On the Rademacher maximal function
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Introduction

Tools known as maximal functions are frequently used in harmonic analysis when studying local behaviour of functions. Typically they measure the suprema of local averages of non-negative functions. It is essential that the size (more precisely, the $L^p$-norm) of the maximal function is comparable to the size of the original function.

When dealing with families of operators between Banach spaces we are often forced to replace the uniform bound with the larger $R$-bound. Hence such a replacement is also needed in the maximal function for functions taking values in spaces of operators. More specifically, the suprema of norms of local averages (i.e. their uniform bound in the operator norm) has to be replaced by their $R$-bound. This procedure gives us the Rademacher maximal function, which was introduced by Hytönen, McIntosh and Portal [15] in order to prove a certain vector-valued Carleson’s embedding theorem. They noticed that the sizes of an operator-valued function and its Rademacher maximal function are comparable for many common range spaces, but not for all. Certain requirements on the type and cotype of the spaces involved are necessary for this comparability, henceforth referred to as the “RMF-property”. It was shown, that other objects and parameters appearing in the definition, such as the domain of functions and the exponent $p$ of the norm, make no difference to this.

After a short introduction to randomized norms and geometry in Banach spaces we study the Rademacher maximal function on Euclidean spaces. The requirements on the type and cotype are considered, providing examples of spaces without RMF. $L^p$-spaces are shown to have RMF not only for $p \geq 2$ (when it is trivial) but also for $1 < p < 2$. A dyadic version of Carleson’s embedding theorem is proven for scalar- and operator-valued functions (Theorems 2.15 and 2.16).

As the analysis with dyadic cubes can be generalized to filtrations on $\sigma$-finite measure spaces, we consider the Rademacher maximal function in this case as well. It turns out that the RMF-property is independent of the filtration and the underlying measure space and that it is enough to consider very simple ones known as Haar filtrations (Theorem 3.7). Scalar- and operator-valued analogues of Carleson’s embedding theorem (Theorems 3.5 and 3.6) are also provided.

With the RMF-property proven independent of the underlying measure space, we can use probabilistic notions and formulate it for martingales. Following a similar result for UMD-spaces, a weak type inequality is shown to be (necessary and) sufficient for the RMF-property (Theorem 4.12). The RMF-property is also studied using concave functions (Theorem 4.15) giving yet another proof of its independence from various parameters.

All Banach spaces can be either real or complex unless otherwise stated and so we speak of scalars without specifying whether they are real or complex. We write $a \lesssim b$ when there exists a constant $C$ such that $a \leq Cb$, with $C$ independent of the indicated variables in expressions $a$ and $b$. By $a \asymp b$ we mean $b \lesssim a \lesssim b$, while the isomorphism of Banach spaces is denoted by $\simeq$. Sets of vectors indexed by a subset of a larger index set are always thought to have zero extension to the whole index set.
1 Preliminaries

All random variables in Banach spaces (functions from a probability space to the Banach space) are assumed to be \( \mathcal{P} \)-strongly measurable, by which we mean that they are \( \mathcal{P} \)-almost everywhere limits of simple functions on the probability space whose measure we denote by \( \mathcal{P} \). Their expectation, denoted by \( \mathbb{E} \), is given by the Bochner integral. By an \( L^p \)-random variable, for \( 1 \leq p < \infty \), we mean random variable \( X \) (in a Banach space) whose \( p \)th moment \( \mathbb{E}\|X\|^p \) is finite. To every random variable \( X \) in a Banach space we associate its distribution - a measure given by \( \mathbb{P}\{X \in B\} \) for Borel sets \( B \) in the Banach space. A random variable \( X \) is said to be symmetric if \( X \) and \( -X \) are identically distributed in the sense that their distributions coincide.

1.1 Randomized norms

It is often very straightforward to generalize analysis of scalar-valued functions to that of functions taking values in a Hilbert space. This is mainly due to identities relying on the inner product such as Pythagoras’ theorem and the Parallelogram law, which guarantee that square sums of vector norms behave nicely. While square sums of norms make perfect sense in the more general setting of Banach spaces as well, it has proven right to replace them by randomized norms.

Let \( (\varepsilon_j)_{j=1}^\infty \) be a sequence of Rademacher variables, more precisely, a sequence of independent random variables attaining values \(-1\) and \(1\) with an equal probability \( \mathbb{P}\{\varepsilon_j = -1\} = \mathbb{P}\{\varepsilon_j = 1\} = 1/2 \). By the independence we have \( \mathbb{E}\{\varepsilon_j \varepsilon_k\} = (\mathbb{E}\varepsilon_j)(\mathbb{E}\varepsilon_k) = 0 \), whenever \( j \neq k \), while (trivially) \( \mathbb{E}\{\varepsilon_j \varepsilon_k\} = 1 \), if \( j = k \). The equality of a randomized norm and a square sum of norms for vectors \( x_1, \ldots, x_N \) in a Hilbert space is thus established by the following calculation:

\[
\mathbb{E}\| \sum_{j=1}^N \varepsilon_j x_j \|^2 = \mathbb{E}\left( \sum_{j=1}^N \varepsilon_j x_j, \sum_{k=1}^N \varepsilon_k x_k \right) = \sum_{j,k=1}^N \mathbb{E}\{\varepsilon_j \varepsilon_k\} \langle x_j, x_k \rangle = \sum_{j=1}^N \|x_j\|^2.
\]

A neat technique of randomization will be applied at times in order to handle randomized norms: If \( (X_j)_{j=1}^N \) is a sequence of independent symmetric random variables in a Banach space and \( (\varepsilon_j)_{j=1}^N \) is a sequence of signs \{-1,1\} or a sequence of Rademacher variables independent from \( (X_j)_{j=1}^N \), then \( (X_j)_{j=1}^N \) and \( (\varepsilon_jX_j)_{j=1}^N \) are identically distributed. In particular, if \( (\varepsilon_j)_{j=1}^N \) and \( (\varepsilon_j')_{j=1}^N \) are independent sequences of Rademacher variables, then for any vectors \( x_1, \ldots, x_N \) in a Banach space the sequences \( (\varepsilon_jx_j)_{j=1}^N \) and \( (\varepsilon_j'x_j)_{j=1}^N \) are identically distributed. In practise this is often applied in the following way: If \( \{1, \ldots, N\} \) is decomposed into disjoint sets \( J_1, \ldots, J_M \), then

\[
\mathbb{E}\left( \left\| \sum_{j=1}^N \varepsilon_j x_j \right\|^p \right) = \mathbb{E}'\left( \left\| \sum_{k=1}^M \varepsilon_k' \sum_{j \in J_k} \varepsilon_j x_j \right\|^p \right),
\]

where \( \mathbb{E}' \) denotes the expectation for \( \varepsilon_j' \)'s.

**Theorem 1.1.** (Kahane’s contraction principle) Let \( 1 \leq p < \infty \) and suppose that \( (X_j)_{j=1}^N \) is a sequence of independent symmetric random variables in a Banach space. Then

\[
\mathbb{E}\left( \left\| \sum_{j=1}^N \lambda_j X_j \right\|^p \right) \leq \left( 2 \max_{1 \leq j \leq N} |\lambda_j| \right)^p \mathbb{E}\left( \left\| \sum_{j=1}^N X_j \right\|^p \right)
\]

for any scalars \( \lambda_1, \ldots, \lambda_N \). If the scalars \( \lambda_j \) are real, the constant 2 may be omitted.

**Proof.** Suppose first that \( \lambda_j \)'s are real and with no loss of generality that each \( |\lambda_j| \leq 1 \). We can write \( (\lambda_1, \ldots, \lambda_N) \) as a convex combination

\[
(\lambda_1, \ldots, \lambda_N) = \sum_{k=1}^{2^N} \alpha^{(k)} \varepsilon^{(k)},
\]

where \( \alpha^{(k)} \) and \( \varepsilon^{(k)} \) are identically distributed according to \( \mathcal{P} \). Let \( (\varepsilon_j')_{j=1}^N \) be a sequence of Rademacher variables independent of \( (\lambda_j)_{j=1}^N \) and \( (X_j)_{j=1}^N \). Then \( (\varepsilon_j')_{j=1}^N \) and \( (\lambda_j \varepsilon_j')_{j=1}^N \) are identically distributed. By the independence of the \( \varepsilon_j' \)'s and the \( \lambda_j \)'s we have

\[
\mathbb{E}\left( \left\| \sum_{j=1}^N \lambda_j \varepsilon_j' X_j \right\|^p \right) = \mathbb{E}'\left( \left\| \sum_{j=1}^N \lambda_j \varepsilon_j' X_j \right\|^p \right),
\]

where \( \mathbb{E}' \) denotes the expectation for the \( \varepsilon_j' \)'s. By another application of the contraction principle

\[
\mathbb{E}'\left( \left\| \sum_{j=1}^N \lambda_j X_j \right\|^p \right) \leq \left( 2 \max_{1 \leq j \leq N} |\lambda_j| \right)^p \mathbb{E}'\left( \left\| \sum_{j=1}^N X_j \right\|^p \right).
\]

Finally, we use the fact that \( \mathbb{P} \)-a.e. \( \lambda_j \) is real to conclude the proof.
where $\varepsilon^{(k)} = (\varepsilon_1^{(k)}, \ldots, \varepsilon_N^{(k)}) \in \{-1, 1\}^N$ are the extreme points of the cube $[-1, 1]^N$. Now
\[
\mathbb{E} \left\| \sum_{j=1}^N \lambda_j X_j \right\|^p = \mathbb{E} \left( \sum_{k=1}^{2N} \alpha^{(k)} \sum_{j=1}^N \varepsilon_j^{(k)} X_j \right)^p \\
\leq \mathbb{E} \left( \sum_{k=1}^{2N} \alpha^{(k)} \left\| \sum_{j=1}^N \varepsilon_j^{(k)} X_j \right\| \right)^p \\
\leq \sum_{k=1}^{2N} \alpha^{(k)} \mathbb{E} \left\| \sum_{j=1}^N \varepsilon_j^{(k)} X_j \right\|^p \\
= \mathbb{E} \left\| \sum_{j=1}^N X_j \right\|^p,
\]
where we used Jensen’s inequality and the fact that each $(\varepsilon_j^{(k)} X_j)_{j=1}^N$ is identically distributed.

Remark. Note that Kahane’s contraction principle can be applied to random variables $X_j = \varepsilon_j x_j$, where $(\varepsilon_j)_{j=1}^N$ is a Rademacher sequence and $x_1, \ldots, x_N$ are vectors in a Banach space. For instance we have
\[
\mathbb{E} \left\| \sum_{j=1}^M \varepsilon_j x_j \right\|^p \leq \mathbb{E} \left\| \sum_{j=1}^N \varepsilon_j x_j \right\|^p
\]
whenever $M \leq N$.

We take for granted the following fundamental result, whose proof can be found in Kahane’s book [17] and perhaps more explicitly in the Lecture notes [26], Section 3.2:

**Theorem 1.2. (The Khintchine-Kahane inequality)** For any $1 \leq p, q < \infty$, there exists a constant $K_{p,q}$ such that
\[
\left( \mathbb{E} \left\| \sum_{j=1}^N \varepsilon_j x_j \right\|^q \right)^{1/q} \leq K_{p,q} \left( \mathbb{E} \left\| \sum_{j=1}^N \varepsilon_j x_j \right\|^p \right)^{1/p},
\]
whenever $x_1, \ldots, x_N$ are vectors in a Banach space.

Remark. For $p \leq q$ the result follows from Hölder’s inequality with $K_{p,q} = 1$. The other direction is however non-trivial.

The scalar-version of this result, known as Khintchine’s inequality, states that
\[
\left( \mathbb{E} \left| \sum_{j=1}^N \varepsilon_j \lambda_j \right|^p \right)^{1/p} \approx \left( \sum_{j=1}^N |\lambda_j|^2 \right)^{1/2}
\]
for any scalars $\lambda_1, \ldots, \lambda_N$. While it can be derived from the vector-version by applying the equality of randomized norms and square sums of norms for scalars, it also has a direct proof (see the result 1.10 in the book [7] by Diestel, Jarchow and Tonge).

**Example 1.3.** Khintchine’s inequality can be applied to associate randomized norms of $L^p$-functions to $L^p$-norms of certain square functions as follows: Suppose that $(\Omega, \mu)$ is a measure space and let $f_1, \ldots, f_N \in L^p(\Omega, \mu)$, where $1 \leq p < \infty$. At any point $\xi \in \Omega$ we have
\[
\mathbb{E} \left| \sum_{j=1}^N \varepsilon_j f_j(\xi) \right|^p \approx \left( \sum_{j=1}^N |f_j(\xi)|^2 \right)^{p/2}
\]
by Khintchine’s inequality and so
\[
\left( \mathbb{E} \int_\Omega \left| \sum_{j=1}^N \varepsilon_j f_j(\xi) \right|^p \, d\mu(\xi) \right)^{1/p} \leq \left( \int_\Omega \left( \sum_{j=1}^N |f_j(\xi)|^2 \right)^{p/2} \, d\mu(\xi) \right)^{1/p}.
\]
In other words,
\[
\left( \mathbb{E} \left\| \sum_{j=1}^N \varepsilon_j f_j \right\|_{L^p}^p \right)^{1/p} \leq \left( \sum_{j=1}^N \left| f_j(\cdot) \right|^2 \right)^{1/2} \left\| \mathbb{E} \right\|_{L^p}.
\]
This remains true even if \( f_j \) take values in a Hilbert space (and absolute values are replaced by norms).

An important fact (found for instance in [7], Theorem 12.3) concerning randomized series guarantees us that for a sequence \((x_j)_{j=1}^\infty\) of vectors in a Banach space \(E\), the series \(\sum_{j=1}^\infty \varepsilon_j x_j\) converges almost surely if and only if it converges in \(L^p\) for one (or equivalently, for each) \(p \in [1, \infty)\). It is also interesting to note that Kolmogorov’s zero-one law applied to a series of independent random variables such as above assures that the probability of convergence is either zero or one (see Stromberg [25], Chapter 3). The space of sequences in \(E\) for which the randomized series converges almost surely is denoted by \(\text{Rad}(E)\).

**Proposition 1.4.** The space \(\text{Rad}(E)\) equipped with any of the equivalent norms
\[
\left\| (x_j)_{j=1}^\infty \right\|_{\text{Rad}_p(E)} = \left( \mathbb{E} \left\| \sum_{j=1}^\infty \varepsilon_j x_j \right\|^p \right)^{1/p}, \quad 1 \leq p < \infty,
\]
is a Banach space.

**Proof.** Suppose that \((x^{(k)})_{k=1}^\infty\) is a Cauchy sequence in \(\text{Rad}(E)\), which by definition means that
\[
\left\| x^{(k)} - x^{(l)} \right\|_{\text{Rad}_1(E)} = \mathbb{E} \left\| \sum_{j=1}^\infty \varepsilon_j (x_j^{(k)} - x_j^{(l)}) \right\| \to 0
\]
as \(k, l \to \infty\). Kahane’s contraction principle tells us that for each index \(j\)
\[
\left\| x_j^{(k)} - x_j^{(l)} \right\| \leq \left\| x^{(k)} - x^{(l)} \right\|_{\text{Rad}_1(E)} \to 0
\]
as \(k, l \to \infty\), implying that the sequence \((x_j^{(k)})_{k=1}^\infty\) is Cauchy. By completeness of \(E\) it converges to a vector \(x_j\), thus providing us with a candidate \(x = (x_j)_{j=1}^\infty\) for the limit.

For all positive integers \(M, N\) and \(k\) we have
\[
\mathbb{E} \left\| \sum_{j=M}^N \varepsilon_j x_j \right\| \leq \mathbb{E} \left\| \sum_{j=M}^N \varepsilon_j (x_j^{(k)} - x_j^{(l)}) \right\| + \mathbb{E} \left\| \sum_{j=M}^N \varepsilon_j x_j^{(k)} \right\|.
\]
Given an \(\varepsilon > 0\), we can choose \(k\) large enough so that
\[
\mathbb{E} \left\| \sum_{j=M}^N \varepsilon_j (x_j - x_j^{(k)}) \right\| \leq \liminf_{l \to \infty} \mathbb{E} \left\| \sum_{j=M}^N \varepsilon_j (x_j^{(l)} - x_j^{(k)}) \right\| \leq \liminf_{l \to \infty} \left\| x^{(l)} - x^{(k)} \right\|_{\text{Rad}_1(X)} \leq \varepsilon.
\]
Since \(x^{(k)}\) is in \(\text{Rad}(E)\) we also have
\[
\mathbb{E} \left\| \sum_{j=M}^N \varepsilon_j x_j^{(k)} \right\| \to 0
\]
as \( M, N \to \infty \). By Cauchy’s criterion \( \sum_{j=1}^{\infty} \varepsilon_j x_j \) converges in \( L^1 \) and thus almost surely. Finally, using (3) with \( M = 1 \) we obtain

\[
\|x - x^{(k)}\|_{\text{Rad}_1(E)} = \mathbb{E}\left[ \sum_{j=1}^{\infty} \varepsilon_j (x_j - x_j^{(k)}) \right] \leq \liminf_{N \to \infty} \mathbb{E}\left[ \sum_{j=1}^{N} \varepsilon_j (x_j - x_j^{(k)}) \right] \leq \varepsilon
\]

for \( k \) large enough. \( \square \)

**Remark.** Although the sequences \((x_j)_{j=1}^{\infty}\) in \( \text{Rad}(E) \) are not in general unconditionally summable (only “almost unconditionally summable”), the sequences \((\varepsilon_j x_j)_{j=1}^{\infty}\) of random variables are unconditionally summable in the \( L^p \)-norm for any \( p \in [1, \infty) \). Thus, even when considering sequences indexed by other sets than natural numbers, the space \( \text{Rad}(E) \) remains the same for different orderings of the indices.

Note that for any Hilbert space \( H \), the equality (1) of square sums of norms and randomized norms guarantees that \( \text{Rad}(H) = l^2(H) \).

### 1.2 Type and cotype of a Banach space

The concepts of type and cotype of a Banach space intend to measure how far the randomized norms are from square sums of norms.

**Definition 1.5.** A Banach space \( E \) is said to have

1. **type** \( p \) for \( 1 \leq p \leq 2 \) if there exists a constant \( C \) such that

\[
\left( \mathbb{E}\left[ \left\| \sum_{j=1}^{N} \varepsilon_j x_j \right\|^2 \right] \right)^{1/2} \leq C \left( \sum_{j=1}^{N} \|x_j\|^p \right)^{1/p}
\]

for any vectors \( x_1, \ldots, x_N \) in \( E \), regardless of \( N \). The smallest constant for which this holds is denoted by \( T_p(E) \).

2. **cotype** \( q \) for \( 2 \leq q < \infty \) if there exists a constant \( C \) such that

\[
\left( \sum_{j=1}^{N} \|x_j\|^q \right)^{1/q} \leq C \left( \mathbb{E}\left[ \left\| \sum_{j=1}^{N} \varepsilon_j x_j \right\|^2 \right] \right)^{1/2}
\]

for any vectors \( x_1, \ldots, x_N \) in \( E \), regardless of \( N \). The smallest constant for which this holds is denoted by \( C_q(E) \).

3. **cotype** \( \infty \) if there exists a constant \( C \) such that

\[
\max_{1 \leq j \leq N} \|x_j\| \leq C \left( \mathbb{E}\left[ \sum_{j=1}^{N} \varepsilon_j x_j \right]^2 \right)^{1/2}
\]

for any vectors \( x_1, \ldots, x_N \) in \( E \), regardless of \( N \). The smallest constant for which this holds is denoted by \( C_{\infty}(E) \).

**Remark.** A few observations can be made.

1. Using \( \text{Rad}(E) \), the requirements for type and cotype can be alternatively written as

\[
\|(x_j)_{j=1}^{N}\|_{\text{Rad}_p(E)} \preceq \|(x_j)_{j=1}^{N}\|_{l^p(E)} \quad \text{and} \quad \|(x_j)_{j=1}^{N}\|_{l^q(E)} \preceq \|(x_j)_{j=1}^{N}\|_{\text{Rad}_q(E)},
\]

i.e. that \( l^p(E) \subset \text{Rad}(E) \) and \( \text{Rad}(E) \subset l^q(E) \), respectively. Note that in the definition above, the quantities \( \mathbb{E}\left[ \sum_{j=1}^{N} \varepsilon_j x_j \right]^2 \) could have been replaced by \( \mathbb{E}\left[ \sum_{j=1}^{N} \varepsilon_j x_j \right]^p \) for any \( 1 \leq p < \infty \) by the Khintchine-Kahane inequality (or by the equivalence of \( \text{Rad}_p(E) \)-norms).
2. Every Banach space has both type 1 and cotype $\infty$ with $T_1 = C_\infty = 1$: That every Banach space has type 1 follows directly from the triangle inequality, as

$$
\left(\mathbb{E}\left[\left|\sum_{j=1}^{N} \varepsilon_j x_j\right|^2\right]\right)^{1/2} \leq \left(\mathbb{E}\left[\left|\sum_{j=1}^{N} \varepsilon_j x_j\right|^p\right]\right)^{1/p} = \left(\mathbb{E}\left[\left|\sum_{j=1}^{N} \varepsilon_j x_j\right|^q\right]\right)^{1/q} = \sum_{j=1}^{N} \|x_j\|.
$$

On the other hand, Kahane’s contraction principle shows that

$$
\|x_k\| = \left(\mathbb{E}\|\varepsilon_k x_k\|^2\right)^{1/2} \leq \left(\mathbb{E}\left[\left|\sum_{j=1}^{N} \varepsilon_j x_j\right|^2\right]\right)^{1/2}
$$

whenever $1 \leq k \leq N$ and so every Banach space has cotype $\infty$. Hence a space is said to have non-trivial type (respectively finite cotype) if it has type $p$ for some $p > 1$ (respectively cotype $q$ for some $q < \infty$).

Furthermore, no Banach space (other than $\{0\}$) can have “type” $p > 2$ nor “cotype” $q < 2$. This is immediate when we choose $x_j = x \neq 0$ for $1 \leq j \leq N$, since then

$$
\left(\mathbb{E}\left[\left|\sum_{j=1}^{N} \varepsilon_j x_j\right|^2\right]\right)^{1/2} = N^{1/2} \|x\|
$$

and

$$
\left(\sum_{j=1}^{N} \|x_j\|^p\right)^{1/p} = N^{1/p} \|x\|
$$

so that choosing $N$ large enough will make both defining inequalities impossible.

3. If a space has type $p$ and cotype $q$ then it also has type $\tilde{p}$ and cotype $\tilde{q}$ for any $1 \leq \tilde{p} \leq p$ and $q \leq \tilde{q} \leq \infty$. This follows immediately by setting $\lambda_j = \|x_j\|$ in the well-known inequality

$$
\max_{1 \leq j \leq N} |\lambda_j| \leq \left(\sum_{j=1}^{N} |\lambda_j|^r\right)^{1/r} \leq \left(\sum_{j=1}^{N} |\lambda_j|^s\right)^{1/s} \leq \sum_{j=1}^{N} |\lambda_j|
$$

for real numbers $\lambda_1, \ldots, \lambda_N$ and $1 \leq s \leq r < \infty$.

4. The equality (1) of randomized norms and square sums of norms in Hilbert spaces means of course that they have both type 2 and cotype 2 with $T_2 = C_2 = 1$. It is evident that isomorphic spaces have same types and cotypes. As a consequence, all finite dimensional spaces have type 2 and cotype 2. A remarkable result of Kwapień’s (see the original paper [18], or the new proof by Pisier in [23]) is that a Banach space with both type 2 and cotype 2 is necessarily isomorphic to a Hilbert space.

**Example 1.6.** We will now show that $L^p$-spaces have type min $\{p, 2\}$ and cotype max $\{p, 2\}$. Suppose that $f_1, \ldots, f_N$ are $L^p$-functions with $1 \leq p < \infty$ on some measure space. Recall that the randomized norm compares to the square function as

$$
\left(\mathbb{E}\left[\left|\sum_{j=1}^{N} \varepsilon_j f_j\right|^p\right]\right)^{1/p} \approx \left(\mathbb{E}\left[\left|\sum_{j=1}^{N} |f_j(\cdot)|^2\right|^p\right]^\frac{1}{2}\right)_{L^p}.
$$

We use a simple manipulation

$$
\left(\sum_{j=1}^{N} \|f_j\|_{L^p}^p\right)^{1/p} = \left(\mathbb{E}\left[\left|\sum_{j=1}^{N} |f_j(\cdot)|^p\right|^\frac{1}{p}\right]\right)_{L^p}
$$
so that for $1 \leq p < 2$, the inequality
\[
\left\| \left( \sum_{j=1}^{N} |f_j(\cdot)|^2 \right)^{1/2} \right\|_{L^p} \leq \left\| \left( \sum_{j=1}^{N} |f_j(\cdot)|^p \right)^{1/p} \right\|_{L^p}
\]
gives type $p$, while for $2 < p < \infty$ the reverse inequality
\[
\left\| \left( \sum_{j=1}^{N} |f_j(\cdot)|^p \right)^{1/p} \right\|_{L^p} \leq \left\| \left( \sum_{j=1}^{N} |f_j(\cdot)|^2 \right)^{1/2} \right\|_{L^p},
\]
guarantees cotype $p$. The rest of the cases rely on the identity
\[
\left\| \left( \sum_{j=1}^{N} |f_j(\cdot)|^2 \right)^{1/2} \right\|_{L^p} = \left\| \sum_{j=1}^{N} |f_j(\cdot)|^2 \right\|_{L^{p/2}}^{1/2}.
\]
If $2 < p < \infty$, then $p/2 > 1$ and we can use Minkowski’s inequality to get
\[
\left\| \sum_{j=1}^{N} |f_j(\cdot)|^2 \right\|_{L^{p/2}}^{1/2} \leq \left( \sum_{j=1}^{N} \left\| |f_j(\cdot)|^2 \right\|_{L^{p/2}} \right)^{1/2} = \left( \sum_{j=1}^{N} \left\| f_j \right\|_{L^p}^p \right)^{1/2},
\]
which gives type 2. If $1 \leq p < 2$, then $p/2 < 1$ and the “reverse” Minkowski inequality implies
\[
\left\| \sum_{j=1}^{N} |f_j(\cdot)|^2 \right\|_{L^{p/2}}^{1/2} \geq \left( \sum_{j=1}^{N} \left\| |f_j(\cdot)|^2 \right\|_{L^{p/2}} \right)^{1/2} = \left( \sum_{j=1}^{N} \left\| f_j \right\|_{L^p}^p \right)^{1/2},
\]
thus showing cotype 2.

Remark. Sequence spaces $l^1$ and $l^\infty$ are examples of spaces with only trivial type.

**Proposition 1.7.** If $E$ has type $p \in (1, 2]$, then $E^*$ has cotype $p'$, where $p'$ is the Hölder conjugate of $p$, i.e. it satisfies $\frac{1}{p} + \frac{1}{p'} = 1$.

**Proof.** Let $x_1^*, \ldots, x_N^* \in E^*$ and pick non-negative real numbers $\lambda_1, \ldots, \lambda_N$ such that
\[
\left( \sum_{j=1}^{N} \|x_j^*\|^p \right)^{1/p'} = \sum_{j=1}^{N} \lambda_j \|x_j^*\| \quad \text{and} \quad \left( \sum_{j=1}^{N} \lambda_j^p \right)^{1/p} \leq 1.
\]
Take for each $x_j^*$ a vector $x_j$ with $\|x_j\| = \lambda_j$ and $\langle x_j, x_j^* \rangle \geq \frac{1}{2} \lambda_j \|x_j^*\|$. Now, since $\mathbb{E}(\varepsilon_j \varepsilon_k) = 0$ for $j \neq k$, we have
\[
\left( \sum_{j=1}^{N} \|x_j^*\|^p \right)^{1/p'} = \sum_{j=1}^{N} \lambda_j \|x_j^*\| \leq 2 \sum_{j=1}^{N} \langle x_j, x_j^* \rangle = 2 \mathbb{E} \left\langle \sum_{j=1}^{N} \varepsilon_j x_j, \sum_{k=1}^{N} \varepsilon_k x_k^* \right\rangle.
\]
Furthermore,
\[
\mathbb{E} \left| \left\langle \sum_{j=1}^{N} \varepsilon_j x_j, \sum_{k=1}^{N} \varepsilon_k x_k^* \right\rangle \right| \leq \mathbb{E} \left( \left\| \sum_{j=1}^{N} \varepsilon_j x_j \right\| \left\| \sum_{k=1}^{N} \varepsilon_k x_k^* \right\| \right)
\]
\[
\leq \left( \mathbb{E} \left\| \sum_{j=1}^{N} \varepsilon_j x_j \right\|^2 \right)^{1/2} \left( \mathbb{E} \left\| \sum_{k=1}^{N} \varepsilon_k x_k^* \right\|^2 \right)^{1/2}
\]
\[
\leq T_p(E) \left( \sum_{j=1}^{N} \|x_j\|^p \right)^{1/p} \left( \mathbb{E} \left\| \sum_{k=1}^{N} \varepsilon_k x_k^* \right\|^2 \right)^{1/2}
\]
\[
\leq T_p(E) \left( \mathbb{E} \left\| \sum_{k=1}^{N} \varepsilon_k x_k^* \right\|^2 \right)^{1/2},
\]
as required. \qed
1.3 Finite dimensional subspaces

This section introduces some basic ideas on how the geometry of a Banach space can be studied using properties of its finite dimensional subspaces. Firstly, two Banach spaces \( E \) and \( E' \) are said to be \( \lambda \)-isomorphic for a \( \lambda \geq 1 \) if there exists a bounded isomorphism \( \Lambda : E \to E' \) such that \( \|\Lambda\|\|\Lambda^{-1}\| \leq \lambda \). In the optimal situation of 1-isomorphism, the spaces are of course more familiarly said to be isometrically isomorphic.

For \( 1 \leq p \leq \infty \) we denote by \( l_p^N \) the space of scalar sequences of length \( N \) equipped with the \( p \)-norm \( \| (\lambda_1, \ldots, \lambda_N) \|_p = \left( \sum_{j=1}^{N} |\lambda_j|^p \right)^{1/p} \) for finite \( p \) and \( \| (\lambda_1, \ldots, \lambda_N) \|_\infty = \max_{1 \leq j \leq N} |\lambda_j| \) in case of \( p = \infty \). A Banach space \( E \) is said to contain \( l_p^N \)'s \( \lambda \)-uniformly if there exist linear subspaces \( E_N \) of \( E \) such that for each positive integer \( N \) the subspace \( E_N \) is \( \lambda \)-isomorphic to \( l_p^N \). Trivially, the sequence space \( l_p^N \) contains \( l_p^N \)'s 1-uniformly.

The following theorem of Maurey and Pisier (see [21] for the original proof, or [7], Theorems 13.3 and 14.1) relates this to the concept of type and cotype:

**Theorem 1.8.** Suppose that \( E \) is a Banach space. Then

1. \( E \) has a non-trivial type if and only if it does not contain \( l_2^N \)'s uniformly (i.e. \( \lambda \)-uniformly for some \( \lambda \geq 1 \)).

2. \( E \) has finite cotype if and only if it does not contain \( l_\infty^N \)'s uniformly.

**Proposition 1.9.** Non-trivial type implies finite cotype.

_Proof._ By the previous theorem, it suffices to show that if a Banach space \( E \) contains \( l_\infty^N \)'s uniformly, then it contains \( l_2^N \)’s uniformly. We may further assume that \( E \) is real: If we prove the claim for real spaces and \( E \) is a complex Banach space containing complex \( l_\infty^N \)'s uniformly, then its real counterpart contains real \( l_2^N \)'s uniformly and has thus only trivial type.

To prove the claim, let us fix a positive integer \( N \) and show that \( l_2^N \hookrightarrow l_\infty^{2^{N-1}} \) isometrically. We define a mapping so that the \( j \)’th unit basis element maps to a sequence of \( 2^{N-1} +1 \)'s and \(-1 \)'s which appear in “blocks” of length \( 2^{N-j} \) in the following fashion:

\[
\begin{align*}
(1,0,\ldots,0) &\mapsto (+1,\ldots,+1,\ldots,+1) \\
(0,1,\ldots,0) &\mapsto (+1,\ldots,+1,-1,\ldots,-1) \\
&\quad \vdots \\
(0,\ldots,1,0) &\mapsto (+1,+,1,-1,\ldots,+,1,+,1,-1) \\
(0,\ldots,0,1) &\mapsto (+1,-1,+,1,-1,\ldots,+,1,-1,+,1,-1)
\end{align*}
\]

Thus a sequence \((\lambda_1, \ldots, \lambda_N)\) maps to

\[
\begin{align*}
(\lambda_1 + \lambda_2 + \ldots + \lambda_{N-1} + \lambda_N, \\
\lambda_1 + \lambda_2 + \ldots + \lambda_{N-1} - \lambda_N, \\
\lambda_1 + \lambda_2 + \ldots - \lambda_{N-1} + \lambda_N, \\
&\quad \vdots \\
\lambda_1 - \lambda_2 - \ldots - \lambda_{N-1} + \lambda_N, \\
\lambda_1 - \lambda_2 - \ldots - \lambda_{N-1} - \lambda_N)
\end{align*}
\]

whose \( \infty \)-norm (that is, the maximum of absolute values of its components) is exactly \(|\lambda_1| + \ldots + |\lambda_N| = \| (\lambda_1, \ldots, \lambda_N) \|_1 \) as all the required combinations of signs are present. \( \square \)

**Proposition 1.10.** If \( E^* \) has non-trivial type, then \( E \) has finite cotype.
Proof. Non-trivial type implies finite cotype for the dual by Proposition 1.7 and thus it follows from the assumption that $E^{**}$ has finite cotype. By Theorem 1.8, $E^{**}$ does not contain $l^{N}_{N}$’s uniformly and the same has to hold for its subspace $E$. This means that $E$ must have finite cotype.

In other words, the two propositions above say that if $E$ has only infinite cotype, then both $E$ and $E^*$ have only trivial type.

Evidently, any infinite dimensional Hilbert space contains $l^{2}_{N}$’s uniformly. Banach spaces satisfy almost the same, as the following theorem (see the original proof by Dvoretzky [9] or a new one by Milman [22]) states:

**Theorem 1.11.** (Dvoretzky) Any infinite dimensional Banach space contains $l^{2}_{N}$’s $(1+\varepsilon)$-uniformly for all $\varepsilon > 0$.

Each of the subspaces $E_{N}$ of a Banach space $E$ that are $(1 + \varepsilon)$-isomorphic to $l^{2}_{N}$ is of course complemented, as it is finite dimensional. Although one may thus define projections from $E$ onto each $E_{N}$, the norms of these projections are not necessarily uniformly bounded. When this is the case, i.e. when there exists a constant $C$ such that the projection onto each $E_{N}$ has norm no greater than $C$, we say that $E_{N}$’s are $C$-complemented in $E$. It is shown in [7] Chapter 19, that this is the case exactly when $E$ enjoys a property known as K-convexity, the definition of which is motivated next.

If $H$ is a Hilbert space and $(x_{j})_{j=1}^{\infty} \in \text{Rad}(H) = l^{2}(H)$, then $X = \sum_{j=1}^{\infty} \varepsilon_{j}x_{j}$ is an $L^{2}$-random variable in $H$. The independence of Rademacher variables and the continuity of the expectation guarantees that $x_{k} = \mathbb{E}(\varepsilon_{k}X)$ and so

$$X = \sum_{k=1}^{\infty} \varepsilon_{k}\mathbb{E}(\varepsilon_{k}X).$$

On the other hand, if $X$ is any $L^{2}$-random variable in $H$, then

$$\sum_{j=1}^{\infty} \|\mathbb{E}(\varepsilon_{j}X)\|^{2} \leq \mathbb{E}\|X\|^{2},$$

which means that $(\mathbb{E}(\varepsilon_{j}X))_{j=1}^{\infty}$ is in $\text{Rad}(H)$. Indeed, whenever $1 \leq k \leq N$,

$$\mathbb{E}\varepsilon_{k} \left( X - \sum_{j=1}^{N} \varepsilon_{j}\mathbb{E}(\varepsilon_{j}X) \right) = \mathbb{E}(\varepsilon_{k}X) - \sum_{j=1}^{N} \mathbb{E}(\varepsilon_{k}\varepsilon_{j})\mathbb{E}(\varepsilon_{j}X) = 0,$$

so that $X$ may be written as an orthogonal sum

$$X = \sum_{j=1}^{N} \varepsilon_{j}\mathbb{E}(\varepsilon_{j}X) + \left( X - \sum_{j=1}^{N} \varepsilon_{j}\mathbb{E}(\varepsilon_{j}X) \right).$$

It follows that

$$\mathbb{E}\|X\|^{2} = \mathbb{E}\left\| \sum_{j=1}^{N} \varepsilon_{j}\mathbb{E}(\varepsilon_{j}X) \right\|^{2} + \mathbb{E}\left\| X - \sum_{j=1}^{N} \varepsilon_{j}\mathbb{E}(\varepsilon_{j}X) \right\|^{2} \geq \sum_{j=1}^{N} \|\mathbb{E}(\varepsilon_{j}X)\|^{2}$$

for all positive integers $N$, which proves the claim. We have thus obtained a bounded projection

$$X \mapsto \sum_{j=1}^{\infty} \varepsilon_{j}\mathbb{E}(\varepsilon_{j}X)$$

of $L^{2}$-random variables (on a fixed probability space) onto a closed subspace of random variables given by $\text{Rad}(H)$-elements.
A Banach space in which something similar holds is said to be K-convex. More precisely, a Banach space $E$ is said to be K-convex if for one (and equivalently for all) $p \in (1, \infty)$ there exists a constant $C$ such that whenever $X$ is an $L^p$-random variable in $E$, the sequence $(\mathbb{E}(\varepsilon_j X))_{j=1}^{\infty}$ is in $\text{Rad}_p(E)$ and satisfies

$$ \mathbb{E}\left\| \sum_{j=1}^{\infty} \varepsilon_j \mathbb{E}(\varepsilon_j X) \right\|^p \leq C \mathbb{E}\|X\|^p. $$

The fundamental fact that a Banach space is K-convex if and only if its dual does (Corollary 13.7 and Theorem 13.5). The assumption on K-convexity thus sharpens Dvoretzky’s theorem as follows:

**Theorem 1.12.** If $E$ is an infinite dimensional K-convex Banach space, there exists a constant $C$ such that for any $\varepsilon > 0$, $E$ contains $C$-complemented $(1+\varepsilon)$-isomorphic copies of $l_2^N$’s.

We then turn to study the type of a space of operators. Suppose that $H$ and $E$ are Banach spaces. For $y \in E$ and $x^* \in H^*$ we write

$$(y \otimes x^*)x = \langle x, x^* \rangle y, \quad x \in H.$$  

Clearly $y \otimes x^* \in \mathcal{L}(H, E)$ and $\|y \otimes x^*\| \leq \|y\\|\|x^*\|$. We can also embed $H^*$ and $E$ isometrically into $\mathcal{L}(H, E)$ by fixing respectively a unit vector $y \in E$ or a functional $x^* \in H^*$ with unit norm and writing

$$H^* \simeq y \otimes H^* := \{y \otimes x^* : x^* \in H^*\} \subset \mathcal{L}(H, E)$$

and

$$E \simeq E \otimes x^* := \{y \otimes x^* : y \in E\} \subset \mathcal{L}(H, E).$$

**Proposition 1.13.** If $H$ and $E$ are infinite dimensional Banach spaces, then $\mathcal{L}(H, E)$ has only trivial type.

**Proof.** Suppose first that $H$ is K-convex and let $\lambda > 1$. By Dvoretzky’s theorem, both $H$ and $E$ contain $l_2^N$’s $\lambda$-uniformly. More precisely, there exist sequences $(H_N)_{N=1}^{\infty}$ and $(E_N)_{N=1}^{\infty}$ of subspaces of $H$ and $E$, such that each $H_N$ and $E_N$ is $\lambda$-isomorphic to $l_2^N$. Now, as $H$ is K-convex, we may further assume that for some constant $C$, each $H_N$ is C-complemented in $H$ so that the projection $P_N$ onto $H_N$ has norm less or equal to $C$. We can then embed $\mathcal{L}(H_N, E_N)$ in $\mathcal{L}(H, E)$ by extending an operator $T \in \mathcal{L}(H_N, E_N)$ to $\tilde{T} = TP_N$ so that $\|\tilde{T}\| \leq C\|T\|$. Fix an $N$ and denote the isomorphisms from $H_N$ and $E_N$ to $l_2^N$ by $\Lambda_N^H$ and $\Lambda_N^E$, respectively. Define

$$\Lambda : \mathcal{L}(l_2^N, l_2^N) \to \mathcal{L}(H_N, E_N)$$

by $\Lambda(T) = (\Lambda_N^E)^{-1} T \Lambda_N^H$. Then $\Lambda^{-1}(S) = \Lambda_N^E S (\Lambda_N^H)^{-1}$ and

$$\|\Lambda\|\|\Lambda^{-1}\| \leq \|(\Lambda_N^E)^{-1}\|\|\Lambda_N^H\|\|\Lambda_N^E\|\|\Lambda_N^H\|^{-1} \leq \lambda^2.$$

As every sequence in $l_2^N$ defines a (diagonal) operator in $\mathcal{L}(l_2^N, l_2^N)$ with same operator norm, we have $l_2^N \hookrightarrow \mathcal{L}(l_2^N, l_2^N)$ isometrically. Thus $\mathcal{L}(H, E)$ contains $l_2^N$’s $C\lambda^2$-uniformly and cannot then by Theorem 1.8 have finite cotype and hence neither non-trivial type.

Suppose then, that $H$ is not K-convex. Then $H^*$ is not K-convex either, has only trivial type and contains $l_1^N$’s uniformly. But $H^* \hookrightarrow \mathcal{L}(H, E)$ isometrically and so $\mathcal{L}(H, E)$ has also only trivial type. \qed
1.4 R-boundedness

In many cases the uniform bound of a family of operators has to be replaced by its R-bound (originally defined by Berkson and Gillespie in [2]).

**Definition 1.14.** A family $\mathcal{T}$ of operators in $L(H,E)$ is said to be **R-bounded** if there exists a constant $C$ such that for any $T_1, \ldots, T_N \in \mathcal{T}$ and any $x_1, \ldots, x_N \in H$, regardless of $N$, we have

$$
\mathbb{E}\left\| \sum_{j=1}^{N} \varepsilon_j T_j x_j \right\|_p \leq C \mathbb{E}\left\| \sum_{j=1}^{N} \varepsilon_j x_j \right\|_p,
$$

for some $p \in [1, \infty)$. The smallest such constant is denoted by $\mathcal{R}_p(\mathcal{T})$. We denote $\mathcal{R}_2$ by $\mathcal{R}$ in short later on.

**Remark.** Next we take a look at some basic properties of R-bounds.

1. Note that Kahane's contraction principle says that the family of operators given by scalar multiplication is R-bounded when the scalars are chosen from a bounded set.

2. By the Khintchine-Kahane inequality, the R-boundedness of a family does not depend on $p$, and the constants $\mathcal{R}_p(\mathcal{T})$ are comparable.

3. Using the triangle inequality, we obtain at once that for any two families $\mathcal{T}$ and $\mathcal{S}$ of bounded linear operators we have $\mathcal{R}_p(\mathcal{T} + \mathcal{S}) \leq \mathcal{R}_p(\mathcal{T}) + \mathcal{R}_p(\mathcal{S})$. Furthermore, applying the definition of R-boundedness first to $\mathcal{T}$ and then to $\mathcal{S}$ we see that $\mathcal{R}_p(\mathcal{T} \mathcal{S}) \leq \mathcal{R}_p(\mathcal{T}) \mathcal{R}_p(\mathcal{S})$ (whenever the compositions make sense).

4. Any operator $T \in L(H,E)$ forms by itself an R-bounded set with $\mathcal{R}_p(\{T\}) = \|T\|$. We may always assume that $0 \in \mathcal{T}$ so that $\mathcal{R}_p(\mathcal{T} \cup \{0\}) = \mathcal{R}_p(\mathcal{T})$: Let $T_1, \ldots, T_N \in \mathcal{T} \cup \{0\}$ and denote by $J$ the indices $j$ for which $T_j \neq 0$. For any vectors $x_1, \ldots, x_N$ in $H$ we now have

$$
\mathbb{E}\left\| \sum_{j=1}^{N} \varepsilon_j T_j x_j \right\|_p = \mathbb{E}\left\| \sum_{j \in J} \varepsilon_j T_j x_j \right\|_p \leq \mathcal{R}_p(\mathcal{T}) \mathbb{E}\left\| \sum_{j \in J} \varepsilon_j x_j \right\|_p \leq \mathcal{R}_p(\mathcal{T}) \mathbb{E}\left\| \sum_{j=1}^{N} \varepsilon_j x_j \right\|_p,
$$

where the last step was obtained using Kahane's contraction principle. Thus the union of R-bounded sets $\mathcal{T}$ and $\mathcal{S}$ (containing 0) is also R-bounded as $\mathcal{T} \cup \mathcal{S} \subset \mathcal{T} + \mathcal{S}$. In particular, any finite family of operators is R-bounded with R-bound at most the sum of their operator norms. Consequently, for any positive integer $N$ the function

$$
L(H,E)^N \to \mathbb{R} : (T_1, \ldots, T_N) \mapsto \mathcal{R}_p(\{T_1, \ldots, T_N\})
$$

is continuous.

By similar reasoning, every summable sequence of operators is R-bounded:

$$
\mathcal{R}_p(\{T_j\}_{j=1}^{\infty}) \leq \sum_{j=1}^{\infty} \|T_j\|.
$$

5. For R-boundedness of a family $\mathcal{T} \subset L(H,E)$ it suffices to check the inequality in the definition for distinct operators in $\mathcal{T}$. Indeed, suppose that the inequality holds for distinct operators and let $T_1, \ldots, T_N \in \mathcal{T}$. Write $J_k$ for the set of indices $j$ for which $T_j = S_k$, $S_1, \ldots, S_M$
being the distinct operators in \(\{T_1, \ldots, T_N\}\). Now randomization gives

\[
\mathbb{E}\left\| \sum_{j=1}^{N} \varepsilon_j T_j x_j \right\|^p = \mathbb{E}\left\| \sum_{k=1}^{M} S_k \sum_{j \in J_k} \varepsilon_j x_j \right\|^p
\]

\[
= \mathbb{E}\mathbb{E}' \left\| \sum_{k=1}^{M} \varepsilon'_k S_k \sum_{j \in J_k} \varepsilon_j x_j \right\|^p
\]

\[
\leq C^p \mathbb{E}\mathbb{E}' \left\| \sum_{k=1}^{M} \varepsilon'_k \sum_{j \in J_k} \varepsilon_j x_j \right\|^p
\]

\[
= C^p \mathbb{E}\left\| \sum_{j=1}^{N} \varepsilon_j x_j \right\|^p,
\]

as required.

6. Boundedly scaled R-bounded families of operators remain R-bounded. More precisely, if a family \(\mathcal{T} \subset \mathcal{L}(H,E)\) is R-bounded, then

\[
\mathcal{R}_p \left( \{ \lambda T : T \in \mathcal{T}, |\lambda| \leq r \} \right) \leq 2 r \mathcal{R}_p(\mathcal{T}).
\]

Indeed, by Kahane’s contraction principle

\[
\mathbb{E}\left\| \sum_{j=1}^{N} \varepsilon_j \lambda_j T_j x_j \right\|^p \leq (2r)^p \mathbb{E}\left\| \sum_{j=1}^{N} \varepsilon_j T_j x_j \right\|^p \leq (2r)^p \mathcal{R}_p(\mathcal{T}) \mathbb{E}\left\| \sum_{j=1}^{N} \varepsilon_j x_j \right\|^p
\]

for any operators \(T_1, \ldots, T_N\) in \(\mathcal{T}\), any vectors \(x_1, \ldots, x_N\) and any scalars \(\lambda_1, \ldots, \lambda_N\) with \(|\lambda_j| \leq r\). As in Kahane’s contraction principle, the factor 2 may be omitted if the scalars \(\lambda\) are real.

7. R-boundedness can be phrased in terms of Rad-spaces: A family \(\mathcal{T} \subset \mathcal{L}(H,E)\) of linear operators is R-bounded if and only if

\[
(x_j)_{j=1}^{\infty} \mapsto (T_j x_j)_{j=1}^{\infty}
\]

defines a bounded linear operator from \(\text{Rad}(H)\) to \(\text{Rad}(E)\) for any sequence \((T_j)_{j=1}^{\infty} \subset \mathcal{T}\).

The operator norm with respect to \(\text{Rad}_p\)-norms then equals \(\mathcal{R}_p(\mathcal{T})\).

Example 1.15. R-boundedness for operators on \(L^p\)-spaces can be formulated using the square function: A family \(\mathcal{T}\) of bounded linear operators from some \(L^q\) to some \(L^p\) with \(1 \leq p, q < \infty\), is R-bounded if and only if

\[
\left\| \left( \sum_{j=1}^{N} |T_j f_j(\cdot)|^2 \right)^{1/2} \right\|_{L^p} \lesssim \left\| \left( \sum_{j=1}^{N} |f_j(\cdot)|^2 \right)^{1/2} \right\|_{L^q}
\]

for any \(T_1, \ldots, T_N \in \mathcal{T}\) and any \(L^q\)-functions \(f_1, \ldots, f_N\).

We will then compare R-boundedness and uniform boundedness. Firstly, any R-bounded set is seen to be uniformly bounded by taking \(N = 1\):

\[
\|T x\| = \left( \mathbb{E}\|\varepsilon T x\|^p \right)^{1/p} \leq \mathcal{R}_p(\mathcal{T}) \left( \mathbb{E}\|\varepsilon x\|^p \right)^{1/p} = \mathcal{R}_p(\mathcal{T}) \|x\|
\]

Thus

\[
\sup_{T \in \mathcal{T}} \|T\|_{\mathcal{L}(H,E)} \leq \mathcal{R}_p(\mathcal{T})
\]

for any \(1 \leq p < \infty\).

In Hilbert spaces also the converse holds. More generally, the following result is proven by Arendt and Bu in [1] (while the authors credit the proof to Pisier):

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Proposition 1.16. Suppose that $H$ and $E$ are Banach spaces. The following conditions are equivalent:

1. $H$ has cotype 2 and $E$ has type 2.

2. Every uniformly bounded family of linear operators in $\mathcal{L}(H, E)$ is R-bounded.

Proof. We begin by showing that the first condition implies the second. Suppose that $\mathcal{T}$ is a family of operators in $\mathcal{L}(H, E)$ with $\sup_{\mathcal{T} \in \mathcal{T}} \|T\| \leq M$. Pick operators $T_1, \ldots, T_N$ from $\mathcal{T}$ and let $x_1, \ldots, x_N$ be vectors in $H$. Now, using the type 2 of $E$, we see that

$$\left( \left\| \sum_{j=1}^{N} \varepsilon_j T_j x_j \right\|^2 \right)^{1/2} \leq T_2(E) \left( \sum_{j=1}^{N} \|T_j x_j\|^2 \right)^{1/2}.$$ 

Uniform boundedness of $\mathcal{T}$ gives

$$\left( \sum_{j=1}^{N} \|T_j x_j\|^2 \right)^{1/2} \leq \left( \sum_{j=1}^{N} \|T_j\|^2 \|x_j\|^2 \right)^{1/2} \leq M \left( \sum_{j=1}^{N} \|x_j\|^2 \right)^{1/2}.$$ 

Finally, since $H$ has cotype 2, we get

$$\left( \sum_{j=1}^{N} \|x_j\|^2 \right)^{1/2} \leq C_2(H) \left( \sum_{j=1}^{N} \|\varepsilon_j y_j\|^2 \right)^{1/2}.$$ 

Combining these gives the desired inequality.

We then turn to the converse. Let us first prove the assertion for the type of $E$. To this end, let $x^* \in H^*$ and $x \in H$ be such that $\|x^*\| = \langle x, x^* \rangle = 1$. The family $\mathcal{T} = \{y \otimes x^* : y \in E, \|y\| = 1\}$ is thus uniformly bounded by 1 and hence also R-bounded. We are now ready to check the condition for type so let us pick vectors $y_1, \ldots, y_N$ from $E$. Choose operators $T_j = \|y_j\|^{-1} y_j \otimes x^*$ from $\mathcal{T}$ and note that $y_i = T_j(\|y_j\| x)$. Now

$$\left( \left\| \sum_{j=1}^{N} \varepsilon_j y_j \right\|^2 \right)^{1/2} = \left( \left\| \sum_{j=1}^{N} \varepsilon_j T_j(\|y_j\| x) \right\|^2 \right)^{1/2} \leq \mathcal{R}(\mathcal{T}) \left( \left\| \sum_{j=1}^{N} \varepsilon_j \|y_j\| x \right\|^2 \right)^{1/2} = \mathcal{R}(\mathcal{T}) \|x\| \left( \sum_{j=1}^{N} \|\varepsilon_j y_j\|^2 \right)^{1/2} = \mathcal{R}(\mathcal{T}) \|x\| \left( \sum_{j=1}^{N} \|y_j\|^2 \right)^{1/2},$$ 

where $x$ can be chosen to have norm arbitrarily close to 1.

We finish by proving the claim for the cotype of $H$. Suppose that $y$ is a unit vector in $E$ and write $\mathcal{S} = \{y \otimes x : x \in H^*, \|x\| = 1\}$. The family $\mathcal{S}$ is uniformly bounded by 1 and thus R-bounded. Let then $x_1, \ldots, x_N$ be vectors in $H$ and choose functionals $x_1^*, \ldots, x_N^* \in H^*$ so that $\langle x_j, x_j^* \rangle = \|x_j\|$ and $\|x_j^*\| = 1$. Pick now operators $S_j = y \otimes x_j^*$ from $\mathcal{S}$ and observe that
Thus
\[
\left( \sum_{j=1}^{N} \|x_j\|^2 \right)^{1/2} = \left( \mathbb{E} \left\| \sum_{j=1}^{N} \varepsilon_j \|x_j\|y \right\| \right)^{1/2} \\
= \left( \mathbb{E} \left\| \sum_{j=1}^{N} \varepsilon_j \|x_j\|y \right\|^2 \right)^{1/2} \\
= \left( \mathbb{E} \left\| \sum_{j=1}^{N} \varepsilon_j S_jx_j \right\|^2 \right)^{1/2} \\
\leq R(\mathcal{S}) \left( \mathbb{E} \left\| \sum_{j=1}^{N} \varepsilon_j x_j \right\|^2 \right)^{1/2},
\]
as required. \(\square\)

Remark. It is clear from above that if \(H\) and \(E\) have cotype 2 and type 2, respectively, and if \(X \subset \mathcal{L}(H, E)\) is a Banach space whose norm dominates the operator norm, then all uniformly \((X-)\) bounded sets are also R-bounded.

There are at least two natural ways to use R-boundedness for sets of vectors in \(E\). One can fix a functional \(x^*\) with unit norm on any Banach space \(H\) and use the embedding \(E \simeq E \otimes x^* \subset \mathcal{L}(H, E)\). Doing so, a set \(S\) of vectors in \(E\) is R-bounded if there exists a constant \(C\) such that
\[
\mathbb{E} \left\| \sum_{j=1}^{N} \varepsilon_j (y_j \otimes x^*) x_j \right\|^p \leq C^p \mathbb{E} \left\| \sum_{j=1}^{N} \varepsilon_j x_j \right\|^p
\]
for any choice of vectors \(y_1, \ldots, y_N \in S\) and \(x_1, \ldots, x_N \in H\).

In particular, one can choose the scalar field for \(H\). As linear operators from the scalars to \(E\) are of the form \(\lambda \mapsto \lambda y\) for some \(y \in E\), it makes sense to call a set \(S\) of vectors in \(E\) R-bounded if there exists a constant \(C\) such that
\[
\mathbb{E} \left\| \sum_{j=1}^{N} \varepsilon_j \lambda_j y_j \right\|^p \leq C^p \mathbb{E} \left\| \sum_{j=1}^{N} \varepsilon_j \lambda_j \right\|^p
\]
for all vectors \(y_1, \ldots, y_N \in S\) and all scalars \(\lambda_1, \ldots, \lambda_N\).

These two conditions are equivalent:

**Lemma 1.17.** Suppose that \(S\) is a subset of a Banach space \(E\) and that \(x^* \in H^*\) has unit norm, where \(H\) is some Banach space. Then for any \(y_1, \ldots, y_N \in S\) and any positive real number \(C\) the inequality (4) holds for any \(x_1, \ldots, x_N \in H\) if and only if the inequality (5) holds for any scalars \(\lambda_1, \ldots, \lambda_N\).

**Proof.** Suppose first that (4) holds for any \(x_1, \ldots, x_N \in H\) and let \(\lambda_1, \ldots, \lambda_N\) be scalars. As \(x^*\) has unit norm, there exists for any \(\varepsilon > 0\) a vector \(x \in H\) with \(\langle x, x^* \rangle = 1\) and \(\|x\| < 1 + \varepsilon\). Put
$x_j = \lambda_j x$. Then

$$\mathbb{E} \left\| \sum_{j=1}^{N} \varepsilon_j \lambda_j y_j \right\|^p = \mathbb{E} \left\| \sum_{j=1}^{N} \varepsilon_j (x_j, x^*) y_j \right\|^p$$

$$= \mathbb{E} \left\| \sum_{j=1}^{N} \varepsilon_j (y_j \otimes x^*) x_j \right\|^p$$

$$\leq C^p \mathbb{E} \left\| \sum_{j=1}^{N} \varepsilon_j x_j \right\|^p$$

$$= C^p \mathbb{E} \left\| \sum_{j=1}^{N} \varepsilon_j \lambda_j x \right\|^p$$

$$= C^p (1 + \varepsilon)^p \mathbb{E} \left\| \sum_{j=1}^{N} \varepsilon_j \lambda_j \right\|^p.$$ 

Suppose then that (5) holds for any scalars $\lambda_1, \ldots, \lambda_N$ and that $x_1, \ldots, x_N$ are vectors in $H$. Applying (5) with $\lambda_j = \langle x_j, x^* \rangle$ we get

$$\mathbb{E} \left\| \sum_{j=1}^{N} \varepsilon_j (y_j \otimes x^*) x_j \right\|^p = \mathbb{E} \left\| \sum_{j=1}^{N} \varepsilon_j (x_j, x^*) y_j \right\|^p$$

$$\leq C^p \mathbb{E} \left\| \sum_{j=1}^{N} \varepsilon_j (x_j, x^*) \right\|^p$$

$$= C^p \mathbb{E} \left\| \sum_{j=1}^{N} \varepsilon_j x_j \right\|^p$$

$$\leq C^p \mathbb{E} \left\| \sum_{j=1}^{N} \varepsilon_j x_j \right\|^p.$$ 

Equipped with sufficient knowledge about Banach space geometry and R-boundedness, we now turn to study functions taking values in spaces of operators.
2 Rademacher maximal function on Euclidean spaces

Suppose that $H$ and $E$ are Banach spaces and that $\mathcal{X} \subset L(H, E)$ is a Banach space whose norm dominates the operator norm. We are mostly interested in the case $\mathcal{X} \simeq E$, i.e. when $\mathcal{X} = E \otimes x^*$ for some $x^* \in H^*$ or $H$ is the scalar field. Another typical choice for $\mathcal{X}$ is $L(H, E)$ itself. Further, when $H$ is a Hilbert space, we can take the so-called $\gamma$-radonifying operators for our $\mathcal{X}$ (for the definition, see Linde and Pietsch [19], Lecture notes [26] Chapter 5 or the book [7] Chapter 12). Their natural norm is not equivalent to the operator norm, thus giving us a non-trivial example of an interesting $\mathcal{X}$. Finally, for Hilbert spaces $H_1$ and $H_2$ one can consider the Schatten - von Neumann classes $S_p(H_1, H_2)$ with $1 \leq p < \infty$ (see [7] Chapter 4).

We equip $\mathbb{R}^n$ with the Lebesgue measure and denote by $L^p(\mathcal{X})$ the Lebesgue-Bochner space consisting of (equivalence classes of) strongly measurable functions from $\mathbb{R}^n$ to $\mathcal{X}$ whose pointwise norm to the $p$th power, $1 \leq p \leq \infty$, is integrable (or in the case of $p = \infty$, the functions whose pointwise norm is essentially bounded). For an integer $j$, we denote by $\mathcal{D}_j$ the collection of dyadic cubes with edges of length $2^{-j}$ in $\mathbb{R}^n$. More precisely, $\mathcal{D}_j = \{2^{-j}([0,1]^n + m) : m \in \mathbb{Z}^n \}$. The family of all dyadic cubes can then be expressed as a union

$$\mathcal{D} = \bigcup_{j \in \mathbb{Z}} \mathcal{D}_j.$$

It is crucial that every $\xi \in \mathbb{R}^n$ is contained in exactly one cube in each $\mathcal{D}_j$ and that two dyadic cubes are either disjoint or one is included in the other. From now on, we will only deal with dyadic cubes and will not stress this specifically every time. We write $\langle f \rangle_Q$ for the average of a locally integrable function $f : \mathbb{R}^n \rightarrow \mathcal{X}$ over a dyadic cube $Q$, that is

$$\langle f \rangle_Q = \frac{1}{|Q|} \int_Q f(\eta) \, d\eta.$$

Recall the definition of the standard dyadic maximal function:

$$Mf(\xi) = \sup_{Q \ni \xi} \|\langle f \rangle_Q\|, \quad \xi \in \mathbb{R}^n.$$

**Proposition 2.1.** The maximal operator $f \mapsto Mf$ is bounded from $L^p(\mathcal{X})$ to $L^p$ whenever $1 < p \leq \infty$, regardless of $\mathcal{X}$.

**Proof.** We first claim that it satisfies for all $f \in L^1(\mathcal{X})$ a so-called weak $(1,1)$-inequality:

$$|\{\xi \in \mathbb{R}^n : Mf(\xi) > \lambda\}| \leq \frac{1}{\lambda} \|f\|_{L^1(\mathcal{X})},$$

whenever $\lambda > 0$. Indeed, decompose $\{\xi \in \mathbb{R}^n : Mf(\xi) > \lambda\}$ into disjoint sets $\bigcup A_j$, where for each $j \in \mathbb{Z}$,

$$A_j = \left\{ Q \in \mathcal{D}_j : \sup_{R \supseteq Q} \|\langle f \rangle_R\| \leq \lambda < \|\langle f \rangle_Q\| \right\}$$

is the collection of “maximal” dyadic cubes in $\mathcal{D}_j$. In other words, for each $\xi$ with $Mf(\xi) > \lambda$, we take in our decomposition the largest dyadic cube $Q$ that contains $\xi$ and satisfies $\|\langle f \rangle_Q\| > \lambda$. This is well-defined since $\|\langle f \rangle_R\|$ tends to zero as $R$ gets larger. Observe that for any $Q \in A_j$ we have

$$|Q| \leq |Q| \frac{\|\langle f \rangle_Q\|}{\lambda} \leq \frac{1}{\lambda} \int_Q \|f(\xi)\| \, d\xi.$$

Thus

$$|\{\xi \in \mathbb{R}^n : Mf(\xi) > \lambda\}| = \sum_{j \in \mathbb{Z}} \sum_{Q \in A_j} |Q| \leq \sum_{j \in \mathbb{Z}} \sum_{Q \in A_j} \frac{1}{\lambda} \int_Q \|f(\xi)\| \, d\xi \leq \frac{1}{\lambda} \|f\|_{L^1(\mathcal{X})}.$$

Since $\|\langle f \rangle_Q\| \leq \|f\|_{L^1(\mathcal{X})}$ is it evident that $M$ maps $L^\infty(\mathcal{X})$ boundedly to $L^\infty$. We may then use the Marcinkiewicz interpolation theorem (see the Appendix) to conclude that $M$ maps $L^p(\mathcal{X})$ boundedly to $L^p$ whenever $1 < p \leq \infty$. \qed
The main object of our study is in the Euclidean case given as follows:

**Definition 2.2.** The Rademacher maximal function is defined by

\[ M_R f(\xi) = R \left( \langle f \rangle_Q : Q \ni \xi \right) =: R \left( \langle f \rangle_Q : Q \ni \xi \right), \quad \xi \in \mathbb{R}^n, \]

for locally integrable \( \mathcal{X} \)-valued functions \( f \).

**Remark.** Some immediate observations are listed below.

1. Each of the truncated versions

\[ M'_R f(\xi) = R \left( \langle f \rangle_Q : Q \ni \xi, Q \in \mathcal{C} \right), \quad \xi \in \mathbb{R}^n, \]

with finite \( \mathcal{C} \subset \mathcal{D} \) is constant on dyadic cubes of some \( \mathcal{D}_j \) (namely the one that contains the smallest cube(s) in \( \mathcal{C} \)). Thus the truncated versions are measurable and as \( M_R f(\xi) = \sup M'_R f(\xi) \), where the supremum is taken over all finite \( \mathcal{C} \subset \mathcal{D} \), also \( M_R f \) is measurable.

2. As R-bounds of finite sets of operators are finite, each truncated maximal function \( M'_R f < \infty \) everywhere for any locally integrable \( f \). Furthermore, if \( f \in L^p(\mathcal{X}) \) for some \( p \in [1, \infty) \), then

\[ R \left( \langle f \rangle_Q : Q \ni \xi, |Q| \geq 2^{nk} \right) \leq \sum_{Q \ni \xi, |Q| \geq 2^{nk}} \|\langle f \rangle_Q\| \leq \|f\|_{L^p(\mathcal{X})} \sum_{j=0}^{\infty} 2^{-nj/p} < \infty, \]

for every \( \xi \in \mathbb{R}^n \) and \( k \in \mathbb{Z} \).

3. By the properties of R-bounds we obtain the pointwise relation \( M f \leq M_R f \). If \( H \) has cotype 2 and \( E \) has type 2 it follows from Proposition 1.16 (and the following remark) that \( M_R f \lesssim M f \). This is the case in particular, when \( H = L^q \) for \( 1 \leq q \leq 2 \) and \( E = L^p \) for \( 2 \leq p < \infty \) over some measure spaces.

We are interested in the boundedness of the Rademacher maximal operator \( f \mapsto M_R f \) from \( L^p(\mathcal{X}) \) to \( L^p \).

**Definition 2.3.** Let \( 1 < p < \infty \). A Banach space \( \mathcal{X} \subset L(H, E) \) is said to have RMF\(_p\) with respect to \( \mathbb{R}^n \) if the Rademacher maximal operator is bounded from \( L^p(\mathcal{X}) \) to \( L^p \).

We will see later on that not every \( \mathcal{X} \) has RMF\(_p\). Observe that this property at least seemingly depends not only on the exponent \( p \) but also on the dimension \( n \). Independence from both of these will be shown.

This maximal function was originally studied by Hytönen, McIntosh and Portal [15] in the case \( \mathcal{X} \simeq E \). Why then, if R-boundedness can be defined for Banach spaces (via the identification above), are we considering spaces of operators? The reason is that if the space happens to have an intrinsic concept of R-boundedness (like a space of operators) then it is more natural than the R-boundedness induced by the identification. Indeed, for infinite dimensional Banach spaces \( H \) and \( E \), the space \( L(H, E) \) has only trivial type and cannot thus have RMF\(_p\) via the identification for it will be shown that spaces with RMF\(_p\) always have non-trivial type.

Trivially, the RMF\(_p\)-property inherits to closed subspaces. In particular, if \( L(H, E) \) has RMF\(_p\), then both \( E \) and \( H^* \) have it.

### 2.1 A weak type inequality

In this section we show that the RMF\(_p\)-property (for a fixed \( n \)) implies a certain weak type inequality for the maximal operator. We start with a dyadic version of Lebesgue’s differentiation theorem.
Suppose first that \( f \) is continuous. Then for any \( \varepsilon > 0 \) we can find a \( \delta > 0 \) so that
\[
\|f(\eta) - f(\xi)\| < \varepsilon, \text{ whenever } |\eta - \xi| < \delta.
\]
Hence
\[
\|\langle f \rangle_Q - f(\xi)\| \leq \frac{1}{|Q|} \int_Q \|f(\eta) - f(\xi)\| \, d\eta \leq \varepsilon
\]
for small enough dyadic cubes \( Q \ni \xi \).

Let then \( f \in L^1(X) \) and approximate it with a continuous function \( g \) in \( L^1 \)-norm. Writing
\[
\langle f \rangle_Q - f(\xi) = (f - g)_Q + (g)_Q - g(\xi) + g(\xi) - f(\xi)
\]
and using the fact that \( \langle g \rangle_Q \to g(\xi) \), as \( Q \to \{\xi\} \) we get
\[
\limsup_{Q \to \{\xi\}} \|\langle f \rangle_Q - f(\xi)\| = \limsup_{Q \to \{\xi\}} \|(f - g)_Q\| + \|g(\xi) - f(\xi)\|.
\]
Thus, for any \( \lambda > 0 \),
\[
\left| \left\{ \xi \in \mathbb{R}^n : \limsup_{Q \to \{\xi\}} \|\langle f \rangle_Q - f(\xi)\| > \lambda \right\} \right| \leq \left| \left\{ \xi \in \mathbb{R}^n : M(f - g)(\xi) > \lambda/2 \right\} \right|
+ \left| \left\{ \xi \in \mathbb{R}^n : \|g(\xi) - f(\xi)\| > \lambda/2 \right\} \right|
\leq \frac{4}{\lambda} \|f - g\|_{L^1},
\]
which can be made as small as we like by choosing a suitable \( g \). Now
\[
\left| \left\{ \xi \in \mathbb{R}^n : \limsup_{Q \to \{\xi\}} \|\langle f \rangle_Q - f(\xi)\| > 0 \right\} \right| \leq \sum_{k=1}^{\infty} \left| \left\{ \xi \in \mathbb{R}^n : \limsup_{Q \to \{\xi\}} \|\langle f \rangle_Q - f(\xi)\| > 1/k \right\} \right| = 0,
\]
as required. Further, the question is of a local nature and thus holds immediately for all locally integrable \( f \).

A well-known decomposition result is needed:

**Proposition 2.4.** If \( f \) is any locally integrable \( X \)-valued function, then for almost every \( \xi \in \mathbb{R}^n \) we have \( \langle f \rangle_Q \to f(\xi) \), as \( Q \to \{\xi\} \) i.e. as \( |Q| \to 0 \) with \( Q \ni \xi \).

**Proof.** Suppose first that \( f \) is continuous. Then for any \( \varepsilon > 0 \) we can find a \( \delta > 0 \) so that
\[
\|f(\eta) - f(\xi)\| < \varepsilon, \text{ whenever } |\eta - \xi| < \delta.
\]
Hence
\[
\|\langle f \rangle_Q - f(\xi)\| \leq \frac{1}{|Q|} \int_Q \|f(\eta) - f(\xi)\| \, d\eta \leq \varepsilon
\]
for small enough dyadic cubes \( Q \ni \xi \).

Let then \( f \in L^1(X) \) and approximate it with a continuous function \( g \) in \( L^1 \)-norm. Writing
\[
\langle f \rangle_Q - f(\xi) = (f - g)_Q + (g)_Q - g(\xi) + g(\xi) - f(\xi)
\]
and using the fact that \( \langle g \rangle_Q \to g(\xi) \), as \( Q \to \{\xi\} \) we get
\[
\limsup_{Q \to \{\xi\}} \|\langle f \rangle_Q - f(\xi)\| = \limsup_{Q \to \{\xi\}} \|(f - g)_Q\| + \|g(\xi) - f(\xi)\|.
\]
Thus, for any \( \lambda > 0 \),
\[
\left| \left\{ \xi \in \mathbb{R}^n : \limsup_{Q \to \{\xi\}} \|\langle f \rangle_Q - f(\xi)\| > \lambda \right\} \right| \leq \left| \left\{ \xi \in \mathbb{R}^n : M(f - g)(\xi) > \lambda/2 \right\} \right|
+ \left| \left\{ \xi \in \mathbb{R}^n : \|g(\xi) - f(\xi)\| > \lambda/2 \right\} \right|
\leq \frac{4}{\lambda} \|f - g\|_{L^1},
\]
which can be made as small as we like by choosing a suitable \( g \). Now
\[
\left| \left\{ \xi \in \mathbb{R}^n : \limsup_{Q \to \{\xi\}} \|\langle f \rangle_Q - f(\xi)\| > 0 \right\} \right| \leq \sum_{k=1}^{\infty} \left| \left\{ \xi \in \mathbb{R}^n : \limsup_{Q \to \{\xi\}} \|\langle f \rangle_Q - f(\xi)\| > 1/k \right\} \right| = 0,
\]
as required. Further, the question is of a local nature and thus holds immediately for all locally integrable \( f \).

A well-known decomposition result is needed:

**Theorem 2.5.** (Calderón-Zygmund decomposition) For any non-negative function \( f \) in \( L^1 \) and any positive real number \( \lambda \) there exists a disjoint family \( \mathcal{Q} \) of dyadic cubes such that

1. \( f(\xi) \leq \lambda \) for almost every \( \xi \in \mathbb{R}^n \setminus \bigcup \mathcal{Q} \),
2. \( |\bigcup \mathcal{Q}| \leq \frac{1}{\lambda} \|f\|_{L^1} \),
3. \( \lambda < \langle f \rangle_Q \leq 2^n \lambda \) for every \( Q \in \mathcal{Q} \).

**Proof.** For \( \mathcal{Q} \) we take the family of maximal cubes introduced earlier (in Proposition 2.1), that is, \( \mathcal{Q} = \bigcup_{j \in \mathbb{Z}} A_j \). Now the second requirement is just the weak \((1,1)\)-inequality for the dyadic maximal operator.

By the definition of \( \mathcal{Q} \) we have \( \langle f \rangle_Q \leq \lambda \) for all dyadic cubes \( Q \not\in \mathcal{Q} \). As the averages approximate the function almost everywhere in the sense of the previous proposition, we must have \( f(\xi) \leq \lambda \) for almost every \( \xi \in \mathbb{R}^n \setminus \bigcup \mathcal{Q} \).

Finally, if \( Q \in \mathcal{Q} \), then by its maximality, we have \( \lambda < \langle f \rangle_Q \) and \( \langle f \rangle_R \leq \lambda \) for every larger dyadic cube \( R \supset Q \). In particular, for the smallest dyadic cube \( \tilde{Q} \) that properly contains \( Q \) we have
\[
\langle f \rangle_Q \leq \frac{|\tilde{Q}|}{|Q|} \frac{1}{|\tilde{Q}|} \int_{\tilde{Q}} f \leq 2^n \lambda,
\]
since \( f \) is non-negative and the sides of \( \tilde{Q} \) are twice as long as the sides of \( Q \).
We then prove the weak type inequality for $M_R$:

**Proposition 2.6.** Suppose that $X \subset \mathcal{L}(H, E)$ has RMF$_p$ with respect to $\mathbb{R}^n$ for some $p \in (1, \infty)$, i.e. that $M_R$ is bounded from $L^p(X)$ to $L^p$. Then there exists a constant $C$ such that for all $f \in L^1(X)$,

$$\{|\xi \in \mathbb{R}^n : M_Rf(\xi) > \lambda|\} \leq \frac{C}{\lambda}\|f\|_{L^1(X)}$$

whenever $\lambda > 0$.

**Proof.** We begin by taking the Calderón-Zygmund decomposition of $\|f(\cdot)\|$ at height $\lambda$ and writing $f$ as a sum $f = g + b$, where

$$g(\xi) = \begin{cases} f(\xi), & \xi \in \mathbb{R}^n \setminus \bigcup Q \\ \langle f \rangle_Q, & \xi \in Q \in \mathcal{Q} \end{cases}$$

and

$$b(\xi) = \sum_{Q \in \mathcal{Q}} b_Q(\xi) = \sum_{Q \in \mathcal{Q}} 1_Q(\xi)(f(\xi) - \langle f \rangle_Q), \quad \xi \in \mathbb{R}^n.$$

Observe that $g$ is in $L^1(X) \cap L^\infty(X)$ and that each $b_Q$ has zero average in $Q$. By the sublinearity of $M_R$, $M_Rf \leq M_Rg + M_Rb$, and so

$$|\{\xi \in \mathbb{R}^n : M_Rf(\xi) > \lambda\}| \leq |\{\xi \in \mathbb{R}^n : M_Rg(\xi) > \lambda/2\}| + |\{\xi \in \mathbb{R}^n : M_Rb(\xi) > \lambda/2\}|.$$

Since $M_R$ is bounded from $L^p(X)$ to $L^p$ and $\|g(\xi)\| \leq 2^n \lambda$ for almost every $\xi \in \mathbb{R}^n$, we may compute

$$|\{\xi \in \mathbb{R}^n : M_Rg(\xi) > \lambda/2\}| \leq \left(\frac{2}{\lambda}\right)^p \int_{\mathbb{R}^n} M_Rg(\xi)^p \, d\xi$$

$$\leq C \left(\frac{2}{\lambda}\right)^p \int_{\mathbb{R}^n} |g(\xi)|^p \, d\xi$$

$$\leq C \left(\frac{2}{\lambda}\right)^p (2^n \lambda)^{p-1} \int_{\mathbb{R}^n} \|g(\xi)\| \, d\xi$$

$$\leq \frac{C}{\lambda}\|f\|_{L^1(X)}.$$

We then show that $M_Rb = 0$ outside $\bigcup \mathcal{Q}$. Suppose that $\xi \in \mathbb{R}^n \setminus \bigcup \mathcal{Q}$ and that $Q$ is any dyadic cube containing $\xi$. If $Q \cap \bigcup \mathcal{Q} = \emptyset$, then $b(\eta) = 0$ for all $\eta \in Q$ and hence $\langle b \rangle_Q = 0$. If, on the other hand, $Q \cap \bigcup \mathcal{Q} \neq \emptyset$, then $Q_j \subset Q$ for each $j \in J$, where $\{Q_j\}_{j \in J}$ denotes the subfamily of cubes in $Q$ intersecting $Q$. Since the cubes $Q_j$ are disjoint, we have

$$\frac{1}{|Q|} \int_Q b(\eta) \, d\eta = \frac{1}{|Q|} \int_{\bigcup_{j \in J} Q_j} b(\eta) \, d\eta = \frac{1}{|Q|} \sum_{j \in J} \int_{Q_j} b_{Q_j}(\eta) \, d\eta = 0.$$

Now

$$|\{\xi \in \mathbb{R}^n : M_Rb(\xi) > \lambda/2\}| \leq |\bigcup \mathcal{Q}| \leq \frac{1}{\lambda}\|f\|_{L^1(X)},$$

as required. \hfill $\Box$

Thus, if $X$ has RMF$_p$ with respect to $\mathbb{R}^n$ for some $p \in (1, \infty)$, then by the Marcinkiewicz interpolation theorem it has RMF$_q$ for all $q \in (1, p)$. We will extend this result to the whole interval $(1, \infty)$ in the next section.

The question whether the weak type inequality is enough to guarantee RMF$_p$ will be studied later. Note that the parameter $p$ is not present in this condition.
2.2 Linearization and interpolation

In this section we show that the RMF$_p$-property is independent of $p$. We begin by introducing a method of linearization. For instance, we may linearize the (truncated) dyadic maximal operator by defining

$$
\mathcal{M}^C f(\xi) = \langle f \rangle_Q, \quad \xi \in \mathbb{R}^n,
$$

for locally integrable $X$-valued functions $f$ and finite subcollections $C \subset D$. Finiteness of $C$ implies that $\mathcal{M}^C f$ is constant on the cubes of some $D_j$, namely the one containing the smallest cube(s) in $C$. The operator $\mathcal{M}^C$ thus maps locally integrable functions linearly to strongly measurable functions with values in $l^\infty(X)$ (indexed by dyadic cubes). As for every $\xi \in \mathbb{R}^n$,

$$
\sup_{c \in D} \parallel \mathcal{M}^C f(\xi) \parallel = \sup_{c \in D} \sup_{\xi \in Q \in C} \parallel \langle f \rangle_Q \parallel = \sup_{Q \ni \xi} \parallel \langle f \rangle_Q \parallel = M f(\xi),
$$

where the $C$'s are finite, we see that the uniform boundedness of $\mathcal{M}^C$'s is equivalent to boundedness of $M$.

We then linearize $M_R$. Again, in order to avoid problems with measurability and convergence, let us fix a positive integer $N$ and work with the truncated version

$$
M^{(N)}_R f(\xi) := \mathcal{R} \left( \langle f \rangle_Q : Q \ni \xi, Q \in D_j, |j| \leq N \right), \quad \xi \in \mathbb{R}^n.
$$

We will write

$$
E_j f = \sum_{Q \in D_j} 1_Q \langle f \rangle_Q
$$

for locally integrable functions $f$. It is crucial to note that if $Q \in D_k$, then $E_j(1_Q f) = 1_Q E_j f$ and $\langle E_j f \rangle_Q = \langle f \rangle_Q$ whenever $j > k$ while $1_Q E_j f = 1_Q \langle E_j f \rangle_Q$ whenever $j \leq k$. Let $f$ be a locally integrable $X$-valued function and note that the $R$-bound of any collection of dyadic averages of $f$ coincides with the norm of the operator they induce from $\text{Rad}(H)$ to $\text{Rad}(E)$. Thus if we define for every $\xi \in \mathbb{R}^n$ a linear operator

$$
\mathcal{M}^{(N)}_R f(\xi) : \text{Rad}(H) \to \text{Rad}(E), \quad (x_j)_{j \in \mathbb{Z}} \mapsto \langle E_j f(\xi) x_j \rangle_{|j| \leq N},
$$

then

$$
\parallel \mathcal{M}^{(N)}_R f(\xi) \parallel = \mathcal{R} \left( \langle f \rangle_Q : Q \ni \xi, Q \in D_j, |j| \leq N \right) = M^{(N)}_R f(\xi)
$$

as only distinct operators (averages over different cubes) need to be considered for the $R$-bound.

We now have linearized $M^{(N)}_R$ using an operator that linearly maps locally integrable $X$-valued functions to functions with values in $\mathcal{Y} := \mathcal{L}(\text{Rad}(H), \text{Rad}(E))$ and so the boundedness of $M_R$ can be deduced from uniform boundedness of the linearized versions $M^{(N)}_R$.

We aim to prove using interpolation, that uniform boundedness of operators $M^{(N)}_R$ from $L^p(X)$ to $L^p(\mathcal{Y})$ for some $p \in (1, \infty)$ implies uniform boundedness from $L^p(X)$ to $L^p(\mathcal{Y})$ for all $p \in (1, \infty)$.

The first step is to study how $M^{(N)}_R$ operates on the dyadic Hardy space $H^1(X)$ consisting of functions $f$ with a representation

$$
f(\xi) = \sum_{j=1}^{\infty} \lambda_j a_j(\xi)
$$

converging for almost every $\xi \in \mathbb{R}^n$, where $(\lambda_j)_{j=1}^{\infty}$ is an absolutely summable sequence (an $l^1$-sequence) of complex numbers and $a_j$ are dyadic atoms. Recall that a strongly measurable $X$-valued function $a$ is a dyadic atom if it is supported in a dyadic cube $Q$, $\parallel a \parallel_{L^\infty(X)} \leq |Q|^{-1}$ and $(a)_{Q} = 0$. As $\parallel a \parallel_{L^1(X)} \leq 1$ for each dyadic atom $a$, the space $H^1(X)$ is contained in $L^1(X)$ and becomes a Banach space when equipped with the norm

$$
\parallel f \parallel_{H^1(X)} = \inf \left\{ \sum_{j=1}^{\infty} |\lambda_j| : f = \sum_{j=1}^{\infty} \lambda_j a_j \right\}.
$$
Proposition 2.7. If $\mathcal{X}$ has RMF$_p$ with respect to $\mathbb{R}^n$ for some $p \in (1, \infty)$, then the operators $\mathcal{M}^{(N)}_R$ are uniformly bounded from $H^1(\mathcal{X})$ to $L^1(\mathcal{Y})$.

Proof. Suppose that $f = \sum_{j=1}^{\infty} \lambda_j a_j$ is in $H^1(\mathcal{X})$. Since the sum converges in $L^1(\mathcal{X})$ and the truncated version $\mathcal{M}^{(N)}_R$ maps $L^1(\mathcal{X})$ boundedly to $L^1(\mathcal{Y})$ (with operator norm depending on $N$), we see that

$$\|\mathcal{M}^{(N)}_R f(\xi)\|_{L^1(\mathcal{Y})} \leq \sum_{j=1}^{\infty} |\lambda_j|\|\mathcal{M}^{(N)}_R a_j(\xi)\|_{L^1(\mathcal{Y})}$$

and thus it suffices to study how $\mathcal{M}^{(N)}_R$ operates on dyadic atoms. In order to do so, suppose that $a$ is such an atom. As the average of $a$ is non-zero only over dyadic cubes contained in $Q$, also $\mathcal{M}^{(N)}_R a$ is supported in $Q$. Thus

$$\int_{\mathbb{R}^n} \|\mathcal{M}^{(N)}_R a(\xi)\|\,d\xi = \int_Q \|\mathcal{M}^{(N)}_R a(\xi)\|\,d\xi$$

$$\leq |Q|^{1/p'} \left( \int_Q \|\mathcal{M}^{(N)}_R a(\xi)\|^p\,d\xi \right)^{1/p}$$

$$\leq C|Q|^{1/p'} \left( \int_Q \|a(\xi)\|^p\,d\xi \right)^{1/p}$$

$$\leq C|Q|^{1/p'} |Q|^{1/p} \|a\|_{L^\infty(\mathcal{X})}$$

$$\leq C,$$

where $C$ is independent of $N$. \hfill \square

Next we consider the space (dyadic) BMO($\mathcal{Y}$) of locally integrable $\mathcal{Y}$-valued functions with bounded mean oscillation, more precisely, functions $g$ for which

$$\|g\|_{\text{BMO}(\mathcal{Y})} = \sup_{Q \in \mathcal{D}} \frac{1}{|Q|} \int_Q \|g(\xi) - \langle g \rangle_Q\|\,d\xi$$

is finite.

Proposition 2.8. If $\mathcal{X}$ has RMF$_p$ with respect to $\mathbb{R}^n$ for some $p \in (1, \infty)$, then the operators $\mathcal{M}^{(N)}_R$ are uniformly bounded from $L^\infty(\mathcal{X})$ to BMO($\mathcal{Y}$).

Proof. Suppose that $f \in L^\infty(\mathcal{X})$. The proof relies on the equality

$$1_Q(\xi)\left(\mathcal{M}^{(N)}_R f(\xi) - \langle \mathcal{M}^{(N)}_R f \rangle_Q\right) = \mathcal{M}^{(N)}_R \left(1_Q(f - \langle f \rangle_Q)\right)(\xi)$$

for any dyadic cube $Q$. To see that it holds it is perhaps easier to view $\mathcal{M}^{(N)}_R f$ as a sequence of functions that take values in $\mathcal{X}$, i.e.

$$\mathcal{M}^{(N)}_R f = (E_j f)_{|j| \leq N}.$$

Take any dyadic cube $Q$ and suppose that $Q \in \mathcal{D}_k$. Since

$$1_Q \left(\mathcal{M}^{(N)}_R f - \langle \mathcal{M}^{(N)}_R f \rangle_Q\right) = 1_Q \left(E_j f - \langle E_j f \rangle_Q\right)_{|j| \leq N},$$

it suffices to prove that

$$1_Q(E_j f - \langle E_j f \rangle_Q) = E_j(1_Q(f - \langle f \rangle_Q))$$

for every $j$.

Let us first look at the right hand side of (6). For $j > k$ we have

$$E_j(1_Q(f - \langle f \rangle_Q)) = 1_Q(E_j f - \langle f \rangle_Q)$$

for every $j$. \hfill \square
while for $j \leq k$

$$E_j(1_Q(f - \langle f \rangle_Q)) = \frac{1}{|Q|} \int_Q 1_Q(f - \langle f \rangle_Q) = 0,$$

where $\tilde{Q}$ denotes the cube in $D_j$ that contains $Q$.

We then turn to the left hand side of (6). For $j > k$

$$1_Q(E_j f - \langle E_j f \rangle_Q) = 1_Q(E_j f - \langle f \rangle_Q)$$

and for $j \leq k$

$$1_Q(E_j f - \langle E_j f \rangle_Q) = 1_Q(\langle E_j f \rangle_Q - \langle E_j f \rangle_Q) = 0.$$

Armed with the equality we calculate

$$\|\mathcal{M}_R^{(N)} f\|_{\text{BMO}(\mathcal{Y})} = \sup_{Q \in \mathcal{D}} \frac{1}{|Q|} \int_Q \|\mathcal{M}_R^{(N)} f(\xi) - \langle \mathcal{M}_R^{(N)} f \rangle_Q\| \, d\xi$$

$$= \sup_{Q \in \mathcal{D}} \int_Q \|\mathcal{M}_R^{(N)} \left(\frac{1}{|Q|}(f - \langle f \rangle_Q)\right)(\xi)\| \, d\xi,$$

where $\frac{1}{|Q|}(f - \langle f \rangle_Q)$ is $2\|f\|_{L^{\infty}(\mathcal{X})}$ times a dyadic atom. What was shown in Proposition 2.7 now implies

$$\|\mathcal{M}_R^{(N)} f\|_{\text{BMO}(\mathcal{Y})} \leq 2C\|f\|_{L^{\infty}(\mathcal{X})}$$

which establishes the (uniform) boundedness of $\mathcal{M}_R^{(N)}$’s. \qed

We can then prove the desired result:

**Proposition 2.9.** If $\mathcal{X}$ has RMF$_p$ with respect to $\mathbb{R}^n$ for some $p \in (1, \infty)$, then it has RMF$_p$ with respect to $\mathbb{R}^n$ for all $p \in (1, \infty)$.

**Proof.** Suppose that $\mathcal{X}$ has RMF$_{p_0}$ for some $p_0 \in (1, \infty)$. This means that $M_R$ is bounded from $L^{p_0}(\mathcal{X})$ to $L^{p_0}$ and so the operators $\mathcal{M}_R^{(N)}$, $N \in \mathbb{Z}_+$, are uniformly bounded from $L^{p_0}(\mathcal{X})$ to $L^{p_0}(\mathcal{Y})$, where $\mathcal{Y} := L(\text{Rad}(H), \text{Rad}(E))$. By Propositions 2.7 and 2.8 we may interpolate (see the Appendix) to see that $\mathcal{M}_R^{(N)}$’s are uniformly bounded from $L^p(\mathcal{X})$ to $L^p(\mathcal{Y})$ for $1 < p < p_0$ and for $p_0 < p < \infty$. Hence $M_R$ is bounded from $L^p(\mathcal{X})$ to $L^p$ for all $1 < p < \infty$ and so $\mathcal{X} \subseteq L(H, E)$ has RMF$_p$ for all $p \in (1, \infty)$ \qed

We may now omit the parameter $p$ from our definition of the RMF-property and say that our space $\mathcal{X} \subseteq L(H, E)$ has RMF with respect to $\mathbb{R}^n$ if it has RMF$_p$ with respect to $\mathbb{R}^n$ for some $p \in (1, \infty)$.

### 2.3 RMF-property, type and cotype

We will now study what kind of restrictions the boundedness of the Rademacher maximal operator puts on the type and cotype of the spaces involved. This will not only give us an example of a Banach space without RMF, but also justify why it is better to consider a space of operators rather than a plain Banach space.

We start by showing that one may restrict to functions supported in the unit cube when studying the RMF-property of a space.

**Lemma 2.10.** Let $1 < p < \infty$. For the boundedness of $M_R : L^p(\mathcal{X}) \to L^p$ it suffices to study functions supported in the unit cube $[0,1]^n$. More precisely, if there exists a constant $C$ such that

$$\int_{[0,1]^n} R \left(\langle f \rangle_Q : Q \ni \xi, Q \subseteq [0,1]^n\right)^p \, d\xi \leq C\|f\|_{L^p(\mathcal{X})}^p$$

for all $f \in L^p(\mathcal{X})$ supported in the unit cube $[0,1]^n$, then $M_R$ is bounded from $L^p(\mathcal{X})$ to $L^p$. 24
Proof. Let $f \in L^p(\mathcal{X})$ and write

$$\mathbb{R}^n = \bigcup_{\alpha \in \{-1,1\}^n} Q_\alpha,$$

where $Q_\alpha = \{\xi \in \mathbb{R}^n : \alpha_j \xi_j \geq 0\}$. Since any two sets $Q_\alpha$ and $Q_\beta$ with $\alpha \neq \beta$ intersect only on a set of measure zero, we may write

$$\|f\|_{L^p(\mathcal{X})} = \sum_{\alpha \in \{-1,1\}^n} \|1_{Q_\alpha} f\|_{L^p(\mathcal{X})}.$$

Furthermore, since $1_{Q_\alpha}(f)Q = (1_{Q_\alpha} f)Q$ for any dyadic cube $Q$, we also have $1_{Q_\alpha} M_R f = M_R (1_{Q_\alpha} f)$ and so

$$\|M_R f\|_{L^p} = \sum_{\alpha \in \{-1,1\}^n} \|M_R (1_{Q_\alpha} f)\|_{L^p(\mathcal{X})}.$$

It is thus enough to study the boundedness of $M_R$ for functions supported in each $Q_\alpha$. By considering a function $\xi \mapsto f(\alpha_1 \xi_1, \ldots, \alpha_n \xi_n)$ instead of $f$, the problem reduces to showing boundedness for functions supported in $Q_{(1, \ldots, 1)}$.

Take a finite set $C \subset \mathcal{D}$ and suppose that $f$ is supported in a cube $2^K [0,1)^n$. Defining $\tilde{f}(\xi) = f(2^K \xi)$ we see that $\tilde{f}$ is supported in $[0,1)^n$ and that $\|f\|_{L^p(\mathcal{X})} = 2^n \|\tilde{f}\|_{L^p(\mathcal{X})}$. Also, if $\xi$ is contained in $Q \in \mathcal{D}_j$, then $2^{-K} \xi$ is in $\tilde{Q} = 2^{-K} Q \in \mathcal{D}_{j+K}$ and $\langle f \rangle_Q = \langle \tilde{f} \rangle_{\tilde{Q}}$. If we write $\tilde{C} = \{\tilde{Q} : \tilde{Q} \in C\}$, then

$$M_R^\alpha f(\xi) = \mathcal{R} \left( \langle f \rangle_Q : Q \ni \xi, Q \in C \right) = \mathcal{R} \left( \langle \tilde{f} \rangle_{\tilde{Q}} : \tilde{Q} \ni 2^{-K} \xi, \tilde{Q} \in \tilde{C} \right) = M_R^{\tilde{f}}(2^{-K} \xi).$$

Thus, if $M_R$ is bounded for functions supported in the unit cube, then

$$\|M_R^{\tilde{f}}\|_{L^p} = 2^{nK} \|M_R \tilde{f}\|_{L^p} \leq C^{2nK} \|\tilde{f}\|_{L^p(\mathcal{X})} = C^n \|f\|_{L^p(\mathcal{X})}.$$

The boundedness of $M_R$ for $L^p$-functions supported in $Q_{(1, \ldots, 1)}$ now follows by standard approximation. Notice also that only averages over cubes contained in $[0,1)^n$ need to be considered, as averages over larger cubes are merely scalings of $\langle f \rangle_{[0,1)^n}$ by real numbers less than one. Thus $\mathcal{X}$ has RMF with respect to $\mathbb{R}^n$ if and only if there exists a constant $C$ such that for some $p \in (1, \infty)$ we have

$$\int_{[0,1)^n} \mathcal{R} \left( \langle f \rangle_Q : \xi \ni 2^{-K} \xi, \xi \ni [0,1)^n \right)^p d\xi \leq C \|f\|_{L^p([0,1)^n; \mathcal{X})}^p$$

for all $f \in L^p([0,1)^n; \mathcal{X})$. \hfill $\Box$

Unlike many other maximal operators, $M_R$ is not in general bounded from $L^\infty(\mathcal{L}(H, E))$ to $L^\infty$. We actually have the following:

**Proposition 2.11.** The Rademacher maximal operator is bounded from $L^\infty(0,1; \mathcal{L}(H, E))$ to $L^\infty(0,1)$ if and only if $H$ has cotype 2 and $E$ has type 2.

**Proof.** If $H$ has cotype 2 and $E$ has type 2, all the uniformly bounded sets are $R$-bounded and $M_R f \leq CM f$ for all $f$ in $L^\infty(0,1; \mathcal{L}(H, E))$. Suppose on the contrary, that $H$ does not have cotype 2 or that $E$ does not have type 2 and fix a $C > 0$. Now there exists a positive integer $N$ and operators $T_1, \ldots, T_N$ in $\mathcal{L}(H, E)$ with at most unit norm such that the $R$-bound of $\{T_1, \ldots, T_N\}$ is greater than $C$. We then construct an $L^\infty$-function on $[0,1]$ that obtains the operators $T_j$ as dyadic averages on an interval. Let us write $I_j = [0, 2^{-j-N}], j = 1, \ldots, N$, so that $I_1 = [0, 2^{1-N})$ is the smallest interval and $I_N = [0, 1)$. We set $S_1 = T_1$ and

$$S_j = 2T_j - T_{j-1}, \quad j = 2, \ldots, N.$$

Now $\|S_j\| \leq 3$ for all $j = 1, \ldots, N$, so that if we define $f(\xi) = S_1$ for $\xi \in I_1$ and $f(\xi) = S_j$ for $\xi \in I_j \setminus I_{j-1}, j = 2, \ldots, N$, we have $f \in L^\infty(0,1; \mathcal{L}(H, E))$. \hfill $\square$
We then look at the averages of \( f \) over the intervals \( I_j \). Obviously

\[
\langle f \rangle_{I_1} = S_1 = T_1,
\]

\[
\langle f \rangle_{I_2} = \frac{S_1 + S_2}{2} = T_1 + \frac{T_2 - T_1}{2} = T_3 \quad \text{and}
\]

\[
\langle f \rangle_{I_3} = \frac{S_1 + S_2 + 2S_3}{4} = \frac{2T_2 + 4T_3 - 2T_2}{4} = T_3.
\]

More generally, observing the telescopic behaviour we calculate

\[
\langle f \rangle_{I_j} = \frac{1}{2^{j-1}} \left( S_1 + \sum_{k=1}^{j} 2^{k-1} S_k \right) = \frac{1}{2^{j-1}} (T_1 + 2^{j-1} T_j - T_1) = T_j,
\]

for \( j = 2, \ldots, N \), as was desired. Thus \( M_R f > C \) on \( I_1 \), where \( C \) was chosen arbitrarily large and the bound 3 for the norm of \( f \) does not depend on \( C \). The operator \( M_R \) cannot therefore be bounded from \( L^\infty(0, 1; \mathcal{L}(H, E)) \) to \( L^\infty(0, 1) \).

\[\square\]

Based on the counterexample from [15] that the sequence space \( l^1 \) does not have RMF we prove the following statement.

**Proposition 2.12.** If \( \mathcal{L}(H, E) \) has RMF with respect to \( \mathbb{R} \), then \( H \) has finite cotype and \( E \) has non-trivial type.

**Proof.** Suppose on the contrary that \( E \) has only trivial type. By Theorem 1.8 it follows that for some \( \lambda \geq 1 \) there exists a sequence \((E_N)_{N=1}^\infty\) of subspaces and a sequence \((\Lambda_N^E)_{N=1}^\infty\) of isomorphisms between each \( E_N \) and \( l^2_N \) such that \( \|\Lambda_N^E\| \|\Lambda_N^E|^{-1}\| \leq \lambda \). Let us then fix an \( N \). It is shown in [15] that there exists a function \( f \in L^p(0, 1; l^1) \) for any \( p \in (1, \infty) \) with the following properties:

1. \( f(\xi) \in l^2_N \) for all \( \xi \in (0, 1) \)
2. \( \|f(\xi)\| = 1 \) for all \( \xi \in (0, 1) \) so that \( \|f\|_{L^p(0, 1; l^1)} = 1 \)
3. \( \|M_R f\|_{L^p(0, 1)} \geq C_1 \log \log N \) where the constant \( C_1 \) does not depend on \( N \).

Define then a function \( g : (0, 1) \to E \) by \( g(\xi) = (\Lambda_N^E)^{-1}(f(\xi)) \) and note that \( \|g\|_{L^p(0, 1; E)} \leq \|\Lambda_N^E\|^{-1} \). Since \( M_R \) is bounded from \( L^p(0, 1; E) \) to \( L^p(0, 1) \) there exists a constant \( C_2 \) such that \( \|M_R g\|_{L^p(0, 1)} \leq C_2 \|g\|_{L^p(0, 1; E)} \). But now, since \( f(\xi) = \Lambda_N^E(g(\xi)) \) we have \( \|M_R f(\xi)\| \leq \|\Lambda_N^E\| \|M_R g(\xi)\| \). Thus

\[
\|M_R f\|_{L^p(0, 1)} \leq \|\Lambda_N^E\| \|M_R g\|_{L^p(0, 1)} \leq C_2 \|\Lambda_N^E\| \|g\|_{L^p(0, 1; E)} \leq C_2 \lambda
\]

which gives a contradiction whenever \( N \) is chosen so large that \( C_1 \log \log N \geq C_2 \lambda \).

The claim on finite cotype is proven similarly. Suppose on the contrary that \( H \) has only infinite cotype. Then \( H^* \) has only trivial type and one can proceed as above by defining a function \( h : (0, 1) \to H^* \) by \( h(\xi) = \Lambda_N^{H^*}(f(\xi)) \). \[\square\]
2.4 RMF-property of $L^p$-spaces

Suppose that $(\Omega, \mu)$ is a $\sigma$-finite measure space.

**Proposition 2.13.** The space $L^p(\Omega)$ has RMF with respect to $\mathbb{R}^n$ whenever $1 < p < \infty$.

**Proof.** We will use the identification $L^p(\mathbb{R}^n; L^p(\Omega)) \simeq L^p(\mathbb{R}^n \times \Omega)$ and write

$$\overline{M}f(\xi, \eta) = \sup_{Q \ni \xi} \frac{1}{|Q|} \left| \int_Q f(\zeta, \eta) \, d\zeta \right|, \quad (\xi, \eta) \in \mathbb{R}^n \times \Omega,$$

for the standard dyadic maximal function of an $f \in L^p(\mathbb{R}^n \times \Omega)$ in the first variable. Observe that for $\mu$-almost every $\eta$ we have

$$\int_{\mathbb{R}^n} \overline{M}f(\xi, \eta)^p \, d\xi \lesssim \int_{\mathbb{R}^n} |f(\xi, \eta)|^p \, d\xi,$$

by the boundedness of the standard dyadic maximal operator. Now

$$\mathbb{E}\left\| \sum_{Q \ni \xi} \varepsilon_Q \lambda_Q(f) \right\|_{L^p(\Omega)}^p = \mathbb{E} \int_{\Omega} \left| \sum_{Q \ni \xi} \varepsilon_Q \lambda_Q \frac{1}{|Q|} \int_Q f(\zeta, \eta) \, d\zeta \right|^p \, d\mu(\eta)$$

$$\simeq \int_{\Omega} \left( \sum_{Q \ni \xi} \lambda_Q \frac{1}{|Q|} \int_Q f(\zeta, \eta) \, d\zeta \right)^2 \frac{p}{2} \, d\mu(\eta)$$

$$\leq \int_{\Omega} \left( \sum_{Q \ni \xi} |\lambda_Q \overline{M}f(\xi, \eta)|^2 \right)^{p/2} \, d\mu(\eta)$$

$$= \int_{\Omega} \overline{M}f(\xi, \eta)^p \, d\mu(\eta) \left( \sum_{Q \ni \xi} |\lambda_Q|^2 \right)^{p/2},$$

and so using the definition of R-boundedness given by inequality (5) we get

$$\mathcal{R} \left( \langle f \rangle_Q : Q \ni \xi \right)^p \lesssim \int_{\Omega} \overline{M}f(\xi, \eta)^p \, d\mu(\eta).$$

Hence

$$\int_{\mathbb{R}^n} M_R f(\xi)^p \, d\xi \lesssim \int_{\Omega} \int_{\mathbb{R}^n} \overline{M}f(\xi, \eta)^p \, d\xi \, d\eta \lesssim \int_{\Omega} \int_{\mathbb{R}^n} |f(\xi, \eta)|^p \, d\xi \, d\mu(\eta),$$

so that $M_R$ is bounded from $L^p(L^p(\Omega))$ to $L^p$.

This holds even more generally. Namely, the RMF-property of $\mathcal{X}$ inherits to $L^p(\mathcal{X})$ as the following theorem shows.

**Proposition 2.14.** Suppose that $\mathcal{X} \subset \mathcal{L}(H, E)$ has RMF with respect to $\mathbb{R}^n$. Then the space $L^p(\Omega; \mathcal{X})$ has RMF with respect to $\mathbb{R}^n$ whenever $1 < p < \infty$.

**Proof.** Again we use the identification $L^p(\mathbb{R}^n; L^p(\Omega; \mathcal{X})) \simeq L^p(\mathbb{R}^n \times \Omega; \mathcal{X})$ and write

$$\overline{M}_R f(\xi, \eta) = \mathcal{R} \left( \frac{1}{|Q|} \int_Q f(\zeta, \eta) \, d\zeta : Q \ni \xi \right)$$

for the Rademacher maximal function in the first variable. By the RMF-property of $\mathcal{X}$ we have for $\mu$-almost every $\eta$ that

$$\int_{\mathbb{R}^n} \overline{M}_R f(\xi, \eta)^p \, d\xi \lesssim \int_{\mathbb{R}^n} \|f(\xi, \eta)\|^p \, d\xi.$$
We then calculate
\[
\mathbb{E}\left\| \sum_{Q \ni z} \varepsilon_Q \lambda_Q(f) Q \right\|_{L^p(\Omega; X)} = \int_{\Omega} \mathbb{E}\left[ \sum_{Q \ni z} \varepsilon_Q \lambda_Q \frac{1}{|Q|} \int_Q f(\xi, \eta) \, d\xi \right]^{p} \, d\mu(\eta) \\
\lesssim \int_{\Omega} \overline{M}_R f(\xi, \eta)^p \, d\mu(\eta) \mathbb{E}\left[ \sum_{Q \ni z} \varepsilon_Q \lambda_Q \right]^{p},
\]
and so
\[
\mathcal{R}\left( (f)_Q : Q \ni z \right) \lesssim \int_{\Omega} \overline{M}_R f(\xi, \eta)^p \, d\mu(\eta).
\]
As before,
\[
\int_{\mathbb{R}^n} M_R(\xi)^p \, d\xi \lesssim \int_{\Omega} \int_{\mathbb{R}^n} \overline{M}_R f(\xi, \eta)^p \, d\xi \, d\mu(\eta) \lesssim \int_{\Omega} \int_{\mathbb{R}^n} \|f(\xi, \eta)\|^p \, d\xi \, d\mu(\eta),
\]
so that $M_R$ is bounded from $L^p((L^p(\Omega; X))$ to $L^p$.

\section{2.5 Carleson's embedding theorem}

According to the classical definition, a non-negative Borel measure $\nu$ on $\mathbb{R}^n \times (0, \infty)$ is a Carleson measure if there exists a constant $C$ such that for every cube $Q$ in $\mathbb{R}^n$
\[
\nu(Q \times (0, l(Q))) \leq C|Q|,
\]
where $l(Q)$ is the side length of $Q$ (see for instance Duoandikoetxea [8], Chapter 9). We discretize this by restricting our attention to $\mathbb{R}^n \times \{2^{-j}\}_{j \in \mathbb{Z}}$ and replacing cubes by dyadic cubes. More specifically, let $\theta = (\theta_Q)_{Q \in \mathcal{D}}$ be a such family of measurable functions that each $\theta_Q$ is supported in $Q$. The family $\theta$ is said to satisfy the Carleson condition if there exists a constant $C$ such that
\[
\int_{\mathbb{R}^n} \sum_{Q \subset R} |\theta_Q(\xi)|^2 \, d\xi \leq C|R|
\]
for every dyadic cube $R$.

\textbf{Theorem 2.15.} (Carleson's embedding theorem for scalar functions) Suppose that $\theta = (\theta_Q)_{Q \in \mathcal{D}}$ is such a family of measurable functions that each $\theta_Q$ is supported in $Q$. Then $\theta$ satisfies the Carleson condition if and only if there exists a constant $C$ such that
\[
\int_{\mathbb{R}^n} \sum_{Q \in \mathcal{D}} |(f)_Q \theta_Q(\xi)|^2 \, d\xi \leq C\|f\|_{L^2}^2,
\]
for every $f \in L^2$.

\textbf{Proof.} The Carleson condition is readily seen to follow from the above inequality: Fix a dyadic cube $R$ and choose $f = 1_R$. Now $\langle f \rangle_Q = 1$ for all dyadic cubes $Q$ contained in $R$ and so the inequality gives
\[
\int_{R} \sum_{Q \subset R} |\theta_Q(\xi)|^2 \, d\xi \leq \int_{\mathbb{R}^n} \sum_{Q \in \mathcal{D}} |(f)_Q \theta_Q(\xi)|^2 \, d\xi \leq C\|1_R\|_{L^2}^2 = C|R|.
\]

Conversely, suppose that $\theta$ satisfies the Carleson condition. Divide the dyadic cubes over which the average of $f$ is non-zero into disjoint collections
\[
\mathcal{A}_k = \{ Q \in \mathcal{D} : 2^k < |\langle f \rangle_Q| \leq 2^{k+1}, \quad k \in \mathbb{Z} \}.
\]
Now
\[
\int_{\mathbb{R}^n} \left( \sum_{Q \in \mathcal{D}} |(f)_{\mathcal{Q}} \theta_Q(\xi)|^2 \right)^{p/2} d\xi \leq C \| f \|^p_{L^p},
\]
whenever \( \theta = (\theta_Q)_{Q \in \mathcal{D}} \) satisfies a certain \( p \)-dependent condition. This will follow from an operator-valued version of Carleson embedding theorem, which we provide next.

In order to avoid problems with convergence, suppose that \( \theta = (\theta_Q)_{Q \in \mathcal{D}} \) is a finitely non-zero family of strongly measurable \( H \)-valued functions, each \( \theta_Q \) supported in \( Q \). If \( H \) was a Hilbert space, we would have for \( p \in (1, 2] \) that
\[
\int_{\mathbb{R}^n} \left( \sum_{Q \in \mathcal{D}} \| \theta_Q(\xi) \|^2 \right)^{p/2} d\xi = \int_{\mathbb{R}^n} E \left( \sum_{Q \in \mathcal{D}} \| \varepsilon_Q \theta_Q(\xi) \|^2 \right)^{p/2} d\xi,
\]
which motivates our definition: The family \( \theta \) is said to satisfy the \( p \)-Carleson condition if there exists a constant \( C \) such that
\[
\int_{R} E \left( \sum_{Q \subset R} \| \varepsilon_Q \theta_Q(\xi) \|^2 \right)^{p/2} d\xi \leq C |R|
\]
for all dyadic cubes \( R \). The smallest such constant is called the \( p \)-Carleson constant \( \| \theta \|_{\text{Car}_p} \) of \( \theta \).

**Theorem 2.16.** (Carleson’s embedding theorem for operator-valued functions) Suppose that \( X \subset \mathcal{L}(H, E) \) has RMF with respect to \( \mathbb{R}^n \) and that \( \theta = (\theta_Q)_{Q \in \mathcal{D}} \) is a finitely non-zero family of strongly measurable \( H \)-valued functions such that each \( \theta_Q \) is supported in \( Q \). If \( E \) has type \( p \in (1, 2] \) and \( \theta \) satisfies the \( p \)-Carleson condition, then
\[
\int_{\mathbb{R}^n} E \left( \sum_{Q \in \mathcal{D}} \| \varepsilon_Q (f) \theta_Q(\xi) \|^2 \right)^{p/2} d\xi \leq \| \theta \|_{\text{Car}_p}^p \| f \|_{L^p(X)}^p
\]
for all \( f \in L^p(X) \).
Proof. Let $f \in L^p(\mathcal{X})$. For every dyadic cube $Q$, the quantity

$$\mathcal{R}\left((f)_S : S \supset Q\right)$$

is finite and approaches 0 as $Q$ grows (recall the second remark after Definition 2.2). Hence we may divide dyadic cubes (over which the average of $f$ is non-zero) into disjoint collections

$$\mathcal{A}_k = \left\{ Q \in \mathcal{D} : 2^k < \mathcal{R}\left((f)_S : S \supset Q\right) \leq 2^{k+1} \right\}, \quad k \in \mathbb{Z}.$$ Note that these $\mathcal{A}_k$ are disjoint by construction.

To decompose the sum we randomize and then use the type of $E$ to estimate

$$\mathbb{E}\left\| \sum_{Q \in \mathcal{D}} \varepsilon_Q(f)q_\theta Q(\xi) \right\|^p = \mathbb{E}\mathbb{E}' \left\| \sum_{k \in \mathbb{Z}} \sum_{Q \in \mathcal{A}_k} \varepsilon_Q(f)q_\theta Q(\xi) \right\|^p \lesssim \sum_{k \in \mathbb{Z}} \mathbb{E}\mathbb{E}' \left\| \sum_{Q \in \mathcal{A}_k} \varepsilon_Q(f)q_\theta Q(\xi) \right\|^p.$$ Every dyadic cube $Q$ in $\mathcal{A}_k$ is contained in a unique maximal cube $R$ in $\mathcal{A}_k$, the set of which is denoted by $\mathcal{A}_k^*$. The maximal cubes are disjoint and hence each $q_\theta Q$ is supported in at most one of them and we have

$$\mathbb{E}\left\| \sum_{Q \in \mathcal{A}_k^*} \varepsilon_Q(f)q_\theta Q(\xi) \right\|^p = \mathbb{E}\left\| \sum_{R \in \mathcal{A}_k^*} \sum_{Q \subset R} \varepsilon_Q(f)q_\theta Q(\xi) \right\|^p \lesssim \sum_{R \in \mathcal{A}_k^*} \mathbb{E}\left\| \sum_{Q \subset R} \varepsilon_Q(f)q_\theta Q(\xi) \right\|^p.$$ As the family $\theta$ is only finitely non-zero, there exists for every $\xi \in \mathbb{R}^n$ a unique minimal cube among the cubes $Q \in \mathcal{A}_k$ for which $q_\theta Q(\xi)$ is non-zero. The definition of $\mathcal{A}_k$ can thus be used (with $Q$ in the role of $S$ and the minimal cube as $Q$) to estimate the averages of $f$ as

$$\mathbb{E}\left\| \sum_{Q \subset R} \varepsilon_Q(f)q_\theta Q(\xi) \right\|^p \leq 2^{(k+1)p} \mathbb{E}\left\| \sum_{Q \subset R} \varepsilon_Q(f)q_\theta Q(\xi) \right\|^p.$$ Observe that if $\xi \in R \in \mathcal{A}_k^*$, then $M_{R,f}(\xi) > 2^k$. Putting the estimates together, integrating over $\mathbb{R}^n$ and using the $p$-Carleson condition we obtain

$$\int_{\mathbb{R}^n} \mathbb{E}\left\| \sum_{Q \in \mathcal{D}} \varepsilon_Q(f)q_\theta Q(\xi) \right\|^p d\xi \lesssim \sum_{k \in \mathbb{Z}} 2^{(k+1)p} \mathbb{E}\left\| \sum_{R \in \mathcal{A}_k} \varepsilon_Q(f)q_\theta Q(\xi) \right\|^p \int_{\mathbb{R}^n} \mathbb{E}\left\| \sum_{Q \subset R} \varepsilon_Q(f)q_\theta Q(\xi) \right\|^p d\xi \leq \sum_{k \in \mathbb{Z}} 2^{(k+1)p} \|\theta\|^{p\mathcal{E}_{p,G}} \left| \bigcup \mathcal{A}_k \right| \lesssim ||\theta||_{\mathcal{C}_p} \left| \mathcal{M}_{R,f} \right|^p_{L^p(\mathcal{X})} \leq ||\theta||_{\mathcal{C}_p} \|f\|^p_{L^p(\mathcal{X})},$$

where the last step follows from the RMF property of $\mathcal{X}$. \qed

Remark. The following “converse” statements hold.

1. If the inequality in the previous theorem is satisfied for a constant $C$ (in place of $||\theta||_{\mathcal{C}_p}$) and there exist operators in $\mathcal{X}$ that are bounded from below, then the $p$-Carleson condition follows from the inequality: Fix a dyadic cube $R$ and choose $f = 1_R \otimes T$, where $T \in \mathcal{X}$ has unit norm and is bounded from below. Now $(f)_Q = T$ for all dyadic cubes $Q$ contained in $R$ and so

$$\int_R \mathbb{E}\left\| \sum_{Q \subset R} \varepsilon_Q q_\theta Q(\xi) \right\|^p d\xi \lesssim \int_R \mathbb{E}\left\| \sum_{Q \subset R} \varepsilon_Q(f)q_\theta Q(\xi) \right\|^p d\xi \leq C^p \|1_R \otimes T\|^p_{L^p(\mathcal{X})} = C^p |R|.$$
2. If $\mathcal{X}$ is such that the Carleson embedding theorem holds for some $p \in (1, 2]$, i.e. whenever $	heta = (\theta_Q)_{Q \in \mathcal{D}}$ is a finitely non-zero family of strongly measurable $H$-valued functions (each $\theta_Q$ supported in $Q$) satisfying the $p$-Carleson condition, we have

$$\int_{\mathbb{R}^n} \mathbb{E} \left\| \sum_{Q \in \mathcal{D}} \varepsilon_Q(f)_Q \theta_Q(\xi) \right\|^p d\xi \lesssim \|\theta\|_{\text{Car}_p}^p \|f\|_{L^p(\mathcal{X})}^p$$

for all $f \in L^p(\mathcal{X})$, then $\mathcal{X}$ has RMF with respect to $\mathbb{R}^n$. Indeed, suppose that $\mathcal{C}$ is a finite collection of dyadic cubes and let $f \in L^p(\mathcal{X})$. There exists for every $\xi \in \mathbb{R}^n$ elements $(x_Q^{(k)})_{Q \ni \xi, Q \in \mathcal{C}}, k \in \mathbb{Z}^+$, of $\text{Rad}(H)$ such that

$$\mathbb{E} \left\| \sum_{Q \ni \xi, Q \in \mathcal{C}} \varepsilon_Q x_Q^{(k)} \right\|^p \leq 1$$

and

$$\mathbb{E} \left\| \sum_{Q \ni \xi, Q \in \mathcal{C}} \varepsilon_Q(f)_Q x_Q^{(k)} \right\|^p \to \mathcal{R}_p \left( (f)_Q : Q \ni \xi, Q \in \mathcal{C} \right)^p$$

as $k$ tends to infinity. Let then $\theta_Q^{(k)}(\xi) = x_Q^{(k)}$, where the sequences of $(x_Q^{(k)})_{Q \ni \xi, Q \in \mathcal{C}}$'s are chosen at each point $\xi$ so that $\theta_Q^{(k)}$ is constant on small enough dyadic cubes (this can be done since $\mathcal{C}$ is finite). Evidently each $\theta_Q^{(k)}$ is strongly measurable, supported in $Q$ and together they satisfy

$$\int_{Q} \mathbb{E} \left\| \sum_{R \subset Q} \varepsilon_R \theta_Q^{(k)}(\xi) \right\|^p d\xi \leq |Q|$$

for every dyadic cube $Q$, i.e. each family $\theta^{(k)} = (\theta_Q^{(k)})_{Q \in \mathcal{D}}$ satisfies $\|\theta^{(k)}\|_{\text{Car}_p} \leq 1$. Thus

$$\int_{\mathbb{R}^n} \mathcal{R}_p \left( (f)_Q : Q \ni \xi, Q \in \mathcal{C} \right)^p d\xi \leq \liminf_{k \to \infty} \int_{\mathbb{R}^n} \mathbb{E} \left\| \sum_{Q \ni \xi, Q \in \mathcal{C}} \varepsilon_Q(f)_Q \theta_Q^{(k)}(\xi) \right\|^p d\xi \lesssim \|f\|_{L^p(\mathcal{X})}^p,$$

and consequently $M_R$ is bounded (remember that $\mathcal{R}_p$ and $\mathcal{R}_2$-bounds are comparable).

3. The corresponding result on type, namely that the validity of the Carleson embedding theorem for some $p \in (1, 2]$ implies type $p$ for $E$, is studied in a forthcoming paper.
3 Rademacher maximal function on measure spaces

We will now generalize the Rademacher maximal function to more abstract spaces. Suppose that $(\Omega, \mathcal{F}, \mu)$ is a $\sigma$-finite measure space, that $H$ and $E$ are Banach spaces and that $\mathcal{X} \subset \mathcal{L}(H, E)$ is a Banach space (whose norm dominates the operator norm). Denote the corresponding Lebesgue-Bochner space of $\mathcal{F}$-measurable $\mathcal{X}$-valued functions by $L^p(\mathcal{F}; \mathcal{X})$ (or $L^p(\mathcal{X})$), $1 \leq p \leq \infty$. The space of “locally” integrable functions, or more precisely, strongly measurable functions $f$ for which $1_A f$ is integrable for every set $A \in \mathcal{F}$ with finite measure, is denoted by $L^1_{\text{loc}}(\mathcal{F}; \mathcal{X})$.

If $\mathcal{G}$ is a sub-$\sigma$-algebra of $\mathcal{F}$ such that $(\Omega, \mathcal{G}, \mu)$ is $\sigma$-finite, there exists for every function $f \in L^1_{\text{loc}}(\mathcal{F}; \mathcal{X})$ a conditional expectation (or an average) $\mathbb{E}(f|\mathcal{G}) \in L^1_{\text{loc}}(\mathcal{G}; \mathcal{X})$ with respect to $\mathcal{G}$ which is the (almost everywhere) unique strongly $\mathcal{G}$-measurable function satisfying

$$\int_A \mathbb{E}(f|\mathcal{G}) \, d\mu = \int_A f \, d\mu$$

for every $A \in \mathcal{G}$ with finite measure. The operator $\mathbb{E}(:|\mathcal{G})$ is a contractive projection from $L^p(\mathcal{F}; \mathcal{X})$ onto $L^p(\mathcal{G}; \mathcal{X})$ for any $p \in [1, \infty]$. This follows immediately, if the vector-valued conditional expectation is constructed as the tensor extension of the scalar-valued conditional expectation, which is a positive operator (see Stein [24] for the scalar-valued case and the Lecture notes [26] for the extension).

Conditional expectations satisfy Jensen’s inequality: If $\phi : \mathcal{X} \to \mathbb{R}$ is a convex function and $f \in L^1_{\text{loc}}(\mathcal{X})$ is such that $\phi \circ f \in L^1_{\text{loc}}(\mathcal{X})$, then

$$\phi \circ \mathbb{E}(f|\mathcal{G}) \leq \mathbb{E}(\phi \circ f|\mathcal{G})$$

for any sub-$\sigma$-algebra $\mathcal{G}$ of $\mathcal{F}$ (for which $(\Omega, \mathcal{G}, \mu)$ is $\sigma$-finite). The proof in the case of a finite measure space can be found in [13].

Suppose then that $(\mathcal{F}_j)_{j \in \mathbb{Z}}$ is a filtration, that is, an increasing sequence of sub-$\sigma$-algebras of $\mathcal{F}$ such that each $(\Omega, \mathcal{F}_j, \mu)$ is $\sigma$-finite. For a function $f \in L^1_{\text{loc}}(\mathcal{F}; \mathcal{X})$, we denote the conditional expectations with respect to this filtration by

$$E_j f := \mathbb{E}(f|\mathcal{F}_j), \quad j \in \mathbb{Z}.$$

Furthermore, the analogue of the dyadic maximal function is given by

$$Mf(\xi) = \sup_{j \in \mathbb{Z}} \|E_j f(\xi)\|, \quad \xi \in \Omega.$$

**Example 3.1.** The Euclidean case $\Omega = \mathbb{R}^n$ is an example of this more abstract setting as can be seen by choosing $\mathcal{F}_j = \sigma(D_j)$ for $j \in \mathbb{Z}$. As $E_j f(\xi) = \langle f, Q_j \rangle$, where $Q$ is the dyadic cube in $D_j$ that contains $\xi$, we see that the maximal functions coincide:

$$Mf(\xi) = \sup_{j \in \mathbb{Z}} \|E_j f(\xi)\| = \sup_{Q \ni \xi} \|\langle f, Q \rangle\|.$$

Observe anyhow that the inclusion $L^1_{\text{loc}}(\mathcal{F}; \mathcal{X}) \subset L^1_{\text{loc}, \text{in}}(\mathcal{X})$ is strict and that the $\sigma$-algebra generated by $\bigcup_{j \in \mathbb{Z}} \mathcal{F}_j$ is merely the Borel $\sigma$-algebra on $\mathbb{R}^n$.

The following Proposition and its proof are just repetitions of the Euclidean case. Yet another proof will be given later on in probabilistic language.

**Proposition 3.2.** The maximal operator $f \mapsto Mf$ is bounded from $L^p(\mathcal{X})$ to $L^p$ whenever $1 < p \leq \infty$, regardless of $\mathcal{X}$.

**Proof.** We first prove the weak $(1, 1)$-inequality. Fix an integer $N$ and decompose

$$\left\{ \xi \in \Omega : \sup_{j \geq N} \|E_j f(\xi)\| > \lambda \right\}$$

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For every $k \geq N$, $A_k \in \mathcal{F}_k$ and so
\[ \mu(A_k) \leq \frac{1}{\lambda} \int_{A_k} \|E_k f(\xi)\| \, d\mu(\xi) \leq \frac{1}{\lambda} \int_{A_k} E_k(\|f(.)\|)(\xi) \, d\mu(\xi) = \frac{1}{\lambda} \int_{A_k} \|f(\xi)\| \, d\mu(\xi). \]

By the disjointness of $A_k$'s we have
\[ \mu\left( \left\{ \xi \in \Omega : \sup_{j \geq N} \|E_j f(\xi)\| > \lambda \right\} \right) = \sum_{k=N}^{\infty} \mu(A_k) \leq \frac{1}{\lambda} \sum_{k=N}^{\infty} \int_{A_k} \|f(\xi)\| \, d\mu(\xi) \leq \frac{1}{\lambda} \|f\|_{L^1(\mathcal{X})} \]
and thus taking the limit as $N \to -\infty$,
\[ \mu\left( \left\{ \xi \in \Omega : Mf(\xi) > \lambda \right\} \right) = \lim_{N \to -\infty} \mu\left( \left\{ \xi \in \Omega : \sup_{j \geq N} \|E_j f(\xi)\| > \lambda \right\} \right) \leq \frac{1}{\lambda} \|f\|_{L^1(\mathcal{X})}. \]

Again $M$ is clearly bounded from $L^\infty(\mathcal{X})$ to $L^\infty$ and so the Marcinkiewicz interpolation theorem gives us boundedness for all $p \in (1, \infty]$. \hfill \Box

**Definition 3.3.** Given a filtration $(\mathcal{F}_j)_{j \in \mathbb{Z}}$ of $\mathcal{F}$, the Rademacher maximal function of a function $f \in L^1_\mu(\mathcal{F}; \mathcal{X})$ is defined by
\[ M_R f(\xi) = \mathcal{R}\left( E_j f(\xi) : j \in \mathbb{Z} \right), \quad \xi \in \Omega. \]

**Remark.** The $\mu$-measurability of $M_R f$ can be seen by studying it as the supremum over $N$ of the truncated versions
\[ M_R^{(N)} f(\xi) = \mathcal{R}\left( E_j f(\xi) : |j| \leq N \right), \quad \xi \in \Omega. \]

Indeed, every $M_R^{(N)} f$ is a composition of a strongly $\mu$-measurable function
\[ \Omega \to \mathcal{X}^{2N+1} : \quad \xi \mapsto (E_j f(\xi))_{j=-N}^{N} \]
and a continuous function (we assume that the norm of $\mathcal{X}$ dominates the operator norm)
\[ \mathcal{X}^{2N+1} \to \mathbb{R} : \quad (T_j)_{j=-N}^{N} \mapsto \mathcal{R}\left( T_j : |j| \leq N \right). \]

We are again interested in the boundedness of $M_R$ from $L^p(\mathcal{X})$ to $L^p$.

**Definition 3.4.** Let $1 < p < \infty$. A Banach space $\mathcal{X} \subset \mathcal{L}(H, E)$ is said to have RMF$_p$ with respect to a filtration on a given $\sigma$-finite measure space if the corresponding Rademacher maximal operator is bounded from $L^p(\mathcal{X})$ to $L^p$.

This property will be shown to be independent of $p$, of the filtration and of the underlying measure space (in the sense of Theorem 3.7). The smallest constant for which the boundedness holds will be called the RMF$_p$-constant for the given filtration on the given measure space.

**Remark.** We already have an example supporting the independence of RMF$_p$ from filtration and the underlying measure space. Indeed, we showed in Section 2.3 that for $\mathcal{X}$ to have RMF$_p$ with respect to $\mathbb{R}^n$ it suffices to study the Rademacher maximal function on $\Omega = [0, 1)^n$ and with respect to filtration given by $\mathcal{F}_j = \sigma(\{Q \in \mathcal{D}_j : Q \subset [0, 1)^n\}), \ j \in \mathbb{N}$. 

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An analogue of Lebesgue's differentiation theorem exists also in this more abstract setting if only the filtration \((\mathcal{F}_j)_{j \in \mathbb{Z}}\) on \((\Omega, \mathcal{F}, \mu)\) generates the whole of \(\mathcal{F}\) in the sense that
\[
\sigma\left( \bigcup_{j \in \mathbb{Z}} \mathcal{F}_j \right) = \mathcal{F}.
\]
When this is the case, we have for \(f \in L^p(\mathcal{F}; X)\) with \(p \in [1, \infty)\) the convergence \(E_j f \to f\), as \(j \to \infty\), both in \(L^p\) and pointwise almost everywhere.

If \(\bigcap_{j \in \mathbb{Z}} \mathcal{F}_j\) contains no sets of finite positive measure, as in the Euclidean case, then all functions \(f \in L^p(\mathcal{F}; X)\) with \(p \in [1, \infty)\) satisfy \(E_j f \to 0\), as \(j \to -\infty\), again both in \(L^p\) and pointwise almost everywhere.

These topics were treated (in the scalar-valued case) in a course 'Martingales and harmonic analysis' by Hytönen (see [12]).

### 3.1 Carleson’s embedding theorem

Suppose that the filtration \((\mathcal{F}_j)_{j \in \mathbb{Z}}\) on a \(\sigma\)-finite measure space \((\Omega, \mathcal{F}, \mu)\) is such that
\[
\sigma\left( \bigcup_{j \in \mathbb{Z}} \mathcal{F}_j \right) = \mathcal{F},
\]
and that \(\bigcap_{j \in \mathbb{Z}} \mathcal{F}_j\) contains no sets of finite positive measure.

A family \(\theta = (\theta_j)_{j \in \mathbb{Z}}\) of \(\mu\)-measurable (scalar-valued) functions on \(\Omega\) is said to satisfy the Carleson condition if there exists a constant \(C\) such that for any integer \(m\) and all sets \(A \in \mathcal{F}_m\) with finite measure we have
\[
\int \sum_{j \geq m} |\theta_j(\xi)|^2 \, d\mu(\xi) \leq C \mu(A).
\]

**Theorem 3.5.** (Carleson's embedding theorem for scalar functions) Suppose that \(\theta = (\theta_j)_{j \in \mathbb{Z}}\) is a family of \(\mu\)-measurable functions. Then \(\theta\) satisfies the Carleson condition if and only if there exists a constant \(C\) such that
\[
\int_{\Omega} \sum_{j \in \mathbb{Z}} |E_j f(\xi)\theta_j(\xi)|^2 \leq C \|f\|^2_{L^2}
\]
for all \(f \in L^2(\mathcal{F}, \mu)\).

**Proof.** The Carleson condition can be seen to follow from the inequality by the following simple argument: Given a set \(A \in \mathcal{F}_m\) with finite measure, let \(f = 1_A\). Now \(E_j f = 1_A\) for all \(j \geq m\) and so
\[
\int_A \sum_{j \geq m} |\theta_j(\xi)|^2 \, d\mu(\xi) \leq \int_{\Omega} \sum_{j \in \mathbb{Z}} |E_j f(\xi)\theta_j(\xi)|^2 \leq C \|1_A\|^2_{L^2} = C \mu(A).
\]

Suppose then that \(\theta\) satisfies the Carleson condition. In order to break the sum into “pointwise” pieces we define
\[\tau_k(\xi) = \min\{j \in \mathbb{Z} : |E_j f(\xi)| > 2^k\}.\]
When these “stopping times” finite and do the disjoint intervals \([\tau_k(\xi), \tau_{k+1}(\xi))\), \(k \in \mathbb{Z}\), cover the whole of \(\mathbb{Z}\)? Since \(E_j f \to 0\) almost everywhere as \(j \to -\infty\), we have that each \(\tau_k > -\infty\) almost everywhere. At points \(\xi \in \Omega\) where \(M f(\xi) \leq 2^k\), we have \(\tau_k(\xi) = \infty\). Hence, as \(M f\) is finite almost everywhere, we have for almost every \(\xi \in \Omega\) that \(\tau_k(\xi) = \infty\) for big enough \(k\).

If for some \(\xi \in \Omega\), \(\tau_k(\xi)\) tends to a finite number \(\tau_{-\infty}(\xi)\) instead of \(-\infty\) as \(k \to -\infty\), then \(\sup_{j < \tau_{-\infty}(\xi)} |E_j f(\xi)| \leq 2^k\) for all \(k \in \mathbb{Z}\) which means that \(E_j f(\xi) = 0\) for all \(j < \tau_{-\infty}(\xi)\) and so that part of the sum can be omitted. Thus we may, for almost every \(\xi \in \Omega\), use the following decomposition of the sum to estimate the conditional expectations:
\[
\sum_{j \in \mathbb{Z}} |E_j f(\xi)\theta_j(\xi)|^2 = \sum_{k \in \mathbb{Z}} \sum_{\tau_k(\xi) \leq j < \tau_{k+1}(\xi)} |E_j f(\xi)\theta_j(\xi)|^2 \leq \sum_{k \in \mathbb{Z}} 2^{2k(k+1)} \sum_{\tau_k(\xi) \leq j < \tau_{k+1}(\xi)} |\theta_j(\xi)|^2.
\]
We then integrate and write $A_m = \{ \xi \in \Omega : \tau_k(\xi) = m \}$ for a fixed $k$ to split the space as $\Omega = \bigcup_{m \in \mathbb{Z}} A_m$. It is crucial that $A_m$ is in $\mathcal{F}_m$ for each integer $m$. Now

$$\int_{\Omega} \sum_{\tau_k(\xi) \leq j < \tau_{k+1}(\xi)} |\theta_j(\xi)|^2 \, d\mu(\xi) \leq \int_{\Omega} \sum_{j \geq \tau_k(\xi)} |\theta_j(\xi)|^2 \, d\mu(\xi) = \sum_{m \in \mathbb{Z} \cup \{\infty\}} \int_{A_m} \sum_{j \geq m} |\theta_j(\xi)|^2 \, d\mu(\xi).$$

For integers $m$ we use the Carleson condition (observe that for $m = \infty$ the sum is empty) to get

$$\sum_{m \in \mathbb{Z}} \int_{A_m} \sum_{j \geq m} |\theta_j(\xi)|^2 \, d\mu(\xi) \leq C \sum_{m \in \mathbb{Z}} \mu(A_m) = C \mu(\{ \xi \in \Omega : \tau_k(\xi) < \infty \}).$$

Finally, $\tau_k(\xi) < \infty$ exactly when $|E_j f(\xi)| > 2^k$ for some integer $j$, i.e. when $Mf(\xi) > 2^k$. Putting it all together gives

$$\int_{\Omega} \sum_{j \in \mathbb{Z}} |E_j f(\xi)\theta_j(\xi)|^2 \, d\mu(\xi) \leq \sum_{k \in \mathbb{Z}} 2^{2(k+1)} \int_{\Omega} \sum_{\tau_k(\xi) \leq j < \tau_{k+1}(\xi)} |\theta_j(\xi)|^2 \, d\mu(\xi) \leq C \sum_{k \in \mathbb{Z}} 2^{2(k+1)} \mu(\{ \xi \in \Omega : Mf(\xi) > 2^k \}) \leq C \|Mf\|_{L^2}^2 \leq C \|f\|_{L^2}^2.$$  

Next we formulate the operator-valued version. Suppose that $\theta = (\theta_j)_{j \in \mathbb{Z}}$ is a finitely non-zero family of strongly $\mu$-measurable $H$-valued functions. The family $\theta$ is said to satisfy the $p$-Carleson condition if there exists a constant $C$ such that for any integer $m$ and all sets $A \in \mathcal{F}_m$ with finite measure we have

$$\int_A \left\| \sum_{j \geq m} \varepsilon_j \theta_j(\xi) \right\|^p \, d\mu(\xi) \leq C^p \mu(A).$$

The smallest such constant is called the $p$-Carleson constant $\|\theta\|_{\text{Car}_p}$ of $\theta$.

**Theorem 3.6. (Carleson’s embedding theorem for operator-valued functions)** Suppose that $X \subset L(H, E)$ has RMF with respect to $(\mathcal{F}_j)_{j \in \mathbb{Z}}$ and that $\theta = (\theta_j)_{j \in \mathbb{Z}}$ is a finitely non-zero family of strongly measurable $H$-valued functions. If $E$ has type $p \in (1, 2]$ and $\theta$ satisfies the $p$-Carleson condition, then

$$\int_{\Omega} \left\| \sum_{j \in \mathbb{Z}} \varepsilon_j E_j f(\xi) \theta_j(\xi) \right\|^p \, d\mu(\xi) \leq \|\theta\|_{\text{Car}_p}^p \|f\|_{L^p(X)}^p$$

for all $f \in L^p(X)$.

**Proof.** Let $f$ be in $L^p(X)$. We start by defining

$$\tau_k(\xi) = \min \left\{ j \in \mathbb{Z} : R(E_j f(\xi) : i \leq j) > 2^k \right\}, \quad k \in \mathbb{Z}, \quad \xi \in \Omega.$$ 

This time we may well have $\tau_k = -\infty$ in a set of positive measure, but this will not pose a problem since $\theta$ is only finitely non-zero. The RMF-property of $X$ implies that $M_R f$ is finite almost everywhere and so for almost every $\xi \in \Omega$, we have $\tau_k(\xi) = \infty$ for big enough $k$. As earlier, $\tau_k(\xi)$ may for some $\xi \in \Omega$ tend to some $\tau_{-\infty}(\xi) > -\infty$ as $k \to -\infty$, but then

$$\sup_{j < \tau_{-\infty}(\xi)} R(E_j f(\xi) : i \leq j) \leq 2^k$$

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As before, we write \( A_m = \{ \xi \in \Omega : \tau_k(\xi) = m \} \) for a fixed \( k \) to split the space as \( \Omega = \bigcup_{m \in \mathbb{Z}} A_m \). Note again that \( A_m \) is in \( F_m \) for each integer \( m \). Now
\[
\int_{A_m} \left\| \sum_{j \geq m} \epsilon_j \theta_j(\xi) \right\|^p d\mu(\xi) \leq \sum_{m \in \mathbb{Z}} \left( \int_{A_m} \left\| \sum_{j \geq m} \epsilon_j \theta_j(\xi) \right\|^p d\mu(\xi) \right) \leq \sum_{m \in \mathbb{Z}} 2^{(k+1)p} \left( \int_{A_m} \left\| \sum_{j \geq m} \epsilon_j \theta_j(\xi) \right\|^p d\mu(\xi) \right).
\]
Using the \( p \)-Carleson condition for integers \( m \) (observe that for \( m = \infty \) the sum is empty and that \( m = -\infty \) can be omitted since \( \theta \) is only finitely non-zero) we obtain
\[
\sum_{m \in \mathbb{Z}} \int_{A_m} \left\| \sum_{j \geq m} \epsilon_j \theta_j(\xi) \right\|^p d\mu(\xi) \leq \| \theta \|^p_{\text{Car}, p} \sum_{m \in \mathbb{Z}} \mu(A_m) = \| \theta \|^p_{\text{Car}, p} \mu(\{ \xi \in \Omega : -\infty < \tau_k(\xi) < \infty \}).
\]
As before, \( \tau_k(\xi) < \infty \) exactly when \( \mathcal{R}(E_i f(\xi) : i \leq j) > 2^k \) for some integer \( j \), i.e. when \( M_R f(\xi) > 2^k \). In conclusion,
\[
\int_{\Omega} \left\| \sum_{j \in \mathbb{Z}} \epsilon_j E_j f(\xi) \theta_j(\xi) \right\|^p d\mu(\xi) \leq \sum_{k \in \mathbb{Z}} 2^{(k+1)p} \int_{A_m} \left\| \sum_{j \geq m} \epsilon_j \theta_j(\xi) \right\|^p d\mu(\xi) \leq \| \theta \|^p_{\text{Car}, p} \sum_{k \in \mathbb{Z}} 2^{(k+1)p} \mu(\{ \xi \in \Omega : M_R f(\xi) > 2^k \}) \approx \| \theta \|^p_{\text{Car}, p} \| M_R f \|^p_{L^p(\Omega)} \lesssim \| \theta \|^p_{\text{Car}, p} |\mathcal{F}_j| \lesssim \| \theta \|^p_{\text{Car}, p} |\mathcal{F}_j| \lesssim \| \theta \|^p_{\text{Car}, p} \| f \|^p_{L^p(\Omega)}.
\]

**Remark.** If the inequality in the previous theorem is satisfied for a constant \( C \) and there exist operators in \( \mathcal{X} \) that are bounded from below, then the \( p \)-Carleson condition follows from the inequality: Fix a set \( A \in F_m \) and choose \( f = 1_A \otimes T \), where \( T \in \mathcal{X} \) has unit norm and is bounded from below. Now \( E_j f = 1_A \otimes T \) for all \( j \geq m \) and so
\[
\int_{A} \left\| \sum_{j \geq m} \epsilon_j \theta_j(\xi) \right\|^p d\mu(\xi) \lesssim \int_{A} \left\| \sum_{j \geq m} \epsilon_j E_j f(\xi) \theta_j(\xi) \right\|^p d\mu(\xi) \leq C \| 1_A \otimes T \|_{L^p(\Omega)} = C \mu(A).
\]

### 3.2 Reduction to Haar filtrations

We will show that the RMF-property is independent of the filtration and the underlying measure space in the following sense:

**Theorem 3.7.** Let \( 1 < p < \infty \). If \( \mathcal{X} \) has RMF \( p \) with respect to the filtration of dyadic intervals on \([0,1)\), then it has RMF \( p \) with respect to any filtration on any \( \sigma \)-finite measure space.

When this is the case, we simply say that \( \mathcal{X} \) has RMF \( p \). The proof the Theorem 3.7 uses ideas from Maurey [20], where a similar result is proven for the UMD-property. We begin with the simplest possible case of filtrations of finite algebras on finite measure spaces and proceed gradually toward more general situations. In order to do so, we first work on measure spaces.
(Ω, F, µ) with µ(Ω) = 1, that are divisible in the sense that any set A ∈ F with positive measure has for all c ∈ (0, 1) a (measurable) subset with measure cµ(A).

By a basis of a finite subalgebra G of F we mean a partition of Ω into disjoint non-empty sets A1, ..., Am ∈ G that generate the subalgebra so that each A ∈ G can be expressed as a union of some of these A_k’s. Such a partition, denoted by bs G, always exists and is unique.

Observe that in a filtration (Fj)∞ j=1 of finite algebras every bs Fj is obtained from bs Fj−1 by splitting a number of sets into smaller ones. In other words, every B ∈ bs Fj can be uniquely written as B = B1 ∪ ... ∪ Bk for some B1, ..., Bk ∈ bs Fj.

Functions measurable with respect to a finite algebra can be identified with functions defined on the basis of this algebra (or any finer algebra). Furthermore, if (Fj)N j=1 is a filtration of finite algebras on (Ω, F, µ) and f is F_N-measurable, then

$$E(f|F_j)(B) = \frac{1}{\mu(B)} \sum_{A \in \text{bs } F_N} \mu(A)f(A), \quad B \in \text{bs } F_j,$$

for any j = 1, ..., N.

A filtration (Fj)∞ j=1 of finite subalgebras of F is called a Haar filtration if bs Fj consists of j + 1 sets of positive measure. We also write F0 = {∅, Ω} so that bs F0 = {∅}. Furthermore, every Fj is obtained from Fj−1 by splitting a set B ∈ bs Fj−1 into two sets B1 and B2 of positive measure. Hence bs Fj consists of the sets in bs Fj−1 with B replaced by B1 and B2. A Haar filtration is said to be dyadic if the splitting is done so that the ratio between µ(B) and either µ(B1) or µ(B2) is an integral multiple of 2m for some m ∈ Z+, and further to be standard each B splits into sets of equal measure.

A typical example of a filtration of finite algebras is of course the dyadic filtration on [0, 1). We denote by Dj the finite algebra of dyadic intervals of length 2−j on [0, 1) and so

$$\text{bs } D_j = \{[(k-1)2^{-j}, k2^{-j}) : k = 1, \ldots, 2^j\}.$$
where $K_j + 1$ is the number of sets in $bs \mathcal{F}_j$.

Note that the RMF$_p$-constant of $\mathcal{X}$ with respect to a filtration $(\mathcal{F}_j)_{j=1}^N$ of finite algebras is at least the RMF$_p$-constant with respect to any “subfiltration” $(\mathcal{F}_{j_k})_{k=1}^M$, where $1 \leq j_k \leq \ldots \leq j_{kM} \leq N$. Indeed, for any $\mathcal{F}_N$-measurable $f$ we have

$$\mathcal{R} \left( \mathbb{E}(f|\mathcal{F}_{j_k})(A) : 1 \leq k \leq M \right) \leq \mathcal{R} \left( \mathbb{E}(f|\mathcal{F}_j)(A) : 1 \leq j \leq N \right), \quad A \in bs \mathcal{F}_N,$$

and the claim follows.

A bijection $b : \mathcal{G} \to \tilde{\mathcal{G}}$, where $(\Omega, \mathcal{G}, \mu)$ and $(\tilde{\Omega}, \tilde{\mathcal{G}}, \tilde{\mu})$ are measure spaces with finite algebras, is called a measure preserving Boolean isomorphism if

1. $b(A_1 \cup A_2) = b(A_1) \cup b(A_2)$
2. $b(A_1 \cap A_2) = b(A_1) \cap b(A_2)$
3. $b(\emptyset) = \emptyset$
4. $b(\Omega) = \tilde{\Omega}$
5. $\tilde{\mu}(b(A)) = \mu(A)$

for all $A_1, A_2$ and $A$ in $\mathcal{G}$. It is plain to see that using such a mapping $b$, the basis of $\tilde{\mathcal{G}}$ can be written as $\{b(A) : A \in bs \mathcal{G}\}$. Furthermore, every bijection $b$ from $bs \mathcal{G}$ to $bs \tilde{\mathcal{G}}$ that preserves measures (i.e. satisfies $\tilde{\mu}(b(A)) = \mu(A)$ for all $A \in bs \mathcal{G}$) extends uniquely to a measure preserving Boolean isomorphism from $\mathcal{G}$ to $\tilde{\mathcal{G}}$.

Two filtrations $(\mathcal{F}_j)_{j=1}^\infty$ and $(\tilde{\mathcal{F}}_j)_{j=1}^\infty$ of finite algebras (possibly on different measure spaces) are said to be equivalent if there exists for every $j \in \mathbb{Z}_+$ a measure preserving Boolean isomorphism between $\mathcal{F}_j$ and $\tilde{\mathcal{F}}_j$. Observe that if $b : \mathcal{F}_N \to \tilde{\mathcal{F}}_N$ is a measure preserving Boolean isomorphism, then for every $\mathcal{F}_N$-measurable $f$ we have

$$\mathbb{E}(f|\mathcal{F}_j) = \mathbb{E}(f \circ b^{-1}|\tilde{\mathcal{F}}_j) \circ b$$

for any $j = 1, \ldots, N$.

In fact, every filtration $(\mathcal{F}_j)_{j=1}^\infty$ of finite algebras on any measure space (of total measure one) is equivalent to a filtration on the unit interval. To see this, suppose that a measure preserving Boolean isomorphism $b$ has been constructed from $\mathcal{F}_{j-1}$ to $\tilde{\mathcal{F}}_{j-1}$. Recall that $bs \mathcal{F}_j$ is obtained from $bs \mathcal{F}_{j-1}$ by splitting a number of sets into smaller ones. For every such $B \in bs \mathcal{F}_{j-1}$ we simply split $b(B)$ in the same ratio and extend $b$ accordingly to $\mathcal{F}_j$.

**Lemma 3.8.** The RMF$_p$-constant of $\mathcal{X}$ (if finite) is the same with respect to equivalent filtrations of finite algebras.

**Proof.** Suppose that $(\mathcal{F}_j)_{j=1}^\infty$ is a filtration of finite algebras on $(\Omega, \mathcal{F}, \mu)$ and fix a positive integer $N$. Then for any $\mathcal{F}_N$-measurable $f$ we have

$$\|f\|_{L^p(\mathcal{X})}^p = \sum_{A \in bs \mathcal{F}_N} \mu(A) \|f(A)\|^p \quad \text{and} \quad \|M^f_{\mathcal{F}_N}\|_{L^p}^p = \sum_{A \in bs \mathcal{F}_N} \mu(A) \mathcal{R} \left( \mathbb{E}(f|\mathcal{F}_j)(A) : 1 \leq j \leq N \right)^p.$$

If now a filtration $(\tilde{\mathcal{F}}_j)_{j=1}^\infty$ on $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mu})$ is equivalent to $(\mathcal{F}_j)_{j=1}^\infty$ and $b$ is a measure preserving Boolean isomorphism from $\mathcal{F}_N$ to $\tilde{\mathcal{F}}_N$, then $bs \tilde{\mathcal{F}}_j = \{b(A) : A \in bs \mathcal{F}_N\}$ and

$$\mu(A) = \tilde{\mu}(b(A)), \quad \mathbb{E}(f \circ b^{-1}|\tilde{\mathcal{F}}_j)(b(A)) = \mathbb{E}(f|\mathcal{F}_j)(A) \quad \text{and} \quad \|f \circ b^{-1}\|_{L^p(\mathcal{X})}^p = \|f\|_{L^p(\mathcal{X})}^p.$$

The claim follows immediately. \[\square\]

The next lemma shows that when dealing with dyadic Haar filtrations, we can choose an equivalent filtration on the unit interval that very much resembles the filtration of dyadic intervals.
Lemma 3.9. Every dyadic Haar filtration on any measure space with total measure one is equivalent to a dyadic Haar filtration \((\mathcal{F}_j)_{j=1}^N\) on the unit interval such that \(\mathcal{F}_j \subset \mathcal{D}_{K_j}\) for some integers \(K_j\)

\[
\mathbb{E}(f|\mathcal{F}_j) = \mathbb{E}(f|\mathcal{D}_{K_j}), \quad 1 \leq j \leq N,
\]

for any \(\mathcal{F}_N\)-measurable \(f\).

**Proof.** Let us first illustrate the construction by a simple example. Suppose that the first algebra of the original filtration results from splitting the measure space into two sets of measures 1/4 and 3/4 and that the second one is obtained by splitting the larger set in half. We start by defining \(bs\mathcal{F}_1 = \{B_1, B\}\), where \(B_1 = [0, 1/4)\) and \(B = [1/4, 1)\), so that \(\mathcal{F}_1 \subset \mathcal{D}_2 (K_1 = 2)\). We are then to define \(bs\mathcal{F}_2\) by dividing \(B\) into two subsets \(B_2\) and \(B_3\) of equal measure so that \(\mathcal{F}_2 \subset \mathcal{D}_2\) and

\[
\mathbb{E}(f|\mathcal{F}_1) = \mathbb{E}(f|\mathcal{D}_2),
\]

for any \(\mathcal{F}_2\)-measurable \(f\). Now \(B\) is the union of three intervals \(I_2, I_3\) and \(I_4\) in \(bs\mathcal{D}_2\) (while \(B_1 = I_1\)). We want to halve each of these (three) intervals and let \(B_2\) and \(B_3\) consist of left and right halves of each interval, respectively. More precisely, we take \(K_2 = 3\) so that \(\mathcal{F}_2 \subset \mathcal{D}_3\) with \(bs\mathcal{F}_2 = \{B_1, B_2, B_3\}\), where

\[
B_2 = [1/4, 3/8) \cup [1/2, 5/8) \cup [3/4, 7/8) \quad \text{and} \quad B_3 = [3/8, 1/2) \cup [5/8, 3/4) \cup [7/8, 1).
\]

![Figure 3: An example of the construction](image)

If now \(f\) is \(\mathcal{F}_2\)-measurable, then clearly \(\mathbb{E}(f|\mathcal{F}_1)(B_1) = f(B_1) = \mathbb{E}(f|\mathcal{D}_2)(I_1)\) and more crucially

\[
\mathbb{E}(f|\mathcal{D}_2)(I_j) = \frac{f(B_2) + f(B_3)}{2}, \quad j = 2, 3, 4,
\]

so that \(\mathbb{E}(f|\mathcal{D}_2)\) is \(\mathcal{F}_1\)-measurable and equal to \(\mathbb{E}(f|\mathcal{F}_1)\).

Suppose then that suitable \(\mathcal{F}_1, \mathcal{F}_2, \ldots, \mathcal{F}_{j-1}\) have been constructed and that we are to split a \(B \in bs\mathcal{F}_{j-1}\) into \(B_1\) and \(B_2\) so that \(|B|/|B_1| = r2^m\). Put \(K_j = K_{j-1} + m\). Now \(B\) is a finite union of dyadic intervals of length \(2^{-K_j}\) (i.e. sets in \(bs\mathcal{D}_{K_j}\)). From each such interval take the \(r\) first subintervals of length \(2^{-K_j}\) (i.e. sets in \(bs\mathcal{D}_{K_j}\)) and denote their union by \(B_1\). The remaining \(B \setminus B_1\) will of course be \(B_2\). Now indeed \(|B|/|B_1| = r2^m\), as required.

We will now check that if \(f\) is \(\mathcal{F}_j\)-measurable, then

\[
\mathbb{E}(f|\mathcal{F}_{j-1}) = \mathbb{E}(f|\mathcal{D}_{K_{j-1}}).
\]
The equality is clear on basis sets of $\mathcal{F}_j$ other than $B_1$ and $B_2$ (which resulted from splitting a set $B \in \text{bs}\mathcal{F}_{j-1}$ as above) since $\mathcal{F}_{j-1} \subset \mathcal{D}_{K_{j-1}}$. Furthermore

$$\mathbb{E}(f|\mathcal{F}_{j-1})(B) = \frac{|B_1|f(B_1) + |B_2|f(B_2)}{|B_1 \cup B_2|}. $$

Now $B = B_1 \cup B_2$ is a union of intervals in $\text{bs}\mathcal{D}_{K_{j-1}}$ each of which intersects $B_1$ and $B_2$ in the same ratio. More precisely, for every $I \in \text{bs}\mathcal{D}_{K_{j-1}}$ that intersects $B$ we have

$$\mathbb{E}(f|\mathcal{D}_{K_{j-1}})(I) = \frac{|B_1|f(B_1) + |B_2|f(B_2)}{|B_1 \cup B_2|}$$

so that $\mathbb{E}(f|\mathcal{D}_{K_{j-1}})$ is constant on $B$ and equals $\mathbb{E}(f|\mathcal{F}_{j-1})$.

Finally, if $f$ is $\mathcal{F}_N$-measurable, we proceed inductively:

$$\mathbb{E}(f|\mathcal{F}_{N-2}) = \mathbb{E}(\mathbb{E}(f|\mathcal{F}_{N-1})|\mathcal{F}_{N-2}) = \mathbb{E}(\mathbb{E}(f|\mathcal{D}_{K_{N-1}})|\mathcal{D}_{K_{N-2}}) = \mathbb{E}(f|\mathcal{D}_{K_{N-2}})$$

and so on. \hfill \Box

We say that $\mathcal{X}$ has RMF$_p$ uniformly with respect to a class of filtrations on a class of measure spaces if the RMF$_p$-constants in question are uniformly bounded.

For the next three lemmas, fix a divisible measure space $(\Omega, \mathcal{F}, \mu)$ with $\mu(\Omega) = 1$. In each of the lemmas we start with a filtration $(\mathcal{F}_j)_{j=1}^\infty$, truncate it at a positive integer $N$ and construct a corresponding more “regular” one, whose $\sigma$-algebras we denote by $\tilde{\mathcal{F}}_j$. Objects corresponding to these are denoted likewise, for instance, conditional expectations are denoted by $\tilde{E}_j$ and $\tilde{E}_j$, respectively.

**Lemma 3.10.** If $\mathcal{X}$ has RMF$_p$ uniformly with respect to dyadic Haar filtrations on $(\Omega, \mathcal{F}, \mu)$, then it has RMF$_p$ uniformly with respect to all Haar filtrations on $(\Omega, \mathcal{F}, \mu)$.

**Proof.** Suppose that $(\mathcal{F}_j)_{j=1}^\infty$ is a Haar filtration, $N$ a positive integer and let $\text{bs}\mathcal{F}_N = \{A_1, \ldots, A_{N+1}\}$. We start by constructing a partition $\{\tilde{A}_1, \ldots, \tilde{A}_{N+1}\}$ of $\Omega$ consisting of dyadic sets, which is arbitrarily close to $\text{bs}\mathcal{F}_N$ in the sense that for each $k$, $\mu(\tilde{A}_k\setminus A_k)$ is small.

We proceed inductively. First, of course, $\text{bs}\mathcal{F}_0 = \{\Omega\} = \text{bs}\tilde{\mathcal{F}}_0$. Also, if $\text{bs}\mathcal{F}_1 = \{B_1, B_2\}$ we may take by divisibility a dyadic set $\tilde{B}_1 \subset B_1$ with $\mu(B_1\setminus \tilde{B}_1) < \varepsilon/3^N$. Defining $\tilde{B}_2 = \Omega \setminus \tilde{B}_1$ we see that also $\tilde{B}_2$ is dyadic and satisfies $\mu(B_2\setminus \tilde{B}_2) < \varepsilon/3^N$, since

$$B_2\Delta \tilde{B}_2 = (\Omega \setminus B_1)\Delta (\Omega \setminus \tilde{B}_1) = B_1 \setminus \tilde{B}_1.$$

Suppose then that for some $j = 1, \ldots, N$, $\text{bs}\mathcal{F}_{j-1} = \{B_1, \ldots, B_{j-1}, B\}$ and that we have constructed $\text{bs}\tilde{\mathcal{F}}_{j-1} = \{\tilde{B}_1, \ldots, \tilde{B}_{j-1}, \tilde{B}\}$ consisting of dyadic sets for which $\mu(B_k\setminus \tilde{B}_k) < \varepsilon/3^{N-j+1}$ (and $\mu(B\setminus \tilde{B}) < \varepsilon/3^{N-j+1}$).

Assume that $\text{bs}\mathcal{F}_j = \{B_1, \ldots, B_j, B_j+1\}$, where $B$ has split into two sets $B_j$ and $B_{j+1}$ so that $\mu(B \setminus B_j) > 0$. It suffices to split $B$ into dyadic sets $\tilde{B}_j$ and $\tilde{B}_{j+1}$ that satisfy $\mu(B_k\setminus \tilde{B}_k) < \varepsilon/3^{N-j}$, where $k = j, j+1$. To do this, take a dyadic set $\tilde{B}_j \subset B \cap B_j$ for which $\mu((B \cap B_j) \setminus \tilde{B}_j) < \varepsilon/3^{N-j+1}$. Then

$$\mu(B_j\setminus \tilde{B}_j) = \mu(B_j \setminus \tilde{B}_j) \leq \mu((B \cap B_j) \setminus \tilde{B}_j) + \mu((B_j \setminus \tilde{B}_j) \setminus \tilde{B}_j) < \frac{2\varepsilon}{3^{N-j+1}}.$$
RMF

Haar filtration will depend on a given function $F^{\mu}$ independent of the filtration $\mathcal{F}$, where we used $\mathcal{F}$.

Now the algebras are then defined by letting $\tilde{F}$ consist of those sets $\bigcup_{k \in K} \tilde{A}_k$ for which $\bigcup_{k \in K} A_k \in \mathcal{F}_j$ and observing that $(\tilde{F}_j)_{j=1}^N$ becomes a dyadic Haar filtration. The actual choice of our dyadic Haar filtration will depend on a given function $f \in L^p(\mathcal{F}; \mathcal{X})$, but this will not matter since or RMF$_\mu$ constants are uniformly bounded!

Now

$$\|M^R_{\mathcal{F}} f\|_{L^p} = \left( \int_{\Omega} \mathcal{R} \left(E_j f(\xi) : 1 \leq j \leq N \right)^p d\mu(\xi) \right)^{1/p}$$

$$\leq \left( \int_{\Omega} \mathcal{R} \left(E_j f(\xi) - \tilde{E}_j f(\xi) : 1 \leq j \leq N \right)^p d\mu(\xi) \right)^{1/p} + \|\tilde{M}_R f\|_{L^p},$$

where the maximal operator $\tilde{M}_R$ satisfies by assumption $\|\tilde{M}_R f\|_{L^p} \leq C \|f\|_{L^p(\mathcal{X})}$ for a constant $C$ independent of the filtration $(\tilde{F}_j)_{j=1}^N$.

Estimating the R-bound in the first term by summing the norms we get

$$\left( \int_{\Omega} \mathcal{R} \left(E_j f(\xi) - \tilde{E}_j f(\xi) : 1 \leq j \leq N \right)^p d\mu(\xi) \right)^{1/p} \leq \left( \int_{\Omega} \left( \sum_{j=1}^N \|E_j f(\xi) - \tilde{E}_j f(\xi)\|^p \right) d\mu(\xi) \right)^{1/p} \leq \sum_{j=1}^N \|E_j f - \tilde{E}_j f\|_{L^p(\mathcal{X})}.$$
Thus

\[ \|E_j f - \tilde{E}_j f\|_{L^p(\mathcal{X})}^p = \int_{\Omega} \|E_j f(\xi) - \tilde{E}_j f(\xi)\|^p d\mu(\xi) \]

\[ = \sum_{k=1}^{j+1} \int_{B_k \cap \tilde{B}_k} \|E_j f(\xi) - \tilde{E}_j f(\xi)\|^p d\mu(\xi) + \sum_{k=1}^{j+1} \int_{B_k \setminus \tilde{B}_k} \|E_j f(\xi) - \tilde{E}_j f(\xi)\|^p d\mu(\xi) \]

and we are left to estimate these two terms separately. We begin with the first one. For \( \xi \in B_k \cap \tilde{B}_k \) we have

\[ E_j f(\xi) = \frac{1}{\mu(B_k)} \int_{B_k} f d\mu \quad \text{and} \quad \tilde{E}_j f(\xi) = \frac{1}{\mu(B_k)} \int_{\tilde{B}_k} f d\mu \]

and thus

\[ \int_{B_k \cap \tilde{B}_k} \|E_j f(\xi) - \tilde{E}_j f(\xi)\|^p d\mu(\xi) = \mu(B_k \cap \tilde{B}_k) \left( \frac{1}{\mu(B_k)} \int_{B_k} f d\mu - \frac{1}{\mu(B_k)} \int_{\tilde{B}_k} f d\mu \right) \]

where

\[ \left\| \frac{1}{\mu(B_k)} \int_{B_k} f d\mu - \frac{1}{\mu(B_k)} \int_{\tilde{B}_k} f d\mu \right\| \]

\[ \leq \frac{1}{\mu(B_k)} \left( \int_{B_k \cap \tilde{B}_k} \|f\| d\mu(\xi) + \int_{B_k \setminus \tilde{B}_k} \|f\| d\mu(\xi) + \int_{\tilde{B}_k \setminus B_k} \|f\| d\mu(\xi) \right) \]

\[ \leq \frac{1}{\mu(B_k)} \left( \|f\|_{L^1(\mathcal{X})} + \frac{1}{\mu(B_k)} \|f\|_{L^\infty(\mathcal{X})} \right) \mu(B_k \Delta \tilde{B}_k). \]

The original partition \{\tilde{A}_k\}_{k=1}^{N+1} can be chosen so that \( |\mu(B_k) - \mu(\tilde{B}_k)| \) and \( \mu(B_k \Delta \tilde{B}_k) \) become arbitrarily small and thus, since the choice of \{\tilde{A}_k\}_{k=1}^{N+1} may depend on \( f \), also

\[ \left\| \frac{1}{\mu(B_k)} \int_{B_k} f d\mu - \frac{1}{\mu(B_k)} \int_{\tilde{B}_k} f d\mu \right\| \]

can be made arbitrarily small. Eventually, the same holds for

\[ \sum_{k=1}^{j+1} \int_{B_k \setminus \tilde{B}_k} \|E_j f(\xi) - \tilde{E}_j f(\xi)\|^p d\mu(\xi) \]

The second term

\[ \sum_{k=1}^{j+1} \int_{B_k \setminus \tilde{B}_k} \|E_j f(\xi) - \tilde{E}_j f(\xi)\|^p d\mu(\xi) \]

is easier to control, as each term can be estimated by

\[ \int_{B_k \setminus \tilde{B}_k} \|E_j f(\xi) - \tilde{E}_j f(\xi)\|^p d\mu(\xi) \leq \mu(B_k \Delta \tilde{B}_k) \|E_j f - \tilde{E}_j f\|_{L^\infty(\mathcal{X})}^p \leq \mu(B_k \Delta \tilde{B}_k) \|f\|_{L^\infty(\mathcal{X})}^p, \]

where \( \mu(B_k \Delta \tilde{B}_k) \) can again be made as small as we like.

All in all, we have established that

\[ \|M_R^{(N)} f\|_{L^p} \leq \sum_{j=1}^{N} \|E_j f - \tilde{E}_j f\|_{L^p(\mathcal{X})} + C \|f\|_{L^p(\mathcal{X})}, \]
Proof. This follows immediately from our earlier observations: Given a filtration \( \mathcal{F} \) and any positive integer \( N \), we can construct a Haar filtration \((\mathcal{F}_j)_{j=1}^N\) so that

\[
\mathcal{F}_1 \subset \mathcal{F}_2 \subset \cdots \subset \mathcal{F}_K = \mathcal{F}_1 \subset \mathcal{F}_{K+1} \subset \cdots \subset \mathcal{F}_N = \mathcal{F}_N.
\]

For any \( \mathcal{F}_N \)-measurable \( f \) we have

\[
\mathcal{R} \left( E_j f(A) : 1 \leq j \leq N \right) \leq \mathcal{R} \left( \tilde{E}_j f(A) : 1 \leq j \leq K_N \right), \quad A \in \text{bs} \mathcal{F}_N,
\]

and the claim follows.

Lemma 3.11. If \( \mathcal{X} \) has RMF \( p \) uniformly with respect to Haar filtrations on \((\Omega, \mathcal{F}, \mu)\), then it has RMF \( p \) uniformly with respect to filtrations of finite algebras on \((\Omega, \mathcal{F}, \mu)\).

Proof. This follows immediately from our earlier observations: Given a filtration \((\mathcal{F}_j)_{j=1}^\infty\) of finite algebras and any positive integer \( N \), we can construct a Haar filtration \((\mathcal{F}_j)_{j=1}^N\) so that

\[
\mathcal{F}_1 \subset \mathcal{F}_2 \subset \cdots \subset \mathcal{F}_K = \mathcal{F}_1 \subset \mathcal{F}_{K+1} \subset \cdots \subset \mathcal{F}_N = \mathcal{F}_N.
\]

For any \( \mathcal{F}_N \)-measurable \( f \) we have

\[
\mathcal{R} \left( E_j f(A) : 1 \leq j \leq N \right) \leq \mathcal{R} \left( \tilde{E}_j f(A) : 1 \leq j \leq K_N \right), \quad A \in \text{bs} \mathcal{F}_N,
\]

and the claim follows.

Lemma 3.12. If \( \mathcal{X} \) has RMF \( p \) uniformly with respect to filtrations of finite algebras then it has RMF \( p \) uniformly with respect to all filtrations.

Proof. Suppose that \((\mathcal{F}_j)_{j=1}^\infty\) is a filtration, \( N \) a positive integer, \( f \) a function in \( L^p(\mathcal{F}_N; \mathcal{X}) \) and that \( \varepsilon > 0 \). We begin by choosing simple functions \( s_j \in L^p(\mathcal{F}_j; \mathcal{X}), \) \( j = 1, \ldots, N \), so that

\[
\|E_j f - s_j\|_{L^p(\mathcal{X})} < \frac{\varepsilon}{2^{j+1}}.
\]

For \( j = 1, \ldots, N \), let \( \tilde{\mathcal{F}}_j \) be the finite algebra generated by \( s_1, \ldots, s_j \) and observe that \( \tilde{\mathcal{F}}_j \subset \tilde{\mathcal{F}}_{j+1} \), i.e. that \((\tilde{\mathcal{F}}_j)_{j=1}^N\) is a filtration. Now

\[
\|M^N_R f\|_{L^p} = \left( \int_{\Omega} \mathcal{R} \left( E_j f(\xi) : 1 \leq j \leq N \right)^p d\mu(\xi) \right)^{1/p} \leq \left( \int_{\Omega} \mathcal{R} \left( E_j f(\xi) - \tilde{E}_j f(\xi) : 1 \leq j \leq N \right)^p d\mu(\xi) \right)^{1/p} + \|M_R f\|_{L^p},
\]

where the maximal operator \( M_R \) satisfies \( \|M_R f\|_{L^p} \leq C\|f\|_{L^p(\mathcal{X})} \) for a constant \( C \) independent of the filtration \((\tilde{\mathcal{F}}_j)_{j=1}^N\). This independence is crucial, as \( \tilde{\mathcal{F}}_j \)'s arise from \( f \).

We then estimate

\[
\left( \int_{\Omega} \mathcal{R} \left( E_j f(\xi) - \tilde{E}_j f(\xi) : 1 \leq j \leq N \right)^p d\mu(\xi) \right)^{1/p} \leq \left( \int_{\Omega} \left( \sum_{j=1}^N \|E_j f(\xi) - \tilde{E}_j f(\xi)\|^p \right) d\mu(\xi) \right)^{1/p} \leq \sum_{j=1}^N \|E_j f - \tilde{E}_j f\|_{L^p(\mathcal{X})} \leq \sum_{j=1}^N \left( \|E_j f - s_j\|_{L^p(\mathcal{X})} + \|\tilde{E}_j f - s_j\|_{L^p(\mathcal{X})} \right).
\]

Furthermore, since

\[
\|\tilde{E}_j f - s_j\|_{L^p(\mathcal{X})} = \|\tilde{E}_j f - \tilde{E}_j s_j\|_{L^p(\mathcal{X})} = \|\tilde{E}_j (E_j f - s_j)\|_{L^p(\mathcal{X})} \leq \|E_j f - s_j\|_{L^p(\mathcal{X})}
\]

we have

\[
\|\tilde{E}_j f - s_j\|_{L^p(\mathcal{X})} \leq \|E_j f - s_j\|_{L^p(\mathcal{X})} < \frac{\varepsilon}{2^{j+1}}.
\]

Hence, we have proven that \( \|M^N_R f\|_{L^p} \) is uniformly small with respect to all filtrations.

\[
\|M^N_R f\|_{L^p} \leq C\|f\|_{L^p(\mathcal{X})}.
\]
we get
\[
\left( \int_{\Omega} R\left( E_j f(\xi) - \tilde{E}_j f(\xi) : 1 \leq j \leq N \right)^p d\mu(\xi) \right)^{1/p} \leq 2 \sum_{j=1}^N \|E_j f - s_j\|_{L^p(\mu)} + \sum_{j=1}^N \frac{\varepsilon}{2^j} < \varepsilon.
\]

We then show that the assumption on divisibility can be dropped.

**Lemma 3.13.** If $\mathcal{X}$ has $\text{RMF}_p$ with respect to any filtration on any divisible measure space with total measure one, then it has $\text{RMF}_p$ with respect to any filtration on any measure space with total measure one.

**Proof.** Suppose that $(\mathcal{F}_j)_{j=1}^\infty$ is a filtration on a not necessarily divisible measure space $(\Omega, \mathcal{F}, \mu)$ with $\mu(\Omega) = 1$. Now the $\sigma$-algebras $\tilde{\mathcal{F}}_j = \{ F \times [0, 1] : F \in \mathcal{F}_j \}$ form a filtration on the product of $(\Omega, \mathcal{F}, \mu)$ and the unit interval with Lebesgue measure, which obviously constitutes a divisible measure space. For a function $f \in L^p(\Omega; \mathcal{X})$ we put $\tilde{f}(\xi, t) = f(\xi), (\xi, t) \in \Omega \times [0, 1]$, and observe that $\|\tilde{f}\|_{L^p(\mathcal{X})} = \|f\|_{L^p(\mathcal{X})}$. Also $\tilde{E}_j \tilde{f}(\xi, t) = E_j f(\xi)$ for all $(\xi, t) \in \Omega \times [0, 1]$, and so $\|\tilde{M}_R \tilde{f}\|_{L^p(\mathcal{X})} = \|M_R f\|_{L^p(\mathcal{X})}$. \(\square\)

The results follow immediately for finite measure spaces: Suppose that $(\Omega, \mathcal{F}, \mu)$ is such. Then $\mu(\Omega)^{-1}\mu$ is a probability measure on $(\Omega, \mathcal{F})$ and evidently the conditional expectations are the same in these two measure spaces. Thus the Rademacher maximal operator remains unaltered and the inequality stating the boundedness is only a matter of scaling by $\mu(\Omega)^{-1}$.

Suppose then that $\mathcal{X}$ has $\text{RMF}_p$ uniformly with respect to any filtration on any finite measure space and let $(\Omega, \mathcal{F}, \mu)$ be a $\sigma$-finite measure space with a filtration $(\mathcal{F}_j)_{j=1}^\infty$. Since $\mathcal{F}_1$ is $\sigma$-finite (by definition), we can write $\Omega$ as a union of disjoint sets $A_k \in \mathcal{F}_1, k \in \mathbb{Z}_+$, each with finite measure. Let us define for positive integers $k$ the finite measures $\mu_k(A) = \mu(A \cap A_k)$ on $\mathcal{F}$. The conditional expectation of a function $f \in L^p(\Omega; \mathcal{X})$ with respect to $\mathcal{F}_j$ and $\mu_k$ is simply the conditional expectation of $1_{A_k} f$ with respect to $\mathcal{F}_j$ which further equals $1_{A_k} E_j f$. In symbols

$$E_j^{(k)} f = 1_{A_k} E_j f,$$

where $E_j^{(k)} f$ denotes the conditional expectation of $f$ with respect to $\mathcal{F}_j$ and $\mu_k$. Thus

$$\|M_R f\|_{L^p} = \sum_{k=1}^\infty \int_{A_k} R\left( E_j f(\xi) : j \in \mathbb{Z}_+ \right)^p d\mu_k(\xi)$$

$$= \sum_{k=1}^\infty \int_{A_k} R\left( E_j^{(k)} f(\xi) : j \in \mathbb{Z}_+ \right)^p d\mu_k(\xi)$$

$$\leq \sum_{k=1}^\infty C^p \int_{A_k} \|f(\xi)\|^p d\mu_k(\xi)$$

$$= C^p \|f\|_{L^p(\mathcal{X})}^p.$$

So far we have only considered filtrations indexed by positive integers. Suppose that $\mathcal{X}$ has $\text{RMF}_p$ with respect to any filtration indexed by $\mathbb{Z}_+$ on any $\sigma$-finite measure space and let $(\mathcal{F}_j)_{j \in \mathbb{Z}}$ be a filtration on $(\Omega, \mathcal{F}, \mu)$. Then $\mathcal{X}$ has $\text{RMF}_p$ with respect to $(\mathcal{F}_j)_{j=-N}^\infty$ with a constant independent of $N$ and thus by monotone convergence theorem with respect to $(\mathcal{F}_j)_{j \in \mathbb{Z}}$.

This concludes the proof of Theorem 3.7.
4 Rademacher maximal function for martingales

We begin by collecting some probabilistic definitions and stating the UMD-condition for Banach spaces.

4.1 Martingales in Banach spaces

A stochastic process (a sequence of random variables on some probability space) \( X = (X_j)_{j=1}^{\infty} \) is always adapted to the filtration \((\mathcal{F}_j)_{j=1}^{\infty}\), where \( \mathcal{F}_j \) is the \( \sigma \)-algebra \( \sigma(X_1, \ldots, X_j) \) generated by \( X_1, \ldots, X_j \), in the sense that each \( X_j \) is \( \mathcal{F}_j \)-measurable. We call a sequence of \( L^1 \)-random variables a martingale if \( \mathbb{E}(X_k|\mathcal{F}_j) = X_j \) whenever \( j \leq k \). A sequence \( \xi = (\xi_j)_{j=1}^{\infty} \) of real \( L^1 \)-random variables is called a submartingale if \( \mathbb{E}(\xi_k|\mathcal{F}_j) \geq \xi_j \) whenever \( j \leq k \). Note that if \( X = (X_j)_{j=1}^{\infty} \) is a martingale in a Banach space, then \( (\|X_j\|)_{j=1}^{\infty} \) is a submartingale.

From any stochastic process \( X = (X_j)_{j=1}^{\infty} \) we can construct a martingale \( \tilde{X} = (\tilde{X}_j)_{j=1}^{\infty} \) by defining

\[
\tilde{X}_j = X_1 + \sum_{k=2}^{j} (X_k - \mathbb{E}(X_k|\mathcal{F}_{k-1})).
\]

Observe, that \( \tilde{X} \) is adapted to the filtration generated by \( X \). Also, if \( X \) itself is a martingale, then \( \tilde{X} = X \).

Note that for any martingale \( X = (X_j)_{j=1}^{\infty} \) we have \( \mathbb{E}X_j = \mathbb{E}X_k \) for all \( j, k \in \mathbb{Z}_+ \). It is customary to write \( \mathcal{F}_0 \) for the trivial \( \sigma \)-algebra and \( X_0 \) for the common expectation of \( X_j \)'s. By defining \( Y_j = X_j - X_0 \) one can restrict to martingales \( Y = (Y_j)_{j=1}^{\infty} \) for which \( Y_0 = \mathbb{E}Y_j = 0 \).

By the definition of a conditional expectation, every submartingale \( \xi = (\xi_j)_{j=1}^{\infty} \) satisfies

\[
\mathbb{E}1_A \xi_j \leq \mathbb{E} \left( 1_A \mathbb{E}(\xi_k|\mathcal{F}_j) \right) = \mathbb{E} 1_A \xi_k
\]

whenever \( j \leq k \) and \( A \in \mathcal{F}_j \). Thus for every martingale \( X = (X_j)_{j=1}^{\infty} \) we have

\[
\mathbb{E}1_A \|X_j\| \leq \mathbb{E}1_A \|X_k\|
\]

whenever \( j \leq k \) and \( A \in \mathcal{F}_j \).

We say that a stochastic process \( X = (X_j)_{j=1}^{\infty} \) is \( L^p \)-bounded for \( p \in [1, \infty) \) if \( \|X\|_p := \sup_{j \in \mathbb{Z}_+} \mathbb{E}\|X_j\|^p < \infty \) and for \( p = \infty \) if the infimum \( \|X\|_\infty \) of all \( C \) for which every \( \|X_j\| \leq C \) almost surely, is finite. A stochastic process \( X = (X_j)_{j=1}^{\infty} \) is said to be simple if the algebras \( \mathcal{F}_j \) are finite (i.e. if the random variables \( X_j \) are simple). A simple martingale is called a (dyadic/standard) Haar martingale if the algebras \( \mathcal{F}_j \) form a (dyadic/standard) Haar filtration.

Given a martingale \( (X_j)_{j=1}^{\infty} \) we define its difference sequence \( (D_j)_{j=1}^{\infty} \) by \( D_j = X_j - X_{j-1} \) for \( j \geq 1 \). If \( (X_j)_{j=1}^{\infty} \) is an \( L^2 \)-martingale in a Hilbert space, then its difference sequence is orthogonal in the sense that \( \mathbb{E}(D_j, D_k) = 0 \) whenever \( j \neq k \) (since conditional expectations are orthogonal projections). Thus for any choice of signs \( \varepsilon_j \in \{-1, 1\} \) we have

\[
\mathbb{E} \left\| \sum_{j=1}^{N} \varepsilon_j D_j \right\|^2 = \mathbb{E} \left\| \sum_{j=1}^{N} D_j \right\|^2,
\]

which motivates the next definition. Observe that here \( \varepsilon_j \)'s are not Rademacher variables but just arbitrary signs.

**Definition 4.1.** Let \( 1 < p < \infty \). A Banach space \( E \) is said to have UMD\(_p\) if there exists a constant \( C \) such that the difference sequence \( (D_j)_{j=1}^{N} \) of any \( L^p \)-martingale in \( E \) satisfies

\[
\mathbb{E} \left\| \sum_{j=1}^{N} \varepsilon_j D_j \right\|^p \leq C \mathbb{E} \left\| \sum_{j=1}^{N} D_j \right\|^p,
\]

whenever \( \varepsilon_j \in \{-1, 1\} \).
This inequality can be viewed as a requirement for uniform boundedness of certain martingale transforms. Indeed, if \( X = (X_j)_{j=1}^{\infty} \) is a martingale in \( E \) and \( v = (v_j)_{j=1}^{\infty} \) is a real \( L^\infty \)-bounded stochastic process (on the same probability space), we define

\[
(v \ast X)_j = \sum_{k=1}^{j} v_k D_k, \quad j \in \mathbb{Z}_+,
\]

where \( (D_j)_{j=1}^{\infty} \) is the difference sequence of \( X \). If \( v \) is predictable with respect to \( X \) in the sense that each \( v_j \) is \( \mathcal{F}_{j-1} \)-measurable (and \( v_1 \) is constant almost surely), then the martingale transform \( v \ast X = ((v \ast X_j)_{j=1}^{\infty} \) is itself a martingale.

For UMD\(_p \) it is thus required that there exists a constant \( C \) such that for every sequence \( \varepsilon = (\varepsilon_j)_{j=1}^{\infty} \) of signs \( \{-1,1\} \) and every martingale \( X = (X_j)_{j=1}^{\infty} \) we have

\[
\mathbb{E}\| (\varepsilon \ast X)_N \|^p \leq C \mathbb{E}\| X_N \|^p.
\]

Taking the supremum over all \( N \) on both sides, we can write this more compactly as

\[
\| \varepsilon \ast X \|_p \leq C \| X \|_p.
\]

This property is independent of \( p \) in the sense that if a Banach space has UMD\(_p \) for one \( p \in (1,\infty) \) then it has UMD\(_p \) for all \( p \in (1,\infty) \) (see Maurey [20]). Thus the parameter \( p \) can be omitted from the definition.

In his paper [4] (and also in [5]), Burkholder shows how UMD-spaces can be characterized by the so-called \( \zeta \)-convexity. A Banach space \( E \) is said to be \( \zeta \)-convex if there exists a biconvex function (a function convex in both variables separately) \( \zeta : E \times E \to \mathbb{R} \) which satisfies \( \zeta(0,0) > 0 \) and \( \zeta(x,y) \leq \|x+y\| \) for all unit vectors \( x \) and \( y \).

One can ask how the RMF-property relates to the UMD-property. First of all, every UMD-space can be shown to be reflexive (see for instance [20]). Our typical example \( \mathcal{L}(H,E) \) is usually non-reflexive, but has RMF at least when \( H \) has cotype 2 and \( E \) has type 2. More interestingly, James constructed in [16] a non-reflexive Banach space \( E \) with type 2. Thus \( E \hookrightarrow \mathcal{L}(H,E) \) can have RMF without being a UMD-space. Bourgain showed in [3] that the Schatten - von Neumann class \( S_p(H_1,H_2) \) is UMD for \( 1 < p < \infty \). As \( H_1 \) and \( H_2 \) are spaces of type and cotype 2, it follows from the third remark after Definition 2.2 that \( S_p(H_1,H_2) \) has RMF as a space of operators. It has also been shown in [15] that \( S_p(H_1,H_2) \) has RMF as \( \mathcal{L}(\mathcal{C}, S_p(H_1,H_2)) \).

### 4.2 Maximal functions for martingales

Let \( \mathcal{X} \subset \mathcal{L}(H,E) \) be a Banach space. For a stochastic process \( X = (X_j)_{j=1}^{\infty} \) in \( \mathcal{X} \) we define the Doob and Rademacher maximal functions by

\[
X^* = \sup_{j \in \mathbb{Z}_+} \| X_j \| \quad \text{and} \quad X^*_R = \mathcal{R}\left( X_j : j \in \mathbb{Z}_+ \right),
\]

respectively.

The boundedness of Doob’s maximal operator is expressed in the following well-known results. The next lemma can be thought to replace Marcinkiewicz interpolation in our earlier proofs of boundedness of maximal operators. In addition, it gives the sharp constant.

**Lemma 4.2.** Let \( \xi \) and \( \eta \) be non-negative \( L^p \)-random variables with \( 1 < p < \infty \). If \( \xi \) and \( \eta \) satisfy

\[
\lambda \mathbb{P}(\xi > \lambda) \leq \mathbb{E}1_{\{\xi > \lambda\}} \eta
\]

whenever \( \lambda > 0 \), then

\[
\mathbb{E}\xi^p \leq (p')^p \mathbb{E}\eta^p,
\]

where \( p' \) is the Hölder conjugate of \( p \).
Proof. We simply calculate
\[
\mathbb{E} \xi^p = \int_0^\infty p \lambda^{p-1} \mathbb{P}(\xi > \lambda) \, d\lambda \\
\leq \int_0^\infty p \lambda^{p-2} \mathbb{E} 1_{\{\xi > \lambda\}} \eta \, d\lambda \\
= \mathbb{E} \int_0^\xi p \lambda^{p-2} \eta \, d\lambda \\
= \mathbb{E} \frac{p}{p-1} \xi^{p-1} \eta \\
\leq p'(\mathbb{E} \xi^{(p-1)p'})^{1/p'}(\mathbb{E} \eta^p)^{1/p} \\
= p'(\mathbb{E} \xi^p)^{(p-1)/p}(\mathbb{E} \eta^p)^{1/p}.
\]
Dividing both sides by \((\mathbb{E} \xi^p)^{(p-1)/p}\) gives the desired inequality. \qed

Proposition 4.3. (Doob’s inequalities)

1. Let \(1 < p < \infty\). Every \(L^p\)-bounded martingale \(X\) satisfies
\[
\mathbb{E} \|X^*\|^p \leq (p')^p \|X\|^p,
\]
where \(p'\) is the Hölder conjugate of \(p\).

2. Every \(L^\infty\)-bounded stochastic process \(X\) satisfies \(X^* \leq \|X\|_\infty\) almost surely.

3. Every \(L^1\)-bounded martingale \(X\) satisfies
\[
\lambda \mathbb{P}(X^* > \lambda) \leq \|X\|_1
\]
whenever \(\lambda > 0\).

Proof. Let \(\lambda > 0\) and consider the event
\[
\left\{ \max_{1 \leq j \leq N} \|X_j\| > \lambda \right\}
\]
for a positive integer \(N\). Decompose it into disjoint events
\[
A_k = \left\{ \max_{1 \leq j \leq k-1} \|X_j\| \leq \lambda < \|X_k\| \right\}, \quad k = 1, \ldots, N,
\]
and note that \(A_k \in \mathcal{F}_k\). Since \(X\) is a martingale we have
\[
\mathbb{P}(A_k) \leq \frac{1}{\lambda} \mathbb{E} 1_{A_k} \|X_k\| \leq \frac{1}{\lambda} \mathbb{E} 1_{A_k} \|X_N\|.
\]
If \(X\) is \(L^p\)-bounded with \(1 < p < \infty\), write
\[
\xi_N = \max_{1 \leq j \leq N} \|X_j\| \quad \text{and} \quad \eta_N = \|X_N\|.
\]
Now
\[
\lambda \mathbb{P}(\xi_N > \lambda) = \lambda \sum_{k=1}^N \mathbb{P}(A_k) \leq \sum_{k=1}^N \mathbb{E} 1_{A_k} \|X_N\| = \mathbb{E} 1_{\{\xi_N > \lambda\}} \eta_N
\]
and we may apply Lemma 4.2 to get
\[
\mathbb{E} \xi_N^p \leq (p')^p \mathbb{E} \eta_N^p.
\]
Now \( \xi_N \rightarrow \|X^*\| \) and \( \eta_N \rightarrow \|X\| \), as \( N \rightarrow \infty \), and so
\[
\mathbb{E}[\|X^*\|^p] = \lim_{N \to \infty} \mathbb{E}[\xi_N^p] \leq (p')^p \lim_{N \to \infty} \mathbb{E}[\eta_N^p] = (p')^p \|X\|^p.
\]

If \( X \) is \( L^\infty \)-bounded stochastic process, then every \( \|X_j\| \leq \|X\|_\infty \) almost surely. After taking a countable supremum we still have \( X^* = \sup_{j \in \mathbb{Z}_+} \|X_j\| \leq \|X\|_\infty \) almost surely.

Finally, if a martingale \( X \) is \( L^1 \)-bounded, then
\[
\lambda \mathbb{P}\left( \max_{1 \leq j \leq N} \|X_j\| > \lambda \right) = \sum_{k=1}^{N} \lambda \mathbb{P}(A_k) \leq \mathbb{E}[\|X_N\|].
\]

Again
\[
\{X^* > \lambda\} = \bigcup_{N=1}^{\infty} \left\{ \max_{1 \leq j \leq N} \|X_j\| > \lambda \right\}
\]
and thus
\[
\lambda \mathbb{P}(X^* > \lambda) = \lim_{N \to \infty} \lambda \mathbb{P}\left( \max_{1 \leq j \leq N} \|X_j\| > \lambda \right) \leq \limsup_{N \to \infty} \mathbb{E}[\|X_N\|] \leq \|X\|_1.
\]

As the boundedness of the Rademacher maximal operator does not depend on the underlying measure space, the following lemma allows us to formulate the RMF\(_p\)-property using probabilistic notions.

**Proposition 4.4.** Let \( 1 < p < \infty \). Then \( \mathcal{X} \) has RMF\(_p\) if and only if there exists a constant \( C \) such that
\[
\mathbb{E}[|X_R|^p] \leq C^p \|X\|^p_p
\]
for any \( L^p \)-bounded martingale \( X \) in \( \mathcal{X} \).

**Proof.** Suppose that \( \mathcal{X} \) has RMF\(_p\) and that \( X = (X_j)_{j=1}^{\infty} \) is defined on a probability space \( (\Omega, \mathcal{F}, \mathbb{P}) \). Since for any positive integer \( N \), \( X_j = E_jX_N \) whenever \( 1 \leq j \leq N \), we see that
\[
\mathbb{E}R\left(X_{1 \leq j \leq N}\right)^p = \int_{\Omega} R\left(X_{1 \leq j \leq N}\right)^p d\mathbb{P}(\omega)
\]
\[
= \int_{\Omega} R\left(E_jX_N\right)^p d\mathbb{P}(\omega)
\]
\[
\leq C^p \int_{\Omega} \|X_N\|^p d\mathbb{P}(\omega)
\]
\[
= C^p \mathbb{E}\|X_N\|^p,
\]
where \( C \) is the RMF\(_p\)-constant of \( \mathcal{X} \) and thus independent of \( N \). Hence
\[
\mathbb{E}[|X_R|^p] = \mathbb{E} \lim_{N \to \infty} R\left(X_{1 \leq j \leq N}\right)^p = \lim_{N \to \infty} \mathbb{E}R\left(X_{1 \leq j \leq N}\right)^p \leq C^p \lim_{N \to \infty} \mathbb{E}\|X_N\|^p = C^p \mathbb{E}\|X\|^p
\]
by the monotone convergence theorem.

Suppose on the other hand, that the inequality is satisfied for all \( L^p \)-bounded martingales. Now every \( f \in L^p(0,1;\mathcal{X}) \) defines a martingale \( X = (E_jf)_{j=1}^{\infty} \), where the conditional expectations are taken with respect to the dyadic filtration on \((0,1)\) and so
\[
\|M_Rf\|^p_{L^p} = \mathbb{E}[|X_R|^p] \leq C^p \|X\|^p_p \leq C^p \|f\|^p_{L^p(\mathcal{X})}.
\]
4.3 The weak RMF-property

Applying ideas from Burkholder [4] we will show that \( X \) has RMF\(^p \) for some \( p \in (1, \infty) \) if and only if it has weak RMF, i.e. if there exists a constant \( C \) such that all \( L^1 \)-bounded martingales \( X \) in \( X \) satisfy
\[
\mathbb{P}(X^*_R > \lambda) \leq \frac{C}{\lambda} \|X\|_1
\]
whenever \( \lambda > 0 \).

To show the necessity of this condition we invoke the Gundy decomposition (see Gundy [11] for the original proof). We will need the concept of a stopping time: We say that a random variable \( \tau \) in \( \mathbb{Z}_+ \cup \{\infty\} \) is a stopping time with respect to a stochastic process \( X \) if \( \{\tau = j\} \) is in \( \mathcal{F}_j \) for every positive integer \( j \). In this case we define
\[
X_{\tau} = \sum_{j=1}^{\infty} 1_{\{\tau = j\}} X_j.
\]
Observe that \( X_{\tau} = 0 \) when \( \tau = \infty \).

**Lemma 4.5.** If \( X \) is an \( L^1 \)-bounded martingale and \( \tau \) is a stopping time with respect to \( X \), then
\[
\mathbb{E}\|X_{\tau}\| \leq \|X\|_1.
\]

**Proof.** For any positive integer \( N \) we have
\[
\sum_{j=1}^{N} \mathbb{E} 1_{\{\tau = j\}} \|X_j\| \leq \sum_{j=1}^{N} \mathbb{E} 1_{\{\tau = j\}} \|X_N\| \leq \mathbb{E}\|X_N\| \leq \|X\|_1.
\]
Now \( \{\tau < \infty\} = \bigcup_{j \in \mathbb{Z}_+} \{\tau = j\} \) and so letting \( N \) go to infinity we get
\[
\mathbb{E}\|X_{\tau}\| \leq \sum_{j=1}^{\infty} \mathbb{E} 1_{\{\tau = j\}} \|X_j\| \leq \|X\|_1.
\]

**Theorem 4.6.** (Gundy decomposition) Suppose that \( X \) is an \( L^1 \)-bounded martingale in \( X \) and that \( \lambda > 0 \). There exists a decomposition \( X = G + H + B \) of \( X \) into martingales \( G, H \) and \( B \) which satisfy
\begin{enumerate}
    \item \( \|G\|_1 \leq 4\|X\|_1 \) and \( \|G\|_\infty \leq 2\lambda \),
    \item \( \mathbb{E}\|H_1\| + \sum_{j=2}^{\infty} \mathbb{E}\|H_j - H_{j-1}\| \leq 4\|X\|_1 \), \( (H = (H_j)^\infty_{j=1}) \),
    \item \( \mathbb{P}(B^* > 0) \leq \frac{3}{X} \|X\|_1 \).
\end{enumerate}

**Proof.** We want to decompose \( X \) into three pieces for a fixed \( \lambda > 0 \) according to the stopping time
\[
\tau = \min\{j \in \mathbb{Z}_+: \|X_j\| > \lambda\}.
\]
Another stopping time \( \sigma \), which will be specified later, is needed in order to make sure our pieces satisfy the required properties even after we modify them into martingales.

Let us write
\[
g_j = 1_{\{\tau > j, \sigma \geq j\}}, \quad h_j = 1_{\{\tau = j, \sigma \geq j\}} \quad \text{and} \quad b_j = 1_{\{\tau \land \sigma < j\}}
\]
so that
\[
X = g \ast X + h \ast X + b \ast X,
\]

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where \( g = (g_j)_{j=1}^{\infty}, h = (h_j)_{j=1}^{\infty} \) and \( b = (b_j)_{j=1}^{\infty} \). We observe that \( b \) is predictable with respect to \( X \) so that \( B = b \ast X \) is a martingale. Note also that separately each \( g_j \) and \( h_j \) is only \( \mathcal{F}_j \)-measurable, but together every \( g_j + h_j \) is \( \mathcal{F}_{j-1} \)-measurable.

Let us first modify \( g \ast X \) into a martingale by

\[
G_j = (g \ast X)_j = g_1D_1 + \sum_{k=2}^{j}(g_kD_k - \mathbb{E}(g_kD_k|\mathcal{F}_{k-1})).
\]

Now

\[
-\mathbb{E}(g_kD_k|\mathcal{F}_{k-1}) = \mathbb{E}(h_kD_k|\mathcal{F}_{k-1})
\]

and so

\[
G_j = \sum_{k=1}^{j}1_{\{\tau > k, \sigma \geq k\}}D_k + \sum_{k=2}^{j}1_{\{\sigma \geq k\}}\mathbb{E}(1_{\{\tau = k\}}D_k|\mathcal{F}_{k-1}) = \sum_{k=1}^{(\tau - 1) \land \sigma \land j}D_k + \sum_{k=2}^{\sigma \land j}\mathbb{E}(1_{\{\tau = k\}}D_k|\mathcal{F}_{k-1}).
\]

If we now take

\[
\sigma = \min \left\{ j \in \mathbb{Z}_+: \sum_{k=2}^{j+1}\mathbb{E}(1_{\{\tau = k\}}\|D_k\||\mathcal{F}_{k-1}) > \lambda \right\}
\]

we get

\[
\|G_j\| \leq \|X_{(\tau - 1) \land \sigma \land j}\| + \left\| \sum_{k=2}^{\sigma \land j}\mathbb{E}(1_{\{\tau = k\}}D_k|\mathcal{F}_{k-1}) \right\| \leq 2\lambda
\]

and so the requirement that \( \|G\|_{\infty} \leq 2\lambda \) is satisfied. To see that \( \|G\|_1 \leq 4\|X\|_1 \) we argue as follows: On the event of \( \tau < \infty \) we have

\[
\|X_{(\tau - 1) \land \sigma \land j}\| \leq \lambda < \|X_{\tau}\|
\]

by the definition of \( \tau \), while when \( \tau = \infty \) we have \((\tau - 1) \land \sigma \land j = \sigma \land j \). Thus

\[
\mathbb{E}\|X_{(\tau - 1) \land \sigma \land j}\| \leq \mathbb{E}\|X_{\tau}\| + \mathbb{E}1_{\{\tau = \infty\}}\|X_{\sigma \land j}\| \leq 2\|X\|_1.
\]

On the other hand, taking the norm inside the conditional expectation we see that

\[
\left\| \sum_{k=2}^{\sigma \land j}\mathbb{E}(1_{\{\tau = k\}}D_k|\mathcal{F}_{k-1}) \right\| \leq \sum_{k=2}^{j}\mathbb{E}(1_{\{\tau = k\}}\|D_k\||\mathcal{F}_{k-1}) = \sum_{k=2}^{j}\mathbb{E}1_{\{\tau = k\}}\|D_k\|.
\]

Using now the fact that \( \|X_{k-1}\| \leq \lambda < \|X_k\| \) when \( \tau = k \) we get

\[
\sum_{k=2}^{j}\mathbb{E}1_{\{\tau = k\}}\|D_k\| \leq 2\sum_{k=2}^{j}\mathbb{E}1_{\{\tau = k\}}\|X_k\| \leq 2\sum_{k=2}^{j}\mathbb{E}1_{\{\tau = k\}}\|X_j\| \leq 2\mathbb{E}\|X_j\| \leq 2\|X\|_1.
\]

Let us then turn to \( h \ast X \) and define

\[
H_j = (h \ast X)_j = h_1D_1 + \sum_{k=2}^{j}(h_kD_k - \mathbb{E}(h_kD_k|\mathcal{F}_{k-1})).
\]

We can forget about \( \sigma \) for a moment and estimate

\[
\|H_j - H_{j-1}\| \leq 1_{\{\tau = j\}}\|D_j\| + \mathbb{E}(1_{\{\tau = j\}}\|D_j\||\mathcal{F}_{j-1}),
\]

and note that the two terms on the right have the same expectation. Now

\[
\mathbb{E}\|H_1\| + \sum_{j=2}^{\infty}\mathbb{E}\|H_j - H_{j-1}\| \leq 2\sum_{j=1}^{\infty}\mathbb{E}1_{\{\tau = j\}}\|D_j\| \leq 4\sum_{j=1}^{\infty}\mathbb{E}1_{\{\tau = j\}}\|X_j\| \leq 4\mathbb{E}\|X\| \leq 4\|X\|_1,
\]

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where we again used the stopping time to see that \( \|X_{j-1}\| \leq \lambda < \|X_j\| \) when \( \tau = j \).

Finally, \( b_j \) is non-zero only in the event of \( \tau \land \sigma < \infty \). Thus

\[
\mathbb{P}(B^* > 0) \leq \mathbb{P}(\tau < \infty) + \mathbb{P}(\sigma < \infty).
\]

By the definition of \( \tau \), Doob’s \( L^1 \)-inequality gives

\[
\mathbb{P}(\tau < \infty) = \mathbb{P}(X^* > \lambda) \leq \frac{1}{\lambda} ||X||_1.
\]

Furthermore

\[
\sum_{j=2}^{\infty} \mathbb{E}(1_{\{\tau = j\}} \|D_j\| |F_{j-1}) \leq 2 \|X\|_1
\]

so that

\[
\mathbb{P}(\sigma < \infty) = \mathbb{P}\left(\sum_{k=2}^{\infty} \mathbb{E}(1_{\{\tau = k\}} \|D_k\| |F_{k-1}) > \lambda\right) \leq \frac{2}{\lambda} ||X||_1,
\]

which concludes the proof.

Lemma 4.7. For any absolutely summable sequence \((T_k)_{k=1}^{\infty}\) of operators in \( X \) we have

\[
\mathcal{R}\left(\sum_{k=1}^{j} T_k : j \in \mathbb{Z}_+\right) \leq \sum_{k=1}^{\infty} \|T_k\|.
\]

Proof. For any positive integer \( N \) and vectors \( x_j \) in \( H \) we may first rearrange the terms and write

\[
\mathbb{E}\left\|\sum_{j=1}^{N} \varepsilon_j \left(\sum_{k=1}^{j} T_k\right) x_j\right\| = \mathbb{E}\left\|\sum_{k=1}^{N} \sum_{j=k}^{N} \varepsilon_j T_k x_j\right\|
\]

\[
= \mathbb{E}\left\|\sum_{k=1}^{N} \sum_{j=k}^{N} \varepsilon_j T_k x_j\right\|
\]

\[
\leq \sum_{k=1}^{N} \mathbb{E}\left\|\sum_{j=k}^{N} \varepsilon_j T_k x_j\right\|
\]

\[
\leq \sum_{k=1}^{N} \|T_k\| \mathbb{E}\left\|\sum_{j=k}^{N} \varepsilon_j x_j\right\|
\]

\[
\leq \sum_{k=1}^{\infty} \|T_k\| \mathbb{E}\left\|\sum_{j=1}^{N} \varepsilon_j x_j\right\|
\]

which proves the claim.

Proposition 4.8. If \( X \) has RMF\( _p \) for some \( p \in (1, \infty) \), then it has weak RMF.

Proof. Taking the Gundy decomposition of \( X \) at height \( \lambda \) we may write

\[
\mathbb{P}(X^*_R > \lambda) \leq \mathbb{P}(B^*_R > \lambda/3) + \mathbb{P}(H^*_R > \lambda/3) + \mathbb{P}(G^*_R > \lambda/3),
\]

and estimate each term separately. Firstly \( \mathbb{P}(B^*_R > 0) = \mathbb{P}(B^* > 0) \), since \( B^*_R = 0 \) if and only if \( B^* = 0 \). Thus

\[
\mathbb{P}(B^*_R > \lambda/3) \leq \mathbb{P}(B^*_R > 0) = \mathbb{P}(B^* > 0) \leq \frac{3}{\lambda} ||X||_1.
\]

Secondly, by Lemma 4.7, we see that

\[
H^*_R = \mathcal{R}\left(H_j : j \in \mathbb{Z}_+\right) = \mathcal{R}\left(\sum_{k=1}^{j} (H_k - H_{k-1}) : j \in \mathbb{Z}_+\right) \leq \sum_{j=1}^{\infty} \|H_j - H_{j-1}\|.
\]

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Hence

\[ \mathbb{P}(H_R > \lambda/3) \leq \mathbb{P}\left( \sum_{j=1}^{\infty} \|H_j - H_{j-1}\| > \frac{\lambda}{3} \right) \]

\[ \leq \frac{3}{\lambda} \mathbb{E} \sum_{j=1}^{\infty} \|H_j - H_{j-1}\| \]

\[ = \frac{3}{\lambda} \sum_{j=1}^{\infty} \mathbb{E}\|H_j - H_{j-1}\| \leq \frac{12}{\lambda} \|X\|_1. \]

Thirdly, by Lemma 4.4,

\[ \mathbb{P}(G_R^* > \lambda/3) \leq \left( \frac{3}{\lambda} \right)^p \mathbb{E}|G_R^*|^p \leq C \left( \frac{3}{\lambda} \right)^p \|G\|_p^p \leq C \frac{3^p 2^{p-1}}{\lambda} \|G\|_1 \leq C \frac{3^p 2^{p+1}}{\lambda} \|X\|_1, \]

where the property \( \|G\|_\infty \leq 2\lambda \) was used to deduce that

\[ \|G\|^p = \sup_{j \in \mathbb{R}_+} \mathbb{E}|G_j|^p \leq \|G\|_\infty^{p-1} \sup_{j \in \mathbb{R}_+} \mathbb{E}|G_j| \leq (2\lambda)^{p-1} \|G\|_1. \]

\[ \square \]

We then turn to the converse. The argument is based on a “good-\(\lambda\) inequality” 4.10 which says roughly that the chance of \( X_R^* \) being large while \( X^* \) diminishes is vanishingly small.

**Lemma 4.9.** If \( X = (X_j)_{j=1}^{\infty} \) is a standard Haar martingale, then \( (\|D_j\|)_{j=1}^{\infty} \) is predictable.

**Proof.** For every \( j \geq 1 \) there is exactly one event \( B \in \mathcal{B}_j \) on which \( X_j - X_{j-1} \) is non-zero. As \( B = B_1 \cup B_2 \) for some \( B_1, B_2 \in \mathcal{B}_j \) with \( \mathbb{P}(B_1) = \mathbb{P}(B_2) \) and \( \mathbb{E}(X_j - X_{j-1}, \mathcal{F}_{j-1}) = 0 \), there exists a \( T \in \mathcal{X} \) such that \( X_j - X_{j-1} = 1_{B_1} T - 1_{B_2} T \). Consequently,

\[ \|D_j\| = \|X_j - X_{j-1}\| = 1_{B_1}\|T\| + 1_{B_2}\|T\| = 1_B\|T\| \]

and so \( \|D_j\| \) is \( \mathcal{F}_{j-1} \)-measurable. \[ \square \]

**Lemma 4.10.** Suppose that \( \mathcal{X} \) has weak RMF. Then for all \( \delta \in (0, 1) \) and \( \beta > 2\delta + 1 \) there exists an \( \alpha(\delta) > 0 \) which tends to zero as \( \delta \searrow 0 \) and which is such that for all \( L^p \)-bounded standard Haar martingales \( X \) in \( \mathcal{X} \) we have

\[ \mathbb{P}(X_R^* > \beta\lambda, X^* \leq \delta\lambda) \leq \alpha(\delta) \mathbb{P}(X_R^* > \lambda), \]

whenever \( \lambda > 0 \).

**Proof.** Let \( X = (X_j)_{j=1}^{\infty} \) be an \( L^p \)-bounded standard Haar martingale in \( \mathcal{X} \). Define the stopping times

\[ \tau_1 = \min\left\{ j \in \mathbb{Z}_+ : \mathcal{R}(X_k : 1 \leq k \leq j) > \lambda \right\}, \]

\[ \tau_2 = \min\left\{ j \in \mathbb{Z}_+ : \mathcal{R}(X_k : 1 \leq k \leq j) > \beta\lambda \right\}, \]

\[ \sigma = \min\left\{ j \in \mathbb{Z}_+ : \|X_j\| > \delta\lambda \ \text{or} \ \|D_{j+1}\| > 2\delta\lambda \right\} \]

and put

\[ v_j = 1_{\{\tau_1 \leq j \leq \tau_2 \wedge \sigma\}}. \]

Now \( v = (v_j)_{j=1}^{\infty} \) is predictable and so \( v \ast X \) is a martingale. When \( \tau_1 < \tau_2 \wedge \sigma \) we calculate

\[ (v \ast X)_j = \sum_{k=1}^{j} v_k D_k = \sum_{\tau_1 < k \leq \tau_2 \wedge \sigma: j} (X_k - X_{k-1}) = \begin{cases} 0, & 1 \leq j \leq \tau_1, \\ X_j - X_{\tau_1}, & \tau_1 < j \leq \tau_2 \wedge \sigma, \\ X_{\tau_2 \wedge \sigma} - X_{\tau_1}, & j > \tau_2 \wedge \sigma. \end{cases} \]

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We first show that
\[ \{ X^*_R > \beta \lambda, \ X^* \leq \delta \lambda \} \subset \{ (v \ast X)^*_R > (\beta - 2\delta - 1)\lambda \}. \]

Suppose that \( X^*_R > \beta \lambda \) and \( X^* \leq \delta \lambda \). Now \( \tau_2 < \infty \) and as \( \| D_{j+1} \| \leq \| X_{j+1} \| + \| X_j \| \leq 2\delta \lambda \) for all \( j \), we also have \( \sigma = \infty \). Since for every \( j \)
\[ \mathcal{R}(X_k : 1 \leq k \leq j) \leq \mathcal{R}(X_k : 1 \leq k \leq j - 1) + \| D_j \|, \]
we have
\[ \mathcal{R}(X_k : 1 \leq k \leq \tau_2 - 1) \geq \mathcal{R}(X_k : 1 \leq k \leq \tau_2) - \| D_{\tau_2} \| > (\beta - 2\delta)\lambda > \lambda. \]
Thus \( \tau_1 < \tau_2 \) and
\[ (v \ast X)_j = \begin{cases} 0, & 1 \leq j \leq \tau_1, \\ X_j - X_{\tau_1}, & \tau_1 < j \leq \tau_2, \\ X_{\tau_2} - X_{\tau_1}, & j > \tau_2. \end{cases} \]
Hence
\[ (v \ast X)^*_R = \mathcal{R}(X_j - X_{\tau_1} : \tau_1 < j \leq \tau_2) \]
\[ \geq \mathcal{R}(X_j : \tau_1 < j \leq \tau_2) - \| X_{\tau_1} \| \]
\[ \geq \mathcal{R}(X_j : 1 \leq j \leq \tau_2) - \mathcal{R}(X_j : 1 \leq j \leq \tau_1) - \| X_{\tau_1} \| \]
\[ \geq \mathcal{R}(X_j : 1 \leq j \leq \tau_2) - \mathcal{R}(X_j : 1 \leq j < \tau_1) - 2\| X_{\tau_1} \| \]
\[ > \beta \lambda - \lambda - 2\delta \lambda \]
\[ > (\beta - 2\delta - 1)\lambda, \]
as required.

We then aim to find a suitable upper bound for \( \| v \ast X \|_1 \). To do this, consider cases \( \{ \tau_1 < \tau_2 \land \sigma \} \) and \( \{ \tau_1 \geq \tau_2 \land \sigma \} \) separately. Assuming the former, an earlier calculation gives
\[ \| (v \ast X)_j \| \leq \| X_{\tau_2 \land \sigma \land j} \| + \| X_{\tau_1} \|, \]
where \( \| X_{\tau_1} \| \leq \delta \lambda \). Furthermore
\[ \| X_{\tau_2 \land \sigma \land j} \| \leq \| X_{\tau_2 \land \sigma \land j - 1} \| + \| D_{\tau_2 \land \sigma \land j} \| \leq \delta \lambda + 2\delta \lambda \]
and so \( \| (v \ast X)_j \| \leq 4\delta \lambda \) for all \( j \in \mathbb{Z}_+ \). In the latter case each \( v_j = 0 \) and so \( (v \ast X)_j = 0 \). This happens in particular on the occasion of \( \{ \tau_1 = \infty \} = \{ X^*_R \leq \lambda \} \). Thus in conclusion
\[ (v \ast X)^*_R \leq 4\delta \lambda \mathbb{1}_{\{ \tau_1 = \infty \}} \]
and so
\[ \| v \ast X \|_1 \leq \mathbb{E}(\| v \ast X \|)_1 \leq 4\delta \lambda \mathbb{P}(X^*_R > \lambda). \]
Putting all these estimates together we get
\[ \mathbb{P}(X^*_R > \beta \lambda, \ X^* \leq \delta \lambda) \leq \mathbb{P}(C, (v \ast X)^*_R > (\beta - 2\delta - 1)\lambda) \]
\[ \leq \frac{C}{(\beta - 2\delta - 1)\lambda} \mathbb{E}(\| v \ast X \|)_1 \]
\[ \leq \frac{4C\delta}{(\beta - 2\delta - 1)} \mathbb{P}(X^*_R > \lambda). \]
Fixing \( \beta > 2\delta + 1 \) we may take
\[ \alpha(\delta) = \frac{4C\delta}{(\beta - 2\delta - 1)}. \]
\[ \square \]
Proposition 4.11. Suppose that $\mathcal{X}$ has weak RMF and let $1 < p < \infty$. Then there exists a constant $C$ such that for any standard Haar martingale $X$ in $\mathcal{X}$ we have $\mathbb{E}|X^*_R|^p \leq C^p\|X\|_p^p$.

Proof. Let $X = (X_j)_{j=1}^N$ be a standard Haar martingale in $\mathcal{X}$ (note that it suffices to prove the claim for finite martingales independently of $N$). We apply the good-$\lambda$ inequality and write
\[
\mathbb{E}|X^*_R|^p = \beta^p \int_0^\infty p\lambda^{p-1}\mathbb{P}(X^*_R > \lambda) \, d\lambda \\
\leq \beta^p \alpha(\delta) \int_0^\infty p\lambda^{p-1}\mathbb{P}(X^*_R > \lambda) \, d\lambda + \beta^p \int_0^\infty p\lambda^{p-1}\mathbb{P}(X^* > \delta\lambda) \, d\lambda \\
= \beta^p \alpha(\delta)\mathbb{E}|X^*_R|^p + \frac{\beta^p}{\delta^p}\mathbb{E}|X^*|^p,
\]
where $\mathbb{E}|X^*|^p \leq C^p\|X\|_p^p$ and $\mathbb{E}|X^*_R|^p$ is finite. Choosing $\delta$ so small that $\beta^p \alpha(\delta) < 1$ we get
\[
\mathbb{E}|X^*_R|^p \leq \frac{\beta^p C^p}{(1 - \beta^p \alpha(\delta))\delta^p}\|X\|_p^p.
\]

We collect our results as follows:

Theorem 4.12. The following are equivalent:

1. $\mathcal{X}$ has RMF$_p$ for all $p \in (1, \infty)$.
2. $\mathcal{X}$ has RMF$_p$ for some $p \in (1, \infty)$.
3. $\mathcal{X}$ has weak RMF.

Proof. Trivially the first condition implies the second. That the third follows from the second was Proposition 4.8. In Proposition 4.11 we showed that the weak RMF-property implies that for any $p \in (1, \infty)$, $\mathbb{E}|X^*_R|^p \leq \|X\|_p^p$ whenever $X$ is an $L^p$-bounded standard Haar martingale in $\mathcal{X}$. As was noted before, the filtration of dyadic intervals on $[0, 1)$ can be "embedded" in a standard Haar filtration. Thus the weak RMF-property is sufficient for the $L^p$-boundedness, $1 < p < \infty$, of the Rademacher maximal operator on the unit interval. By Theorem 3.7 this implies RMF$_p$ for all $p \in (1, \infty)$.

4.4 RMF-property and concave functions

The existence of a biconcave function $v : E \times E \to \mathbb{R}$ for which
\[
v(x, y) \geq \left\| \frac{x + y}{2} \right\|^p - C\left\| \frac{x - y}{2} \right\|^p
\]
can be shown to be equivalent with $E$ being a UMD-space (see [6]). These ideas have been applied (again in [6]) to prove the boundedness of Doob’s maximal operator and we will now use them to study the Rademacher maximal function. More precisely, we will show that for a fixed $p \in (1, \infty)$, a constant $C$ is such that $\mathbb{E}|X^*_R|^p \leq C\|X\|_p^p$ for all finite simple martingales $X = (X_j)_{j=1}^N$ in $\mathcal{X}$ if and only if there exists a suitable majorant for the real-valued function
\[
u(T, T) = \mathcal{R}(T)^p - C\|T\|^p,
\]
defined for finite subsets $T$ of operators in $\mathcal{X}$ and $T \in \mathcal{X}$. Observe that $\mathbb{E}|X^*_R|^p - C\|X\|_p^p \leq 0$ can equivalently be written as
\[
\mathbb{E}u\left(\{X_j\}_{j=1}^N, X_N\right) \leq 0,
\]
since $\|X\|_p^p = \mathbb{E}\|X_N\|^p$. 

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Proposition 4.13. The estimate
\[ \mathbb{E} u \left( \{ X_j \}_{j=1}^N, X_N \right) \leq 0 \]
holds for all finite simple martingales \( X = (X_j)_{j=1}^N \) in \( \mathcal{X} \) if and only if there exists a function \( v \) satisfying

1. \( u(\mathcal{T}, T) \leq v(\mathcal{T}, T) \)
2. \( v(\{ T \}, T) \leq 0 \)
3. \( v(\mathcal{T} \cup \{ T \}, T) = v(\mathcal{T}, T) \)
4. \( v(\mathcal{T}, \cdot) \) is concave

for all finite subsets \( \mathcal{T} \) of \( \mathcal{X} \) and all \( T \in \mathcal{X} \).

The proof of sufficiency is based on the following lemma.

Lemma 4.14. Suppose that \( v \) is as above and that \( (X_j)_{j=1}^N \) is a simple martingale in \( \mathcal{X} \). Then, for all \( 2 \leq k \leq N \), we have
\[ \mathbb{E} v \left( \{ X_j \}_{j=1}^k, X_k \right) \leq \mathbb{E} v \left( \{ X_j \}_{j=1}^{k-1}, X_{k-1} \right). \]

Proof. Let us fix a \( k \) and write \( \mathcal{F}_j \) for the \( \sigma \)-algebra generated by \( X_1, \ldots, X_j \). By the simplicity of \( (X_j)_{j=1}^N \), the set \( \{ X_j \}_{j=1}^{k-1} \) has a finite number \( s \) of different possibilities \( T_1, \ldots, T_s \subseteq \mathcal{X} \) so that the event \( A_r \) of \( T_r \) happening is in \( \mathcal{F}_{k-1} \). Now, using the third property of \( v \) we get
\[ v \left( \{ X_j \}_{j=1}^k, X_k \right) = v \left( \{ X_j \}_{j=1}^{k-1} \cup \{ X_k \}, X_k \right) = v \left( \{ X_j \}_{j=1}^{k-1}, X_k \right) = \sum_{r=1}^s 1_{A_r} v(T_r, X_k) \]
and so the fourth property with the aid of Jensen’s inequality implies
\[ \mathbb{E} \left( v(T_r, X_k) \, | \, \mathcal{F}_{k-1} \right) \leq v(T_r, \mathbb{E}(X_k \, | \, \mathcal{F}_{k-1})). \]
Thus
\[ \mathbb{E} v \left( \{ X_j \}_{j=1}^k, X_k \right) = \sum_{r=1}^s \mathbb{E} \left( 1_{A_r} v(T_r, X_k) \right) \]
\[ = \sum_{r=1}^s \mathbb{E} \left( 1_{A_r} \mathbb{E} \left( v(T_r, X_k) \, | \, \mathcal{F}_{k-1} \right) \right) \]
\[ \leq \sum_{r=1}^s \mathbb{E} \left( 1_{A_r} v(T_r, X_{k-1}) \right) \]
\[ = \mathbb{E} v \left( \{ X_j \}_{j=1}^{k-1}, X_{k-1} \right). \]

With the aid of the above lemma, the existence of a desired \( v \) is now readily seen to imply that
\[ \mathbb{E} u \left( \{ X_j \}_{j=1}^N, X_N \right) \leq \mathbb{E} v \left( \{ X_j \}_{j=1}^N, X_N \right) \leq \mathbb{E} v \left( \{ X_j \}_{j=1}^{N-1}, X_{N-1} \right) \leq \ldots \leq \mathbb{E} v \left( \{ X_1 \}, X_1 \right) \leq 0. \]

On the other hand, the validity of \( \mathbb{E} u \left( \{ X_j \}_{j=1}^N, X_N \right) \leq 0 \) for finite simple martingales enables us to construct the auxiliary function \( v \) with the desired properties by defining
\[ v(\mathcal{T}, T) = \sup \mathbb{E} u \left( \{ X_j \}_{j=1}^N \cup \mathcal{T}, X_N \right), \]
where the supremum is taken over all finite and simple martingales \((X_j)_{j=1}^N\) (where \(N\) is allowed to vary) for which \(X_1 = T\) almost surely. Let us check that the required properties are satisfied. For the first property, take \(N = 1\) and \(X_1 = T\) almost surely to see that
\[
u(T, T) = R(T)^p - C\|T\|^p \leq R(T \cup \{T\})^p - C\|T\|^p = \mathbb{E}\left(R(T \cup \{X_1\})^p - C\|X_1\|^p\right) \leq \nu(T, T).
\]
For the third one, it suffices to note that if \(X_1 = T\) almost surely, then \(\{T\} \subset \{X_j\}_{j=1}^N\) almost surely and so \(\nu(T \cup \{T\}, T) = \nu(T, T)\). The second property follows from the assumption and the third property: Let \(X = (X_j)_{j=1}^N\) be a simple martingale with \(X_1 = T\) almost surely. Now
\[
\mathbb{E}u(\{X_j\}_{j=1}^N \cup \emptyset, X_N) \leq 0
\]
and so \(\nu(\emptyset, T) \leq 0\). By the third property,
\[
\nu(\{T\}, T) = \nu(\emptyset, T) \leq 0.
\]
To see that \(\nu(T, \cdot)\) is concave, take operators \(T_1\) and \(T_2\) and put \(T = \alpha T_1 + (1 - \alpha)T_2\) for some \(0 < \alpha < 1\). We need to show that \(\nu(T, T) \geq \alpha \nu(T, T_1) + (1 - \alpha)\nu(T, T_2)\). To do this, take \(m_1\) and \(m_2\) such that \(m_1 < \nu(T, T_i)\). Now there exist finite simple martingales \((X_j^{(i)})_{j=1}^N\) (defined on the unit interval) such that \(X^{(i)}_1 = T_i\) almost surely and
\[
\mathbb{E}u(\{X_j^{(i)}\}_{j=1}^N \cup T, X_N^{(i)}) > m_i.
\]
Let \(X_1 = T\) almost surely and define
\[
X_j(t) = \begin{cases} X_j^{(1)} \left( \frac{t}{\alpha} \right), & t \in [0, \alpha) \\ X_j^{(2)} \left( \frac{t - \alpha}{1 - \alpha} \right), & t \in [\alpha, 1) \end{cases}
\]
for \(j = 2, \ldots, N + 1\).

A moment’s reflection assures us that \((X_j)_{j=1}^{N+1}\) is also a simple martingale. Now
\[
\nu(T, T) \geq \mathbb{E}u(\{X_j\}_{j=1}^{N+1} \cup T, X_{N+1})
\]
\[
\geq \mathbb{E}u(\{X_j\}_{j=2}^{N+1} \cup T, X_{N+1})
\]
\[
= \int_0^\alpha u(\{X_j^{(1)} \left( \frac{t}{\alpha} \right)\}_{j=1}^N \cup T, X_j^{(1)} \left( \frac{t}{\alpha} \right)) \mathrm{d}t
\]
\[
+ \int_\alpha^1 u(\{X_j^{(2)} \left( \frac{t - \alpha}{1 - \alpha} \right)\}_{j=1}^N \cup T, X_j^{(2)} \left( \frac{t - \alpha}{1 - \alpha} \right)) \mathrm{d}t
\]
\[
= \alpha \int_0^1 u(\{X_j^{(1)}(s)\}_{j=1}^N \cup T, X_j^{(1)}(s)) \mathrm{d}s + (1 - \alpha) \int_0^1 u(\{X_j^{(2)}(s)\}_{j=1}^N \cup T, X_j^{(2)}(s)) \mathrm{d}s
\]
\[
> \alpha m_1 + (1 - \alpha)m_2.
\]
Letting \(m_i \to \nu(T, T_i)\) we get concavity. The proof of Proposition 4.15 is now complete.
If we assume \( \mathbb{E} u \left( \{ X_j \}_{j=1}^N, X_N \right) \leq 0 \) only for standard Haar martingales, we obtain midpoint concavity of \( v(T, \cdot) \) (when restricted to standard Haar martingales). Indeed, suppose that the supremum in the definition of \( v \) is taken over finite standard Haar martingales and observe that properties other than concavity follow exactly as above. In the proof of midpoint concavity, let \( T = (T_1 + T_2)/2 \) and define \((X_j)_{j=1}^{2N+1}\) as follows:

\[
\begin{align*}
X_1 &= T \quad \text{almost surely,} \\
X_2(t) &= \begin{cases} 
X_1^{(1)}(2t) = T_1, & t \in [0, 1/2), \\
x_1^{(2)}(2t - 1) = T_2, & t \in [1/2, 1), 
\end{cases} \\
X_{2j-1}(t) &= \begin{cases} 
X_j^{(1)}(2t), & t \in [0, 1/2), \\
x_{2j-2}(t), & t \in [1/2, 1), 
\end{cases} \\
X_{2j}(t) &= \begin{cases} 
X_{2j-1}(t), & t \in [0, 1/2), \\
x_j^{(2)}(2t - 1), & t \in [1/2, 1). 
\end{cases}
\end{align*}
\]

This way \((X_j)_{j=1}^{2N+1}\) becomes a standard Haar martingale and calculations similar as above give us \( v(T, T) \geq v(T, T_1)/2 + v(T, T_2)/2 \).

For conclusion we state:

**Theorem 4.15.** Let \( 1 < p < \infty \). Then \( X \) has RMF \( p \) if and only if there exists a function \( v \) such that for some constant \( C \),

1. \( v(T, T) \geq \mathcal{R}(T)^p - C\|T\|^p \),
2. \( v(\{T\}, T) \leq 0 \),
3. \( v(T \cup \{T\}, T) = v(T, T) \),
4. \( v(T, \cdot) \) is midpoint concave,

for all finite subsets \( T \) of \( \mathcal{X} \) and all \( T \in \mathcal{X} \).

**Proof.** If \( X \) has RMF \( p \), there exists a constant \( C \) such that \( \mathbb{E} |X_R|^p \leq C\|X\|^p \) especially for all standard Haar martingales \( X = (X_j)_{j=1}^N \) in \( \mathcal{X} \). Equivalently,

\[
\mathbb{E} \left( \mathcal{R} \left( X_j : 1 \leq j \leq N \right)^p - C\|X_N\|^p \right) \leq 0
\]

for standard Haar martingales \( X = (X_j)_{j=1}^N \), which by Proposition 4.13 enables us to construct a desired \( v \).

Suppose conversely that there exists such a function \( v \). Concavity of functions defined on linear spaces reduces to concavity of real functions in the sense that concavity on a linear space is equivalent to concavity along any one-dimensional affine subspace. According to a well-known result, midpoint concave functions that are locally bounded from below are actually concave. That
\( v(T, \cdot) \) is locally bounded from below follows easily: Take \( N = 1 \) and \( X_1 = T \) almost surely to see that

\[
v(T, T) \geq u(\{T\} \cup T, T) \geq \mathcal{R}(T)^p + (1 - C)\|T\|^p.
\]

Hence \( v \) is concave and by Proposition 4.13 we have \( \mathbb{E}|X^*_R|^p \leq C\|X\|^p \) especially for all finite simple martingales \( X = (X_j)_{j=1}^N \). By Theorem 3.7 (or just by Lemma 3.12) \( \mathcal{X} \) has RMF\(_p\). \( \square \)

Observe that this is another way to see that to have the condition \( \mathbb{E}|X^*_R|^p \leq C\|X\|^p \) for finite simple martingales it suffices to check it for standard Haar martingales.
Summary

The RMF-property of a Banach space $X$ is by definition a requirement that a certain maximal operator is bounded with respect to $L^p$-norms. Originally (in the paper [15] by Hytönen, McIntosh and Portal) the question was studied via the embedding $X \hookrightarrow \mathcal{L}(C, X)$. Here we have adopted a more general viewpoint by assuming that $X \subset \mathcal{L}(H, E)$, where $H$ and $E$ are Banach spaces and the norm of $X$ dominates the operator norm.

We began our study of the RMF-property from the Euclidean case, where it initially depended on both the exponent $p \in (1, \infty)$ and the dimension $n$. Using interpolation we first showed the independence from $p$. After that we proved using a simple scaling argument that it suffices to study the Rademacher maximal function on a finite measure space, namely the unit cube, and with respect to a “one-sided” filtration of “small” dyadic cubes. After generalizing the definition of the maximal function to filtrations on arbitrary measure spaces, we set out to study different kinds of simple filtrations on spaces with finite (unit) measure. It turned out that it suffices to study the RMF-property with respect to standard Haar filtrations on the unit interval (or just with respect to the filtration of dyadic intervals).

Another question was the sufficiency of the weak type inequality, i.e. the weak RMF-property. In the beginning, we used the Calderón-Zygmund decomposition to derive the weak type inequality from $L^p$-boundedness of $M_R$. A same kind of reasoning was carried out for martingales with the aid of Gundy’s decomposition. We were able to prove that the weak RMF suffices for RMF by showing that standard Haar martingales satisfy a certain good-$\lambda$ inequality.

Versions of Carleson’s embedding theorem were provided both in the Euclidean (dyadic) case and in the more general case of filtrations on measure spaces. These theorems are typically used (at least in the scalar case) when studying $L^2$-boundedness of the paraproduct operators

$$f \mapsto \sum_{j \in \mathbb{Z}} E_j f(E_{j+1}b - E_j b),$$

where $b$ is a BMO-function. Convergence of the series as well as boundedness of the operator follow when we choose $\theta_j = E_{j+1}b - E_j b$ in Carleson’s embedding theorem.

The Rademacher maximal function has already found applications also in the more general setting of a $\sigma$-finite measure space, see Hytönen [14].

How the RMF-property of a space $X \subset \mathcal{L}(H, E)$ relates to properties such as UMD and type (or cotype) of the Banach spaces $H$ and $E$, is still not completely understood. We showed that if $H$ has cotype 2 and $E$ has type 2, then the RMF-property of $X$ follows trivially. On the other hand, if $H^* \hookrightarrow X$ and $E \hookrightarrow X$ (for instance if $X = \mathcal{L}(H, E)$), then $H$ has finite cotype and $E$ has non-trivial type provided that $X$ has RMF. Actually, $L^p$ has RMF whenever $1 < p < \infty$. How about $\mathcal{L}(L^q, L^p)$? We noted also that $E$ does not have to be UMD even if $X$ has RMF. Does every UMD-space have RMF?
Appendix

Formulae for $L^p$-norms

Suppose that $\mathcal{X}$ is a Banach space and that $1 \leq p < \infty$. Using Fubini’s theorem we obtain a useful formula for $L^p$-norms of $\mathcal{X}$-valued functions on any $\sigma$-finite measure space $(\Omega, \mu)$:

$$\|f\|_{L^p(\mathcal{X})}^p = \int_0^{\infty} p\lambda^{p-1}\mu(\{\xi \in \Omega : \|f(\xi)\| > \lambda\})\,d\lambda.$$  

Dividing the above integral in pieces we also see that

$$\|f\|_{L^p(\mathcal{X})}^p \approx \sum_{k \in \mathbb{Z}} 2^{kp}\mu(\{\xi \in \Omega : \|f(\xi)\| > 2^k\}).$$

Interpolation

Suppose that $\mathcal{X}$ and $\mathcal{Y}$ are Banach spaces and that $(\Omega, \mu)$ is a $\sigma$-finite measure space.

Let $1 \leq p < \infty$. An operator $T$ from $L^p(\mathcal{X})$ to strongly measurable $\mathcal{Y}$-valued functions is said to be weak $(p, p)$ if there exists a constant $C$ such that for all $f \in L^p(\mathcal{X})$ we have

$$\mu(\{\xi \in \Omega : \|Tf(\xi)\| > \lambda\}) \leq \frac{C}{\lambda^p}\|f\|_{L^p(\mathcal{X})}.$$  

whenever $\lambda > 0$. For $p = \infty$ such an operator is customarily said to be weak $(\infty, \infty)$ if it is bounded from $L^\infty(\mathcal{X})$ to $L^\infty(\mathcal{Y})$. Furthermore, an operator $T$ acting on strongly measurable functions is said to be sublinear if for $\mu$-almost every $\xi \in \Omega$ we have

$$\|T(f + g)(\xi)\| \leq \|Tf(\xi)\| + \|Tg(\xi)\| \quad \text{and} \quad \|T(\lambda f)(\xi)\| \leq |\lambda|\|Tf(\xi)\|$$  

whenever $f$ and $g$ are strongly measurable functions and $\lambda$ is a scalar.

The following results can be found (in the scalar case) in the book [10] by García-Cuerva and Rubio de Francia. The dyadic Hardy and BMO spaces were defined in Section 2.2.

**Theorem.** (Marcinkiewicz interpolation theorem) Let $1 \leq p_0 < p_1 \leq \infty$ and suppose that $T$ is a sublinear operator from $L^{p_0}(\mathcal{X}) + L^{p_1}(\mathcal{X})$ to strongly measurable $\mathcal{Y}$-valued functions on $\Omega$. If $T$ is weak $(p_0, p_0)$ and weak $(p_1, p_1)$, then it is bounded from $L^p(\mathcal{X})$ to $L^p(\mathcal{Y})$ for all $p_0 < p < p_1$.

**Theorem.** Let $1 < p_0 < \infty$ and suppose that a linear operator $T$ maps $L^{p_0}(\mathcal{X})$ boundedly to $L^{p_0}(\mathcal{Y})$ and $L^{\infty}(\mathcal{X})$ boundedly to the (dyadic) BMO($\mathcal{Y}$). Then $T$ is bounded from $L^p(\mathcal{X})$ to $L^p(\mathcal{Y})$ for all $p_0 < p < \infty$.

**Theorem.** Let $1 < p_0 < \infty$ and suppose that a sublinear operator $T$ is weak $(p_0, p_0)$ and maps the (dyadic) $H^1(\mathcal{X})$ boundedly to $L^1(\mathcal{Y})$. Then $T$ is bounded from $L^p(\mathcal{X})$ to $L^p(\mathcal{Y})$ for all $1 < p < p_0$.  

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References


