Dirichlet problems for mean curvature and p-harmonic equations on Cartan-Hadamard manifolds

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Academic dissertation

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Helsinki, September 2017

Esko Heinonen

List of included articles

This dissertation consists of an introductory part and the following articles:

[A] J.-B. Casteras, E. Heinonen, and I. Holopainen, *Solvability of minimal graph equation under pointwise pinching condition for sectional curvatures*, J. Geom. Anal. 27(2):1106–1130 (2017).

[B] E. Heinonen, *Asymptotic Dirichlet problem for $\mathcal{A}$-harmonic functions on manifolds with pinched curvature*, Potential Anal., 46(1):63-74 (2017).

[C] J.-B. Casteras, E. Heinonen, and I. Holopainen, *Dirichlet problem for f-minimal graphs.* J. Anal. Math., to appear. Preprint: arXiv:1605.01935 (2016).

[D] J.-B. Casteras, E. Heinonen, and I. Holopainen, *Existence and non-existence of minimal graphic and p-harmonic functions.* Preprint: arXiv:1701.00953 (2017).

In the joint articles [A], [C] and [D], the author made significant contributions to the analysis and had a central role in the writing process and verifying the details of the proofs.

The author acknowledges J. Geom. Anal. as the original publisher of the article [A] and Potential Anal. as the original publisher of the article [B].

Contents

1. Overview

The unifying theme of the four articles, [A,B,C,D], forming this dissertation is the existence and non-existence of continuous entire non-constant solutions for nonlinear differential operators on a Riemannian manifold M. The existence results of such solutions are proved by studying the asymptotic Dirichlet problem under different assumptions on the geometry of the manifold.

Minimal graphic functions are studied in articles [A] and [D]. Article [A] deals with an existence result whereas in [D] we give both existence and non-existence results with respect to the curvature of M. Moreover p-harmonic functions are studied in [D].

Article [B] deals with the existence of $\mathcal{A}$-harmonic functions under similar curvature assumptions as in [A]. In article [C] we study the existence of f-minimal graphs, which are generalisations of usual minimal graphs. In contrast to the other articles, here we also consider the existence in the case of bounded domains.

Before turning to the ideas and results of the research articles, we present some key concepts of the thesis and give a brief history of the development of the asymptotic Dirichlet problem. Due to the similarity of the techniques in [A] and [B], we treat them together in Section 4. Article [C] is treated in Section 5 and article [D] in Section 6. At the beginning of the Sections 4 – 6 we briefly give the background of the methods and techniques used in the articles.

2. Preliminaries

This section is devoted to defining the key concepts of this thesis. Throughout the thesis we assume that M is an n-dimensional, $n \geq 2$, connected, non-compact orientable Riemannian manifold equipped with a Riemannian metric $\langle \cdot, \cdot \rangle$. The tangent space at each point $x \in M$ will be denoted by T_xM and the norm with respect to the Riemannian metric by $|\cdot|$. Unless otherwise specified, the integration will be with respect to the Riemannian volume form dm.

In the case of smooth functions $u\colon M \to \mathbb{R}$, the covariant derivation will be denoted by D or semicolon. The first covariant derivative agrees with the usual partial derivative and for the second covariant derivative we have

$$D_iD_ju = u_{i;j} = u_{ij} - \Gamma_{ij}^k u_k = u_{j;i} = D_jD_iu,$$

with $u_k = \partial u/\partial x^k$. The third covariant derivative is no more symmetric with respect to the last indices. If the Riemannian metric is given by $ds^2 = \sigma_{ij}dx^idx^j$ in local coordinates with inverse matrix (σ^{ij}), we will use a short hand notation $u^i = \sigma^{ij}D_ju$.

A *Cartan-Hadamard* (also *Hadamard*) manifold M is a simply connected Riemannian manifold whose all sectional curvatures satisfy

$$K_M \leq 0.$$

Basic examples of such manifolds are the Euclidean space $\mathbb{R}^n$, with zero curvature, and the hyperbolic space $\mathbb{H}^n$, with constant negative curvature. The name of these manifolds has its origin in the Cartan-Hadamard theorem which states that the exponential map is a diffeomorphism in the whole tangent space at every point of M.

Given a smooth function $k\colon [0,\infty) \to [0,\infty)$ we denote by $f_k\colon [0,\infty) \to \mathbb{R}$ the smooth non-negative solution to the initial value problem (Jacobi equation)

$$\begin{cases} f_k(0) = 0, \\ f_k'(0) = 1, \\ f_k'' = k^2 f_k. \end{cases}$$

These functions play an important role in estimates involving curvature bounds since they result to rotationally symmetric manifolds that can be used in various comparison theorems, e.g. Hessian and Laplace comparison (see [40]).

Recall that a rotationally symmetric manifold, also a model manifold, M_f is $\mathbb{R}^n$ equipped with a metric of the form $g^2 = dr^2 + f(r)^2 d\theta^2$, where r is the distance to a pole o and $d\theta$ is the standard metric on the unit sphere $\mathbb{S}^{n-1}$. The sectional curvatures of a model manifold can be obtained from the radial curvature function, namely we have

$$K_{M_f}(P_x) = -\frac{f''\big(r(x)\big)}{f\big(r(x)\big)} \cos^2\alpha + \frac{1 - f'\big(r(x)\big)^2}{f\big(r(x)\big)^2} \sin^2\alpha, \tag{2.1}$$

where α is the angle between $\nabla r(x)$ and the 2-plane $P_x \subset T_x M$, and hence these manifolds offer examples of Cartan-Hadamard manifolds when $f'' \ge 0$. In the case of the radial sectional curvature the formula simplifies to

$$K_{M_f} = -\frac{f''}{f}.$$

For the verification of these formulae one could see e.g. [85].

2.1. **Mean curvature equation and minimal surfaces.** In 2-dimensional case we have a nice and simple interpretation. Let $\Omega \subset \mathbb{R}^2$ be an open set and $u\colon \Omega \to \mathbb{R}$ a C^2 function with graph $\Sigma_u = \{(x, u(x))\colon x \in \Omega\}$. Keeping the boundary $\partial\Sigma_u$ fixed and making a smooth variation of the graph, we get that the critical points of the area functional

$$\int_\Omega \sqrt{1 + |\nabla u|^2}$$

are solutions to the *minimal graph equation*

$$\operatorname{div} \frac{\nabla u}{\sqrt{1 + |\nabla u|^2}} = 0. \tag{2.2}$$

The graphs of solutions of (2.2) have the minimal area among all graphs with fixed boundary $\partial\Sigma_u$.

More generally we define minimal graphic functions as follows. Let $\Omega \subset M$ be an open set. Then a function $u \in W^{1,1}_{\text{loc}}(\Omega)$ is a *(weak) solution of the minimal graph equation* if

$$\int_\Omega \frac{\langle \nabla u, \nabla \varphi \rangle}{\sqrt{1 + |\nabla u|^2}} = 0$$

for every $\varphi \in C_0^\infty(\Omega)$. Note that the integral is well-defined since

$$\sqrt{1 + |\nabla u|^2} \ge |\nabla u| \quad \text{a.e.},$$

and thus

$$\int_\Omega \frac{|\langle \nabla u, \nabla \varphi \rangle|}{\sqrt{1+|\nabla u|^2}} \le \int_\Omega \frac{|\nabla u||\nabla \varphi|}{\sqrt{1+|\nabla u|^2}} \le \int_\Omega |\nabla \varphi| < \infty.$$

The operator in (2.2) gives also the mean curvature of the graph Σ_u. Namely, if $\bar{N}$ is the unit normal vector field of Σ_u, then the mean curvature vector at point x is given by

$$\left(\operatorname{div} \frac{\nabla u}{\sqrt{1+|\nabla u|^2}} \right) \bar{N}(x) = \bar{H}(x)$$

and the (scalar) mean curvature is

$$\operatorname{div} \frac{\nabla u}{\sqrt{1+|\nabla u|^2}} = H(x). \tag{2.3}$$

Therefore it is also called the mean curvature operator. Recall that the mean curvature of a submanifold is the trace of the second fundamental form and general minimal (hyper) surfaces (not necessarily graphs of functions) are the surfaces having zero mean curvature.

Instead of minimal surfaces, one can also consider surfaces of constant mean curvature (CMC surfaces) or surfaces of prescribed mean curvature. In the latter case one considers solutions of (2.3) and H is a function defined on M or in more general situation in $M \times \mathbb{R}$, see Section 5 and [C].

It is well known that under certain conditions there exists a (strong) solution of (2.2) with given boundary values. Namely, let $\Omega \subset\subset M$ be a smooth relatively compact open set whose boundary has positive mean curvature with respect to inwards pointing unit normal. Then for each $\theta \in C^{2,\alpha}(\bar{\Omega})$ there exists a unique $u \in C^\infty(\Omega) \cap C^{2,\alpha}(\bar{\Omega})$ that solves the minimal graph equation (2.2) in Ω and has the boundary values $u|\partial\Omega = \theta|\partial\Omega$. Similar existence result holds also for the case of prescribed mean curvature equation (2.3) but with an assumption that the lower bound for the mean curvature of the boundary $\partial\Omega$ depends on the function H.

The standard strategy to prove these type of results is to obtain a priori height and gradient estimates for the solutions and then apply the continuity or Leray-Schauder method. For the proofs in the Euclidean case one should see the original papers by Jenkins and Serrin [59] and by Serrin [80] or the book [39] where also more general equations are considered. For the Riemannian case see e.g. [81] and [32]. In [C] we treat the more general case where the prescribed mean curvature depends also on the $\mathbb{R}$-variable of the product space $M \times \mathbb{R}$. Good references for the general theory of minimal surfaces are e.g. [26] and [63].

It is also useful to write the minimal graph equation in a non-divergence form

$$\frac{1}{W} g^{ij} D_i D_j u = 0,$$

where $W = \sqrt{1+|\nabla u|^2}$,

$$g^{ij} = \sigma^{ij} - \frac{u^i u^j}{W^2} \tag{2.4}$$

and $u^i = \sigma^{ij} D_j u$. The induced metric on the graph of u is given by

$$g_{ij} = \sigma_{ij} + u_i u_j$$

with inverse (2.4). Similarly the mean curvature of the graph is given by

$$\frac{1}{W} g^{ij} D_i D_j u = nH.$$

For the derivation of these formulae, see e.g. [77].

2.2. **$\mathcal{A}$-harmonic functions.** The weak solutions of the quasilinear elliptic equation

$$\mathcal{Q}[u] = -\operatorname{div} \mathcal{A}_x(\nabla u) = 0 \tag{2.5}$$

are called $\mathcal{A}$-*harmonic* functions. Here the $\mathcal{A}$-harmonic operator (of type p), $\mathcal{A}\colon TM \to TM$, is subject to certain conditions; for instance $\langle \mathcal{A}(V), V\rangle \approx |V|^p$, $1 < p < \infty$, and $\mathcal{A}(\lambda V) = \lambda|\lambda|^{p-2}\mathcal{A}(V)$ for all $\lambda \in \mathbb{R} \setminus \{0\}$ (see [B] for the precise definition). The set of all such operators is denoted by $\mathcal{A}^p(M)$

To be more precise what we mean by a weak solution, let $\Omega \subset M$ be an open set and $\mathcal{A} \in \mathcal{A}^p(M)$. A function $u \in C(\Omega) \cap W^{1,p}_{\text{loc}}(\Omega)$ is $\mathcal{A}$-harmonic in Ω if it satisfies

$$\int_\Omega \langle \mathcal{A}(\nabla u), \nabla \varphi\rangle = 0 \tag{2.6}$$

for every test function $\varphi \in C_0^\infty(\Omega)$. If $|\nabla u| \in L^p(\Omega)$, then it is equivalent to require (2.6) for all $\varphi \in W_0^{1,p}(\Omega)$ by approximation. In the special case $\mathcal{A}(v) = |v|^{p-2}v$, yielding an equation

$$-\operatorname{div}\big(|\nabla u|^{p-2}\nabla u\big) = 0, \tag{2.7}$$

$\mathcal{A}$-harmonic functions are called *p-harmonic* and, in particular, if $p = 2$, we obtain the usual harmonic functions. Therefore we see that $\mathcal{A}$-harmonic functions are really a generalisation of harmonic functions.

As the properties of the harmonic functions can be studied with superharmonic functions, the $\mathcal{A}$-superharmonic functions play a similar role for the $\mathcal{A}$-harmonic functions. A lower semicontinuous function $u\colon \Omega \to (-\infty, \infty]$ is called $\mathcal{A}$-*superharmonic* if $u \not\equiv \infty$ in each component of Ω, and for each open $D \subset\subset \Omega$ and for every $h \in C(\bar{D})$, $\mathcal{A}$-harmonic in D, $h \le u$ on ∂D implies $h \le u$ in D. In the case of equation (2.7) these functions are called *p-superharmonic*. A very good standard reference for the study of nonlinear potential theory in the Euclidean case is the book [46] by Heinonen, Kilpeläinen and Martio. For the Riemannian setting see [48].

The question about the solvability of the Dirichlet problem (also the asymptotic one, see Section 2.3) for $\mathcal{A}$-harmonic functions can be approached via the Perron's method which reduces the problem to the question about the regularity of the boundary points. Recall that a boundary point x_0 is regular if

$$\lim_{x \to x_0} \overline{H}_f(x) = f(x_0)$$

for every continuous boundary data f. Here $\overline{H}_f$ is the upper Perron solution. For precise definitions see [B] and for a complete treatment [46].

2.3. **Asymptotic Dirichlet problem on Cartan-Hadamard manifolds.** Cartan-Hadamard manifolds can be compactified by adding the *asymptotic boundary* (also *sphere at infinity*) $\partial_\infty M$ and equipping the resulting space $\bar{M} := M \cup \partial_\infty M$ with the *cone topology*, making $\bar{M}$ diffeomorphic to the closed unit ball. The asymptotic boundary $\partial_\infty M$ consists of equivalence classes of geodesic rays under the equivalence relation

$$\gamma_1 \sim \gamma_2 \quad \text{if} \quad \sup_{t \geq 0} \operatorname{dist}\big(\gamma_1(t), \gamma_2(t)\big) < \infty.$$

Equivalently it can be considered as the set of geodesic rays emitting from a fixed point $o \in M$, which justifies the name sphere at infinity.

The basis for the cone topology in $\bar{M}$ is formed by cones

$$C(v, \alpha) := \{y \in M \setminus \{x\} \colon \sphericalangle(v, \dot{\gamma}_0^{x,y}) < \alpha\}, \quad v \in T_x M,\ \alpha > 0,$$

truncated cones

$$T(v, \alpha, R) := C(v, \alpha) \setminus \bar{B}(x, R), \quad R > 0,$$

and all open balls in M. Cone topology was first introduced in [37].

This construction allows us to define the main concept of this thesis, namely the *asymptotic Dirichlet problem* (also *Dirichlet problem at infinity*) for a quasilinear elliptic operator Q:

Problem. *Let $\theta \colon \partial_\infty M \to \mathbb{R}$ be a continuous function. Does there exist a continuous function $u \colon \bar{M} \to \mathbb{R}$ with*

$$\begin{cases} Q[u] = 0 & \text{in } M; \\ u | \partial_\infty M = \theta, \end{cases}$$

and if yes, is the function u unique?

In the case such function u exists for every $\theta \in C(\partial_\infty M)$, we say that the asymptotic Dirichlet problem in $\bar{M}$ is *solvable*. As we will see, the solvability of this problem depends on the geometry of the manifold M, but the uniqueness of the solutions depends also on the operator Q. For the usual Laplace, $\mathcal{A}$-harmonic and minimal graph operators we have the uniqueness but more complicated operators may not satisfy maximum principles and hence also the uniqueness of solutions will be lost (see Section 5).

3. Background of the asymptotic Dirichlet problem

In this section we give a brief history of the asymptotic Dirichlet problem and developments before [A],[B],[C] and [D]. We will denote by M a Cartan-Hadamard manifold with sectional curvature K_M. Point $o \in M$ will be a fixed point and $r = d(o, \cdot)$ is the distance to o. By P_x we denote a 2-dimensional subspace of $T_x M$.

3.1. **Harmonic functions.** The study of the harmonic functions on Cartan-Hadamard manifolds has its origin in [40] where they proposed the conjecture that if the sectional curvatures of the manifold M satisfy

$$K_M \leq -\frac{C}{r^2}, \quad C > 0,$$

outside a compact set, then there exists a bounded non-constant harmonic function on M. One way to show the existence of such functions is to try to solve the asymptotic Dirichlet problem with continuous boundary data on $\partial_\infty M$.

The study of the asymptotic Dirichlet problem began in the beginning of 1980's when Choi [23] gave a definition of the problem and showed that it can be solved on a general n-dimensional Cartan-Hadamard manifold by assuming that the sectional curvatures have an upper bound $K_M \le -a^2$, for some constant $a > 0$, and that any two points on the boundary $\partial_\infty M$ can be separated by convex neighbourhoods. In [6] Anderson showed that such neighbourhoods can be constructed by assuming that the sectional curvatures are bounded between two negative constants, resulting to the following.

Theorem. *Assume that the sectional curvatures of M satisfy*

$$-b^2 \le K_M \le -a^2, \tag{3.1}$$

where $0 < a \le b$ are arbitrary constants. Then the asymptotic Dirichlet problem is uniquely solvable.

Sullivan [82] solved the asymptotic Dirichlet problem independently at the same time by assuming (3.1) and using probabilistic methods. In [7] Anderson and Schoen gave an identification of the Martin boundary of M under the assumption (3.1).

A slightly different setting was considered by Ballmann [9], and Ballmann and Ledrappier [10] when studying the Dirichlet problem on negatively curved rank 1 manifolds. Ancona considered Gromov hyperbolic graphs [3] and Gromov hyperbolic manifolds [4]. In [2] he solved the asymptotic Dirichlet problem by assuming an upper bound for the sectional curvatures and that balls up to a fixed radius are L-bi-Lipschitz equivalent to an open set in $\mathbb{R}^n$.

In [19] Cheng introduced the pointwise pinching condition

$$|K_M(P_x)| \le C_K |K_M(P'_x)| \tag{3.2}$$

for the sectional curvatures, and solved the problem assuming (3.2) and positive bottom spectrum for the Laplacian. Here $C_K > 0$ is a constant and $P_x, P'_x \subset T_x M$ are any 2-dimensional subspaces containing the radial vector field. It is worth noting that (3.2) allows the curvature to behave very freely along different geodesic rays.

Trying to relax the assumption (3.1), the first result allowing the curvature to approach zero was due to Hsu and March [57] with assumption

$$-b^2 \le K_M \le -C/r^2$$

for some constants $b > 0$ and $C > 2$. On the other hand, Borbély [14] allowed the curvature to decay with assumption

$$-be^{\lambda r} \le K_M \le -a$$

for some constants $b \ge a > 0$ and $\lambda < 1/3$.

In 2003 Hsu [56] solved the asymptotic Dirichlet problem already under very general curvature assumptions, namely his first result allowed the upper bound behave like $K_M \le -\alpha(\alpha - 1)/r^2$ for $\alpha > 0$ and instead of a lower bound for the sectional curvatures, he assumed a Ricci lower bound $-r^{2\beta} \le \mathrm{Ric}$ with

$\beta < \alpha - 2$. His second result assumed a constant sectional curvature upper bound $-a$ but allowed the Ricci lower bound to decay as

$$-h(r)^2 e^{2ar} \leq \mathrm{Ric},$$

where h is a function satisfying $\int_0^\infty r h(r)\, dr < \infty$.

3.2. **$\mathcal{A}$- and p-harmonic functions.** Investigation of the nonlinear setting was started by Pansu [73] in 1988 when he showed the existence of non-constant bounded p-harmonic functions, with $p > (n-1)b/a$ and gradients in L^p, under the curvature assumption (3.1). His proof was based on study of the L^p-cohomology and it also gave non-existence for $p \leq (n-1)a/b$.

In [50] Holopainen showed that the direct approach by Anderson and Schoen in [7] can be generalised to work also in the case of p-harmonic functions under the assumption (3.1). Few years later Holopainen, Lang and Vähäkangas [53] proved the existence of non-constant bounded p-harmonic functions in Gromov hyperbolic metric measure spaces X equipped with a Borel regular locally doubling measure.

Vähäkangas [86] replaced Cheng's [19] assumption on the spectrum of the Laplacian by a curvature upper bound $K_M \leq -\phi(\phi-1)/r^2$ and was able to generalise the techniques used by Cheng to show the existence of non-constant bounded $\mathcal{A}$-harmonic functions assuming also (3.2).

Holopainen and Vähäkangas [55] (see also the unpublished licentiate thesis [85]) generalised the approach of [50] and [7] even further to allow very general curvature bounds

$$-(b \circ r)^2 \leq K_M \leq -(a \circ r)^2,$$

where a and b are functions satisfying assumptions [55, (A1)-(A7)] (see also [C, Section 4]). As a special case they obtain e.g. the following.

Theorem. *Let M be a Cartan-Hadamard manifold of dimension $n \geq 2$. Suppose that*

$$-r(x)^{2(\phi-2)-\varepsilon} \leq K(P_x) \leq -\frac{\phi(\phi-1)}{r(x)^2}, \tag{3.3}$$

$r(x) \geq R_0$, for some constants $\phi > 1$ and $\varepsilon, R_0 > 0$. Then the asymptotic Dirichlet problem for p-Laplacian is solvable for every $p \in (1, 1+(n-1)\phi)$.

And assuming a constant curvature upper bound $-k$, they can also allow the curvature to decay exponentially. Namely under the curvature bounds

$$-r(x)^{-2-\varepsilon} e^{2kr(x)} \leq K_M(P_x) \leq -k \tag{3.4}$$

they solve the Dirichlet problem for every $p \in (1, \infty)$.

In the unpublished preprint [87] Vähäkangas proved the existence of $\mathcal{A}$-harmonic functions under curvature assumptions similar to (3.3) and (3.4). His technique adapted the method of Cheng [19] using Sobolev and Caccioppoli-type inequalities together with complementary Young functions. Recently Casteras, Holopainen and Ripoll [17] refined the methods of [87] and improved the curvature upper bound to (almost) optimal, assuming

$$-\frac{\big(\log r(x)\big)^{2\tilde{\varepsilon}}}{r(x)^2} \leq K_M(P_x) \leq -\frac{1+\varepsilon}{r(x)^2 \log r(x)} \tag{3.5}$$

for some constants $\varepsilon > \tilde{\varepsilon} > 0$.

3.3. **Minimal graphic functions.** In this subsection we mention also some results that do not concern directly the asymptotic Dirichlet problem but are still related to the study of this thesis. Readers interested in the general theory of minimal surfaces could see e.g. the survey [69].

The theory of minimal surfaces is very classical and has its origin in the 18th century. One of the most interesting questions was the Plateau's problem raised originally by Lagrange [62] in 1760, named after the Belgian physicist Joseph Plateau (1801-1883), and finally solved independently by Douglas [36] and Radó [74] in the beginning of 1930's. Another interesting aspect is the Bernstein-type problem that deals with minimal hypersurfaces in $\mathbb{R}^n$. The 3-dimensional case was proved by Bernstein [13] in 1915-1917.

In 1968 Jenkins and Serrin [59] proved the solvability of the Dirichlet problem on bounded domains $\Omega \subset \mathbb{R}^n$ whose boundary has non-negative mean curvature. Serrin [80] gave a classical existence result for the prescribed mean curvature graphs in $\mathbb{R}^n$ and more recently Guio and Sa Earp [45] considered similar Dirichlet problem in the hyperbolic space.

Nelli and Rosenberg [72] constructed catenoids, helicoids and Scherk-type surfaces in $\mathbb{H}^2 \times \mathbb{R}$ and they also proved the solvability of the asymptotic Dirichlet problem in $\mathbb{H}^2$.

Theorem. *Let Γ be a continuous rectifiable Jordan curve in $\partial_\infty \mathbb{H}^2 \times \mathbb{R}$, that is a vertical graph. Then, there exists a minimal vertical graph on $\mathbb{H}^2$ having Γ as asymptotic boundary. The graph is unique.*

In 2005 Meeks and Rosenberg [68] developed the theory of properly embedded minimal surfaces in $N \times \mathbb{R}$, where N is a closed orientable Riemannian surface but the existence of entire minimal surfaces in product spaces $M \times \mathbb{R}$ really draw attention after the papers by Collin and Rosenberg [27] and Gálvez and Rosenberg [38]. In [27] Collin and Rosenberg constructed a harmonic diffeomorphism from $\mathbb{C}$ onto $\mathbb{H}$ and hence disproved the conjecture of Schoen and Yau [79]. Gálvez and Rosenberg generalised this result to Hadamard surfaces whose curvature is bounded from above by a negative constant. A key tool in their constructions was to solve the Dirichlet problem on unbounded ideal polygons with alternating boundary values $\pm\infty$ on the sides of the ideal polygons.

Spruck [81] established a priori gradient estimates and existence results for graphs of constant positive mean curvature in product spaces $N \times \mathbb{R}$, where N is n-dimensional simply connected and complete Riemannian manifold. Many of these results apply also to the case of zero mean curvature and especially the gradient estimate has been used in later works considering the asymptotic Dirichlet problem.

Sa Earp and Toubiana [78] constructed minimal vertical graphs over unbounded domains in $\mathbb{H}^2 \times \mathbb{R}$ taking prescribed boundary data. Espírito-Santo and Ripoll [35] considered the existence of solutions to the exterior Dirichlet problem on simply connected manifolds with negative sectional curvature. Here the idea is to find minimal hypersurfaces on unbounded domains with compact boundary assuming zero boundary values.

Espírito-Santo, Fornari and Ripoll [34] proved the solvability of the asymptotic Dirichlet problem with negative constant upper bound for the sectional curvature and an assumption on the isometry group of the manifold.

Rosenberg, Schulze and Spruck [77] studied minimal hypersurfaces in $N\times\mathbb{R}_+$ with N complete Riemannian manifold having non-negative Ricci curvature and sectional curvatures bounded from below. They proved so-called half-space properties both for properly immersed minimal surfaces and for graphical minimal surfaces. In the latter, a key tool was a gradient estimate for solutions of the minimal graph equation.

Ripoll and Telichevesky [76] showed the existence of entire bounded non-constant solutions for slightly larger class of operators, including minimal graph operator, by studying the strict convexity (SC) condition of the manifold. Similar class of operators was studied also by Casteras, Holopainen and Ripoll [18] but instead of considering the SC condition, they solved the asymptotic Dirichlet problem by using similar barrier functions as in [55]. Both of these gave the existence of minimal graphic functions under the assumption (3.4) and the latter also included (3.3).

The method of Cheng adapted also to the case of minimal graphs and in [17] Casteras, Holopainen and Ripoll proved the following.

Theorem. *Let M be a Cartan-Hadamard manifold of dimension $n\ge 3$ and suppose that*

$$-\frac{\big(\log r(x)\big)^{2\tilde{\varepsilon}}}{r(x)^2}\le K_M(P_x)\le -\frac{1+\varepsilon}{r(x)^2\log r(x)} \tag{3.6}$$

holds for some constants $\varepsilon>\tilde{\varepsilon}>0$ and r large enough. Then the asymptotic Dirichlet problem is uniquely solvable.

Telichevesky [83] considered the Dirichlet problem on unbounded domains Ω proving the existence of solutions provided that $K_M\le -1$, the ordinary boundary of Ω is mean convex and that Ω satisfies the SC condition at infinity. The SC condition was studied by Casteras, Holopainen and Ripoll also in [16] and they proved that the manifold M satisfies the SC condition under very general curvature assumption. As special cases they obtain the bound (3.6) and

$$-ce^{(2-\varepsilon)r(x)}e^{e^{r(x)/e^3}}\le K_M\le -\phi e^{2r(x)}$$

for some constants $\phi>1/4$, $\varepsilon>0$ and $c>0$.

3.4. **Rotationally symmetric manifolds.** The situation on rotationally symmetric manifolds is slightly different from the general n-manifolds and hence we decided to treat them separately, although the problems on these manifolds has been studied at the same time as on the general manifolds. In [23] Choi gave also a definition of the asymptotic Dirichlet problem with respect to a pole on model manifolds and in the case of a Cartan-Hadamard model, it coincides with the previous definition.

As in the case of general manifolds, the study of the existence results begun with the harmonic functions. In 1977 Milnor [70] proved that a 2-dimensional rotationally symmetric surface M_f possess non-constant harmonic functions if and only if $\int_1^\infty 1/f(s)\,ds<\infty$. In terms of curvature bounds this gives the

existence when
$$K_{M_f} \leq -\frac{1+\varepsilon}{r^2 \log r}. \tag{3.7}$$
Choi [23] extended this result and proved that if (3.7) holds outside a compact set, then the asymptotic Dirichlet problem with respect to a pole is solvable for all $n \geq 2$.

March [66] studied the behaviour of the Brownian motion and used the invariant σ-field to characterise the existence of harmonic functions in terms of the curvature function, obtaining the following result.

Theorem. *Let M_f be a model manifold with negative radial curvature. Then there exist non-constant bounded harmonic functions if and only if*
$$\int_1^\infty \Big(f(s)^{n-3} \int_s^\infty f(t)^{1-n} dt \Big) ds < \infty. \tag{3.8}$$

In 2-dimensional case this corresponds to the curvature bound (3.7) and when $n \geq 3$, (3.8) is equivalent to
$$K_{M_f} \leq -\frac{1/2+\varepsilon}{r^2 \log r}.$$
Murata [71] gave an analytic proof that (3.8) is equivalent to either (i) M_f does not have strong Liouville property or (ii) the asymptotic Dirichlet problem is solvable. A simple analytic proof for the existence part of March's result can be found from Vähäkangas' licentiate thesis [85].

In 2012 Ripoll and Telichevesky [75] considered the asymptotic Dirichlet problem for the minimal graph equation. They proved the existence of entire non-constant bounded minimal graphic functions on 2-dimensional Hadamard surfaces assuming (3.8), i.e. the curvature upper bound (3.7). Idea in the proofs in [85] and [75] is to use (3.8) to construct barriers at infinity.

3.5. **Non-existence of solutions.** By the non-existence results in $\mathbb{R}^n$, it is already clear that the curvature upper bound must be strictly negative but the discussion about the rotationally symmetric case and the theorems replacing the sectional curvature lower bound with the pinching condition (3.2) raise a question about the necessity of the lower bound. However, when M is a general n-dimensional Cartan-Hadamard manifold it is not enough to assume only the curvature upper bound.

Concerning results in this direction, Ancona [5] proved in 1994 the following.

Theorem. *There exists a 3-dimensional Cartan-Hadamard manifold with $K_M \leq -1$ such that the asymptotic Dirichlet problem for the Laplacian is not solvable.*

His construction of such manifold was based on probabilistic methods. Namely, he proved the non-solvability of the asymptotic Dirichlet problem by showing that Brownian motion almost surely exits M at a single point on the asymptotic boundary.

Borbély [15] constructed similar manifold using analytic arguments and later Ulsamer [84] showed that Borbély's manifold can be constructed also with probabilistic methods, and generalised the Anconas result to higher dimensions. Arnaudon, Thalmaier and Ulsamer [8] continued the probabilistic study of these manifolds.

Holopainen [52] generalised Borbély's example to cover also the p-harmonic functions and then Holopainen and Ripoll [54] proved that the same example works also for the minimal graph equation. These results show that apart from the 2-dimensional or the rotationally symmetric setting, one really needs to have a control also on the lower bound.

It is also worth pointing out two closely related results by Greene and Wu [41] that partly answer the question about the optimal curvature upper bound. Firstly, in [41, Theorem 2 and Theorem 4] they showed that an n-dimensional, $n \neq 2$, Cartan-Hadamard manifold with asymptotically non-negative sectional curvature is isometric to $\mathbb{R}^n$. Secondly, in [41, Theorem 2] they showed that an odd dimensional Riemannian manifold with a pole $o \in M$ and everywhere non-positive or everywhere non-negative sectional curvature is isometric to $\mathbb{R}^n$ if $\liminf_{s\to\infty} s^2 k(s) = 0$, where $k(s) = \sup\{|K(P_x)| \colon x \in M,\, d(o,x) = s, P_x \in T_xM \text{ two-plane}\}$.

4. Pointwise pinching condition for the sectional curvatures

4.1. **Background.** To solve the asymptotic Dirichlet problem for the Laplacian, Anderson and Schoen [7] solve the problem

$$\begin{cases} \Delta u_R = -\Delta f & \text{in } B(o,R), \\ u_R = 0 & \text{on } \partial B(o,R) \end{cases} \tag{4.1}$$

in geodesic balls and then construct a barrier function to be able to extract a converging subsequence from $(u_R + f)$ when $R \to \infty$. This process relays highly on the curvature assumption $-b^2 \le K_M \le -a^2$.

Assuming only a pointwise pinching condition

$$|K_M(P_x)| \le C_K |K_M(P_x')| \tag{4.2}$$

and positivity of the first eigenvalue of the Laplacian, Cheng [19] was able to relax the curvature assumptions of Anderson and Schoen. To prove the claim, it is still necessary to extract the converging subsequence and show the correct boundary values at infinity but for this end Cheng's approach did not use barriers. His proof of convergence is based on an L^p-norm estimate, namely, he proves an upper bound for the L^p-norm of a solution in compact subsets in terms of the L^p-norm of $|\nabla f|$.

In order to show the correct boundary values of u at infinity, Cheng uses the assumption $|\nabla f| \in L^p$ and Moser iteration technique to prove that the supremum of $|u|^p$ on a ball $B(x,(1-\varepsilon)R)$, $\varepsilon \in (0,1)$, is bounded in terms of the integral of $|u|^p$ over $B(x,R)$. The last step is to show that the gradient of radially constant function is in L^p and this is the step requiring condition (4.2).

Vähäkangas [86] replaced the assumption on the eigenvalue by a curvature upper bound

$$K_M(P_x) \le -\frac{\phi(\phi-1)}{r(x)^2}, \quad \phi > 1,$$

and showed that the same result holds also for the p-Laplacian. The approach in his proof was essentially the same as Cheng's. In [87] Vähäkangas refined this argument with help of Young functions and was able to prove the solvability result for $\mathcal{A}$-harmonic functions under the curvature assumptions of [55].

These ideas involving Young functions was also used in [17] where Casteras, Holopainen and Ripoll solved the asymptotic Dirichlet problem for the minimal graph equation and the $\mathcal{A}$-harmonic equation under the assumption (3.6).

4.2. **Articles [A] and [B] revisited.** In [A] we generalise the result of Vähäkangas [86] and prove that under the same curvature assumptions the asymptotic Dirichlet problem is solvable also for the minimal graph equation. To be more precise, our main theorem is the following.

Theorem 4.3 ([A, Theorem 1.3])**.** *Let M be a Cartan-Hadamard manifold of dimension $n \geq 2$ and let $\phi > 1$. Assume that*

$$K(P) \leq -\frac{\phi(\phi-1)}{r(x)^2}, \tag{4.4}$$

where $K(P)$ is the sectional curvature of any two-dimensional subspace $P \subset T_xM$ containing the radial vector $\nabla r(x)$, with $x \in M \setminus B(o, R_0)$. Suppose also that there exists a constant $C_K < \infty$ such that

$$|K(P)| \leq C_K |K(P')|$$

whenever $x \in M \setminus B(o, R_0)$ and $P, P' \subset T_xM$ are two-dimensional subspaces containing the radial vector $\nabla r(x)$. Moreover, suppose that the dimension n and the constant ϕ satisfy the relation

$$n > \frac{4}{\phi} + 1. \tag{4.5}$$

Then the asymptotic Dirichlet problem for the minimal graph equation is uniquely solvable for any boundary data $f \in C(\partial_\infty M)$.

We notice that if we choose the constant ϕ in the curvature assumption to be bigger than 4, then our theorem holds in every dimension $n \geq 2$. Similarly, if we let the dimension n to be at least 5, we can take the constant ϕ to be as close to 1 as we wish.

In [B] we improve the results of Vähäkangas [86, 87] and Casteras, Holopainen and Ripoll [17] and show that in the case of $\mathcal{A}$-harmonic functions it is possible to solve the asymptotic Dirichlet problem assuming only the pinching condition (4.2) and a weaker curvature upper bound. A localised argument proving the $\mathcal{A}$-regularity of points $x_0 \in \partial_\infty M$ leads to the main theorem of [B].

Theorem 4.6 ([B, Theorem 1.3])**.** *Let M be a Cartan-Hadamard manifold of dimension $n \geq 2$. Assume that*

$$K(P) \leq -\frac{1+\varepsilon}{r(x)^2 \log r(x)}, \tag{4.7}$$

for some constant $\varepsilon > 0$, where $K(P)$ is the sectional curvature of any two-dimensional subspace $P \subset T_xM$ containing the radial vector $\nabla r(x)$, with $x \in M \setminus B(o, R_0)$. Suppose also that there exists a constant $C_K < \infty$ such that

$$|K(P)| \leq C_K |K(P')| \tag{4.8}$$

whenever $x \in M \setminus B(o, R_0)$ and $P, P' \subset T_xM$ are two-dimensional subspaces containing the radial vector $\nabla r(x)$. Then the asymptotic Dirichlet problem for the $\mathcal{A}$-harmonic equation is uniquely solvable for any boundary data $f \in C(\partial_\infty M)$ provided that $1 < p < n\alpha/\beta$.

In the case of usual Laplacian we have $\alpha = \beta = 1$ and $p = 2$. Hence we obtain the following special case.

Corollary 4.9 ([B, Corollary 1.6]). *Let M be a Cartan-Hadamard manifold of dimension $n \geq 3$ and assume that the assumptions* (4.7) *and* (4.8) *are satisfied. Then the asymptotic Dirichlet problem for the Laplace operator is uniquely solvable for any boundary data $f \in C(\partial_\infty M)$.*

It is also worth pointing out that in dimension $n = 2$ the condition (4.2) is trivially satisfied since at any point $x \in M$ there exists only one tangent plane P_x. Therefore it is enough to assume only a curvature upper bound and we obtain the following corollaries.

Corollary 4.10. *Let M be a 2-dimensional Cartan-Hadamard manifold and let $\phi > 4$. Assume that*

$$K(P) \leq -\frac{\phi(\phi-1)}{r(x)^2},$$

where $K(P)$ is the sectional curvature of a two-dimensional subspace $P \subset T_xM$ containing the radial vector $\nabla r(x)$, with $x \in M \setminus B(o, R_0)$. Then the asymptotic Dirichlet problem for the minimal graph equation is uniquely solvable for any boundary data $f \in C(\partial_\infty M)$.

Corollary 4.11. *Let M be a 2-dimensional Cartan-Hadamard manifold or n-dimensional rotationally symmetric Cartan-Hadamard manifold satisfying the curvature upper bound* (4.7)*. Then the asymptotic Dirichlet problem for the $\mathcal{A}$-harmonic equation is uniquely solvable for any boundary data $f \in C(\partial_\infty M)$ provided that $1 < p < n\alpha/\beta$.*

As it was pointed out in [17] (see also [D, Theorem 5.1]), the curvature upper bound and the range of p, $1 < p < n\alpha/\beta$, in Theorem 4.6 are in a sense optimal. Namely, if we assume that

$$K(P) \geq -\frac{1}{r(x)^2 \log r(x)}$$

and consider $\mathcal{A}$-harmonic operator of type $p \geq n$, it follows that M is p-parabolic, i.e. every bounded $\mathcal{A}$-harmonic function (of type p) is constant.

In Cheng's proof one of the key points was to show the L^p-bound for a solution u and in the proofs of Theorems 4.3 and 4.6 we need a similar estimate. However, instead of just considering the norm of u, we take an auxiliary smooth function $\varphi\colon [0,\infty) \to [0,\infty)$, related to Young functions, and show the bound for $\varphi(|u-\theta|/c)$. In [A] θ is a radial extension of the boundary data function and in [B] it is a certain continuous function that can also be thought as a boundary data. Once we have the integral estimate, it remains to show that we can bound the supremum of $\varphi(|u-\theta|)$ in $B(x, s/2)$ in terms of the integral of $\varphi(|u-\theta|)$ over $B(x, s)$. Together these estimates guarantee that $u(x) \to \theta(x_0)$ as $x \to x_0 \in \partial_\infty M$.

4.2.1. *Integral bounds for solutions.* Vähäkangas [87, Lemma 2.17] proved an integral estimate for $\mathcal{A}$-harmonic functions under the curvature assumption $K_M \leq -\phi(\phi-1)/r^2$. Clever idea in his proof was to use a Caccioppoli-type inequality, special type of Young functions F and G, and Young's inequality.

Taking certain smooth homeomorphism $H\colon [0,\infty)\to[0,\infty)$ he defined $G(t)=\int_0^t H(s)ds$ and $F(t)=\int_0^t H^{-1}(s)ds$. Then

$$\psi(t)=\int_0^t \frac{ds}{G^{-1}(s)}$$

and $\varphi=\psi^{-1}$ are homeomorphisms so that $G\circ\varphi'=\varphi$. For the functions F and G we have the Young's inequality

$$ab\le F(a)+G(b)$$

and the idea is to reduce the integrability of $\varphi(|u-\theta|)$ to the integrability of $F(|\nabla\theta|w)$ for some Lipschitz weight function w. In order to do this, a Caccioppoli-type inequality [87, Lemma 2.15]

$$\begin{aligned}\left(\int_U \eta^p\psi'(h)|\nabla u|^p\right)^{1/p} &\le \frac{\beta}{\alpha}\left(\int_U \eta^p\psi'(h)|\nabla\theta|^p\right)^{1/p}\\ &\quad+\frac{p\beta}{\alpha}\left(\int_U \frac{\psi^p}{(\psi')^{p-1}}(h)|\nabla\eta|^p\right)^{1/p}, \qquad (4.12)\end{aligned}$$

$h=|u-\theta|$, plays a central role.

Refining this idea Casteras, Holopainen and Ripoll proves the L^p-estimate for $\mathcal{A}$-harmonic functions under the curvature assumption (4.7).

Lemma 4.13. [16, Lemma 16] *Let M be a Cartan-Hadamard manifold satisfying (4.7). Suppose that $U\subset M$ is an open relatively compact set and that u is an $\mathcal{A}$-harmonic function in U with $u-\theta\in W_0^{1,p}(U)$, where $\mathcal{A}\in\mathcal{A}^p(M)$ with*

$$1<p<\frac{n\alpha}{\beta},$$

and $\theta\in W^{1,\infty}(M)$ is a continuous function with $||\theta||_\infty\le 1$. Then there exists a bounded C^1-function $\mathcal{C}\colon[0,\infty)\to[0,\infty)$ and a constant $c_0\ge 1$, that is independent of θ, U and u, such that

$$\begin{aligned}&\int_U \varphi\big(|u-\theta|/c_0\big)^p\big(\log(1+r)+\mathcal{C}(r)\big)\\ &\qquad\le c_0+c_0\int_U F\left(\frac{c_0|\nabla\theta| r\log(1+r)}{\log(1+r)+\mathcal{C}(r)}\right)\big(\log(1+r)+\mathcal{C}(r)\big).\end{aligned}$$

In [A] also the second derivative φ'' appears in the estimates and hence we need also another pair, F_1 and G_1, of Young functions so that $G_1\circ\varphi''\approx\varphi$. Then, with the Caccioppoli-type inequality [A, Lemma 3.1]

$$\begin{aligned}\int_U \eta^2\varphi'(|u-\theta|/\nu)\frac{|\nabla u|^2}{\sqrt{1+|\nabla u|^2}} &\le C_\varepsilon\int_U \eta^2\varphi'(|u-\theta|/\nu)|\nabla\theta|^2\\ &\quad+(4+\varepsilon)\nu^2\int_U \frac{\varphi^2}{\varphi'}(|u-\theta|/\nu)|\nabla\eta|^2, \qquad (4.14)\end{aligned}$$

we are able to obtain similar estimate if the gradient of θ is bounded in terms of the infimum $j(x)$ of the norms $|V(x)|$ of the Jacobi fields V along geodesic $\gamma^{o,x}$.

Lemma 4.15 ([A, Lemma 3.3]). *Let M be a Cartan-Hadamard manifold satisfying* (4.4) *and* (4.5). *Let $U = B(o,R)$, with $R > 0$ big enough, and suppose that $u \in C^2(U) \cap C(\bar{U})$ is the unique solution to the minimal graph equation in U, with $u|\partial U = \theta|\partial U$, where $\theta\colon M \to \mathbb{R}$ is a Lipschitz function, with $|\nabla\theta(x)| \le 1/j(x)$ almost everywhere. Then there exists a constant c independent of u such that*

$$\int_U \varphi(|u-\theta|/c) \le c + c\int_U F(r|\nabla\theta|) + c\int_U F_1(r^2|\nabla\theta|^2).$$

The integrability of functions F and F_1 in the previous lemmata follows from their construction and from the assumptions on the curvature and function θ.

4.2.2. *Pointwise estimates.* The last major step is to pass from the integral estimates to pointwise estimates. Together with the Caccioppoli-type inequality (4.12), the Sobolev inequality (see e.g. [47])

$$\left(\int_{B(x,r_S)} |\eta|^{n/(n-1)}\right)^{(n-1)/n} \le C_S \int_{B(x,r_S)} |\nabla\eta|, \tag{4.16}$$

$\eta \in C_0^\infty(B(x,r_S))$, and a Moser iteration procedure Vähäkangas obtains the supremum estimate [87, Lemma 2.20]

$$\operatorname*{ess\,sup}_{B(x,s/2)} \varphi\big(|u-\theta|\big)^{p(n-1)} \le c\int_{B(x,s)} \varphi\big(|u-\theta|\big)^p$$

for $\mathcal{A}$-harmonic functions $u \in W^{1,p}_{\text{loc}}(M)$, with $u - \theta \in W^{1,p}_0(\Omega)$, $\inf_M \theta \le u \le \sup_M \theta$, and $u = \theta$ a.e. in $M \setminus \Omega$.

In [A] we prove a similar estimate for the minimal graphic functions and, again, the Caccioppoli-type inequality (4.14), the Sobolev inequality (4.16) and a Moser iteration procedure are the main tools.

Lemma 4.17 ([A, Lemma 3.4]). *Let $\Omega = B(o,R)$ and suppose that $\theta\colon \Omega \to \mathbb{R}$ is a bounded Lipschitz function with $|\theta|, |\nabla\theta| \le C_1$. Let $u \in C^2(\Omega)$ be a solution of the minimal graph equation in Ω such that u has the boundary values θ and $\inf_\Omega \theta \le u \le \sup_\Omega \theta$. Fix $s \in (0, r_S)$, where r_S is the radius of the Sobolev inequality* (4.16)*, and suppose that $B = B(x,s) \subset \Omega$. Then there exists a positive constant $\nu_0 = \nu_0(\varphi, C_1)$ such that for all fixed $\nu \ge \nu_0$*

$$\sup_{B(x,s/2)} \varphi\big(|u-\theta|/\nu\big)^{n+1} \le c\int_B \varphi\big(|u-\theta|/\nu\big),$$

where c is a positive constant depending only on n, ν, s, C_S, C_1 and φ.

4.2.3. *Further questions.* It remains open whether the curvature upper bound (4.4) could be relaxed to

$$K(P_x) \le -\frac{1+\varepsilon}{r(x)^2 \log r(x)} \tag{4.18}$$

since the methods used in [A] or in [17] do not apply to this case. In [17] they have the upper bound (4.18) but they also assume a lower bound for the sectional curvatures, which enables to have an a priori gradient estimate. This is needed to obtain [17, Lemma 22].

Another question concerns the condition (4.5). It is a technical assumption coming from the Caccioppoli-type inequality (4.14) and there should not be a deeper reason requiring it.

5. f-minimal graphs

Let M be an n-dimensional Riemannian manifold with a Riemannian metric given by $ds^2 = \sigma_{ij} dx^i dx^j$ in local coordinates. Assume that $f\colon N \to \mathbb{R}$ is a smooth function, where $N = M \times \mathbb{R}$ is equipped with the product metric $ds^2 + dt^2$. Then f-minimal graphs are special type of surfaces with prescribed mean curvature, namely graphs of functions $u\colon \Omega \to \mathbb{R}$ that are solutions to the f-minimal graph equation

$$\begin{cases} \operatorname{div} \dfrac{\nabla u}{\sqrt{1+|\nabla u|^2}} = \langle \bar{\nabla} f, \nu \rangle & \text{in } \Omega; \\ u|\partial\Omega = \varphi, \end{cases} \tag{5.1}$$

where $\Omega \subset M$ is a bounded domain, $\bar{\nabla} f$ is the gradient of f with respect to the product Riemannian metric, and ν denotes the downward unit normal to the graph of u, i.e.

$$\nu = \frac{(\nabla u, -1)}{\sqrt{1+|\nabla u|^2}}. \tag{5.2}$$

More generally an f-minimal hypersurface Σ is an immersed hypersurface of a Riemannian manifold (N, g) whose mean curvature satisfies

$$H = \langle \bar{\nabla} f, \nu \rangle$$

at every point of Σ. To get some interpretation of f-minimal surfaces we mention the following examples:

(a) minimal hypersurfaces if f is identically constant,
(b) self-shrinkers in $\mathbb{R}^{n+1}$ if $f(x) = |x|^2/4$,
(c) minimal hypersurfaces of weighted manifolds $M_f = \big(M, g, e^{-f} d\,\mathrm{vol}_M\big)$, where (M, g) is a complete Riemannian manifold with the Riemannian volume element $d\,\mathrm{vol}_M$.

A reader interested in recent studies on self-shrinkers and f-minimal hypersurfaces should see [89], [25], [24], [20], [21], [22], [58], and references therein.

As a remark we point out that we cannot ask for the uniqueness of a solution of (5.1) if the function $f\colon M \times \mathbb{R} \to \mathbb{R}$ depends on the t-variable since comparison principles fail to hold, see [39, Theorem 10.1]. A simple counter example is obtained if one considers the function $f\colon \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}$, $f(x,t) = |(x,t)|^2/4$ and the open disk $B(0,2) \subset \mathbb{R}^2$. Namely, then both the upper and lower hemispheres and the disk $B(0,2)$ itself are f-minimal hypersurfaces with zero boundary values on the circle $\partial B(0,2)$.

5.1. **Background.**

5.1.1. *Barrier method.* A priori estimates and the barrier method goes back to the work of Bernstein [11, 12] and has been widely used to solve Dirichlet problems for different PDEs. A classical way to construct barriers on bounded domains is to use the distance function to the boundary and combine it with some auxiliary function h satisfying i.a. $h(0) = 0$. For a comprehensive description of the method one should see e.g. [39]. For more recent research, with similar choice of the function h as in [C], see [81] and [32].

To obtain a priori interior gradient estimates, at least for the mean curvature equation, is not as straightforward as the cases of height and boundary gradient estimates. In 1986 Korevaar [61] introduced two different approaches to obtain the estimate. His idea is to use a carefully chosen cutoff function η and then prove a priori bound for

$$\eta\big(x, u(x)\big)W(x),$$

$W = \sqrt{1 + |\nabla u|^2}$, in a ball $B(0,1) \subset \mathbb{R}^n$. The first approach is "Standard form calculation", suggested by L. Simon, that is based on direct computations at a maximum point of ηW. The second approach is to perturb the surface along its downward normal and then lift the perturbed surface in order to try to get a barrier.

More recently, and in the manifold setting, Korevaar's (also Korevaar-Simon) method has been used for example in [81], [77], [29].

5.1.2. *Barrier at infinity.* The approach of Anderson and Schoen [7] was based on the idea of extending a continuous boundary value function $\varphi\colon \partial_\infty M \to \mathbb{R}$ radially to the whole $\bar{M}$. Then after a suitable smoothening procedure they obtain sub- and superharmonic functions that can be used as barriers. Holopainen [50] used similar technique to prove the solvability of the asymptotic Dirichlet problem for p-Laplacian under the same curvature assumption

$$-b^2 \le K \le -a^2$$

for some constants $b \ge a > 0$.

Holopainen and Vähäkangas [55] generalised this approach to cover the more general curvature conditions (3.3) and (3.4) for the p-Laplacian. In order to allow the more general bounds, their smoothening procedure depends also on the curvature lower bound. This difference to the earlier proofs results to very technical and long computations. However, the barrier function that they obtained has appeared to be very flexible and suit also other PDEs, like the minimal graph equation which was considered by Casteras, Holopainen and Ripoll [18]. We will use their constructions also in [C].

5.2. **Article [C].** The article [C] is divided roughly into two parts: In the first part we study the existence of f-minimal graphs over bounded domains Ω with continuous boundary values on $\partial\Omega$ and in the second part we prove the existence of entire f-minimal graphs by solving the asymptotic Dirichlet problem. In the first part, under a technical assumption that $f \in C^2(\bar{\Omega} \times \mathbb{R})$ is of the form

$$f(x,t) = m(x) + r(t), \tag{5.3}$$

we obtain the following existence result.

Theorem 5.4 ([C, Theorem 1.2])**.** *Let $\Omega \subset M$ be a bounded domain with $C^{2,\alpha}$ boundary $\partial\Omega$. Suppose that $f \in C^2(\bar{\Omega} \times \mathbb{R})$ satisfies* (5.3)*, with*

$$F = \sup_{\bar{\Omega}\times\mathbb{R}} |\bar{\nabla} f| < \infty, \quad \operatorname{Ric}_\Omega \ge -\frac{F^2}{n-1}, \quad \textit{and} \quad H_{\partial\Omega} \ge F.$$

Then, for all $\varphi \in C(\partial\Omega)$, there exists a solution $u \in C^{2,\alpha}(\Omega) \cap C(\bar{\Omega})$ to the equation (5.1) *with boundary values φ.*

A standard way to obtain solutions for PDEs with $C^{2,\alpha}$ boundary values is to use the Leray-Schauder method [39, Theorem 13.8], that we have chosen, or the continuity method [39, Theorem 5.2, Theorem 17.8]. Both of these options reduces the question of the solvability of the Dirichlet problem to the existence of a priori height and gradient (both boundary and interior) estimates. Finally the reduction of the smoothness of boundary data is obtained via similar approximation as in [29]. This is possible since the local interior gradient estimate [C, Lemma 2.3] does not depend on the gradient of the boundary data.

In the second part of the article, applying this existence result above, we are able to generalise the result of Holopainen and Vähäkangas [55] and show that, under the same very general curvature assumptions, the asymptotic Dirichlet problem is solvable for the f-minimal graph equation.

Before stating the main results, we need to give some technical definitions and assumptions on the function f. We assume that there exists an auxiliary smooth function $a_0\colon [0,\infty) \to (0,\infty)$ such that

$$\int_1^\infty \left(\int_r^\infty \frac{ds}{f_a^{n-1}(s)} \right) a_0(r) f_a^{n-1}(r) dr < \infty,$$

for the discussion about the choice of a_0 see [C, Example 4.5] and [C, Example 4.6]. Then we define $g\colon [0,\infty) \to [0,\infty)$ by

$$g(r) = \frac{1}{f_a^{n-1}(r)} \int_0^r a_0(t) f_a^{n-1}(t) dt. \tag{5.5}$$

The function g was introduced in [67] where they studied elliptic and parabolic equations with asymptotic Dirichlet boundary conditions on Cartan-Hadamard manifolds. In addition to (5.3), we assume that the function $f \in C^2(\bar{\Omega} \times \mathbb{R})$ satisfies

$$\sup_{\partial B(o,r)\times\mathbb{R}} |\bar{\nabla} f| \le \min \left\{ \frac{a_0(r) + (n-1)\frac{f_a'(r)}{f_a(r)} g^3(r)}{\big(1+g^2(r)\big)^{3/2}}, (n-1)\frac{f_a'(r)}{f_a(r)} \right\}, \tag{5.6}$$

for every $r > 0$, and

$$\sup_{\partial B(o,r)\times\mathbb{R}} |\bar{\nabla} f| = o\left(\frac{f_a'(r)}{f_a(r)} r^{-\varepsilon-1} \right) \tag{5.7}$$

for some $\varepsilon > 0$ as $r \to \infty$. Then, as special cases of the main result [C, Theorem 1.3], we obtain the following corollaries.

Corollary 5.8 ([C, Corollary 1.34]). *Let M be a Cartan-Hadamard manifold of dimension $n \ge 2$. Suppose that there are constants $\phi > 1$, $\varepsilon > 0$, and $R_0 > 0$ such that*

$$-\rho(x)^{2(\phi-2)-\varepsilon} \le K(P_x) \le -\frac{\phi(\phi-1)}{\rho(x)^2}, \tag{5.9}$$

for all 2-dimensional subspaces $P_x \subset T_xM$ and for all $x \in M$, with $\rho(x) \ge R_0$. Assume, furthermore, that $f \in C^2(M \times \mathbb{R})$ satisfies (5.3), (5.6), *and* (5.7), *with $f_a(t) = t$ for small $t \ge 0$ and $f_a(t) = c_1 t^\phi + c_2 t^{1-\phi}$ for $t \ge R_0$. Then the* *asymptotic Dirichlet problem for equation* (5.1) *is solvable for any boundary data* $\varphi \in C(\partial_\infty M)$.

In another special case we assume that sectional curvatures are bounded from above by a negative constant $-k^2$ but allow the lower bound to decrease even exponentially.

Corollary 5.10 ([C, Corollary 1.5])**.** *Let M be a Cartan-Hadamard manifold of dimension $n \geq 2$. Assume that*

$$-\rho(x)^{-2-\varepsilon} e^{2k\rho(x)} \leq K(P_x) \leq -k^2 \tag{5.11}$$

for some constants $k > 0$ and $\varepsilon > 0$ and for all 2-dimensional subspaces $P_x \subset T_xM$, with $\rho(x) \geq R_0$. Assume, furthermore, that $f \in C^2(M \times \mathbb{R})$ satisfies (5.3), (5.6), *and* (5.7), *with $f_a(t) = t$ for small $t \geq 0$ and $f_a(t) = c_1 \sinh(kt) + c_2 \cosh(kt)$ for $t \geq R_0$. Then the asymptotic Dirichlet problem for the equation* (5.1) *is solvable for any boundary data* $\varphi \in C(\partial_\infty M)$.

The proof of the solvability of the asymptotic Dirichlet problem follows the usual path of solving the problem in a sequence of geodesic balls, hence obtaining a sequence of solutions. Then the last part is to show the existence of a limit that is a solution with correct boundary values on $\partial_\infty M$. In order to extract the converging subsequence, we have to prove a uniform height estimate [C, Lemma 4.4] for the sequence of solutions. The correct behaviour at infinity can be then proved with suitable barrier functions.

5.2.1. *A priori estimates.* The usual way to obtain a priori height and boundary gradient estimates for solutions u in bounded domains Ω is to construct upper and lower barriers using the distance function $d(\cdot) = \operatorname{dist}(\cdot, \partial\Omega)$ to the boundary. Then these barriers, together with the comparison principle, implies the desired estimates. This procedure requires two key assumptions: The (inward) mean curvature of the level sets of d is bounded from below by the prescribed mean curvature of the graph of u in some neighbourhood of $\partial\Omega$ and, of course, that the distance function is smooth enough.

The mean curvature assumption in the neighbourhood of $\partial\Omega$ can be replaced by an assumption on the boundary and by a lower bound for the Ricci curvature. Namely, denoting by $\Omega_0 \subset \Omega$ the open set of points that can be joined to $\partial\Omega$ by unique minimising geodesic, it follows that if

$$H_{\partial\Omega} \geq F \quad \text{and} \quad \operatorname{Ric}_\Omega \geq -F^2/(n-1)$$

then $H(x_0) \geq F$ for all $x_0 \in \Omega_0$. Here $H(x_0)$ denotes the mean curvature of the level set of d passing through x_0. This is done in [C, Lemma 3.1] (see also [81, Lemma 4.2] and [32, Lemma 5]) and the proof is based on a Riccati equation for the shape operator. The smoothness of the distance function in Ω_0 was proved in [65], to wit, in Ω_0 d has the same regularity as the boundary $\partial\Omega$.

In order to use the comparison principle we have to "freeze" the mean curvature term $\langle \bar\nabla f, \nu \rangle$ in (5.1). More precisely, if u is a solution of (5.1),

$$Q[u] = \frac{1}{W}\left(\sigma^{ij} - \frac{u^i u^j}{W^2}\right) u_{i;j} - \langle \bar\nabla f, \nu_u \rangle ,$$

we define an operator

$$\tilde{Q}[v] = \frac{1}{W}\left(\sigma^{ij} - \frac{v^i v^j}{W^2}\right) v_{i;j} - b,$$

where $W = \sqrt{1+|\nabla v|^2}$ and

$$b(x) = \left\langle \bar{\nabla} f\big((x,u(x)\big), \nu(x)\right\rangle.$$

The reason for this is that the operator Q need not satisfy the required assumptions of comparison principles, see e.g. [39, Theorem 10.1], whereas $\tilde{Q}$ does. The desired height estimate is finally obtained in [C, Lemma 2.1] and the boundary gradient estimate in [C, Lemma 2.2].

The interior gradient estimate is obtained in [C, Lemma 2.3] and the proof is based on the method due to Korevaar and Simon [61], see also [29] in the case of Killing graphs. The estimate is localised to balls $B(o,r) \subset \Omega$ and if the solution is $C^1(\bar{\Omega})$ we have also a global gradient estimate with upper bound depending also on the gradient on the boundary. Idea is to have an auxiliary smooth function η vanishing outside $B(o,r)$ and then consider a function

$$h = \eta W$$

with $W = \sqrt{1+|\nabla u|^2}$. It follows that the function h attains its maximum at some point $p \in B(o,r)$ and this permits to prove an upper bound for W, and hence also for $|\nabla u|$. It is in this part of the paper where we need the asumption (5.3), namely, for technical reasons we need to assume that all the "space derivatives"

$$f_i = \frac{\partial}{\partial x_i}, \quad i = 1, \ldots, \dim M$$

are independent of t, i.e. $f_{it} = f_{ti} = 0$.

5.2.2. *Entire f-minimal graphs.* First step of solving the asymptotic Dirichlet problem is to consider an exhaustion of M and obtain a sequence of solutions. A natural exhaustion is, of course, the sequence of geodesic balls $B(o,k)$, $k \in \mathbb{N}$, for which the boundary mean curvature assumption of Theorem 5.4 is satisfied. More precisely, we have

$$H(x) = \Delta r(x) \geq (n-1)\frac{f_a'\big(r(x)\big)}{f_a\big(r(x)\big)} \geq \sup_{\partial B(o,r(x))\times\mathbb{R}} |\bar{\nabla} f|,$$

where $H(x)$ denotes the inward mean curvature of the level set $\{y \in \bar{B}(o,R) \colon d(y) = d(x)\} = \partial B(o,r(x))$ and the last estimate follows from the assumption (5.6). This implies that we can even drop the assumption on the Ricci curvature. This step is done in [C, Lemma 4.7].

In order to obtain the uniform height estimate [C, Lemma 4.4] we use a function V,

$$\begin{aligned} V(x) = V\big(r(x)\big) &= \left(\int_{r(x)}^{\infty} \frac{ds}{f_a^{n-1}(s)}\right)\left(\int_0^{r(x)} a_0(t) f_a^{n-1}(t)dt\right) \\ &\quad - \int_0^{r(x)} \left(\int_t^{\infty} \frac{ds}{f_a^{n-1}(s)}\right) a_0(t) f_a^{n-1}(t)dt - H + ||\varphi||_{\infty}, \end{aligned}$$

$$H := \limsup_{r\to\infty} \Big\{ \int_r^\infty \frac{ds}{f_a^{n-1}(s)} \int_0^r a_0(t) f_a^{n-1}(t) dt - \int_0^r \int_t^\infty \frac{ds}{f_a^{n-1}(s)} a_0(t) f_a^{n-1}(t) dt \Big\} \le 0,$$

constructed in [67]. There it was used as a supersolution for an elliptic equation but it turns out that under the assumption (5.6) V works also as an upper barrier for the f-minimal equation. Then, replacing V by $-V$, we obtain a lower barrier and together these imply the desired height estimate. Even though (5.6) seems a very technical assumption, it is not more restrictive than (5.7), see [C, Example 4.5] and [C, Example 4.6].

Final crucial step is to prove the correct boundary values on $\partial_\infty M$ and this requires barriers at infinity. It turns out that the barrier function

$$\psi = A(R_3^\delta r^{-\delta} + h)$$

used by Holopainen and Vähäkangas [55] is very flexible and it suits also the case of f-minimal graphs, see [C, Lemma 4.3]. The assumption (5.7) for the asymptotic behaviour of the gradient $\bar\nabla f$ is required in this part of the article.

6. Optimality of the curvature bounds

6.1. **Background.** The background of the Korevaar-Simon method for obtaining interior gradient estimates was discussed in Section 5.1.1. However, we mention two articles that are closely related to our results. In [77] Rosenberg, Schulze and Spruck proved a gradient estimate for minimal graphic functions $M \times \mathbb{R} \to \mathbb{R}$ assuming non-negative Ricci curvature and negative constant lower bound for the sectional curvatures. This estimate was applied to prove a half-space property for non-negative solutions of the minimal graph equation.

Dajczer and Lira [30] extended this result for the Killing graphs in warped products $M \times_\varrho \mathbb{R}$ proving that, under certain assumptions on the manifold, any bounded entire Killing graph with constant mean curvature must be a slice. The key ingredient of their proof was a global gradient estimate that extended the result in [77].

Harnack's inequalities has been studied so widely that it is impossible to give a brief background about the developments in different settings so we just mention the works of Grigor'yan and Saloff-Coste [44], Holopainen [49], and Li and Tam [64] that are closely related to [D]. Concerning the background of the asymptotic Dirichlet problems on rotationally symmetric cases, see Section 3.4.

6.2. **Article [D].** The motivation for the study of the article [D] was to show that the curvature upper bound

$$K_M \le -\frac{C}{r^2 \log r} \tag{6.1}$$

really is the best that one can hope in order to show the existence of entire bounded non-constant solutions for the minimal graph equation. The article [D] consists of two parts, namely, the first part deals with non-existence type results and the latter with existence on rotationally symmetric manifolds. In order to prove these non-existence results we assume that the manifold has only one end and asymptotically non-negative sectional curvature, that is

Definition 6.2. Manifold M has asymptotically non-negative sectional curvature (ANSC) if there exists a continuous decreasing function $\lambda\colon [0,\infty) \to [0,\infty)$ such that

$$\int_0^\infty s\lambda(s)\,ds < \infty,$$

and that $K_M(P_x) \geq -\lambda(d(o,x))$ at any point $x \in M$.

The main theorem of the first part is the following.

Theorem 6.3 ([D, Theorem 1.1])**.** *Let M be a complete Riemannian manifold with asymptotically non-negative sectional curvature and only one end. If $u\colon M \to \mathbb{R}$ is a solution to the minimal graph equation that is bounded from below and has at most linear growth, then it must be a constant. In particular, if M is a Cartan-Hadamard manifold with asymptotically non-negative sectional curvature, the asymptotic Dirichlet problem is not solvable.*

It is worth pointing out that we do not assume, differing from previous results into this direction, the Ricci curvature to be non-negative; see e.g. [77], [33], [30], [31]. In terms of concrete curvature bounds, our theorem gives immediately the following corollary that answers the question about the optimality of (6.1).

Corollary 6.4 ([D, Corollary 1.2])**.** *Let M be a complete Riemannian manifold with only one end and assume that the sectional curvatures of M satisfy*

$$K(P_x) \geq -\frac{C}{r(x)^2\big(\log r(x)\big)^{1+\varepsilon}}$$

for sufficiently large $r(x)$ and for some $C > 0$ and $\varepsilon > 0$. Then any solution $u\colon M \to [a,\infty)$ with at most linear growth to the minimal graph equation must be constant.

The proof of Theorem 6.3 is based on an application of a gradient estimate Proposition 6.11 that enables us to prove a global Harnack's inequality for $u - \inf_M u$. By well-known methods, see [46, Theorem 6.6], the global Harnack's inequality can be iterated to yield Hölder continuity estimates and a Liouville (or Bernstein) type result when the solution has controlled growth. More precisely, we obtain the following corollary.

Corollary 6.5 ([D, Corollary 1.3])**.** *Let M be a complete Riemannian manifold with asymptotically non-negative sectional curvature and only one end. Then there exists a constant $\kappa \in (0,1]$, depending only on n and on the function λ in the (ANSC) condition such that every solution $u\colon M \to \mathbb{R}$ to the minimal graph equation with*

$$\lim_{d(x,o)\to\infty} \frac{|u(x)|}{d(x,o)^\kappa} = 0$$

must be constant.

Before turning to the latter part of [D], we point out that our results differ from the theorems of Greene and Wu [41] (besides the methods) mentioned in Section 3.5 since we do not assume the existence of a pole or the manifold to be simply connected, and the (ANSC) condition allows the sectional curvature to change a sign. Moreover, in Theorems 6.7 and 6.8 we will see that, in order to get the result [41, Theorem 2], it is necessary to assume $\liminf_{s\to\infty} s^2k(s) = 0$ for all of the sectional curvatures and not only for the radial ones (recall formula (2.1)).

The goal of the latter part of [D] is to prove the solvability of the asymptotic Dirichlet problem, and hence also the existence of entire bounded non-constant solutions, for the minimal graphic and p-harmonic equations assuming the optimal curvature upper bound (6.1). The main idea is to assume

$$\int_1^\infty \Big(f(s)^\beta \int_s^\infty f(t)^\alpha dt \Big) ds < \infty, \tag{6.6}$$

with an appropriate choice of α and β, and then use this condition to construct barriers at infinity. This results to very elementary proofs when compared to the proofs in the general case that was considered for example in [18], [17], [A] and [B].

Noticing that, on manifold M_f, the condition (6.6) implies the desired curvature upper bound, we obtain the following results.

Theorem 6.7 ([D, Corollary 4.2])**.** *Let M_f be a rotationally symmetric n-dimensional Cartan-Hadamard manifold whose radial sectional curvatures outside a compact set satisfy the upper bounds*

$$K(P_x) \le -\frac{1+\varepsilon}{r(x)^2 \log r(x)}, \quad \text{if } n = 2$$

and

$$K(P_x) \le -\frac{1/2+\varepsilon}{r(x)^2 \log r(x)}, \quad \text{if } n \ge 3.$$

Then the asymptotic Dirichlet problem for the minimal graph equation is solvable with any continuous boundary data on $\partial_\infty M_f$.

Theorem 6.8 ([D, Corollary 4.4])**.** *Let M_f be a rotationally symmetric n-dimensional Cartan-Hadamard manifold, $n \ge 3$, whose radial sectional curvatures satisfy the upper bound*

$$K(P_x) \le -\frac{1/2+\varepsilon}{r(x)^2 \log r(x)}. \tag{6.9}$$

Then the asymptotic Dirichlet problem for the p-Laplace equation, with $p \in (2, n)$, is solvable with any continuous boundary data on $\partial_\infty M_f$.

We point out that the case $p = 2$ in Theorem 6.8 reduces to the case of usual harmonic functions and was covered by March [66].

Finally, in the last section of [D], we show that in Theorem 6.8 the assumption $p < n$ on the range of p is also optimal. Note also that (ANSC) implies global Harnack's inequality for $\mathcal{A}$-harmonic functions ([49, Examples 3.1]).

Theorem 6.10 ([D, Theorem 5.1])**.** *Let $\alpha > 0$ be a constant and assume that M is a complete n-dimensional Riemannian manifold whose radial sectional curvatures satisfy*

$$K_M(P_x) \ge -\frac{\alpha}{r(x)^2 \log r(x)}$$

for every x outside some compact set and every 2-dimensional subspace $P_x \subset T_xM$ containing $\nabla r(x)$. Then M is p-parabolic

(a) *if* $p = n$ *and* $0 < \alpha \leq 1$*; or*
(b) $p > n$ *and* $\alpha > 0$.

6.2.1. *Gradient estimate for minimal graphic functions.* It is well-known that the (ANSC) assumption implies a volume doubling condition and a Poincaré inequality (see [D] for short discussion) and these can be used to prove a local Harnack's inequality for uniformly elliptic operators. Then the assumption that M has only one end yields a global Harnack's inequality, see e.g. [1], [60] and [49, Examples 3.1]. Therefore the question reduces to interpreting the minimal graph operator as a uniformly elliptic operator

$$\frac{1}{A(x)} \operatorname{div} \big(A(x)\nabla u\big),$$

where

$$A(x) = \frac{1}{\sqrt{1+|\nabla u|^2}}.$$

Note that if $|\nabla u|$ is uniformly bounded, then there exists a constant c such that $c \leq \sqrt{A} \leq 1$. This uniform gradient bound can be obtained from the following proposition, whose proof is based on the method due to Korevaar.

Proposition 6.11 ([D, Proposition 3.1])**.** *Assume that the sectional curvature of M has a lower bound $K(P_x) \geq -K_0^2$ for all $x \in B(p,R)$ for some constant $K_0 = K_0(p,R) \geq 0$. Let u be a positive solution to the minimal graph equation in $B(p,R) \subset M$. Then*

$$\begin{aligned} |\nabla u(p)| \leq & \left(\frac{2}{\sqrt{3}} + \frac{32u(p)}{R}\right) \qquad (6.12)\\ & \cdot \left(\exp\left[64u(p)^2\left(\frac{2\psi(R)}{R^2} + \sqrt{\frac{4\psi(R)^2}{R^4} + \frac{(n-1)K_0^2}{64u(p)^2}}\right)\right] + 1\right), \end{aligned}$$

where $\psi(R) = (n-1)K_0 R \coth(K_0 R) + 1$ if $K_0 > 0$ and $\psi(R) = n$ if $K_0 = 0$.

In order to allow at most linear growth for u in [D, Corollary 3.2], we apply Proposition 6.11 to points $p \in M \setminus B(o,R_0)$, for some $R_0 > 0$ large, and use the fact that (ANSC) implies

$$K(P_x) \geq -\frac{c}{d(x,o)^2}$$

for all $x \in M \setminus B(o,R_0/2)$.

6.2.2. *Optimal curvature upper bound on the rotationally symmetric case.* In order to obtain barriers from (6.6) we first define a function

$$\eta(r) = k\int_r^\infty f(t)^\alpha \int_1^t f(s)^\beta \, ds dt, \qquad (6.13)$$

$k > 0$, and then consider the function $\eta + B$, where $B\colon M \setminus \{o\}$,

$$B\big(\exp(r\vartheta)\big) = B(r,\vartheta) = b(\vartheta), \quad \vartheta \in \mathbb{S}^{n-1} \subset T_oM,$$

is a radial extension of the boundary data function $b\colon \mathbb{S}^{n-1} \to \mathbb{R}$. In the case of the minimal graph equation we choose $\alpha = -n+1$ and $\beta = n-3$, as in [66], and for the p-Laplacian we choose $\alpha = -(n-1)/(p-1)$ and $\beta = (n-2p+1)/(p-1)$. Note that in both cases $\alpha + \beta = -2$ and hence they correspond to the same curvature bound. Then a straightforward computation shows that $\eta + B$ is a supersolution in $M \setminus B(o, R_0)$ for $R_0 > 0$ large enough and we can define global super- and subsolutions that work as barriers.

6.2.3. *p-parabolicity when $p \geq n$.* To be more precise we recall that

Definition 6.14. Riemannian manifold N is p-parabolic, $1 < p < \infty$, if

$$\operatorname{cap}_p(K, N) = 0$$

for every compact set $K \subset N$. Here the p-capacity of the pair (K, N) is

$$\operatorname{cap}_p(K, N) = \inf_{\substack{u \in C_0^\infty(N) \\ u|K \geq 1}} \int_N |\nabla u|^p.$$

In order to prove Theorem 6.10, and to show that the curvature bound (6.9) in Theorem 6.8 is optimal and the upper bound $p < n$ necessary, we apply Bishop-Gromov volume comparison together with (6.9). The proof is a direct application of the following condition that, for $p = 2$, was proved by Varopoulos [88] and Grigor'yan [42, 43]. Keselman and Zorich [90] proved the case $p = n$ and their proof applies also to other values of p, see also [49], [28], [51].

Proposition 6.15. *A complete Riemannian manifold M is p-parabolic if*

$$\int^\infty \left(\frac{t}{V(t)} \right)^{1/(p-1)} dt = \infty. \tag{6.16}$$

We point out that converse of this proposition is not always true, namely, there exists a manifold such that the integral of (6.16) is finite but M is p-parabolic, see [88].

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