

RECONSTRUCTION OF RIEMANNIAN MANIFOLD FROM BOUNDARY AND INTERIOR DATA

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Academic dissertation

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The thesis consists of this overview and the following articles:

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1 Introduction

In this overview we focus on the inverse problems of medical and seismic imaging. Although these two imaging problems may seem very different, the underlying mathematics is similar in both of them. We consider two medical imaging techniques, namely *Electric impedance tomography* (EIT) and *obstetric sonography*, more commonly known as *ultrasound imaging*. In seismic imaging, we are interested in seeing inside the Earth with echoes of the *seismic waves*. We describe both the physical measurement setup and the related inverse problem. In addition to measurements, to solve the inverse problem, we also need some *a priori* information about the measured object. In our case including the *a priori* information means fixing the mathematical background.

In the EIT-scan, a doctor attaches electrical sources and receivers on the skin of the patient. Then, by inserting different voltage patterns to the sources, the receivers measures the corresponding current patterns [23]. The inverse problem consist of finding the electrical conductivity inside the patient. Medically this is highly interesting, since different tissues have very different conductivities. For instance, a breast tumor can have four times higher conductivity than the healthy tissue [24], since to grow the tumor needs lots of blood that has a high conductivity.

Often, ultrasound imaging is used to see the fetus in its mother's uterus. The ultrasonic scanner produces a sound wave by using a transducer: the wave propagates in the body and reflects from the tissue boundaries back to the transducer that records the echo. In this imaging problem, we want to recover the wave speed of the ultrasound as a function of the position inside the mother's body. In this case, the medical relevance is that different wave speeds correspond to the different mixture of tissues, and, thus a solution of this inverse problem is the image of the fetus. Unfortunately, the current sonographic devices do not recover the wave speed, but produce the image of the fetus as a function with the following variables: the points of transducer, the travel time of the sonic wave from the transducer to the reflection point inside the body and back to transducer, and the strength of the echo (see [39] Section Abdominal ultrasound). We hope that better understanding of the related inverse problem would lead to design of better sonograms. In article [II] we have studied a similar problem. The key difference to the previous example is that in medical ultrasound imaging the transducer is on the skin of the patient. In [II] we assume that the wave sources are located inside the object of interest, and we do measurements in this same area.

The interest in seismic prospecting lies in obtaining information about the structure of the crust of the Earth. For instance, to find oil or gas pockets. A widely used indirect method to see inside the crust is to detonate charges or shake the ground with powerful blows on some area, and, then to listen to the echoes of seismic waves produced by these interferences in some other area [58]. As in

ultrasound imaging, the inverse problem is to find the wave speed of the seismic wave.

For instance earthquakes produce seismic waves. These waves are so strong that they travel through the mantle and even the core. This is a way to obtain information about the deep structures of our planet [54]. Since we cannot control where and when the earthquake occurs, we cannot do similar measurements as in the two previous examples. Nevertheless, there is a large network of seismometers that record the earthquakes constantly. Thus, the typical measurement is to record the travel time difference of the seismic waves produced by an earthquake. Suppose that at the North pole occurs an earthquake, and it produces a seismic wave that propagates through the Earth. Say that there is a seismometer in London and in Tokyo which record the time of arrival of the seismic wave. Then, we can compute the travel time difference of the seismic wave. If the network of seismometers is dense enough and they do measure a large amount of earthquakes, we can hope to recover the wave speed of the seismic waves. In Article [I] we have studied this kind of inverse problem.

This thesis is about geometric inverse problems. By this we mean that the mathematical framework is the Riemannian geometry and the objects of interest are smooth Riemannian manifolds with or without boundary. EIT, ultrasound imaging and seismic imaging are examples of geometric inverse problems that have been studied extensively. In the next Section we consider the mathematics behind these problems and list some literature.

To summarize, in inverse problems one tries to obtain more information about the object of interest by doing indirect measurements and combining this additional information with the *a priori* information. The *a priori* information and the measurements together are called the data. From the point of view of a geometric inverse problem, the data must be invariant under Riemannian isometries. This means that two Riemannian manifolds that are Riemannian isometric must admit the same data. We want to show that Riemannian manifolds with the same data have also some other geometric properties in common. For instance, two Riemannian manifolds admit the data \mathbf{A} if and only if they are Riemannian isometric.

In this thesis we focus, on the uniqueness questions of the geometric inverse problems. In addition also the stability questions can be asked. Roughly speaking this means the following. Suppose that we know *a priori* that our model space (X, d_X) and the measurement space (Y, d_Y) are complete metric spaces and the forward map

$$F : X \rightarrow Y$$

is given. The forward map is the mathematical way to connect the unknown object of interest to the corresponding measurements. For instance in the case of EIT

the space X is the space of conductivities and Y is the space of current-to-voltage measurements. The uniqueness problem is to prove that the map F is one-to-one. The corresponding stability problem is to show that if $y_1, y_2 \in F(X)$ are close to each other then also x_1 and x_2 are close to each other, where $F(x_i) = y_i$. In this overview we do not consider the stability of the forward operators of our example cases.

2 Geometric inverse problems

Through out this Section we denote (N, g) a smooth, connected, complete and oriented n -dimensional Riemannian manifold without boundary. We also use the notation (\bar{M}, g) for a smooth, connected, compact and oriented n -dimensional Riemannian manifold with boundary. Without loss of generality, we always assume that $\bar{M} \subset N$ and the metric tensor on \bar{M} is a restriction of a metric tensor g that is defined on N . Next, we provide a short list of some important objects of Riemannian manifold (N, g) . For a thorough review for the differential and Riemannian geometry we refer to monographs [36, 37].

We call a smooth curve $\gamma : \mathbb{R} \rightarrow N$ a geodesic, if it satisfies the geodesic equation

$$\nabla_{\dot{\gamma}(t)} \dot{\gamma}(t) = 0,$$

where ∇ is the Riemannian connection of metric tensor g . For any $(p, \xi) \in SN$, where SN is the unit tangent bundle of g , there exists a unique geodesic $\gamma_{p,\xi} : \mathbb{R} \rightarrow N$ that satisfies the initial conditions

$$\gamma_{p,\xi}(0) = p, \text{ and } \dot{\gamma}_{p,\xi}(0) = \xi.$$

The Riemannian distance function $d_g : N \times N \rightarrow [0, \infty)$ is defined via the lengths of geodesics. More precisely,

$$d_g(p, q) = \min\{a \in [0, \infty) : \text{there exists } \xi \in S_p N \text{ such that } \gamma_{p,\xi}(a) = q\}.$$

Let $f \in C^\infty(N)$. The g -gradient $\nabla_g f$ of function f is defined by the equation

$$\langle \nabla_g f, V \rangle_g = Vf = df(V), \text{ for all smooth vector fields } V \text{ on } N.$$

We write $\mathcal{A}^k(N)$ for the space of smooth k -forms of N . On a Riemannian manifold there is also a counter part for the classical divergence operator $\text{div} : \mathcal{T}(N) \rightarrow C^\infty(N)$, where $\mathcal{T}(N)$ is the vector space of all smooth vector fields of N . The div operator is defined by the following formula

$$\text{div} X dV_g = d(i_X dV_g), \quad X \in \mathcal{T}(N),$$

where $dV_g \in \mathcal{A}^n(N)$ is the Riemannian volume form related to metric tensor g , $d : \mathcal{A}^k(N) \rightarrow \mathcal{A}^{k+1}(N)$ is the exterior derivative operator and $i_X : \mathcal{A}^k(N) \rightarrow \mathcal{A}^{k-1}(N)$ is the interior multiplication operator with respect to vector field $X \in \mathcal{T}(N)$.

The most important partial differential operator of a Riemannian manifold (N, g) is the Laplace–Beltrami operator $\Delta_g : C^\infty(N) \rightarrow C^\infty(N)$, defined by

$$\Delta_g f := -\operatorname{div} \nabla_g f, \quad f \in C^\infty(N).$$

In local coordinates $(x^j)_{j=1}^n$ the Laplace–Beltrami operator is given by

$$\Delta_g f := -\frac{1}{\sqrt{|g|}} \frac{\partial}{\partial x^i} (g^{ij} \sqrt{|g|} \frac{\partial}{\partial x^j} f),$$

where $|g| = \det([g_{ij}]_{i,j=1}^n)$ and $[g^{ij}]_{i,j=1}^n$ is the inverse matrix of g . The Laplace–Beltrami operator Δ_g is an elliptic second order partial differential operator.

Next we will consider the mathematics of EIT, ultrasound imaging and seismic imaging. We will also relate these problems to other geometric inverse problems.

2.1 Calderón problem

In this subsection we consider the mathematical model behind EIT, that is, the Calderón problem. There are many ways to formulate this problem. However since the focus of this thesis is in geometric inverse problems, we start with the Riemannian version. We recall the Dirichlet problem for the Laplace–Beltrami operator reads as:

$$\begin{cases} \Delta_g u = 0, & \text{in } M \\ u = f, & \text{on } \partial M. \end{cases} \quad (1)$$

Since Δ_g is an elliptic second order differential operator and \overline{M} is a compact smooth manifold with boundary, the problem (1) has a unique solution $u \in H^1(\overline{M})$ provided that $f \in H^1(\overline{M})$. Moreover if $f \in C^\infty(\overline{M})$, then $u \in C^\infty(\overline{M})$. For a complete survey on the theory of elliptic second order operators, we refer to [17, 60]. For $f \in H^1(\overline{M})$, we denote u^f the unique solution of (1).

Mathematically the EIT-measurement is given by the Dirichlet-to-Neumann (DN) mapping of problem (1), namely

$$\Lambda_g : H^{1/2}(\partial M) \rightarrow H^{-1/2}(\partial M), \quad \langle \Lambda_g f, h \rangle = \int_{\overline{M}} \langle \nabla_g u^f, \nabla_g h \rangle_g dV_g, \quad h \in H^{1/2}(\partial M). \quad (2)$$

The space $H^{1/2}(\partial M)$ is the quotient space of $H^1(\overline{M})$ where the quotient map is the trace map. Notice that, if f is smooth then by Greens formula and (1) the

following formula holds true:

$$\Lambda_g f = \langle \nu, \nabla_g u^f \rangle_g =: \frac{\partial}{\partial \nu} u^f,$$

where ν is the outward pointing unit normal vector field on ∂M . The Calderón inverse problem on Riemannian manifold reads as follows:

$$\text{Does } \Lambda_g = \Lambda_{\tilde{g}} \quad \text{imply the existence of a diffeomorphism } \Phi : \overline{M} \rightarrow \overline{M} \quad (3) \\ \text{such that } \Phi|_{\partial M} = id \text{ and } g = \Phi^* \tilde{g}?$$

The problem (3) has a long history and the first versions of it was proposed by Calderón in [11]. The version given in (3) is a conjecture by Lee and Uhlmann in dimension three and higher [38]. We will next survey results related to problem (3). We will split the survey into two parts, depending on the dimension.

2.1.1 Calderón problem in dimension 3 and higher.

We assume that $\Omega \subset \mathbb{R}^n$, $n \geq 3$ is an open, connected, bounded domain with smooth boundary. Let $\sigma = [\sigma^{ij}]_{i,j=1}^n$ be a smooth, symmetric and positive definite matrix valued mapping on $\overline{\Omega}$. We define the corresponding *anisotropic conductivity operator* by

$$L_\sigma u := \nabla \cdot (\sigma \nabla u), \quad u \in H^1(\overline{\Omega}), \quad (4)$$

and the related conductivity equation:

$$\begin{cases} L_\sigma u = 0, & \text{in } \Omega \\ u = f, & \text{on } \partial\Omega. \end{cases} \quad (5)$$

If we define a Riemannian metric g on $\overline{\Omega}$ with respect to Euclidean coordinates as

$$g = (\det \sigma)^{\frac{1}{n-2}} \sigma^{-1} \quad \text{equivalently } \sigma = (\det g)^{\frac{1}{2}} g^{-1},$$

then the problems (1) and (5) are equivalent, since $\Delta_g = (\det \sigma)^{\frac{1}{2-n}} L_\sigma$. Therefore also the DN-map Λ_g of (1) and the DN-map Λ_σ of (5) coincide. We conclude that the Calderón problem (3) can be equivalently stated with the conductivity operator L_σ or the Laplace-Beltrami operator Δ_g , if $\Omega \subset \mathbb{R}^n$ is an open, connected, bounded domain with smooth boundary and dimension $n \geq 3$. By this we mean that problem (3) is equivalent to

$$\text{Does } \Lambda_\sigma = \Lambda_{\tilde{\sigma}} \quad \text{imply the existence of a diffeomorphism } \Phi : \overline{\Omega} \rightarrow \overline{\Omega} \quad (6) \\ \text{such that } \Phi|_{\partial\Omega} = id \text{ and } \sigma = \Phi^* \tilde{\sigma}?$$

The first global uniqueness solution for the problem (6) is by Sylvester and Uhlmann in [57], where they assume that the conductivity is smooth and *isotropic*. This means that the conductivity $\sigma \in C^\infty(\overline{\Omega})$ is a positive function. For smooth anisotropic conductivities the problem (6) is still open.

Initially, Calderón considered in [11] the problem (6) in the context of the domain $\Omega \subset \mathbb{R}^n$, $n \geq 2$ with Lipschitzian boundary and L^∞ -isotropic conductivity σ , for which there exists $c > 0$ such that $0 < c < \sigma(x)$, $x \in \overline{\Omega}$. This is a more realistic formulation of the problem (6) from the point of view of medical imaging, since there are jumps in the conductivity when moving from a soft tissue to bones. The problem (6) as originally formulated by Calderón is still open for higher dimensions. However in [21, 20] the problem (6) is proved under the assumptions that the boundary is Lipschitzian and the isotropic conductivities belong to $W^{s,p}(\overline{\Omega})$, where the indices s and p depend on dimension n . In [12] the problem (6) is proved for the Lipschitzian boundary and Lipschitz-conductivity σ .

Let us return back to the Riemannian setting. In [38, 35, 34] the problem (3) is solved under the assumption that Riemannian manifold (\overline{M}, g) is real analytic. It is worth to mention that in [35, 34] the authors assume only that the DN-map is known on an open subset $\Gamma \subset \partial M$. More precisely, they assume that the operator

$$\Lambda_{g,\Gamma} f := \Lambda_g f|_\Gamma, \quad f \in C_0^\infty(\Gamma)$$

is given. In [15, 16] the authors have solved the problem (3) under the assumptions that $(\overline{M}, g) \subset\subset (\mathbb{R} \times \overline{M}_0, g)$ and $g = e \oplus g_0$, where (\overline{M}_0, g_0) is a compact $(n-1)$ -dimensional manifold with boundary.

2.1.2 Calderón problem in dimension two.

The two dimensional case is very different from the higher dimensional case. We start again with the Euclidean version. We assume that $\Omega \subset \mathbb{R}^2$ is an open, connected, bounded domain with smooth boundary and Riemannian metric tensor g . In Euclidean coordinates we define

$$\sigma^{ij} := \det(g)^{1/2} g^{ij}. \tag{7}$$

Notice that $\det(\sigma) \equiv 1$. This implies that the map $g \mapsto \sigma$ given by (7) is not one-to-one nor onto. Therefore the problems (1) and (5) are not equivalent, on $\Omega \subset \mathbb{R}^2$ in the same way as in the higher dimensional cases.

The first positive answer for the Calderón problem (6) with C^2 -smooth isotropic conductivity is by Nachman [46]. In [9] the problem (6) is proved under the assumption, of Lipschitzian boundary and C^1 -smooth isotropic conductivity. The Calderón problem as originally formulated in [11] was solved by Astala and Päivärinta in [2].

Next, we consider the anisotropic version of problem (6). In [56], Sylvester solves the C^3 -smooth anisotropic case of the problem by reducing it to the isotropic one, by using the isothermal coordinates [1]. Combining the techniques of [56, 46], the C^2 -smooth anisotropic case can be proven. In [55], the version of the problem (6) with C^1 -smooth anisotropic conductivity is solved. The anisotropic L^∞ -conductivity case was solved in [3] by Astala, Lassas and Päiväranta.

Finally, we consider the Riemannian case. We note that the problem (3) does not have a positive answer, since for any metric tensor g on \overline{M} the DN-maps for g and σg , $\sigma \in C^\infty(\overline{M})$ coincide, if $\sigma|_{\partial M} \equiv 1$ (see [38]). However, there is also a version of problem (3) in dimension two,

$$\begin{aligned} \text{Does } \Lambda_g = \Lambda_{\tilde{g}} \quad & \text{imply the existence of a diffeomorphism } \Phi : \overline{M} \rightarrow \overline{M} \\ & \text{such that } \Phi|_{\partial M} = id \text{ and } g = \sigma \Phi^* \tilde{g} \\ & \text{for some positive } \sigma \in C^\infty(\overline{M}) \text{ such that } \sigma|_{\partial M} \equiv 1? \end{aligned} \quad (8)$$

The Problem (8) was solved in [35]. See [19] for a survey on the Calderón problem in dimension two.

2.2 The inverse boundary value problem of the wave equation

This thesis considers geometric inverse problems that are related to the wave equation. We recall first the Dirichlet-problem for the Riemannian wave equation for a fixed time interval $(0, T)$, $T > 0$, namely,

$$\begin{cases} (\partial_t^2 - \Delta_g)u(t, x) = 0, & \text{in } (0, T) \times M \\ u = f, & \text{on } (0, T) \times \partial M \\ u(0, x) = \partial_t u(0, x) = 0, & \text{for every } x \in \overline{M}. \end{cases} \quad (9)$$

The direct problem is to solve (9) for a given f . By [30] it holds that for every $f \in H^1((0, T) \times \partial M)$ such that $f(0, x) = \frac{\partial}{\partial t} f(0, x) = 0$, there exists a unique solution for the problem (9), that satisfies:

$$u^f \in C([0, T]; H^1(\overline{M})) \cap C^1([0, T]; L^2(\overline{M})).$$

Moreover, by standard regularity arguments for hyperbolic partial differential operators, it holds that $u \in C^\infty([0, T] \times \overline{M})$, if $f \in C_0^\infty((0, T) \times \partial M)$. Therefore, the Hyperbolic Dirichlet-to-Neumann map (HDN), $\Lambda_g^T : C_0^\infty((0, T) \times \partial M) \rightarrow C^\infty((0, T) \times \partial M)$ given by

$$\Lambda_g^T f(t, x) = \partial_\nu u^f(t, x), \quad (t, x) \in (0, T) \times \partial M \quad (10)$$

is well defined. The inverse problem for (9) consist of proving the following:

$$\Lambda_g^T = \Lambda_{\tilde{g}}^T \quad \text{implies the existence of a diffeomorphism } \Phi : \overline{M} \rightarrow \overline{M} \quad (11)$$

such that $\Phi|_{\partial M} = id$ and $g = \Phi^*\tilde{g}$.

This problem was first solved by Belishev and Kurylev [7], under the assumption $T > \max_{x \in \overline{M}} dist_g(x, \partial M)$, using similar methods to Belishev's paper [6] for the isotropic case in the Euclidean setup. The proofs in [7] are based on the control theoretic method called the *boundary control method* (BC). We review this method briefly below. We refer to [26] for a more detailed introduction. The key ingredient of BC-method is the unique continuation theorem for the wave equation proven by Tataru [59].

An important version of problem (11) is the case of partial data. Suppose that $\mathcal{S}, \mathcal{R} \subset \partial M$ are open and consider an operator

$$\Lambda_{g, \mathcal{S}, \mathcal{R}}^T : C_0^\infty((0, T) \times \mathcal{S}) \rightarrow C^\infty((0, T) \times \mathcal{R}), \quad \Lambda_{g, \mathcal{S}, \mathcal{R}}^T f := \Lambda_g^T f|_{(0, T) \times \mathcal{R}}. \quad (12)$$

The inverse problem related to $\Lambda_{g, \mathcal{S}, \mathcal{R}}^T$ consist of proving the following:

$$\text{Does } \Lambda_{g, \mathcal{S}, \mathcal{R}}^T = \Lambda_{\tilde{g}, \mathcal{S}, \mathcal{R}}^T \quad \text{imply the existence of a diffeomorphism } \Phi : \overline{M} \rightarrow \overline{M} \quad (13)$$

such that $\Phi|_{\partial M} = id$ and $g = \Phi^*\tilde{g}$?

The first proof for (13) in the case of $\mathcal{S} = \mathcal{R}$ was given in [25]. The case of $\mathcal{S} \cap \mathcal{R} = \emptyset$ is still open in a general setting, but, for instance, the case $\overline{\mathcal{S}} \cap \overline{\mathcal{R}} \neq \emptyset$ is proven in [31]. In [32, 45] the problem (13) is proven in the case of $\mathcal{S} \cap \mathcal{R} = \emptyset$, but in both some stronger assumptions for g or Δ_g are required. We emphasize that problem (13) is the mathematical background of the ultrasound imaging and the seismic prospecting described above. The ultrasound imaging corresponds to the case where $\mathcal{S} = \mathcal{R}$, since both ultrasound source and receiver are implemented in the same transducer. The seismic prospecting is closely connected to the version of problem (13), where the sources and the receivers are located far away from each other. In article [II] we considered a problem related to problem (13). The key difference is that we assumed $\mathcal{S} = \mathcal{R} \subset N$ is open and instead of HDN-map we studied the local source-to-solution operator of the interior source problem for the Riemannian wave equation. See Section 3.2 for more details.

To solve problems (11) and (13) we have to assume that the measurement time T is large enough. Due to the finite speed of wave propagation, it is natural to assume, for instance, that

$$T > \max_{x \in \overline{M}} dist_g(x, \partial M).$$

The BC-method is commonly used to solve problems (11) and (13). The first important tool of BC-method is the approximate controllability. This means that the following set is dense

$$\{u^f(t, \cdot) : f \in C^\infty((0, 2t) \times \mathcal{S})\} \subset L^2(M(\mathcal{S}, t)), \quad (14)$$

where $M(\mathcal{S}, t) := \{x \in \overline{M} : \text{dist}_g(x, \mathcal{S}) \leq t\}$ is so called the *domain of influence*. The proof of (14) requires the Tataru's unique continuation theorem of [59]. The second important tool is the Blagoveshchenskii identity:

$$\langle u^f(T, \cdot), u^h(T, \cdot) \rangle_{L^2(\overline{M})} = \langle f, Kh \rangle_{L^2((0, 2T) \times \partial M)}, \quad (15)$$

that is valid for every $f, h \in C_0^\infty((0, 2T) \times \partial M)$. Here the operator K is given by

$$K := J\Lambda_g^{2T} - R\Lambda_g^{2T}RJ,$$

where

$$Rf(t, x) := f(2T - t, x) \quad \text{and} \quad Jf(t, x) := 1/2 \int_t^{2T-t} f(s, x) ds.$$

The identity (15) is based on d'Alembert's formula, that is, the solution of (1 + 1)-dimensional wave equation, and integration by parts. The identity (15) was first introduced in [8] and it gives the exact relationship between the boundary measurement (12) and the $L^2(\overline{M})$ -norm of the solution $u^f(t, \cdot)$.

Let $p \in \overline{M}$. There exists a boundary source $h \in C_0^\infty((0, 2T) \times \partial M)$ such that the corresponding solution $x \mapsto u^h(T, x)$ of (9) is localized near p . These waves are called Gaussian beams and they were introduced in [4]. Next we consider a strategy, that was introduced in [5], to obtain more information on (\overline{M}, g) by using (14), (15) and the Gaussian beams.

Let $q \in \partial M$ and $\mathcal{S} \subset \partial M$ be an open neighborhood of q . Let $t > 0$. Using (14) and (15) we can check, whether $u^h(T, \cdot)$ is orthogonal to $L^2(M(\mathcal{S}, t))$. If this is the case, then, roughly, $\text{dist}_g(p, \mathcal{S}) > t$. By letting \mathcal{S} converge to q , we see that the HDN-map determines $d_q(y, p)$. Choosing a different $q \in \partial M$, we will eventually determine the map $d_g(p, \cdot)|_{\partial M}$.

Therefore, the HDN map (10) determines the boundary distance functions, namely the collection

$$\mathcal{R}(\overline{M}) := \{d_g(p, \cdot) : \partial M \rightarrow \mathbb{R}_+ \mid p \in \overline{M}\}. \quad (16)$$

In [29] it was shown that the data

$$(\partial M, \mathcal{R}(\overline{M})) \quad (17)$$

determines the topological and Riemannian structures of (\overline{M}, g) . The construction of the smooth structure was introduced in [26]. In article [I] we studied a geometric inverse problem where instead of data (17) we considered the family of distance difference functions on an open set $M \subset N$. See Section 3.1 for more details.

2.3 Inverse problems related to the Riemannian distance function

The Riemannian distance function d_g is also an interesting object by its own right. In the following we introduce few geometric inverse problems where the data is related to the boundary distance function

$$d_{\partial M} := d_g|_{\partial M \times \partial M} : \partial M \times \partial M \rightarrow \mathbb{R}. \quad (18)$$

We emphasize that the function $d_{\partial M}$ measures the distance of points $p, q \in \partial M$ along the shortest path on \overline{M} . Generally, this path is not contained in ∂M . In [62] it was shown that, if (\overline{M}, g) is simple, the singularities of the HDN (10) determine the boundary distance function $d_{\partial M}$. A compact Riemannian manifold (\overline{M}, g) is simple, if the boundary ∂M is strictly convex and any two points of \overline{M} can be joined by a unique distance minimizing geodesic.

We define the inward and outward pointing vectors of the unit boundary bundle ∂SM by

$$\partial_{\pm} SM := \{(x, \xi) \in \partial SM : \pm \langle \xi, \nu(x) \rangle_g < 0\}.$$

For each $(p, \xi) \in S\overline{M}$ we associate an exit time

$$\tau_g(p, \xi) = \inf\{t > 0 : \gamma_{p, \xi}(t) \in \partial M\}. \quad (19)$$

Notice that it is possible that for some $(p, \xi) \in S\overline{M}$ the exit time $\tau_g(p, \xi)$ is infinite. In such a case, we say that the geodesic $\gamma_{p, \xi}$ is trapped. To make the analysis easier we assume that $\tau_g(p, \xi) < \infty$ for every $(p, \xi) \in S\overline{M}$. Therefore, the *scattering relation*

$$L_g : \overline{\partial_+ SM} \rightarrow \overline{\partial_- SM}, \quad L_g(x, \xi) := \left(\gamma_{x, \xi}(\tau_g(x, \xi)), \dot{\gamma}_{x, \xi}(\tau_g(x, \xi)) \right) \quad (20)$$

is well defined. There are two closely connected inverse problems related to mappings τ_g and L_g . We first formulate the *lens rigidity problem*. This problem reads as follow:

$$\begin{aligned} \text{Do } \tau_g = \tau_{\tilde{g}}, g|_{\partial M} = \tilde{g}|_{\partial M} \text{ and } L_g = L_{\tilde{g}} \text{ imply the existence of} \\ \text{a diffeomorphism } \Phi : \overline{M} \rightarrow \overline{M} \text{ such that } \Phi|_{\partial M} = id \text{ and } g = \Phi^* \tilde{g}? \end{aligned} \quad (21)$$

The other problem is only related to the scattering relation L_g . It is called the *scattering rigidity problem* and it reads as follows:

$$\begin{aligned} \text{Do } g|_{\partial M} = \tilde{g}|_{\partial M} \text{ and } L_g = L_{\tilde{g}} \text{ imply the existence of a diffeomorphism} \\ \Phi : \overline{M} \rightarrow \overline{M} \text{ such that } \Phi|_{\partial M} = id \text{ and } g = \Phi^* \tilde{g}? \end{aligned} \quad (22)$$

In article [III] we studied an inverse problem related to the scattering rigidity problem. A key difference is that instead of L_g we assume that the collections

$$R_{\partial M}^E(p) := \{(\gamma_{p, \xi}(\tau_g(p, \xi)), \dot{\gamma}_{p, \xi}(\tau_g(p, \xi))) \in \partial SM : \xi \in S_p M\}, \quad p \in M \quad (23)$$

are given. See Section 3.3 for more details.

Problems (21) and (22) are closely related to the boundary rigidity questions. We say that (\overline{M}, g) is boundary rigid if every compact Riemannian manifold $(\widetilde{M}, \widetilde{g})$ with the same boundary and the same boundary distance function is isometric to (\overline{M}, g) via a boundary-preserving isometry. In particular, the scattering rigidity, lens rigidity and boundary rigidity problem are equivalent on simple manifolds [43, 50]. Michel conjectured in [43] that simple manifolds are boundary rigid. This problem has been studied intensively. See, for instance, [48, 14, 33, 10]. So far it is known that simple 2-dimensional manifolds are boundary rigid [49]. More recently, boundary rigidity results are established on manifolds of dimension 3 and higher that satisfy certain global convex foliation condition [52, 53]. See [51, 63] for recent surveys on these topics.

3 The review of the results of the thesis

3.1 Article [I]

In this article we study an inverse problem related to the example of obtaining information about the deep structures of the Earth from the travel time differences of seismic waves produced by earthquakes. Let (N, g) be a closed, connected and smooth Riemannian manifold with an open set M , that has a smooth boundary. We denote a compact set $F := N \setminus M$. For a given point $p \in N$ we define a *distance difference function* of p as

$$D_p(x, y) := d_g(p, x) - d_g(p, y), \quad x, y \in F. \quad (24)$$

In Article [I] we ask if the family

$$\mathcal{D}(M) := \{D_p : p \in M\}$$

of distance difference functions with $(F, g|_F)$ determine the closed Riemannian manifold (N, g) . The main result of this paper is the following.

Theorem 3.1. *Let (N_i, g_i) , $i = 1, 2$ be a closed and connected Riemannian manifold of dimension 2 or higher. Let $M_i \subset N_i$ be an open set, with a smooth boundary. We denote a compact set $F_i := N_i \setminus M_i$ and assume that $F_i^{\text{int}} \neq \emptyset$. If the distance difference data of (N_1, g_1) and (N_2, g_2) are the same, that is*

$$\text{there exists a diffeomorphism } \phi : F_1 \rightarrow F_2 \text{ such that } g_1|_{F_1} = \phi^*(g_2|_{F_2}) \quad (25)$$

and

$$\{D_p(\cdot, \cdot) : p \in M_1\} = \{D_q(\phi(\cdot), \phi(\cdot)) : q \in M_2\}, \quad (26)$$

then there exists a Riemannian isometry $\Psi : (N_1, g_1) \rightarrow (N_2, g_2)$ such that $\Psi|_{F_1} = \psi$.

We also show the necessity of the assumption $F^{int} \neq \emptyset$ by an example with two non-isometric Riemannian manifolds that satisfy (25)–(26).

To prove Theorem 3.1, we use techniques similar to [26, 29], especially for the reconstruction of topological and smooth structure. Then, we show that Riemannian metrics g and \tilde{g} of a smooth compact manifold N , for which (25)–(26) holds, are geodesically equivalent. This means that images of geodesics of g coincide with images of geodesics of \tilde{g} and vice versa. The properties of geodesically equivalent metrics have been studied by Matveev and Topalov in [40, 41, 42, 61]. Especially in [61], the authors show that if g and \tilde{g} are geodesically equivalent, then there are several invariants related to the (1, 1)–tensor field G , given in local coordinates $(x^j)_{j=1}^n$ by

$$G(x) = g^{ji}(x)\tilde{g}_{ik}(x)\frac{\partial}{\partial x^j} \otimes dx^k,$$

that are constants along the integral curves of the geodesic flow of g . We use this to prove Theorem 3.1.

In [I] we consider an application of Theorem 3.1 for an inverse problem of a wave equation with spontaneous point sources.

$$\begin{cases} (\partial_t^2 - \Delta_g)G(\cdot, \cdot, s, y) = \kappa(s, y)\delta_{s,y}(\cdot, \cdot), & \text{in } \mathcal{N} \\ G(t, x, s, y) = 0, & \text{for } t < s, x \in N. \end{cases} \quad (27)$$

where $\mathcal{N} = \mathbb{R} \times N$ is the space-time. The solution $G(t, x, s, y)$ is the wave produced by a point source located at the point $y \in M$ and time $s \in \mathbb{R}$ having the magnitude $\kappa(s, y) \in \mathbb{R} \setminus \{0\}$. Above, $\delta_{s,y}(t, x) := \delta_s(t)\delta_y(x)$ corresponds to a point source at $(s, y) \in \mathcal{N}$.

Assume that there are two manifolds (N_1, g_1) and (N_2, g_2) satisfying the assumptions given in Theorem 3.1 and that

$$\text{There exists an isometry } \phi : F_1 \rightarrow F_2 \quad (28)$$

$$W_1 = W_2 \quad (29)$$

where W_1 and W_2 are collections of supports of waves produced by point sources taking place at unknown points at unknown time, that is,

$$W_1 = \{\text{supp}(G^1(\cdot, \cdot, s_1, y_1)) \cap (\mathbb{R} \times F_1); y_1 \in M_1, s_1 \in \mathbb{R}\} \subset 2^{\mathbb{R} \times F_1},$$

and

$$W_2 = \{\text{supp}(G^2(\cdot, \phi(\cdot), s_2, y_2)) \cap (\mathbb{R} \times F_1); y_2 \in M_2, s_2 \in \mathbb{R}\} \subset 2^{\mathbb{R} \times F_1},$$

where functions G^j , $j = \{1, 2\}$ solve equation (27) on manifold N_j . Here, $2^{\mathbb{R} \times F_j} = \{V; V \subset \mathbb{R} \times F_j\}$ is the power set of $\mathbb{R} \times F_j$. Roughly W_j corresponds to the data that one measures by observing, in the set F_j , the waves that are produced by spontaneous point sources that that go off, at an unknown time and at an unknown location, in the set M_j .

Proposition 3.2. *Let (N_j, g_j) , $j = 1, 2$ be a closed compact Riemannian n -manifold, $n \geq 2$ and $M_j \subset N_j$ be an open set such that $F_j = N_j \setminus M_j$ have non-empty interior. If the spontaneous point source data of these manifolds coincide, that is, we have (28)–(29), then (N_1, g_1) and (N_2, g_2) are isometric.*

3.2 Article [II]

In this article we consider

$$\begin{cases} \partial_t^2 u(t, x) - \Delta_g u(t, x) = \chi(t, x)W(t, x) & \text{in } \mathbb{R}_+^{1+n} = (0, \infty) \times \mathbb{R}^n, \\ u|_{t=0} = \partial_t u|_{t=0} = 0. \end{cases} \quad (30)$$

Here the vanishing initial conditions in (30) are interpreted in the sense that u is supported in $[0, \infty) \times \mathbb{R}^n$.

We assume that the source W is a realization of a Gaussian white noise random variable on \mathbb{R}^{1+n} . Moreover, χ stands for a smooth function

$$\chi(t, x) = \chi_0(t)\kappa(x),$$

such that $\chi_0 \in C^\infty(\mathbb{R})$ and defined by

$$\chi_0(t) = \begin{cases} 0, & t \leq 0 \\ 1, & t \geq 1, \end{cases}$$

and $\kappa \in C_0^\infty(\mathbb{R}^n)$. We assume that there exists an open and non-empty set $\mathcal{X} \subset \mathbb{R}^n$ where κ is non-vanishing. The spatial structure of our noise model coincides, *e.g.*, with the typical choice (2.17) in the monograph [18].

The source χW can be modelled as a random variable taking values in a local Sobolev space with negative index, and the same holds true for the solution u . Contrary to papers such as [44, 47, 13], we do not consider $t \mapsto u(t, \cdot)$ as a random process.

The problem we study is the following: suppose we can record the empirical correlation

$$C_T(t_1, x_1, t_2, x_2) = \frac{1}{T} \int_0^T u(t_1 + s, x_1)u(t_2 + s, x_2)ds, \quad (31)$$

for $t_1, t_2 > 0$, $x_1, x_2 \in \mathcal{X}$ and $T > 0$. What information does this data yield regarding the metric g ? For any finite T , the correlation C_T is random in the sense that it depends on the realization of the source. A fundamental part of our result below is to show that this data becomes *statistically stable*, *i.e.*, independent of the realization, as T increases. More precisely, we show that the limit

$$\lim_{T \rightarrow \infty} \langle C_T, f \otimes h \rangle_{\mathcal{D}' \times C_0^\infty(\mathbb{R}^{2+2n})}, \quad f, g \in C_0^\infty(\mathbb{R}^{1+n}),$$

is deterministic. Thereafter, the paper is devoted to showing that this stability enables the recovery of g :

Theorem 3.3. *Let $n \geq 3$. Suppose that g is non-trapping and that g coincides with the Euclidean metric outside a compact set. Let $u = \mathbb{U}(\omega)$ be the solution of (30) where $W = \mathbb{W}(\omega)$ is a realization of the white noise \mathbb{W} on \mathbb{R}^{1+n} . Then with probability one, the empirical correlations (31) defined in the sense of generalized random variables in $\mathcal{D}'((\mathbb{R} \times \mathcal{X})^2)$ for $T > 0$, determine the Riemannian manifold (\mathbb{R}^n, g) up to an isometry.*

Recall that a metric tensor g on \mathbb{R}^n is non-trapping if for each compact $K \subset \mathbb{R}^n$ there exists $T > 0$ such that for each $(p, \xi) \in T\mathbb{R}^n$, $p \in K$, $\|\xi\|_g = 1$, it holds that $\gamma_{p,\xi}(t) \notin K$ when $t \geq T$.

Note that the covariance data (31) is determined by the measurement $u|_{(0,\infty) \times \mathcal{X}}$. This implies the following corollary:

Corollary 3.4. *The measurement $u|_{(0,\infty) \times \mathcal{X}}$, with a single realization of the white noise source, determines the Riemannian manifold (\mathbb{R}^n, g) , up to an isometry, with probability one under the assumptions of Theorem 3.3.*

The statistical stability of C_T , $T > 0$, allows us to reduce the passive imaging problem to a deterministic inverse problem, that we then solve. As the deterministic inverse problem is of independent interest, we solve it in a more general geometric setting.

Let (N, g) be a complete, oriented smooth Riemannian manifold. The second problem studied in this article is related to the problem (13). Consider the interior source problem for the Riemannian wave equation

$$\begin{cases} (\partial_t^2 - \Delta_g)u = f, & \text{in } (0, \infty) \times N \\ u(0, x) = \partial_t u(0, x) = 0, & x \in N. \end{cases} \quad (32)$$

If $f \in C_0^\infty((0, \infty) \times N)$, then there exists a unique $u^f \in C^\infty((0, \infty) \times N)$ that solves (32) (see [60], Chapter 6). Let $\mathcal{X} \subset N$ be open, and consider the following local source-to-solution operator

$$\Lambda_{\mathcal{X}} : C_0^\infty((0, \infty) \times \mathcal{X}) \rightarrow C^\infty((0, \infty) \times \mathcal{X}), \quad \Lambda_{\mathcal{X}} f = u^f|_{(0,\infty) \times \mathcal{X}}.$$

The second main theorem of [II] is the following.

Theorem 3.5. *Let (N, g) be a smooth and complete Riemannian manifold of dimension $n \geq 2$. Let $\mathcal{X} \subset N$ be an open and nonempty set. Then the data $(\mathcal{X}, \Lambda_{\mathcal{X}})$ determines (N, g) up to an isometry. More precisely this means the following.*

Let (N_i, g_i) , $i = 1, 2$, be a smooth and complete Riemannian manifold. Let $\mathcal{X}_i \subset N_i$ be open and nonempty, and assume that there exists a diffeomorphism

$$\phi : \mathcal{X}_1 \rightarrow \mathcal{X}_2 \quad (33)$$

that satisfies

$$\phi^*(\Lambda_{\mathcal{X}_2} f) = \Lambda_{\mathcal{X}_1}(\phi^* f), \quad \text{for all } f \in C_0^\infty((0, \infty) \times \mathcal{X}_2), \quad (34)$$

where $(\phi^* f) \in C_0^\infty((0, \infty) \times \mathcal{X}_1)$ is defined by

$$(\phi^* f)(t, x) = f(t, \phi(x)).$$

Then (N_1, g_1) and (N_2, g_2) are Riemannian isometric.

We also point out the relationship between Theorem 3.5 and the following Inverse spectral problem of the Laplace–Beltrami operator.

Corollary 3.6. *Let (N, g) be a smooth and compact Riemannian manifold of dimension $n \geq 2$ without boundary. Let $\mathcal{X} \subset N$ be an open and nonempty set. Let $(\varphi_k)_{k=1}^\infty \subset C^\infty(N)$ be the collection of L^2 -orthonormal eigenfunctions of operator Δ_g . Let $(\lambda_k)_{k=1}^\infty$ be the collection of corresponding eigenvalues of Δ_g . Then, the Spectral data*

$$(\mathcal{X}, (\varphi_k|_{\mathcal{X}}, \lambda_k)_{k=1}^\infty) \quad (35)$$

determines (N, g) up to isometry.

The Corollary 3.6 is connected to the main result of [7], where Belishev and Kurylev provide a proof for the Gel’fand inverse boundary spectral problem for the Laplace–Beltrami operator. They showed that the boundary spectral data

$$\{\partial M, (\partial_\nu \varphi_k|_{\partial M}, \lambda_k)_{k=1}^\infty\} \quad (36)$$

determines a compact Riemannian manifold (\overline{M}, g) up to an isometry. Here, the set $(\varphi_k)_{k=1}^\infty$ is the collection of L^2 -orthonormal eigenfunctions of Dirichlet–Laplace–Beltrami operator, and the collection $(\lambda_k)_{k=1}^\infty$ is the set of corresponding eigenvalues. This means that for every $k \in \mathbb{N}$ the function φ_k solves the equation $\Delta_g \varphi_k = \lambda_k \varphi_k$ and $\varphi_k|_{\partial M} = 0$.

Finally, we will introduce an inverse problem that is a natural generalization of the Gel’fand inverse boundary spectral problem. Suppose that $\Sigma \subset N$ is a smooth $(n-1)$ -dimensional submanifold of N such that $\Sigma = \partial \mathcal{X}$ for some open set $\mathcal{X} \subset N$. Let ν be the outward pointing unit normal of Σ . Then, the *Cauchy spectral data* of Laplace–Beltrami operator is

$$(\Sigma, ((\varphi_k|_{\Sigma}, \partial_\nu \varphi_k|_{\Sigma}, \lambda_k)_{k=1}^\infty)). \quad (37)$$

In [27] it was shown that data (37) determines the manifold (N, g) up to an isometry. We emphasize that Corollary 3.6 follows from the results of [27], since the data (35) determines the data (37).

3.3 Article [III]

In article [III] we study an inverse problem of a reconstruction of a compact Riemannian manifold (\overline{M}, g) with smooth boundary from the scattering data of internal sources. We assume that (\overline{M}, g) is embedded in a closed smooth Riemannian manifold (N, g) . We define the first exit time function by

$$\tau_{exit}(p, \xi) = \inf\{t > 0 : \gamma_{p, \xi}(t) \in N \setminus \overline{M}\}, \quad (p, \xi) \in S\overline{M}, \quad (38)$$

We also assume that (\overline{M}, g) is non-trapping, namely:

$$\tau_{exit}(p, \xi) < \infty, \quad \text{for all } (p, \xi) \in S\overline{M}. \quad (39)$$

For each point $q \in \overline{M}$ we define a *scattering set of a point source* q as:

$$\begin{aligned} R_{\partial M}(q) = \{ & (p, \eta^T) \in T\partial M : \text{there exist } \xi \in S_q N \\ & \text{and } t \in [0, \tau_{exit}(q, \xi)] \text{ such that} \\ & p = \gamma_{q, \xi}(t), \eta = \dot{\gamma}_{q, \xi}(t) \} \in 2^{T\partial M}. \end{aligned} \quad (40)$$

Here $\eta^T \in T\partial M$ is the tangential component of $\eta \in \partial SM$, and 2^S denotes the power set of the set S . Let $R_{\partial M}(M) = \{R_{\partial M}(q); q \in M\} \subset 2^{T\partial M}$, and consider the following collection:

$$(\partial M, R_{\partial M}(M)). \quad (41)$$

We refer to (41) as the *scattering data of internal sources* that depends on the Riemannian manifold (\overline{M}, g) .

The inverse problem considered in paper [III] is the determination of the Riemannian manifold (\overline{M}, g) from the data (41). More precisely, this means the following. Let (N_j, g_j) , $j = 1, 2$, and M_j be similar to (N, g) and M .

We say that the scattering data of internal sources of (\overline{M}_1, g_1) is equivalent to (\overline{M}_2, g_2) , if

$$\text{there exists a diffeomorphism } \phi : \partial M_1 \rightarrow \partial M_2 \text{ such that} \quad (42)$$

$$\{D\phi(R_{\partial M_1}(q)); q \in M_1\} = \{R_{\partial M_2}(p); p \in M_2\}. \quad (43)$$

Here $D\phi : T\partial M_1 \rightarrow T\partial M_2$ is the differential of ϕ , *i.e.*, the tangential mapping of ϕ .

To prove the inverse problem of data (41), we need some additional information about the geometry of (N, g) . This is described in the following. We denote the set of all smooth Riemannian metrics on N by $\mathcal{M}et(N)$ (we will assign the smooth Whitney topology (see [22]) on $\mathcal{M}et(N)$ to make it into a topological space), it turns out that there exists a generic subset (a set that contains a countable

intersection of open and dense sets) $G \subset \mathcal{M}et(N)$ such that, for $g \in G$, the manifold $(\overline{M}, g|_{\overline{M}})$ is determined by the data (41) up to an isometry.

Given $g \in \mathcal{M}et(N)$, $p, q \in N$ and $\ell > 0$, we denote the number of g -geodesics connecting p and q of length ℓ by $I(g, p, q, \ell)$. We also define:

$$I(g) := \sup_{p, q, \ell} I(g, p, q, \ell).$$

Kupka, Peixoto and Pugh showed in [28] that there exists a generic set $G \subset \mathcal{M}et(N)$, such that, for all $g \in G$,

$$I(g) \leq 2n + 2. \tag{44}$$

Next, we define the collection of admissible Riemannian manifolds. Let

$$\mathcal{G} = \{(N, g) : N \text{ is a connected, closed, smooth Riemannian } n\text{-manifold}; \\ g \text{ satisfies (44)}\}. \tag{45}$$

The main result of [III] is the following.

Theorem 3.7. *Let $(N_i, g_i) \in \mathcal{G}$, $i = 1, 2$ be a smooth, closed, connected Riemannian n -manifold, $n \geq 2$, $M_i \subset N_i$ an open set with smooth strictly convex boundary with respect to g_i . Suppose that (\overline{M}_i, g_i) , $i = 1, 2$, is non-trapping, in the sense of (39).*

If properties (42)–(43) hold true, then $(\overline{M}_1, g|_{\overline{M}_1})$ is Riemannian isometric to $(\overline{M}_2, g_2|_{\overline{M}_2})$.

The problem studied in [III] has a natural connection to the scattering rigidity problem (22). In [III] we show that data (41) determines the scattering relation L_g (see (20)). It is worth of mentioning that data (41) is much larger than (22), since in the case of strictly convex boundary functions τ_g (see (19)) and τ_{exit} (see (38)) coincide. On the other hand, we did not have as strong geometric assumptions as in [49] since every simple manifold $(\overline{M}, g) \in \mathcal{G}$.

3.4 Article [IV]

Article [IV] is a conference paper related to article [I]. In [IV] we generalize the result of [I]. We say that a set $A \subset N$ is convex if for any two $p, q \in A$ holds that every distance minimizing unit speed geodesic γ from p to q satisfies

$$\gamma([0, d_g(p, q)]) \subset A.$$

The main theorem of [IV] is the following.

Theorem 3.8. *Let $n \geq 2$ and (N_i, g_i) , $i = 1, 2$ be complete Riemannian manifolds. Also, let $U_i \subset N_i$ be a relatively compact open set with smooth boundary. Let \bar{U}_i be convex and $M_i \subset U_i$ be an open subset whose boundary is a smooth submanifold of dimension $(n - 1)$ and $\bar{M}_i \subset U_i$. Denote $F_i = \bar{U}_i \setminus M_i$ and suppose that $F_i^{\text{int}} \neq \emptyset$.*

Assume that there exists a diffeomorphism $\phi : F_1 \rightarrow F_2$ such that

$$g_1|_{F_1} = \phi^* g_2|_{F_2}.$$

Moreover, assume that the distance difference data for manifolds M_1 and M_2 are the same, in the sense that

$$\{D_x^1 \in C(F_1 \times F_1) : x \in M_1\} = \{D_y^2(\phi(\cdot), \phi(\cdot)) \in C(F_1 \times F_1) : y \in M_2\}.$$

Here, $D_x^i(z_1, z_2) = d_{g_i}(x, z_1) - d_{g_i}(x, z_2)$ for $x \in N_i$ and $z_1, z_2 \in F_i$.

Then, the Riemannian manifolds with boundary $(\bar{U}_1, g_1|_{\bar{U}_1})$ and $(\bar{U}_2, g_2|_{\bar{U}_2})$ are isometric.

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