On the Geometry of Infinite-Dimensional Grassmannian Manifolds and Gauge Theory

Vesa Tähtinen

Academic dissertation

To be presented, with the permission of the Faculty of Science of the University of Helsinki, for public criticism in Auditorium XII, the Main Building of the University, on January 23rd 2010 at 10 a.m.

Faculty of Science
Department of Mathematics and Statistics
University of Helsinki
2010

ISBN 978-952-92-6725-5 (Paperback)
ISBN 978-952-10-5998-8 (PDF)
Yliopistopaino
Helsinki 2009

On this thesis

This thesis consists of an introductory part and two articles or preprints. The articles are

I Vesa Tähtinen: "*Anomalies in Gauge Theory and Gerbes over Quotient Stacks*", Journal of Geometry and Physics **58** (2008) 1080–1100.

II Vesa Tähtinen: "*Dirac Operator on the Restricted Grassmannian Manifold*", Preprint. 81 pages.

Abstract

This thesis is about the geometry and representation theory related to certain infinite dimensional Grassmannian manifolds and their relations with gauge theory.

The first paper. In gauge theories one is interested in lifting the action of the gauge transformation group $\mathcal{G}$ on the space of connection one-forms $\mathcal{A}$ to the total space of the Fock bundle $\mathcal{F} \longrightarrow \mathcal{A}$ in a consistent way with the second quantized Dirac operators $\hat{\not{D}}_A$, $A \in \mathcal{A}$. In general, there is an obstruction to this and one has to introduce a Lie group extension $\hat{\mathcal{G}}$, not necessarily S^1-central, of $\mathcal{G}$ that acts in the Fock bundle.

It was first noticed in the works of J. Mickelsson, [**Mi**] and L. Faddeev, [**Fad**] that in dimensions greater than one the group multiplication in $\hat{\mathcal{G}}$ depends also on the elements $A \in \mathcal{A}$. We give a new interpretation of this phenomenon and show that $\hat{\mathcal{G}}$ can be replaced with a *Lie groupoid* extension of the action groupoid $\mathcal{A} \rtimes \mathcal{G}$. Viewed this way the extension now proves out to be an S^1-*central* extension so that one may apply the general theory of these extension developed by K. Behrend and P. Xu in [**BeXu**].

In particular, one knows then that the S^1-groupoid central extension of $\mathcal{A} \rtimes \mathcal{G}$ corresponds to an S^1-gerbe over the quotient stack $[\mathcal{A}/\mathcal{G}]$. Moreover, it is known that when the action of $\mathcal{G}$ on $\mathcal{A}$ is free and transitive, the stack $[\mathcal{A}/\mathcal{G}]$ is isomorphic to the manifold $\mathcal{A}/\mathcal{G}$ and on the other hand one knows from D. Stevenson's PhD thesis [**Steve**] that S^1-gerbes over manifolds corrensopond to *bundle gerbes* which are geometric objects studied by A. Carey, J. Mickelsson and M. Murray in [**CaMiMu**] to give a geometric interpretation of Hamiltonian anomalies in Yang-Mills theory.

The second paper. In the second paper we construct a Dirac like operator on an *infinite-dimensional* Kähler manifold called the *restricted Grassmannian manifold.* The restricted Grassmannian manifold is a very important infinite-dimensional manifold being related to the representation theory of loop groups as well as to second quantization of fermions [**PreSe**].

The restricted Grassmannian manifold determined by a complex separable polarized Hilbert space $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$ is a homogeneous manifold of the form $\mathrm{Gr}_{\mathrm{res}}(\mathcal{H}, \mathcal{H}_+) \cong \mathcal{U}_{\mathrm{res}}(\mathcal{H}, \mathcal{H}_+)/(U(\mathcal{H}_+) \times U(\mathcal{H}_-))$ where $\mathcal{U}_{\mathrm{res}}$ is the infinite-dimensional *restricted unitary group* defined in [**PreSe**]. We make use of infinite-dimensional wedge representations of the central extension $\hat{\mathcal{U}}_{\mathrm{res}}$ on the *fermionic Fock space* $\mathcal{F}(\mathcal{H}, \mathcal{H}_+)$ to construct a well-defined Dirac like operator acting on a relevant Hilbert space of spinors on the restricted Grassmannian manifold. As our main result we show that our operator is an unbounded symmetric operator with finite-dimensional kernel.

Our Dirac operator construction in infinite-dimensions is motivated by Mickelsson's program to construct new *twisted K-theory classes* related to Yang-Mills theory [**Mi4**]. However, we do not construct any twisted K-theory classes in this paper; the existence of a good Dirac like operator on the restricted Grassmannian manifold is the first step in Mickelsson's philosophy.

Acknowledgments

First and foremost I wish to express my sincere gratitude to my supervisor Jouko Mickelsson for introducing me to the exciting subject this thesis is about. I am also grateful for all his support and encouragement as I pursued towards the solutions of my research problems. Mickelsson's group has provided a pleasant and extremely inspiring environment for research.

I am grateful to the pre-examiners of this thesis, Varghese Mathai (University of Adelaide, Australia) and Martin Schlichenmaier (University of Luxembourg), who carefully read through the manuscript of my PhD Thesis and made suggestions how to improve it.

I would like to thank Kari Vilonen (Northwestern University, USA) for his encouragement and career advice as well as for explaining to me some of the ideas behind geometric representation theory.

I would also like to thank all my previous teachers from whom I have learnt a lot during my studies. In particular, I would like to mention Tauno Metsänkylä, Matti Jutila, Jyrki Lahtonen and Kari Ylinen at the University of Turku.

I also thank Hannu Härkönen and Antti Veilahti for many useful conversations on mathematics, especially on number theory and algebraic geometry, and Juha Loikkanen for explaining me a lot of theoretical physics from the perspective of a mathematician.

Special thanks go to Jan Cristina, who carefully read through a preprint of my second research article "Dirac operator on the restricted Grassmannian manifold" and made suggestions how to correct the grammatic errors it contained.

For financial assistance I am indebted to the Academy of Finland and the Finnish Academy of Science and Letters, Vilho, Yrjö and Kalle Väisälä Foundation.

Furthermore, I would like to thank the Department of Mathematics and Statistics in Helsinki for providing a pleasant working environment.

Finally, I wish to thank my family and Miina for their love and endless support during this process!

Introduction

Contents

CHAPTER 1

Stacks

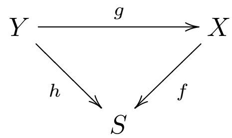
Our main references in this chapter are [**BeXu, Gom, Hein, Sor, Met**]

1. Motivation: Moduli problems in algebraic geometry

Given a scheme M over S, define its (contravariant) *functor of points* $\mathrm{Hom}_S(-, M)$,

$$\mathrm{Hom}_S(-, M) : (Sch/S) \longrightarrow (Sets),$$
$$B \mapsto \mathrm{Hom}_S(B, M).$$

Here (Sch/S) is the category of S-schemes, whose objects are scheme morphisms $f : X \longrightarrow S$ and morphisms are commuting diagrams



THEOREM 1.1. *The functor of points $Hom_S(-, M)$ defines the scheme M uniquely and $Hom_S(-, M)$ is a sheaf (in the étale topology).*

By definition

$$\text{Moduli problems} \leftrightsquigarrow \text{Contravariant functors } F : (Sch/S) \longrightarrow (Sets)$$

DEFINITION 1.2. A (contravariant) functor $F : (Sch/S) \longrightarrow (Sets)$ is *representable* by a scheme M, if there exists an isomorphism of functors $F \cong \mathrm{Hom}_S(-, M)$. The scheme M is then called the *fine moduli space* of F.

Essentially, the existence of a fine moduli space M means that for every S-scheme B, there exists a bijection of sets

$$\text{Families parametrized by } B \overset{\cong}{\longleftrightarrow} \text{Morphisms } B \longrightarrow M$$

EXAMPLE 1.3 (Algebraic vector bundles). Recall, that for any scheme X a *vector bundle* of rank r over X is a scheme Y and a morphism of schemes $f : Y \longrightarrow X$, together with additional data consisting of an open covering $\{U_i\}$ of X and isomorphisms $\psi_i : f^{-1}(U_i) \longrightarrow \mathbb{A}^r_{\mathbb{Z}} \times_{\mathrm{Spec}\,\mathbb{Z}} U_i := \mathbb{A}^r_{U_i}$, such that for any open *affine* subset $V = \mathrm{Spec}\, A \subseteq U_i \cap U_j$, the automorphism $\psi =: \psi_j \circ \psi_i^{-1}$ of

$$\begin{aligned} \mathbb{A}^r_V &= \mathbb{A}_{\mathbb{Z}} \times_{\mathrm{Spec}\,\mathbb{Z}} \mathrm{Spec}\, A = \mathrm{Spec}\, \mathbb{Z}[x_1, \ldots, x_r] \times_{\mathrm{Spec}\,\mathbb{Z}} \mathrm{Spec}\, A \\ &\cong \mathrm{Spec}\, (\mathbb{Z}[x_1, \ldots x_r] \otimes_{\mathbb{Z}} A) \cong \mathrm{Spec}\, A[x_1, \ldots, x_r] \end{aligned}$$

is given by a linear automorphism θ of A, i.e. $\theta(a) = a$ for any $a \in A$ and $\theta(x_i) = \sum a_{ij} x_j$ for suitable $a_{ij} \in A$.

You should compare this definition with the analytic definition of a vector bundle: Over a manifold M, every vector bundle of rank r is locally of the form $U_i \times K^r$, where $K = \mathbb{R}$ or $\mathbb{C}$ and $U_i \subseteq M$ is open, while over a scheme X every vector bundle of rank r is locally of the form $U_i \times_{\mathrm{Spec}\,\mathbb{Z}} \mathbb{A}^r_{\mathbb{Z}}$, where $\mathbb{A}^r_{\mathbb{Z}} := \mathrm{Spec}\, \mathbb{Z}[x_1, \ldots x_r]$ is the affine space over $\mathbb{Z}$.

Now, let X be a projective scheme over $\mathbb{C}$, i.e. a closed subscheme of $\mathbb{P}^n_{\mathbb{C}} = \operatorname{Proj} \mathbb{C}[x_0, x_1, \dots, x_n]$, the projective n-space over $\mathbb{C}$, for some n. Define

$$\underline{\mathfrak{M}}_{r,c_i} : (Sch/\mathbb{C}) \longrightarrow (Sets)$$

to be the *moduli functor of vector bundles* on X of fixed rank r and Chern classes c_i so that

$$\underline{\mathfrak{M}}_{r,c_i}(B) =: \Big\{ \text{Vector bundles } f : \mathcal{E} \longrightarrow B \times X \,\Big|\, f \text{ is flat over } B,\ \operatorname{rk} \mathcal{E} = r, \\ c_i(\mathcal{E}|_{X \times \{b\}}) = c_i, \quad \text{for all } b \in B \Big\} / \cong .$$

The intuition is that

$$\underline{\mathfrak{M}}(B) \leftrightsquigarrow \{\textsf{Families of vector bundles parametrized by } B\}/\cong .$$

For a morphism $f : B' \longrightarrow B$, the corresponding map of sets

$$\underline{\mathfrak{M}}_{r,c_i}(f) = f^* : \underline{\mathfrak{M}}_{r,c_i}(B) \longrightarrow \underline{\mathfrak{M}}_{r,c_i}(B')$$

is induced by the pullback of vector bundles.

Example 1.4 (Curves/$\mathbb{C}$). We set $\underline{\mathrm{M}}_g : (Sch/\mathbb{C}) \longrightarrow (Sets)$ to be the moduli functor of smooth curves of genus g over $\mathbb{C}$,

$$\underline{\mathrm{M}}_g(B) =: \Big\{ \text{'Algebraic families' } \phi : C \longrightarrow B \,\Big|\, \phi^{-1}(b) \text{ is a (geometrically) connec-} \\ \text{ted curve of genus g for all } b \in B \Big\} / \cong .$$

Here by an algebraic family we mean smooth and proper morphisms $\phi : C \longrightarrow B$. For a morphism $B' \longrightarrow B$, $\underline{\mathrm{M}}_g(f) : \underline{\mathrm{M}}_g(B) \longrightarrow \underline{\mathrm{M}}_g(B')$ is the map of sets induced by the pullback f^*,

$$\begin{array}{ccc} B' \times_B C & \longrightarrow & C \\ {\scriptstyle f^*(\phi)} \downarrow & & \downarrow {\scriptstyle \phi} \\ B' & \xrightarrow{\ f\ } & B \end{array}$$

None of these examples are representable, because of the presence of automorphisms. We give here some heuristic reasoning to explain why this happens. For example, given a curve C over $\mathbb{C}$ with a nontrivial automorphism, one can show that there exists an algebraic family of curves $\phi : \mathscr{C} \longrightarrow B$ such that $\phi^{-1}(b) \cong C$ for all $b \in B$ but $\mathscr{C} \ncong B \times C$, where $\mathrm{pr}_B : B \times C \longrightarrow B$ is the trivial family of curves with every fiber equal to C. Suppose now that the fine moduli scheme M_g would exist. The ($\mathbb{C}$-valued) points of M_g would then correspond to morphisms $f : \operatorname{Spec} \mathbb{C} \longrightarrow M_g$ (notice that as a topological space $\operatorname{Spec} \mathbb{C}$ is just a point), which by the definition of a fine moduli space correspond to isomorphism classes of algebraic families of curves $\varphi : \mathscr{C}' \longrightarrow \operatorname{Spec} \mathbb{C}$. Since $\operatorname{Spec} \mathbb{C}$ is just a point we see that this is the same thing as an isomorphism class of a curve/$\mathbb{C}$. Hence

$$\textsf{Points of } M_g \leftrightsquigarrow \textsf{Isomorphism classes of genus } g \textsf{ curves over } \mathbb{C}.$$

This implies that one can think of every mapping $g \in \operatorname{Hom}_{\mathbb{C}}(B, M)$ associating an isomorphism class $g(b) \in M$ of curves of genus g over $\mathbb{C}$ for all $b \in B$. The intuition is then that $\operatorname{Hom}_{\mathbb{C}}(B, M)$ remembers only the isomorphism classes of the fibers of each family of curves $\varphi : \mathscr{C}' \longrightarrow B$, when the associated $g_\varphi \in \operatorname{Hom}_{\mathbb{C}}(B, M_g)$ is given by $g_\varphi(b) = [\varphi^{-1}(b)]$. Now the fibers of $\phi : \mathscr{C} \longrightarrow B$ and $\mathrm{pr}_B : B \times C \longrightarrow B$ are isomorphic for every $b \in B$ but the families themselves are different so that there cannot exist a bijection

$$\underline{\mathrm{M}}_g(B) \cong \operatorname{Hom}_{\mathbb{C}}(B, M_g),$$

a contradiction.

Question. *How to deal with the automorphisms?*
Answer (Grothendieck). *Keep them!*

More precisely, thinking in terms of our example of the moduli of vector bundles: Instead of modding out the isomorphisms, replace the set of isomorphism classes of vector bundles over $B \times X$ by the *category* $\mathcal{M}(B)$,

$$\begin{aligned} \mathrm{Ob}\,(\mathcal{M}(B)) =: \quad & \Big\{ \text{Vector bundles } f : \mathcal{E} \longrightarrow B \times X \,\Big|\, f \text{ is flat over } B,\ \mathrm{rk}\,\mathcal{E} = r, \\ & \qquad c_i(\mathcal{E}|_{X \times \{b\}}) = c_i, \quad \text{for all } b \in B \Big\}, \\ \mathrm{Mor}(\mathcal{M}(B)) =: \quad & \Big\{ \text{Isomorphisms of vector bundles on } B \times X \Big\}. \end{aligned}$$

Hence we have a 'functor'

$$\mathcal{M} : (Sch/\mathbb{C}) \longrightarrow (groupoids),$$

$$B \mapsto \mathcal{M}(B).$$

This is not really a functor but a *2-functor* (see the Appendix). If $f : B' \longrightarrow B$ is a morphism, the pullback defines a *functor* $F(f) = f^* : \mathcal{M}(B) \longrightarrow \mathcal{M}(B')$ and for every diagram

$$B'' \xrightarrow{\ g\ } B' \xrightarrow{\ f\ } B$$

it gives a natural transformation of functors (a 2-isomorphism)

$$\epsilon_{g,f} : g^* \circ f^* \longrightarrow (f \circ g)^*.$$

2. Grothendieck topologies and sheaves on a site

2.1. Historical motivation. Grothendieck topology is a generalization of the concept of a topological space. Its original motivation were mainly:

(1) Study of *algebraic* principal G-bundles over an algebraic variety (or more generally over a scheme) with G an algebraic group;
(2) The proof of the Weil conjectures concerning the zeta functions $Z(t)$ of smooth projective varieties $X/\mathbb{F}_q$ of dimension n, where $q = p^m$ for some m and $p > 0$ is the characteristic of the finite field $\mathbb{F}_q$.

By definition

$$Z(t) = Z(X;t) = \exp(\sum_{r=1}^{\infty} N_r \frac{t^r}{r}),$$

where

$$N_r = \text{The number of points of } \overline{X} = X \times_{\mathbb{F}_q} \overline{\mathbb{F}}_q \text{ with coordinates in } \mathbb{F}_{q^r} \subset \overline{\mathbb{F}}_q.$$

The Weil conjectures state roughly that the zeta function $Z(t)$ satisfies

(1) Rationality ($Z(t)$ is a rational function in the variable t),
(2) A functional equation $Z(\frac{1}{q^n t}) = \pm q^{nE/2} t^E Z(t)$,
(3) An analog of the Riemann hypothesis.

Weil himself noted, that if one had a cohomology theory of varieties/$\mathbb{F}_q$ with coefficients in $\mathbb{Q}_\ell$, where $\ell \neq p$, satisfying the usual properties of a topological cohomology theory (i.e. functoriality, finite dimensionality, cup product, Poincaré duality, Lefschetz fixed point formula etc.), then one could prove his conjectures (or at least rationality and the functional equation). This led Grothendieck to develop his theory of ℓ-adic cohomology. By definition

$$H^i(X, \mathbb{Q}_\ell) =: (\varprojlim_n H^i_{\acute{e}t}(X, \mathbb{Z}/\ell^n\mathbb{Z})) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell,$$

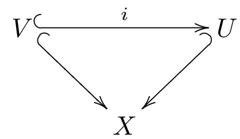
where the *constant sheaves* $\mathbb{Z}/\ell^n\mathbb{Z}$ are now (generalized) sheaves in the *étale topology* of X.

2.2. Grothendieck topologies. To categorify the notion of topology, we consider the following example.

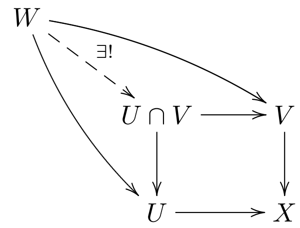
EXAMPLE 1.5 (Topological spaces over X). Let X be a topological space. Denote by (Top/X) the category with

$$\begin{aligned} \mathrm{Ob}\,(Top/X) &=: \{U \subseteq X \mid U \text{ open}\} \\ \mathrm{Mor}(Top/X) &=: \{\text{inclusions } V \subseteq U\} \end{aligned}$$

Hence, thinking inclusions as maps, the morphisms in (Top/X) are commuting diagrams



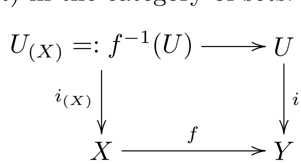
Let then $U, V \in \mathrm{Ob}\,(Top/X)$. By definition of the intersection of two sets, $U \cap V$ is the fiber product $U \times_X V$ in (Top/X):



Hence

Intersections $U \cap V$, $\quad U, V \subseteq X$ open $\longleftrightarrow$ Fiber products $U \times_X V$ in (Top/X).

Let next f be a continuous mapping and $U \subseteq Y$ open. There exists a Cartesian diagram (i.e. a fiber product) in the category of sets:



We conclude that

Inverse images $f^{-1}(U) \subseteq X \longleftrightarrow$ Base extensions $i_{(X)} : U_{(X)} = X \times_Y U \longrightarrow X$

so that the inverse images induce a functor $f^* : (Top/Y) \longrightarrow (Top/X)$.

Finally, we consider open coverings $(U_i)_{i\in I}$ of $U \in \mathrm{Ob}\,(Top/X)$. We make the following remarks:

(1) $U \subseteq U$ is a covering of U;
(2) For any open covering $(U_i)_{i\in I}$ of U and open $V \subseteq U$, $(U_i \cap V)_{i\in I}$ is an open covering of V (restriction);
(3) If $(U_i)_{i\in I}$ is an open covering of U, and $(V_{ij})_{j\in J_i}$ is an open covering of U_i for all $i \in I$, then $(V_{ij})_{i,j}$ is an open covering of U (refinement).

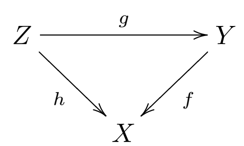
Definition 1.6. Let $\mathfrak{C}$ be a category with fiber products. Suppose that for each $U \in \operatorname{Ob}\mathfrak{C}$, there exists a distinguished family of maps $(U_i \longrightarrow U)_{i\in I}$, the *coverings* of U, satisfying

(1) For any $U \in \operatorname{Ob}(\mathfrak{C})$, the family $(U \xrightarrow{\text{id}} U)$ consisting of a single map is a covering of U.
(2) For any covering $(U_i \longrightarrow U)_{i\in I}$ and any morphism $V \longrightarrow U$ in $\mathfrak{C}$, $(U_i \times_U V \longrightarrow V)_{i\in I}$ is a covering of V ('restriction').
(3) If $(U_i \longrightarrow U)$ is a covering of U, and $(V_{ij} \longrightarrow U_i)_{j\in J_i}$ is a covering of U_i, then $(V_{ij} \longrightarrow U)_{i,j}$ is a covering of U ('refinement').

The system of coverings is then called a *Grothendieck topology*, and the category $\mathfrak{C}$ with a Grothendieck topology is called a *site* and is denoted $\mathbf{T}$. The underlying category of a site is denoted by $Cat(\mathbf{T})$.

Example 1.7 (The site $\mathbf{C}_X$). This is a variant of (Top/X), where we replace inclusions by local homeomorphisms:

Let X be a fixed topological space and define the category $\mathbf{C}_X$, whose objects are local homeomorphisms $f : Y \longrightarrow X$, and whose morphisms from $h : Z \longrightarrow X$ to $f : Y \longrightarrow X$ are commuting diagrams



The coverings of $\mathbf{C}_X$ are families of morphisms $(U_i \xrightarrow{f_i} U)_{i\in I}$ such that f_i is a local homeomorphism for all $i \in I$ and the total map $\coprod_{i\in I} U_i \longrightarrow U$ is surjective.

Example 1.8 (The site $\mathfrak{S}$). Define the category $\mathfrak{S}$ so that

$$\begin{aligned} \operatorname{Ob}(\mathfrak{S}) &= \{\text{all } C^\infty\text{-manifolds } X\} \\ \operatorname{Mor}(\mathfrak{S}) &= \{C^\infty\text{-maps } X \longrightarrow X'\} \end{aligned}$$

Notice, that there are no commutative diagrams in this definition! The coverings of $\mathfrak{S}$ consist of families of maps $(U_i \xrightarrow{f_i} U)_{i\in I}$ such that f_i is a local diffeomorphism for all $i \in I$ and the total map $\coprod_{i\in I} U_i \longrightarrow U$ is surjective.

Remark 1.9. Note that not all fiber products exist in $\mathfrak{S}$, but if at least one of the two morphisms $U \longrightarrow X$ or $V \longrightarrow X$ is submersive (i.e. the derivative of the map is surjective), then the fiber product exists in $\mathfrak{S}$.

2.3. Sheaves on a site. Next, we need to recall the notion of a sheaf on a topological space.

Definition 1.10. Suppose X is a topological space. A *presheaf* of abelian groups on X is a pair $(\mathcal{F}, \text{res})$ consisting of

(1) An abelian group $\mathcal{F}(U)$ for every open subset $U \subseteq X$;
(2) A group homomorphism

$$\text{res}^U_V : \mathcal{F}(U) \longrightarrow \mathcal{F}(V)$$

for every open $V \subseteq U$, satisfying:

$$\text{res}^U_U = \text{id}_{\mathcal{F}(U)}, \quad \text{for every open } U \subseteq X;$$

$$\text{res}^V_W \circ \text{res}^U_V = \text{res}^U_W, \quad \text{for every open } W \subseteq V \subseteq U.$$

Definition 1.11. A presheaf $\mathcal{F}$ on a topological space X is called a *sheaf* if for every open set $U \subseteq X$ and every family of open subsets $U_i \subseteq U$, where $i \in I$, such that $U = \bigcup_{i \in I} U_i$ the following conditions, which we call the Sheaf Axioms, are satisfied:

(1) If $f, g \in \mathcal{F}(U)$ are elements such that $f|U_i = g|U_i$ for every $i \in I$, then $f = g$;
(2) Given elements $f_i \in \mathcal{F}(U_i)$, $i \in I$, such that

$$f_i|U_i \cap U_j = f_j|U_i \cap U_j \quad \text{for all } i, j \in I,$$

then there exists an $f \in \mathcal{F}(U)$ such that $f|U_i = f_i$ for every $i \in I$.

Hence, condition (1) says that locally equal section are equal, and condition (2) says that a local family of compatible sections can be glued to a 'global' section.

Example 1.12. Suppose X is a Riemann surface and $\mathcal{O}(U)$ is the ring of holomorphic functions defined on the open set $U \subseteq X$. Taking the usual restriction mapping $\mathcal{O}(U) \longrightarrow \mathcal{O}(V)$ for $V \subseteq U$ one gets the sheaf $\mathcal{O}$ of holomorphic functions on X. The sheaf $\mathcal{M}$ of meromorphic functions on X is defined analogously.

It is now easy to define the categorified versions of a sheaf:

Definition 1.13. A *presheaf of sets* on a site $\mathbf{T}$ is a contravariant functor

$$\mathcal{F} : Cat(\mathbf{T}) \longrightarrow (Sets)$$

Definition 1.14. A *sheaf of sets* on $\mathbf{T}$ is a presheaf satisfying the sheaf condition

$$(S) : \mathcal{F}(U) \rightarrowtail \prod_{i \in I} \mathcal{F}(U_i) \rightrightarrows \prod_{(i,j) \in I \times I} \mathcal{F}(U_i \times_U U_j)$$

is exact for every covering $(U_i \longrightarrow U)$. Thus a presheaf $\mathcal{F}$ is a sheaf iff the map

$$\mathcal{F}(U) \longrightarrow \prod_{i \in I} \mathcal{F}(U_i), \quad f \mapsto (f|U_i)$$

idetifies $\mathcal{F}(U)$ with the subset of the product consisting of families (f_i) such that $f_i|U_i \times_U U_j = f_j|U_i \times_U U_j$.

Similarly, one defines the notions of a preasheaf of abelian groups and a sheaf of abelian groups.

Equiping the category (Top/X) with its natural structure of a site described earlier, it is easy to see that the above definition gives the usual definition of a sheaf on a topological space X.

3. Stacks as 2-functors

Recall now the following correspondeces from the first lecture:

$$\text{Representing objects for moduli problems, when exist} \leftrightsquigarrow \text{Schemes } M \in (Sch/S),$$

and

$$\text{Schemes } M \in (Sch/S) \leftrightsquigarrow \text{Functors } \mathrm{Hom}_S(-, M) : (Sch/S) \longrightarrow (Sets),$$

where $\mathrm{Hom}_S(-, M)$ is a sheaf of sets in the étale topology of (Sch/S). Moreover, the presence of nontrivial automorphisms of objects led us to consider 2-functors.

Idea. In order to make moduli problems behave better, replace the concept of a fine moduli space (a scheme) by a 2-functor (a presheaf) with topological conditions, making it a sheaf of some kind.

But before giving the formal definition, we consider the following example where we glue spaces and maps:

EXAMPLE 1.15 (Grothendieck's fiber spaces, **[Gro]**). Recall, that a *fiber space* over a topological space X is a triple (X, E, p) of the space X, a space E and a continuous map $p : E \longrightarrow X$. Hence, a general fiber space doesn't need to have a structure group or to be locally trivial. Maps, inverse images, subspaces, quotients, products etc. of fiber spaces are defined analogously with their special cases of vector bundles.

Let now X be a topological space, (U_i) an open covering of X, for each index i let E_i be a fiber space over U_i, and for any couple of indices i, j such that $U_{ij} = U_i \cap U_j \neq \emptyset$, let f_{ij} be a U_{ij}-isomorphism

$$f_{ij} : E_j|U_{ij} \xrightarrow{\cong} E_i|U_{ij}$$

On the topological sum

$$\mathcal{E} =: \coprod_i E_i$$

we consider the relation

$$y_i \in E_i|U_{ij} \sim y_j \in E_j|U_{ij} \Longleftrightarrow y_i = f_{ij}(y_j).$$

This is an equivalence relation iff for each triple of indices (i, j, k) such that $U_{ijk} = U_i \cap U_j \cap U_k \neq \emptyset$ the isomorphisms f_{ij} satisfy the cocycle condition

$$f_{ik} = f_{ij} \circ f_{jk}$$

(where we have written simply f_{ik} instead of the isomorphism of $E_k|U_{ijk}$ onto $E_i|U_{ijk}$ induced by f_{ik}, and likewise for f_{ij} and f_{jk}). Suppose, this condition is satisfied and define

$$E =: \mathcal{E}/\sim .$$

The projections $p_i : E_i \longrightarrow U_i$ define a continuous map on the topological sum $\mathcal{E}$ into X, and this map is compatible with the equivalence relation in $\mathcal{E}$ (the maps f_{ij} are fiber preserving as maps of fiber spaces), so that there is a continuous map $p : E \longrightarrow X$.

The indentity map of E_i into $\mathcal{E}$ defines a map $\phi_i : E_i \longrightarrow E$, which gives a U_i-isomorphism $\phi_i : E_i \xrightarrow{\sim} E|U_i$ satisfying

$$f_{ij} = \phi_i^{-1} \circ \phi_j$$

(where again, we have written ϕ_i instead of the restriction of ϕ_i to $E_i|U_{ij}$, etc.).

The reader should have in mind the definition of a presheaf in groupoids given in §6 Appendix before reading the following definition.

DEFINITION 1.16 (Stack). A *stack* is a *sheaf of groupoids*, i.e. a 2-functor

$$\mathcal{F} : (Sch/S) \longrightarrow (groupoids)$$

that satisfies the following sheaf axioms. Let $(U_i \longrightarrow U)_{i \in I}$ be a covering of U in a site on (Sch/S). Then

(1) *(Glueing of morphisms).* If $X, Y \in \operatorname{Ob} \mathcal{F}(U)$ and $\phi_i : X|_i \longrightarrow Y|_i$ are morphisms in $\mathcal{F}(U_i)$ such that $\phi_i|_{ij} = \phi_j|_{ij}$ in $\mathcal{F}(U_i \times_U U_j)$, there exists a morphism $\eta : X \longrightarrow Y$ in $\mathcal{F}(U)$ such that $\eta|_i = \phi_i$ for all $i \in I$.

(2) *(Monopresheaf).* If $X, Y \in \operatorname{Ob} \mathcal{F}(U)$ and $\phi, \psi : X \longrightarrow Y$ are morphisms in $\mathcal{F}(U)$ such that $\phi|_i = \psi|_i$ in $\mathcal{F}(U_i)$ for all $i \in I$, then $\phi = \psi$.

(3) *(Gluing of objects).* If $X_i \in \operatorname{Ob} \mathcal{F}(U_i)$ and $\phi_{ij} : X_j|_{ij} \longrightarrow X_i|_{ij}$ are morphisms in $\mathcal{F}(U_i \times_U U_j)$ satisfying the cocycle condition

$$\phi_{ij}|_{ijk} \circ \phi_{jk}|_{ijk} = \phi_{ik}|_{ijk}$$

in $\mathcal{F}(U_i \times_U U_j \times_U U_k)$, then there exists $X \in \mathrm{Ob}\,\mathcal{F}(U)$ and $\phi_i : X|_i \xrightarrow{\cong} X_i$ such that

$$\phi_{ji} \circ \phi_i|_{ij} = \phi_j|_{ij}$$

as morphisms $X|_{ij} \longrightarrow X_j|_{ij}$ in $\mathcal{F}(U_i \times_U U_j)$.

EXAMPLE 1.17. The 2-functor $\mathcal{M} : (Sch/\mathbb{C}) \longrightarrow (groupoids)$ is a stack, called the *moduli stack of vector bundles over* X.

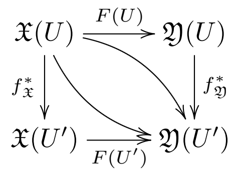
REMARK 1.18. Replacing the site (Sch/S) in the above definition with $\mathbf{C}_X$ leads to topological stacks over X, and replacing it with $\mathfrak{S}$, we get stacks over the site $\mathfrak{S}$ used by Behrend and Xu. From now on we will concentrate on differential geometric stacks and our site will be $\mathfrak{S}$.

3.1. Morphisms of stacks. Since stacks live in the world of 2-categories, they have two kinds of morphisms, namely the 1- and 2-morphisms.

DEFINITION 1.19 (1-morphisms). Let $\mathfrak{X}$ and $\mathfrak{Y}$ be stacks. A *1-morphism* $F : \mathfrak{X} \longrightarrow \mathfrak{Y}$ will associate for every $U \in \mathrm{Ob}\,(\mathfrak{S})$ (i.e. a C^∞-manifold) a functor

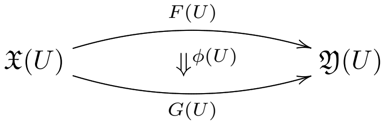
$$F(U) : \mathfrak{X}(U) \longrightarrow \mathfrak{Y}(U)$$

and for every arrow $U' \xrightarrow{f} U$ an isomorphism of functors $\alpha(f) : f^*_{\mathfrak{X}} \circ F(U') \xrightarrow{\sim} F(U) \circ f^*_{\mathfrak{Y}}$



satisfying the natural compatibility conditions.

DEFINITION 1.20 (2-morphisms). Let $F, G : \mathfrak{X} \longrightarrow \mathfrak{Y}$ be 1-morphisms of stacks. A *2-morphism* $\phi : F \longrightarrow G$ associates for every $U \in \mathrm{Ob}\,(\mathfrak{S})$ an isomorphism of functors $\phi(U) : F(U) \xrightarrow{\sim} G(U)$:



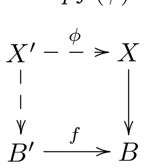
satisfying the necessary compatibility conditions.

4. Stacks as categories

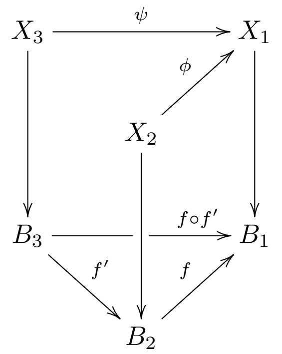
DEFINITION 1.21. A *category over* $\mathfrak{S}$ is a category $\mathcal{F}$ and a covariant functor $p_{\mathcal{F}} : \mathcal{F} \longrightarrow \mathfrak{S}$. If X is an object (resp. ϕ is a morphism) of $\mathcal{F}$, and $p_{\mathcal{F}}(X) = B$ (resp. $p_{\mathcal{F}}(\phi) = f$), then we say that X lies over B (resp. ϕ lies over f).

DEFINITION 1.22 (Groupoid fibration). A category $\mathcal{F}$ over $\mathfrak{S}$ is called a *category fibered on groupoids* if

(1) For every $f : B' \longrightarrow B$ in $\mathfrak{S}$ and every object $X \in \mathrm{Ob}\,(\mathcal{F})$ with $p_{\mathcal{F}}(X) = B$, there exists at least one object $X' \in \mathrm{Ob}\,(\mathcal{F})$ and a morphism $\phi : X' \longrightarrow X$ such that $p_{\mathcal{F}}(X') = B'$ and $p_{\mathcal{F}}(\phi) = f$.



(2) For every diagram



(where $p_{\mathcal{F}}(X_i) = B_i$, $p_{\mathcal{F}}(\phi) = f$, $p_{\mathcal{F}}(\psi) = f \circ f'$), there exists a unique $\varphi : X_3 \longrightarrow X_2$ satisfying $\psi = \phi \circ \varphi$ and $p_{\mathcal{F}}(\varphi) = f'$.

It follows from the definition, that the object X' whose existence is asserted in condition (1) is unique up to a canonical isomorphism. For each X and f we choose once and for all such an X' and call it f^*X. Moreover ϕ is an isomorphism iff $p_{\mathcal{F}}(\phi) = f$ is an isomorphism.

Definition 1.23. Let $\mathcal{F}$ be a category fibered on groupoids over $\mathfrak{S}$ and let B be an object of $\mathfrak{S}$. Define $\mathcal{F}_B$, the *fiber over* B, to be the subcategory of $\mathcal{F}$ whose objects lie over B and whose morphisms lie over id_B.

Remark 1.24. Since the identity map is an isomorphism in $\mathfrak{S}$ and as we noted, morphisms over isomorphisms are isomorphisms, the fiber categories $\mathcal{F}_B$ are groupoids.

Next, we are going to show that 2-functors (presheaves) $\mathcal{F} : \mathfrak{S} \longrightarrow (groupoids)$ define groupoid fibrations $\underline{\mathcal{F}} \longrightarrow \mathfrak{S}$ and conversely:

- *(From 2-functors to groupoid fibrations)* Suppose, we are given a 2-functor $\mathcal{F} : \mathfrak{S} \longrightarrow (groupoids)$. Define

$$\mathrm{Ob}\,(\underline{\mathcal{F}}) =: \coprod_{U \in \mathrm{Ob}\,\mathfrak{S}} \mathrm{Ob}\,\mathcal{F}(U).$$

Since this is a disjoint union, we may define all morphisms of $\underline{\mathcal{F}}$ by defining the morphisms going from $x \in \mathrm{Ob}\,\mathcal{F}(U)$ to $y \in \mathrm{Ob}\,\mathcal{F}(V)$. By definition these are pairs (α, f) with $f : U \longrightarrow V$ an arrow in $\mathfrak{S}$ and α an arrow in $\mathcal{F}(U)$ from x to f^*y. We encode this as

$$x \xrightarrow{\ \alpha\ } f^*y \overset{f}{\cdots\cdots} y$$

With these notations, the composite of two arrows

$$x \xrightarrow{\ \alpha\ } f^*y \overset{f}{\cdots\cdots} y \xrightarrow{\ \beta\ } g^*z \overset{g}{\cdots\cdots} z$$

is defined to be

$$x \xrightarrow{\ \alpha\ } f^*y \xrightarrow{\ f^*\beta\ } f^*g^*z \xrightarrow{\ \sim\ } (gf)^*z \overset{gf}{\cdots\cdots} z.$$

The functor $p_{\underline{\mathcal{F}}} : \underline{\mathcal{F}} \longrightarrow \mathfrak{S}$ is defined to send an object of $\mathcal{F}(U)$ to U and an arrow (α, f) to f. Since $(\alpha, \mathrm{id}_U) \mapsto \alpha$, where α is a morphism of $\mathcal{F}(U)$, is a bijection, we see that we may identify each $\mathcal{F}(U)$ via $p_{\underline{\mathcal{F}}}$ with the fiber category $\underline{\mathcal{F}}_U$.

- *(From groupoid fibrations to 2-functors)* Let $p_{\underline{\mathcal{F}}} : \underline{\mathcal{F}} \longrightarrow \mathfrak{S}$ be a groupoid fibration. For every $U \in \mathrm{Ob}\,(\mathfrak{S})$ define the groupoid $\mathcal{F}(U)$ to be the fiber category

$$\mathcal{F}(U) := \underline{\mathcal{F}}_U$$

Let $U' \xrightarrow{f} U$ be an arrow in $\mathfrak{S}$. We would like to map objects and morphisms $\mathcal{F}(U) \longrightarrow \mathcal{F}(U')$ in a functorial way.

Objects: Let $x \in \mathrm{Ob}\,\underline{\mathcal{F}}_U$. We map this to the pullback $x \mapsto f^*x \in \mathrm{Ob}\,\underline{\mathcal{F}}_{U'}$.

Morphisms: For every arrow $x' \xrightarrow{u} x$ in $\underline{\mathcal{F}}_U$ we denote by $f^*(u)$ the unique arrow in $\underline{\mathcal{F}}_{U'}$ making the following diagram commutative

$$\begin{array}{ccc} f^*x' & \text{———} & x' \\ {\scriptstyle f^*(u)}\Big| & & \Big|{\scriptstyle u} \\ f^*x & \text{———} & x \end{array}$$

We get a functor $f^* : \underline{\mathcal{F}}_U \longrightarrow \underline{\mathcal{F}}_{U'}$, and one can show that for a composition

$$U'' \xrightarrow{g} U' \xrightarrow{f} U$$

we get an isomorphism of functors $g^* \circ f^* \xrightarrow{\sim} (f \circ g)^*$ satisfying the conditions of a 2-functor (see the Appendix).

Example 1.25. Manifolds X give groupoid fibrations. Consider the category $\underline{X} =: \mathfrak{S}/X$ (differentiable manifolds over X), a variant of $\mathfrak{S}$. Define the functor $p_{\underline{X}} : \mathfrak{S}/X \longrightarrow \mathfrak{S}$ so that on objects $p_{\underline{X}}(S \xrightarrow{f} X) \mapsto S$ and for morphisms $g : (T \xrightarrow{h} X) \longrightarrow (S \xrightarrow{f} X)$

$$\begin{array}{ccc} T & \xrightarrow{\quad g \quad} & S \\ & {\scriptstyle h}\searrow \quad \swarrow{\scriptstyle f} & \\ & X & \end{array}$$

$p_{\underline{X}}(g) = g$, where on the right hand side g is a morphism $g : T \longrightarrow S$ in $\mathfrak{S}$ without reference to any commutative diagram.

Definition 1.26. Let $\mathcal{F} \longrightarrow \mathfrak{S}$ be a category fibered in groupoids. Then $\mathcal{F}$ is called a *stack* over $\mathfrak{S}$ if the following three axioms are satisfied:

(1) For any C^∞ manifold $X \in \mathrm{Ob}\,(\mathfrak{S})$, any two objects $x, y \in \mathrm{Ob}\,(\mathcal{F})$ lying over X, and any two isomorphims $\phi, \psi : x \longrightarrow y$ over X such that $\phi|U_i = \psi|U_j$ for all U_i in a covering family $(U_i \longrightarrow X)$, then $\phi = \psi$.

(2) For any $X \in \mathrm{Ob}\,(\mathfrak{S})$, any two objects $x, y \in \mathrm{Ob}\,(\mathcal{F})$ lying over X, a covering family $(U_i \longrightarrow X)$, and a collection of isomorphisms $\phi_i : x|U_i \longrightarrow y|U_i$ such that $\phi_i|U_{ij} = \phi_j|U_{ij}$ for all i, j, there exists an isomorphism $\phi : x \longrightarrow y$ such that $\phi|U_i = \phi_i$ for all i.

(3) For every $X \in \mathrm{Ob}\,(\mathfrak{S})$, every covering family $(U_i \longrightarrow X)$, every family $\{x_i\}$ of objects x_i in the fibre $\mathcal{F}_{U_i}$, and every family of morphisms $\{\phi_{ij}\}$, $\phi_{ij} : x_i|U_{ij} \longrightarrow x_j|U_{ij}$ satisfying the cocycle condition $\phi_{jk} \circ \phi_{ij} = \phi_{ik}$ in the fibre $\mathfrak{X}_{U_{ijk}}$, there exists an object x over X, together with isomorphisms $\phi_i : x|U_i \longrightarrow x_i$ such that $\phi_{ij} \circ \phi_i = \phi_j$ over U_{ij}.

Example 1.27. The groupoid fibrations $p_{\underline{X}} : \underline{X} \longrightarrow \mathfrak{S}$ associated to manifolds X are stacks.

Example 1.28 (Classifying stack). Let G be a Lie group. Let $\mathfrak{X} = BG$ be the category of pairs (S, P), where $S \in \mathrm{Ob}\,(\mathfrak{S})$ is a C^∞-manifold and P is a principal G-bundle over S. A morphism $(f, \alpha) : (S, P) \longrightarrow (T, Q)$ is a commutative diagram

$$\begin{CD} P @>\alpha>> Q \\ @V{\pi_P}VV @VV{\pi_Q}V \\ S @>f>> T \end{CD}$$

where $P \longrightarrow Q$ is G-equivariant. The functor $p_{BG} : BG \longrightarrow \mathfrak{S}$ is defined by $(S, P) \mapsto S$ and $(f, \alpha) \mapsto f$. In the light of Example 1.15, it is easy to believe that BG is a stack.

Actually, BG is an example of a rather specific class of stacks. Recall the two elementary properties of principal bundles:

(1) Evere space S has at least one principal G-bundle over it, namely the trivial bundle.
(2) Any two principal G-bundles are locally isomorphic.

These facts lead to the definition of a gerbe:

Definition 1.29 (Gerbe). Let $p_{\mathcal{G}} : \mathcal{G} \longrightarrow \mathfrak{S}$ be a stack. Then $\mathcal{G}$ is called a *gerbe* over $\mathfrak{S}$ if the following two conditions hold:

(1) For any object S of $\mathfrak{S}$ there exists a covering $(S_i \longrightarrow S)_{i \in I}$ such that the fiber $\mathcal{G}_{S_i}$ is nonempty for all $i \in I$.
(2) For any object S of $\mathfrak{S}$ and any two objects x_1, x_2 of $\mathcal{G}_S$ there exists a covering family $(S_i \longrightarrow S)_{i \in I}$ such that $x_1|S_i$ and $x_2|S_i$ are isomorphic for all $i \in I$.

Remark 1.30. Condition (1) says that objects locally exists (note this is weaker than the global existence satisfied by BG), and condition (2) says that any two objects are locally isomorphic.

Example 1.31 (Quotient stack). Suppose that a Lie group G acts on a manifold X. Suppose moreover that the action is free. Then X/G exists as a manifold and the quotient morphism $\pi : X \longrightarrow X/G$ is actually a principal G-bundle. Recall, that in Grothendieck's philosophy a space is determined by its S-valued points. i.e. maps from spaces S to the space. What are the S-valued points of X/G? If $S \xrightarrow{f} X/G$, we get (by pullback) a Cartesian diagram

$$\begin{CD} X' @>\alpha>> X \\ @V{G}VV @VV{G}V \\ S @>f>> X/G \end{CD} \tag{1.1}$$

So f defines a G bundle $X' \xrightarrow{G} S$ and a G-equivariant map α . If the action is not free, the quotient X/G does not, in general, exist as a manifold, howerever we consider the following groupoid fibration:

Define the category $[X/G]$ whose objects are principal G-bundles $\pi : P \longrightarrow S$ together with a G-equivariant morphism $\alpha : P \longrightarrow X$. A morphism is a Cartesian diagram

$$\begin{CD} P' @>p>> P \\ @V{\pi'}VV @VV{\pi}V \\ S' @>>> S \end{CD}$$

such that $\alpha \circ p = \alpha'$. The functor $p_{[X/G]} : [X/G] \longrightarrow \mathfrak{S}$ is defined analogously with p_{BG} described above. One can show that this is a stack. This definition makes sense for any action of G on X and the "quotient map" $X \longrightarrow [X/G]$ (we will see in a moment what this means) behaves like a G-bundle.

Note that choosing $X = * =$ a point, all equivariant morphisms $\alpha : P \longrightarrow *$ are trivial and hence by definition

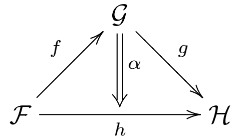
$$[*/G] = BG.$$

4.1. Morphisms of stacks. When we consider stacks as fibered categories instead of sheaf of groupoids, the 1- and 2-morphisms get a more elegant form:

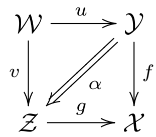
Definition 1.32 (1-morphisms). Let $\mathcal{F}$ and $\mathcal{G}$ be stacks. A 1-morphism is a functor $F : \mathcal{F} \longrightarrow \mathcal{G}$ such that $p_{\mathcal{F}} = p_{\mathcal{G}} \circ F$ (notice that here we have a strict equality of functors!). If F is an equivalence of categories, we say that the stacks $\mathcal{F}$ and $\mathcal{G}$ are isomorphic.

Definition 1.33 (2-morphisms). Let $F, G : \mathcal{F} \longrightarrow \mathcal{G}$ be 1-morphisms of stacks. A 2-morphism from F to G is an isomorphism of functors $\phi : F \xrightarrow{\sim} G$.

Definition 1.34. A (2-)commutative triangle of stacks is a diagram



such that f, g and h are 1-morphisms of stacks and $\alpha : g \circ f \xrightarrow{\sim} h$ is a 2-morphism. Similarly, we say that a diagram



(2-)commutes, if f, g, u and v are 1-morphisms of stacks and $\alpha : f \circ u \xrightarrow{\sim} g \circ v$ is a 2-morphism.

Lemma 1.35. *1-morphisms $x : X \longrightarrow \mathfrak{X}$ from a manifold to a stack correspond bijectively to objects x in the fiber category $\mathfrak{X}_X$.*

Remark 1.36. Actually this already holds when $\mathfrak{X}$ is just a groupoid fibration and the 1- and 2-morphisms are defined similarly as for stacks.

Example 1.37. According to the previous lemma, 1-morphisms $M \longrightarrow BG$ correspond to objects in the fiber category BG_M, which by definition are the G-bundles over M, hence the name "classifying stack".

Example 1.38. 1-morphisms $S \longrightarrow [X/G]$ correspond to principal G bundles P over S together with a G-equivariant map $P \longrightarrow X$. Especially, we define the quotient map $X \longrightarrow [X/G]$ to be the map corresponding to the trivial G-bundle $X \times G$ over X with the G-equivariant map to X being the action of G on X.

Remark 1.39. Of course, the above lemma holds, if B is an S-scheme and $\mathfrak{X}$ is a stack over (Sch/S). Hence for moduli problems, the corresponding *moduli stack* (when it exists) behaves like a fine moduli space should. For example $\mathbb{C}$-morphisms

$$B \longrightarrow \mathcal{M},$$

where $\mathcal{M}$ is the moduli stack of vector bundles introduced earlier, correspond bijectively to objects in the fiber category $\mathcal{M}_B = \mathcal{M}(B)$, which by definition are vector bundles over $B \times X$, where X was our fixed projective $\mathbb{C}$-scheme. Especially, the $\mathbb{C}$-valued points $\mathcal{M}(\mathbb{C})$ of the moduli stack $\mathcal{M}$, i.e, scheme morphisms

$$\operatorname{Spec}\mathbb{C} \longrightarrow \mathcal{M},$$

correspond to vector bundles over X, the objects we wanted to classify.

DEFINITION 1.40 (Fibre products of stacks). Given two 1-morphisms $f_1 : \mathcal{F}_1 \longrightarrow \mathcal{G}$ and $f_2 : \mathcal{F}_2 \longrightarrow \mathcal{G}$ of stacks, we define a new stack $\mathcal{F}_1 \times_{\mathcal{G}} \mathcal{F}_2$ (with projections π_i to $\mathcal{F}_1$ and $\mathcal{F}_2$) as follows. Objects of $\mathcal{F}_1 \longrightarrow \mathcal{G}$ are triples (X_1, X_2, α) where X_1 and X_2 are objects of $\mathcal{F}_1$ and $\mathcal{F}_2$ respectively that lie over the same manifold U, and $\alpha : f_1(X_1) \longrightarrow f_2(X_2)$ is an isomorphism in $\mathcal{G}$ (equivalently, $p_{\mathcal{G}}(\alpha) = \mathrm{id}_U$). A morphism from (X_1, X_2, α) to (Y_1, Y_2, β), where Y_i lie over V, is a pair (ϕ_1, ϕ_2) of morphisms $\phi_i : X_i \longrightarrow Y_i$ in $\mathcal{F}_i$ that lie over the same map of manifolds $f : U \longrightarrow V$, and such that $\beta \circ f_1(\phi_1) = f_2(\phi) \circ \alpha$:

$$\begin{array}{ccc} f_1(X_1) & \xrightarrow{\alpha} & f_2(X_2) \\ {\scriptstyle f_1(\phi_1)}\downarrow & & \downarrow{\scriptstyle f_2(\phi_2)} \\ f_1(Y_1) & \xrightarrow{\beta} & f_2(Y_2) \end{array}$$

where $\alpha, \beta, f_1(\phi_1)$ and $f_2(\phi_2)$ are morphisms in the category $\mathcal{G}$.

The projection functor $p_{\mathcal{F}_1 \times_{\mathcal{G}} \mathcal{F}_2} : \mathcal{F}_1 \times_{\mathcal{G}} \mathcal{F}_2 \longrightarrow \mathfrak{S}$ is defined so that for objects (X_1, X_2, α) like above $p_{\mathcal{F}_1 \times_{\mathcal{G}} \mathcal{F}_2}(X_1, X_2, \alpha) = U$ and for morphisms $p_{\mathcal{F}_1 \times_{\mathcal{G}} \mathcal{F}_2}(\phi_1, \phi_2) = f$.

The projection 1-morphisms of stacks $\pi_i : \mathcal{F}_1 \times_{\mathcal{G}} \mathcal{F}_2 \longrightarrow \mathcal{F}_i$ are defined so that for objects (X_1, X_2, α),

$$\pi_i(X_1, X_2, \alpha) = X_i$$

and for morphisms (ϕ_1, ϕ_2)

$$\pi_i(\phi_1, \phi_2) = \phi_i : X_i \longrightarrow Y_i.$$

These fit into a 2-commutative diagram

$$\begin{array}{ccc} \mathcal{F}_1 \times_{\mathcal{G}} \mathcal{F}_2 & \xrightarrow{\pi_2} & \mathcal{F}_2 \\ {\scriptstyle \pi_1}\downarrow & \swarrow\!\!\!\swarrow & \downarrow{\scriptstyle f_2} \\ \mathcal{F}_1 & \xrightarrow{f_1} & \mathcal{G} \end{array}$$

which satisfies the 2-categorified version of the usual universal property of fibre products.

EXAMPLE 1.41. Let $u : X \to [X/G]$ be the quotient map. Let $f : S \longrightarrow [X/G]$ be a 1-morphism and $\pi : X' \longrightarrow S$ be the corresponding G-bundle over S with an equivariant map $\alpha : X' \longrightarrow X$. One can show, that $S \times_{[X/G]} X$ is isomorphic to the manifold X', and moreover that we have a 2-Cartesian diagram

$$\begin{array}{ccc} X' & \xrightarrow{\alpha} & X \\ {\scriptstyle \pi}\downarrow & \swarrow\!\!\!\swarrow & \downarrow{\scriptstyle u} \\ S & \xrightarrow{f} & [X/G] \end{array}$$

which should be compared with diagram (1.1).

4.2. Differentiable stacks.

Definition 1.42. A stack $\mathfrak{X}$ is said to be *representable* if there exists a differentiable manifold X such that the stack $\underline{X}$ associated to X is isomorphic to $\mathfrak{X}$.

From now on we use the same symbol X to denote both the manifold and the associated stack $\underline{X}$.

Definition 1.43. A morphism of stacks $\mathfrak{X} \longrightarrow \mathfrak{Y}$ is called a *representable submersion*, if for *every* manifold U and *every* morphism $U \longrightarrow \mathfrak{Y}$ the fiber product $V =: \mathfrak{X} \times_{\mathfrak{Y}} U$ is representable and the induced morphism of manifolds $V \longrightarrow U$ is a submersion.

Definition 1.44. A morphism of stacks $f : \mathfrak{X} \longrightarrow \mathfrak{Y}$ is an epimorphism if for any object y in $\mathfrak{Y}$ over $S \in \mathrm{Ob}\,(\mathfrak{S})$ there exists a covering $(S_i \longrightarrow S)_{i\in I}$ and objects x_i in $\mathfrak{X}$ over S_i such that $f(x_i) = y|_{S_i}$.

In practice, we would like to replace the word "every" in the definition of a representable submersion with the word "some". This is possible if the morphism $U \longrightarrow \mathfrak{Y}$ is an epimorphism as the following lemma states:

Lemma 1.45. *Let $f : \mathfrak{X} \longrightarrow \mathfrak{Y}$ be a morphism of stacks over $\mathfrak{S}$. Suppose given a manifold U and a morphism $U \longrightarrow \mathfrak{Y}$ which is an epimorphism. If the fibered product $V = \mathfrak{X} \times_{\mathfrak{Y}} U$ is representable and $V \longrightarrow U$ is a submersion, then f is a representable submersion.*

When we know that $f : \mathfrak{X} \longrightarrow \mathfrak{Y}$ is a representable submersion, there exists the following criteria to decide when $f : \mathfrak{X} \longrightarrow \mathfrak{Y}$ itself is surjective (i.e. an epimorphism).

Lemma 1.46. *Let $f : \mathfrak{X} \longrightarrow \mathfrak{Y}$ be a representable submersion of stacks over $\mathfrak{S}$. Then the following conditions are equivalent:*

(1) f is an epimorphism;
(2) For every manifold $U \longrightarrow \mathfrak{Y}$ the submersion $V = \mathfrak{X} \times_{\mathfrak{Y}} U \longrightarrow U$ is surjective;
(3) For some manifold $U \longrightarrow \mathfrak{Y}$, where $U \longrightarrow \mathfrak{Y}$ is an epimorphism, the submersion $V \longrightarrow U$ is surjective.

We are now ready to give the definition of a differentiable stack:

Definition 1.47. A stack $\mathfrak{X}$ over $\mathfrak{S}$ is called *differentiable* or a C^∞-stack, if there exists a manifold X and a surjective representable submersion $x : X \longrightarrow \mathfrak{X}$. In this case X together with the structure morphism is called an *atlas* for $\mathfrak{X}$ or a *presentation* of $\mathfrak{X}$.

Example 1.48 (An atlas for BG). Recall, that $[*/G] = BG$, the classifying stack. Let $u : * \longrightarrow BG$ be the quotient map. We are going to show that this is an atlas for BG.

(1) *($u : * \longrightarrow BG$ is surjective)* Recall, that for a manifold X, the objects of the corresponding stack $\underline{X}$ are maps $S \longrightarrow X$ of manifolds. Hence, for a point $*$ the objects of the corresponding stack correspond bijectively to all manifolds S and the objects of the S-fiber of $*$ is presicely S. On the other hand the S-fibers of BG consist of all principal G-bundles over S. One can show that the map u restricted to S-fibers is such that it sends a manifold S to the trivial G-bundle over S. Let now $P \in \mathrm{Ob}\,(BG_S)$, i.e. a principal G-bundle over S. Next, choose a covering $(S_i \longrightarrow S)_{i\in I}$ such that $P_i =: P|_{S_i}$ is a trivial bundle over S_i for every $i \in I$. Then $u(S_i) = P_i$ for every $i \in I$, which shows that u is an epimorphism.

(2) *($u : * \longrightarrow BG$ is a representable submersion)* It follows from Example 1.41 and the definition of the quotient map, that the fibre product $* \times_{BG} *$ is equal to G (the total space of the trivial G-bundle over $*$). Hence, we have a 2-commutative diagram

$$\begin{array}{ccc} G & \xrightarrow{\pi_2} & * \\ {\scriptstyle \pi_1}\big\downarrow & \swarrow\!\!\!\!\swarrow & \big\downarrow{\scriptstyle u} \\ * & \xrightarrow[u]{} & BG \end{array}$$

We want to apply Lemma 1.45 to show that u is a representable submersion. Now $*$ is a manifold and the vertical arrow $u : * \longrightarrow BG$ on the right hand side of the diagram is an epimorphism as we just saw above. Since $V := * \times_{BG} * = G$, the fiber product $* \times_{BG} *$ is representable, and moreover the map $\pi_2 : V = G \longrightarrow *$ is clearly a submersion, the claim follows from the lemma.

Remark 1.49. For differentiable stacks, one can develop sheaf theory, cohomology theories such as deRham cohomology, (twisted) K-theory, tangent stacks exist, etc. [**BeXu**], [**Be**], [**L-GTuXu**], and therefore differentiable stacks should be thought of as generalizations of differentiable manifolds, whose points may have nontrivial automorphisms.

Remark 1.50. The algebraic analog of a differentiable stack is an *algebraic stack*. This is a stack $\mathcal{F}$ over $(Sch/\mathbb{C})$ with an atlas $u : U \longrightarrow \mathcal{F}$, where U is a scheme instead of a manifold, and u is a surjective étale/smooth morphism.

There are two important classes of algebraic stacks, depending on the properties of the diagonal morphism $\triangle_{\mathcal{F}} : \mathcal{F} \longrightarrow \mathcal{F} \times_S \mathcal{F}$, called $Deligne-Mumford$ stacks and $Artin$ stacks. One can show, that the stack $\mathcal{M}$ of vector bundles on a projective scheme X is an Artin stack. An example of a Deligne-Mumford stack is provided by the moduli stack of stable genus g curves, denoted by $\overline{\mathcal{M}}_g$. This is an algebraic stack parametrizing stable genus g curves, where the word stable means essentially that instead of considering families of nonsingular genus g curves, we allow the curves in our families to have ordinary double points (i.e. nodes) as singularities, and require the automorphism groups of these curves to be finite.

5. Appendix: Natural transformations

Suppose that $\mathfrak{C}$ and $\mathfrak{D}$ are categories. Let $F, G : \mathfrak{C} \longrightarrow \mathfrak{D}$ be functors. A *natural transformation* $\eta : F \Rightarrow G$ is a rule that associates a morphism $\eta_C : F(C) \longrightarrow G(C)$ in $\mathfrak{D}$ to every object C of $\mathfrak{C}$ in such a way that for every morphism $f : C \longrightarrow C'$ in C the following diagram commutes:

$$\begin{array}{ccc} F(C) & \xrightarrow{Ff} & F(C') \\ {\scriptstyle \eta_C}\big\downarrow & & \big\downarrow{\scriptstyle \eta_{C'}} \\ G(C) & \xrightarrow{Gf} & G(C') \end{array}$$

If each η_C is an isomorphism, we say that η is a *natural isomorphism* and write $\eta : F \cong G$.

Definition 1.51 (Equivalence). We call a functor $F : \mathfrak{C} \longrightarrow \mathfrak{D}$ an *equivalence of categories* if there exists a functor $G : \mathfrak{D} \longrightarrow \mathfrak{C}$ and natural isomorphisms $\mathrm{id}_{\mathfrak{C}} \cong GF$, $\mathrm{id}_{\mathfrak{D}} \cong FG$.

6. Appendix: 2-categories and 2-functors

6.1. 2-categories. A 2-category $\mathfrak{C}$ consists of the following data:

(1) A class of objects $\operatorname{Ob}\mathfrak{C}$
(2) For each pair $X, Y \in \operatorname{Ob}\mathfrak{C}$, a category $\operatorname{Hom}(X, Y)$
(3) *Horizontal composition of 1-morphisms and 2-morphisms.* For each triple $X, Y, Z \in \operatorname{Ob}\mathfrak{C}$, a functor

$$\mu_{X,Y,Z} : \operatorname{Hom}(X, Y) \times \operatorname{Hom}(Y, Z) \longrightarrow \operatorname{Hom}(X, Z)$$

satisfying some compatibility conditions, which we shall soon describe.

An object f of the category $\operatorname{Hom}(X, Y)$ is called a 1-morphism of $\mathfrak{C}$ and is represented with a diagram

$$X \xrightarrow{f} Y$$

and a morphism α of the category $\operatorname{Hom}(X, Y)$ is called a 2-morphism of $\mathfrak{C}$, and is represented pictorially as

$$X \underset{f'}{\overset{f}{\underset{\Downarrow\alpha}{\rightrightarrows}}} Y$$

The axioms of a 2-category are given now as follows:

(1) (*Composition of 1-morphisms*) Given a diagram

$$X \xrightarrow{f} Y \xrightarrow{g} Z \qquad \text{there exist} \qquad X \xrightarrow{g \circ f} Z$$

and this composition is associative: $(h \circ g) \circ f = h \circ (g \circ f)$.
(2) (*Identity for 1-morphisms*) For each object X there is a 1-morphism id_X such that $f \circ \mathrm{id}_y = \mathrm{id}_X \circ f = f$.
(3) (*Vertical composition of 2-morphisms*) Given a diagram

$$X \begin{array}{c} \xrightarrow{f} \\ \Downarrow\alpha \\ \xrightarrow{g} \\ \Downarrow\beta \\ \xrightarrow[h]{} \end{array} Y \qquad \text{there exists} \qquad X \underset{h}{\overset{f}{\underset{\Downarrow\beta\circ\alpha}{\rightrightarrows}}} Y$$

and this compostion is associative $(\gamma \circ \beta) \circ \alpha = \gamma \circ (\beta \circ \alpha)$.
(4) (*Horizontal composition of 2-morphisms*) Given a diagram

$$X \underset{f'}{\overset{f}{\underset{\Downarrow\alpha}{\rightrightarrows}}} Y \underset{g'}{\overset{g}{\underset{\Downarrow\beta}{\rightrightarrows}}} Z$$

there exists

$$X \underset{g' \circ f'}{\overset{g \circ f}{\underset{\Downarrow\beta*\alpha}{\rightrightarrows}}} Y$$

and it is associative $(\gamma * \beta) * \alpha = \gamma * (\beta * \alpha)$.
(5) (*Identity for 2-morphisms*) For every 1-morphism f there is a 2-morphism id_f such that $\alpha \circ \mathrm{id}_g = \mathrm{id}_f \circ \alpha = \alpha$. More over, $\mathrm{id}_g * \mathrm{id}_f = \mathrm{id}_{g \circ f}$.
(6) (*Compatibility between horizontal and vertical composition of 2-morphisms*) Given a diagram

$$X \begin{array}{c} \xrightarrow{f} \\ \Downarrow\alpha \\ \xrightarrow{f'} \\ \Downarrow\alpha' \\ \xrightarrow[f'']{} \end{array} Y \begin{array}{c} \xrightarrow{g} \\ \Downarrow\beta \\ \xrightarrow{g'} \\ \Downarrow\beta' \\ \xrightarrow[g'']{} \end{array} Z$$

then $(\beta' \circ \beta) * (\alpha' \circ \alpha) = (\beta' * \alpha') \circ (\beta * \alpha)$.

Two objects of a 2-category are equivalent if there exists two 1-morphisms $f : X \longrightarrow Y$, $g : Y \longrightarrow X$ and two 2-isomorphisms (invertible 2-morphism) $\alpha : g \circ f \longrightarrow \mathrm{id}_X$ and $\beta : f \circ g \longrightarrow \mathrm{id}_Y$.

A commutative diagram of 1-morphisms in a 2-category is a diagram

$$\begin{array}{ccccc} & & Y & & \\ & \overset{f}{\nearrow} & \Downarrow \alpha & \overset{g}{\searrow} & \\ X & & \xrightarrow[h]{\qquad\qquad} & & Z \end{array}$$

such that $\alpha : g \circ f \longrightarrow h$ is a 2-isomorphism.

On the other hand, a diagram of 2-morphisms will be called commutative only if the compositions are actually equal.

6.2. 2-functors. A covariant 2-functor F between two 2-categories $\mathfrak{C}$ and $\mathfrak{C}'$ is a law that for each $X \in \mathrm{Ob}\,(\mathfrak{C})$ gives an object $F(X) \in \mathrm{Ob}\,(\mathfrak{C}')$. For each 1-morphism $f : X \longrightarrow Y$ in $\mathfrak{C}$ gives a 1-morphism $F(f) : F(X) \longrightarrow F(Y)$ in $\mathfrak{C}'$, and for each 2-morphism $\alpha : f \Rightarrow g$ in $\mathfrak{C}$ gives a 2-morphism $F(\alpha) : F(f) \Rightarrow F(g)$ in $\mathfrak{C}'$ such that

(1) *(Respects identity 1-morphisms)* $F(\mathrm{id}_X) = \mathrm{id}_{F(X)}$,
(2) *(Respects identity 2-morphisms)* $F(\mathrm{id}_f) = \mathrm{id}_{F(f)}$,
(3) *(Respects composition of 1-morphisms up to a 2-isomorphism)* For every diagram

$$X \xrightarrow{f} Y \xrightarrow{g} Z$$

there exists a 2-isomorphism $\epsilon_{g,f} : F(g) \circ F(f) \longrightarrow F(g \circ f)$

$$\begin{array}{ccccc} & & F(Y) & & \\ & \overset{F(f)}{\nearrow} & \Downarrow \epsilon_{g,f} & \overset{F(g)}{\searrow} & \\ F(X) & & \xrightarrow[F(g\circ f)]{\qquad\qquad} & & F(Z) \end{array}$$

(a) $\epsilon_{f,\mathrm{id}_X} = \epsilon_{\mathrm{id}_Y,f} = \mathrm{id}_{F(f)}$
(b) ϵ *is associative.* The following diagram is commutative

$$\begin{array}{ccc} F(h) \circ F(g) \circ F(f) & \xRightarrow{\epsilon_{h,g} \times \mathrm{id}} & F(h \circ g) \circ F(f) \\ {\scriptstyle \mathrm{id} \times \epsilon_{g,f}} \Downarrow & & \Downarrow {\scriptstyle \epsilon_{h\circ g,f}} \\ F(h) \circ F(g \circ f) & \xRightarrow{\epsilon_{h,g\circ f}} & F(h \circ g \circ f) \end{array}$$

(4) *(Respects vertical composition of 2-morphisms)* For every pair of 2-morphisms $\alpha : f \longrightarrow f'$ and $\beta : g \longrightarrow g'$, we have $F(\beta \circ \alpha) = F(\beta) \circ F(\alpha)$.
(5) *(Respects horizontal composition of 2-morphisms)* For every pair of 2-morphisms $\alpha : f \longrightarrow f'$ and $\beta : g \longrightarrow g'$, the following diagram commutes

$$\begin{array}{ccc} F(g) \circ F(f) & \xRightarrow{F(\beta)*F(\alpha)} & F(g') \circ F(f') \\ {\scriptstyle \epsilon_{g,f}} \Downarrow & & \Downarrow {\scriptstyle \epsilon_{g',f'}} \\ F(g \circ f) & \xRightarrow{F(\beta*\alpha)} & F(g' \circ f') \end{array}$$

DEFINITION 1.52. Let (groupoids) be the 2-category, whose objects are groupoids, 1-morphisms are functors between groupoids, and 2-morphisms are natural transformations between these functors.

Example 1.53. A *presheaf in groupoids* (also called a *quasi-functor* or a *lax 2-functor*) is a contravariant 2-functor $\mathcal{F} : (Sch/S) \longrightarrow (groupoids)$, where the 1-category (Sch/S) is extended trivially to a 2-category by declaring that the set of 2-morphisms of (Sch/S) consists of the identity 2-morphism alone.

Hence for each S-scheme B we have a groupoid $\mathcal{F}(B)$. For each 1-morphism $f : B' \longrightarrow B$ in (Sch/S), we have a functor $\mathcal{F}(f) = f^* : \mathcal{F}(B) \longrightarrow \mathcal{F}(B')$ such that for every 1-morphism $g : B'' \longrightarrow B'$ in (Sch/S) there exists a natural transformation of functors (a 2-isomorphism) $\epsilon_{g,f} : g^* \circ f^* \longrightarrow (f \circ g)^*$. These 2-isomorphisms need to satisfy the following compatibility relation: For every 1-morphism $h : B''' \longrightarrow B''$ in (Sch/S) the following diagram commutes:

$$\begin{array}{ccc} h^* \circ g^* \circ f^* & \longrightarrow & h^* \circ (f \circ g)^* \\ \downarrow & & \downarrow \\ (g \circ f)^* \circ f & \longrightarrow & (f \circ g \circ h)^*. \end{array}$$

7. Lie groupoids

Definition 1.54. A *Lie groupoid* $\Gamma = X_1 \rightrightarrows X_0$ consists of

- Two smooth manifolds X_1 (the *morphisms* or *arrows*) and X_0 (the *objects* or *points*);
- Two smooth surjective submersions $s : X_1 \longrightarrow X_0$ the *source* map and $t : X_1 \longrightarrow X_0$ the *target* map;
- A smooth embedding $e : X_0 \longrightarrow X_1$ (the *identities* or *constant arrows*);
- A smooth involution $i : X_1 \longrightarrow X_1$, (the *inversion*) also denoted $x \mapsto x^{-1}$;
- A multiplication
$$m : \Gamma^{(2)} \longrightarrow \Gamma,$$
$$(x, y) \mapsto x \cdot y,$$
where $\Gamma^{(2)} = X_1 \times_{s,t} X_1 = \{(x, y) \in X_1 \times X_1 \mid s(x) = t(y)\}$. Notice, that $\Gamma^{(2)}$ is a smooth manifold, since s and t are submersions. We require the multiplication map m to be smooth and that
  1. $s(x \cdot y) = s(y), \quad t(x \cdot y) = t(x)$,
  2. $x \cdot (y \cdot z) = (x \cdot y) \cdot z$,
  3. e is a section of both s and t,
  4. $e(t(x)) \cdot x = x = x \cdot e(s(x))$,
  5. $s(x^{-1}) = t(x), \quad t(x^{-1}) = s(x)$,
  6. $x \cdot x^{-1} = e(t(x)), \quad x^{-1} \cdot x = e(s(x))$,

  whenever (x, y) and (y, z) are in $\Gamma^{(2)}$.

Definition 1.55. A morphism of Lie groupoids $(\Psi, \psi) : [X_1' \rightrightarrows X_0'] \longrightarrow [X_1 \rightrightarrows X_0]$ are the following commutative diagrams:

$$\begin{array}{ccc} X_1' & \xrightarrow{\Psi} & X_1 \\ s' \downdownarrows t' & & s \downdownarrows t \\ X_0' & \xrightarrow{\psi} & X_0 \end{array} \qquad \begin{array}{ccc} X_1 & \xrightarrow{\Psi} & X_1' \\ e' \uparrow & & \uparrow e \\ X_0' & \xrightarrow{\psi} & X_0 \end{array}$$

$$\begin{array}{ccc} X_1' \times_{s',t'} X_1' & \xrightarrow{\Psi \times \Psi} & X_1 \times_{s,t} X_1 \\ m' \downarrow & & \downarrow m \\ X_1' & \xrightarrow{\Psi} & X_1 \end{array} \qquad \begin{array}{ccc} X_1' & \xrightarrow{\Psi} & X_1 \\ i' \downarrow & & \downarrow i \\ X_1' & \xrightarrow{\Psi} & X_1 \end{array}$$

Example 1.56. A Lie group G is a Lie groupoid over a point, $G \rightrightarrows \bullet$.

EXAMPLE 1.57. Let M be a differentiable manifold and G a Lie group acting smoothly on M from the right. The action groupoid $M \times G \rightrightarrows M$, denoted by $M \rtimes G$, is defined by the following data:

- $s(x,g) = x$;
- $t(x,g) = xg$, so that a pair $\big((x,g),(x',g')\big)$ is decomposable iff $x' = xg$;
- $m\Big((x,g),(xg,g')\Big) = (x, gg')$;
- $i(x,g) = (xg, g^{-1})$;
- $e(x) = (x, \mathbf{1}_G)$.

DEFINITION 1.58. Let $\Gamma = X_1 \rightrightarrows X_0$ be a Lie groupoid. A right action of Γ on a manifold N consists of two smooth maps $a : N \longrightarrow X_0$ (*the anchor or the moment map*), $m : N \times_{X_0,a,t} X_1 = \{(n,x) \in N \times X_1 \mid a(n) = t(x)\} \longrightarrow N$ (*the action*), such that, denoting $m(n,x) = nx$,

$$(nx)y = n(xy), \quad n1 = n, \quad a(nx) = s(x).$$

DEFINITION 1.59. Let $\Gamma = X_1 \rightrightarrows X_0$ be a Lie groupoid and S a manifold. A Γ *torsor over* S is a manifold P, together with a surjective submersion $\pi : P \longrightarrow S$ and a right action of Γ on P, such that for all $p, p' \in P$ in the same fibre $\pi^{-1}(s)$, there exists a unique $\gamma \in X_1$, such that $p \cdot \gamma$ is defined and $p \cdot \gamma = p'$.

DEFINITION 1.60. Let π and $\rho : Q \longrightarrow T$ be Γ torsors. A *morphism* of Γ-torsors from Q to P is given by a commutative diagram of differentiable maps

$$\begin{array}{ccc} Q & \xrightarrow{\phi} & P \\ \downarrow & & \downarrow \\ T & \longrightarrow & S \end{array} \tag{1.2}$$

such that ϕ is Γ-equivariant.

EXAMPLE 1.61 (Trivial torsors). Let $\Gamma = X_1 \rightrightarrows X_0$ be a Lie groupoid and $f : S \longrightarrow X_0$ be a smooth map. The *trivial* Γ-torsor P over S induced by f is by definition $P = S \times_{f,X_0,s} \Gamma$, and the action of Γ is defined so that

$$(s,\gamma) \cdot \delta = (s, \gamma \cdot \delta).$$

The structure map $\pi : P \longrightarrow S$ is the first projection, and the anchor map of the Γ-action is the second projection followed by the target map t. In analogy with principal bundles, it is showed in [**BeXu**] that every Γ-torsor is *locally trivial.*

Hence Γ-torsors form a category with respect to the the above notion of morphism, which we denote by $B\Gamma$. There is a natural functor $B\Gamma \longrightarrow \mathfrak{S}$, which assigns to a Γ-torsor $P \longrightarrow S$ the base manifold S. The following proposition is proved in [**BeXu**]:

PROPOSITION 1.62. *For every Lie gropoid* $\Gamma = X_1 \rightrightarrows X_0$*, the category of* Γ*-torsors* $B\Gamma$ *is a differentiable stack.*

If $x : X \longrightarrow \mathfrak{X}$ is a differentiable stack, then for any two morphisms $f_i : Y_i \longrightarrow \mathfrak{X}$, where Y_i is a manifold for $i = 1,2$, the fibered product $Y_1 \times_{\mathfrak{X}} Y_2$ is again a manifold. This can be seen as follows: Notice that $Y_1 \times_{\mathfrak{X}} Y_2 \cong (Y_1 \times Y_2) \times_{\mathfrak{X}\times\mathfrak{X}} \mathfrak{X}$ where the map $\Delta : \mathfrak{X} \longrightarrow \mathfrak{X} \times \mathfrak{X}$ is the diagonal map. One can show that since by definition the atlas $x : X \longrightarrow \mathfrak{X}$ is a representable morphism, it follows that the the diagonal is also representable.

Now recall that the morphisms $f_i : Y_i \longrightarrow \mathfrak{X}$ can equivalently be considered as objects y_i in the fibre category $\mathfrak{X}_{Y_i}$. The fibre product $Y_1 \times_{\mathfrak{X}} Y_1$ represents the following functor

$$\begin{aligned}\underline{\mathrm{Isom}}(f_1, f_2)(T) \quad = \quad & \{\text{Pairs } (g, \phi_g) \mid g : T \longrightarrow Y_1 \times Y_2 \text{ a smooth map,} \\ & \phi_g : g^* \circ \mathrm{pr}_1^*(y_1) \xrightarrow{\sim} g^* \circ \mathrm{pr}_2^*(y_2) \text{ an isomorphism in } \mathfrak{X}_T\}.\end{aligned}$$

In particular it follows from this that the automorphism of a map $f : Y \longrightarrow \mathfrak{X}$ are given by sections of the map $\mathrm{Aut}(f) =: (Y \times_{\mathfrak{X}} Y) \times_{Y \times Y} Y \longrightarrow Y$.

Next, consider the following (2-)fibre product

$$\begin{array}{ccc} \underline{\mathrm{Isom}}(x,x) & \longrightarrow & X \\ \downarrow & \swarrow\!\!\!\!\swarrow & \downarrow{\scriptstyle x} \\ X & \xrightarrow{x} & \mathfrak{X} \end{array}$$

Since $x : X \longrightarrow \mathfrak{X}$ is an atlas, $\underline{\mathrm{Isom}}(x,x)$ is a manifold and the projections $\underline{\mathrm{Isom}}(x,x) \longrightarrow X$ are surjective submersions. Hence we can make $\underline{\mathrm{Isom}}(x,x) \rightrightarrows X$ into a Lie groupoid by defining s and t to be the projection mappings, and the multiplication being defined as follows:

The points (i.e. objects over a point $\bullet \in \mathfrak{S}$) of $\underline{\mathrm{Isom}}(x,x)$ are by definition of the stacky fibre product, triples (x, ϕ, x'), where $x, x' \in X$ and $\phi : \pi(x) \longrightarrow \pi(x')$ is a morphism in the groupoid $\mathfrak{X}_*$ (the fibre of $\mathfrak{X}$ over $\bullet \in \mathfrak{S}$). Thus it is clear how to define the composition:

$$(x, \phi, x') \circ (x', \psi, x'') = (x, \psi \circ \phi, x'').$$

Different presentations $x : X \longrightarrow \mathfrak{X}$ of a differentiable stack $\mathfrak{X}$ may give non-isomorphic Lie groupoids. However, the associated Lie groupoids $\underline{\mathrm{Isom}}(x,x)$ are always *Morita equivalent* as we will see in the next section. On the other hand, starting from a differentiable stack and any presentation $x : X \longrightarrow \mathfrak{X}$, the stack of $\underline{\mathrm{Isom}}(x,x)$-torsors is always isomorphic to $\mathfrak{X}$ so that we end up where we began:

PROPOSITION 1.63. *Let $\mathfrak{X}$ be a differentiable stack and $x : X \longrightarrow \mathfrak{X}$ its atlas. Then*

$$\mathfrak{X} \cong B\underline{Isom}(x,x).$$

EXAMPLE 1.64. It is known that quotient stacks $[X/G]$ map to action groupoids $X \rtimes G$ and vice versa under the presented correspondence between stacks and groupoids, [**Be**].

8. Morita equivalence

DEFINITION 1.65 (Morita equivalence). Two Lie groupoids $X_\bullet$ and $Y_\bullet$ are *Morita equivalent* if there exists a manifold Q, such that $X_\bullet$ and $Y_\bullet$ act on Q from the left and right respectively with moment maps $a_X : Q \longrightarrow X_0$ and $a_Y : Q \longrightarrow Y_0$ and the two actions commute, making Q a left $X_\bullet$ torsor and a right $Y_\bullet$ torsor. Such a Q is called a *Morita bibundle* of $X_\bullet$ and $Y_\bullet$.

THEOREM 1.66. *Let $X_\bullet$ and $Y_\bullet$ be Lie groupoids and let $\mathfrak{X}$ and $\mathfrak{Y}$ be the associated stacks, i.e. $\mathfrak{X}$ is the stack of $X_\bullet$ torsors and $\mathfrak{Y}$ is the stack of $Y_\bullet$ torsors. Then the following are equivalent:*

(1) The differentiable stacks $\mathfrak{X}$ and $\mathfrak{Y}$ are isomorphic;
(2) Lie groupoids $X_\bullet$ and $Y_\bullet$ are Morita equivalent.

9. Gerbes and S^1-central extensions of Lie groupoids

EXAMPLE 1.67. Let G be a Lie group and BG its classifying stack. As we have seen, this is a stack, but it is in fact a rather special stack. This is because

(1) Every manifold X has at least one principal G bundle over it, namely the trivial G bundle;
(2) Any two principal G bundles are locally isomorphic.

These two facts lead to the definition of a gerbe.

DEFINITION 1.68. Let $\mathfrak{X} \longrightarrow \mathfrak{S}$ be a stack. Then $\mathfrak{X}$ is called a *gerbe* over (the site) $\mathfrak{S}$, if it satisfies the two following conditions:

(1) For any object S of $\mathfrak{S}$ there exists a covering family $\{S_i \longrightarrow S\}$ such that the fibre $\mathfrak{X}_{S_i}$ is nonempty for every i.
(2) For any object S of $\mathfrak{S}$ and any two objects $x, y \in \mathfrak{X}_S$ there exists a covering family $S_i \longrightarrow S$ such that $x|_{S_i}$ and $y|_{S_i}$ are isomorphic for every i.

REMARK 1.69. Here condition (1) says that objects locally exists, which is a weaker condition than the global existence satisfied by BG. Condition (2) says that any two objects are locally isomorphic.

Next we make the following generalization of the definition of a gerbe.

DEFINITION 1.70. Let $\mathfrak{X}$ and $\mathfrak{R}$ be stacks over $\mathfrak{S}$ and $\pi : \mathfrak{R} \longrightarrow \mathfrak{X}$ a morphism of stacks. Then $\pi : \mathfrak{R} \longrightarrow \mathfrak{X}$ is called a *gerbe* over (the stack) $\mathfrak{X}$, if

(1) π has local sections, i.e. there is an atlas $p : X \longrightarrow \mathfrak{X}$ and a section $s : X \longrightarrow \mathfrak{R}$ of $\pi|_X$, where by a section we mean there exists a natural isomorphism $\phi : \pi \circ s \Rightarrow p$ of functors.
(2) Locally over $\mathfrak{X}$ all objects of $\mathfrak{R}$ are isomorphic, i.e. for any two objects $t_1, t_2 \in \mathfrak{X}_T$ and lifts $s_1, s_2 \in \mathfrak{R}_T$ with $\pi(s_i) \cong t_i$, there is a covering $\{T_i \longrightarrow T\}$ such that $s_1|_{T_i} \cong s_2|_{T_i}$.

A gerbe $\pi : \mathfrak{R} \longrightarrow \mathfrak{X}$ is *trivial*, if it admits a global section, i.e. if there exists a morphism of stacks $\sigma : \mathfrak{X} \longrightarrow \mathfrak{R}$ satisfying $\pi \circ \sigma \cong \mathrm{id}_{\mathfrak{X}}$.

DEFINITION 1.71. A gerbe $\mathfrak{R} \longrightarrow \mathfrak{X}$ is called an S^1-gerbe if there is an atlas $p : X \longrightarrow \mathfrak{X}$ and a section $s : X \longrightarrow \mathfrak{R}$ such that there is an isomorphism

$$\Phi : \mathrm{Aut}(s/p) := (X \times_{\mathfrak{R}} X) \times_{X \times_{\mathfrak{X}} X} X \cong S^1 \times X$$

as a family of groups over X such that on $X \times_{\mathfrak{X}} X$ the diagram

$$\begin{array}{ccc} \mathrm{Aut}(s \circ \mathrm{pr}_1/p \circ \mathrm{pr}_1) & \xrightarrow{\cong} & \mathrm{Aut}(s \circ \mathrm{pr}_2/p \circ \mathrm{pr}_2) \\ & {\scriptstyle \mathrm{pr}_1^*\Phi} \searrow \quad \swarrow {\scriptstyle \mathrm{pr}_2^*\Phi} & \\ & X \times_{\mathfrak{X}} X \times S^1 & \end{array}$$

where the horizontal map is the isomorphism given by the universal property of the fibre product, commutes. This means that the automorphism groups of objects of $\mathfrak{R}$ are central extensions of those of $\mathfrak{X}$ by S^1.

DEFINITION 1.72. Let $\Gamma = X_1 \rightrightarrows X_0$ be a Lie groupoid. An S^1-*central extension* of $X_1 \rightrightarrows X_0$ consists of

(1) a Lie groupoid $R_1 \rightrightarrows X_0$ and a morphism of Lie groupoids $(\pi, \mathrm{id}) : [R_1 \rightrightarrows X_0] \longrightarrow [X_1 \rightrightarrows X_0]$,
(2) a left S^1 action on R_1, making $\pi : R_1 \longrightarrow X_1$ a left principal S^1 bundle. The action must satisfy $(s \cdot x)(t \cdot y) = st \cdot (xy)$, for all $s, t \in S^1$ and $(x, y) \in R_1 \times_{X_0} R_1$.

When $R_1 \longrightarrow X_1$ is topologically trivial, then $R_1 \cong X_1 \times S^1$ and the central extension is determined by a *groupoid 2-cocycle* of $X_1 \rightrightarrows X_0$ with values in S^1. This is a smooth map

$$c : \Gamma^{(2)} = \Big\{(x, y) \in X_1 \times X_1 \mid s(x) = t(y)\Big\} \longrightarrow S^1$$

satisfying the cocycle condition

$$c(x,y)c(xy,z)c(x,yz)^{-1}c(y,z)^{-1} = 1$$

for all $(x,y,z) \in \Gamma^{(3)}$. The groupoid structure on $R_1 \rightrightarrows X_0$ is given by

$$(x,\lambda_1)\cdot(y,\lambda_2) = (xy, \lambda_1\lambda_2 c(x,y)),$$

for all $(x,y) \in \Gamma^{(2)}$ and $\lambda_1, \lambda_2 \in S^1$.

PROPOSITION 1.73. *Let $X_1 \rightrightarrows X_0$ be a Lie groupoid and $\mathfrak{X}$ its corresponding differential stack of $X_\bullet$-torsors. There is one-to-one correspondence between S^1-central extensions of $X_1 \rightrightarrows X_0$ and S^1-gerbes $\mathfrak{R}$ over $\mathfrak{X}$ whose restriction to X_0 : $\mathfrak{R}|_{X_0}$ admits a trivialization.*

10. Sheaf cohomology on differentiable stacks

Let $\pi : \mathfrak{X} \longrightarrow \mathfrak{S}$ be a differentiable stack. The category $\mathfrak{X}$ has a natural structure of a site inherited from $\mathfrak{S}$. More precisely, call a set of morphisms $\{x_i \longrightarrow x\}$ in $\mathfrak{X}$ a covering family if the image family $\{U_i \longrightarrow U\}$ in $\mathfrak{S}$ is a covering family in $\mathfrak{S}$.

DEFINITION 1.74. Let $\pi : \mathfrak{X} \longrightarrow \mathfrak{S}$ be a differentiable stack. By a sheaf of Abelian groups over $\mathfrak{X}$ we mean a sheaf on the induced site on $\mathfrak{X}$.

We denote the category of Abelian sheaves on $\mathfrak{X}$ by $\mathfrak{Ab}(\mathfrak{X})$.

REMARK 1.75. There is an equivalent definition of a sheaf over a stack which is often used, [**Laum**], [**Hein**]. In this picture a sheaf $\mathcal{F}$ on a stack is a determined by the following data

(1) For each morphism $X \longrightarrow \mathfrak{X}$ where X is a manifold, a sheaf $\mathcal{F}_{X\longrightarrow\mathfrak{X}}$ on X.
(2) For any 2-commuting triangle

$$\begin{array}{ccc} X & \xrightarrow{\quad f \quad} & Y \\ & {\scriptstyle h}\searrow \;\overset{\varphi}{\Longrightarrow}\; \swarrow{\scriptstyle g} & \\ & \mathfrak{X} & \end{array} \tag{1.3}$$

with an isomorphism $\varphi : g \circ f \longrightarrow h$ of functors, there exists a morphism of sheaves $\Phi_{f,\varphi} : f^*\mathcal{F}_{Y\longrightarrow\mathfrak{X}} \longrightarrow \mathcal{F}_{X\longrightarrow\mathfrak{X}}$ (often denoted simply by Φ_f) compatible for $X \longrightarrow Y \longrightarrow Z$. We require that Φ_f is an isomorphism, whenever f is an open covering.

The sheaf $\mathcal{F}$ is called *Cartesian* if all Φ_f are isomorphisms.

DEFINITION 1.76. A sheaf $\mathcal{I} \in \mathrm{Ob}(\mathfrak{Ab}(\mathfrak{X}))$ is called *injective* if it satisfies the following universal lifting property: Given an injection $f : \mathcal{F} \longrightarrow \mathcal{G}$ and a map $\alpha : \mathcal{F} \longrightarrow \mathcal{I}$ in $\mathfrak{Ab}(\mathfrak{X})$ there exists at least one map $\beta : \mathcal{G} \longrightarrow \mathcal{I}$ such that $\alpha = \beta \circ f$:

$$\begin{array}{ccccc} 0 & \longrightarrow & \mathcal{F} & \xrightarrow{\;f\;} & \mathcal{G} \\ & & {\scriptstyle\alpha}\downarrow & \swarrow{\scriptstyle \exists\beta} & \\ & & \mathcal{I} & & \end{array}$$

PROPOSITION 1.77. *The category $\mathfrak{Ab}(\mathfrak{X})$ is an Abelian category with enough injective objects, i.e. for every object $\mathcal{F} \in \mathrm{Ob}(\mathfrak{Ab}(\mathfrak{X}))$ there exists an injection $0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{I}$ with $\mathcal{I}$ injective.*

Corollary 1.78. *Every object $\mathcal{F} \in \mathrm{Ob}(\mathfrak{Ab}(\mathfrak{X}))$ admits an injective resolution, i.e. an exact cochain complex*

$$0 \longrightarrow \mathcal{F} \xrightarrow{\varepsilon} \mathcal{I}^0 \xrightarrow{\delta} \mathcal{I}^1 \xrightarrow{\delta} \mathcal{I}^2 \xrightarrow{\delta} \cdots .$$

Definition 1.79. Let U be a manifold. A sheaf in the usual sense (i.e. defined only on open subsets of U) is called a *small* sheaf on U.

Definition 1.80. Let $\mathfrak{X}$ be a stack over $\mathfrak{S}$ and $\mathcal{F}$ a sheaf over $\mathfrak{X}$. Let $x \in \mathrm{Ob}(\mathfrak{X}_U)$, where $U \in \mathrm{Ob}(\mathfrak{S})$ is a manifold. The small sheaf on U, which maps the open subset $V \subseteq U$ to $\mathcal{F}(x \mid V)$ is called the small sheaf *induced* by $\mathcal{F}$ via $x : U \longrightarrow \mathcal{F}$ on U. We denote it by $\mathcal{F}_{x,U}$ or simply $\mathcal{F}_U$, if there is no risk of confusion.

Given a morphism in $\theta : y \longrightarrow x$ in $\mathfrak{X}$ lying over a C^∞ map $f : V \longrightarrow U$ in $\mathfrak{S}$, there is an induced morphism of small sheaves over V

$$\theta^* : f^{-1}\mathcal{F}_{x,U} \longrightarrow \mathcal{F}_{y,V}.$$

The cohomology of a sheaf $\mathcal{F} \in \mathrm{Sh}(\mathfrak{X})$ is defined in the same way as it is defined for manifolds: One first defines the *global section* functor

$$\Gamma(\mathfrak{X}, \cdot) : \mathfrak{Ab}(\mathfrak{X}) \longrightarrow \mathfrak{Ab},$$

where now

$$\Gamma(\mathfrak{X}, \mathcal{F}) := \varprojlim \Gamma(X, \mathcal{F}_{X \longrightarrow \mathfrak{X}})$$

and the limit is taken over all atlases $X \longrightarrow \mathfrak{X}$, the transition functions for a 2-commutative diagram

$$\begin{array}{ccc} X' & \xrightarrow{\quad f \quad} & X \\ & {\scriptstyle h}\searrow \;\overset{\varphi}{\Longrightarrow}\; \swarrow{\scriptstyle g} & \\ & \mathfrak{X} & \end{array}$$

are given by the restriction maps $\Phi_{f,\varphi}$.

Next one chooses an injective resolution $0 \longrightarrow \mathcal{F} \xrightarrow{\varepsilon} \mathcal{I}^\bullet$ and sets

$$H^i(\mathfrak{X}, \mathcal{F}) = h^i(\Gamma(\mathfrak{X}, \mathcal{I}^\bullet)).$$

Remark 1.81. For a Cartesian sheaf $\mathcal{F}$ over $\mathfrak{X}$ the global section functor can be defined by choosing an atlas $X \longrightarrow \mathfrak{X}$ and then setting

$$\Gamma(\mathfrak{X}, \mathcal{F}) := \ker\Big(\Gamma(X, \mathcal{F}) \rightrightarrows \Gamma(X \times_{\mathfrak{X}} X)\Big).$$

This is known to be independent of the chosen atlas $X \longrightarrow \mathfrak{X}$ and moreover it coincides with the previous definition, [**Hein**].

Theorem 1.82 (Giraud). *Isomorphism classes of S^1-gerbes over $\mathfrak{X}$ are in one-to-one correspondence with $H^2(\mathfrak{X}, \underline{S}^1)$.*

The above extends naturally to the derived category of Abelian sheaves on $\mathfrak{X}$, giving us the *total derived functor*

$$\mathbf{R}\Gamma(\mathfrak{X}, \cdot) : D^+(\mathfrak{X}) \longrightarrow D^+(\mathfrak{Ab}).$$

For a complex $\mathcal{F}^\bullet \in D^+(\mathfrak{X})$ of Abelian sheaves on $\mathfrak{X}$, the homology groups of the complex $\mathbf{R}\Gamma(\mathfrak{X}, \mathcal{F}^\bullet)$ are denoted by

$$\mathbb{H}^i(\mathfrak{X}, \mathcal{F}^\bullet) = h^i(\mathbf{R}\Gamma(\mathfrak{X}, \mathcal{F}^\bullet))$$

and called the *hypercohomology groups* of $\mathfrak{X}$ with values in $\mathcal{F}^\bullet$.

The following is a straight-forward generalization of a classical result in sheaf theory:

PROPOSITION 1.83. *Let $\mathcal{F}^\bullet$ be a bounded below complex of Abelian sheaves on a differentiable stack $\mathfrak{X}$. Then there exists a convergent spectral sequence*

$$E_1^{p,q} = H^q(\mathfrak{X}, \mathcal{F}^p) \Longrightarrow \mathbb{H}^{p+q}(\mathfrak{X}, \mathcal{F}^\bullet).$$

PROOF. See Prop. 1.2.10., [**Bry2**]. □

COROLLARY 1.84. *Let $\mathcal{F}^\bullet$ be a bounded below complex of Abelian sheaves on a differentiable stack $\mathfrak{X}$. Assume that each sheaf $\mathcal{F}^p$ is* acyclic, *i.e. satisfies $H^q(\mathfrak{X}, \mathcal{F}^p) = 0$ for $q > 0$. Then there exists a canonical isomorphism between the hypercohomology groups $\mathbb{H}^n(\mathfrak{X}, \mathcal{F}^\bullet)$ and the cohomology of the complex*

$$\cdots \longrightarrow \Gamma(\mathfrak{X}, \mathcal{F}^p) \longrightarrow \Gamma(\mathfrak{X}, \mathcal{F}^{p+1}) \longrightarrow \cdots .$$

EXAMPLE 1.85 (de Rham cohomology of $\mathfrak{X}$). Let $\mathfrak{X}$ be a differentiable stack. We want to define the sheaf $\Omega^i_{\mathfrak{X}}$ of i:th differentiable forms on $\mathfrak{X}$. This is done as follows: for an object $x \in \mathrm{Ob}(\mathfrak{X})$ lying over $U \in \mathrm{Ob}(\mathfrak{S})$, set $\Omega^i_{\mathfrak{X}}(x) = \Omega^i(U)$, the $\mathbb{R}$-vector space of ($\mathbb{R}$-valued) differentiable i-forms on U. For a morphism $y \longrightarrow x$ lying over the C^∞-map $f : V \longrightarrow U$, define the restriction map $\Omega^i_{\mathfrak{X}}(x) \longrightarrow \Omega^i_{\mathfrak{X}}(y)$ to be the pullback map $f^* : \Omega^i(U) \longrightarrow \Omega^i(V)$. It follows from the definition that the sheaves $\Omega^i_{\mathfrak{X}}$ are sheaves of $\mathbb{R}$-vector spaces.

The sheaf $\Omega^0_{\mathfrak{X}}$ is called the *structure sheaf* of $\mathfrak{X}$ and is denoted by $\mathcal{O}_{\mathfrak{X}}$.

The exterior derivative $d : \Omega^i(U) \longrightarrow \Omega^{i+1}(U)$, where U is a manifold, commutes with the pullback of differentiable forms via any C^∞-map, i.e. with our restriction maps, and so d induces a homomorphism of sheaves $d : \Omega^i_{\mathfrak{X}} \longrightarrow \Omega^{i+1}_{\mathfrak{X}}$, for all $i \geq 0$. Clearly, $d^2 = 0$, and so we have a complex $\Omega^\bullet_{\mathfrak{X}}$ of $\mathbb{R}$-vector spaces over $\mathfrak{X}$. The complex $\Omega^\bullet_{\mathfrak{X}}$ is called the *de Rham* complex of $\mathfrak{X}$ and its hypercohomology is called the *de Rham* cohomology of $\mathfrak{X}$:

$$H^i_{DR}(\mathfrak{X}) = \mathbb{H}^i(\mathfrak{X}, \Omega^\bullet_{\mathfrak{X}}).$$

Now we need to recall two results concerning manifolds (possible infinite-dimensional).

LEMMA 1.86 (Poincare's lemma). *Let U be a convex open subset of a topological vector space E. Then the de Rham complex*

$$\cdots \xrightarrow{d} \Omega^p(U) \xrightarrow{d} \Omega^{p+1}(U) \xrightarrow{d} \cdots$$

has $H^p(\Omega^\bullet(U)) = 0$ for $p > 0$, and $H^0(\Omega^\bullet(U)) = \mathbb{R}$.

This has the following well-known consequence:

PROPOSITION 1.87 (Prop. 1.4.3., [**Bry2**]). *Let M be a smooth manifold modelled on a topological vector space E. Then the de Rham (sheaf) complex $\Omega^\bullet_M$ is a resolution of the constant sheaf $\mathbb{R}_M$.*

Next, let $\mathbb{R}_{\mathfrak{X}}$ (or just $\mathbb{R}$, for short) denote the sheaf over $\mathfrak{X}$ defined by

$$\mathbb{R}_{\mathfrak{X}}(x) = \{f : U \longrightarrow \mathbb{R} \mid f \text{ is locally constant}\}$$

for an object x of $\mathfrak{X}$ lying over $U \in \mathrm{Ob}(\mathfrak{S})$. The sheaf $\mathbb{R}_{\mathfrak{X}}$ is a subsheaf of $\Omega^0_{\mathfrak{X}}$, and moreover it follows from the definition and the previous proposition that the deRham complex $\Omega^\bullet_{\mathfrak{X}}$ is a resolution of $\mathbb{R}_{\mathfrak{X}}$. Hence we conclude

COROLLARY 1.88. *Let $\mathfrak{X}$ be a differentiable stack over $\mathfrak{S}$ such that the sheaves $\Omega^i_{\mathfrak{X}}$, $i \geq 0$, are all acyclic. Then*

$$H^i_{DR}(\mathfrak{X}) = H^i(\mathfrak{X}, \mathbb{R}),$$

for all $i \geq 0$.

PROOF. Follows directly from above and Corollary 1.84 applied to the sheaf complex $\Omega^\bullet_{\mathfrak{X}}$. □

11. Čech and simplicial cohomology of stacks

DEFINITION 1.89. Let Δ be the category whose objects are finite ordered sets $[n] = \{0 < 1 < \cdots < n\}$, and whose morphisms are nondecreasing monotone functions.

DEFINITION 1.90. Let $\mathcal{A}$ be a category. A *simplicial object* A in $\mathcal{A}$ is a contravariant functor $A : \Delta^{\text{op}} \longrightarrow \mathcal{A}$

DEFINITION 1.91. A morphism of simplicial objects is a natural transformation between the corresponding functors, and the category $\mathcal{SA}$ of all simplicial objects in $\mathcal{A}$ is just the functor category $\mathcal{A}^{\Delta^{\text{op}}}$.

PROPOSITION 1.92. *To give a simplicial object A in a category $\mathcal{A}$, it is necessary and sufficient to give a sequence of objects $A_0, A_1, A_2, \ldots$ together with* face operators $\partial_i : A_p \longrightarrow A_{p-1}$ *and* degeneracy operators $\sigma_i : A_p \longrightarrow A_{p+1}$, *where* $i = 0, 1, \ldots, p$, *satisfying the so called* simplicial identities*:*

$$\begin{aligned}
\partial_i \partial_j &= \partial_{j-1} \partial_i, && \text{if } i < j \\
\sigma_i \sigma_j &= \sigma_{j+1} \sigma_i, && \text{if } i \leq j \\
\partial_i \sigma_j &= \begin{cases} \sigma_{j-1} \partial_i, & \text{if } i < j \\ id, & \text{if } i = j \text{ or } i = j+1 \\ \sigma_j \partial_{i-1}, & \text{if } i > j+1. \end{cases}
\end{aligned}$$

PROOF. Omitted. See [**Weib**], Prop. 8.1.3. □

If one dualizes the concept of simplicial objects, one obtains cosimplicial objects and the following proposition:

PROPOSITION 1.93. *To give a cosimplicial object A in a category $\mathcal{A}$, it is necessary and sufficient to give a sequence of objects $A^0, A^1, \ldots$ together with* coface operators $\partial^i : A^{p-1} \longrightarrow A^p$ *and* codegeneracy operators $\sigma^i : A^{p+1} \longrightarrow A^p$, *where* $i = 0, 1, \ldots, p$, *which satisfy the* cosimplicial identities

$$\begin{aligned}
\partial^j \partial^i &= \partial^i \partial^{j-1}, && \text{if } i < j \\
\sigma^j \sigma^i &= \sigma^i \sigma^{j+1}, && \text{if } i \leq j \\
\sigma^j \partial^i &= \begin{cases} \partial^i \sigma^{j-1}, & \text{if } i < j \\ id, & \text{if } i = j \text{ or } i = j+1 \\ \partial^{i-1} \sigma^j, & \text{if } i > j+1. \end{cases}
\end{aligned}$$

PROOF. Omitted. See [**Weib**], Cor. 8.1.4. □

REMARK 1.94. It is clear by the above, that if we have a contravariant funtor $F : \mathcal{A} \longrightarrow \mathcal{B}$, then F maps simplicial objects in $\mathcal{A}$ to cosimplicial objects in $\mathcal{B}$. In the same way, a covariant functor F maps simplicial objects to simplicial objects, etc.

DEFINITION 1.95. Let A be a simplicial object in an *Abelian* category $\mathcal{A}$. The *associated*, or *unnormalized, chain complex* $C(A)$ has its objects $C_p = A_p$, and its boundary morphism $d : C_p \longrightarrow C_{p-1}$ is the alternating sum of the face operators $\partial_i : C_p \longrightarrow C_{p-1}$:

$$d = \partial_0 - \partial_1 + \cdots + (-1)^p \partial_p.$$

The simplicial identities for $\partial_i \partial_j$ imply that $d^2 = 0$, so that we indeed have a complex.

We now come back to our original situation and define for all $p \geq 0$

$$X_p = \underbrace{X \times_{\mathfrak{X}} \ldots \times_{\mathfrak{X}} X}_{p+1 \text{ times}}.$$

Since $X \longrightarrow \mathfrak{X}$ is a representable submersion, all X_p are manifolds. We want to make $X_\bullet = \{X_p\}$ into a simplicial manifold, i.e. a simplicial object in the category of manifolds:

$$\cdots \ X_2 \ X_1 \Longrightarrow X_0. \tag{1.4}$$

First, note that X_p corresponds to the space of chains of composable p arrows in the groupoid $X_1 \rightrightarrows X_0$. Define the face and degeneracy maps so that

$$\partial_i(g_1, \ldots, g_p) = \begin{cases} (g_2, \ldots, g_p), & \text{if } i = 0 \\ (g_1, \ldots, g_i g_{i+1}, \ldots, g_n), & \text{if } 0 < i < p \\ (g_1, \ldots, g_{p-1}), & \text{if } i = p, \end{cases}$$

$$\sigma_i(g_1, \ldots, g_p) = (g_1, \ldots, g_i, 1, g_{i+1}, \ldots, g_p).$$

Remark 1.96. The simplicial manifold constructed above is called the *nerve* of $\Gamma = X_1 \rightrightarrows X_0$ and can actually be defined for any Lie groupoid, as can be seen from the construction. This should be compared with the simplicial set BG in topology, whose geometric realization $|BG|$ is called the *classifying space* of G. Multiplication operation in a group is replaced in the groupoid case by composing arrows, which is a more general operation.

Example 1.97. We claim that for a quotient stack $[X/G]$ with the natural atlas $X \longrightarrow [X/G]$

$$X_p = \underbrace{X \times_{\mathfrak{X}} \ldots \times_{\mathfrak{X}} X}_{p+1 \text{ times}} \cong X \times \prod_{i=1}^{p} G.$$

This can be seen as follows. By definition $X_0 = X$ and the product on the right hand side is empty, thus the claim is true when $p = 0$. Next note that by [**Hein**] we have $X \times_{\mathfrak{X}} X \cong X \times G$. This implies that

$$\begin{aligned} X \times_{\mathfrak{X}} X \times_{\mathfrak{X}} X &\cong (X \times_{\mathfrak{X}} X) \times_X (X \times_{\mathfrak{X}} X) \cong (X \times G) \times_X (X \times G) \\ &\cong X \times G \times G. \end{aligned}$$

Here the last isomorphism follows since

$$(X \times G) \times_X (X \times G) = \Big\{ \Big((x_1, g_1), (x_2, g_2) \Big) \in (X \times G) \times (X \times G) \,\Big|\, x_1 = x_2 \Big\}.$$

More generally, one may write for $p > 2$

$$\begin{aligned} X_{p+1} &= \underbrace{X \times_{\mathfrak{X}} \ldots \times_{\mathfrak{X}} X}_{p+2 \text{ times}} \cong \underbrace{\Big(X \times_{\mathfrak{X}} \ldots \times_{\mathfrak{X}} X \Big)}_{p+1 \text{ times}} \times_X \Big(X \times_{\mathfrak{X}} X \Big) \\ &\cong X_p \times_X (X \times G) \cong X_p \times G \end{aligned}$$

and the claim follows from this by induction.

Now, let $\mathcal{F}$ be a sheaf of Abelian groups on $\mathfrak{X}$. Every X_p has $p+1$ canonical projections $X_p \longrightarrow \mathfrak{X}$, which are all canonically isomorphic to each other. We choose one of them and call it $\pi_p : X_p \longrightarrow \mathfrak{X}$. Recall that π_p as a map from a manifold to a stack can be identified with an object of $\mathfrak{X}$ lying over X_p. We denote the Abelian group $\mathcal{F}(\pi_p)$ associated to the object π_p by the contravariant sheaf functor $\mathcal{F}$ by $\mathcal{F}(X_p)$. By Remark 1.94 we have then a cosimplicial Abelian group

$$\mathcal{F}(X_0) \Longrightarrow \mathcal{F}(X_1) \ \mathcal{F}(X_2) \ \cdots. \tag{1.5}$$

Since the category of Abelian groups is an Abelian category, we may form the associated cochain complex to $\mathcal{F}(X_\bullet)$:

$$C(\mathcal{F}(X_\bullet)): \qquad \mathcal{F}(X_0) \xrightarrow{\partial} \mathcal{F}(X_1) \xrightarrow{\partial} \mathcal{F}(X_2) \xrightarrow{\partial} \cdots \tag{1.6}$$

Definition 1.98. The homology groups of the complex (1.6) are denoted by

$$\check{H}^i(X_\bullet, \mathcal{F}) = h^i(\mathcal{F}(X_\bullet))$$

and called the *Čech* cohomology groups of F with respect to the covering $X \longrightarrow \mathfrak{X}$.

As usual, there exists also a map $\check{H}^i(X_\bullet, \mathcal{F}) \longrightarrow H^i(\mathfrak{X}, \mathcal{F})$. Moreover, we have the following proposition

Proposition 1.99. *Let $\mathcal{F}$ be a Cartesian sheaf of Abelian groups on a differentiable stack $\mathfrak{X}$. Let $X \longrightarrow \mathfrak{X}$ be an atlas and $\mathcal{F}^\bullet$ the induced simplicial sheaf on the simplicial manifold $X_\bullet$. Then there is an E_1-spectral sequence:*

$$E_1^{p,q} = H^q(X_p, \mathcal{F}_p) \Longrightarrow H^{p+q}(\mathfrak{X}, \mathcal{F}).$$

Moreover,

$$H^i(\mathfrak{X}, \mathcal{F}) \cong H^i(X_\bullet, \mathcal{F}^\bullet)$$

for all $i \geq 0$, where the latter cohomology group is the simplicial cohomology of $\mathcal{F}^\bullet$ (see Appendix B).

Proof. See [**De**], [**Hein**]. □

Corollary 1.100. *Let $\mathfrak{X}$ be a differentiable stack with an atlas $X \longrightarrow \mathfrak{X}$. Then*

$$H^i(\mathfrak{X}, S^1) \cong H^i(X_\bullet, \underline{S}^1)$$

for all $i \geq 0$.

Example 1.101. Let again $\mathfrak{X} = [X/G]$ be the quotient stack and $\mathcal{F} = \underline{S}^1_{\mathfrak{X}}$. By Example 1.97 $X_p \cong X \times \prod_{i=1}^p G$. Hence for each $p \geq 0$ the induced small sheaves of $\underline{S}^1$ on X_p are the sheaves $\underline{S}^1_{,X\times G^p}$. It follows now from Corollary 1.100 and Appendix B that the cohomology groups $H^i([X/G], \underline{S}^1)$ are isomorphic to the G-equivariant cohomology groups of X. Especially, the group

$$H^2([X/G], \underline{S}^1) \cong H^2(X \times G^\bullet, S^1_{X\times G^\bullet})$$

classifies the isomorphism classes of G-equivariant gerbes on X.

Next, let $(\mathcal{F}^\bullet, d)$ be a complex of Abelian sheaves over $\mathfrak{X}$, bounded below. Let $X \longrightarrow \mathfrak{X}$ be an atlas. As we saw, for every i we get a Čech complex $\mathcal{F}^i(X_\bullet, \partial)$. Because d and ∂ commute, denoting by $\mathcal{F}^{i,p} = \mathcal{F}^i(X_p)$ (Abelian groups as sections of $\mathcal{F}^i$) we obtain a double complex $(\{\mathcal{F}^{i,p}\}_{i,p}, d, \partial)$,

$$\begin{array}{ccccccc}
 & & \uparrow & & \uparrow & & \\
\longrightarrow & & \mathcal{F}^i(X_p) & \xrightarrow{d} & \mathcal{F}^{i+1}(X_p) & \longrightarrow & \\
 & & \uparrow \partial & & \uparrow \partial & & \\
\longrightarrow & & \mathcal{F}^i(X_{p-1}) & \xrightarrow{d} & \mathcal{F}^{i+1}(X_{p-1}) & \longrightarrow & \\
 & & \uparrow & & \uparrow & &
\end{array} \tag{1.7}$$

One has the associated *total complex* $\mathrm{Tot}(\mathcal{F}^{\bullet\bullet})$, with $\mathrm{Tot}^n = \bigoplus_{i+p=n} \mathcal{F}^{i,p}$ and total differential $\delta = d + (-1)^i \partial$ in bidegree (i,p). The homology groups of the associated double complex are denoted by

$$\check{\mathbb{H}}^i(X_\bullet, \mathcal{F}^{\bullet\bullet}) = h^i\big(\mathrm{Tot}(\mathcal{F}^\bullet(X_\bullet))\big)$$

and called the *Čech hypercohomology* groups of $\mathcal{F}^\bullet$ with respect to the covering $X \longrightarrow \mathfrak{X}$. So a degree n cocycle c consists of a finite family $c_i \in \mathcal{F}^{i,n-i} = \mathcal{F}^i(X_{n-i})$ such that

$$d(c_i) = (-1)^i \partial_{i+1} \in \mathcal{F}^{i+1,n-i} = \mathcal{F}^{i+1}(X_{n-i}).$$

PROPOSITION 1.102. *Assume that for every i and every p the small sheaf $\mathcal{F}^i_p$ induced by $\mathcal{F}^i$ on X_p is asyclic. Then*

$$\check{\mathbb{H}}^i(X_\bullet, \mathcal{F}^{\bullet\bullet}) = \mathbb{H}^i(\mathfrak{X}, \mathcal{F}^\bullet).$$

COROLLARY 1.103. *Suppose $X \longrightarrow \mathfrak{X}$ is an atlas such that all manifolds X_p admit a smooth partition of unity. Then*

$$H^i_{DR}(\mathfrak{X}) = \check{\mathbb{H}}^i(X_\bullet, \Omega^\bullet_{\mathfrak{X}}) = h^i\big(\mathit{Tot}(\Omega^\bullet(X_\bullet))\big)$$

PROOF. By standard arguments in sheaf theory a smooth partition of unity on X_p gives a smooth partition of unity for the sheaves $\Omega^i_{X_p}$ for all $i \geq 0$ forcing them to be asyclic. □

EXAMPLE 1.104. Let G be an I.L.H. Lie group acting from the right on an I.L.H. manifold X. Suppose that X and G satisfy (A.1). This implies that X and G admit a smooth partition of unity. Thus all the products $X \times \prod_{i=1}^p G$ admit a smooth partition of unity. Consider the standard atlas $X \longrightarrow [X/G]$. We saw in Example 1.97 that $X_p = X \times \prod_{i=1}^p G$ so that

$$H^i_{DR}([X/G]) \cong \check{\mathbb{H}}^i(X \times G^\bullet, \Omega^\bullet_{X \times G^\bullet}).$$

REMARK 1.105. One can use a similar construction to define (Lie groupoid) cohomology groups $H^k(\Gamma^\bullet, \mathcal{F})$ associated to a Lie groupoid $\Gamma = X_1 \rightrightarrows X_0$ and an Abelian sheaf $\mathcal{F}$ on the category of differentiable manifolds, see [**L-GTuXu**].

CHAPTER 2

Basics on noncommutative differential geometry

Here we follow [**G-BV**] and [**Con**].

1. C^*-algebras and the Gel'fand-Neimark theorem

Let $\mathscr{A}$ be an associative not necessarily commutative $\mathbb{C}$-algebra with a unit.

The algebra $\mathscr{A}$ is called a **-algebra* if it admits an (antilinear) involution $^* : \mathscr{A} \longrightarrow \mathscr{A}$ such that

$$\begin{aligned} a^{**} &= a \\ (ab)^* &= b^* a^* \\ (\alpha a + \beta b)^* &= \overline{\alpha} a^* + \overline{\beta} b^*, \end{aligned} \tag{2.1}$$

for all $a, b \in \mathscr{A}$ and $\alpha, \beta \in \mathbb{C}$. Here the bar denotes the usual complex conjugation in $\mathbb{C}$.

A *normed algebra* $\mathscr{A}$ is an algebra with a norm $\|\cdot\| : \mathscr{A} \longrightarrow \mathbb{R}$ satisfying

$$\begin{aligned} \|a\| &\geq 0, \qquad \|a\| = 0 \iff a = 0, \\ \|\alpha a\| &= |\alpha| \, \|a\| , \\ \|a + b\| &\leq \|a\| + \|b\| , \\ \|ab\| &\leq \|a\| \, \|b\| , \end{aligned} \tag{2.2}$$

for any $a, b \in \mathscr{A}$ and $\alpha \in \mathbb{C}$. The topology defined by the norm is called is called the *norm* or *uniform* topology.

A *Banach algebra* is a norm algebra, which is complete in the norm topology.

A *Banach *-algebra* is a normed *-algebra, which is complete and satisfies

$$\|a^*\| = \|a\| ,$$

for all $a \in \mathscr{A}$.

A C^*-algebra is a Banach algebra such that

$$\|a^* a\| = \|a\|^2 ,$$

for all $a \in \mathscr{A}$.

Example 2.1. Let M be a compact Hausdorff topological space. The commutative algebra $\mathrm{C}(M)$ of continuous functions on M is a C^*-algebra. The involution * is given by complex conjugation and the norm is the supremum norm,

$$\|f\|_\infty = \sup_{x \in M} |f(x)|.$$

Example 2.2. The noncommutative algebra $\mathcal{B}(\mathcal{H})$ of bounded linear operators on an infinite dimensional Hilbert space $\mathcal{H}$ is a C^*-algebra. The involution * is given by the adjoint and the norm is the operator norm,

$$\|B\| = \sup\{\|B\chi\| \,|\chi \in \mathcal{H}, \, \|\chi\| \leq 1\}.$$

Definition 2.3. Let $\mathscr{A}$ and $\mathscr{B}$ be two C^*-algebras. A *-morphism is a $\mathbb{C}$-linear map $\pi : \mathscr{A} \longrightarrow \mathscr{B}$ satisfying

$$\begin{aligned} \pi(ab) &= \pi(a)\pi(b) \\ \pi(a^*) &= \pi(a)^*, \end{aligned}$$

for all $a, b \in \mathscr{A}$.

Definition 2.4. A representation of a C^*-algebra $\mathscr{A}$ is a pair $(\mathcal{H}, \pi)$, where $\mathcal{H}$ is a Hilbert space, and π is a *-morphism

$$\pi : \mathscr{A} \longrightarrow \mathcal{B}(\mathcal{H}).$$

Definition 2.5. Two representations $(\mathcal{H}_1, \pi_1)$ and $(\mathcal{H}_2, \pi_2)$ are said to be (unitary) *equivalent* if there exists a unitary operator $U : \mathcal{H}_1 \longrightarrow \mathcal{H}_2$ such that

$$\pi_1(a) = U^*\pi_2(a)U,$$

for every $a \in \mathscr{A}$.

Let now $\mathscr{C}$ be a commutative C^*-algebra with a unit. The space of all equivalence classes of all irreducible representations of $\mathscr{C}$ is called the *structure space* of $\mathscr{C}$ and is denoted by $\hat{\mathscr{C}}$. The trivial representation $\mathscr{C} \longrightarrow \{0\}$ is not included. Since $\mathscr{C}$ is commutative, every irreducible representation is one-dimensional. It follows that $\hat{\mathscr{C}}$ is the space of characters of $\mathscr{C}$, i.e. the space of *-linear functionals $\phi : \mathscr{C} \longrightarrow \mathbb{C}$, which are *multiplicative* in the sense that $\phi(ab) = \phi(a)\phi(b)$ for all $a, b \in \mathscr{C}$. Clearly, $\phi(1_{\mathscr{C}}) = 1$ for all $\phi \in \hat{\mathscr{C}}$.

The space $\hat{\mathscr{C}}$ can be made a topological space, called the *Gel'fand space* of $\mathscr{C}$ by endowing it with the *Gel'fand topology*. This is the topology defined by pointwise convergence on $\mathscr{C}$. Hence a sequence $\{\phi_\lambda\}_{\lambda\in\Lambda}$ of elements of $\hat{\mathscr{C}}$, where Λ is any directed set, converges to $\phi \in \hat{\mathscr{C}}$ if and only if for any $c \in \mathscr{C}$ the sequence $\{\phi_\lambda(c)\}_{\lambda\in\Lambda}$ converges to $\phi(c)$ in the topology of $\mathbb{C}$. One can show that, since $\mathscr{C}$ has a unit, $\hat{\mathscr{C}}$ is a compact Hausdorff space.

Definition 2.6. Let $\mathscr{C}$ be a commutative C^*-algebra with a unit. If $c \in \mathscr{C}$, its *Gel'fand transformation* $\hat{c}$ is the complex-valued function on the topological space $\hat{\mathscr{C}}$, $\hat{c} : \hat{\mathscr{C}} \longrightarrow \mathbb{C}$, given by

$$\hat{c}(\phi) = \phi(c),$$

for all $\phi \in \hat{\mathscr{C}}$.

It is clear from the definition of Gelfand topology, that the Gelfand transformation is continuous. Hence $\hat{c} \in \mathrm{C}(\hat{\mathscr{C}})$ for every $c \in \mathscr{C}$.

The Gel'fand-Neimark theorem now states, that all commutative C^*-algebras arise from algebras of continuous functions on some compact Hausdorff topological spaces:

Theorem 2.7 (Gel'fand-Neimark). *Let $\mathscr{C}$ be a commutative C^*-algebra with a unit. Then the Gel'fand transformation $c \mapsto \hat{c}$ is an isometric $*$-isomorphism of $\mathscr{C}$ onto $C(\hat{\mathscr{C}})$. Here isometric means that*

$$\|\hat{c}\|_\infty = \|c\|,$$

for all $c \in \mathscr{C}$, with $\|\cdot\|_\infty$ being the supremum norm as in Example 2.1.

2. Noncommutative vector bundles

Let M be a compact finite dimensional manifold. We shall consider a (real or complex) finite dimensional vector bundle $E \longrightarrow M$. We denote by $\mathcal{E} := \Gamma(E, M)$ the smooth sections of E. It has a natural structure of a right $\mathrm{C}^\infty(M)$-module defined in an obvious way. The $\mathrm{C}^\infty(M)$-module $\mathcal{E}$ proves out to be finite projective, and moreover the converse holds:

THEOREM 2.8 (Serre-Swan). *A $C^\infty(M)$-module $\mathscr{P}$ is isomorphic to a module of the form $\Gamma(E, M)$ iff it is finite projective.*

This motivates the following generalization:

DEFINITION 2.9. Let $\mathscr{A}$ be a *-algebra. A *vector bundle* over $\mathscr{A}$ is a finite projective $\mathscr{A}$-module $\mathcal{E}$.

REMARK 2.10. Recall, that $\mathcal{E}$ is a finite projective module over $\mathscr{A}$ iff there exists $m \in \mathbb{N}$ and $p \in \mathscr{A}^{m\times m}$ such that $p^2 = p$ and $\mathcal{E} = p\mathscr{A}^m$. If one can choose $p = \mathrm{id}_{\mathscr{A}^{m\times m}}$, then $\mathcal{E}$ is a trivial vector bundle. The elements of $\mathrm{End}_{\mathscr{A}}(\mathcal{E})$ are matrices $v \in \mathscr{A}^{m\times m}$, that satisfy $pv = vp$.

Next, we want to define a noncommutative analog of a Hermitean vector bundle on a manifold. So, suppose that the vector bundle $E \longrightarrow M$ is also endowed with a Hermitean structure. The Hermitean product $(\cdot,\cdot)_m$ on each fibre E_m gives a $\mathrm{C}^\infty(M)$-sesquilinear map on the module $\mathcal{E} = \Gamma(E, M)$ of smooth sections of E,

$$\begin{aligned} (\cdot,\cdot) \quad &: \quad \mathcal{E}\times\mathcal{E} \longrightarrow \mathrm{C}^\infty(M), \\ (\eta_1,\eta_2)(m) \quad &:= \quad (\eta_1(m),\eta_2(m))_m, \end{aligned} \tag{2.3}$$

for all $\eta_1, \eta_2 \in \Gamma(E, M)$. The map (2.3) satisfies

$$\begin{aligned} (\eta_1 a, \eta_2 b) &= a^*(\eta_1,\eta_2)b \\ (\eta_1,\eta_2)^* &= (\eta_2,\eta_1) \\ (\eta,\eta) &\geq 0, \qquad (\eta,\eta) = 0 \iff \eta = 0, \end{aligned} \tag{2.4}$$

for any $\eta_1, \eta_2, \eta \in \mathcal{E}$ and $a, b \in \mathrm{C}^\infty(M)$. It is then natural to give the following generalization:

DEFINITION 2.11. Let $\mathscr{A}$ be a *-algebra and $\mathcal{E}$ a vector bundle over $\mathscr{A}$. Then $\mathcal{E}$ is called a *Hermitean* vector bundle if there exists a sesquilinear map $\mathcal{E}\times\mathcal{E} \longrightarrow \mathscr{A}$ satisfying:

$$\begin{aligned} (\eta_1 a, \eta_2 b) &= a^*(\eta_1,\eta_2)b \\ (\eta_1,\eta_2)^* &= (\eta_2,\eta_1) \\ (\eta,\eta) \text{ is positive for all } \eta\in\mathcal{E}, \qquad & (\eta,\eta) = 0 \iff \eta = 0, \end{aligned} \tag{2.5}$$

for any $\eta_1, \eta_2, \eta \in \mathcal{E}$ and $a, b \in \mathscr{A}$.

Here we need to recall that an element $a \in \mathscr{A}$ is positive, if it can be written in the form $a = b^*b$ for some $b \in \mathscr{A}$.

DEFINITION 2.12. Let $\mathcal{E}$ be a Hermitean vector bundle over a *-algebra $\mathscr{A}$. Then the group $\mathcal{U}(\mathcal{E})$ of *gauge transformations* of $\mathcal{E}$ is given by

$$\mathcal{U}(\mathcal{E}) := \{u \in \mathrm{End}_{\mathscr{A}}(\mathcal{E}) \mid uu^* = u^*u = 1\}.$$

3. Noncommutative differential forms

We first assume until further notice, that A is a commutative k-algebra. Kähler differentials for A are formal A-linear combinations of the symbols db, where $d \in A$ and $d(ab) = a\,db + b\,da$. More formally:

Definition 2.13. Set $\Omega^1_{\text{com}}(A) := A \otimes A/(ab \otimes c - a \otimes bc + ac \otimes b)$. The elements of $\Omega^1_{\text{com}}(A)$ are called the (commutative) *Kähler differentials* of A (heuristically, an element $a \otimes b$ should be thought as a differential form $a\,db$).

Proposition 2.14. *There is a canonical isomorphism of A-modules*

$$\Omega^1_{com}(A) \cong I/I^2,$$

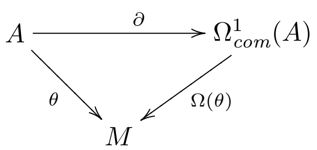
where I is the kernel of the multiplication map, i.e. $I := \ker[m : A \otimes A \longrightarrow A]$.

Commutative Kähler differentials can be characterized by their universal property, which we will now discuss.

Definition 2.15. Let A be a k-algebra and M an A-module. Then a map $\theta : A \longrightarrow M$ is a k-*derivation* if it satisfies (1) θ is additive, (2) $\theta(ab) = a\theta(b) + b\theta(a)$, and (3) $\theta(\alpha \cdot 1_A) = 0$ for all $\alpha \in k$.

Define $\partial : A \longrightarrow \Omega^1_{\text{com}}(A)$ by $\partial a = a \otimes 1_A - 1_A \otimes a$. Then ∂ is a derivation, which is often denoted symbolically by $a \mapsto da$.

Theorem 2.16. *Let A be a k algebra, M an A-module, and $\theta : A \longrightarrow M$ a derivation. Then the assigment $\Omega^1_{com}(\theta) : (a\,db) \longrightarrow a\theta(b)$ gives an A-module map $\Omega^1_{com}(\theta) : \Omega^1_{com}(A) \longrightarrow M$, which is uniquely defined by the requirement that the following diagram commutes:*



Thus, for an associative *commutative* k-algebra A and a left A-module M, we have seen that the functor $M \mapsto \text{Der}(A, M)$ is representable by the A-module $\Omega^1_{\text{com}}(A)$ of Kähler differentials.

If A is an associative, but not necessarily *commutative* k-algebra, the k-vector space $\text{Der}(A, M)$ is still defined provided M is an A-*bimodule*. It proves out, that the functor $M \mapsto \text{Der}(A, M)$, defined in the category of A-bimodules is again representable.

Definition 2.17. Let $m : A \otimes A \longrightarrow A$ denote the multiplication map viewed as a map of A-bimodules. Set $I = \ker[m : A \otimes A \longrightarrow A]$ and define $\Omega^1_{\text{nc}}(A) := I/I^2$. We call $\Omega^1_{\text{nc}}(A)$ the A-bimodule of *noncommutative* 1-*forms* on A.

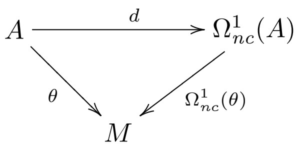
Proposition 2.18. *For every $M \in A - \text{bimod}$, there exists a canonical isomorphism*

$$Der(A, M) \cong Hom_{\text{A-bimod}}(\Omega^1_{nc}(A), M).$$

Thus, the functor $M \longrightarrow Der(A, M)$ is representable by the A-bimodule $\Omega^1_{nc}(A)$.

Next, notice that the map $d : A \longrightarrow \Omega^1_{\text{nc}}(A), a \mapsto da = a \otimes 1_A - 1_A \otimes a$ is a derivation. We have the following analog of Theorem 2.16:

Proposition 2.19. *Let A be an associative not nesessarily commutative k-algebra with unit and M an A-bimodule. For any derivation $\theta : A \longrightarrow M$, the* *assignment* $x\,dy \longrightarrow x\cdot\theta(y)$ *gives a well-defined* A*-bimodule map* $\Omega^1_{nc}(\theta)$ *that makes the following diagram commute:*



Definition 2.20. The algebra $\Omega^\bullet_{\mathrm{nc}}(A)$ of *noncommutative differential forms* on A is defined to be the tensor algebra (over A) of the bimodule $\Omega^1_{\mathrm{nc}}(A)$, i.e.

$$\Omega^\bullet_{\mathrm{nc}}(A) := T_A\Omega^1_{\mathrm{nc}}(A) = A \oplus \Omega^1_{\mathrm{nc}}(A) \oplus T^2_A\Omega^1_{\mathrm{nc}}(A) \oplus T^3_A\Omega^1_{\mathrm{nc}}(A) \oplus \cdots .$$

Definition 2.21. Let $\mathcal{E}$ be a Hermitean vector bundle over a *-algebra $\mathscr{A}$. A (universal) *connection* on $\mathcal{E}$ is a linear map $\nabla : \mathcal{E} \longrightarrow \mathcal{E} \otimes_{\mathscr{A}} \Omega^1_{\mathrm{nc}}(\mathscr{A})$ satisfying

$$\nabla(\eta a) = (\nabla\eta)a + \eta \otimes da,$$

for all $a \in \mathscr{A}$ and $\eta \in \mathcal{E}$. The connection ∇ is said to be *compatible* with the Hermitean structure if

$$d(\eta, \eta') = (\nabla\eta, \eta') + (\eta, \nabla\eta'),$$

for all $\eta, \eta' \in \mathcal{E}$.

One may extend ∇ to a derivation of $\mathcal{E}$-valued forms,

$$\nabla : \mathcal{E} \otimes_{\mathscr{A}} \Omega^\bullet_{\mathrm{nc}}(\mathscr{A}) \longrightarrow \mathcal{E} \otimes_{\mathscr{A}} \Omega^{\bullet+1}_{\mathrm{nc}}(\mathscr{A}),$$

by requiring

$$\nabla(\eta\omega) = (\nabla\eta) \otimes \omega + \eta \otimes d\omega,$$

for all $\eta \in \mathcal{E}$ and $\omega \in \Omega^\bullet_{\mathrm{nc}}(\mathscr{A})$. The *curvature* $\theta^\nabla \in \Omega^2_{\mathrm{nc}}(\mathscr{A}) \otimes_{\mathscr{A}} \mathrm{End}(\mathcal{E})$ of ∇ is then defined via the square $\theta^\nabla = \nabla^2 = \nabla \circ \nabla$. More precisely, one sets

$$\nabla^2 x = \theta^\nabla \cdot x,$$

for all $x \in \Omega^\bullet_{\mathrm{nc}}(\mathscr{A}) \otimes_{\mathscr{A}} \otimes\mathcal{E}$. Here the product $\cdot$ is defined so that for every $T \in \mathrm{End}(\mathcal{E})$, $\alpha \in \Omega^2_{\mathrm{nc}}(\mathscr{A})$, $\beta \in \Omega^p_{\mathrm{nc}}(\mathscr{A})$ and $\eta \in \mathcal{E}$

$$(\alpha \otimes T) \cdot (\beta \otimes \eta) := \alpha \wedge \beta \otimes T(\eta).$$

The gauge group $\mathcal{U}(\mathcal{E})$ operates naturally on the space of compatible connections:

$$\gamma_u(\nabla) := u\nabla u^* : \eta \mapsto u\nabla(u^*\eta),$$

so that $\gamma_u(\nabla)$ has curvature $u\theta u^*$.

4. Ideals of operators and Dixmier traces

4.1. Weak-$\mathcal{L}^p$ spaces.

Theorem 2.22. *Suppose* $\mathcal{H}$ *is a separable infinite dimensional Hilbert space and* $A \in \mathcal{K}$, *where* $\mathcal{K} \subseteq \mathcal{B}(\mathcal{H})$ *consists of all compact bounded operators in* $\mathcal{H}$. *Then* A *has a uniformly convergent expansion*

$$A = \sum_{j\geq 0} s_j(A)|\psi_j\rangle\langle\phi_j|,$$

where each $s_j(A) > 0$ *with* $s_0(A) \geq s_1(A) \geq \ldots$ *and* $\{\psi_j\}$, $\{\phi_j\}$ *are orthonormal sets.*

The $s_j(A)$ are called the *singular values* of A. Also of importance are their partial sums

$$\sigma_n(A) := \sum_{j=0}^{n-1} s_j(A).$$

We consider a special two-parameter family of ideals in $\mathcal{B}(\mathcal{H})$ introduced by Connes [**Con**], denoted $\mathcal{L}^{p,q}(\mathcal{H})$ or simply $\mathcal{L}^{p,q}$.

DEFINITION 2.23. Suppose $1 < p < \infty$ and $1 \leq q \leq \infty$. Then for $q < \infty$ an operator $A \in \mathcal{K}$ belongs to $\mathcal{L}^{p,q}$ if

$$\sum_{n=1}^{\infty} \frac{\sigma_n(A)^q}{n^{1+q/p'}} < \infty,$$

where $p' := p/(p-1)$. For $q = \infty$ an operator $A \in \mathcal{K}$ belongs to $\mathcal{L}^{p,\infty}$ if the sequence $n^{-1/p'}\sigma_n(A)$ is bounded — which is equivalent to $s_n(A) = O(n^{-1/p})$.

PROPOSITION 2.24. *Each* $\mathcal{L}^{p,q}$ $(1 < p < \infty,\ 1 \leq q \leq \infty)$ *is a two-sided ideal in* $\mathcal{B}(\mathcal{H})$ *and the (strict) inclusion*

$$\mathcal{L}^{p_1,q_1} \subseteq \mathcal{L}^{p_2,q_2}$$

holds if $p_1 < p_2$ *or if* $p_1 = p_2$ *and* $q_1 \leq q_2$. *Moreover, the ideal* $\mathcal{L}^{p,p}$ *is the standard Schatten class* $\mathcal{L}^p$.

DEFINITION 2.25. The ideals $\mathcal{L}^{p,\infty} =: \mathcal{L}^{p+}$ $(p > 1)$ are called *weak-*$\mathcal{L}^p$ spaces. These have a natural norm defined by

$$\|A\|_{p+} := \sup_n \frac{\sigma_n(A)}{n^{(p-1)/p}}.$$

DEFINITION 2.26. Set $\mathcal{L}^{\infty,\infty} := \mathcal{L}^\infty$ and $\mathcal{L}^{1,1} := \mathcal{L}^1$. The space $\mathcal{L}^{1+}$ is called the *Dixmier ideal* and is defined to consist of operators $A \in \mathcal{K}$ such that

$$\|A\|_{1+} := \sup_n \frac{\sigma_n(A)}{\log n} < \infty.$$

4.2. Dixmier traces. Let A be a *positive* operator, $A \in \mathcal{L}^{1+}(\mathcal{H})$; one would like to define a positive functional Tr^+ by

$$\mathrm{Tr}^+(A) = \lim_{n\to\infty} \frac{\sigma_n(A)}{\log n}.$$

There are two problems with the above formula: its linearity and its convergence.

To handle linearity, for $A_i > 0$ and $A_i \in \mathcal{L}^{1+}$, one has to compare

$$\frac{1}{\log n}\sum_{j=0}^{n-1} s_j(A_1 + A_2) =: \gamma_n$$

with

$$\frac{1}{\log n}\sum_{j=0}^{n-1} s_j(A_1) + \frac{1}{\log n}\sum_{j=0}^{n-1} s_j(A_2) =: \alpha_n + \beta_n.$$

One can prove the following inequalities

$$\alpha_n + \beta_n \leq \left(\frac{\log 2n}{\log n}\right)\gamma_n, \quad \gamma_n \leq \alpha_n + \beta_n.$$

Since $\frac{\log 2n}{\log n} \to 1$ as $n \to \infty$, we see that linearity would follow if we had convergence. Now, it follows from the assumption $A_i \in \mathcal{L}^{1+}$ that the sequences α_n, β_n and γ_n are *bounded* and thus even wihout convergence, we get a unitarily invariant positive trace on (the positive cone of) $\mathcal{L}^{1+}$ for each linear form $\lim_\omega = \ell$ on the space $\ell^\infty(\mathbb{N})$ of bounded sequences that satisfies

(1) $\lim_\omega(\alpha_n) \geq 0$, if $\alpha_n \geq 0$;
(2) $\lim_\omega(\alpha_n) = \lim(\alpha_n)$ if α_n is convergent;
(3) $\lim_\omega(\alpha_1, \alpha_1, \alpha_2, \alpha_2, \alpha_3, \alpha_3, \ldots) = \lim_\omega(\alpha_n)$.

To produce such ℓ one proceeds as follows. To any bounded sequence $(\alpha_n)_{n\in\mathbb{N}}$ we assign the bounded function $f_\alpha(\lambda)$ given by

$$f_\alpha(\lambda) = \alpha_n \quad \text{for } \lambda \in]n-1, n[.$$

Next, we replace f by its Cèsaro mean with respect to the multiplicative group $\mathbb{R}_+^*$ with Haar measure $d\lambda/\lambda$

$$M(f)(\lambda) := \frac{1}{\log \lambda} \int_1^\lambda f(u) \frac{du}{u}.$$

One can show (cf. **[Con]**) that if one chooses a positive linear functional L on the vector space $\mathrm{L}^\infty(\mathbb{R}_+^*)$ of bounded continuous functions on $\mathbb{R}_+^*$, such that $L(1) = 1$ and which is zero on the subspace $C_0(\mathbb{R}_+^*)$ of functions vanishing at ∞, then the assigment

$$(\alpha_n) \mapsto L(M(f_\alpha)) =: \ell((\alpha_n))$$

satisfies all the conditions (1)–(3) above.

Thus, we choose one such L and define

$$\lim{}_\omega(\alpha_n) := L(M(f_\alpha)).$$

Definition 2.27. For $A \geq 0, A \in \mathcal{L}^{1+}(\mathcal{H})$, we set

$$\mathrm{Tr}^+(A) := \lim_\omega \frac{1}{\log n} \sum_{j=0}^{n-1} s_j(A).$$

Since Tr^+ is additive:

$$\mathrm{Tr}^+(A_1 + A_2) = \mathrm{Tr}^+(A_1) + \mathrm{Tr}^+(A_2), \quad \text{for all } A_i \geq 0,\ A_i \in \mathcal{L}^{1+}$$

Tr^+ extends uniquely by linearity to the whole ideal $\mathcal{L}^{1+}$.

5. Connes' K-cycles

Definition 2.28. A K-cycle $(\mathcal{H}, D)$ on the *-algebra $\mathscr{A}$ consists of a unitary representation $\pi : \mathscr{A} \longrightarrow U(\mathcal{H})$ of $\mathscr{A}$ on a Hilbert space $\mathcal{H}$, together with an (unbounded) selfadjoint operator D on $\mathcal{H}$ with compact resolvent $(D-\lambda)^{-1}$, where $\lambda \notin \mathbb{R}$, such that $[D, \pi(a)] \in \mathcal{B}(\mathcal{H})$ for all $a \in \mathscr{A}$. In many cases, $\mathcal{H}$ is a $\mathbb{Z}_2$-graded Hilbert space, equipped with a grading operator $\Gamma : \mathcal{H} \longrightarrow \mathcal{H}$ such that

(1) $\Gamma^2 = 1$
(2) $\mathscr{A}$ acts on $\mathcal{H}$ by even operators, and D is on odd operator, i.e. $a\Gamma = \Gamma a$, and $D\Gamma = -\Gamma D$.

Example 2.29. Let (M, g^M) be a compact smooth n-dimensional spin manifold. Take $\mathscr{A} = \mathrm{C}^\infty(M)$ and let $\mathcal{H} := \mathrm{L}^2(M, S)$, the space of square integrable sections of the irreducible spinor bundle S, and $D = \not{D}$, the standard Dirac operator. The scalar product on $\mathcal{H}$ is the usual one induced by the metric g^M,

$$(\psi, \phi) := \int_M d\mu(g) \overline{\psi(x)} \phi(x),$$

and $\mathscr{A}$ acts on $\mathcal{H}$ as multiplication operators, i.e.

$$(f \cdot \psi)(x) := f(x)\phi(x),$$

for all $f \in \mathscr{A}$, and $\psi \in \mathcal{H}$. One can show that the operator $[\not{D}, a]$ is densely defined and that $[\not{D}, a] \in \mathcal{B}(\mathcal{H})$ for all $a \in \mathscr{A}$. Hence $(\mathrm{L}^2(M, S), \not{D})$ is a K-cycle over the *algebra $\mathrm{C}^\infty(M)$, called the *Dirac K-cycle*.

The Dirac K-cycle actually encodes the Riemannian geometry of M:

PROPOSITION 2.30. *The geodesic distance between two points p, q of M is given by*

$$d(p,q) = \sup\{|a(p) - a(q)| | a \in \mathscr{A},\ \|[\not{D}, a]\| \leq 1\}.$$

Moreover, the metric can be recovered from the Dirac cocycle,

THEOREM 2.31. *For all $a \in C^\infty(M)$*

$$C_n \int_M a(x)\mu(dx) = \mathrm{Tr}^+(a|\not{D}|^{-n}),$$

where

$$\mu(dx) = \sqrt{\det g}\, dx^1 \wedge \cdots \wedge dx^n, \qquad n = \dim M,$$

is the canonical volume measure of the Riemannian manifold M, Tr^+ is the Dixmier trace *and the constant C_n is defined so that, $C_{2k} = (2\pi)^{-k}/k!$ and $C_{2k+1} = (2\pi)^{-k-1}/(2k+1)!$.*

DEFINITION 2.32. A K-cycle $(\mathcal{H}, D)$ is called n^+ summable if $|D|^{-1} \in \mathcal{L}^{n+}(\mathcal{H})$.

Equivalently, a K-cycle is n^+ summable if $|D|^{-n}$ belongs to the Dixmier ideal $\mathcal{L}^{1+}(\mathcal{H}) \subseteq \mathcal{B}(\mathcal{H})$.

PROPOSITION 2.33. *Let $(\mathcal{H}, D)$ be a K-cycle over an algebra $\mathscr{A}$. Then the following equality defines a *-representation of $\Omega^\bullet_{\mathrm{nc}}(\mathscr{A})$ in $\mathcal{H}$:*

$$\pi(a_0\, da_1\, da_2 \ldots da_n) := i^n a_0 [D, a_1] \ldots [D, a_n].$$

DEFINITION 2.34. Let $(\mathcal{H}, D)$ be a K-cycle over an algebra $\mathscr{A}$. The graded differential algebra of *D-forms* on $\mathscr{A}$ is defined to be

$$\Omega^\bullet_D(\mathscr{A}) := \pi(\Omega^\bullet_{\mathrm{nc}}(\mathscr{A}))/J,$$

where $J := J_0 + dJ_0$ is a graded differential two-sided ideal of $\Omega^\bullet_{\mathrm{nc}}(\mathscr{A})$, where J_0 is the graded two-sided ideal of $\Omega^\bullet_{\mathrm{nc}}(\mathscr{A})$ given by $J_0^k := \{b \in \Omega^k_{\mathrm{nc}}(\mathscr{A}) | \pi(b) = 0\}$.

We denote by π_D the canonical projection $\pi_D : \Omega^\bullet_{\mathrm{nc}}(\mathscr{A}) \longrightarrow \Omega^\bullet_D(\mathscr{A})$.

DEFINITION 2.35. An n^+ summable K-cycle $(\mathcal{H}, D)$ on an algebra $\mathscr{A}$ is called *tame*, if

$$\mathrm{Tr}^+(ST|D|^{-n}) = \mathrm{Tr}^+(S|D|^{-n}T),$$

for any $T \in \pi(\Omega^\bullet_{\mathrm{nc}})$ and $S \in \mathcal{B}(\mathcal{H})$.

Thus, for a tame K-cycle, the following three traces coincide and define an inner product on $\pi(\Omega^\bullet_{\mathrm{nc}})$:

$$\langle S \mid T \rangle := \mathrm{Tr}^+(S^\dagger T|D|^{-n}) = \mathrm{Tr}^+(S^\dagger|D|^{-n}T) = \mathrm{Tr}^+(T|D|^{-n}S^\dagger).$$

It is then natural to form the Hilbert space completion of $\pi(\Omega^k_{\mathrm{nc}}(\mathscr{A}))$ with respect to this inner product, which we denote by $\widetilde{\mathcal{H}}_k$. Now, let P be the orthogonal projector on $\widetilde{\mathcal{H}}_k$, whose range is the orthogonal complement of $\pi(dJ_0^{k-1})$, and define $\mathcal{H}_k := P\widetilde{\mathcal{H}}_k$. It can be shown, that $\Omega^k_D(\mathscr{A})$ is a dense subspace of $\mathcal{H}_k$.

6. Noncommutative Yang-Mills action

Any skew form $\alpha \in \Omega^1_{\mathrm{nc}}(\mathscr{A})$, determines a (universal) connection $\nabla = d + \alpha$ on the trivial bundle $\mathcal{E} = \mathscr{A}$, whose curvature is $\theta := d\alpha + \alpha^2$. Let then $(\mathcal{H}, D)$ be an n^+ summable tame K-cycle. One then defines the *pre- Yang-Mills functional* to be

$$I(\nabla) := \mathrm{Tr}^+(\pi(\theta)^2 |D|^{-n})$$

It follows, that $I(\nabla) \geq 0$, since it is the square of the norm of $\pi(\theta)$ in $\widetilde{\mathcal{H}}_2$. Moreover, since the curvature of $\gamma_u(\theta)$ is $u\theta u^*$, one sees using tameness, that this action is also *gauge invariant*:

$$I(\gamma_u(\nabla)) = \mathrm{Tr}^+(\pi(\theta)^2 u^* |D|^{-n} u) = \mathrm{Tr}^+(\pi(\theta)^2 u^* u |D|^{-n}) = I(\nabla).$$

For the general action, see **[G-BV]**.

CHAPTER 3

Gerbes in Yang-Mills theory

1. Obstruction to canonical quantization of fermions in classical Yang-Mills theory

1.1. Dirac operators. Suppose that (M, g^M) is a compact oriented Riemannian manifold of dimension $d = 2n+1$ without boundary. Let FM be the $\mathrm{SO}(d)$ bundle over M consisting of oriented orthonormal frames in the tangent bundle TM. Let $C\ell(d) := C\ell(\mathbb{R}^d)$ be the *Clifford algebra* associated to the real Euclidean vector space $\mathbb{R}^d$.

By definition the *spin group* $\mathrm{Spin}(d) := \mathrm{Spin}(\mathbb{R}^d)$ is the group generated by elements in $C\ell_0(d)$ with norm 1. The *complexified* Clifford algebra $\mathbb{C}\ell(d)$ is defined as the tensor product $\mathbb{C}\ell(d) := C\ell(d) \otimes \mathbb{C}$. We have $\mathrm{Spin}(d) \subseteq C\ell(d) \subseteq \mathbb{C}\ell(d)$. Moreover, it is known that for odd d all irreducible complex representations $\mathbb{C}\ell(d) \longrightarrow \mathrm{End}_{\mathbb{C}}(V_S)$ restrict to a unique irreducible representation $\rho_{spin} : \mathrm{Spin}(d) \longrightarrow \mathrm{Aut}_{\mathbb{C}}(V_S)$, [**Pay**].

We shall assume that M has a *spin structure*, i.e. there exists a principal $\mathrm{Spin}(d)$ bundle P_S over M and a covering map

$$\phi : P_S \longrightarrow FM, \qquad \phi(pg) = \phi(p)\pi_S(g),$$

where $\pi_S : \mathrm{Spin}(d) \longrightarrow \mathrm{SO}(d)$ is the double covering homomorphism, $p \in P_S$ and $g \in \mathrm{Spin}(d)$ are arbitrary. Let $S = P_S \times_{\rho_{spin}} V_S$ be the associated vector bundle over M. It is called the *spin bundle* of the spin manifold M.

Let G be a finite dimensional semi-simple compact Lie group and $\rho : G \longrightarrow \mathrm{Aut}_{\mathbb{C}}(V)$ a unitary complex representation of G with respect to an inner product $(\cdot,\cdot)_V$ on V, i.e. $(\rho(g)x, \rho(g)y) = (x,y)$ for all $g \in G$ and $x, y \in V$. Next suppose that $\pi : P \longrightarrow M$ is an arbitrary principal G bundle and form the associated vector bundle $E = P \times_\rho V$. One can show that since ρ is unitary the associated vector bundle E is a Hermitean vector bundle with Hermitean metric h^E.

Denote by $\mathcal{A}$ the space of $\mathfrak{g} = \mathrm{Lie}(G)$ valued connection 1-forms on P and by $\mathcal{G}_e$ the based gauge transformation group (see Appendix A). It is known that $\mathcal{A}/\mathcal{G}_e$ is a smooth infinite dimensional Fréchet manifold, [**Pay**]. To each $A \in \mathcal{A}$ one can associate a Hermitean connection

$$\nabla'_A : \Gamma(E) \longrightarrow \Gamma(E \otimes T^*M)$$

on (E, h^E), i.e. a connection satisfying

$$d\,h^E(\xi,\eta) = h^E(\nabla'_A\xi, \eta) + h^E(\xi, \nabla'_A\eta)$$

for all $\xi, \eta \in \Gamma(E)$. On the other hand, since $\mathrm{Spin}(d)$ is a finite covering of $\mathrm{SO}(d)$ the Levi-Civita connection ∇ on FM lifts to a connection on the spinor bundle S. This yields a Clifford connection

$$\nabla_A := \nabla \otimes 1 + 1 \otimes \nabla'_A$$

on $\mathcal{E} := S \otimes E$. One may now define the Dirac operator $\not{D}_A : \Gamma(\mathcal{E}) \longrightarrow \Gamma(\mathcal{E})$ as the composition

$$\Gamma(\mathcal{E}) \xrightarrow{\nabla_A} \Gamma(\mathcal{E} \otimes T^*M) \xrightarrow{\sim} \Gamma(\mathcal{E} \otimes TM) \xrightarrow{c} \Gamma(\mathcal{E}),$$

where c is the Clifford multiplication. This extends to an operator on $\mathcal{H} = \mathrm{L}^2(\mathcal{E})$, the Hilbert space of square integrable sections of the vector bundle $\mathcal{E}$. The domain of $\not{D}_A$ in $\mathcal{H}$ is known to be $H^1(M;S)$, the first Sobolev space, [**Boss**]. More over, the leading symbol σ_L of $\not{D}_A^2$ satisfies

$$\sigma_L(\not{D}_A^2)(x,\xi) = |\xi|^2 \cdot \mathrm{id} \quad \text{for all } x \in M,\ \xi \in T_x^*M.$$

This is invertible for every $\xi \in T_x^*M \setminus \{0\}$, hence $\not{D}_A^2$ is an *elliptic* operator by definition. Since generally

$$\sigma_L(AB) = \sigma_L(A)\sigma_L(B)$$

for two differential operators A and B, one concludes that $\not{D}_A$ is elliptic, also. Finally, one knows from functional analysis that $\not{D}_A$ is a *Fredholm* operator since it is elliptic and the manifold M is compact. Thus $\dim\ker \not{D}_A < \infty$ and $\dim\operatorname{coker} \not{D}_A < \infty$. Moreover, the gauge transformation group $\mathcal{G}$ acts on $\mathcal{H}$ and the Dirac operator $\not{D}_A$ satisfies the following equivariance condition

$$g\not{D}_A g^{-1} = \not{D}_{A^g}$$

for all $g \in \mathcal{G}$.

1.2. Fock bundle. For each $A \in \mathcal{A}$ s.t. $0 \notin \operatorname{spec}(\not{D}_A)$ the operator $\not{D}_A$ produces a decomposition

$$\mathcal{H} = \mathcal{H}_+(A) \oplus \mathcal{H}_-(A),$$

where the spaces $\mathcal{H}_\pm$ are the corresponding eigenspaces to the positive and negative eigenvalues of the Dirac operator $\not{D}_A$, respectively. Corresponding to this decomposition there exists an irreducible Dirac representation of the representation of the algebra $\mathrm{CAR}(\mathcal{H}) =: \mathbb{C}\ell(\mathcal{H}\oplus\bar{\mathcal{H}})$ (the algebra of *canonical anticommutation relations* or the algebra of *fermion fields*) on the *Fock space*

$$\begin{aligned}\mathcal{F}_A \quad &:= \quad \bigwedge\Big(\mathcal{H}_+(A) \oplus \bar{\mathcal{H}}_-(A)\Big) = \bigwedge \mathcal{H}_+(A) \otimes \bigwedge \bar{\mathcal{H}}_-(A)\\ &= \quad \bigoplus_{p,q}\Big(\bigwedge^p \mathcal{H}_+(A) \otimes \bigwedge^q \bar{\mathcal{H}}_-(A)\Big),\end{aligned}$$

where physically the subspace $\bigwedge^p \mathcal{H}_+(A) \otimes \bigwedge^q \bar{\mathcal{H}}_-(A)$ consists of the states with p particles and q antiparticles, all of positive energy. [1] A CAR-representation $\psi_A : \mathrm{CAR} \longrightarrow \mathrm{End}(\mathcal{F}_A)$ is determined by giving a *vacuum* vector $|0_A\rangle \in \mathcal{F}_A$ characterized by the property that

$$\psi_A^*(u)|0_A\rangle = 0 = \psi_A(v)|0_A\rangle, \quad \text{for all } u \in \mathcal{H}_-(A),\ v \in \mathcal{H}_+(A).$$

Two representations of the CAR-algebra are said to be equivalent if it is possible to represent them in the same Fock space, in a way that both vacuum vectors will be of finite norm.

THEOREM 3.1. *Two different polarizations $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_- = W_+ \oplus W_-$ define equivalent Dirac representations of the CAR-algebra if and only if the projections $\mathrm{pr}^-_{W_+} : W_+ \longrightarrow \mathcal{H}_-$ and $\mathrm{pr}^+_{W_-} : W_- \longrightarrow \mathcal{H}_+$ are Hilbert-Schmidt.*

THEOREM 3.2 (Shale-Stinespring). *Two Dirac representation of the CAR-algebra defined by a pair of polarizations $\mathcal{H}_+$ and $\mathcal{H}'_+$ are equivalent if and and only if there is $g \in \mathcal{U}_{res}(\mathcal{H})$ such that $\mathcal{H}'_+ = g \cdot \mathcal{H}_+$. In addition, in order that an element*

[1] Here $\bar{\mathcal{H}}_-$ denotes the abstract complex conjugate space to $\mathcal{H}_-$. It is a copy of $\mathcal{H}_-$ with the scalars acting in a conjugate way: $\lambda \cdot \bar{\xi} = (\lambda \cdot \xi)^-$; we don't suppose that there is a complex conjugation operation defined inside the Hilbert space $\mathcal{H}$.

$g \in \mathcal{U}(\mathcal{H})$ *is* implementable *in the Fock space, i.e. there is a unitary operator* $\hat{g} \in \mathcal{U}(\mathcal{F})$ *such that*

$$\hat{g}\psi^*(v)\hat{g}^{-1} = \psi^*(gv), \quad \textit{for all } v \in \mathcal{H},$$

and similarly for the $\psi(v)$*'s, one must have* $g \in \mathcal{U}_{res}(\mathcal{H})$*.*

Here $\mathcal{U}_{res}(\mathcal{H})$ is the group of unitary operators g in the polarized Hilbert space $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$ such that the off-diagonal blocks are Hilbert-Schmidt operators.

One would like to glue somehow the different CAR-algebra representations $\mathcal{F}_A$ into an infinite-dimensional Hilbert bundle $\mathcal{F}$ over $\mathcal{A}$ with a continuous section $s_{\mathcal{F}} : \mathcal{A} \longrightarrow \mathcal{F}$ such that $s_{\mathcal{F}}(A) = |0_A\rangle$ (a Dirac representation if fixed by a given vacuum vector so this way it is possible to define what we mean by a continuously varying family of CAR-representations). First, to construct a bundle of Fock spaces one can use the following trick: One replaces the operator $\not{D}_A$ with the operator $\not{D}_A - \lambda$, where $\lambda \in \mathbb{R}, \lambda \notin \operatorname{spec}(\not{D}_A)$. This way, one obtains a decomposition

$$\mathcal{H} = \mathcal{H}_+(A, \lambda) \oplus \mathcal{H}_-(A, \lambda),$$

with the corresponding (irreducible) Fock space representation

$$\rho_{A,\lambda} : \mathrm{CAR}(\mathcal{H}) \longrightarrow \mathrm{End}(\mathcal{F}_{A,\lambda})$$

of the CAR-algebra.

The Fock spaces $\mathcal{F}_{A,\lambda}$ depend on the choice of the *vacuum level* λ. However, for $\lambda, \mu \notin \operatorname{spec}(\not{D}_A)$ there exists a natural projective isomorphism

$$\mathcal{F}_{A,\lambda} \equiv \mathcal{F}_{A,\mu} \mod \mathbb{C}^\times, \tag{3.1}$$

allowing us to glue the different Fock spaces $\mathcal{F}_{A,\lambda}$ together into an infinite dimensional *projective* Fock bundle $\mathbb{P}\mathcal{F}$ over $\mathcal{A}$, [**Ara**]. One can show that since $\mathcal{A}$ is contractible as an affine space, there exists a trivial vector bundle $\mathcal{F} = \mathcal{A} \times \mathcal{F}_0$ over $\mathcal{A}$ whose projectivization is projectively isomorphic to $\mathbb{P}\mathcal{F}$.

Now the fibre of $\mathcal{F}$ at $A \in \mathcal{A}$ is equal to $\mathcal{F}_A \cong \mathcal{F}_0$ but unfortunately for the energy polarization $\mathcal{H} = \mathcal{H}_+(A) \oplus \mathcal{H}_-(A)$ the map $A \mapsto |0_A\rangle$ does *not* define a continuous section of $\mathcal{F}$ (or equivalently the map $\mathcal{A} \longrightarrow \mathrm{Gr}(\mathcal{H}) : A \mapsto \mathcal{H}_+(A)$ isn't continuous). This problem is resolved by intoducing another family $W(A)$ of polaritations $\mathcal{H} = W(A) \oplus W(A)^\perp$ parametrized by $A \in \mathcal{A}$ such that

(1) The map $\mathcal{A} \longrightarrow \mathrm{Gr}(\mathcal{H}) : A \mapsto W(A)$ *is* continuous;
(2) The corresponding CAR-algebra representations ρ_A and $\rho_{W(A)}$ induced by the two polarizations are *equivalent.*

To construct such a family of polarizations one proceeds as follows: Each $A \in \mathcal{A}$ defines a Grassmannian manifold $\mathcal{G}r_{res}(A)$ consisting of all closed subspaces $W \subseteq \mathcal{H}$ such that the difference $\mathrm{pr}_{\mathcal{H}_+(A)} - \mathrm{pr}_W \in \mathcal{L}(\mathcal{H})$ is a Hilbert-Schmidt operator. One can show that these spaces can be glued together to form a locally trivial fibre bundle over $\mathcal{A}$, called the *Grasmannian* bundle $\mathcal{G}r$. The question now is that does this bundle admit a global section $A \mapsto W(A)$? If it does the $W(A)$'s give us a family of polarizations with the required properties.

Luckily, the answer to our question is "*yes*". This is because $\mathcal{G}r$ happens to be of the form

$$\mathcal{G}r = P \times_{\mathcal{U}_{res}(\mathcal{H})} \mathrm{Gr}_{res}(\mathcal{H}),$$

where the fibre

$$P_A = \{g \in \mathcal{U}(\mathcal{H}) \mid g \cdot \mathcal{H}_+ \in \mathcal{G}r_A\}$$

and $\mathrm{Gr}_{res}(\mathcal{H})$ is the *restricted Grassmannian manifold* of Segal and Wilson (see Appendix A). Now

$$\mathrm{Gr}_{res}(\mathcal{H}) \cong \mathcal{U}_{res}(\mathcal{H})/(\mathcal{U}(\mathcal{H}_+) \times \mathcal{U}(\mathcal{H}_-))$$

and by a result of N. Kuiper the subgroup $\mathcal{U}(\mathcal{H}_+) \times \mathcal{U}(\mathcal{H}_-)$ is contractible and so $\mathcal{G}r$ has a global section if and only if P is trivial. This happens to be the case since $\mathcal{A}$ is contractible as an affine space.

1.3. Second quantizing gauge transformations. After a certain necessary renormalization process, introduced by Mickelsson in [**Mi3**], on operations on the one-particle Hilbert space $\mathcal{H}$ (e.g. the action of gauge transformation group) one would hope to lift the action of $\mathcal{G}$ on $\mathcal{A}$ to an action on $\mathcal{F}$ so that the diagram

$$\begin{array}{ccc} \mathcal{F} & \xrightarrow{\Gamma_A(g)} & \mathcal{F} \\ \downarrow & & \downarrow \\ \mathcal{A} & \xrightarrow{g} & \mathcal{A} \end{array}$$

commutes and

$$\Gamma_A(g)\hat{\not{D}}_A\Gamma_A^{-1}(g) = \hat{\not{D}}_{A^g},$$

where $\hat{\not{D}}_A$ is the second quantized Dirac operator. Unfortunately, there is an obstruction to this. To study this, it is useful to switch to the Lie algebra picture.

Definition 3.3. Second quantization of an infinitesimal gauge transformation is the map $d\Gamma_A : \mathscr{D}(A) \subseteq \mathrm{Lie}(\mathcal{G}) \longrightarrow \mathrm{End}(\mathcal{F}_A)$ characterized by

$$[d\Gamma_A(X), \psi_A^*(v)] = \psi_A^*(X \cdot v), \quad \text{for all } v \in \mathcal{H}, \tag{3.2}$$

$$\langle 0_A | d\Gamma_A(X) | 0_A \rangle = 0. \tag{3.3}$$

Here we may choose the domain $\mathscr{D}(A)$ of $d\Gamma_A(X)$ to be the set

$$\mathscr{D}(A) = \{X \in \mathrm{Lie}(\mathcal{G}) \mid [\epsilon_A, X] \text{ is Hilbert-Schmidt}\},$$

where $\epsilon_A = \pm$ on $\mathcal{H}_\pm(A)$. Moreover, supposing there exists a described lift $\Gamma_A : \mathcal{G} \longrightarrow \mathrm{End}(\mathcal{F})$ we should have

$$\Gamma_A(e^{iX}) = e^{id\Gamma_A(X)}, \quad \text{for all } X \in \mathrm{Lie}(\mathcal{G}).$$

In view of this, equation (3.2) can be written as

$$\Gamma_A(e^{iX})\psi_A^*(v)\Gamma_A^{-1}(e^{iX}) = \psi_A^*(e^{iX} \cdot v), \quad \text{for all } X \in \mathrm{Lie}(\mathcal{G}),\ v \in \mathcal{H}$$

relating Definition 3.3 to Theorem 3.2.

Next, we introduce the so called *Gauss law generators* acting on (Schrödinger wave) functions $\phi : \mathcal{A} \longrightarrow \mathcal{H}$,

$$G_A(X) = X + \mathcal{L}_X,$$

where $A \in \mathcal{A}$, $X \in \mathrm{Lie}(\mathcal{G})$ and the *Lie derivative* $\mathcal{L}_X$ is defined so that

$$\Big(\mathcal{L}_X\phi\Big)(A) = \frac{d}{dt}\phi(A^{e^{tX}})\Big|_{t=0}$$

Their second quantization is defined to be

$$d\Gamma(G_A(X)) = d\Gamma_A(X) + \mathcal{L}_X,$$

where $X \in \mathrm{Lie}(\mathcal{G})$. The renormalization procedure makes it possible to consider $d\Gamma_A(X)$ acting on $\mathcal{F}_0$ instead of $\mathcal{F}_A$. Now the second quantized Gauss law generators do not have anymore the same Lie algebra bracket as $\mathrm{Lie}(\mathcal{G})$ but instead

$$[d\Gamma(G_A(X)), d\Gamma(G_A(Y))] = d\Gamma([G_A(X), G_A(Y)]) + c(X, Y; A),$$

where $c(X, Y; A)$ is a $\mathrm{Map}(\mathcal{A}, \mathbb{R})$-valued Lie algebra cocycle of $\mathrm{Lie}(\mathcal{G})$ called the *Schwinger term*. This is the sought obstruction term. The connection with *gerbes* comes from a transgression map

$$H^3_{DR}(\mathcal{A}/\mathcal{G}_e) \longrightarrow H^2(\mathrm{Lie}(\mathcal{G}), \mathrm{Map}(\mathcal{A}, \mathbb{R}))$$

studied in [**CaMuWa**].

In [**CaMiMu**] Carey, Mickelsson and Murray construct explicitly the gerbe in question as a collection of local line bundles over the manifold $\mathcal{A}/\mathcal{G}_e$ that satisfy certain compatibility conditions. Let us recall this construction briefly.

Define for all $\lambda \in \mathbb{R}$ the open subsets

$$U_\lambda = \{A \in \mathcal{A} \mid \lambda \notin \mathrm{spec}(\not{D}_A)\} \subseteq \mathcal{A}.$$

These form an open cover for $\mathcal{A}$. Over each intersection $U_{\lambda\mu} =: U_\lambda \cap U_\mu$ there exists a line bundle $\mathrm{Det}_{\lambda\nu}$, whose fibre $\mathrm{Det}_{\lambda\nu}(A)$ at $A \in \mathcal{A}$ is related to (3.1) by the equation

$$\mathcal{F}_{A,\lambda} = \mathrm{Det}_{\lambda\mu}(A) \otimes \mathcal{F}_{A,\mu}$$

(thus giving the phase) and defined so that

$$\mathrm{Det}_{\lambda\mu}(A) = \bigwedge^{max} \big(\mathcal{H}_+(A,\lambda) \cap \mathcal{H}_-(A,\mu)\big)$$

for $\lambda < \mu$ and $\mathrm{Det}_{\mu\lambda} := \mathrm{Det}_{\lambda\mu}^{-1}$. The phase is related to the arbitrariness in filling the Dirac sea between vacuum levels λ and μ. Such a filling corresponds to an exterior product $v_1 \wedge v_2 \wedge \ldots \wedge v_m$ of a complete orthonormal set of eigenvectors $\not{D}_A v_i = \lambda_i v_i$ with $\lambda < \lambda_i < \mu$. A rotation of the eigenvector basis gives a multiliplication of the exterior product by the determinant of the rotation. Now, since the exterior product satisfies the 'exponential law'

$$\bigwedge^{max} (V \oplus W) = \bigwedge^{max} V \otimes \bigwedge^{max} W$$

for finite dimensional vector spaces V and W, one sees that over the triple intersections $U_{\lambda\lambda'\lambda''} := U_\lambda \cap U_{\lambda'} \cap U_{\lambda''}$

$$\mathrm{Det}_{\lambda\lambda'} \otimes \mathrm{Det}_{\lambda'\lambda''} = \mathrm{Det}_{\lambda\lambda''},$$

so that the collection $\{\mathrm{Det}_{\lambda\mu}\}$ of local line bundles define a *bundle gerbe* on $\mathcal{A}$. These local determinant line bundles are actually $\hat{\mathcal{G}}$-equivariant, where $\hat{\mathcal{G}}$ is the group extension of $\mathcal{G}$ integrating the Lie algebra extension of $\mathrm{Lie}(\mathcal{G})$ determined by the Scwhinger term, and so descend to the moduli space $\mathcal{A}/\mathcal{G}_e$ giving us the gerbe whose Dixmier-Douady class transgresses to the Schwinger term.

Readers interested to learn more about the subject are adviced to consult e.g. [**Ek**].

APPENDIX A

I.L.H. manifolds and Lie groups

Our references are [**Bry2**] and [**Pay**].

Definition A.1. A topological vector space E is called an I.L.H. vector space if $E = \varprojlim_n \mathcal{H}_n$ is an inverse limit of separable Hilbert spaces $\mathcal{H}_n$.

Hence, the topology of an I.L.H. vector space E is the inverse limit topology. This is the coarsest topology which makes all the projection maps $p_n : E \longrightarrow \mathcal{H}_n$ continuous. Often one wants to impose the following extra condition in the definition of an I.L.H. vector space:

- For every open ball B in $\mathcal{H}_n$, we have

$$p_n^{-1}(\overline{B}) = \overline{p_n^{-1}(B)}. \tag{A.1}$$

Theorem A.2. *Let X be a paracompact manifold, modelled on an I.L.H. vector space E satisfying (A.1). Then for any open covering $\mathcal{U} = \{U_i\}_{i\in I}$ of X there exists a smooth partition of unity subordinate to $\mathcal{U}$.*

Theorem A.3. *Let X be a paracompact manifold, modelled on an I.L.H. vector space E satisfying (A.1). Then the sheaves Ω_X^p of p:th order differential forms on X are soft, and we have canonical isomorphisms*

$$\check{H}^p(X,\mathbb{R}) \xrightarrow{\sim} H^p(X,\mathbb{R}) \xrightarrow{\sim} H^p_{DR}(X),$$

where $\mathbb{R}$ is the constant sheaf on X, $\check{H}^p(X,\mathbb{R}) = \varinjlim_{\mathcal{U}} \check{H}^p(\mathcal{U},\mathbb{R})$ and the de Rham cohomology $H^p_{DR}(X)$ is the p:th hypercohomology group of the complex

$$0 \longrightarrow \Omega_X^0 \xrightarrow{d} \Omega_X^1 \xrightarrow{d} \cdots .$$

Example A.4. The space $C^\infty(S^1)$ with the topology defined by the family of semi-norms $\|\cdot\|_n$, where

$$\|f\|_n^2 = \int_0^1 (\|f(x)\|^2 + \cdots + ||\frac{d^n}{dx^n}f(x)||^2)\, dx$$

is the inverse limit of the Hilbert spaces $\mathcal{H}^n(S^1)$, where $\mathcal{H}^n(S^1)$ is the completion of $C^\infty(S^1)$ for the norm $\|\cdot\|_n$. Moreover, the condition (A.1) is satisfied by the projection map $p_j : C^\infty(S^1) \longrightarrow \mathcal{H}^j(S^1)$.

Definition A.5. An I.L.H. topological group G is called an *I.L.H. Lie group* if it is a smooth I.L.H. manifold with the group operations given by smooth I.L.H. maps.

Definition A.6. Let P, B be smooth I.L.H. manifolds modelled on I.L.H. vector spaces E and F respectively, $\pi : P \longrightarrow B$ a smooth I.L.H. map and G an I.L.H. Lie group. Then (P, B, G, π) is an I.L.H. principal bundle if the transition maps are smooth I.H.L. maps.

Let (P, M, G, π) be a smooth principal G-bundle on a closed manifold M, where we assume all the manifolds to be finite dimensional and that G is compact. Let $E = \operatorname{ad} P := P \times_G \mathrm{Lie}(G)$, where G acts on $\mathrm{Lie}(G)$ by the adjoint action, and $F := T^*M \otimes \operatorname{ad} P$.

EXAMPLE A.7. The space $\mathcal{A}(P)$ of smooth connections on P is an affine I.L.H space with tangent vector space $\mathrm{C}^\infty(F)$. Since G is compact, $\mathrm{Lie}(G)$ can be equipped with a positive definite inner product which is invariant under the adjoint action. This way the bundle $\mathrm{ad}\,P$ inherits an inner product structure and choosing a Riemannian metric on M yields an inner product on $F = T^*M \otimes \mathrm{ad}\,P$. Hence $\mathcal{A}(P)$ which is modelled on $\mathrm{C}^\infty(F)$ can be equipped with an L^2-metric obtained by integrating along M the inner product on F.

EXAMPLE A.8. Let $E_G = \mathrm{Ad}\,P := P \times_G G$ where G acts on itself by the adjoint action. Then the set $\mathcal{G}(P) := \mathrm{C}^\infty(E_G)$ is an I.L.H. Lie group modelled on $\mathrm{C}^\infty(E)$. It corresponds to the group of *gauge transformations* of the principle G-bundle P, i.e. the group of automorphisms of P that cover the identity.

EXAMPLE A.9 (Infinite dimensional Grassmannian of Segal and Wilson). Let $\mathcal{H}$ be a separable Hilbert space with an orthogonal decomposition $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$. Recall that for any two Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$ the space $H.S.(\mathcal{H}_1, \mathcal{H}_2)$ of Hilbert-Schmidt operators $T : \mathcal{H}_1 \longrightarrow \mathcal{H}_2$ is a Hilbert space with norm $\|T\|_2 = \sqrt{\mathrm{Tr}(T^*T)}$. Let $\mathrm{Gr}_{res}(\mathcal{H})$ denote the set of closed subspaces $W \subseteq \mathcal{H}$ such that

(1) The orthogonal projection onto $\mathcal{H}_+$, $\mathrm{pr}_W^+ : W \longrightarrow \mathcal{H}_+$ is Fredholm;

(2) The orthogonal projection onto $\mathcal{H}_-$, $\mathrm{pr}_W^- : W \longrightarrow \mathcal{H}_-$ is Hilbert-Schmidt.

Then $\mathrm{Gr}_{res}(\mathcal{H})$ is a Hilbert manifold modelled on $H.S.(\mathcal{H}_+, \mathcal{H}_-)$.

APPENDIX B

Equivariant cohomology

1. Group actions on topological spaces

DEFINITION B.1. Let G be a topological group and X a topological space. Then G *acts* (continuously) on X if there exists a continuous function $\Phi : G \times X \longrightarrow X$, denoted by $(g, x) \mapsto gx$, such that

(1) $(gg')x = g(g'y)$,
(2) $1_G \cdot x = x$.

One defines similarly a smooth action of a Lie group G on a manifold X.

DEFINITION B.2. A topological group G acts on a topological space X *freely* if for every $x \in X$, $gx = x$ implies $g = 1_G$.

2. Equivariant cohomology

Here we follow [**Gomi**]. Let G be a topological group, X a G-space and let $EG \longrightarrow BG$ be the universal fibration such that EG is contractible and G acts freely on EG. We let G act diagonally on $X \times EG$ and define

$$X_G = (X \times EG)/G.$$

DEFINITION B.3. (Equivariant cohomology)

$$H^*_G(X, \mathbb{Z}) := H^*(X_G, \mathbb{Z}).$$

One defines similarly the equivariant cohomology group $H^*_G(X, \mathbb{R})$.

There is another way to define equivariant cohomology groups using simplicial manifolds which we now introduce. From now on, we assume that G is a Lie group acting on a smooth manifold X. Thus we have a simplicial manifold $\{G^\bullet \times X\} = \{G^p \times X\}_{p \geq 0}$, where the face maps $\partial_i : G^{p+1} \times X \longrightarrow G^p \times X$, $(i = 0, \dots, p+1)$ are given by

$$\partial_i(g_1, \dots, g_{p+1}, x) = \begin{cases} (g_2, \dots, g_{p+1}, x), & i = 0 \\ (g_1, \dots, g_{i-1}, g_i g_{i+1}, g_{i+2}, \dots g_{p+1}, x), & i = 1, \dots, p \\ (g_1, \dots, g_p, g_{p+1}, x), & i = p+1, \end{cases}$$

and the degeneracy maps $s_i : G^p \times X \longrightarrow G^{p+1} \times X$, $(i = 0, \dots, p)$ by

$$s_i(g_1, \dots, g_p, x) = (g_1, \dots, g_i, 1_G, g_{i+1}, \dots, g_p, x).$$

These satisfy

$$\partial_i \circ \partial_j = \partial_{j-1} \circ \partial_i, \quad (i < j), \tag{B.1}$$

$$s_i \circ s_j = s_{j+1} \circ s_i, \quad (i \leq j), \tag{B.2}$$

$$\partial_i \circ s_j = \begin{cases} s_{j-1} \circ \partial_i, & (i < j), \\ \text{id}, & (i = j, j+1), \\ s_j \circ \partial_{i-1}, & (i > j+1). \end{cases} \tag{B.3}$$

Definition B.4. A *simplicial sheaf* of Abelian groups on a simplicial manifold $X_\bullet$ is a family of sheaves $\mathcal{F}^\bullet = \{\mathcal{F}^p\}_{p\geq 0}$, where each $\mathcal{F}^p$ is a sheaf of Abelian groups on X_p such that there are sheaf homomorfisms $\tilde{\partial}_i : \partial_i^{-1}\mathcal{F}^p \longrightarrow \mathcal{F}^{p+1}$ and $\tilde{s}_i : s_i^{-1}\mathcal{F}^{p+1} \longrightarrow \mathcal{F}^p$ satisfying the same relations as (B.1), (B.2) and (B.3).

For each $p \geq 0$, let $\mathcal{I}^{p,\bullet}$ be an injective resolution of the sheaf $\mathcal{F}^p$ on X_p, i.e. an exact sequence

$$0 \longrightarrow \mathcal{F}^p \xrightarrow{\varepsilon} \mathcal{I}^{p,0} \xrightarrow{\delta} \mathcal{I}^{p,1} \xrightarrow{\delta} \mathcal{I}^{p,2} \xrightarrow{\delta} \cdots$$

where for each $q \geq 0$ the sheaf $\mathcal{I}^{p,q}$ is injective (i.e. the functor $\mathrm{Hom}(\cdot, \mathcal{I}^{p,q}) : \mathfrak{Ab}(X) \longrightarrow \mathfrak{Ab}$ is exact) and $\delta^2 = 0$ so that $\mathcal{I}^{p,\bullet}$ is a complex. We denote this collection of injective resolutions by $\mathcal{I}^{\bullet\bullet}$ and call it an injective resolution of the simplicial sheaf $\mathcal{F}^\bullet$. The sheaf homomorphism $\tilde{\partial}_i : \partial_i^{-1}\mathcal{F}^p \longrightarrow \mathcal{F}^{p+1}$ induces a homomorphism of Abelian groups

$$\partial_i^* : \Gamma(X_p, \mathcal{I}^{p,q}) \longrightarrow \Gamma(X_{p+1}, \mathcal{I}^{p+1,q}).$$

These can be combined into a homomorphism

$$\partial : \Gamma(X_p, \mathcal{I}^{p,q}) \longrightarrow \Gamma(X_{p+1}, \mathcal{I}^{p+1,q}), \quad \partial = \sum_{j=0}^{p+1} (-1)^j \partial_j^*$$

satisfying $\partial^2 = 0$. One defines similarly the homomorphism δ satisfying $\delta^2 = 0$. The system $\Gamma^{\bullet\bullet} = (\Gamma(G^i \times X, \mathcal{I}^{i,j}), \partial, \delta)$ forms a double complex and we define the *hypercohomology of the simplicial sheaf* $\mathcal{F}^\bullet$ to be the cohomology of this double complex,

$$\mathbb{H}^k(X_\bullet, \mathcal{F}^\bullet) = h^k(\mathrm{Tot}(\Gamma^{\bullet\bullet})).$$

Example B.5 (Deligne [**De**]). Let G be a Lie group acting on a smooth manifold X. Then the family of sheaves $\{\underline{S}^1_{G^p\times X}\}_{p\geq 0}$ gives rise to a simplicial sheaf over $G^\bullet \times X$, where $\underline{S}^1_{G^p\times X}$ is the sheaf of germs of S^1-valued smooth functions on $G^p \times X$. We denote the corresponding (hyper)cohomology groups by

$$H^k(G^\bullet \times X, \underline{S}^1) := \mathbb{H}^k(G^\bullet \times X, \underline{S}^1_{G^p\times X}).$$

Let G be a Lie group acting on a smooth manifold X.

Definition B.6. A *G-equivariant principal S^1-bundle* is a principal S^1-bundle $P \longrightarrow X$ together with a lift of the G-action on X to that on P by bundle isomorphisms.

For a principal S^1-bundle $P \longrightarrow X$, we define a principal S^1-bundle $\partial P \longrightarrow G \times X$ by $\partial P = \partial_0^* P \otimes \partial_1^* P^{\otimes -1}$.

Lemma B.7. *For a principal S^1-bundle P over X, the following notions are equivalent:*

1. *$P \longrightarrow X$ is a G-equivariant principal S^1-bundle;*
2. *There exists a section $\sigma \in \Gamma(G \times X, \partial P)$ such that $\partial\sigma = 1$ on $G^2 \times X$, where $\partial\sigma := \partial_0^*\sigma \otimes \partial_1^*\sigma^{\otimes -1} \otimes \partial_2^*\sigma$.*

Next, let $\mathfrak{R}$ be an S^1-gerbe over X. We define an S^1-gerbe over $G \times X$ by setting $\partial\mathfrak{R} = \partial_0^*\mathfrak{R} \otimes \partial_1^*\mathfrak{R}^{\otimes -1}$.

Definition B.8. A *G-equivariant gerbe* over X is defined to be gerbe $\mathfrak{R}$ over X together with the following data:

1. A global object $R \in \Gamma(G \times X, \partial\mathfrak{R})$;
2. An isomorphism $\psi : \partial R \longrightarrow \mathbf{1}$ of global objects in the trivial gerbe over $G^2 \times X$ which satisfies $\partial\psi = 1$ on $G^3 \times X$.

Here we regard the trivial gerbe over $G^2 \times X$ as the sheaf of categories of S^1-bundles. So the global object **1** means simply the trivial S^1-bundle over $G^2 \times X$.

PROPOSITION B.9 (Brylinski [**Bry1**]). *Let G be a Lie group acting on a manifold X.*

(1) *The isomorphism classes of G-equivariant principal S^1-bundles over X are classified by $H^1(G^\bullet \times X, \underline{S}^1)$.*
(2) *The isomorphism classes of G-equivariant gerbes over X are classified by $H^2(G^\bullet \times X, \underline{S}^1)$.*

For *compact* Lie groups, we have the following result.

LEMMA B.10 ([**Bry1**]). *If G is a finite dimensional compact Lie group acting on a finite dimensional manifold X, then there exists a natural isomorphism*

$$H^k(G^\bullet \times X, \underline{S}^1) \cong H_G^{k+1}(X, \mathbb{Z}),$$

for all $k > 0$.

APPENDIX C

Highest weight representations

Here we follow [**ShiUe**].

1. Highest weight representations of simple Lie algebras

Let $\mathfrak{g}$ be a finite dimensional *simple* Lie algebra over $\mathbb{C}$, i.e. $\mathfrak{g}$ is nonabelian and has no proper nonzero ideals, and $\mathfrak{h}$ Cartan subalgebra of $\mathfrak{g}$, that is a maximal abelian subspace of $\mathfrak{g}$ in which every ad $H, H \in \mathfrak{h}$, is diagonable (here for a fixed $X \in \mathfrak{g}$, ad $X : \mathfrak{g} \longrightarrow \mathfrak{g}$ is the linear transformation of Lie defined so that $(\text{ad } X)Z = [X, Z]$ for all $Z \in \mathfrak{g}$). Then we have the *root space decomposition*

$$\mathfrak{g} = \mathfrak{h} \oplus \sum_{\alpha \in \Delta} \mathfrak{g}_\alpha,$$

where the elements $\alpha \in \mathfrak{h}^* \setminus \{0\}$ are such that

$$\mathfrak{g}_\alpha = \{X \in \text{Lie} \mid [H, X] = \alpha(H)X \quad \text{for all } H \in \mathfrak{h}\} \neq 0$$

The set $\Delta \subseteq \mathfrak{h}^*$ of all such α is called the *root system* of $(\mathfrak{g}, \mathfrak{h})$ and the elements $\alpha \in \Delta$ are called *roots*. The set Δ is finite. Let $\mathfrak{h}^*_\mathbb{R}$ be the vector space spanned over $\mathbb{R}$ by Δ. We fix a base of $\mathfrak{h}^*_\mathbb{R}$, which determines a lexiographic order on $\mathfrak{h}^*_\mathbb{R}$. This order determines the decomposition $\Delta = \Delta_+ \sqcup \Delta_-$ The elements of Δ_+ (resp. of Δ_-) are called *positive* (resp. *negative*) roots. The space $\mathfrak{h}^* = \mathfrak{h}^*_\mathbb{R} \otimes_\mathbb{R} \mathbb{C}$ is spanned ovet $\mathbb{C}$ by Δ. The *Killing form* is the symmetric bilinear for on Lie defined by $B(X, Y) = \text{Tr}(\text{ad } X \text{ ad } Y)$. A (non-trivial) constant multiple of Cartan-Killing form will be denoted by (,). Any Cartan-Killing form satisfies the identity

$$([X, Y], Z) + (Y, [X, Z]) = 0$$

for all $X, Y, Z \in \mathfrak{g}$.

To each element $\lambda \in \mathfrak{h}^*$, one can associate the unique element $H_\lambda \in \mathfrak{h}$ such that the equality

$$\lambda(H) = (H_\lambda, H)$$

holds for every $H \in \mathfrak{h}$. For a root $\alpha \in \Delta$, the element H_α is called the *root vector* corresponding to α.

We define an inner product on $\mathfrak{h}^*$ by

$$(\lambda, \mu) = (H_\lambda, H_\mu).$$

Let θ denote the longest root, i.e the root for which

$$\theta \in \Delta_+, \quad \theta + \alpha_i \notin \Delta.$$

Then we normalize our Cartan-Killing form so that

$$(\theta, \theta) = 2.$$

One can show that it is possible to choose elements $X_\theta \in \mathfrak{g}_\theta$ and $X_{-\theta} \in \mathfrak{g}_{-\theta}$ satisfying

$$[H_\theta, X_\theta] = 2X_\theta, \quad [H_\theta, X_{-\theta}] = -2X_{-\theta}, \quad [X_\theta, X_{-\theta}] = -H_\theta,$$

so that $\{H_\theta, X_\theta, X_{-\theta}\}$ generate a Lie subalgebra of $\mathfrak{g}$ isomorphic to $\mathfrak{sl}(2, \mathbb{C})$.

The *weight lattice* $P \subseteq \mathfrak{h}^*$ is the group of linear forms $\lambda \in \mathfrak{h}^*$ such that

$$\lambda(H_\alpha) \in \mathbb{Z}$$

for all roots α. A weight λ is *dominant* if $\lambda(H_\alpha) \geq 0$ for all positive roots α; we denote by P_+ the set of dominant weights. Dominant weights characterize all simple finite-dimensional $\mathfrak{g}$-modules:

THEOREM C.1. *To each dominant weight $\lambda \in P_+$ is associated a simple $\mathfrak{g}$-module V_λ, unique up to isomorphism, containing a* highest weight vector v_λ *with weight λ (i.e. v_λ is annihilated by $\mathfrak{g}_\alpha$ for every $\alpha > 0$, $Hv_\lambda = \lambda(H)v_\lambda$ for all $H \in \mathfrak{h}$, and $U(\mathfrak{n}_-)v_\lambda = V_\lambda$, where $\mathfrak{n}_- := \oplus_{\alpha<0}\mathfrak{g}_\alpha$). The map $\lambda \longrightarrow [V_\lambda]$ is a bijection of P_+ onto the set of isomorphism classes of finite-dimensional simple $\mathfrak{g}$-modules.*

2. Highest weight representations of affine Lie algebras

Now, put

$$\widehat{\mathfrak{g}} = \mathfrak{g} \otimes_{\mathbb{C}} \mathbb{C}((\xi)) \oplus \mathbb{C} \cdot c,$$

where $\mathbb{C}((\xi))$ denotes the field of Laurent power series over $\mathbb{C}$ in one variable. This has the following Lie algebra structure:

(1) c belongs to the center of $\widehat{\mathfrak{g}}$;
(2) For $X, Y \in$ Lie and $f(\xi), g(\xi) \in \mathbb{C}((\xi))$,

$$[X \otimes f(\xi), Y \otimes g(\xi)] = [X,Y] \otimes f(\xi)g(\xi) + (X,Y)\mathrm{Res}_{\xi=0}(g(\xi)f'(\xi)d\xi) \cdot c.$$

Note that $\widehat{\mathfrak{g}}$ is infinite dimensional and is a central extension of the *loop algebra* $L\mathfrak{g} = \mathfrak{g} \otimes_{\mathbb{C}} \mathbb{C}((\xi))$. We call $\widehat{\mathfrak{g}}$ the *affine Lie algebra* associated to the simple Lie algebra Lie. We define the following Lie subalgebras of $\widehat{\mathfrak{g}}$ by

$$\widehat{\mathfrak{g}}_+ = \mathfrak{g} \otimes \mathbb{C}[[\xi]]\xi, \quad \widehat{\mathfrak{g}}_- = \mathfrak{g} \otimes \mathbb{C}[\xi^{-1}]\xi^{-1}.$$

This gives us the decomposition

$$\widehat{\mathfrak{g}} = \widehat{\mathfrak{g}}_- \oplus \mathfrak{g} \oplus \mathbb{C} \cdot c \oplus \widehat{\mathfrak{g}}_+.$$

Moreover, for $X \in \mathfrak{g}, n \in \mathbb{Z}$ we define $X(n) = X \otimes \xi^n \in \widehat{\mathfrak{g}}$.

Next, let us fix a positive integer ℓ, called the *level*; we are interested in the irreducible representations of $\widehat{\mathfrak{g}}$ which are *of level* ℓ, i.e. such that the central element $c \in \widehat{\mathfrak{g}}$ acts as multiplication by c. Define the set P_ℓ as

$$P_\ell = \{\lambda \in P_+ \mid \lambda(H_\theta) \leq \ell\}.$$

THEOREM C.2. *For each $\lambda \in P_\ell$ there exists a left $\widehat{\mathfrak{g}}$-module $\mathcal{H}_\lambda$ characterized up to isomorphism by the following properties*

(1) $V_\lambda = \{|v\rangle \in \mathcal{H}_\lambda \mid \widehat{\mathfrak{g}}_+|v\rangle = 0\}$ is an irreducible left $\mathfrak{g}$-module with highest weight λ.
(2) The central element c of $\widehat{\mathfrak{g}}$ acts on $\mathcal{H}_\lambda$ by $\ell \cdot id$.
(3) $\mathcal{H}_\lambda$ is generated by V_λ over $\widehat{\mathfrak{g}}$ with the fundamental relation

$$X_\theta(-1)^{\ell-(\theta,\lambda)+1}|\theta\rangle = 0.$$

Here $|\lambda\rangle$ denotes the highest weight vector of V_λ and θ is the longest root, $(o \neq)X_\theta \in \mathfrak{g}_\theta$.

Concretely, $\mathcal{H}_\lambda$ can be given by the formula

$$\mathcal{H}_\lambda = U(\widehat{\mathfrak{g}}_-)V_\lambda / U(\widehat{\mathfrak{g}}_-)X_\theta(-1)^{\ell-(\theta,\lambda)+1}|\lambda\rangle.$$

We call $\mathcal{H}_\lambda$ the *integrable left $\widehat{\mathfrak{g}}$-module of level ℓ with highest weight λ.*

Integrable representations are characterized by the next theorem

THEOREM C.3. *Given a left $\widehat{\mathfrak{g}}$-module $\mathcal{H}$ satisfying the properties* (1), (2) *of Theorem C.2. For $\mathcal{H}$ to be integrable it is necessary and sufficient that for any $|\varphi\rangle \in \mathcal{H}, \alpha \in \Delta$ and any integer $n \in \mathbb{Z}$, there exists a positive integer m such that the equality*

$$X_\alpha(n)^k|\varphi\rangle = 0$$

holds whenever $k \geq m$. Here the integer m may depend on $|\varphi\rangle$.

The condition in the theorem can be expressed by saying that $X_\alpha(n)$ acts on $\mathcal{H}$ *locally nilpotently.* In general, a left $\widehat{\mathfrak{g}}$-module $\mathcal{H}$ is called an integrable $\widehat{\mathfrak{g}}$-module of level ℓ if

(1) It decomposes into the direct sum of the eigenspaces for the action of $\widehat{\mathfrak{h}} = \mathfrak{h} + \mathbb{C}$,
(2) $X_\alpha(n)$, $X_\alpha \in \mathfrak{g}_\alpha$ $(\forall \alpha \in \Delta)$, acts locally nilpotently,
(3) The central element c acts as $\ell \cdot \mathrm{id}$.

THEOREM C.4. *An integrable $\widehat{\mathfrak{g}}$-module of level ℓ can be decomposed into a finite direct sum of irreducible integrable modules $\mathcal{H}_\lambda$ $(\in P_\ell)$.*

3. Sugawara's construction

3.1. Constructing Virasoro algebras from affine Lie algebras. We define the normal ordering $\circ\atop\circ$ $\circ\atop\circ$ by

$${\circ\atop\circ} X(n)Y(m) {\circ\atop\circ} = \begin{cases} X(n)Y(m) & \text{if } n < m \\ \frac{1}{2}(X(n)Y(m) + Y(m)X(n)) & \text{if } n = m \\ Y(m)X(n) & \text{if } n > m\ . \end{cases}$$

Now let $\{J^a\}_{1 \leq a \leq \dim \mathfrak{g}}$ be an orthogonal basis of $\mathfrak{g}$ with respect to the normalized Cartan-Killing form (,) and set

$$L_n = \frac{1}{2(h^\vee + \ell)} \sum_{m \in \mathbb{Z}} \sum_{a=1}^{\dim \mathfrak{g}} {\circ\atop\circ} J^a(m)J^a(n-m) {\circ\atop\circ},$$

where $h^\vee$ is the dual Coxeter number.

Note that even though L_n is a sum infinite both in positive and negative powers, its action on an integrable highest weight left $\widehat{\mathfrak{g}}$-module $\mathcal{H}_\lambda$ of level ℓ is well-defined. This is because, for any $|\varphi\rangle \in \mathcal{H}_\lambda$ and for any $X \in \mathfrak{g}$, there exists a positive integer M such that $X(n)|\varphi\rangle = 0$ whenever $n \geq M$. Now the quadratic term ${\circ\atop\circ} J^a(m)J^a(n-m) {\circ\atop\circ}$ is equal to $J^a(n-m)J^a(m)$ if m is sufficiently large and to $J^a(m)J^a(n-m)$ if $-m$ is sufficiently large. Thus, for each $|\varphi\rangle \in \mathcal{H}_\lambda$, the terms acting non-trivially in the right hand side of

$$L_n|\varphi\rangle = \frac{1}{2(h^\vee + \ell)} \sum_{m \in \mathbb{Z}} \sum_{a=1}^{\dim \mathfrak{g}} {\circ\atop\circ} J^a(m)J^a(n-m) {\circ\atop\circ} |\varphi\rangle$$

are only finitely many.

Next, regard $L_n, X(m)$ as operators on $\mathcal{H}_\lambda$. Then the following holds.

PROPOSITION C.5. *For any $X \in \mathfrak{g}$, one has the equality*

$$[L_n, X(m)] = -mX(n+m), \tag{C.1}$$

as well as the equality

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{c_v}{12}(n^3 - n)\delta_{n+m,0}. \tag{C.2}$$

Here c_v is defined as

$$c_v = \frac{\ell \dim \mathfrak{g}}{h^\vee + \ell}.$$

The Lie algebra spanned by $\{L_n\}$ and (a scalar operator) 1 with the fundamental commutation relations (C.2) in the Proposition is called the *Virasoro algebra* with *central charge* $c_v = \frac{\ell \dim \mathfrak{g}}{h^\vee + \ell}$.

3.2. Diagonalizing L_0. Define the subspace $\mathcal{H}_\lambda(d)$ of $\mathcal{H}_\lambda$ for each non-negative integer $d \in \mathbb{Z}_{\geq 0}$ by

$$\mathcal{H}_\lambda(d) := \{|v\rangle \in \mathcal{H}_\lambda\} \mid L_0|v\rangle = (d + \Delta_\lambda)|v\rangle, \tag{C.3}$$

where

$$\Delta_\lambda = \frac{(\lambda, \lambda) + (\lambda, \rho)}{2(h^\vee + \ell)}, \quad \rho = \frac{1}{2}\sum_{\alpha \in \Delta_+} \alpha.$$

It follows that L_0 acts on $V_\lambda \subset \mathcal{H}_\lambda$ as

$$L_0|v\rangle = \frac{1}{2(h^\vee + \ell)} \sum_{a=1}^{\dim \mathfrak{g}} J^a J^a |v\rangle = \Delta_\lambda |v\rangle$$

for $|v\rangle \in V$. Namely, the action of L_0 on V_λ coincides with that of the Casimir operator of $\mathfrak{g}$. Hence for any positive integer m and any $v \in V_\lambda$

$$\begin{aligned} L_0 X(-m)|v\rangle &= X(-m)L_0|v\rangle + mX(-m)|v\rangle \\ &= (\Delta_\lambda + m)X(-m)|v\rangle. \end{aligned}$$

Therefore we see directly from the definition (C.3) that $X(-m)|v\rangle \in \mathcal{H}_\lambda(m)$, Similarly, for any positive integers $m_1, \ldots m_k$,

$$L_0 X_1(-m_1) \cdots X_k(-m_k)|v\rangle = (\Delta_\lambda + m_1 + \cdots + m_k) X_1(-m_1) \cdots X_k(-m_k)|v\rangle,$$

where $|v\rangle \in V_\lambda$. From this one can conclude that each $\mathcal{H}_\lambda(d)$ is a finite dimensional vector space and that

$$\mathcal{H}_\lambda = \bigoplus_{d=0}^{\infty} \mathcal{H}_\lambda(d).$$

Bibliography

[Ara] H. Araki: *Bogoliubov automorphisms and Fock representations of canonical anticommutation relations*, Contemporary mathematics, AMS, vol. **62** (1987).

[Be] K. Behrend: *Cohomology of stacks*, http://www.math.ubc.ca/ behrend/preprints.html

[BeXu] K. Behrend, P. Xu: *Differentiable stacks and gerbes*, arXiv:math.DG/0605694.

[Boss] B. Booss-Bavnbek, K. Wojciechowski: *Elliptic boundary problems for Dirac operators*, Birkhauser, Boston, 1993.

[Bry1] J-L. Brylinski: *Gerbes on complex reductive Lie groups*, arXiv:math.DG/0002158.

[Bry2] J-L. Brylinski: *Loop spaces, characteristic classes and geometric quantization*, Birkhäuser Boston, Inc., Boston, MA (1993).

[CaMiMu] A. Carey, J. Mickelsson, M. Murray: *Index theory, gerbes, and Hamiltonian quantization*, Commun. Math. Phys. 183 (1997) 707–722.

[CaMuWa] A. Carey, M. Murray, B. Wang: *Higher bundle gerbes and cohomology classes in gauge theories*, arXiv:hep-th/9511169v1.

[Con] A. Connes: *Noncommutative geometry*, Academic Press, San Diego (1994).

[De] P. Deligne: *Théorie de Hodge,* III, Inst. Hautes Études Sci. Publ. Math. (1974), 5–77.

[Ek] C. Ekstrand: *Schwinger terms from external field problems*, PhD Thesis (Royal Institute of Technology, Stockholm), 1999.

[Fad] L. Faddeev: *Operator anomaly for the Gauss law*, Phys. Lett. 145B, 1984.

[Gom] T. Gómez: *Algebraic stacks*, arXiv:math.AG/9911199v1.

[Gomi] K. Gomi: *Equivariant smooth Deligne cohomology*, Osaka J. Math **42** (2005), 309–337.

[G-BV] J. M. Gracia-Bondía, J. C. Várilly: *Connes' noncommutative differential geometry and the standard model*, Journal of geometry and physics **12** (1993), 223–301.

[Gro] A. Grothendieck, *A general theory of fibre spaces with structure sheaf*, National science foundation researh project on Geometry of function spaces, Report No. 4, University of Kansas, Department of mathematics (1955).

[Hein] J. Heinloth: *Some notes on differentiable stacks*, Mathematisches Institut Seminars (Y. Tschinkel, ed.), p. 1–32, Universität Göttingen (2004–05). ¨

[Laum] G. Laumon, L. Moret-Bailly: *Champs algébrigues*, volume **39**

[Met] D. Metzler: *Topological and smooth stacks*, arXiv:mathDG/0306176v1.

[Mi] J. Mickelsson: *Chiral anomalies in even and odd dimensions*, Commun. Math. Phys. **97**, 1985.

[Mi3] J. Mickelsson: *Regularization of current algebra*, Constraint theory and quantization methods (Montepulciano 1993), 72–79, World Sci. Publ. River Edge, NJ (1994).

[Mi4] J. Mickelsson: *From gauge anomalies to gerbes and gerbal actions*, arXiv: math.DG/0812.1640v1.

[Pay] S. Paycha: *Basic prerequisities in differential geometry and operation theory in view of applications to quantum field theory* http://math.univ-bpclermont.fr/ paycha/publications.html

[PreSe] A. Pressley, G. Segal: *Loop groups*, Oxford mathematical monographs, Clarendon Press, 1986.

[ShiUe] Y. Shimizu, K Ueno: *Advances in moduli theory.* Iwanami series in modern mathematics, Translations of mathematical monographs, Vol **206**. American mathematical society (2002).

[Sor] C. Sorger: *Lectures on moduli of principal G-bundles over algebraic curves*, http://www.math.sciences.univ-nantes.fr/ sorger/publications.html.

[Steve] D. Stevenson: *The geometry of bundle gerbes*, PhD Thesis (University of Adeleide), 2000, arXiv:math.DG/0004117 v1.

[Weib] C. Weibel: *An introduction to homological algebra*, Cambridge studies in advanced mathematics, volume **38**.

[L-GTuXu] J-L. Tu, P. Xu, C. Laurent-Gengoux: *Twisted K-theory of differentiable stacks.* arXiv:math.KT/0306138.