, n and k witness the periodicity of Q.

Let S be the set of atomic 1-types on τ and let Let C be the set of all sequences in $\mathbb{N}^S$ such that if $c \in C$ and $\mathfrak{A}$ has c_t realizations of each type $t \in S$, then $\mathfrak{A} \in \mathrm{Q}$. We can express Q in $\mathrm{FO}(D_k)$ now by the sentence

$$\bigvee_{c \in C \cap (n+k)^S} \bigwedge_{t \in S} D_{n,k,c_t} x(t(x)).$$

Thus $\mathrm{FO}(\mathrm{Q}) \leq \mathrm{FO}(D_k)$.

Assume then that FO(Q) is regular. Given $c \in \mathbb{N}^S$, $t_0 \in S$ and a sequence $\overline{x}$ of constant symbols such that $|\overline{x}| \geq \sum_{t \in S} c_t$, we can define interpretations $\psi_{c,t_0}, \psi'_{c,t_0} \in I(\mathrm{FO}, \{U, \overline{x}\}, \tau \cup \{V\}\}$ such that if $\mathfrak{A}$ is a $\{U, \overline{x}\}$-structure and $x_i^{\mathfrak{A}}$ are distinct constants outside of $U^{\mathfrak{A}}$, then $\langle V^{\psi^*(\mathfrak{A})} \rangle^{\psi^*(\mathfrak{A})}$ is a τ-structure having for all $t \in S$, c_t realizations of t, and $\langle V^{(\psi')^*(\mathfrak{A})} \rangle^{(\psi')^*(\mathfrak{A})}$ is similar, but it has $|U^{\mathfrak{A}}| + c_{t_0}$ realizations of t_0.

Let θ be a FO(Q) sentence defining the relativization of Q. If $\mathfrak{A}$ is a $\{U\}$-structure, $|U^{\mathfrak{A}}| \equiv 0 \pmod{k}$ and $\overline{a} \in (\mathrm{Dom}(\mathfrak{A}) \setminus U^{\mathfrak{A}})^{|\overline{x}|}$ are distinct element, then $(\mathfrak{A}, \overline{a}/\overline{x}) \models (\theta \circ \psi_{c,t_0}) \leftrightarrow (\theta \circ \psi'_{c,t_0})$ for all c such that $c_{t_0} \geq n$. If $|U^{\mathfrak{A}}| \not\equiv 0 \pmod{k}$, this has to fail for some $t_0 \in S$ and $c \in (n+k)^S$ with $c_{t_0} \geq n$, because otherwise k would not be the smallest period of Q.

This gives us a way to express D_k in FO(Q): We first quantify distinct values for variables $\overline{x}$ with $|\overline{x}| = (n+2k)^{|S|}$ so that the elements are not in $U^{\mathfrak{A}}$. If $\mathrm{Dom}(\mathfrak{A}) \setminus U^{\mathfrak{A}}$ is too small, we can redefine $U^{\mathfrak{A}}$ by removing some multiple of k elements from it. Then we test that sentence $(\theta \circ \psi_{c,t_0}) \leftrightarrow (\theta \circ \psi'_{c,t_0})$ holds for all $t_0 \in S$ and $c \in (n+2k)^{|S|}$ with $c_{t_0} \geq n$. We have to define D_k separately for structures with less than $(n+2k)^{|S|} + k$ elements. □

The logic $\mathrm{FO}(D_k)_{k \in \mathbb{N}}$ is by the previous propositions a candidate for the strongest extension of FO with unary quantifiers that has weak uniform reduction. It has also all other nice properties of FO. Before showing them, we give a characterization of the logic. Because every $\mathrm{FO}(D_k)_{k \in \mathbb{N}}$-sentence can be expressed in $\mathrm{FO}(D_k)$ for some k, it suffices to characterize $\mathrm{FO}(D_k)$.

Let $\mathrm{FO}_n(D_k)$ be a fragment defined recursively so that $\mathrm{FO}_0(D_k) \equiv \mathrm{QF}$ and $\mathrm{FO}_{n+1}(D_k) \equiv \mathrm{QF} \circ (\mathrm{FO}_n(D_k) \cup \bigcup_{i<k} \mathcal{L}_{D_{1,k,i}} \circ \mathrm{FO}_n(D_k))$. It is easy to see that $\bigcup_{n \in \mathbb{N}} \mathrm{FO}_n(D_k) \equiv \mathrm{FO}(D_k)$. Let $S_n^k[\tau]$ be the set of all 1-types of $\mathrm{FO}_n(D_k)$ on vocabulary τ and for all $t \in S_n^k[\tau]$, let $t(\mathfrak{A}) = \{a \in \mathrm{Dom}(\mathfrak{A}) \mid \mathfrak{A} \models t(a)\}$. We have now

$$\begin{aligned}
\mathfrak{A} \equiv_{\mathrm{FO}_{n+1}(D_k)} \mathfrak{B} \quad &\Longleftrightarrow \quad \mathfrak{A} \equiv_{\mathrm{QF} \circ (\mathrm{FO}_n(D_k) \cup \bigcup_{i<k} \mathcal{L}_{D_{1,k,i}} \circ \mathrm{FO}_n(D_k))} \mathfrak{B} \\
&\Longleftrightarrow \quad \mathfrak{A} \equiv_{\mathrm{FO}_n(D_k) \cup \bigcup_{i<k} \mathcal{L}_{D_{1,k,i}} \circ \mathrm{FO}_n(D_k)} \mathfrak{B} \\
&\Longleftrightarrow \quad \bigwedge_{t \in S_n^k[\tau]} \mathrm{cut}_{1,k}(|t(\mathfrak{A})|) = \mathrm{cut}_{1,k}(|t(\mathfrak{B})|),
\end{aligned}$$

because every 1-type of $\mathrm{FO}_n(D_k)$ already determines the $\equiv_{\mathrm{FO}_n(D_k)}$-equivalence class of the structure.

Proposition 6.19. *The logic* $\mathrm{FO}(D_k)_{k\in K}$ *is vectorized regular and has uniform reduction for all* $K \subseteq \mathbb{Z}_+$.

Proof. We show first vectorized regularity. It suffices to show that the relativization and all vectorizations of the query D_k are definable in $\mathrm{FO}(D_k)$. The relativization is defined by

$$(A, U^{\mathfrak{A}}, R^{\mathfrak{A}}) \in D_k^U \iff (A, U^{\mathfrak{A}}, R^{\mathfrak{A}}) \models D_k x(Ux \wedge Rx).$$

For vectorization we need some auxiliary sentences. First, the $\{R\}$-query $D_{k,l}^n$ with semantics

$$\mathfrak{A} \in D_{k,l}^n \iff |R^{\mathfrak{A}}| \equiv l \pmod{k},$$

where $\mathrm{ar}(R) = n$, is definable in $\mathrm{FO}(D_k^{(n)})$. The definition is

$$\exists \overline{x}_0 \ldots \overline{x}_{l-1} \left(\bigwedge_{i<l} R\overline{x}_i \wedge \bigwedge_{i<j<l} \overline{x}_i \neq \overline{x}_j \wedge D_k^{(n)} \overline{y}(R\overline{y} \wedge \bigwedge_{1\leq i\leq l} \overline{y} \neq \overline{x}_i) \right).$$

Now nth vectorization of D_k can be defined recursively

$$\mathfrak{A}^{(n)} \in D_k \iff \mathfrak{A} \in D_k^{(n)}$$

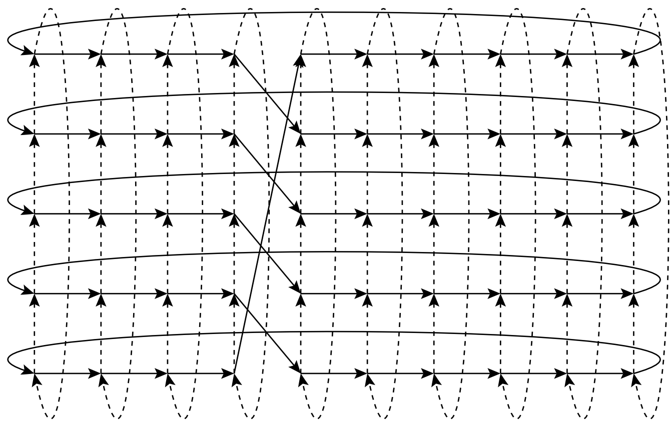
$$\iff \mathfrak{A} \models \bigvee_{\substack{\bar{l}\in\{0,\ldots,k-1\}^k \\ \sum i l_i \equiv 0 \pmod{k}}} (D_{k,l_1} x D_{k,1}^{n-1} \overline{y} Rx\overline{y} \wedge \cdots \wedge D_{k,l_{k-1}} x D_{k,k-1}^{n-1} \overline{y} Rx\overline{y}).$$

We show then that the logic has uniform reduction by showing that $\mathrm{FO}_n(D_k)[\tau]$ has uniform reduction for all $n \in \mathbb{N}$. Because every $\mathrm{FO}(D_k)$-sentence can access only finitely many relations of the structure, we may assume that the vocabulary τ is finite. Then the set $\mathrm{FO}_n(D_k)[\tau]$ is also finite and uniform reduction follows if we can show that for all $\mathfrak{A} \equiv_{\mathrm{FO}_n(D_k)} \mathfrak{B}$ and $\mathfrak{C}$, we have $\mathfrak{A} \sqcup \mathfrak{C} \equiv_{\mathrm{FO}_n(D_k)} \mathfrak{B} \sqcup \mathfrak{C}$.

This is clearly true for $n = 0$. Assume it is true for n and let us show it for $n+1$. If $\mathfrak{A} \equiv_{\mathrm{FO}_{n+1}(D_k)} \mathfrak{B}$, then for all $t \in S_n^k[\tau]$ $\mathrm{cut}_{1,k}(|t(\mathfrak{A})|) = \mathrm{cut}_{1,k}(|t(\mathfrak{B})|)$. If $a \in \mathrm{Dom}(\mathfrak{A})$ and $b \in \mathrm{Dom}(\mathfrak{B})$ and $\mathrm{tp}^{\mathfrak{A}}_{\mathrm{FO}_n(D_k)}(a) = \mathrm{tp}^{\mathfrak{B}}_{\mathrm{FO}_n(D_k)}(b)$, then $(\mathfrak{A}, a/x) \equiv_{\mathrm{FO}_n(D_k)} (\mathfrak{B}, b/x)$ and by induction hypothesis $(\mathfrak{A} \sqcup \mathfrak{C}, a/x) \equiv_{\mathrm{FO}_n(D_k)} (\mathfrak{B} \sqcup \mathfrak{C}, b/x)$, i.e., $\mathrm{tp}^{\mathfrak{A}\sqcup\mathfrak{C}}_{\mathrm{FO}_n(D_k)}(a) = \mathrm{tp}^{\mathfrak{B}\sqcup\mathfrak{C}}_{\mathrm{FO}_n(D_k)}(b)$. Similarly, for all $c \in \mathrm{Dom}(\mathfrak{C})$, $\mathrm{tp}^{\mathfrak{A}\sqcup\mathfrak{C}}_{\mathrm{FO}_n(D_k)}(c) = \mathrm{tp}^{\mathfrak{B}\sqcup\mathfrak{C}}_{\mathrm{FO}_n(D_k)}(c)$. Thus for all $t \in S_n^k[\tau]$,

$$\begin{aligned}
\mathrm{cut}_{1,k}(|t(\mathfrak{A} \sqcup \mathfrak{C})|) &= \mathrm{cut}_{1,k}(|t(\mathfrak{A} \sqcup \mathfrak{C}) \cap \mathrm{Dom}(\mathfrak{A})| + |t(\mathfrak{A} \sqcup \mathfrak{C}) \cap \mathrm{Dom}(\mathfrak{C})|) \\
&= \mathrm{cut}_{1,k}(|t(\mathfrak{B} \sqcup \mathfrak{C}) \cap \mathrm{Dom}(\mathfrak{B})| + |t(\mathfrak{B} \sqcup \mathfrak{C}) \cap \mathrm{Dom}(\mathfrak{C})|) \\
&= \mathrm{cut}_{1,k}(|t(\mathfrak{B} \sqcup \mathfrak{C})|),
\end{aligned}$$

where we use the facts that $\mathrm{cut}_{1,k}(|t(\mathfrak{A}\sqcup\mathfrak{C})\cap\mathrm{Dom}(\mathfrak{A})|) = \mathrm{cut}_{1,k}(|t(\mathfrak{B}\sqcup\mathfrak{C})\cap\mathrm{Dom}(\mathfrak{B})|)$ and $|t(\mathfrak{A} \sqcup \mathfrak{C}) \cap \mathrm{Dom}(\mathfrak{C})| = |t(\mathfrak{B} \sqcup \mathfrak{C}) \cap \mathrm{Dom}(\mathfrak{C})|$. This shows that $\mathfrak{A} \sqcup \mathfrak{C} \equiv_{\mathrm{FO}_{n+1}(D_k)} \mathfrak{B} \sqcup \mathfrak{C}$. ∎



6.4. **Locality of** $\mathrm{FO}_<(D_k)$**.** Grohe and Schwentick have proven in [GS00] that $\mathrm{FO}_<$ is Gaifman-local. We consider in this section how this can be extended to stronger logics.

Assume our logic $\mathcal{L}$ is a regular order-invariant extension of FO. We saw in Section 6.2 that $\mathrm{FO}_<$ can express the query Q_k, which means that $\mathrm{FO}_< \circ \mathrm{QF}_\infty$ is not Gaifman-local. By Theorem 4.16, if $\mathcal{L}$ is Hanf-local, then $\mathcal{L}$ has to have weak uniform reduction. In fact, we could prove this also for Gaifman-locality by replacing $\leftrightarrows_r$ by $\leftrightsquigarrow_{2,r}$ in Lemma 4.15. This shows that if we hope for $\mathcal{L}$ to be local, it should have weak uniform reduction.

By Propositions 6.17 and 6.18, if FO(Q) is regular, has uniform reduction and Q is a unary quantifier, then $\mathrm{FO(Q)} \equiv \mathrm{FO}(D_k)$ for some k. Thus it is natural to consider the locality of the logics $\mathrm{FO}_<(D_k)$. However, these logics fail to have even Gaifman-locality.

Proposition 6.20. $\mathrm{FO}_<(D_k)$ *is not Gaifman-local for any* $k \geq 2$.

Proof. Let $\tau = \{S, T\}$, where both S and T are binary relation symbols. Let M_k be a class of τ-structures defined by the following sentences

$$\forall x(\exists^1 ySxy \wedge \exists^1 yTxy \wedge \exists^1 ySyx \wedge \exists^1 yTyx)$$

$$\forall v_0 \exists v_1 \dots v_{k-1} \left(\bigwedge_{0\leq i<j\leq k-1} v_i \neq v_j \wedge Tv_0v_1 \wedge \dots \wedge Tv_{k-2}v_{k-1} \wedge Tv_{k-1}v_0 \right)$$

$$\forall x \exists y x' y'(Sxx' \wedge Syy' \wedge Txy \wedge Tx'y').$$

These sentences express that S and T are graphs of permutations s and t, every orbit of t contains exactly k elements and $s \circ t = t \circ s$.

Now, consider the following $\mathrm{FO}(D_k)[\tau \cup \{<\}]$ sentence

$$\varphi \equiv D_k z \psi(z),$$

where

$$\begin{aligned}
\theta_i(x,y) &\equiv \exists v_0 \dots v_i (v_0 = x \wedge v_i = y \wedge Tv_0v_1 \wedge \cdots \wedge Tv_{i-1}v_i) \\
\theta(x,y) &\equiv \bigvee_{0 \le i < k} \theta_i(x,y) \\
\rho(x) &\equiv \forall y(\theta(x,y) \to y \ge x) \\
\psi(z) &\equiv \exists xyx' \left(\rho(x) \wedge Sxy \wedge \rho(x') \wedge \bigvee_{0 \le i < j < k-1} \theta_i(y,z) \wedge \theta_j(y,x') \right).
\end{aligned}$$

Formula $\theta(x,y)$ expresses that x and y belong to the same orbit of t. Formula $\rho(x)$ says that x is the smallest element of its t-orbit. We explain formula ψ next.

Given $\mathfrak{A} \in M_k$ and a set $X \subseteq \mathrm{Dom}(\mathfrak{A})$ such that X contains exactly one element from every orbit of t, let $f_X^{\mathfrak{A}} \colon X \to k$ be the unique function such that $t^{f_X^{\mathfrak{A}}(x)}(s(x)) \in X$ for every $x \in X$. Formula ψ is defined such that $|\psi(\mathfrak{A})| = \sum_{x \in \rho(\mathfrak{A})} f_{\rho(\mathfrak{A})}^{\mathfrak{A}}(x)$ and so $(\mathfrak{A}, <) \models \varphi$, if and only if $\sum_{x \in \rho(\mathfrak{A})} f_{\rho(\mathfrak{A})}^{\mathfrak{A}}(x) \equiv 0 \pmod{k}$.

Let $a, b \in X$ such that $s(a)$ and b belong to the same t-orbit. Let $Y = (X \setminus \{b\}) \cup t(b)$. Then $f_X^{\mathfrak{A}} \restriction (X \setminus \{a,b\}) = f_Y^{\mathfrak{A}} \restriction (X \setminus \{a,b\})$, $f_X^{\mathfrak{A}}(a) \equiv f_Y^{\mathfrak{A}}(a) + 1 \pmod{k}$ and $f_X^{\mathfrak{A}}(b) \equiv f_Y^{\mathfrak{A}}(t(b)) - 1 \pmod{k}$. Thus $\sum_{x \in X} f_X^{\mathfrak{A}}(x) \equiv \sum_{x \in Y} f_Y^{\mathfrak{A}}(x) \pmod{k}$. Because the sum is invariant under changing one element of the set X, it is invariant under any changes. This shows that $\sum_{x \in \rho(\mathfrak{A})} f_{\rho(\mathfrak{A})}^{\mathfrak{A}}(x)$ does not depend modulo k on the linear order $<$ and the sentence φ is order-invariant.

Let $\mathfrak{A}_{n,k,h}$ be a τ-structure with universe $n \times k$, so that

$$T^{\mathfrak{A}_{n,k,h}} = \{\langle (i,j),(i,j') \rangle \mid i \in n, j \in k, j' \in k, j' \equiv j + 1 \pmod{k}\}$$

and

$$\begin{aligned}
S^{\mathfrak{A}_{n,k,h}} = &\{\langle (i,j),(i+1,j) \rangle \mid i \in (n-1), j \in k\} \\
&\cup \{\langle (n-1,j),(0,j') \rangle \mid j \in k, j' \in k, j' \equiv j + h \pmod{k}\}.
\end{aligned}$$

Because $\sum_{x \in n \times \{0\}} f_{n \times \{0\}}^{\mathfrak{A}_{n,k,h}} = h$, for all $n \in \mathbb{Z}_+$, $\mathfrak{A}_{n,k,0} \not\equiv_\varphi \mathfrak{A}_{n,k,1}$. However it is easy to see that $\mathfrak{A}_{4r+2,k,0} \leftrightsquigarrow_{2,r} \mathfrak{A}_{4r+2,k,1}$ by dividing the domain of $\mathfrak{A}_{4r+2,k,0}$ to two parts both containing the same number of consecutive t-orbits. Hence $\mathrm{FO}_<(D_k)$ is not Gaifman-local. □

We will see in the next section a weak version of Gaifman-locality that explicitly excludes the equivalence we used in the previous example and so applies to $\mathrm{FO}(D_k)$. This leads however to a definition that is hard to use in practice. A simple idea to weaken Gaifman-locality using the original definition is to forbid that the neighborhoods of the constants intersect when Gaifman-locality is applied.

Definition 6.21. $\mathcal{L}$ is *symmetrically Gaifman-local*, if for every $\varphi \in \mathcal{L}[\tau \cup \{x,y\}]$, there exists $r \in \mathbb{N}$ such that, if $\mathfrak{A} \in \mathrm{Mod}(\tau)$, $a, b \in \mathrm{Dom}(\mathfrak{A})$, $N_r^{\mathfrak{A}}(a) \cap N_r^{\mathfrak{A}}(b) = \emptyset$ and $\mathfrak{N}_r^{\mathfrak{A}}(a) \cong \mathfrak{N}_r^{\mathfrak{A}}(b)$, then $(\mathfrak{A}, a/x, b/y) \equiv_\varphi (\mathfrak{A}, b/x, a/y)$.

The logic $\mathrm{FO}_<(D_k)$ satisfies this definition, if k is odd, but fails to satisfy it, if k is even as we next show.

Let $\tau = \{F\}$, where F is a binary relation symbol. Let EvenP be a query containing all τ-structures $\mathfrak{A}$ where $F^{\mathfrak{A}}$ is a graph of an even permutation $f\colon \mathrm{Dom}(\mathfrak{A}) \to \mathrm{Dom}(\mathfrak{A})$.

Proposition 6.22. *The query* EvenP *is definable in* $\mathrm{FO}_<(D_2)$.

Proof. We can express in FO that $F^{\mathfrak{A}}$ is a graph of a permutation on $\mathrm{Dom}(\mathfrak{A})$, so we have to find only a $\mathrm{FO}_<(D_2)$-sentence separating even and odd permutations.

Let f be a permutation of a finite set A ordered by $<$. A pair $(a,b) \in A^2$ is called *an inversion pair*, if $a < b$ but $f(a) > f(b)$. It is well known fact that f is an even permutation if and only if it has even number of inversion pairs. We can express this definition in $\mathrm{FO}_<(D_2)$:

$$D_2 x \neg D_2 y (x < y \wedge \exists x' y' (Fxx' \wedge Fyy' \wedge y' < x')).$$

□

Proposition 6.23. FO(EvenP) *is not symmetrically Gaifman-local.*

Proof. Let $\tau = \{E, U, V\}$, where E is binary and U and V are unary relation symbols. Let $\mathfrak{A}_n$ be a τ-structure with universe $n \times 2$, $E^{\mathfrak{A}_n} = \{((i,j),(i+1,j)) \mid 0 \le i < n-1, 0 \le j \le 1\}$, $U^{\mathfrak{A}_n} = \{(0,0)\}$ and $V^{\mathfrak{A}_n} = \{(0,1)\}$. If $n > r$, then $\mathfrak{N}_r^{\mathfrak{A}_n}((n-1,0)) \cong \mathfrak{N}_r^{\mathfrak{A}_n}((n-1,1))$ and these neighborhoods do not intersect.

Let $\psi \in I(\mathrm{QF}, \tau \cup \{u,v\}, \{F\})$, where $\psi_F(x,y) = Exy \vee (x = u \wedge Uy) \vee (x = v \wedge Vy)$. If FO(EvenP) were symmetrically Gaifman-local, then for big enough n, $(\mathfrak{A}_n, (n-1,0)/u, (n-1,1)/v) \equiv_{\mathrm{EvenP}\circ\psi} (\mathfrak{A}_n, (n-1,1)/u, (n-1,0)/v)$. However, F is a graph of an even permutation in only one of the structures $\psi^*((\mathfrak{A}_n, (n-1,0)/u, (n-1,1)/v))$ and $\psi^*((\mathfrak{A}_n, (n-1,1)/u, (n-1,0)/v))$. Hence FO(EvenP) cannot be symmetrically Gaifman-local. □

In order to avoid the previous example, we weaken the definition of symmetrical Gaifman-locality further.

Definition 6.24. The logic $\mathcal{L}$ is *alternatingly Gaifman-local*, if for every $\varphi \in \mathcal{L}[\tau \cup \{x,y,z\}]$, there exists $r \in \mathbb{N}$ such that, if $\mathfrak{A} \in \mathrm{Mod}(\tau)$, $a,b,c \in \mathrm{Dom}(\mathfrak{A})$, $N_r^{\mathfrak{A}}(a) \cap N_r^{\mathfrak{A}}(b) = N_r^{\mathfrak{A}}(b) \cap N_r^{\mathfrak{A}}(c) = N_r^{\mathfrak{A}}(c) \cap N_r^{\mathfrak{A}}(a) = \emptyset$ and $\mathfrak{N}_r^{\mathfrak{A}}(a) \cong \mathfrak{N}_r^{\mathfrak{A}}(b) \cong \mathfrak{N}_r^{\mathfrak{A}}(c)$, then $(\mathfrak{A}, a/x, b/y, c/z) \equiv_\varphi (\mathfrak{A}, b/x, c/y, a/z)$.

It is easy to see that symmetrical Gaifman-locality implies alternating Gaifman-locality. We show in the following sections that $\mathrm{FO}_<(D_k)_{k \in \mathbb{Z}_+}$ has the property.

6.5. **Ribbons.** We define in this section some machinery for proving symmetrical and alternating Gaifman-locality of the logics $\mathrm{FO}(D_k)$. The basic idea of representing relevant information of structures by word models is from [GS00].

Definition 6.25. A $\tau \cup \{<, \sim\}$-structure $\mathfrak{A}$ is *a τ-ribbon*, if $<^{\mathfrak{A}}$ is a linear order, $\sim^{\mathfrak{A}}$ is an equivalence relation, and its equivalence classes can be enumerated as

$C_0^{\mathfrak{A}}, \ldots, C_{m-1}^{\mathfrak{A}}$ such that for all $0 \leq i < j < m$, $a \in C_i^{\mathfrak{A}}$ and $b \in C_j^{\mathfrak{A}}$, we have $a < b$ and for all $R \in \tau$ and $\overline{a} \in R^{\mathfrak{A}}$, either $[\overline{a}] \subseteq C_i^{\mathfrak{A}} \cup C_{i+1}^{\mathfrak{A}}$ for some $i < m-1$ or $[\overline{a}] \subseteq C_0^{\mathfrak{A}} \cup C_{m-1}^{\mathfrak{A}}$

If $\mathfrak{A}$ is a τ-ribbon, there is clearly only one way to enumerate the classes $C_i^{\mathfrak{A}}$ satisfying the definition.

Given a τ-ribbon $\mathfrak{A}$ with m equivalence classes and $0 \leq i < m$, let $\mathfrak{T}_r^{\mathfrak{A}}(i) = \left\langle \bigcup_{i-r \leq j \leq i+r} C_j^{\mathfrak{A}} \right\rangle^{(\mathfrak{A}, C_{i-r}^{\mathfrak{A}}/S_{-r}, \ldots, C_{i+r}^{\mathfrak{A}}/S_r)}$, where the subscripts of $C^{\mathfrak{A}}$ are counted modulo m. Note that $\mathfrak{T}_r^{\mathfrak{A}}(i)$ is a partial $\tau \cup \{S_{-r}, \ldots, S_r\}$-structure with some constants possibly undefined.

Let $T_n^k[\tau]$ be the set of all $\equiv_{\mathrm{FO}_n(D_k)}$-equivalence classes on $\bigcup_{X \subseteq \mathrm{Con}(\tau)} \mathrm{Mod}(\mathrm{Rel}(\tau) \cup \{S_{-r}, \ldots, S_r\} \cup X)$.

An n,k-reduct $\mathfrak{R}_n^k(\mathfrak{A})$ of a τ-ribbon $\mathfrak{A}$ with m equivalence classes is a $\{<\} \cup \{U_t \mid t \in T_n^k[\tau \cup \{S_{-3^n}, \ldots, S_{3^n}\}]\}$-structure. Its universe is m, $<^{\mathfrak{R}_n^k(\mathfrak{A})}$ is a natural ordering of m and $U_t^{\mathfrak{R}_n^k(\mathfrak{A})} = \{i < n \mid \mathfrak{T}_{3^n}^{\mathfrak{A}}(i) \in t\}$, where we identify a partial $\tau \cup \{S_{-3^n}, \ldots, S_{3^n}\}$-structure with constants $X \subseteq \mathrm{Con}(\tau)$ defined with a complete $\mathrm{Rel}(\tau) \cup \{S_{-3^n}, \ldots, S_{3^n}\} \cup X$-structure.

Let $i^{\mathfrak{A}} \colon \mathrm{Dom}(\mathfrak{A}) \to m$ be a function such that for all $a \in \mathrm{Dom}(\mathfrak{A})$, $a \in C_{i^{\mathfrak{A}}(a)}^{\mathfrak{A}}$. As usual $i^{\mathfrak{A}}(\overline{a}) = (i^{\mathfrak{A}}(a_j))_{j<|\overline{a}|}$. We can prove now that the n,k-reduct completely characterizes the $\equiv_{\mathrm{FO}_n(D_k)}$-class of the structure.

Lemma 6.26. *Let $\mathfrak{A}$ and $\mathfrak{B}$ be τ-ribbons, $\overline{a} \in \mathrm{Dom}(\mathfrak{A})^l$ and $\overline{b} \in \mathrm{Dom}(\mathfrak{B})^l$. If*

$$(\mathfrak{R}_n^k(\mathfrak{A}), i^{\mathfrak{A}}(\overline{a})/\overline{x}) \equiv_{\mathrm{FO}_{2n}(D_k)} (\mathfrak{R}_n^k(\mathfrak{B}), i^{\mathfrak{B}}(\overline{b})/\overline{x})$$

and for all $j < l$,

$$\mathfrak{T}_{3^n}^{(\mathfrak{A},\overline{a}/\overline{x})}(i^{\mathfrak{A}}(a_j)) \equiv_{\mathrm{FO}_n(D_k)} \mathfrak{T}_{3^n}^{(\mathfrak{B},\overline{b}/\overline{x})}(i^{\mathfrak{B}}(b_j)),$$

then $(\mathfrak{A}, \overline{a}/\overline{x}) \equiv_{\mathrm{FO}_n(D_k)} (\mathfrak{B}, \overline{b}/\overline{x})$.

Proof. The proof is by induction. Assume first $n = 0$ and that the conditions of the lemma hold. Then $\mathrm{FO}_n(D_n) \equiv \mathrm{QF}$ and $3^n = 1$. All tuples in $R^{\mathfrak{A}}$ are present in some $\mathfrak{T}_1^{\mathfrak{A}}(i)$ and so the second condition implies that $\mathrm{atp}^{\mathfrak{A} \restriction \tau \cup \{\sim\}}(\overline{a}) = \mathrm{atp}^{\mathfrak{B} \restriction \tau \cup \{\sim\}}(\overline{b})$ and that the elements in the same $\sim^{\mathfrak{A}}$-class are ordered in the same way as the corresponding elements in $\mathfrak{B}$. The first condition implies that also elements in different classes are ordered similarly in both structures. Therefore $(\mathfrak{A}, \overline{a}/\overline{x}) \equiv_{\mathrm{QF}} (\mathfrak{B}, \overline{b}/\overline{x})$.

Assume then that $n > 0$ and the lemma has been proven for all smaller values of n. Let $(\mathfrak{A}, \overline{a}/\overline{x})$ and $(\mathfrak{B}, \overline{b}/\overline{x})$ be structures satisfying the conditions of the lemma.

Claim. If $\mathrm{tp}_{\mathrm{FO}_{2n-1}(D_k)}^{(\mathfrak{R}_n^k(\mathfrak{A}), i^{\mathfrak{A}}(\overline{a})/\overline{x})}(p) = \mathrm{tp}_{\mathrm{FO}_{2n-1}(D_k)}^{(\mathfrak{R}_n^k(\mathfrak{B}), i^{\mathfrak{B}}(\overline{b})/\overline{x})}(q)$, then for all $t \in T_{n-1}^k[\tau \cup \{<, \sim\}]$, we have $\mathrm{cut}_{1,k}(|t((\mathfrak{A}, \overline{a}/\overline{x})) \cap C_p^{\mathfrak{A}}|) = \mathrm{cut}_{1,k}(|t((\mathfrak{B}, \overline{b}/\overline{x})) \cap C_q^{\mathfrak{B}}|)$.

Proof. Assume $\mathrm{tp}_{\mathrm{FO}_{2n-1}(D_k)}^{(\mathfrak{R}_n^k(\mathfrak{A}), i^{\mathfrak{A}}(\overline{a})/\overline{x})}(p) = \mathrm{tp}_{\mathrm{FO}_{2n-1}(D_k)}^{(\mathfrak{R}_n^k(\mathfrak{B}), i^{\mathfrak{B}}(\overline{b})/\overline{x})}(q)$. If for some $j < |\overline{x}|$, $|i^{\mathfrak{A}}(a_j) - p| \leq 3^{n-1}$, then $i^{\mathfrak{A}}(a_j) - p = i^{\mathfrak{B}}(b_j) - q$.

We claim first that $\mathfrak{T}^{(\mathfrak{A},\overline{a}/\overline{x})}_{2\cdot 3^{n-1}}(p) \equiv_{\mathrm{FO}_n(D_k)} \mathfrak{T}^{(\mathfrak{B},\overline{b}/\overline{x})}_{2\cdot 3^{n-1}}(q)$. If the structures do not contain constants from $\overline{x}$, this follows from the fact that p and q satisfy the same unary relations on $\mathfrak{R}^k_n(\mathfrak{A})$ and $\mathfrak{R}^k_n(\mathfrak{B})$ and they encode the $\equiv_{\mathrm{FO}_n(D_k)}$ class of the $\mathfrak{T}$-structures. Otherwise, $\mathfrak{T}^{(\mathfrak{A},\overline{a}/\overline{x})}_{2\cdot 3^{n-1}}(p)$ is a substructure of $\mathfrak{T}^{(\mathfrak{A},\overline{a}/\overline{x})}_{3^n}(i^{\mathfrak{A}}(a_j))$ for some $j < |\overline{x}|$ and because the former is $\mathrm{FO}_n(D_k)$-equivalent to $\mathfrak{T}^{(\mathfrak{A},\overline{a}/\overline{x})}_{3^n}(i^{\mathfrak{A}}(b_j))$, and we can define the substructure using unary relations S_r coding different equivalence classes, we also get the claimed equivalence.

Now, for all $t \in T^k_{n-1}[\tau \cup \{<, \sim\}]$,

$$\mathrm{cut}_{1,k}(|t(\mathfrak{T}^{(\mathfrak{A},\overline{a}/\overline{x})}_{2\cdot 3^{n-1}}(p)) \cap C^{\mathfrak{A}}_p|) = \mathrm{cut}_{1,k}(|t(\mathfrak{T}^{(\mathfrak{B},\overline{b}/\overline{x})}_{2\cdot 3^{n-1}}(q)) \cap C^{\mathfrak{B}}_q|).$$

We have proven the claim, if we can show that for all $a' \in C^{\mathfrak{A}}_p$ and $b' \in C^{\mathfrak{B}}_q$ such that

$$\mathrm{tp}^{\mathfrak{T}^{(\mathfrak{A},\overline{a}/\overline{x})}_{2n}(p)}_{\mathrm{FO}_{n-1}(D_k)}(a') = \mathrm{tp}^{\mathfrak{T}^{(\mathfrak{B},\overline{b}/\overline{x})}_{2n}(q)}_{\mathrm{FO}_{n-1}(D_k)}(b'),$$

we have

$$\mathrm{tp}^{(\mathfrak{A},\overline{a}/\overline{x})}_{\mathrm{FO}_{n-1}(D_k)}(a') = \mathrm{tp}^{(\mathfrak{B},\overline{b}/\overline{x})}_{\mathrm{FO}_{n-1}(D_k)}(b').$$

This can be shown using the induction hypothesis. By the assumption of the claim,

$$(\mathfrak{R}^k_n(\mathfrak{A}), i^{\mathfrak{A}}(\overline{a}a')/\overline{x}) \equiv_{\mathrm{FO}_{2n-1}(D_k)} (\mathfrak{R}^k_n(\mathfrak{B}), i^{\mathfrak{B}}(\overline{b}b')/\overline{x}),$$

which implies

$$(\mathfrak{R}^k_{n-1}(\mathfrak{A}), i^{\mathfrak{A}}(\overline{a}a')/\overline{x}) \equiv_{\mathrm{FO}_{2(n-1)}(D_k)} (\mathfrak{R}^k_{n-1}(\mathfrak{B}), i^{\mathfrak{B}}(\overline{b}b')/\overline{x}).$$

Clearly,

$$\mathfrak{T}^{(\mathfrak{A},\overline{a}a'/\overline{x}y)}_{3^{n-1}}(p) \equiv_{\mathrm{FO}_{n-1}(D_k)} \mathfrak{T}^{(\mathfrak{B},\overline{b}b'/\overline{x}y)}_{3^{n-1}}(q).$$

For all $j < |\overline{x}|$, either the new constant a' is not in $\mathfrak{T}^{(\mathfrak{A},\overline{a}a'/\overline{x}y)}_{3^{n-1}}(i^{\mathfrak{A}}(a_j))$ or it is a substructure of $\mathfrak{T}^{(\mathfrak{A},\overline{a}/\overline{x})}_{2\cdot 3^{n-1}}(p)$. In both cases, we get the equivalence

$$\mathfrak{T}^{(\mathfrak{A},\overline{a}a'/\overline{x}y)}_{3^{n-1}}(i^{\mathfrak{A}}(a_j)) \equiv_{\mathrm{FO}_{n-1}(D_k)} \mathfrak{T}^{(\mathfrak{B},\overline{b}b'/\overline{x}y)}_{3^{n-1}}(i^{\mathfrak{B}}(b_j)).$$

So the conditions of the lemma are satisfied for $n-1$. □

Because

$$(\mathfrak{R}^k_n(\mathfrak{A}), i^{\mathfrak{A}}(\overline{a})/\overline{x}) \equiv_{\mathrm{FO}_{2n}(D_k)} (\mathfrak{R}^k_n(\mathfrak{B}), i^{\mathfrak{B}}(\overline{b})/\overline{x}),$$

we have for all $t \in T^k_{2n-1}[\{<\} \cup \{U_t\}]$,

$$\mathrm{cut}_{1,k}(|t((\mathfrak{R}^k_n(\mathfrak{A}), i^{\mathfrak{A}}(\overline{a})/\overline{x}))|) = \mathrm{cut}_{1,k}(|t((\mathfrak{R}^k_n(\mathfrak{B}), i^{\mathfrak{B}}(\overline{b})/\overline{x}))|).$$

Combined with the previous claim, this implies for all $t \in T^k_{n-1}[\tau \cup \{<, \sim\}]$,

$$\mathrm{cut}_{1,k}(|t((\mathfrak{A}, \overline{a}/\overline{x}))|) = \mathrm{cut}_{1,k}(|t((\mathfrak{B}, \overline{b}/\overline{x}))|),$$

which proves the lemma. □

Corollary 6.27. *If* $\mathfrak{R}^k_n(\mathfrak{A}) \equiv_{\mathrm{FO}_{2n}(D_k)} \mathfrak{R}^k_n(\mathfrak{B})$*, then* $\mathfrak{A} \equiv_{\mathrm{FO}_n(D_k)} \mathfrak{B}$.

Definition 6.28. Let $\mathfrak{A}$ be an ordered structure. A subset $I \subseteq \mathrm{Dom}(\mathfrak{A})$ is $FO_n(D_k)$-*indiscernible*, if for all $<^{\mathfrak{A}}$-ordered sequences $\overline{a}, \overline{b} \in I^{<\omega}$ such that $k = |\overline{a}| = |\overline{b}|$, we have $\mathrm{tp}^{\mathfrak{A}}_{FO_n(D_k)}(\overline{a}) = \mathrm{tp}^{\mathfrak{A}}_{FO_n(D_k)}(\overline{b})$.

Lemma 6.29. *For all $m, n \in \mathbb{N}$ and $k \in \mathbb{Z}_+$, there exists n_0 such that if $\mathfrak{A}$ is an ordered structure and $J \subseteq \mathrm{Dom}(\mathfrak{A})$ with $|J| \geq n_0$, then there is a $\mathrm{FO}_n(D_k)$-indiscernible set $I \subseteq J$ with $|I| = m$.*

Proof. A standard argument using Ramsey's theorem. □

It is well known that first-order logic cannot separate two long enough linear orders. Sentences of $\mathrm{FO}_n(D_k)$ cannot separate them, if they have additionally the same length modulo k^n [Nur00].

Lemma 6.30. *Let $\mathfrak{A}, \mathfrak{B} \in \mathrm{Ord}(\emptyset)$. If $\mathrm{cut}_{k^n,k^n}(\mathfrak{A}) = \mathrm{cut}_{k^n,k^n}(\mathfrak{B})$, then $\mathfrak{A} \equiv_{\mathrm{FO}_n(D_k)} \mathfrak{B}$.* □

Lemma 6.31. *Let I be a $\mathrm{FO}_n(D_k)$-indiscernible set in a word model $\mathfrak{A}$. Then for all $Y_0, Y_1 \subseteq I$, such that $\mathrm{cut}_{2k^{2n},k^{2n}}(|Y_0|) = \mathrm{cut}_{2k^{2n},k^{2n}}(|Y_1|)$, we have $(\mathfrak{A}, Y_0/U) \equiv_{\mathrm{FO}_n(D_k)} (\mathfrak{A}, Y_1/U)$.*

Proof. Define an equivalence relation $\sim_i$ on $(\mathfrak{A}, Y_i/U)$ such that $a \sim_i b$, if there are no elements of Y_i between a and b and if $a \in Y_i$, then $a \leq^{\mathfrak{A}} b$. and if $b \in Y_i$ then $b \leq^{\mathfrak{A}} a$. The structure $(\mathfrak{A}, Y_i/U, \sim_i/\sim)$ is a ribbon (excluding $<$, the structure $\mathfrak{A}$ contains only unary relations and $\sim_i$ is defined so that its equivalence classes are continuous), so it suffices to show that $\mathfrak{R}^k_n((\mathfrak{A}, Y_0/U, \sim_0/\sim)) \equiv_{\mathrm{FO}_{2n}(D_k)} \mathfrak{R}^k_n((\mathfrak{A}, Y_1/U, \sim_1/\sim))$.

Let $\overline{y} = y_{-3^n} \dots y_{3^n+1}$, $\theta(x) \equiv y_{-3^n} \leq x < y_{3^n+1}$ and $\psi \in I(\mathrm{QF}, \tau \cup [\overline{y}], \tau \cup \{\sim, U\})$ such that $\psi_U(x) \equiv \bigvee_{-3^n \leq j \leq 3^n} x = y_j$, for all $R \in \tau$, $\psi_R(\overline{x}) \equiv R\overline{x}$, and $\psi_\sim(x, x') \equiv \bigvee_{-3^n \leq j \leq 3^n} y_j \leq x < y_{j+1} \land y_j \leq x' < y_{j+1}$. Let $\overline{b}^i$ be a sequence enumerating Y_i in ascending order. If $k^n < j < |Y_i| - k^n$, we have now $\mathfrak{T}^{(\mathfrak{A}, Y_i/U, \sim_i/\sim)}_{3^n}(j) = \langle \theta(\mathfrak{A}, b^i_{j-3^n} \dots b^i_{j+3^n+1}/\overline{y}) \rangle^{\psi^*(\mathfrak{A}, b^i_{j-3^n} \dots b^i_{j+3^n+1}/\overline{y})}$

The structures $(\mathfrak{A}, b^i_{j-3^n} \dots b^i_{j+3^n+1}/\overline{y})$ and $(\mathfrak{A}, b^{i'}_{j'-3^n} \dots b^{i'}_{j'+3^n+1}/\overline{y})$ are $\mathrm{FO}_n(D_k)$-equivalent for all $i, i' \in \{0, 1\}$, $k^n < j < |Y_i| - k^n$ and $k^n < j' < |Y_{i'}| - k^n$, since Y_0 and Y_1 are subsets of the same indiscernible set. Because a relativization of any $\mathrm{FO}_n(D_k)$-sentence is still in $\mathrm{FO}_n(D_k)$ and $\mathrm{FO}_n(D_k) \circ \mathrm{QF} \leq \mathrm{FO}_n(D_k)$, this gives $\mathfrak{T}^{(\mathfrak{A}, Y_i/U, \sim_i/\sim)}_{3^n}(j) \equiv_{\mathrm{FO}_n(D_k)} \mathfrak{T}^{(\mathfrak{A}, Y_{i'}/U, \sim_{i'}/\sim)}_{3^n}(j')$.

We can show similarly that

$$\mathfrak{T}^{(\mathfrak{A}, Y_0/U, \sim_0/\sim)}_{3^n}(j) \equiv_{\mathrm{FO}_n(D_k)} \mathfrak{T}^{(\mathfrak{A}, Y_1/U, \sim_1/\sim)}_{3^n}(j)$$

and

$$\mathfrak{T}^{(\mathfrak{A}, Y_0/U, \sim_0/\sim)}_{3^n}(|Y_0| - j) \equiv_{\mathrm{FO}_n(D_k)} \mathfrak{T}^{(\mathfrak{A}, Y_1/U, \sim_1/\sim)}_{3^n}(|Y_1| - j)$$

for all $j \leq 3^n$. Thus $\mathfrak{R}^k_n((\mathfrak{A}, Y_0/U, \sim_0/\sim))$ and $\mathfrak{R}^k_n((\mathfrak{A}, Y_1/U, \sim_1/\sim))$ have common prefix and suffix of length 3^n and all elements between them have the same atomic type. If $\mathrm{cut}_{2k^{2n},k^{2n}}(|Y_0|) = \mathrm{cut}_{2k^{2n},k^{2n}}(|Y_1|)$, Lemma 6.30 gives equivalence between the words without suffix and prefix. Concatenating $\mathrm{FO}_n(D_k)$-equivalent structures with a fixed structure keeps them $\mathrm{FO}_n(D_k)$-equivalent and so we get $\mathfrak{R}^k_n((\mathfrak{A}, Y_0/U, \sim_0/\sim)) \equiv_{\mathrm{FO}_{2n}(D_k)} \mathfrak{R}^k_n((\mathfrak{A}, Y_1/U, \sim_1/\sim))$. □

Lemma 6.32. *Let $n \in \mathbb{N}$ and $k, s \in \mathbb{Z}_+$ such that $\gcd(k, s) = 1$. Fix also a vocabulary τ. There exists an integer m such that if $\mathfrak{A}$ is a word model with vocabulary τ and $X \subseteq \mathrm{Dom}(\mathfrak{A})$ with $|X| = m$, then there are $Y_0, Y_1 \subseteq X$ such that $(\mathfrak{A}, Y_0/U) \equiv_{\mathrm{FO}_n(D_k)} (\mathfrak{A}, Y_1/U)$, $|Y_0| \equiv 0 \pmod s$ and $|Y_1| \equiv 1 \pmod s$.*

Proof. Choosing m big enough, we find a $\mathrm{FO}_n(D_k)$-indiscernible subset $I \subseteq X$ such that we can choose $Y_0, Y_1 \subseteq I$ satisfying $|Y_0| \equiv 0 \pmod s$, $|Y_1| \equiv 1 \pmod s$ and $\mathrm{cut}_{2k^{2n},k^{2n}}(|Y_0|) = \mathrm{cut}_{2k^{2n},k^{2n}}(|Y_1|)$. □

Given a τ-structure $\mathfrak{A}$, we call a partial structure $\mathfrak{B}$ *a change on* $\mathfrak{A}$ if $\mathrm{Dom}(\mathfrak{B}) \subseteq \mathrm{Dom}(\mathfrak{A})$ and the same constant symbols are defined in $\mathfrak{B}$ and $\langle \mathrm{Dom}(\mathfrak{B})\rangle^{\mathfrak{A}}$. We get a structure $\mathrm{Apply}_{\mathfrak{B}}(\mathfrak{A})$ from $\mathfrak{A}$ by applying the change $\mathfrak{B}$, where $\mathrm{Dom}(\mathrm{Apply}_{\mathfrak{B}}(\mathfrak{A})) = \mathrm{Dom}(\mathfrak{A})$, for all $R \in \mathrm{Rel}(\tau)$, $R^{\mathrm{Apply}_{\mathfrak{B}}(\mathfrak{A})} = R^{\mathfrak{B}} \cup \{\overline{a} \in R^{\mathfrak{A}} \mid [\overline{a}] \not\subseteq \mathrm{Dom}(\mathfrak{B})\}$ and for all $c \in \mathrm{Con}(\tau)$, if $c \in \mathrm{Dom}(\mathfrak{B})$, $c^{\mathrm{Apply}_{\mathfrak{B}}(\mathfrak{A})} = c^{\mathfrak{B}}$ and otherwise $c^{\mathrm{Apply}_{\mathfrak{B}}(\mathfrak{A})} = c^{\mathfrak{A}}$.

Two changes $\mathfrak{B}$ and $\mathfrak{B}'$ are disjoint, if $\mathrm{Dom}(\mathfrak{B}) \cap \mathrm{Dom}(\mathfrak{B}') = \emptyset$. If Y is a set of pairwise disjoint changes on $\mathfrak{A}$, we can apply them all in any order and get the same structure. We denote the structure got in this way by $\mathrm{Apply}_Y(\mathfrak{A})$.

If $\mathfrak{A}$ is a τ-ribbon, we call a change $\mathfrak{B}$ on $\mathfrak{A}$ *local*, if $(\sim)^{\mathfrak{B}} = (\sim)^{\langle \mathrm{Dom}(\mathfrak{B})\rangle^{\mathfrak{A}}}$, $(<)^{\mathfrak{B}} = (<)^{\langle \mathrm{Dom}(\mathfrak{B})\rangle^{\mathfrak{A}}}$ and $\mathrm{Dom}(\mathfrak{B}) = C^{\mathfrak{A}}_i \cup C^{\mathfrak{A}}_{i+1}$ for some i.

Proposition 6.33. *Fix $n \in \mathbb{N}$, $k, s \in \mathbb{Z}_+$ such that $\gcd(k, s) = 1$ and a vocabulary τ. Then there exists an integer m such that if $\mathfrak{A}$ is a τ-ribbon and Y is a set of disjoint local changes on $\mathfrak{A}$ with $|Y| \geq m$, there are $Y_0, Y_1 \subseteq Y$ such that $|Y_0| \equiv 0 \pmod s$, $|Y_1| \equiv 1 \pmod s$ and $\mathrm{Apply}_{Y_0}(\mathfrak{A}) \equiv_{\mathrm{FO}_n(D_k)} \mathrm{Apply}_{Y_1}(\mathfrak{A})$.*

Proof. If m is big enough, we can choose the sets Y_0 and Y_1 as follows. First, enumerate Y as $\{\mathfrak{B}_i \mid i \in J_0\}$ such that $\mathrm{Dom}(\mathfrak{B}_i) = C_i \cup C_{i+1}$. Choose $J_1 \subseteq J_0$ such that for all $i, j \in J_1$, $|i-j| > 2\cdot 3^n$. We can make the choice so that $|J_1| \geq |J_0|/(2\cdot 3^n)$.

The reducts $\mathfrak{R}^k_n(\mathfrak{A})$ and $\mathfrak{R}^k_n(\mathrm{Apply}_{\mathfrak{B}_i}(\mathfrak{A}))$ differ only on elements $\{i - 3^n, \ldots, i + 3^n + 1\}$. We have chosen J_1 so that these changes on reducts do not overlap. Now, choose $J_2 \subseteq J_1$ so that for all $i, j \in J_2$,

$$\langle\{i - 3^n, \ldots, i + 3^n + 1\}\rangle^{\mathfrak{R}^k_n(\mathrm{Apply}_{\mathfrak{B}_i}(\mathfrak{A}))} \cong \langle\{j - 3^n, \ldots, j + 3^n + 1\}\rangle^{\mathfrak{R}^k_n(\mathrm{Apply}_{\mathfrak{B}_j}(\mathfrak{A}))}.$$

This choice can be made so that $|J_2| \geq |J_1|/t^{1+2\cdot 3^n}$, where t is the number of atomic 1-types occurring on reducts and depends only on τ, n and k.

By the previous lemma, if m is big enough, there exists $K_0, K_1 \subseteq J_2$ such that $K_0 \equiv 0 \pmod s$, $K_1 \equiv 1 \pmod s$ and $(\mathfrak{R}^k_n(\mathfrak{A}), K_0/U) \equiv_{\mathrm{FO}_{4n}(D_k)} (\mathfrak{R}^k_n(\mathfrak{A}), K_1/U)$. Let $Y_0 = \{\mathfrak{B}_i \mid i \in K_0\}$ and $Y_1 = \{\mathfrak{B}_i \mid i \in K_1\}$.

Now, there exists a FO_{2n}-interpretation ψ satisfying for all $J' \subseteq J_2$,

$$\mathfrak{R}^k_n(\mathrm{Apply}_{\{\mathfrak{B}_i \mid i \in J'\}}(\mathfrak{A})) = \psi^*((\mathfrak{R}^k_n(\mathfrak{A}), J'/U)).$$

Thus

$$\mathfrak{R}^k_n(\mathrm{Apply}_{Y_0}(\mathfrak{A})) \equiv_{\mathrm{FO}_{2n}(D_k)} \mathfrak{R}^k_n(\mathrm{Apply}_{Y_1}(\mathfrak{A})),$$

which gives us $\mathrm{Apply}_{Y_0}(\mathfrak{A}) \equiv_{\mathrm{FO}_n(D_k)} \mathrm{Apply}_{Y_1}(\mathfrak{A})$. □

Given a set K of positive integers, let $\mathrm{PFC}(K)$ be the set of all positive integers whose prime factors occur as prime factors of some elements in K.

Proposition 6.34. *The logic* $\mathrm{FO}(D_k)_{k\in K}$ *is uniformly* $\mathrm{PFC}(K), g_{\tau,<}$*-periodic for all vocabularies* τ*. In particular,* $\mathrm{FO}_<(D_k)_{k\in K}$ *is uniformly* $\mathrm{PFC}(K), g_\tau$*-periodic.*

Proof. The proposition can be proved using Lemma 6.30 in the same way as in the proof of Proposition 6.33: Concatenation of many copies of the same structure gives us a ribbon whose almost all neighborhoods are isomorphic with each other. □

6.6. **Locality proofs.** As Proposition 6.20 showed, $\mathrm{FO}_<(D_k)$ is not Gaifman-local. We define next a weakening of Gaifman-locality, based on the characterization we gave in Section 5, that will avoid the example in Proposition 6.20.

Let $\mathfrak{A}$ and $\mathfrak{B}$ be τ-structures such that $\mathfrak{A} \leftrightsquigarrow_{2,r} \mathfrak{B}$. By the definition, there then exist partitions $\mathrm{Dom}(\mathfrak{A}) = A_0 \cup A_1$ and $\mathrm{Dom}(\mathfrak{B}) = B_0 \cup B_1$ and isomorphisms, $\alpha_0\colon \mathfrak{N}_r^{\mathfrak{A}}(A_0) \cong \mathfrak{N}_r^{\mathfrak{B}}(B_0)$ and $\alpha_1\colon \mathfrak{N}_r^{\mathfrak{A}}(A_1) \cong \mathfrak{N}_r^{\mathfrak{B}}(B_1)$. Denote this situation by $(A_0, A_1, \alpha_0, \alpha_1)\colon \mathfrak{A} \leftrightsquigarrow_{2,r} \mathfrak{B}$.

Given a permutation p, let $\mathrm{ord}(p)$ be the order of p, i.e., the least $k \in \mathbb{Z}_+$ such that $p^k = \mathrm{id}_{\mathrm{dom}(p)}$. If $(A_0, A_1, \alpha_0, \alpha_1)\colon \mathfrak{A} \leftrightsquigarrow_{2,r} \mathfrak{B}$, then $\alpha_1^{-1} \circ \alpha_0$ is a permutation of the set $N_r^{\mathfrak{A}}(A_0) \cap N_r^{\mathfrak{A}}(A_1)$.

Theorem 6.35. *For every* $\varphi \in \mathrm{FO}_<(D_k)[\tau]$ *and* s *such that* $\gcd(k, s) = 1$*, there exists* $r \in \mathbb{Z}_+$ *such that if* $(A_0, A_1, \alpha_0, \alpha_1)\colon \mathfrak{A} \leftrightsquigarrow_{2,r} \mathfrak{B}$ *and* $\mathrm{ord}(\alpha_1^{-1} \circ \alpha_0) = s$*, then* $\mathfrak{A} \equiv_\varphi \mathfrak{B}$*.*

Proof. Suppose that φ is an order-invariant sentence in $\mathrm{FO}_n(D_k)[\tau \cup \{<\}]$. Let m be as given by Proposition 6.33 for n, k, s and the vocabulary τ. Define $r = 2m$.

Assume $(A_0, A_1, \alpha_0, \alpha_1)\colon \mathfrak{A} \leftrightsquigarrow_{2,r} \mathfrak{B}$. We expand $\mathfrak{A}$ into a ribbon $\mathfrak{A}'$ by adding a linear order and an equivalence relation such such that $C_0^{\mathfrak{A}'} = A_0$, for all $1 \le i \le r$, $C_i^{\mathfrak{A}'} = N_i^{\mathfrak{A}}(A_0) \setminus N_{i-1}^{\mathfrak{A}}(A_0)$ and $C_{r+1}^{\mathfrak{A}'} = \mathrm{Dom}(\mathfrak{A}) \setminus N_r^{\mathfrak{A}}(A_0)$. We may choose the order between elements in the same equivalence classes arbitrarily.

For all $1 \le i \le r$, $(\alpha_1^{-1} \circ \alpha_0) \restriction C_i^{\mathfrak{A}'}$ is an automorphism of $\langle C_i^{\mathfrak{A}'} \rangle^{\mathfrak{A}}$. Define for all $0 \le i < r$ a local change $\mathfrak{B}_i$ of $\mathfrak{A}$ by setting $\mathrm{Dom}(\mathfrak{B}_i) = C_i^{\mathfrak{A}'} \cup C_{i+1}^{\mathfrak{A}'}$, $\sim^{\mathfrak{B}_i} = \sim^{\langle \mathrm{Dom}(\mathfrak{B}_i) \rangle^{\mathfrak{A}}}$, $<^{\mathfrak{B}_i} = <^{\langle \mathrm{Dom}(\mathfrak{B}_i) \rangle^{\mathfrak{A}}}$ and for all $R \in \tau$,

$$R^{\mathfrak{B}_i} = (\mathrm{id}_{C_i^{\mathfrak{A}'}} \cup (\alpha_1^{-1} \circ \alpha_0) \restriction C_{i+1}^{\mathfrak{A}'})(R^{\langle \mathrm{Dom}(\mathfrak{B}_i) \rangle^{\mathfrak{A}}}).$$

Let $Y = \{\mathfrak{B}_{2i} \mid i < m\}$. By the choice of m, there exist subsets $Y_0, Y_1 \subseteq Y$ such that $\mathrm{Apply}_{Y_0}(\mathfrak{A}') \equiv_{\mathrm{FO}_n(D_k)} \mathrm{Apply}_{Y_1}(\mathfrak{A}')$, $|Y_0| \equiv 0 \pmod{s}$ and $|Y_1| \equiv 1 \pmod{s}$. We claim now that $\mathrm{Apply}_{Y_0}(\mathfrak{A}') \restriction \tau \cong \mathfrak{A}$ and $\mathrm{Apply}_{Y_1}(\mathfrak{A}') \restriction \tau \cong \mathfrak{B}$.

Define $J_0, J_1 \subseteq r$ such that $Y_i = \{\mathfrak{B}_j \mid j \in J_i\}$. Let

$$f_0 = \mathrm{id}_{C_0^{\mathfrak{A}'}} \cup \bigcup_{1 \le i \le r} ((\alpha_1^{-1} \circ \alpha_0)^{|J_0 \cap i|} \restriction C_i^{\mathfrak{A}'}) \cup \mathrm{id}_{C_{r+1}^{\mathfrak{A}'}}.$$

The function is clearly a bijection $\mathrm{Dom}(\mathfrak{A}') \to \mathrm{Dom}(\mathfrak{A})$. For all $i \in J_0$,

$$f_0 \restriction (C_i^{\mathfrak{A}'} \cup C_{i+1}^{\mathfrak{A}'})\colon \langle C_i^{\mathfrak{A}'} \cup C_{i+1}^{\mathfrak{A}'} \rangle^{\mathfrak{A}} \cong \mathfrak{B}_i \restriction \tau = \langle C_i^{\mathfrak{A}'} \cup C_{i+1}^{\mathfrak{A}'} \rangle^{\mathrm{Apply}_{Y_0}(\mathfrak{A}')\restriction\tau}$$

and for all $i \in r \setminus J_0$,

$$f_0 \restriction (C_i^{\mathfrak{A}'} \cup C_{i+1}^{\mathfrak{A}'})\colon \langle C_i^{\mathfrak{A}'} \cup C_{i+1}^{\mathfrak{A}'} \rangle^{\mathfrak{A}} \cong \langle C_i^{\mathfrak{A}'} \cup C_{i+1}^{\mathfrak{A}'} \rangle^{\mathfrak{A}} = \langle C_i^{\mathfrak{A}'} \cup C_{i+1}^{\mathfrak{A}'} \rangle^{\mathrm{Apply}_{Y_0}(\mathfrak{A}')\restriction\tau}.$$

Since $|J_0| \equiv 0 \pmod{\mathrm{ord}(\alpha_1^{-1} \circ \alpha_0)}$, we have $(\alpha_1^{-1} \circ \alpha_0)^{|J_0 \cap r|} = \mathrm{id}_{N_r^{\mathfrak{A}}(A_0) \cap N_r^{\mathfrak{A}}(A_1)}$ and so

$$f_0 \restriction (C_r^{\mathfrak{A}'} \cup C_{r+1}^{\mathfrak{A}'})\colon \langle C_r^{\mathfrak{A}'} \cup C_{r+1}^{\mathfrak{A}'} \rangle^{\mathfrak{A}} \cong \langle C_r^{\mathfrak{A}'} \cup C_{r+1}^{\mathfrak{A}'} \rangle^{\mathrm{Apply}_{Y_0}(\mathfrak{A}')\restriction\tau}.$$

This shows $f_0\colon \mathfrak{A} \cong \mathrm{Apply}_{Y_0}(\mathfrak{A}') \restriction \tau$.

Let

$$f_1 = (\alpha_0 \restriction C_0^{\mathfrak{A}'})^{-1} \cup \bigcup_{1 \le i \le r} ((\alpha_1^{-1} \circ \alpha_0)^{|J_1 \cap i|} \circ (\alpha_0 \restriction C_i^{\mathfrak{A}'})^{-1}) \cup (\alpha_1 \restriction C_{r+1}^{\mathfrak{A}'})^{-1}$$

We can show as above that $f_1\colon \mathfrak{B} \cong \mathrm{Apply}_{Y_1}(\mathfrak{A}') \restriction \tau$.

The isomorphisms f_0 and f_1 induce linear orders on $\mathfrak{A}$ and $\mathfrak{B}$ such that the resulting structures are $\mathrm{FO}_n(D_k)$-equivalent. Because φ is order-invariant, this shows $\mathfrak{A} \equiv_\varphi \mathfrak{B}$. □

Symmetrical Gaifman-locality is an easy consequence of Theorem 6.35. We generalize it a little for the further discussion.

Lemma 6.36. *For every* $\varphi \in \mathrm{FO}_<(D_k)[\tau \cup \{x_0, \dots, x_{s-1}\}]$, *where* $\gcd(k, s) = 1$, *there exists* $r \in \mathbb{Z}_+$ *such that, if* $\mathfrak{A} \in \mathrm{Mod}(\tau)$, $a_0, \dots, a_{s-1} \in \mathrm{Dom}(\mathfrak{A})$, *for all* $i < j < s$, $N_r^{\mathfrak{A}}(a_i) \cap N_r^{\mathfrak{A}}(a_j) = \emptyset$ *and* $\mathfrak{N}_r^{\mathfrak{A}}(a_i) \cong \mathfrak{N}_r^{\mathfrak{A}}(a_j)$, *then* $(\mathfrak{A}, a_0 \dots a_{s-1}/\overline{x}) \equiv_\varphi (\mathfrak{A}, a_1 \dots a_{s-1} a_0/\overline{x})$.

Proof. Let r_0 be the radius that Theorem 6.35 requires for φ and s and put $r = 2r_0$. Let $\mathfrak{A}$ and $\overline{a}$ be as in the lemma. We may choose isomorphisms $\beta_i\colon \mathfrak{N}_r^{\mathfrak{A}}(a_i) \cong \mathfrak{N}_r^{\mathfrak{A}}(a_{i+1})$, where we identify a_s and a_0, such that $\beta_{s-1} \circ \dots \circ \beta_0 = \mathrm{id}_{N_r^{\mathfrak{A}}(a_0)}$. Let $A_0 = N_{r_0}^{\mathfrak{A}}(\overline{a})$, $A_1 = \mathrm{Dom}(\mathfrak{A}) \setminus A_0$, $\alpha_0 = \bigcup_{i<s} \beta_i$ and $\alpha_1 = \mathrm{id}_{N_{r_0}^{\mathfrak{A}}(A_1)}$. Now

$$(A_0, A_1, \alpha_0, \alpha_1)\colon (\mathfrak{A}, a_0 \dots a_{s-1}/\overline{x}) \leftrightsquigarrow_{2,r_0} (\mathfrak{A}, a_1 \dots a_{s-1} a_0/\overline{x})$$

and $\mathrm{ord}(\alpha_1^{-1} \circ \alpha_0) = s$ and so, by Theorem 6.35,

$$(\mathfrak{A}, a_0 \dots a_{s-1}/\overline{x}) \equiv_\varphi (\mathfrak{A}, a_1 \dots a_{s-1} a_0/\overline{x}).$$

□

Corollary 6.37. $\mathrm{FO}_<(D_k)$ *is symmetrically Gaifman-local, if* k *is odd.* □

Lemma 6.36 gives us also alternating Gaifman-locality in some cases. If six is not a factor of k, then either k is odd and symmetrical Gaifman-locality implies alternating Gaifman-locality or three is not a factor of k and the lemma gives alternating Gaifman-locality directly.

We can also prove alternating Gaifman-locality on structures that have enough disjoint neighborhoods isomorphic to the neighborhoods of the constants we want to permutate. Suppose that the elements $a_0, \dots, a_{s-1}$ have pairwise disjoint isomorphic

r-neighborhoods on $\mathfrak{A}$. If $\gcd(k,s)=1$ and $\varphi \in \mathrm{FO}_<(D_k)$ and r is big enough, then for every s-cycle γ on s we have $(\mathfrak{A}, \overline{a}/\overline{x}) \equiv_\varphi (\mathfrak{A}, a_{\gamma(0)} \dots a_{\gamma(s-1)}/\overline{x})$. Because all s-cycles generate the alternating group on s, in particular, any permutation of three elements is possible.

In the general case, we do not have the extra elements we could use as above. We can, however, arrange a similar configuration in another way.

An *s-ribbon* $\mathfrak{A}$ is a τ-ribbon with the following extra requirement: If $\mathfrak{A}$ has m equivalence classes $C_0^{\mathfrak{A}}, \dots, C_{m-1}^{\mathfrak{A}}$, there exists for every $1 \le i < m-1$ a partition $C_i^{\mathfrak{A}} = D_{i,0}^{\mathfrak{A}} \cup \dots \cup D_{i,s-1}^{\mathfrak{A}}$ such that if $R \in \mathrm{Rel}(\tau)$, $\overline{a} \in R$ and $[\overline{a}] \subseteq C_i^{\mathfrak{A}} \cup C_{i+1}^{\mathfrak{A}}$ for some $1 \le i < m-2$, then for some $j < s$, $[\overline{a}] \subseteq D_{i,j}^{\mathfrak{A}} \cup D_{i+1,j}^{\mathfrak{A}}$. Informally, s-ribbon consists of s parallel subribbons of the same length that are connected only at the endpoints. For all $j < s$, let $\mathfrak{D}_j^{\mathfrak{A}} = \left\langle D_{1,j}^{\mathfrak{A}} \cup \dots \cup D_{m-2,j}^{\mathfrak{A}} \right\rangle^{\mathfrak{A}}$ be the jth subribbon. Note that $\mathfrak{D}_j^{\mathfrak{A}}$ is a τ-ribbon.

Let $\overline{p} = (p_i)_{i<m-2}$ be a sequence of permutations on s. Define another sequence of permutations $(q_i^{\overline{p}})_{i<m-1}$ as $q_0^{\overline{p}} = \mathrm{id}_s$ and $q_{i+1}^{\overline{p}} = p_i \circ q_i$. If we have for all $j < s$ an isomorphism $\beta_j \colon \mathfrak{D}_j^{\mathfrak{A}} \cong \mathfrak{D}_{q_{m-2}(j)}^{\mathfrak{A}}$ we define the structure $\mathrm{Order}_{\overline{p}}(\mathfrak{A})$ that has the same universe and constants as $\mathfrak{A}$ and relations are defined as follows: $\sim^{\mathrm{Order}_{\overline{p}}(\mathfrak{A})} = \sim^{\mathfrak{A}}$ and for all $R \in \tau$,

$$\begin{aligned} R^{\mathrm{Order}_{\overline{p}}(\mathfrak{A})} \quad = \quad & \Bigg\{ \overline{a} \in (\mathrm{Dom}(\mathfrak{A}))^{\mathrm{ar}(R)} \mid ([\overline{a}] \not\subseteq D_{m-2}^{\mathfrak{A}} \cup D_{m-1}^{\mathfrak{A}} \wedge \overline{a} \in R^{\mathfrak{A}}) \\ & \vee \left([\overline{a}] \subseteq D_{m-2}^{\mathfrak{A}} \cup D_{m-1}^{\mathfrak{A}} \wedge \left(\bigcup_{j<s} \beta_j \cup \mathrm{id}_{C_{m-1}^{\mathfrak{A}}} \right) (\overline{a}) \in R^{\mathfrak{A}} \right) \Bigg\}. \end{aligned}$$

If $a \in D_{i,j_0}^{\mathfrak{A}}$ and $b \in D_{i,j_1}^{\mathfrak{A}}$ for some $1 \le i < m-1$ and $j_0, j_1 \in s$, then $a <^{\mathrm{Order}_{\overline{p}}(\mathfrak{A})} b$ if and only if $q_i^{\overline{p}}(j_0) < q_i^{\overline{p}}(j_1)$. Otherwise, $a <^{\mathrm{Order}_{\overline{p}}(\mathfrak{A})} b$ if and only if $a <^{\mathfrak{A}} b$.

Lemma 6.38. *Let $\mathfrak{E}, \mathfrak{F} \in \mathrm{Ord}(\tau)$ and let $\mathrm{Dom}(\mathfrak{E}) = E_0 \cup \dots \cup E_{s-1}$ and $\mathrm{Dom}(\mathfrak{F}) = F_0 \cup \dots \cup F_{s-1}$ be partitions. Denote $\langle E_i \rangle^{\mathfrak{E}}$ by $\mathfrak{E}_i$ and $\langle F_i \rangle^{\mathfrak{F}}$ by $\mathfrak{F}_i$. Assume the following:*

a) $\mathfrak{E} \restriction \tau = \bigsqcup_{i<s} \mathfrak{E}_i \restriction \tau$ and $\mathfrak{F} \restriction \tau = \bigsqcup_{i<s} \mathfrak{F}_i \restriction \tau$.
b) For all $i < s$, $\mathfrak{E}_i \equiv_{\mathrm{FO}_n(D_k)} \mathfrak{F}_i$.
c) If $a \in E_i$, $b \in E_j$, $a' \in F_i$, $b' \in F_j$, $i \ne j$, $\mathrm{atp}^{\mathfrak{E}_i}(a) = \mathrm{atp}^{\mathfrak{F}_i}(a')$ and $\mathrm{atp}^{\mathfrak{E}_j}(b) = \mathrm{atp}^{\mathfrak{F}_j}(b')$, then $a <^{\mathfrak{E}} b$ if and only if $a' <^{\mathfrak{F}} b'$.

Then $\mathfrak{E} \equiv_{\mathrm{FO}_n(D_k)} \mathfrak{F}$.

Proof. The proof is by induction. In the case $n = 0$, the conditions (a) and (b) imply that the constants of $\mathfrak{E}$ and $\mathfrak{F}$ satisfy the same relations $R \in \tau$. They are also ordered similarly by the condition (c). Thus the constants have the same atomic type and $\mathfrak{E} \equiv_{\mathrm{QF}} \mathfrak{F}$.

Assume then that the lemma is true for $\mathrm{FO}_{n-1}(D_k)$. We prove it for $\mathrm{FO}_n(D_k)$.

Let $c \in E_i$ and $c' \in F_i$ such that $\mathrm{tp}^{\mathfrak{E}_i}_{\mathrm{FO}_{n-1}(D_k)}(c) = \mathrm{tp}^{\mathfrak{F}_i}_{\mathrm{FO}_{n-1}(D_k)}(c')$. This is equivalent to $(\mathfrak{E}_i, c/x) \equiv_{\mathrm{FO}_{n-1}(D_k)} (\mathfrak{F}_i, c'/x)$. The structures $(\mathfrak{E}, c/x)$ and $(\mathfrak{F}, c'/x)$ and partitions $E_0 \cup \cdots \cup E_{s-1}$ and $F_0 \cup \cdots \cup F_{s-1}$ satisfy now the conditions of the lemma for $\mathrm{FO}_{n-1}(D_k)$. This is clear for the conditions (a) and (b) and because the number of atomic types increases on $(\mathfrak{E}, c/x)$ and $(\mathfrak{F}, c'/x)$ the condition (c) requires, in fact, less than before. By the induction hypothesis, $(\mathfrak{E}, c/x) \equiv_{\mathrm{FO}_{n-1}(D_k)} (\mathfrak{F}, c'/x)$.

The assumption $\mathfrak{E}_i \equiv_{\mathrm{FO}_n(D_k)} \mathfrak{F}_i$ is equivalent with the condition that for all $t \in T^k_{n-1}[\tau \cup \{<\}]$, $\mathrm{cut}_{1,k}(|t(\mathfrak{E}_i)|) = \mathrm{cut}_{1,k}(|t(\mathfrak{F}_i)|)$. This implies by the paragraph above that the same holds also on $\mathfrak{E}$ and $\mathfrak{F}$. Hence $\mathfrak{E} \equiv_{\mathrm{FO}_n(D_k)} \mathfrak{F}$. □

Lemma 6.39. *Assume that for all $j_0 < j_1 < s$, $\mathfrak{R}^k_n(\mathfrak{D}^{\mathfrak{A}}_{j_0}) = \mathfrak{R}^k_n(\mathfrak{D}^{\mathfrak{A}}_{j_1})$. Let $\overline{p}$ and $\overline{p}'$ be two sequences of permutations on s. If $i < m$ and $p_{i+d} = p'_{i+d}$ for all $|d| \leq 3^n$ with $0 \leq i + d < m - 2$, then i has the same atomic type on $\mathfrak{R}^k_n(\mathrm{Order}_{\overline{p}}(\mathfrak{A}))$ and $\mathfrak{R}^k_n(\mathrm{Order}_{\overline{p}'}(\mathfrak{A}))$.*

Proof. The conclusion of the lemma is equivalent to the claim $\mathfrak{T}^{\mathrm{Order}_{\overline{p}}(\mathfrak{A})}_{3^n}(i) \equiv_{\mathrm{FO}_n(D_k)} \mathfrak{T}^{\mathrm{Order}_{\overline{p}'}(\mathfrak{A})}_{3^n}(i)$. If $i \leq 3^n$, this is trivial, because assumptions then imply that $\mathfrak{T}^{\mathrm{Order}_{\overline{p}}(\mathfrak{A})}_{3^n}(i) \cong \mathfrak{T}^{\mathrm{Order}_{\overline{p}'}(\mathfrak{A})}_{3^n}(i)$. This is also the case if $i \geq m - 1 - 3^n$, since the modifications we made on relations $R \in \tau$ look locally like we had changed the order of the subribbons according to $q^{\overline{p}}_{m-2}$.

If $3^n < i < m - 1 - 3^n$, the structures $\mathfrak{T}^{\mathrm{Order}_{\overline{p}}(\mathfrak{A})}_{3^n}(i)$ and $\mathfrak{T}^{\mathrm{Order}_{\overline{p}'}(\mathfrak{A})}_{3^n}(i)$ are not necessarily isomorphic. Let $\gamma = (q^{\overline{p}'}_i)^{-1} \circ q^{\overline{p}}_{i+d}$. Then $q^{\overline{p}}_{i+d} = q^{\overline{p}'}_{i+d} \circ \gamma$ for all $|d| \leq 3^n$. Define partitions $\mathrm{Dom}(\mathfrak{T}^{\mathrm{Order}_{\overline{p}}(\mathfrak{A})}_{3^n}(i)) = E_0 \cup \cdots \cup E_{s-1}$ and $\mathrm{Dom}(\mathfrak{T}^{\mathrm{Order}_{\overline{p}'}(\mathfrak{A})}_{3^n}(i)) = F_0 \cup \cdots \cup F_{s-1}$ such that $E_j = D_{i-3^n,j} \cup \cdots \cup D_{i+3^n,j}$ and $F_j = D_{i-3^n,\gamma(j)} \cup \cdots \cup D_{i+3^n,\gamma(j)}$. Now the structures and the partitions satisfy the conditions of Lemma 6.38 and so $\mathfrak{T}^{\mathrm{Order}_{\overline{p}}(\mathfrak{A})}_{3^n}(i) \equiv \mathfrak{T}^{\mathrm{Order}_{\overline{p}'}(\mathfrak{A})}_{3^n}(i)$. □

Let γ_m be an m-cycle on s that maps every $j < m-1$ to $j+1$ and $m-1$ to 0. If X and Y are disjoint subsets of M, let $\overline{p}(X,Y)$ be a sequence of permutations such that $p_i(X,Y) = \gamma_s^{-1}$, if $i \in X$, $p_i(X,Y) = \gamma_s$, if $i \in Y$ and otherwise $p_i(X,Y) = \mathrm{id}_s$.

Lemma 6.40. *Fix $n \in \mathbb{N}$, a vocabulary τ and $k, s \in \mathbb{Z}_+$ such that $\gcd(k,s) = 1$. There exists m_0 with the following property. Assume $\mathfrak{A}$ is an s-ribbon, for all $j_0 < j_1 < s$, $\mathfrak{R}^k_n(\mathfrak{D}^{\mathfrak{A}}_{j_0}) = \mathfrak{R}^k_n(\mathfrak{D}^{\mathfrak{A}}_{j_1})$ and $i_0 < i_1 < u_0 < i_2 < i_3 < u_1 < m$ such that $i_1 - i_0 \geq m_0$ and $i_3 - i_2 \geq m_0$. Then there exist $X_0, X_1 \subseteq [i_0, i_1]$ and $Y_0, Y_1 \subseteq [i_2, i_3]$ such that*

$$\mathfrak{R}^k_n(\mathrm{Order}_{\overline{p}(X_0,Y_0)}(\mathfrak{A}), u_0/x, u_1/y) \equiv_{\mathrm{FO}_{4n}(D_k)} \mathfrak{R}^k_n(\mathrm{Order}_{\overline{p}(X_1,Y_1)}(\mathfrak{A}), u_0/x, u_1/y),$$

$|X_0| \equiv |Y_0| \equiv 0 \pmod{s}$ *and* $|X_1| \equiv |Y_1| \equiv 1 \pmod{s}$

Proof. The proof is essentially the same as the proof of Proposition 6.33. Instead of one pair of sets, we have to find two pairs (X_0, X_1) and (Y_0, Y_1) and we have to use Lemma 6.39 to show that every element included into one of the sets modifies the reduct only locally. □

Lemma 6.41. *For all $n \in \mathbb{N}$, a vocabulary τ and $k, s \in \mathbb{Z}_+$ such that $\gcd(k,s) = 1$, there exists m_1 with the following property. Assume $\mathfrak{A}$ is an s-ribbon with at least m_1*

equivalence classes, for all $j_0 < j_1 < s$, $\mathfrak{R}^k_n(\mathfrak{D}^{\mathfrak{A}}_{j_0}) = \mathfrak{R}^k_n(\mathfrak{D}^{\mathfrak{A}}_{j_1})$ *and* $\mathfrak{D}^{\mathfrak{A}}_0 \cong \mathfrak{D}^{\mathfrak{A}}_1 \cong \mathfrak{D}^{\mathfrak{A}}_2$. *Then there exist sequences* $\overline{p}$ *and* $\overline{p}'$ *of permutations such that* $q^{\overline{p}}_{m-2} = \mathrm{id}_s$, $q^{\overline{p}'}_{m-2} = \gamma_3$ *and* $\mathrm{Order}_{\overline{p}}(\mathfrak{A}) \equiv_{\mathrm{FO}_n(D_k)} \mathrm{Order}_{\overline{p}'}(\mathfrak{A})$.

Proof. Let m_0 be the number given by Lemma 6.40. Choose m_1 so large that given $\mathfrak{A}$ with $m \geq m_1$ equivalence classes, we may choose integers i_0, i_1, i_2, i_3, u_0 and u_1 satisfying $1 \leq i_0 \leq i_1 + 2 \cdot 3^n < u_0 < i_2 - 2 \cdot 3^n < i_3 + 2 \cdot 3^n < u_1 < m - 1 - 3^m$, $i_1 - i_0 \geq m_0$ and $i_3 - i_2 \geq m_0$. Then the assumptions of Lemma 6.40 hold. Let X_0, X_1, Y_0 and Y_1 be as in the conclusion of the lemma.

We define $\overline{p}$ and $\overline{p'}$ such that $p_{u_0} = p'_{u_0} = p_{u_1} = p'_{u_1} = \gamma_2$ and for all $i < m$ with $i \notin \{u_0, u_1\}$, $p_i = p_i(X_0, Y_0)$ and $p'_i = p_i(X_1, Y_1)$. Since all elements of X_0 and X_1 are before u_0 and all elements of Y_0 and Y_1 are between u_0 and u_1, $q^{\overline{p}}_{m-2} = \gamma_2 \circ \mathrm{id}_s \circ \gamma_2 \circ \mathrm{id}_s = \mathrm{id}_s$ and $q^{\overline{p}'}_{m-2} = \gamma_2 \circ \gamma_s \circ \gamma_2 \circ \gamma_s^{-1} = \gamma_3$.

No elements of X_0, X_1, Y_0 or Y_1 are within $2 \cdot 3^n$ from u_0 or u_1 and so their neighborhoods in $\mathfrak{R}^k_n(\mathrm{Order}_{\overline{p}(X_0,Y_0)}(\mathfrak{A}), u_0/x, u_1/y)$ and $\mathfrak{R}^k_n(\mathrm{Order}_{\overline{p}(X_1,Y_1)}(\mathfrak{A}), u_0/x, u_1/y)$ are isomorphic. By Lemma 6.39, modifications we made on permutation sequences $\overline{p}$ and $\overline{p}'$ on u_0 and u_1 modify the reducts only at the 3^n-neighborhoods of the elements and in the same way in both reducts. Thus there exists a FO_{2n}-interpretation ψ such that

$$\psi^*(\mathfrak{R}^k_n(\mathrm{Order}_{\overline{p}(X_0,Y_0)}(\mathfrak{A}), u_0/x, u_1/y)) = \mathfrak{R}^k_n(\mathrm{Order}_{\overline{p}}(\mathfrak{A}))$$

and

$$\psi^*(\mathfrak{R}^k_n(\mathrm{Order}_{\overline{p}(X_1,Y_1)}(\mathfrak{A}), u_0/x, u_1/y)) = \mathfrak{R}^k_n(\mathrm{Order}_{\overline{p}'}(\mathfrak{A})).$$

This shows that $\mathfrak{R}^k_n(\mathrm{Order}_{\overline{p}}(\mathfrak{A})) \equiv_{\mathrm{FO}_{2n}(D_k)} \mathfrak{R}^k_n(\mathrm{Order}_{\overline{p}'}(\mathfrak{A}))$ and so

$$\mathrm{Order}_{\overline{p}}(\mathfrak{A}) \equiv_{\mathrm{FO}_n(D_k)} \mathrm{Order}_{\overline{p}'}(\mathfrak{A}).$$

□

We discussed earlier how alternating Gaifman-locality can be shown, if the neighborhood type of the constants to be permuted have enough realizations in the structure. We have now established that it is enough to have many disjoint parts of the structure that are equivalent with the neighborhood. Such parts fortunately always arrangeable.

Theorem 6.42. *The logic* $\mathrm{FO}_<(D_k)$ *is alternatingly Gaifman-local for all* $k \in \mathbb{Z}_+$.

Proof. Fix $n \in \mathbb{N}$, $k \in \mathbb{Z}_+$ and a vocabulary τ. Let s be the smallest prime that is not a factor of k and let m_1 be as in Lemma 6.41. Because we have already proven alternating Gaifman-locality for values of k not divisible by 6, we may assume that $s > 3$. Let l be the number of different atomic types occurring on n, k-reducts. Set $r = s l^{m_1} m_1$.

Let $\mathfrak{A}$ be a τ-structure and $a_0, a_1, a_2 \in \mathrm{Dom}(\mathfrak{A})$ such that the neighborhoods $\mathfrak{N}^{\mathfrak{A}}_r(a_0) \cong \mathfrak{N}^{\mathfrak{A}}_r(a_1) \cong \mathfrak{N}^{\mathfrak{A}}_r(a_2)$ are disjoint.

Define a τ-ribbon $\mathfrak{B}_j$ such that $\mathfrak{B}_j \restriction \tau = \langle N_r^{\mathfrak{A}}(a_j) \setminus \{a_j\}\rangle^{\mathfrak{A}}$, the ribbon has r equivalence classes $C_i^{\mathfrak{B}_j} = N_{i+1}^{\mathfrak{A}}(a_j) \setminus N_i^{\mathfrak{A}}(a_j)$. The linear order can be chosen on the ribbons such that $\mathfrak{B}_0 \cong \mathfrak{B}_1 \cong \mathfrak{B}_2$.

The reduct $\mathfrak{R}_n^k(\mathfrak{B}_0)$ is a word model of length r with l symbols in the alphabet. By the pigeon hole principle, it contains $s-2$ disjoint isomorphic subwords of length m_0. Suppose these words begin at the positions $w_0 < \cdots < w_{s-3}$.

We define now a s-ribbon $\mathfrak{C}$ with m_0+1 equivalence classes such that $\mathfrak{C} \restriction \tau = \mathfrak{A}$, $\overline{x}^{\mathfrak{C}} = \overline{a}$, for all $0 \leq i < m_0$, $C_{i+1}^{\mathfrak{C}} = C_{w_0+i}^{\mathfrak{B}_0} \cup C_{w_0+i}^{\mathfrak{B}_1} \cup C_{w_0+i}^{\mathfrak{B}_2} \cup \bigcup_{1 \leq j < s} C_{w_j+i}^{\mathfrak{B}_0}$ and $C_0^{\mathfrak{A}}$ contains all elements not in the other equivalence classes. The linear order is chosen so that $\mathfrak{C}$ is a ribbon and $\langle \mathrm{Dom}(\mathfrak{B}_j)\rangle^{\mathfrak{C}\restriction\tau\cup\{<\}} = \mathfrak{B}_j \restriction \tau \cup \{<\}$. Finally, $D_{i,0}^{\mathfrak{C}} = C_{w_0+i}^{\mathfrak{B}_0}$, $D_{i,1}^{\mathfrak{C}} = C_{w_0+i}^{\mathfrak{B}_1}$, $D_{i,2}^{\mathfrak{C}} = C_{w_0+i}^{\mathfrak{B}_2}$ and for all $2 < j < s$, $D_{i,j}^{\mathfrak{C}} = C_{w_{j-2}+i}^{\mathfrak{B}_0}$.

The s-ribbon $\mathfrak{C}$ satisfies the conditions of Lemma 6.41 and so there are sequences $\overline{p}$ and $\overline{p}'$ with $\mathrm{Order}_{\overline{p}}(\mathfrak{C}) \equiv_{\mathrm{FO}_n(D_k)} \mathrm{Order}_{\overline{p}'}(\mathfrak{C})$, $q_{m-2}^{\overline{p}} = \mathrm{id}_s$ and $q_{m-2}^{\overline{p}'} = \gamma_3$. Since $\mathrm{Order}_{\overline{p}}(\mathfrak{C}) \restriction \tau \cup [\overline{x}] = (\mathfrak{A}, a_0a_1a_2/\overline{x})$ and $\mathrm{Order}_{\overline{p}'}(\mathfrak{C}) \restriction \tau \cup [\overline{x}] \cong (\mathfrak{A}, a_1a_2a_0/\overline{x})$ and the structures $\mathrm{Order}_{\overline{p}}(\mathfrak{C})$ and $\mathrm{Order}_{\overline{p}'}(\mathfrak{C})$ contain linear orders, we have

$$(\mathfrak{A}, a_0a_1a_2/\overline{x}) \equiv_{(\mathrm{FO}_n(D_k))_<} (\mathfrak{A}, a_1a_2a_0/\overline{x}).$$

This shows that $\mathrm{FO}_<(D_k)$ is alternatingly Gaifman-local. □

7. Trees

Let τ be a vocabulary containing one binary relation symbol E and some unary relation symbols. A τ-structure is a tree if E forms a connected directed graph having a specific element called root having no incoming edges and every other element has exactly one incoming edge.

Denote the root of the tree $\mathfrak{A}$ by $r_{\mathfrak{A}}$. We assume that the vocabulary contains a relation symbol that marks the root of the tree. Given an element $a \neq r_{\mathfrak{A}}$, the unique element $p_{\mathfrak{A}}(a)$ such that $(p_{\mathfrak{A}}(a), a) \in E^{\mathfrak{A}}$ is called the parent of a. The set of the children of an element a is defined as $c_{\mathfrak{A}}(a) = \{b \in \mathrm{Dom}(\mathfrak{A}) \mid (a,b) \in E^{\mathfrak{A}}\}$. A k-level subtree $S_k^{\mathfrak{A}}(a)$ of the tree is defined as

$$S_0^{\mathfrak{A}}(a) = \{a\}$$

and

$$S_{k+1}^{\mathfrak{A}}(a) = \{a\} \cup \bigcup_{x \in S_k^{\mathfrak{A}}(a)} c_{\mathfrak{A}}(x),$$

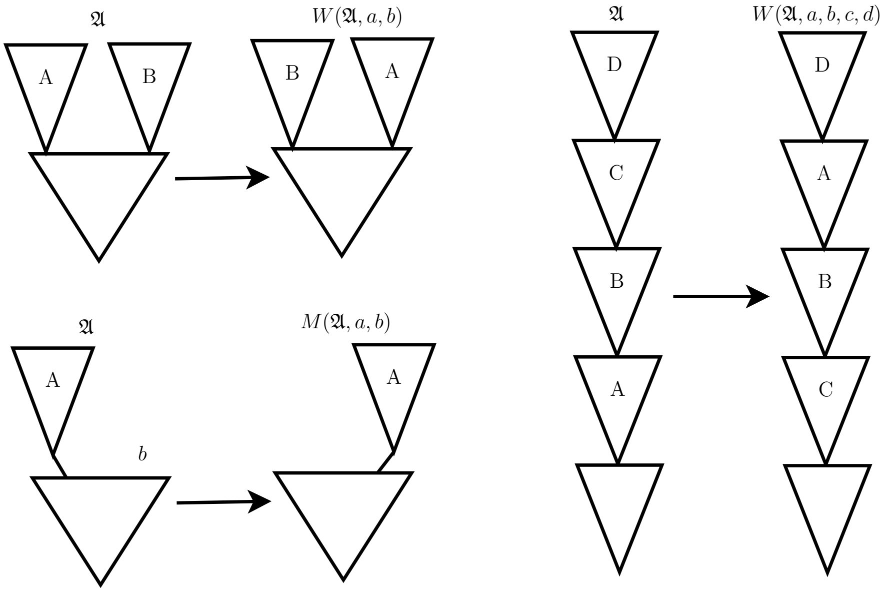
and a subtree $S^{\mathfrak{A}}(a)$ as

$$S^{\mathfrak{A}}(a) = \bigcup_{i \in \mathbb{N}} S_i^{\mathfrak{A}}(a).$$

Let $\mathfrak{S}_k^{\mathfrak{A}}(a) = \langle S_k^{\mathfrak{A}}(a)\rangle^{(\mathfrak{A}, a/x)}$ and $\mathfrak{S}^{\mathfrak{A}}(a) = \langle S^{\mathfrak{A}}(a)\rangle^{(\mathfrak{A}, a/x)}$.

Given two elements $a, b \in \mathrm{Dom}(\mathfrak{A})$, we let $W(\mathfrak{A}, a, b)$ be a τ-structure with the same universe as $\mathfrak{A}$ and the same unary relations. Relation symbol E has interpretation $(E^{\mathfrak{A}} \setminus \{(p_{\mathfrak{A}}(a), a), (p_{\mathfrak{A}}(b), b)\}) \cup \{(p_{\mathfrak{A}}(b), a), (p_{\mathfrak{A}}(a), b)\}$. We also define $W(\mathfrak{A}, a, b, c, d) = W(W(\mathfrak{A}, a, c), b, d)$. (see Figure 2).

FIGURE 2. Swap operations



Let $M(\mathfrak{A}, a, b)$ be a τ-structure with the same universe as $\mathfrak{A}$ and the same unary relations. Relation symbol E has interpretation $(E^{\mathfrak{A}} \setminus \{(p_{\mathfrak{A}}(a), a)\}) \cup \{(b, a)\}$. With this notation $W(\mathfrak{A}, a, b) = M(M(\mathfrak{A}, a, p_{\mathfrak{A}}(b)), b, p_{\mathfrak{A}}(a))$.

7.1. **Hanf -locality.** We show in this section that symmetrically Gaifman-local extensions of first-order logic with generalized quantifiers are Hanf-local on trees. The same method is then applied in subsequent sections to show stronger versions of Hanf-locality.

We use Gaifman-locality to show that certain local changes to trees are possible while keeping the tree Φ-equivalent to the original one. Repeatedly using these operations, two r-Hanf-equivalent trees can then be shown Φ-equivalent if Φ is a given finite set of sentences and r is big enough.

Instead of neighborhoods, it is technically easier to speak about k-level subtrees. We generalize therefore the definition of Hanf-equivalence. Let $\sim$ be an equivalence relation on $\mathrm{Mod}(\tau \cup \{x\})$. We write $\alpha \colon \mathfrak{A} \leftrightarrows^{\sim} \mathfrak{B}$, if α is a bijection $\mathrm{Dom}(\mathfrak{A}) \to \mathrm{Dom}(\mathfrak{B})$ and for all $a \in \mathrm{Dom}(\mathfrak{A})$, $(\mathfrak{A}, a/x) \sim (\mathfrak{B}, \alpha(a)/x)$.

Let $\Phi \subseteq \mathcal{L}[\tau]$.

Definition 7.1. A pair $(\sim, \Phi)$ admits swapping if the following properties hold for all trees $\mathfrak{A}$:

- If $a, b \in \mathrm{Dom}(\mathfrak{A})$, $(\mathfrak{A}, a/x) \sim (\mathfrak{A}, b/x)$ and $S^{\mathfrak{A}}(a) \cap S^{\mathfrak{A}}(b) = \emptyset$ then $\mathfrak{A} \equiv_{\Phi} W(\mathfrak{A}, a, b)$ and for all $c \in \mathrm{Dom}(\mathfrak{A})$, $(\mathfrak{A}, c/x) \sim (W(\mathfrak{A}, a, b), c/x)$.

- If $a, b, c, d \in \text{Dom}(\mathfrak{A})$, $S^{\mathfrak{A}}(a) \supseteq S^{\mathfrak{A}}(b) \supsetneq S^{\mathfrak{A}}(c) \supsetneq S^{\mathfrak{A}}(d)$, $(\mathfrak{A}, a/x) \sim (\mathfrak{A}, c/x)$ and $(\mathfrak{A}, b/x) \sim (\mathfrak{A}, d/x)$ then $\mathfrak{A} \equiv_\Phi W(\mathfrak{A}, a, b, c, d)$ and for all $e \in \text{Dom}(\mathfrak{A})$, $(\mathfrak{A}, e/x) \sim (W(\mathfrak{A}, a, b, c, d), e/x)$.

We define next a new equivalence relation $\approx$ based on $\sim$. $(\mathfrak{A}, a/x) \approx (\mathfrak{A}', a'/x)$ if and only if $(\mathfrak{A}, a/x) \sim (\mathfrak{A}', a'/x)$ and there is a bijection $\alpha\colon c_{\mathfrak{A}}(a) \to c_{\mathfrak{A}'}(a')$ such that for all $b \in c_{\mathfrak{A}}(a)$, $(\mathfrak{A}, b/x) \sim (\mathfrak{A}', \alpha(b)/x)$. If $(\sim, \Phi)$ admits swapping then $(\approx, \Phi)$ admits swapping.

Lemma 7.2. *If a pair $(\sim, \Phi)$ admits swapping and $\mathfrak{A} \leftrightarrows^{\approx} \mathfrak{A}'$ then $\mathfrak{A} \equiv_\Phi \mathfrak{A}'$.*

Proof. By the definition of the $\leftrightarrows^{\approx}$ -relation, there is a bijection $\alpha\colon \text{Dom}(\mathfrak{A}) \to \text{Dom}(\mathfrak{A}')$ such that for all $a \in \text{Dom}(\mathfrak{A})$, $(\mathfrak{A}, a/x) \approx (\mathfrak{A}', \alpha(a)/x)$. Let $\Psi(\mathfrak{A}, \mathfrak{A}', \alpha) = \{a \in \text{Dom}(\mathfrak{A}) \setminus \{r_{\mathfrak{A}}\} \mid \alpha(p_{\mathfrak{A}}(a)) \neq p_{\mathfrak{A}'}(\alpha(a))\}$.

The proof proceeds by induction on the size of the set $\Psi(\mathfrak{A}, \mathfrak{A}', \alpha)$. If $\Psi(\mathfrak{A}, \mathfrak{A}', \alpha) = \emptyset$, α is an isomorphism and $\mathfrak{A} \equiv_\Phi \mathfrak{A}'$ is clear. Suppose then that $\Psi(\mathfrak{A}, \mathfrak{A}', \alpha) \neq \emptyset$ and the lemma has been proved in all cases where the cardinality of the set is smaller.

Fix a bijection $\beta\colon \text{Dom}(\mathfrak{A}) \setminus \{r_{\mathfrak{A}}\} \to \text{Dom}(\mathfrak{A}') \setminus \{r_{\mathfrak{A}'}\}$ such that for all $a \in \text{Dom}(\mathfrak{A}) \setminus \{r_{\mathfrak{A}}\}$,

- $\alpha(p_{\mathfrak{A}}(a)) = p_{\mathfrak{A}'}(\beta(a))$,
- $(\mathfrak{A}, a/x) \sim (\mathfrak{A}', \beta(a)/x)$, and
- if $a \notin \Psi(\mathfrak{A}, \mathfrak{A}', \alpha), \beta(a) = \alpha(a)$.

The bijection can be constructed by first defining it on $\text{Dom}(\mathfrak{A}) \setminus \Psi(\mathfrak{A}, \mathfrak{A}', \alpha)$ and then patching it from the bijections $c_{\mathfrak{A}}(a) \to c_{\mathfrak{A}'}(\alpha(a))$ existing by the definition of $\approx$ -relation.

Let $h\colon \text{Dom}(\mathfrak{A}) \to \text{Dom}(\mathfrak{A})$, where $h(a) = \alpha^{-1}(\beta(a))$ for all $a \in \text{Dom}(\mathfrak{A}) \setminus \{r_{\mathfrak{A}}\}$ and $h(r_{\mathfrak{A}}) = r_{\mathfrak{A}}$. Because of the third property of β, h is the identity function on $\text{Dom}(\mathfrak{A}) \setminus \Psi(\mathfrak{A}, \mathfrak{A}', \alpha)$. On the other hand, if $h(a) = a$, then $\alpha(a) = \beta(a)$ and $p_{\mathfrak{A}}(\alpha(a)) = p_{\mathfrak{A}}(\beta(a)) = \alpha(p_{\mathfrak{A}}(a))$ and thus $a \notin \Psi(\mathfrak{A}, \mathfrak{A}', \alpha)$. Moreover, h is bijective and for all $a \in \text{Dom}(\mathfrak{A})$, $(\mathfrak{A}, a/x) \sim (\mathfrak{A}, h(a)/x)$.

If there exists some element $a \in \Psi(\mathfrak{A}, \mathfrak{A}', \alpha)$ such that $S^{\mathfrak{A}}(a) \cap S^{\mathfrak{A}}(h(a)) = \emptyset$ then $W(\mathfrak{A}, a, h(a)) \equiv_\Phi \mathfrak{A}$, because $(\sim, \Phi)$ admits swapping and $(\mathfrak{A}, a/x) \sim (\mathfrak{A}, h(a)/x)$. Since for all $b \in \text{Dom}(\mathfrak{A})$, $(\mathfrak{A}, b/x) \sim (W(\mathfrak{A}, a, h(a)), b/x)$, we also have $(\mathfrak{A}, b/x) \approx (W(\mathfrak{A}, a, h(a)), b/x)$ and therefore α still has the property $(W(\mathfrak{A}, a, h(a)), b/x) \approx (\mathfrak{A}', \alpha(b)/x)$ for all $b \in \text{Dom}(\mathfrak{A})$. This shows that $W(\mathfrak{A}, a, h(a)) \leftrightarrows^{\approx} \mathfrak{A}$.

All elements but a and $h(a)$ have the same parents in $\mathfrak{A}$ and $W(\mathfrak{A}, a, h(a))$ so $\Psi(W(\mathfrak{A}, a, h(a)), \mathfrak{A}', \alpha) \subseteq \Psi(\mathfrak{A}, \mathfrak{A}', \alpha)$. However, $\alpha(p_{W(\mathfrak{A},a,h(a))}(h(a))) = \alpha(p_{\mathfrak{A}}(a)) = p_{\mathfrak{A}'}(\beta(a)) = p_{\mathfrak{A}'}(\alpha(h(a)))$ and so $h(a) \notin \Psi(W(\mathfrak{A}, a, h(a)), \mathfrak{A}', \alpha)$. This implies $|\Psi(W(\mathfrak{A}, a, h(a)), \mathfrak{A}', \alpha)| < |\Psi(\mathfrak{A}, \mathfrak{A}', \alpha)|$ and we can use the induction hypothesis to show that $W(\mathfrak{A}, a, h(a)) \equiv_\Phi \mathfrak{A}'$. Hence we may assume in the following that for all $a \in \Psi(\mathfrak{A}, \mathfrak{A}', \alpha)$, $S^{\mathfrak{A}}(a) \cap S^{\mathfrak{A}}(h(a)) \neq \emptyset$, i.e either $a \in S^{\mathfrak{A}}(h(a))$ or $h(a) \in S^{\mathfrak{A}}(a)$.

Let $h'\colon \text{Dom}(\mathfrak{A}') \to \text{Dom}(\mathfrak{A}')$, where $h'(a) = \alpha(\beta^{-1}(a))$ for all $a \in \text{Dom}(\mathfrak{A}') \setminus \{r_{\mathfrak{A}'}\}$ and $h'(r_{\mathfrak{A}'}) = r_{\mathfrak{A}'}$. The function is defined as h but $\mathfrak{A}$ and $\mathfrak{A}'$ changed. As before we can show that if $a \in \text{Dom}(\mathfrak{A}')$ and $h'(a)$ are not comparable, the trees have to be equivalent. So for all $a \in \mathrm{Dom}(\mathfrak{A}')$, $S^{\mathfrak{A}'}(a) \cap S^{\mathfrak{A}'}(h'(a)) \neq \emptyset$. Note that $h' = \alpha \circ h^{-1} \circ \alpha^{-1}$.

Choose now an element $a \in \Psi(\mathfrak{A}, \mathfrak{A}', \alpha)$ such that no predecessor ($(p_{\mathfrak{A}})^i(a)$ for some $i \in \mathbb{Z}_+$) of a belongs to $\Psi(\mathfrak{A}, \mathfrak{A}', \alpha)$. This implies $h(a) \in S^{\mathfrak{A}}(a)$.

Let $a' = \beta(a) = \alpha(h(a))$. By induction, $(p_{\mathfrak{A}'})^i(a') = \alpha((p_{\mathfrak{A}})^i(a))$ and thus $a' \notin S^{\mathfrak{A}'}(h'(a'))$. Thus $\alpha(a) = h'(a') \in S^{\mathfrak{A}'}(a')$.

Choose $b' \in S^{\mathfrak{A}'}(a')$ be such that $\alpha^{-1}(b') \notin S^{\mathfrak{A}}(h(a))$ but $\alpha^{-1}(p_{\mathfrak{A}'}(b')) \in S^{\mathfrak{A}}(h(a))$. This kind of element exists because $\alpha^{-1}(a') = h(a) \in S^{\mathfrak{A}}(h(a))$ but $\alpha^{-1}(h(a')) = a \notin S^{\mathfrak{A}}(h(a))$.

Let $b = \beta^{-1}(b')$. Since $\alpha^{-1}(p_{\mathfrak{A}'}(b')) = p_{\mathfrak{A}}(b) \in S^{\mathfrak{A}}(h(a))$, $b \in S^{\mathfrak{A}}(h(a))$ and $b \neq h(a)$. On the other hand, $h(b) = \alpha^{-1}(\beta(b)) = \alpha^{-1}(b') \notin S^{\mathfrak{A}}(h(a))$. Combining these inclusions, we get $S^{\mathfrak{A}}(a) \supseteq S^{\mathfrak{A}}(h(b)) \supsetneq S^{\mathfrak{A}}(h(a)) \supsetneq S^{\mathfrak{A}}(b)$ Hence $W(\mathfrak{A}, a, h(b), h(a), b) \equiv_\Phi \mathfrak{A}$, $W(\mathfrak{A}, a, h(b), h(a), b) \leftrightarrows^{\approx} \mathfrak{A}$ and thus

$$\Psi(W(\mathfrak{A}, a, h(b), h(a), b), \mathfrak{A}', \alpha) \subseteq \Psi(\mathfrak{A}, \mathfrak{A}', \alpha) \setminus \{h(a)\}.$$

The induction hypothesis gives now $W(\mathfrak{A}, a, h(b), h(a), b) \equiv_\Phi \mathfrak{A}'$. □

Let $\sim_r$ be an equivalence relation $(\mathfrak{A}, a/x) \sim_r (\mathfrak{A}', a'/x) \iff \mathfrak{S}^{\mathfrak{A}}_r(a) \cong \mathfrak{S}^{\mathfrak{A}'}_r(a')$. Let $\approx_r$ be the respective $\approx$ -relation.

Lemma 7.3. *Let $\mathcal{L}$ be a logic such that $\mathcal{L} \circ \mathrm{QF}$ is symmetrically Gaifman-local. Then for all $\varphi \in \mathcal{L}[\tau]$, there is $r \in \mathbb{N}$ such that if $(\mathfrak{A}, a/x) \sim_r (\mathfrak{A}, b/x)$ and $S^{\mathfrak{A}}_{r+1}(a) \cap S^{\mathfrak{A}}_{r+1}(b) = \emptyset$ then $W(\mathfrak{A}, a, b) \equiv_\varphi \mathfrak{A}$.*

Proof. Note that if $S^{\mathfrak{A}}(a) \cap S^{\mathfrak{A}}(b) \neq \emptyset$, $W(\mathfrak{A}, a, b)$ is not a tree. However, we can still speak about equivalence between two structures.

Let ψ be a QF interpretation on vocabulary $\tau \cup \{v_0, v_1, v_2, v_3\}$ such that it preserves all unary relations and $\psi_E(x, y) = Exy \vee (x = v_2 \wedge y = v_0) \vee (x = v_1 \wedge y = v_3)$. Let $\varphi' = \varphi \circ \psi$ and choose $r \in \mathbb{N}$ witnessing symmetrical Gaifman-locality of φ'.

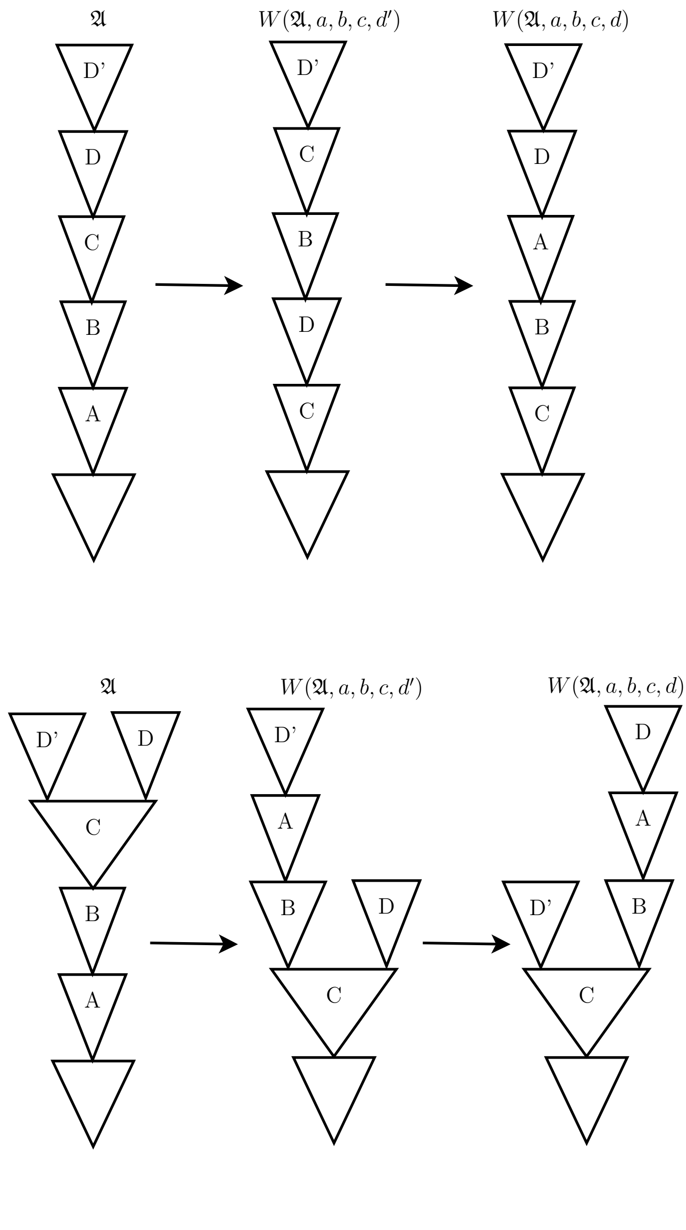
Assume $\mathfrak{A}$, a and b are as in the lemma. Let $c = p_{\mathfrak{A}}(a)$ and $d = p_{\mathfrak{A}}(b)$. Let $\mathfrak{A}'$ be a τ-structure with unary relations having the same interpretation as in $\mathfrak{A}$, but $E^{\mathfrak{A}'} = E^{\mathfrak{A}} \setminus \{(c, a), (d, b)\}$. Because $(\mathfrak{A}, a/x) \sim_r (\mathfrak{A}, b/x)$, $\mathfrak{N}^{\mathfrak{A}'}_r(a) \cong \mathfrak{N}^{\mathfrak{A}'}_r(b)$ and the sets $N^{\mathfrak{A}'}_{r+1}(a)$, $N^{\mathfrak{A}'}_{r+1}(b)$ and $N^{\mathfrak{A}'}_r(cd)$ are disjoint $\mathfrak{N}^{\mathfrak{A}'}_r(abcd) \cong \mathfrak{N}^{\mathfrak{A}'}_r(bacd)$. By symmetrical Gaifman-locality, $(\mathfrak{A}', abcd/\overline{v}) \equiv_{\Phi'} (\mathfrak{A}', bacd/\overline{v})$. Because $\psi^*((\mathfrak{A}', abcd/\overline{v})) \cong \mathfrak{A}$ and $\psi^*((\mathfrak{A}', bacd/\overline{v})) \cong W(\mathfrak{A}, a, b)$, this implies $\mathfrak{A} \equiv_\Phi W(\mathfrak{A}, a, b)$. □

Lemma 7.4. *Let $\mathcal{L}$ be a logic such that $\mathcal{L} \circ \mathrm{QF}$ is symmetrically Gaifman-local. Then for all finite $\Phi \subseteq \mathcal{L}[\tau]$, there is $r \in \mathbb{N}$ such that $(\sim_r, \Phi)$ admits swapping.*

Proof. Let r' be the constant given by the previous lemma for the sentences in Φ. The first condition of swapping follows if $r \geq r'$. For the second condition, let $r \geq 3r'$.

Suppose $\mathfrak{A}$ is a structure and a, b, c and d are elements such that $S^{\mathfrak{A}}(a) \supseteq S^{\mathfrak{A}}(b) \supsetneq S^{\mathfrak{A}}(c) \supsetneq S^{\mathfrak{A}}(d)$, $(\mathfrak{A}, a/x) \sim_r (\mathfrak{A}, c/x)$ and $(\mathfrak{A}, b/x) \sim_r (\mathfrak{A}, d/x)$. Let $\mathfrak{A}' = W(\mathfrak{A}, a, b, c, d)$. Then $W(\mathfrak{A}, a, c) = W(\mathfrak{A}', b, d)$. Note that $d^{\mathfrak{A}}(a, c) = d^{\mathfrak{A}'}(b, d)$, so

FIGURE 3. The case $d^{\mathfrak{A}}(a, c) \leq r'$ and $d^{\mathfrak{A}}(b, d) \leq r'$



if $d^{\mathfrak{A}}(a,c) > r'$, by the previous lemma, $\mathfrak{A} \equiv_\Phi W(\mathfrak{A},a,c)$ and $\mathfrak{A}' \equiv_\Phi W(\mathfrak{A}',b,d)$. If $d^{\mathfrak{A}}(b,d) > r'$ and $a \neq b$, $W(\mathfrak{A},b,d) = W(\mathfrak{A}',a,c)$ and $d^{\mathfrak{A}'}(a,c) = d^{\mathfrak{A}}(b,d)$ so $\mathfrak{A} \equiv_\Phi W(\mathfrak{A},b,d)$ and $\mathfrak{A}' \equiv_\Phi W(\mathfrak{A}',a,c)$. If $d^{\mathfrak{A}}(b,d) > r'$ and $a = b$, $W(\mathfrak{A},b,d) = W(\mathfrak{A}',c,d)$ and $d^{\mathfrak{A}'}(c,d) = d^{\mathfrak{A}}(b,d)$ so we proceed as before.

Thus we have to consider only the case where $d^{\mathfrak{A}}(a,c) \le r'$ and $d^{\mathfrak{A}}(b,d) \le r'$. Fix an isomorphism $\alpha\colon \mathfrak{S}^{\mathfrak{A}}_r(a) \to \mathfrak{S}^{\mathfrak{A}}_r(c)$. Let $d_0 = d$ and $d_{i+1} = \alpha(d_i)$ for all i such that $d_i \in S^{\mathfrak{A}}_r(a)$. Note that $(\mathfrak{A}, d_i/x) \sim_{r-d(a,d_i)} (\mathfrak{A}, d_{i+1}/x)$ Let d' be the first d_i such that $d^{\mathfrak{A}}(a,d) > 2r'$. Then $(\mathfrak{A}, d'/x) \sim_{r'} (\mathfrak{A}, d/x)$.

Now $S^{\mathfrak{A}}_{r'+1}(d) \cap S^{\mathfrak{A}}_{r'+1}(d') = \emptyset$ implies $W(\mathfrak{A},a,b,c,d') \equiv_\Phi \mathfrak{A}$ and together with $S^{W(\mathfrak{A},a,b,c,d')}_{r'+1}(d) \cap S^{W(\mathfrak{A},a,b,c,d')}_{r'+1}(b) = \emptyset$, we get $\mathfrak{A} \equiv_\Phi W(W(\mathfrak{A},a,b,c,d'),d,d,b,d') \cong W(\mathfrak{A},a,b,c,d)$ (see Figure 3). $\square$

Lemma 7.5. *Let $\mathcal{L}$ a logic such that $\mathcal{L} \circ \mathrm{QF}$ is symmetrically Gaifman-local. Then for all finite $\Phi \subseteq \mathcal{L}[\tau]$, there is $r \in \mathbb{N}$ such that $\mathfrak{A} \leftrightarrows^{\sim_r} \mathfrak{A}'$ implies $\mathfrak{A} \equiv_\Phi \mathfrak{A}'$.*

Proof. By the previous lemma, there is $r \in \mathbb{N}$ such that $(\sim_r, \Phi)$ admits swapping. Therefore $\mathfrak{A} \leftrightarrows^{\approx_r} \mathfrak{A}'$ implies $\mathfrak{A} \equiv_\Phi \mathfrak{A}'$ by Lemma 7.2. Now, this lemma follows because $\approx_r$ and $\sim_{r+1}$ are the same equivalence relations. $\square$

Theorem 7.6. *Symmetrically Gaifman-local logics with $\mathcal{L} \circ \mathrm{QF} \le \mathcal{L}$ are Hanf-local on trees.*

Proof. Follows from the previous lemma and the fact that $\mathfrak{N}^{\mathfrak{A}}_r(a) \cong \mathfrak{N}^{\mathfrak{A}'}_r(a')$ implies $(\mathfrak{A}, a/x) \sim_r (\mathfrak{A}', a'/x)$. $\square$

7.2. *$FO(D_k)$* **-Hanf -locality.** For this section fix a logic $\mathcal{L}$ that satisfies $\mathcal{L} \circ \mathrm{QF} \le \mathcal{L}$, is symmetrically Gaifman-local and has uniform reduction. Let $K \subseteq \mathbb{Z}_+$ be such that $\mathcal{L}$ is uniformly K, g_τ-periodic for all vocabularies τ (possibly $K = \mathbb{N}$) and K is closed under lcm, i.e., if $k, k' \in K$, then $\mathrm{lcm}(k,k') \in K$.

We begin by proving some auxiliary lemmas.

Lemma 7.7. *Suppose $\varphi \in \mathcal{L}[\tau \cup \{U\}]$. There is $r \in \mathbb{N}$ such that if $\mathfrak{A}$ is a structure and $Y \subseteq \mathrm{Dom}(\mathfrak{A})$ a set of elements such that for all $a, b \in Y$, $\mathfrak{N}^{\mathfrak{A}}_r(a) \cong \mathfrak{N}^{\mathfrak{A}}_r(b)$ and $a \neq b \Rightarrow d^{\mathfrak{A}}(a,b) > 2r$, then for all $Z, Z' \subseteq Y$, $|Z| = |Z'|$ implies $(\mathfrak{A}, Z/U) \equiv_\varphi (\mathfrak{A}, Z'/U)$.*

Proof. Let $\psi \in I(\mathrm{QF}, \tau \cup \{U,x,y\}, \tau \cup \{U\})$ that maps identically all other relation symbols than U and $\psi_U(z) = Uz \vee z = x$. Let r be the radius witnessing symmetrical Gaifman-locality of $\varphi \circ \psi$.

If $\mathfrak{A}$ and Y satisfy the conditions of the lemma, $Z \subset Y$ and $a, b \in Y \setminus \{a,b\}$, $a \neq b$, then

$$\begin{aligned} (\mathfrak{A}, Z \cup \{a\}/U) &= \psi^*(\mathfrak{A}, Z \cup /U, a/x, b/y) \\ \equiv_\varphi \quad & \psi^*(\mathfrak{A}, Z \cup /U, b/x, a/y) = (\mathfrak{A}, Z \cup \{b\}/U) \end{aligned}$$

Changing elements one by one as above, we can prove for all $Z, Z' \subseteq Y$ satisfying $|Z| = |Z'|$ that $(\mathfrak{A}, Z/U) \equiv_\varphi (\mathfrak{A}, Z'/U)$. $\square$

Lemma 7.8. *If $\Phi \subseteq \mathcal{L}[\tau]$, there are natural numbers m and $k \in K$ such that if M is a collection of disjoint relational structures and $|M| > m$, there is $M' \subseteq M$, $|M'| = |M| - k$, such that $\bigsqcup M' \equiv_\Phi \bigsqcup M$.*

Proof. Let $\equiv_{\Phi'}$, $\Phi' \subseteq \mathcal{L}[\tau]$, be a finite equivalence relation witnessing uniform reduction of all sentences in Φ and let s be the number of its equivalence classes. Set $m = (n + 2k)s$.

Because $\mathcal{L}$ is uniformly K, g_τ-periodic and K is closed under lcm, there are $k \in K$ and $n \in \mathbb{N}$ such that for all τ-structures $\mathfrak{A}$ and $i \in \mathbb{N}$, $g_\tau(i, \mathfrak{A}) \equiv_\Phi g_\tau(\mathrm{cut}_{n,k}(i), \mathfrak{A})$.

If $|M| > m$, some equivalence class $X \subseteq M$ of $\equiv_{\Phi'}$ in M has at least $n + 2k$ elements. Choose $\mathfrak{A} \in X$. We can show by uniform reduction that

$$\begin{aligned} \bigsqcup M &\equiv_\Phi g_\tau(|X|, \mathfrak{A}) \sqcup \bigsqcup (M \setminus X) \\ \equiv_\Phi \quad & g_\tau(|X| - k, \mathfrak{A}) \sqcup \bigsqcup (M \setminus X) \equiv_\Phi \bigsqcup M \setminus Y, \end{aligned}$$

where Y is any subset of X with k structures. Now $M' = M \setminus Y$ satisfies the conclusion of the lemma. □

Lemma 7.9. *Suppose $\Phi \subseteq \mathcal{L}[\tau \cup \{U\}]$ is finite. There are $r, l \in \mathbb{N}$ and $k \in K$ such that if $\mathfrak{A}$ is a structure and $Y \subseteq \mathrm{Dom}(\mathfrak{A})$ a set of elements such that for all $a, b \in Y$, $a \neq b$, $\mathfrak{N}_r^{\mathfrak{A}}(a) \cong \mathfrak{N}_r^{\mathfrak{A}}(b)$ and a and b are in different components of the structure, then for all $Z, Z' \subseteq Y$, $l \leq |Z| \leq |Z'| \leq |Y| - l$ and $|Z| \equiv |Z'| \pmod{k}$ imply $(\mathfrak{A}, Z/U) \equiv_\Phi (\mathfrak{A}, Z'/U)$.*

Proof. Let $\Phi' \subseteq \mathcal{L}[\tau \cup \{U\}]$ be a finite set of sentences that witnesses uniform reduction of all sentences in Φ. Let l_0 be the number of equivalence classes of $\equiv_{\Phi'}$ and let m and k be constants given by Lemma 7.8 for the set Φ'. Put $l = l_0^2 m + 1$.

By Lemma 7.7, when r is chosen sufficiently big, $|Z| = |Z'|$ implies $(\mathfrak{A}, Z/U) \equiv_\Phi (\mathfrak{A}, Z'/U)$. Hence it suffices to prove that for all $l \leq i \leq |Y| - l - k$ there are some sets $Z, Z' \subseteq Y$ such that $|Z| = i$, $|Z'| = i + k$ and $(\mathfrak{A}, Z/U) \equiv_\Phi (\mathfrak{A}, Z'/U)$. Applying this repeatedly gives the lemma.

For all $a \in Y$, let C_a be the component of $\mathfrak{A}$ to which a belongs. Since $|Y| \geq l$, there exists a subset $Y_0 \subseteq Y$ such that $|Y_0| = m + 1$ and for all $a, b \in Y_0$, $(\langle C_a \rangle^{\mathfrak{A}}, \emptyset/U) \equiv_{\Phi'} (\langle C_b \rangle^{\mathfrak{A}}, \emptyset/U)$ and $(\langle C_a \rangle^{\mathfrak{A}}, \{a\}/U) \equiv_{\Phi'} (\langle C_b \rangle^{\mathfrak{A}}, \{b\}/U)$.

Choose some a_0 and let $\mathfrak{C} = \langle C_{a_0} \rangle^{\mathfrak{A}}$. Let $\mathfrak{A}' = \langle \mathrm{Dom}(\mathfrak{A}) \setminus \bigcup_{a \in Y_0} C_a \rangle^{\mathfrak{A}}$. For all $Z, Z' \subseteq Y \setminus Y_0$ such that $|Z'| = |Z| - |Y_0| + k$,

$$\begin{aligned} (\mathfrak{A}, Z/U) &\equiv_\Phi (\mathfrak{A}', Z/U) \sqcup g_{\tau \cup \{U\}}(|Y_0|, (\mathfrak{C}, \emptyset/U)) \\ &\equiv_\Phi (\mathfrak{A}', Z/U) \sqcup g_{\tau \cup \{U\}}(|Y_0| - k, (\mathfrak{C}, \emptyset/U)) \\ &\equiv_\Phi (\mathfrak{A}', Z'/U) \sqcup g_{\tau \cup \{U\}}(|Y_0| - k, (\mathfrak{C}, \{a\}/U)) \\ &\equiv_\Phi (\mathfrak{A}', Z'/U) \sqcup g_{\tau \cup \{U\}}(|Y_0|, (\mathfrak{C}, \{a\}/U)) \\ &\equiv_\Phi (\mathfrak{A}, (Z' \cup Y_0)/U). \end{aligned}$$

If $|Y_0| \leq i \leq |Y| - |Y_0|$ such sets with $i = |Z|$ can be found and we are done. □

Lemma 7.10. *For every finite $\Phi \subseteq \mathcal{L}[\tau]$, there are $m \in \mathbb{N}$ and $k \in K$ such that if $\mathfrak{A}$ is a tree (possibly with some edges removed), $a \in \mathrm{Dom}(\mathfrak{A})$, $Y \subseteq \mathrm{Dom}(\mathfrak{A})$ is a set of children of a, and $|Y| \geq m$, then for some $Y' \subseteq Y$, $|Y'| = k$, $\mathfrak{A} \equiv_\Phi \langle \mathrm{Dom}(\mathfrak{A}) \setminus \bigcup_{y' \in Y'} S^{\mathfrak{A}}(y') \rangle^{\mathfrak{A}}$.*

Proof. Let $\psi \in I(\mathrm{QF}, \tau \cup \{z, U\}, \tau)$ such that $\psi_E(x, y) = Exy \vee (x = z \wedge Uy)$ and ψ^* maps other relations identically. Define $\Phi' = \{\varphi \circ \psi \mid \varphi \in \Phi\}$ and choose $m \in \mathbb{N}$ and $k \in K$ as in Lemma 7.8 for Φ'.

Assume $\mathfrak{A}$, $a \in \mathrm{Dom}(\mathfrak{A})$ and $Y \subseteq \mathrm{Dom}(\mathfrak{A})$ satisfy the conditions of the lemma. Define a new structure $\mathfrak{A}'$ so that every relation except E has the same interpretation as in $\mathfrak{A}$ and $E^{\mathfrak{A}'} = E^{\mathfrak{A}} \setminus (\{a\} \times Y)$. Then $\mathfrak{A} = \psi^*((\mathfrak{A}', a/z, Y/U))$.

Because we assumed that $\mathfrak{A}$ is a tree, for every $a \in Y$, $S^{\mathfrak{A}}(a)$ is a component of $\mathfrak{A}'$. By the choice of m and k, there exists $Y' \subseteq Y$ such that $|Y'| = k$ and $\mathfrak{A}' \equiv_{\Phi'} \left\langle \mathrm{Dom}(\mathfrak{A}) \setminus \bigcup_{a \in Y'} S^{\mathfrak{A}}(a) \right\rangle^{\mathfrak{A}'}$. This implies $\mathfrak{A} \equiv_\Phi \left\langle \mathrm{Dom}(\mathfrak{A}) \setminus \bigcup_{a \in Y'} S^{\mathfrak{A}}(a) \right\rangle^{\mathfrak{A}}$ and so the lemma has been proven. □

We define next new equivalence relations $\sim_{r,m,k}$. For all r, m and k, let $T_{r,m,k}$ be the set of equivalence classes of $\sim_{r,m,k}$ and let $G^{\mathfrak{A}}_{r,m,k}$ be a function $\mathrm{Dom}(\mathfrak{A}) \to \mathbb{N}^{T_{r,m,k}}$ such that for all $a \in \mathrm{Dom}(\mathfrak{A})$ and $t \in T_{r,m,k}$, $G^{\mathfrak{A}}_{r,m,k}(a)(t) = |\{x \in c_{\mathfrak{A}}(a) \mid x \in t\}|$.

The equivalence relations are defined recursively:

$$(\mathfrak{A}, a/x) \sim_{0,m,k} (\mathfrak{A}', a'/x) \iff \langle \{a\} \rangle^{\mathfrak{A}} \cong \langle \{a'\} \rangle^{\mathfrak{A}'}$$

and

$$\begin{aligned} &(\mathfrak{A}, a/x) \sim_{r+1,m,k} (\mathfrak{A}', a'/x) \\ \iff\ &(\mathfrak{A}, a/x) \sim_{r,m,k} (\mathfrak{A}', a'/x) \text{ and for all } t \in T_{r,m,k}, \\ &\mathrm{cut}_{m,k}(G^{\mathfrak{A}}_{r,m,k}(a)(t)) = \mathrm{cut}_{m,k}(G^{\mathfrak{A}'}_{r,m,k}(a')(t)). \end{aligned}$$

Let $\approx_{r,m,k}$ be the respective $\approx$-relation. Note that $(\mathfrak{A}, a/x) \approx_{r,m,k} (\mathfrak{A}', a'/x)$ if and only if $G^{\mathfrak{A}}_{r,m,k}(a) = G^{\mathfrak{A}'}_{r,m,k}(a')$.

Lemma 7.11. *For every finite $\Phi \subseteq \mathcal{L}[\tau]$, there are $r, m \in \mathbb{N}$ and $k \in K$ such that if $(\mathfrak{A}, a/x) \sim_{r,m,k} (\mathfrak{A}, b/x)$ and $S^{\mathfrak{A}}_{r+1}(a) \cap S^{\mathfrak{A}}_{r+1}(b) = \emptyset$ then $W(\mathfrak{A}, a, b) \equiv_\Phi \mathfrak{A}$.*

Proof. Let ψ and ψ' be interpretations in $I(\mathrm{QF}, \tau \cup \{v_0, v_1, v_2, v_3\}, \tau)$ that map all other relations in τ identically, but $\psi_E(x, y) = Exy \vee (x = v_2 \wedge y = v_0) \vee (x = v_1 \wedge y = v_3)$ and $\psi'_E(x, y) = Exy \vee (x = v_2 \wedge y = v_1) \vee (x = v_0 \wedge y = v_3)$. Let $\Phi' = \{\varphi \circ \psi \mid \varphi \in \Phi\}$ and $\Phi'' = \{\varphi \circ \psi' \mid \varphi \in \Phi\}$. Choose a radius r so big that it witnesses symmetrical Gaifman-locality of all sentences in Φ'. Let m and $k \in K$ be constants given by Lemma 7.10 for the set $\Phi' \cup \Phi''$.

Assume $\mathfrak{A}$ and $a, b \in \mathrm{Dom}(\mathfrak{A})$ are as in the lemma. Let $c = p_{\mathfrak{A}}(a)$ and $d = p_{\mathfrak{A}}(b)$. Let $\mathfrak{A}'$ be a τ-structure with unary relations having the same interpretations as in $\mathfrak{A}$, but $E^{\mathfrak{A}'} = E^{\mathfrak{A}} \setminus \{(c, a), (d, b)\}$.

By applying Lemma 7.10 repeatedly at every element $p \in S^{\mathfrak{A}}(a)$ such that $l = d^{\mathfrak{A}}(a, p) < r$ and $G^{\mathfrak{A}}_{r-l,m,k}(p)(t) \geq m + k$ to the subtrees of the type t, we find a $\Phi \cup \Phi'$-equivalent substructure $\mathfrak{B}$ of $\mathfrak{A}'$ so that the $\sim_{r,m,k}$-class of a is same as in

$\mathfrak{A}'$ and every element $p \in S^{\mathfrak{B}}_{r-1}$ with $l = d^{\mathfrak{B}}(a,p) < r$ satisfies $G^{\mathfrak{B}}_{r-l,m,k}(p)(t) = \mathrm{cut}_{m,k}(G^{\mathfrak{B}}_{r-l,m,k}(p)(t))$. By doing the same thing in the subtree of b, we find a substructure $\mathfrak{C}$ of $\mathfrak{A}'$ such that $(\mathfrak{C}, abcd/\overline{v}) \equiv_{\Phi' \cup \Phi''} (\mathfrak{A}', abcd/\overline{v})$ and $\mathfrak{S}^{\mathfrak{C}}_r(a) \cong \mathfrak{S}^{\mathfrak{C}}_r(b)$.

Now, $(\mathfrak{A}', abcd/\overline{v}) \equiv_{\Phi'} (\mathfrak{C}, abcd/\overline{v}) \equiv_{\Phi'} (\mathfrak{C}, bacd/\overline{v}) \equiv_{\Phi'} (\mathfrak{A}', bacd/\overline{v})$. The last equivalence is an implication of $(\mathfrak{A}'', abcd/\overline{v}) \equiv_{\Phi''} (\mathfrak{A}', abcd/\overline{v})$ and the middle one follows from symmetrical Gaifman-locality. Because $\psi^*((\mathfrak{A}', abcd/\overline{v})) \cong \mathfrak{A}$ and $\psi^*((\mathfrak{A}', bacd/\overline{v})) \cong W(\mathfrak{A}, a, b)$, this implies $\mathfrak{A} \equiv_{\Phi} W(\mathfrak{A}, a, b)$. □

Lemma 7.12. *For every finite $\Phi \subseteq \mathcal{L}[\tau]$, there are $r, m \in \mathbb{N}$ and $k \in K$ such that if*

- $t \in T_{r,m,k}$,
- $Q \subseteq \mathrm{Dom}(\mathfrak{A})$, $|Q| = k$, *and* $(\mathfrak{A}, a/x) \in t$ *for all* $a \in Q$,
- $p_{\mathfrak{A}}(Q) = \{b\}$,
- $c \in \mathrm{Dom}(\mathfrak{A})$ *such that* $c \notin S^{\mathfrak{A}}(Q)$,
- $G^{\mathfrak{A}}_{r,m,k}(b)(t) \geq m + k$ *and* $G^{\mathfrak{A}}_{r,m,k}(c)(t) \geq m$,

then $M(\mathfrak{A}, Q, c) \equiv_{\Phi} \mathfrak{A}$, where $M(\mathfrak{A}, Q, c)$ is a structure with the same unary relations as $\mathfrak{A}$ and $E^{M(\mathfrak{A},Q,c)} = (E^{\mathfrak{A}} \setminus (\{b\} \times Q)) \cup (\{c\} \times Q)$.

Proof. Let $\psi \in I(\mathrm{QF}, \tau \cup \{v_0, v_1, U_0, U_1\}$ be an interpretation mapping all relations of τ identically except $\psi_E(x,y) = Exy \vee (x = v_0 \wedge U_0 y) \vee (x = v_1 \wedge U_1 y)$. Let $\Phi' = \{\varphi \circ \psi \mid \varphi \in \Phi\}$. Let Φ'' be as Φ' but every occurrence of U_0 replaced by U_0' and U_1 replaced by U_1'. Choose r and l so that the conditions of Lemma 7.9 hold for the set Φ'.

Let m' and k be the constants Lemma 7.10 gives for the set $\Phi' \cup \Phi''$ (note that the vocabulary is $\tau \cup \{U_0, U_1, U_0', U_1', v_0, v_1\}$) and put $m = \max\{m', l\}$.

Let $\mathfrak{A}$, t, Q, b and c be as in the lemma. Define $R_0 = \{a \in c_{\mathfrak{A}}(b) \mid (\mathfrak{A}, a/x) \in t\}$ and $R_1 = \{a \in c_{\mathfrak{A}}(c) \mid (\mathfrak{A}, a/x) \in t\}$ and let $\mathfrak{A}'$ be a τ-structure with $E^{\mathfrak{A}'} = E^{\mathfrak{A}} \setminus ((\{b\} \times R_0) \cup (\{c\} \times R_1))$ and unary relations as in $\mathfrak{A}$. Then if we manage to establish $(\mathfrak{A}', R_0/U_0, R_1/U_1, b/v_0, c/v_1) \equiv_{\Phi'} (\mathfrak{A}', (R_0 \setminus Q)/U_0, (R_1 \cup Q)/U_1, b/v_0, c/v_1)$, we get $\mathfrak{A} \equiv_{\Phi} M(\mathfrak{A}, Q, c)$.

By the choice of m, we find a substructure $\mathfrak{A}''$ of $\mathfrak{A}'$, as in the proof of Lemma 7.11, such that $R_0 \cup R_1 \subseteq \mathrm{Dom}(\mathfrak{A}'')$,

$$(\mathfrak{A}'', R_0/U_0, R_1/U_1, b/v_0, c/v_1) \equiv_{\Phi'} (\mathfrak{A}', R_0/U_0, R_1/U_1, b/v_0, c/v_1),$$

$$(\mathfrak{A}'', (R_0 \setminus Q)/U_0, (R_1 \cup Q)/U_1, b/v_0, c/v_1) \equiv_{\Phi'} (\mathfrak{A}', (R_0 \setminus Q)/U_0, (R_1 \cup Q)/U_1, b/v_0, c/v_1)$$

and for all $a, a' \in R_0 \cup R_1$, $\mathfrak{N}^{\mathfrak{A}''}_r(a) \cong \mathfrak{N}^{\mathfrak{A}''}_r(a')$. Now, by Lemma 7.9,

$$(\mathfrak{A}'', R_0/U_0, R_1/U_1, b/v_0, c/v_1) \equiv_{\Phi'} (\mathfrak{A}'', (R_0 \setminus Q)/U_0, (R_1 \cup Q)/U_1, b/v_0, c/v_1).$$

□

Lemma 7.13. *For every finite $\Phi \subseteq \mathcal{L}[\tau]$, there are $r, m \in \mathbb{N}$ and $k \in K$ such that $(\sim_{r,m,k}, \Phi)$ admits swapping.*

Proof. The proof of this lemma is similar to the proof of Lemma 7.4. The first condition of the swapping follows directly from the lemma 7.11. The second condition is easy to prove in the case $d^{\mathfrak{A}}(a,c) > r'$ or $d^{\mathfrak{A}}(b,d) > r'$.

Let $r \geq 3r'$ and suppose $d^{\mathfrak{A}}(a,c) \leq r'$ and $d^{\mathfrak{A}}(b,d) \leq r'$. We construct a sequence of elements. Let $d_0 = d$ and for every i let d_{i+1} be an element such that $(\mathfrak{A}, d_{i+1}) \sim_{r',m,k} (\mathfrak{A}, d_i)$ and $d^{\mathfrak{A}}(a, d_i) = d^{\mathfrak{A}}(c, d_{i+1})$. This kind of element can be found if $(\mathfrak{A}, a) \sim_{r,m,k} (\mathfrak{A}, c)$ and $d^{\mathfrak{A}}(a, d_i) \leq 2r'$. Let d' be the first d_i such that $d^{\mathfrak{A}}(a, d_i) > 2r'$. The rest of the proof goes as in Lemma 7.4. □

Lemma 7.14. *For every finite $\Phi \subseteq \mathcal{L}[\tau]$, there are $r, m \in \mathbb{N}$ and $k \in K$ such that $\mathfrak{A} \leftrightarrows^{\sim_{r+1,m,k}} \mathfrak{A}'$ implies $\mathfrak{A} \equiv_\Phi \mathfrak{A}'$.*

Proof. Let $r, m \in \mathbb{N}$ and $k \in K$ be integers such that $(\sim_{r,m,k}, \Phi)$ admits swapping and the lemma 7.12 holds.

Let $\alpha\colon \mathrm{Dom}(\mathfrak{A}) \to \mathrm{Dom}(\mathfrak{A}')$ be a bijection such that for all $a \in \mathrm{Dom}(\mathfrak{A})$, $(\mathfrak{A}, a/x) \sim_{r+1,m,k} (\mathfrak{A}', \alpha(a)/x)$. The number of elements belonging to each equivalence class $t \in T_{r,m,k}$ is the same in $\mathfrak{A}$ and $\mathfrak{A}'$. Therefore if for some $a \in \mathrm{Dom}(\mathfrak{A})$, $G^{\mathfrak{A}}_{r,m,k}(a)(t) < G^{\mathfrak{A}'}_{r,m,k}(\alpha(a))(t)$, there has to be an element $b \in \mathrm{Dom}(\mathfrak{A})$ such that $G^{\mathfrak{A}}_{r,m,k}(b)(t) > G^{\mathfrak{A}'}_{r,m,k}(\alpha(b))(t)$. This implies $G^{\mathfrak{A}}_{r,m,k}(a)(t) \geq m$ and $G^{\mathfrak{A}}_{r,m,k}(b)(t) \geq m + k$. For any $Q \subseteq c_{\mathfrak{A}}(b)$ satisfying $S^{\mathfrak{A}}(Q) \cap S^{\mathfrak{A}}(a) = \emptyset$ and $|Q| = k$, we have $M(\mathfrak{A}, Q, a) \equiv_\Phi \mathfrak{A}$. Applying this transformation repeatedly, we find a structure $\mathfrak{A}'' \equiv_\Phi \mathfrak{A}$ such that for all $a \in \mathrm{Dom}(\mathfrak{A})$, $G^{\mathfrak{A}''}_{r,m}(a) = G^{\mathfrak{A}'}_{r,m}(\alpha(a))$ i.e. $\mathfrak{A}'' \leftrightarrows^{\approx_{r,m}} \mathfrak{A}'$. By Lemma 7.2, $\mathfrak{A}'' \equiv_\Phi \mathfrak{A}$. □

7.3. **Threshold Hanf-locality.** Assume $\mathcal{L}$ and K satisfy the same conditions as in the previous section, i.e., $\mathcal{L} \circ \mathrm{QF} \leq \mathcal{L}$, $\mathcal{L}$ is symmetrically Gaifman-local and has uniform reduction, $K \subseteq \mathbb{Z}_+$ such that $\mathcal{L}$ is uniformly K, g_τ-periodic for all vocabularies τ and K is closed under lcm.

We begin by defining a new periodicity condition. Define a function $g_{\tau,S}$ in the same way as $g_{\tau,<}$, but for structures containing a successor relation. If $\mathfrak{A}$ is a structure with a successor relation, $g_{\tau,S}(i, \mathfrak{A})$ is a union of i copies of $\mathfrak{A}$ with successor relations concatenated.

Proposition 7.15. *$\mathcal{L}$ is uniformly $K, g_{\tau,S}$-periodic for all vocabularies τ.*

Proof. It suffices to prove that $\{i \in \mathbb{N} \mid g_{\tau,S}(i, \mathfrak{A}) \models \varphi\}$ is k-periodic for all τ-structures $\mathfrak{A}$ and sentences $\varphi \in \mathcal{L}[\tau]$, since $\mathcal{L}$ has uniform reduction and K is closed under lcm and so for any fixed φ, the proposition can be reduced to periodicity of $g_{\tau,S}(\cdot, \mathfrak{A})$ for some finite set of structures $\mathfrak{A}$.

Fix φ and $\mathfrak{A}$ and let $n \in \mathbb{N}$ and $k \in K$ witness uniform $K, g_{\tau\cup\{S\}}$-periodicity of φ. Denote a loop of i copies of $\mathfrak{A}$, i.e., the structure $g_{\tau,S}(i, \mathfrak{A})$ with the last and the first element added to the interpretation of S, by $\mathfrak{C}_i$. By symmetrical Gaifman-locality, for big enough m and i, and any j,

$$g_{\tau,S}(m + ij, \mathfrak{A}) \equiv_\varphi g_{\tau,S}(m, \mathfrak{A}) \sqcup g_\tau(j, \mathfrak{C}_i).$$

If $j \geq n$

$$g_{\tau,S}(m,\mathfrak{A}) \sqcup g_\tau(j,\mathfrak{C}_i) \equiv_\varphi g_{\tau,S}(m,\mathfrak{A}) \sqcup g_\tau(j+k,\mathfrak{C}_i)$$

and so $g_{\tau,S}(m+ij,\mathfrak{A}) \equiv_\varphi g_{\tau,S}(m+ij+ik,\mathfrak{A})$. In the same way, assuming m is big enough, $g_{\tau,S}((m-j)+(i+1)j,\mathfrak{A}) \equiv_\varphi g_{\tau,S}((m-j)+(i+1)j+(i+1)k,\mathfrak{A})$. This implies $g_{\tau,S}(m+ij+ik,\mathfrak{A}) \equiv_\varphi g_{\tau,S}((m+ij+ik)+k,\mathfrak{A})$ and so $g_{\tau,S}(i,\mathfrak{A}) \equiv g_{\tau,S}(i+k,\mathfrak{A})$ for all big enough i. □

Our goal in this section is to prove the following theorem.

Theorem 7.16. *For all $\varphi \in \mathcal{L}[\tau]$, there exists $\Phi \subseteq \mathrm{FO}(D_k)_{k\in K}[\tau \cup \{v_0\}]$, $n, r \in \mathbb{N}$ and $k \in K$ such that if $\mathfrak{A}$ and $\mathfrak{B}$ are τ-trees and for all $\equiv_\Phi$-equivalence classes t,*

$$\mathrm{cut}_{n,k}(|\{a \in \mathrm{Dom}(\mathfrak{A}) \mid \mathfrak{N}_r^{\mathfrak{A}}(a) \in t\}|) = \mathrm{cut}_{n,k}(|\{a \in \mathrm{Dom}(\mathfrak{B}) \mid \mathfrak{N}_r^{\mathfrak{B}}(a) \in t\}|),$$

then $\mathfrak{A} \equiv_\varphi \mathfrak{B}$. In particular, $\mathcal{L} \leq \mathrm{FO}(D_k)_{k\in K}$.

The theorem is, in a way, a generalization of Hanf's theorem that combines both game-based notions of locality [ABL04] and threshold Hanf-locality used for example in [FSV95]. The theorem is not true even for $\mathrm{FO}(D_k)_{k\in K}$ without the restriction to trees.

Fix a finite set Φ of formulas and let r, m and $k \in K$ be integers such that $\mathfrak{A} \leftrightarrows^{\sim_{r+1,m,k}} \mathfrak{A}'$ implies $\mathfrak{A} \equiv_\Phi \mathfrak{A}'$.

To simplify notations, let $C(\mathfrak{A}) = C^{\sim_{r+1,m,k}}(\mathfrak{A})$. We show that if l and $k' \in K$ are big enough $\mathrm{cut}_{l,k'}(C(\mathfrak{A})) = \mathrm{cut}_{l,k'}(C(\mathfrak{A}'))$ implies $\mathfrak{A} \equiv_\Phi \mathfrak{A}'$. (Also m has to be increased.)

Our next goal is to find when a given vector $c \in \mathbb{N}^{T_{r+1,m,k}}$ has a realization, i.e., a structure $\mathfrak{A}$ such that $C(\mathfrak{A}) = c$. A necessary condition for this is clearly that for every occurrence of some type, there are also elements that realize the types of its children. We formalize next this idea.

In what follows, we use $c \leq c'$ for vectors to mean that $c(t) \leq c'(t)$ for all t, and $c < c'$ for $c \leq c'$ and $c \neq c'$. Also $\operatorname{supp} c = \{t \in T_{r,m,k} \mid c(t) \neq 0\}$.

Let $u \colon T_{r+1,m,k} \to T_{r,m,k}$ be the unique function such that if an element realizes a $T_{r+1,m,k}$-type t, then it realizes $T_{r,m,k}$-type $u(t)$. Let $v \colon T_{r+1,m,k} \to \mathbb{N}^{T_{r,m,k}}$ be a function such that $v(t)(t')$ tells how many children of a realization of $T_{r+1,m,k}$-type t realize the $T_{r,m,k}$-type t', i.e., $(\mathfrak{A}, a/x) \in t$ implies $v(t)(t') = \mathrm{cut}_{m,k}(G^{\mathfrak{A}}_{r,m,k}(a)(t'))$.

Let H be a $\mathbb{Z}$-homomorphism $: \mathbb{Z}^{T_{r+1,m,k}} \to \mathbb{Z}^{T_{r,m,k}}$:

$$H(c) = \sum_{t \in T_{r+1,m,k}} (\chi_{u(t)} - v(t))c(t),$$

where $\chi_t \in \mathbb{N}^{T_{r,m,k}}$ is the characteristic function of the singleton $\{t\}$. For a given c, $H(c)$ tells the number of realizations of each $T_{r,m,k}$-type minus the number of $T_{r,m,k}$ -types needed to realize all $T_{r+1,m,k}$ -types. Hence it is required that $H(c) \geq \overline{0}$ for c to be realizable.

If $H(c)(t) > \overline{0}$ and t is not the type of the root, then if we can realize c, some element of the realization must have more children of type t than is necessary to realize the element. Let $F(c) = \bigcup_{t \in \text{supp } (c)} \{t' \in T_{r,m,k} \mid v(t)(t') \geq m\}$. If c can be realized, $H(c)(t) > \overline{0}$ implies $t \in F(c)$ or that t is the type of the root.

We next consider how to add new elements to trees. We call $c \in \mathbb{N}^{T_{r+1,m,k}}$ *minimal* if $c > \overline{0}$, $H(c) \geq \overline{0}$ and if $c' \in \mathbb{N}^{T_{r+1,m,k}}$ is another vector such that $c \geq c' > \overline{0}$ and $H(c) \geq H(c') \geq \overline{0}$, then $c' = c$.

Lemma 7.17. *If $c \in \mathbb{N}^{T_{r+1,m,k}}$ is minimal $H(c) = \overline{0}$ or $H(c) = \chi_t$ for some $t \in T_{r,m,k}$.*

Proof. Suppose that $c \in \mathbb{N}^{T_{r+1,m,k}}$ is a counterexample. Choose some element $a \in$ supp $H(c)$. We define a sequence $c_0, \ldots, c_j$ such that $c_0 = c$, $c_i > c_{i+1} > \overline{0}$ for all $i < j$ and $H(c_j) = \chi_a$.

If c_i has been defined, let $t \in$ supp c_i be an element such that $u(t) \in$ supp $(H(c_i) - \chi_a)$ and put $c_{i+1} = c_i - \chi_t$. Then $c_i > c_{i+1} > \overline{0}$ and $H(c_{i+1}) = H(c_i) - \chi_{u(t)} + v(t) \geq \chi_a$.

Now the element c_j contradicts the minimality of c. □

We construct for every minimal element $c \in \mathbb{N}^{T_{r+1,m,k}}$ a τ-structure $\mathfrak{D}_c$. These structures will be building blocks we attach to trees when pumping new elements to them. The universe of the structure, $\text{Dom}(\mathfrak{D}_c)$, has cardinality $\sum_{t \in T_{r+1,m,k}} c(t)$. We assign to every element $a \in \text{Dom}(\mathfrak{D}_c)$ a type $t^c_a \in T_{r+1,m,k}$ such that for every $t \in T_{r+1,m,k}$, $|\{a \in \text{Dom}(\mathfrak{D}_c) \mid t^c_a = t\}| = c(t)$. We define the unary relations of τ so that every element $a \in \text{Dom}(\mathfrak{D}_c)$ is in the same relations as if a were a realization of t^c_a.

Relation $E^{\mathfrak{D}_c}$ is defined so that a is connected to exactly $v(t^c_a)(t)$ many elements b with $u(t_b) = t$ and there is at most one element connected to each element. Self-loops are allowed. The existence of the relation follows from $H(c) \geq \overline{0}$.

In this way, we have constructed a structure that realizes c, but is not necessarily a tree. The structure is connected because if it were not, the types realized by one of its components would contradict the minimality of c. If $H(c) = 0$, then the structure contains one cycle, otherwise $H(c) = \chi_t$ and $\mathfrak{D}_c$ is a tree whose root has type t.

If $\mathfrak{D}_c$ is a tree, let q_c be its root. Otherwise, choose q_c to be one of the elements on its cycle. In that case, let q'_c be the predecessor of q_c, i.e., $(q'_c, q_c) \in E^{\mathfrak{D}_c}$.

Let $\mathfrak{A}$ be a tree, $a \in \text{Dom}(\mathfrak{A}) \setminus \{r_{\mathfrak{A}}\}$ and c still a minimal vector. We define a new structure $\mathfrak{A}' = I(\mathfrak{A}, c, a)$ in the following way: The universe of $\mathfrak{A}'$ is disjoint union of the sets $\text{Dom}(\mathfrak{A})$ and $\text{Dom}(\mathfrak{D}_c)$ and the unary relations are also defined as unions of respective relations on $\mathfrak{A}$ and $\mathfrak{D}_c$. Let a' be the predecessor of a. If $H(c) \neq \overline{0}$, put

$$E^{\mathfrak{A}'} = E^{\mathfrak{A}} \cup E^{\mathfrak{D}_c} \cup \{(a', q_c)\}.$$

If $H(c) = \overline{0}$, put

$$E^{\mathfrak{A}'} = (E^{\mathfrak{A}} \cup E^{\mathfrak{D}_c} \cup \{(a', q_c), (q'_c, a)\}) \setminus \{(a', a), (q'_c, q_c)\}.$$

Let $I_1(\mathfrak{A}, c, a) = I(\mathfrak{A}, c, a)$ and $I_{n+1}(\mathfrak{A}, c, a) = I(I_n(\mathfrak{A}, c, a), c, a)$.

Lemma 7.18. *If $k|n$, $(\mathfrak{A}, a/x) \in t_{q_c}$ and in the case $H(c) \neq \overline{0}$, a' has at least m children of type t_{q_c}, then $C(I_n(\mathfrak{A}, c, a)) = C(\mathfrak{A}) + nc$.*

Proof. First thing to note is that $I_n(\mathfrak{A}, c, a)$ is always a tree. The lemma is proved showing by induction on $r' \leq r + 1$ that $\sim_{r',m,k}$ -type of every element in $\mathrm{Dom}(\mathfrak{A})$ is preserved and if y is an element in $\mathrm{Dom}(\mathfrak{D}_c)$ and y' is corresponding element in some copy of $\mathfrak{D}_c$, then $\sim_{r',m,k}$ -type of y' is t_y. For $r' = 0$, the only thing to check out is that unary relations are defined properly. The induction step is clear for elements in $\mathrm{Dom}(\mathfrak{A}) \setminus \{a'\}$. If $H(c) = \overline{0}$, $\sim_{r',m,k}$ -type of a' is preserved, because $(I_n(\mathfrak{A}, c, a), q_c/x) \sim_{r',m,k} (\mathfrak{A}, a/x)$. Preservation of the types of other elements follows similarly. If $H(c) \neq \overline{0}$, type of x' is preserved, because it already has m children of the type of a_c and we add n new elements, where $k|n$. □

If $\mathrm{supp}\ c \subseteq \mathrm{supp}\ C(\mathfrak{A})$ and $\mathrm{supp}\ H(c) \subseteq F(C(\mathfrak{A}))$ we always find an element a satisfying the conditions of the lemma. So we may drop the parameter a, $I_n(\mathfrak{A}, c) = I_n(\mathfrak{A}, c, a)$. The next lemma allows us to pump elements to trees while keeping them Φ-equivalent.

Lemma 7.19. *There are $l_0 \in \mathbb{N}$ and $k' \in K$, $k|k'$, $k'|l_0$, such that if* $\mathrm{supp}\ c \subseteq \mathrm{supp}\ C(\mathfrak{A})$ *and* $\mathrm{supp}\ H(c) \subseteq F(C(\mathfrak{A}))$*, then $I_{l_0}(\mathfrak{A}, c) \equiv_\Phi I_{l_0+k'}(\mathfrak{A}, c)$.*

Proof. If $H(c) \neq \overline{0}$, this follows from Lemma 7.10 when we apply it to the structure $I_{l_0+k'}(\mathfrak{A}, c, x)$. We choose x to be a in the lemma and Y the set of all children of x not belonging to $\mathrm{Dom}(\mathfrak{A})$.

If $H(c) = \overline{0}$, we use the fact that $\mathcal{L}$ is uniformly $K, g_{\tau,S}$-periodic. The structures $I_n(\mathfrak{A}, c)$ can be divided to two parts where one contains the elements in $\mathrm{Dom}(\mathfrak{A})$ and another is interpretable in $g_{\tau,S}(n, S(\mathfrak{D}_c, S))$. Using uniform reduction, we can find l_0 and $k' \in K$ such that $I_{l'}(\mathfrak{A}, c) \equiv_\Phi I_{l'+k'}(\mathfrak{A}, c)$ for all $l' \geq l_0$. □

We proved the lemmas assuming that c is minimal. If c is not minimal, but $c \geq \overline{0}$ and $H(c) \geq \overline{0}$, it can be written as a sum of minimal elements. We then notice that by applying the lemmas for every term of the sum, we get the results also in the general case.

Lemma 7.20. *If A is a finite set and $B \subseteq \mathbb{N}^A$ is an arbitrary subset it has a finite number of minimal elements.*

Proof. Suppose that B has an infinite set of minimal elements M. Let $<$ be an arbitrary linear order on M. For each pair $(a, b) \in M^2$ such that $a < b$, assign a color $\{i \in A \mid a(i) > b(i)\}$. The color is never $\emptyset$ because all elements in M are minimal. By Ramsey's theory, there is an infinite sequence of elements $a_0 < a_1 < \cdots$ such that $B = \{i \in A \mid a_j(i) > a_{j+1}(i)\}$ does not depend on j. But if $i \in B$, $a_0(i) > a_1(i) > \cdots$ is a strictly decreasing infinite sequence of natural numbers, that is a contradiction. □

Now we obtain necessary and sufficient conditions for c to be realizable. We use abbreviations $a \equiv b \pmod{k}$ and $k|a$ for vectors a and b to mean that vectors are componentwise equal modulo k or k divides every component of the vector.

Lemma 7.21. *There is a finite set $V \subseteq \mathbb{N}^{T_{r,m,k}}$ such that for every $c \in \mathbb{N}^{T_{r+1,m,k}}$:*

There is a tree $\mathfrak{A}$ such that $C(\mathfrak{A}) = c$ if and only if there exists $x \in V$ such that $x \leq c$, $x \equiv c \pmod{k}$, $\operatorname{supp} x = \operatorname{supp} c$, $H(c-x) \geq 0$ and $\operatorname{supp} H(c-x) \subseteq F(x)$.

Proof. For every $c \in \mathbb{N}^{T_{r,m,k}}$, let M_c be the set of all trees $\mathfrak{A}$ such that $\operatorname{supp} C(\mathfrak{A}) = \operatorname{supp} c$ and $C(\mathfrak{A}) \equiv c \pmod{k}$. Let V_c be the set of elements $x \in \mathbb{N}^{T_{r+1,m,k}}$ such that $(x, H(x))$ is a minimal element in the set $\{(C(\mathfrak{A}), H(C(\mathfrak{A}))) \mid \mathfrak{A} \in M_c\}$. By the previous lemma each set V_c is finite. Because M_c depends only on $\operatorname{supp} c$ and c modulo k the union $V = \bigcup_{c \in \mathbb{N}^{T_{r,m,k}}} V_c$ is also finite.

Assume $\mathfrak{A}$ is a tree and $c = C(\mathfrak{A})$. Let $x \in V_c \subseteq V$ be an element such that $x \leq c$ and $H(x) \leq H(c)$. This is possible, because V_c contains all minimal elements realizable on some structure in M_c and $c \in M_c$. Now the first four conditions are satisfied. Because c and x are realizable, $\operatorname{supp} H(c) \subseteq F(c)$ and $\operatorname{supp} H(x) \subseteq F(x)$. $F(x)$ depends only on $\operatorname{supp} x$ so $F(c) = F(x)$. Thus $\operatorname{supp} H(c-x) \subseteq \operatorname{supp} H(c) \cup \operatorname{supp} H(x) \subseteq F(x)$.

Assume then that c satisfies all conditions of the lemma. Let $x \in V$ be as in the condition. Because all elements in V are realizable, we find a structure $\mathfrak{A}_0$ such that $x = C(\mathfrak{A}_0)$. Let $b = \frac{c-x}{k}$. We know that $b \geq 0$, $H(b) \geq 0$, $\operatorname{supp} b \subseteq \operatorname{supp} C(\mathfrak{A})$ and $\operatorname{supp} H(b) \subseteq F(x)$. Thus $C(I_k(\mathfrak{A}_0, b)) = c$. $\square$

Lemma 7.22. *There is $l_1 \in \mathbb{N}$ such that if*

$$\operatorname{cut}_{l_1,k'}(C(\mathfrak{A})) = \operatorname{cut}_{l_1,k'}(C(\mathfrak{A}')),$$

$$\operatorname{cut}_{l_1,k'}(H(C(\mathfrak{A})))) = \operatorname{cut}_{l_1,k'}(H(C(\mathfrak{A}'))),$$

$C(\mathfrak{A}) \leq C(\mathfrak{A}')$ and $H(C(\mathfrak{A})) \leq H(C(\mathfrak{A}'))$, then $\mathfrak{A} \equiv_\Phi \mathfrak{A}'$.

Proof. If $X \subseteq \mathbb{N}^I$, denote $||X|| = \max\{x_i \mid x \in X, i \in I\}$. Let M be the set of all minimal $c \in \mathbb{N}^{T_{r+1,m,k}}$. We put $l_1 = ||V|| + l_0||M||$. (Note that if $x \in V$ then $(H(x))_t \leq ||V||$.) Because every $c \in \mathbb{N}^{T_{r+1,m,k}}$ was a sum of minimal elements, we can assume without loss of generality that $b = \frac{C(\mathfrak{A}')-C(\mathfrak{A})}{k'}$ is minimal. Let $x \in V$ be any element satisfying the conditions of the previous lemma with $C(\mathfrak{A})$. If $t \in \operatorname{supp} b$, $C(\mathfrak{A})(t) \geq l_1$. Hence $c = C(\mathfrak{A}) - l_0 b \geq x$ also satisfies the conditions of the previous lemma and there is a tree $\mathfrak{A}''$ such that $C(\mathfrak{A}'') = c$. This gives

$$\mathfrak{A} \equiv_\Phi I_{l_0}(\mathfrak{A}'', b) \equiv_\Phi I_{l_0+k'}(\mathfrak{A}'', b) \equiv_\Phi \mathfrak{A}'.$$

$\square$

Now the last obstacle is to prove the equivalence also in the case where $C(\mathfrak{A})$ and $C(\mathfrak{A}')$ are incomparable. This is done by finding a new tree $\mathfrak{A}''$ bigger then $\mathfrak{A}$ or $\mathfrak{A}'$ such that we can use the previous lemma to prove equivalences $\mathfrak{A} \equiv_\Phi \mathfrak{A}'' \equiv_\Phi \mathfrak{A}'$.

If we can find $e \geq 0$, $H(e) \geq 0$ such that $\operatorname{supp}(C(\mathfrak{A}) - C(\mathfrak{A}')) \subseteq \operatorname{supp} e$, $\operatorname{supp} H(C(\mathfrak{A}) - C(\mathfrak{A}')) \subseteq \operatorname{supp} H(e)$, but $t \in \operatorname{supp} e \Rightarrow \min\{C(\mathfrak{A})(t), C(\mathfrak{A}')(t)\} \geq l_1$ and $t \in \operatorname{supp} H(e) \Rightarrow \min\{H(C(\mathfrak{A}))(t), H(C(\mathfrak{A}'))(t)\} \geq l_1$ then $\mathfrak{A}'' = I_{nk'}(\mathfrak{A}, e)$ will be suitable when n is chosen large enough.

Lemma 7.23. *Let A be a finite set and $B \subseteq \mathbb{Z}^A$ a submodule. Then there is $\alpha \geq 1$ such that if $a \in B$ and $a \geq \mathbf{0}$, $M/m > \alpha$, $X = \{x \in A \mid a(x) \geq M\}$ and $A \setminus X = \{x \in A \mid a(x) \leq m\}$ there is $e \in B$, $e \geq 0$, such that $X = \operatorname{supp}(e)$.*

Proof. Let $I \subseteq A$ be an arbitrary subset. The projection $p_I \colon \mathbb{Z}^A \to \mathbb{Z}^I$ maps submodule B to submodule $B' \subseteq \mathbb{Z}^I$.

This is a free module, because it is a submodule of $\mathbb{Z}^I$. Therefore it has a basis B. Let $b \colon B \to \mathbb{Z}^A$ such that $p_I \circ b = \mathrm{id}_B$. We can extend this map to linear map $q_I \colon \mathbb{Z}^I \to \mathbb{Z}^A$ such that $p_I \circ q_I = \mathrm{id}_{\mathbb{Z}^I}$.

Let $\alpha = \max\{\|q_I\|_\infty \ : \ I \subseteq A\}$ and suppose a, X, M and m are as in the lemma. Let $I = A \setminus X$. Let $e = a - q_I(p_I(a))$. Because $p_I(e) = p_I(a - q_I(p_I(a))) = p_I(a) - p_I(a) = 0$, supp $e \subseteq X$. Because $\|p_I(a)\|_\infty \leq m$, $\|q_I(p_I(a))\|_\infty \leq \alpha m < M$ and $X \subseteq$ supp e. □

Apply the lemma for the submodule $\{(c, H(c)) \mid c \in \mathbb{Z}^T_{r+1,m,k}\} \subseteq \mathbb{Z}^{T_{r+1,m,k}} \times \mathbb{Z}^{T_{r,m,k}}$. Then put $l = l_1 \alpha^{|T_{r+1,m,k}|+|T_{r,m,k}|}$. Suppose $\mathrm{cut}_{l,k'}(C(\mathfrak{A})) = \mathrm{cut}_{l,k'}(C(\mathfrak{A}'))$ and $\mathrm{cut}_{l,k'}(H(C(\mathfrak{A})))) = \mathrm{cut}_{l,k'}(H(C(\mathfrak{A}')))$. Then we can find m and M such that $l_1 \leq m \leq \alpha m < M \leq l$ and each $C(\mathfrak{A})(t)$ and $H(C(\mathfrak{A}))(t)$ is either greater than M or smaller than m. The lemma gives then e with desired properties. Thus we have proved:

Lemma 7.24. *There is $l \in \mathbb{N}$ such that if $\mathrm{cut}_{l,k'}(C(\mathfrak{A})) = \mathrm{cut}_{l,k'}(C(\mathfrak{A}'))$ and $\mathrm{cut}_{l,k'}(H(C(\mathfrak{A})))) = \mathrm{cut}_{l,k'}(H(C(\mathfrak{A}')))$, then $\mathfrak{A} \equiv_\Phi \mathfrak{A}'$.* □

Now by increasing m we can also drop the second condition. Let $m' = l + m + k'$ and suppose $\mathrm{cut}_{l,k'}(C^{\sim_{r+1,m',k}}(\mathfrak{A})) = \mathrm{cut}_{l,k'}(C^{\sim_{r+1,m',k}}(\mathfrak{A}'))$. Clearly $\mathrm{cut}_{l,k'}(C(\mathfrak{A})) = \mathrm{cut}_{l,k'}(C(\mathfrak{A}'))$. If $H(C(\mathfrak{A}))(t) \neq H(C(\mathfrak{A}'))(t)$ for some $t \in T_{r+1,m,k}$ there is an extension of t, $t' \in T_{r+1,m',k}$, such that $H(C^{\sim_{r+1,m',k}}(\mathfrak{A}))(t') \neq H(C^{\sim_{r+1,m',k}}(\mathfrak{A}'))(t')$ and hence $t' \in F(C^{\sim_{r+1,m',k}}(\mathfrak{A}))$. This means that some element of $\mathfrak{A}$ has at least m' children realizing t'. Thus $H(C(\mathfrak{A})) \geq m' - m - k' = l$. $H(C(\mathfrak{A})) \equiv H(C(\mathfrak{A}')) \pmod{k'}$ follows from the fact $C(\mathfrak{A}) \equiv C(\mathfrak{A}') \pmod{k'}$ and linearity of H.

In this way, we get Lemma 7.24 into the form of Theorem 7.16, where additionally r-level subtrees are replaced by neighborhoods of radius r.

7.4. **Trees and** $\mathrm{FO}_<(D_k)$**.** As a corollary of Theorem 7.16 and propositions we showed in Section 6, we get now:

Corollary 7.25. *If K contains only odd integers, $\mathrm{FO}_<(D_k)_{k \in K} \equiv \mathrm{FO}(D_k)_{k \in \mathrm{PFC}(K)}$ on trees.*

By Proposition 6.22, query EvenP is definable in $\mathrm{FO}_<(D_2)$. The corollary does not hold, if K contains even integers, because of the following proposition and Gaifman-locality of $\mathrm{FO}(D_k)_{k \in \mathbb{Z}_+}$.

Proposition 7.26. *Logic FO(EvenP) is not Gaifman-local on trees.*

Proof. The example used in the proof of Proposition 6.23 is a tree, if we add an edge between elements $(0,0)$ and $(0,1)$ in $\mathfrak{A}_n$ and remove the edge in the interpretation. □

It seems, however, conceivable that EvenP is the only new thing expressible in the order-invariant logic with D_2.

Conjecture 7.27. *If K contains even integers, then*

$$\mathrm{FO}_<(D_k)_{k\in K} \equiv \mathrm{FO}(\mathrm{EvenP}, D_k)_{k\in \mathrm{PFC}(K)}$$

on trees.

If one tries to prove the conjecture using the same proof structure as used here, the first question is, what is the right replacement of Hanf-locality in this case. Because alternating Gaifman-locality allows cycling three up to some level isomorphic and disjoint subtrees, one possibility would call trees alternatingly Hanf-local, if the bijection between trees can be composed from alternating permutations of up to some level isomorphic subtrees.

This seems to work well with different pumping constructions used in the proof, but showing an analogue of Lemma 7.2 is not trivial. Every alternating permutation can be composed from 3-cycles, but the problem is how to order the cycles so that when applying the cycle, subtrees of the elements do not intersect.

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