HELSINGIN YLIOPISTO — HELSINGFORS UNIVERSITET — UNIVERSITY OF HELSINKI

Tiedekunta/Osasto — Fakultet/Sektion — Faculty	Laitos — Institution — Department
Matemaattis-luonnontieteellinen	Matematiikan ja tilastotieteen laitos

Tekijä — Författare — Author

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Työn nimi — Arbetets titel — Title

Probabilistic Liouville field theory on the two-dimensional sphere

Oppiaine — Läroämne — Subject

Matematiikka

Työn laji — Arbetets art — Level	Aika — Datum — Month and year	Sivumäärä — Sidoantal — Number of pages
Pro gradu -tutkielma	Toukokuu 2017	105 s.

Tiivistelmä — Referat — Abstract

In this thesis we construct the probabilistic Liouville field theory on the two-dimensional sphere. We prove some of the symmetry properties of the theory and define the correlation functions of the vertex operators. Finally, we define the Liouville quantum gravity measure. The thesis also contains a discussion on how the theory is related to quantum field theory and scaling limits of random planar maps.

Essential building block of the theory is the Gaussian free field, which can be thought of as a random Gaussian field with the covariance operator given by the inverse of the Laplacian. Another important aspect of the Liouville field theory is the exponential of the Gaussian free field. Defining this requires some work, since the Gaussian free field will turn out to be a random generalized function, and the exponential of such an object is not defined in general. We will define the exponential by using the theory of Gaussian multiplicative chaos.

The thesis contains a self-contained exposition of the definitions and basic properties of the Gaussian free field and its exponential. Some basic background in analysis, probability and geometry is assumed.

Avainsanat — Nyckelord — Keywords

Liouville field theory, Liouville quantum gravity, Gaussian free field, Gaussian multiplicative chaos

Säilytyspaikka — Förvaringsställe — Where deposited

Kumpulan tiedekirjasto

Muita tietoja — Övriga uppgifter — Additional information

University of Helsinki

Department of Mathematics and Statistics

Master's thesis

Probabilistic Liouville field theory on the two-dimensional sphere

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22nd May 2017

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Chapter 1

Introduction

In this thesis we construct the probabilistic Liouville field theory on the two-dimensional sphere. It is a statistical field theory defined by the Liouville action and the theory is related to scaling limits of random surfaces and it can also be used for constructing the Liouville quantum field theory. We begin by introducing a simpler statistical field theory, the Gaussian free field. Then we define the probabilistic Liouville field by modifying the Gaussian free field in a certain way, resulting in a non-Gaussian field. To define this modification we need the notion of an exponential of the Gaussian free field, which we define by using techniques from the theory of Gaussian multiplicative chaos. We also prove some symmetry properties of the Liouville field theory and define the Liouville quantum gravity measure.

1.1 Field theory

In this section we introduce some basic concepts from field theory to provide context for the subject of this thesis. Field theory is a mathematical framework for building physical theories describing systems with an infinite amount of degrees of freedom. We work on a heuristic level and the precise mathematical treatment will begin later.

1.1.1 Classical field theory

A classical field theory is a deterministic model of a system that has an infinite amount of degrees of freedom. One expresses the state of the system in terms of a field which we assume to be a real valued function $X : D \to \mathbb{R}$, where D is a suitable domain depending on the system, often just a subset of $\mathbb{R}^d$. When we want to differentiate between the time coordinate and the space coordinates we write $D = D_T \times D_S$ where $D_T \subset \mathbb{R}$ is some interval and $D_S \subset \mathbb{R}^{d-1}$. In the Lagrangian formalism we define a classical field theory by specifying a real valued action functional S on a suitable space of fields. Then we postulate that the behaviour of the physical system is described by the field X for which $S(X)$ is a stationary value of the action functional (for example a local minimum or a local maximum), assuming that S and the space it acts on is defined in such a way that there is only one stationary value. By applying variational methods on S one can derive a partial differential equation called the Euler–Lagrange equation corresponding to S, which is solved by the stationary X.

Often the action functional takes the form

$$S(X) = \int_{D_T \times D_S} \left(\frac{1}{2} \dot{X}(t,s)^2 - \frac{1}{2} |\nabla X(t,x)|^2 - \frac{1}{2} m^2 X(t,x) - V(X(t,x)) \right) dx\, dt\,, \quad (1.1.1)$$

where $\dot{X}$ denotes the time derivative, ∇X the gradient with respect to x and $m \geqslant 0$ is a constant called mass. The term $\dot{X}^2$ corresponds to kinetic energy density of the field, the term $|\nabla X|^2$ to gradient energy density and the term $m^2 X^2$ to energy density caused by mass m of the field. The function $V(X(t,x))$ describes interactions of the field. The case $V \equiv 0$ is called the Klein–Gordon action

$$S_{KG}(X) = \frac{1}{2} \int_{D_T \times D_S} \left(\dot{X}(t,x)^2 - |\nabla X(t,x)|^2 - m^2 X(t,s)^2 \right) dx\, dt\,. \quad (1.1.2)$$

The Euler–Lagrange equation of this action is the Klein–Gordon equation

$$(\partial_t^2 - \Delta + m^2) X(t,x) = 0\,,$$

where Δ is the Laplacian. Note that the $V \equiv 0$ case corresponds to a linear Euler–Lagrange equation, and actually this equation describes a "free field". In the case of a non-trivial V we get a non-linear equation which is characteristic of interacting fields.

Another example is the classical Liouville field theory. This example requires some knowledge of differential geometry. We fix a two-dimensional manifold M and we consider fields $X : M \to \mathbb{R}$ and metrics g (Riemannian or Lorentzian) on the surface M. The Liouville action on a space of fields and metrics is defined by

$$S_L(X,g) = \frac{1}{4\pi} \int_M \left(|\nabla_g X(x)|^2 + Q R_g(x) X(x) + 4\pi\mu e^{\gamma X(x)} \right) \sqrt{\det g(x)} dx\,. \quad (1.1.3)$$

Here ∇_g is the gradient, R_g is the scalar curvature and γ, μ and Q are constants. The name Liouville field comes from the fact that the Euler–Lagrange equation of this action is

$$\Delta_g X(t,x) = \frac{Q}{2} R_g(x) + 2\pi\gamma\mu e^{\gamma X(t,x)}\,,$$

where Δ_g is either the d'Alembert operator in the Lorentzian case or the Laplace–Beltrami operator in the Riemannian case. This is known as the Liouville equation and if g is Riemannian it can be used for finding metrics with constant negative curvature on M, which is related to uniformization of Riemann surfaces.

1.1.2 Statistical field theory

In statistical field theory we also consider systems with an infinite amount of degrees of freedom. However, we do not consider the time-evolution of the system as in classical field theory. Rather, we look for a natural probability distribution on the space of configurations of the field.

As in classical field theory, we define a specific statistical field theory by specifying an action functional[1] on the space of fields. Usually the action takes the form

$$S(X) = \int_D \left(\frac{1}{2} |\nabla X(x)|^2 + \frac{1}{2} m^2 X(x)^2 + V(X(x)) \right) dx \,, \tag{1.1.4}$$

which closely resembles (1.1.1). The constant m is again called mass and the function V the potential. In classical field theory this action with $V \equiv 0$ yields the Euler–Lagrange equation

$$(-\Delta + m^2) X(x) = 0 \,,$$

which is the Klein–Gordon equation without the time derivative, also called the Helmholtz equation. Using the action we define a probability distribution $\mathbb{P}_S$ on the space of fields by setting (heuristically)

$$\mathbb{P}_S(DX) := \frac{1}{Z} e^{-S(X)} \mathcal{D}X \,, \tag{1.1.5}$$

where Z is a normalization constant and $\mathcal{D}X$ denotes a formal uniform measure on the space of fields. The negative exponential ensures that the distribution of X is highly concentrated around the minimal value of S. Note that the distribution (1.1.5) is of the form of a canonical ensemble[2].

The distribution (and corresponding random fields) corresponding to the $V \equiv 0$ case is usually called massive free field (if $m > 0$) or massless free field (if $m = 0$). The massless case is also called the Gaussian free field. Free refers to the absence of the potential term V and Gaussian to the fact that the probability distribution $\mathbb{P}_S$ is Gaussian. To see this, assume that X vanishes on the boundary ∂D (other boundary conditions suffice too), then by Green's formula the distribution takes the form

$$\frac{1}{Z} e^{-\frac{1}{2} \int_D (|\nabla X(x)|^2 + m^2 X(x)^2) \, dx} \mathcal{D}X = \frac{1}{Z} e^{\frac{1}{2} \int_D X(x)(\Delta - m^2) X(x) \, dx} \mathcal{D}X \,.$$

This is a Gaussian probability distribution[3] with covariance operator $(-\Delta + m^2)^{-1}$. In Chapter 3 we define the Gaussian free field rigorously.

[1] In statistical field theory the action functional actually tells the energy of a field configuration, which is not the case in classical field theory, but we still use the term action to highlight the analogy between these two types of functionals.

[2] In equilibrium statistical mechanics of finite systems, the canonical ensemble arises as an entropy-maximizing probability distribution (when the particle number and temperature is fixed). It is stationary in the sense that if we apply the time-evolution given by the Hamilton's equations on the distribution, the distribution remains the same.

[3] Recall that in the finite-dimensional case centered Gaussian distributions take the form

$$\frac{1}{\sqrt{(2\pi)^d \det C}} \exp\left(-\frac{x \cdot C^{-1} x}{2} \right) dx \,,$$

where C is the covariance matrix.

From the mathematical point of view it is customary to write the distribution for a general action (1.1.4) as

$$\mathbb{P}_S(DX) = \frac{1}{Z} e^{-\int_D V(X(x))\,dx} \mu_m(DX)\,,$$

where μ_m is the distribution of the $V \equiv 0$ action with mass m. Now if we have managed to define μ_m rigorously, then one could troy to define $\mathbb{P}_S$ to be the distribution absolutely continuous with respect to μ_m with Radon–Nikodym derivative $e^{-\int_D V(X(x))\,dx}$. As simple as it sounds, this task can be quite hard, as we will see in the case of the Liouville field. The difficulty arises from the fact that the measures μ_m do not exist in any function space, but rather in a space of generalized functions (also called distributions, not to be confused with probability distributions). Then the expression $V(X)$ does not necessarily make sense a priori, and one needs to apply a renormalization procedure to make sense of the potential.

The Liouville action (1.1.3) in the case of a Riemannian metric g defines a statistical field theory on a Riemannian manifold (M, g) which we call the probabilistic Liouville field theory. The probability distribution is heuristically given by

$$\frac{1}{Z} e^{-S_L(X)}\,\mathcal{D}X = \frac{1}{Z} e^{-\frac{1}{4\pi}\int_M (QR_g(x)X(x) + 4\pi e^{\gamma X(x)})\sqrt{\det g(x)}\,dx} \nu_{\mathrm{GFF}}(DX)\,,$$

where $\nu_{\mathrm{GFF}} := \mu_0$ is the Gaussian free field on M. In Chapter 4 we define the object $e^{\gamma X}$, where X is distributed according to ν_{GFF}, as a random measure and in Chapter 5 we give a rigorous definition of the probabilistic Liouville field theory in the case $M = \mathbb{S}^2$. The Liouville field exhibits conformal invariance and thus it gives an example of a conformal field theory. We will prove a global conformal invariance result in Chapter 5. Local conformal invariance is discussed in [29].

1.1.3 Quantum field theory

One of the main motivations for the study of probabilistic Liouville theory comes from quantum field theory. One way to construct a quantum field theory is to take a classical field theory and its Euler–Lagrange equation and then apply a process called quantization to interpret the classical solutions of the Euler–Lagrange equation as operator valued fields satisfying certain algebraic properties. In the physics literature this is called canonical quantization. We will not give details of this process. Another way to construct quantum field theories is to begin with a statistical field theory and then identify the correlation functions of the statistical field theory with correlation functions of a certain quantum field theory. This is called the path integral formulation[4]. We give a short description of this process. For a thorough discussion see for example [19].

Given a statistical field theory, that is, a probability measure μ on some space of maps $X : D \to \mathbb{R}$, we define the correlation (or Schwinger) functions of the theory by

$$S_n(\xi_1, \ldots, \xi_n) := \mathbb{E}\left[X(\xi_1) \ldots X(\xi_n)\right] = \int X(\xi_1) \ldots X(\xi_n)\, \mu(DX)$$

[4]The actual path integration is done in the statistical field theory and then we analytically continue, or "Wick rotate", the results to the quantum theory.

where $\xi_i \in D$. If we define X as a random generalized function, then

$$\int S_n(\xi_1, \dots, \xi_n) \prod_{i=1}^{n} f(\xi_i)\, d\xi_i := \mathbb{E}\left[X(f_1) \dots X(f_n)\right] .$$

where f_i are some test functions on which X acts.

Analogously in quantum field theory one defines the correlation (or Wightman) functions W_n by

$$W_n(x_1, \dots, x_n) := \langle \Omega, Y(x_1) \dots Y(x_n)\Omega \rangle ,$$

if the quantum field Y is defined pointwise and by

$$\int W_n(\xi_1, \dots, \xi_n) \prod_{i=1}^{n} f(\xi_i)\, d\xi_i := \langle \Omega, Y(f_1) \dots Y(f_n)\Omega \rangle ,$$

if Y is not defined pointwise. Here $\langle \cdot, \cdot \rangle$ denotes an inner product in the Hilbert space in which the operator valued field Y operates and Ω is an element of this Hilbert space called the vacuum state. Also, $\xi_i = (t_i, x_i) \in \mathbb{R} \times \mathbb{R}^{d-1}$ are points in space-time.

Now the connection between the statistical and quantum theories is that we can analytically continue the Wightman functions to get the correspondence

$$W_n((-it_1, x_1), \dots, (-it_n, x_n)) = S_n((t_1, x_1), \dots, (t_n, x_n)) \tag{1.1.6}$$

where on the left-hand side we have the Wightman functions of some quantum field Y and on the right-hand side the Schwinger functions of some random field X. On the other hand, if the Schwinger functions satisfy a condition called the Reflection Positivity, then one can reconstruct the Wightman functions, Hilbert space and the quantum field of the quantum field theory that produce these Schwinger functions as in the above equation, see [19]. Hence by passing to imaginary time in the quantum field theory we jump into the realm of statistical field theory.

This is why the study of the corresponding statistical field theory is sometimes called Euclidean quantum field theory, at least when the real interest is in the quantum side. In the title of this thesis we use the term probabilistic (Liouville) field theory to highlight the fact that while we construct the theory using probabilistic methods, that is, we define the theory as a statistical field theory, we also want to keep in mind the relation to the quantum field theory. Thus it would feel wrong to only use the term "statistical field theory" or "quantum field theory".

A simple example of a quantum field is acquired by applying the quantization process on the Klein–Gordon equation $(\partial_t^2 - \Delta + m^2)Y(t, x) = 0$. The result is a massive free quantum field. The Schwinger functions given by the correspondence (1.1.6) are the correlation functions of the massive free fields described in section 1.1.2.

Using the random Liouville field one can construct a two-dimensional quantum field theory called the Liouville quantum field theory (LQFT). This is one of the main motivations for the subject of this thesis. For an outline of this construction see [28]. The LQFT was first introduced by Polyakov in 1981 in the context of string theory and two-dimensional quantum gravity [40]. Polyakov established a link between noncritical bosonic string theory and a 2-dimensional quantum gravity model called Liouville Quantum Gravity. The LQFT is a major building block of Liouville Quantum Gravity, see [20] for a mathematical discussion. Major work on the relation to random surfaces was done by Knizhnik, Polyakov and Zamolodchikov in 1988 [26]. The mathematically rigorous treatment of the theory in the probabilistic setting has begun during the last few years [7, 8, 9, 20, 21, 28, 29]. A physics review is given in [39].

1.2 Random surfaces

Next we glimpse into the theory of random surfaces which provides another source of motivation for the study of probabilistic Liouville theory. The reason for this is that the random Liouville field can be used to describe scaling limits of certain models of random discrete surfaces. The connection comes via the object $e^{\gamma X}$ where now X is the random Liouville field. We will give the precise definition of this exponential in Chapter 5. More on random surfaces with the topology of the sphere can be found in [17, 7, 28, 42]. Different topologies are discussed in [9, 20, 21]

1.2.1 Scaling limit of uniformly random planar maps

The Donsker theorem states that the scaling limit of suitably rescaled simple random walks in $\mathbb{R}^d$ is the Brownian motion. Heuristically this can be interpreted in the following geometric way. We look at random discrete one-dimensional manifolds (the simple random walk, extended to continuous time by linear interpolation), and the scaling limit of these random manifolds satisfies the conditions of Brownian motion (when we identify the graph of a realization of Brownian motion with a one-dimensional manifold). Formally one could describe the probability distribution that the Brownian motion on $[0,1]$ induces on the space of continuous maps $X : [0,1] \to \mathbb{R}$ as

$$\exp\left(-\frac{1}{2}\int_0^1 |X'(t)|^2 \, dt\right) \mathcal{D}X \,,$$

but given the fact that a realization of a Brownian motion is almost surely nowhere differentiable, it is not immediately clear how to interpret this measure. However, recalling the discussion in the previous section, we see that this distribution has the form of a one-dimensional Gaussian free field. In Chapter 3 we demonstrate that in the one-dimensional case our definition of the Gaussian free field will satisfy the conditions of Brownian motion.

The natural question now is that does there exist something analogous to the Donsker theorem in two-dimensions. As in the one-dimensional case, we ought to look at certain random discrete two-dimensional manifolds and try to pass to the scaling limit. We fix our manifolds to have the topology of the sphere. Then the discrete two-dimensional manifolds are the planar maps and the scaling limit is the Brownian map.

A *planar map* is a finite and connected graph with an embedding into the two-dimensional sphere. A *p-angulation* is a planar map consisting of p-gons. All our planar maps will be p-angulations of the sphere and in the following discussion planar map and p-angulation will be synonymous. Two planar maps are isomorphic if there exists an orientation preserving homeomorphism from the sphere to itself such that it induces a graph isomorphism. Let M_n^p be the (finite) set of all p-angulations of the sphere, up to isomorphism, with n faces. Let m_n be a uniformly distributed random element of M_n^p. One could then think of m_n as a metric space equipped with (a rescaled version of) the graph distance. Then we have a random compact metric space (m_n, d_{m_n}). There exists a metric, the so-called Gromov–Hausdorff metric, in the space of all compact metric spaces modulo isometries. Then one can show that for $p \in \{3\} \cup \{2\mathbb{N}_0 + 4\}$ the sequence of random metric spaces (m_n, d_{m_n}) converges weakly in distribution to a random metric space (m_∞, d), the *Brownian map*, see [30, 35]. The Brownian map is almost surely homeomorphic to the sphere $\mathbb{S}^2$ and has fractal dimension 4. It can be thought of as a uniformly distributed random metric space that has the topology of the sphere.

Another approach for taking the scaling limit is to think of planar maps as Riemann surfaces. Thus we take our planar map and extend it to a simplex. Then we define a conformal structure on this simplex in the following manner. On each face we define a chart such that the face maps to a unit area polygon. These charts can be extended to the edges by the Schwarz reflection principle. Defining charts on vertices is a bit trickier and we will not address this problem here (see [17]). Altogether the process yields a finite atlas (indexed by the edges and vertices) on the simplex so that it becomes a complex manifold of dimension one (real dimension two), homeomorphic to the sphere. Then we get an embedding into the sphere by the Riemann uniformization theorem[5]. Let n be the number of faces of our planar map. Next we define a measure on the planar map in such a way that the total mass is one and each polygon has area $1/n$. Now we can put the push-forward measure μ_n on the sphere[6]. The Radon–Nikodym derivative of this measure with respect to the Lebesgue measure gets rougher as we increase n.

What happens to the measures $(\mu_n)_{n\in\mathbb{N}}$ in the scaling limit is still an open question. However, Duplantier and Sheffield have identified a candidate for the scaling limit, which is the exponential of (a multiple of) the Gaussian free field X. Thus the limiting measure is heuristically of the form

$$\mu = e^{\gamma X(x)}\, dx$$

and in this particular case $\gamma = \sqrt{8/3}$. These measures are called the *critical Liouville Quantum Gravity measures*, see [14, 17]. The case $\gamma = \sqrt{8/3}$ is called pure quantum gravity because the only thing we are dealing with is the geometry of the manifold and there are no "matter fields" coupled to the system.

[5]This embedding is unique up to Möbius transformations, and thus to obtain a unique embedding we could mark a face on the planar map and three points in $\mathbb{S}^2$, and then use the embedding that maps the vertices of the marked face to the three marked points in $\mathbb{S}^2$.

[6]We will briefly discuss the significance of these measures in section 1.2.4.

Now we have two different approaches for taking the scaling limit of random planar maps. One yields the Brownian map and the other a sphere with the measure

$$e^{\sqrt{8/3}X(x)}\,dx\,.$$

In a recent series of papers Miller and Sheffield showed that these two limiting objects encode the same information, see [36, 37, 38].

1.2.2 Random planar maps coupled with Ising model

Next we discuss a discrete model where there is a "matter field" coupled to the system. This model leads to the measure $e^{\gamma X(x)}\,dx$ corresponding to a different value of γ. The material in this section is based on [17].

In statistical physics, the Ising model is a lattice model of ferromagnetic matter. We fix a lattice $\Lambda \subset \mathbb{Z}^2$, put a particle on every point on the lattice and then define the Ising Hamiltonian (without an external magnetic field) $H_{\Lambda,\beta} : \{-1,1\}^\Lambda \to \mathbb{R}$ to be

$$H_{\Lambda,\beta}(\sigma) = -\beta \sum_{x\sim y} \sigma(x)\sigma(y)\,.$$

The sum is over nearest neighbours. Physically speaking the Hamiltonian gives the interaction energy of a spin configuration σ. The parameter $\beta > 0$ is the inverse of the temperature. The probabilities of the spin configurations σ are then assumed to be distributed according to the *canonical ensemble*. This means that

$$\mathbb{P}_{\Lambda,\beta}(\sigma) = \frac{1}{Z_{\Lambda,\beta}} e^{-H_{\Lambda,\beta}(\sigma)}\,,$$

where the normalizing constant $Z_{\Lambda,\beta}$ is called the *partition function* and is given by

$$Z_{\Lambda,\beta} = \sum_{\sigma\in\{-1,1\}^\Lambda} e^{-H_{\Lambda,\beta}(\sigma)}\,.$$

We could also define the Ising model on a random lattice and couple the randomness of the lattice to the randomness of the spin configuration. By this we mean that the matter itself should affect the geometry of the lattice. Thus we do not just pick a random lattice and study the Ising model on it, but rather we do a more intricate construction. We put a spin configuration on the faces of a p-angulation $m_n \in M_n^p$. Then the probability of the pair (m_n, σ) is defined to be

$$\mathbb{P}_{n,\beta}(m_n,\sigma) = \frac{1}{\widetilde{Z}_{n,\beta}} \exp\left(-H_{m_n,\beta}(\sigma)\right)\,,$$

$$\widetilde{Z}_{n,\beta} = \sum_{\substack{m_n\in M_n^p \\ \sigma\in\{-1,1\}^{m_n}}} \exp\left(-H_{m_n,\beta}(\sigma)\right)\,.$$

Now the marginal distribution of m_n defined by this probability distribution satisfies

$$\mathbb{P}_{n,\beta}(m_n) = \sum_{\sigma \in \{-1,1\}^{m_n}} \mathbb{P}_{n,\beta}(m_n, \sigma) = \frac{Z_{m_n,\beta}}{\widetilde{Z}_{n,\beta}}\,.$$

Hence we see that probability for getting a certain planar map m_n is weighted by the Ising partition function $Z_{m_n,\beta}$, which means that m_n is no longer uniformly distributed. Thus the "matter field" and geometry are coupled.

Now we could embed this random model of planar maps into the sphere and consider the corresponding measure, just like in the uniformly random case. The scaling limit of these measures for the critical temperature $\beta = \beta_c$ [7] is conjectured to be closely related to the measure $e^{\gamma X(x)}\,dx$ with $\gamma = \sqrt{3}$, where again X is the GFF. See Section 6 in [47] for the precise conjecture.

1.2.3 Random surface models for general γ and μ

Now we investigate random surface models parametrized by two numbers γ and μ. For additional details and references see [7, 28].

The two previous examples correspond to the cases $(\gamma,\mu) = (\sqrt{8/3}, 0)$ and $(\gamma,\mu) = (\sqrt{3}, 0)$. Let M_n^3 be the set of all triangulations (p-angulations with $p = 3$) of $\mathbb{S}^2$ with n faces. We put a probability measure $\mathbb{P}_{\mu_0,\gamma}$ on

$$M^3 := \bigcup_{n=3}^{\infty} M_n^3$$

by defining for $T \in M^3$

$$\mathbb{P}_{\mu_0,\gamma}(T) = \frac{1}{Z_{\mu_0,\gamma}} e^{-\mu_0 |T|} Z_\gamma(T)\,,$$

where $|T|$ denotes the number of faces in T, $Z_\gamma(T)$ is a partition function of a critical lattice model on T and $Z_{\mu_0,\gamma}$ is a normalizing constant defined by

$$Z_{\mu_0,\gamma} = \sum_{n=3}^{\infty} \sum_{T \in M_n^3} e^{-\mu_0 |T|} Z_\gamma(T)\,.$$

We have such a model for each $\gamma \in [\sqrt{2}, 2]$; for example the case $\gamma = \sqrt{8/3}$ corresponds to "pure gravity", that is, $Z_{\sqrt{8/3}}(T) \equiv 1$ and the case $\gamma = \sqrt{3}$ corresponds to the Ising model. Thus we have (using notation from the previous section)

$$\begin{aligned}
\mathbb{P}_{\mu_0,\sqrt{3}}(T) &= \frac{1}{Z_{\mu_0,\sqrt{3}}} e^{-\mu_0 |T|} \mathbb{P}_{|T|,\beta_c}(T) \\
&= \frac{1}{Z_{\mu_0,\sqrt{3}}} e^{-\mu_0 |T|} \sum_{\sigma \in \{-1,1\}^T} e^{-H_{T,\beta_c}(\sigma)}\,,
\end{aligned}$$

[7] For example, if $p = 4$, then $\beta_c = \log 2$.

or, if we want to at the probabilities of the pairs (T,σ), we set

$$\mathbb{P}_{\mu_0,\sqrt{3}}(T,\sigma) = \frac{1}{Z_{\mu_0,\sqrt{3}}} e^{-\mu_0|T|} e^{-H_{T,\beta_c}(\sigma)} .$$

This is the *grand canonical ensemble* of the Ising model, defined on a random triangulation at the critical temperature.

Physicists have conjectured that

$$Z_n := \sum_{T\in M_n^3} Z_\gamma(T) = n^{1-\frac{4}{\gamma^2}} e^{\bar{\mu}n}(1+o(1))$$

for large n, where $\bar{\mu} > 0$, see [2]. This implies that

$$\begin{aligned} Z_{\mu_0,\gamma} &= \sum_{n=3}^{\infty} \sum_{T\in M_n^3} e^{-\mu_0|T|} Z_\gamma(T) \\ &= \sum_{n=3}^{\infty} e^{n(\bar{\mu}-\mu_0)} n^{1-\frac{4}{\gamma^2}} (1+o(1)) , \end{aligned}$$

which converges for $\mu_0 > \bar{\mu}$ and thus $\mathbb{P}_{\mu_0,\gamma}$ is well-defined as a probability measure for $\mu_0 > \bar{\mu}$. The behaviour of the limit $\lim_{\mu_0 \searrow \bar{\mu}} Z_{\mu_0,\gamma}$ is governed by the term n^{1-4/γ^2} and we see that for $1-4/\gamma^2 \geqslant -1$, which corresponds to $\gamma \geqslant \sqrt{2}$, the limit is infinite. This means that in the $\gamma \geqslant \sqrt{2}$ case the measure samples large triangulations as $\mu_0 \to \bar{\mu}$.

Next we embed the model into the sphere. Let M_T be the Riemann surface corresponding to the triangulation T (as discussed before). Now if we mark three faces on a triangulation T, then there exists a unique conformal map $\psi : M_T \to \mathbb{S}^2$ such that the middle point of the marked faces map to some chosen points $z_1, z_2, z_3 \in \mathbb{S}^2$. Let λ_T be the Lebesgue measure in local coordinates of the faces on M_T. The push-forward measure $\nu_T := (\psi_T)_* \lambda_T$ defines a measure on $\mathbb{S}^2$ and when we pick T according to $\mathbb{P}_{\mu_0,\gamma}$, we obtain a random measure on $\mathbb{S}^2$ which we will denote by $\nu_{\mu_0,\gamma}$.

The natural question now is that what are the scaling limits of these measures. Recall that as μ_0 gets closer to $\bar{\mu}$, the typical size of T grows (assuming $\gamma \geqslant \sqrt{2}$). We define

$$\rho_{\mu,\gamma}^{(\varepsilon)} := \varepsilon \nu_{\bar{\mu}+\varepsilon\mu,\gamma}$$

and investigate the limit of this renormalized measure as $\varepsilon \to 0$. Let us first look at the area of the whole space $\rho_{\mu,\gamma}^{(\varepsilon)}(\mathbb{S}^2)$. For a given triangulation we have $\varepsilon\nu_T(\mathbb{S}^2) = \varepsilon|T|$, so it follows that

$$\begin{aligned} \mathbb{E}[F(\rho_{\mu,\gamma}^{(\varepsilon)}(\mathbb{S}^2))] &= \sum_{n=3}^{\infty} \sum_{T\in M_n^3} F(\varepsilon\nu_T(\mathbb{S}^2)) \mathbb{P}_{\mu_0,\gamma}(T) \\ &= \frac{1}{Z_{\bar{\mu}+\varepsilon\mu,\gamma}} \sum_{n=3}^{\infty} F(\varepsilon n) e^{-\varepsilon\mu n} n^{2-\frac{4}{\gamma^2}} \frac{1}{n} \\ &\xrightarrow{\varepsilon\to 0} \int_0^\infty F(x) e^{-\mu x} x^{2-\frac{4}{\gamma^2}}\, dx \Big/ \int_0^\infty e^{-\mu x} x^{2-\frac{4}{\gamma^2}}\, dx . \end{aligned}$$

Thus the total mass has distribution of a gamma variable with parameters $2 - 4/\gamma^2$ and μ. In Chapter 5 we will construct a family of continuum measures on $\mathbb{S}^2$ parametrized by μ and γ which have the same distribution for the total mass. These measures are constructed by exponentiating the random Liouville field, and the measures are also called the *non-critical Liouville Quantum Gravity measures*. They are conjectured to describe the scaling limits of $\rho^{(\varepsilon)}_{\mu,\gamma}$ [7].

1.2.4 Uniformization and isothermal coordinates

Next we comment briefly on why in the embedded random surface models we study certain measures instead of the manifold itself. An important tool in the study of Riemann surfaces is the Riemann uniformization theorem. It states that every two-dimensional sufficiently regular simply connected manifold M equipped with a Riemannian metric can be conformally mapped to either $\mathbb{D}$, $\mathbb{S}^2$ or $\mathbb{C}$. Another way to put this is that we can parametrize the manifold by points $z = x + iy$ in $\mathbb{D}, \mathbb{S}^2$ or $\mathbb{C}$ in such a way that the metric of the manifold takes the form

$$g(z) = e^{\lambda(z)}(dx \otimes dx + dy \otimes dy)\,.$$

These coordinates are called the *isothermal coordinates* for M. In isothermal coordinates the area $\mathcal{A}(E)$ of a subset $E \subset D$ is given by

$$\mathcal{A}(E) = \int_E e^{\lambda(z)}\,dz\,,$$

where dz is the Lebesgue measure in D. The scalar curvature of M is then given by

$$R(z) = -e^{-\lambda(z)}\Delta\lambda(z)\,.$$

Then

$$\int_E R(z)e^{\lambda(z)}\,dz = -\int_E \Delta\lambda(z)\,dz\,.$$

In other words, the function $-\Delta\lambda(z)$ gives the density of the scalar curvature. Thus the manifold M is flat if and only if λ is harmonic.

The important point here is that the function λ characterizes the manifold M. Hence if we want to study random surfaces, we could try to study the random function $\lambda : D \to \mathbb{R}$. In our context the natural choice for the random function (actually, a generalized function) λ will be a function that is a "Gaussian perturbation" of a harmonic function. In Chapter 3 we will show that the Gaussian free field X has this property in some sense. Then certain regularized versions X_ε, to be discussed later, of the Gaussian free field yield regularized versions M_ε of the manifold M. Thus a manifold with a metric of the form

$$e^{\gamma X(z)}\,(dx \otimes dx + dy \otimes dy)$$

can be thought of as a limit (in some sense) of the regularized manifolds M_ε. So when we embed our random surface models into the sphere, this limit is analogous to taking the scaling limit of random planar maps with respect to the Gromov–Hausdorff metric, which yields the Brownian map. Now notice that areas of sets in this manifold will formally be given by

$$\mathcal{A}(E) = \int_E e^{\gamma X(z)}\, dz\,,$$

which tells us that the measure $e^{\gamma X(z)}\, dz$ is the natural way to compute areas on M. A problem arises from the fact that even if we manage to describe the measure $e^{\gamma X(z)}dz$ on the limit manifold M, it is unclear how the measure relates to the notion of distance on this highly irregular manifold. Recall that on a smooth manifold, a two-point distance function, a measure and a Riemannian metric can all be used to characterize the same structure. For some additional details on the previous discussion see [14].

Chapter 2

Preliminaries

In this chapter we list some preliminary results that we will later use. Some proofs are given and for the rest we point out a reference. We assume that the reader is familiar with basic probability theory and analysis.

2.1 Notation

If not otherwise stated, all our random variables will be defined on a common probability space $(\Omega, \mathcal{F}, \mathbb{P})$ where Ω is the underlying set, $\mathcal{F}$ a sigma-algebra and $\mathbb{P} : \mathcal{F} \to \mathbb{R}_{\geqslant 0}$ a probability measure. If X and Y are random variables, then $X \overset{d}{=} Y$ denotes equality in distribution. Conditional expectation is denoted by $\mathbb{E}(X|\mathcal{G})$ where X is a random variable and $\mathcal{G}$ is a sub sigma-algebra of $\mathcal{F}$. Sometimes we may define another probability measure on the same measurable space $(\Omega, \mathcal{F})$ and in these situations we write $\mathbb{E}_{\mathbb{P}}$ for the expectation with respect to the probability measure $\mathbb{P}$. The Borel sets in a domain $D \subset \mathbb{R}^d$ are denoted by $\mathcal{B}(D)$. We say that random variables X and Y are iid if they are independent and they have identical distributions. The notation $\mathcal{N}(\mu, V)$ denotes the Gaussian distribution with mean μ and variance V.

An inner product in a Hilbert space H is denoted by $\langle x, y\rangle_H$. The notation $\langle x, y\rangle$ denotes the dual bracket. Inner product of finite-dimensional vectors is also denoted by $x \cdot y$.

Let $D \subset \mathbb{R}^d$ be an open subset. We define the following spaces of real-valued functions on D.

- $L^1_{\text{loc}}(D) =$ the space of locally integrable functions on D.
- $C(D) =$ the space of continuous functions on D.
- $C^n(D) =$ the space of n-times continuously differentiable functions on D.
- $C^\infty(D) = \cap_{n \geqslant 1} C^n(D)$.

  If E is some space of functions from D to $\mathbb{R}$, then we define

- $E_0 = \{f \in E \mid f|_{\partial D} = 0\}$.

- $E_b = \{f \in E \mid f \text{ is bounded }\}$.
- $E_c = \{f \in E \mid f \text{ has compact support }\}$.

Integration by parts yields the Green's formula (assuming g is sufficiently differentiable)

$$\int_D \nabla f(x) \cdot \nabla g(x)\, dx = -\int_D f(x)\Delta g(x)\, dx + \int_{\partial D} f(x)\nabla g(x) \cdot \hat{n}(x)\, dS(x)$$

where Δ denotes the Laplacian and in the boundary integral $\hat{n}(x)$ denotes the unit normal vector of ∂D at x. The boundary term vanishes whenever f or g vanishes on ∂D and indeed we will often apply the formula in the reduced form

$$\int_D \nabla f(x) \cdot \nabla g(x)\, dx = -\int_D f(x)\Delta g(x)\, dx\,, \quad (f \in C^1(D), g \in C_0^2(D))\,.$$

This formula also holds for weakly differentiable functions, which we will discuss in section 2.3.

2.2 Functional analysis

In this section X will denote a Banach space.

Definition 2.2.1. *The dual space of X, denoted by $X^\star$, is the set of all bounded linear functionals on X. We use the dual bracket notation*

$$\langle \Lambda, x\rangle := \Lambda(x) \quad (\Lambda \in X^\star, x \in X)\,.$$

Proposition 2.2.2. *Assume that $\Lambda \in X^\star$. The expression*

$$\|\Lambda\|_{X^\star} = \sup_{\|x\|=1} |\langle \Lambda, x\rangle|$$

defines a norm on $X^\star$ called the operator norm. Equipped with this norm, $X^\star$ is a Banach space.

Proof. See Theorem 4.3 in [44]. □

Definition 2.2.3. *The topology generated by the collection of maps $(f_x : X^\star \to \mathbb{R})_{x\in X}$ defined by*

$$f_x(\Lambda) = \langle \Lambda, x\rangle\,,$$

is called the weak-$\star$ topology on $X^\star$.

Proposition 2.2.4. *A sequence $(\Lambda_n)_{n\in\mathbb{N}}$ of elements of $X^\star$ converge to $\Lambda \in X^\star$ with respect to the weak-$\star$ topology if and only if*

$$\lim_{n\to\infty} \langle \Lambda_n, x\rangle = \langle \Lambda, x\rangle$$

for all $x \in X$. In other words, the weak-$\star$ topology is the topology of pointwise convergence.

Proof. This follows from observing that the weak-$\star$ topology of $X^\star$ agrees with the topology it inherits from $\mathbb{R}^X$ where $\mathbb{R}^X$ is endowed with the product topology. Indeed, the product topology is generated by the projections $\mathrm{pr}_x : \mathbb{R}^X \to \mathbb{R}$, $\mathrm{pr}_x(f) = f(x)$, and then we can define the corresponding relative topology on $X^\star$ by the restrictions $\mathrm{pr}_x|_{X^\star}$. Now the claim follows from observing that the family of functions $(\mathrm{pr}_x|_{X^\star})_{x\in X}$ is the same as the family that generates the weak-$\star$ topology on $X^\star$. □

Theorem 2.2.5. (Banach–Alaoglu theorem) *The closed unit ball of $X^\star$ is weak-$\star$ compact.*

Proof. Theorem 3.15 in [44]. □

Definition 2.2.6. *A Radon measure μ on $D \subset \mathbb{R}^d$ is a Borel measure satisfying the following conditions:*

1. *Inner regularity: For any Borel set $B \subset D$ we have*

$$\mu(B) = \sup\{\mu(K) : K \subset B,\ K \text{ compact }\}.$$

2. *Locally finite: Every point $x \in D$ has a neighbourhood $U \subset D$ such that $\mu(U) < \infty$.*

The space of continuous compactly supported functions on $D \subset \mathbb{R}^d$, denoted by $C_c(D)$, becomes a Banach space when we equip it with the sup-norm.

Theorem 2.2.7. (Riesz representation theorem) *Let $\Lambda \in (C_c(D))^\star$ be such that $\langle \Lambda, f\rangle \geqslant 0$ whenever $f \in C_c(D)$ and $f \geqslant 0$. Then there exists a unique Radon measure on D such that*

$$\langle \Lambda, f\rangle = \int_D f(x)\,\mu(dx)\,, \quad (f \in C_c(D))\,.$$

Proof. Theorem 2.14 in [45]. □

Remark 2.2.8. *While most of the spaces that we will consider will be Banach spaces, the space $C_c^\infty(D)$ does not have a natural Banach space structure. This is not a real problem since $C_c^\infty(D)$ admits a topology which makes it a complete locally convex topological vector space, which means that we can define $(C_c^\infty(D))^\star$ and it is "nice enough". For details see Chapter 6 of [44].*

2.3 Sobolev spaces

Let $D \subset \mathbb{R}^d$ be an open set. We define $H^n(D) \subset L^2(D)$ to be the subset of n-times weakly differentiable functions on D with the weak derivatives belonging to $L^2(D)$. Equipped with the Sobolev inner product

$$\langle f, g\rangle_{H^n} = \sum_{|\alpha|\leqslant n} \int_D \partial^\alpha f(x) \partial^\alpha g(x)\,dx\,,$$

$H^n(D)$ becomes a Hilbert space. By $H_0^n(D)$ we denote the completion of $C_c^\infty(D)$ with respect to the norm of $H^n(D)$. If D is bounded and $f \in H_0^1(D)$, we have the *Poincaré inequality*

$$\|f\|_{L^2}^2 \leqslant C \int_D |\nabla f(x)|^2 \, dx$$

where C depends only on the domain D (Corollary 9.19 in [5]). This implies that the Dirichlet inner product

$$\langle f, g \rangle_{H_0^1} := \int_D \nabla f(x) \cdot \nabla g(x) \, dx \,, \quad (f, g \in H_0^1(D)) \,,$$

induces the same topology on $H_0^1(D)$ as the inner product of $H^1(D)$. From now on we will endow $H_0^1(D)$ with the Dirichlet inner product. The following theorem is fundamental.

Theorem 2.3.1. (Eigenfunctions and eigenvalues of the Dirichlet Laplacian) *Assume that $D \subset \mathbb{R}^d$ is open, bounded and has C^1 boundary. Then the following claims hold.*

1. *There exists a sequence of functions $(e_n)_{n\in\mathbb{N}} \subset H_0^1(D) \cap C^\infty(\overline{D})$ and a sequence $(\lambda_n)_{n\in\mathbb{N}}$ of positive real numbers such that*

$$-\Delta e_n(x) = \lambda_n e_n(x) \,, \quad n \in \mathbb{N} \,,$$

and the sequence $(e_n)_{n\in\mathbb{N}}$ forms an orthonormal basis of $H_0^1(D)$.

2. *(Weyl's law) When $d = 2$, the asymptotic behaviour of the eigenvalues defined above is given by*

$$\lim_{n\to\infty} \frac{\lambda_n}{n} = \frac{4\pi}{|D|} \,,$$

where $|D|$ denotes the Lebesgue measure of D.

Proof. For the first statement, see Theorem 1 in section 6.5 of [16]. For the second statement, see Theorem 20 in Section VI.4 in [6]. □

Definition 2.3.2. *The eigenbasis $(e_n)_{n\in\mathbb{N}}$ given by the previous proposition will from now on be called the Laplace-eigenbasis of $H_0^1(D)$.*

The Sobolev spaces $H^s(D)$ can also be defined for an arbitrary real number $s \in \mathbb{R}$. In the case $D = \mathbb{R}^d$ and $s \geqslant 0$ we can set

$$\|f\|_{H^s} := \left(\int_{\mathbb{R}^d} |\widehat{f}(x)|^2 (1 + |x|^2)^s \, dx \right)^{1/2}$$

where $\widehat{f}$ denotes the Fourier transform of a L^2-function f. For negative s the definition remains the same when we assume that f is a tempered distribution. These norms are equivalent to the regular $H^s(\mathbb{R}^d)$ norms when s is a positive integer, and $H^s(\mathbb{R}^d)$ is a Hilbert space for each $s \in \mathbb{R}$.

For proper subdomains we can use Theorem 2.3.1. The only negative Sobolev space we will need is $H^{-1}(D)$ and it can be defined in the following way. Let $(e_n)_{n\in\mathbb{N}}$ be the Laplace-eigenbasis of $H_0^1(D)$ with eigenvalues $(\lambda_n)_{n\in\mathbb{N}}$ given by Theorem 2.3.1. Suppose that $(a_n)_{n\in\mathbb{N}}$ is a sequence of real numbers such that

$$\sum_{n=1}^{\infty} \frac{a_n^2}{\lambda_n} < \infty \,.$$

Then the series

$$\rho := \sum_{n=1}^{\infty} a_n \sqrt{\lambda_n} e_n \tag{2.3.1}$$

converges in $(C_c^\infty(D))^\star$ (with respect to the weak-$\star$ topology). We define $H^{-1}(D)$ to be the space of all such ρ.

The family of Sobolev spaces has the duality property that $(H_0^s(D))^\star$ is isomorphic to $H^{-s}(D)$[1]. We choose to identify $H^{-s}(D)$ with $H_0^s(D)$ and thus define the following notation:

Definition 2.3.3. *We denote $H_0^1(D)^\star$ by $H^{-1}(D)$.*

Since $H_0^1(D)$ is a Hilbert space, the Riesz–Fréchet representation theorems yields us a canonical isomorphism between $H_0^1(D)$ and $H^{-1}(D)$. The canonical isomorphism is the distributional Laplacian (which we will define soon). Even though this isomorphism exists, we decide to think of $H_0^1(D)$ and $H^{-1}(D)$ as different spaces. Thus our philosophy will be ($\simeq$ denotes identification)

$$H_0^1(D) \subset L^2(D) \simeq L^2(D)^\star \subset H_0^1(D)^\star \simeq H^{-1}(D) \,,$$

so that both the inclusions $H_0^1(D) \subset L^2(D)$ and $L^2(D) \subset H^{-1}(D)$[2] are continuous, injective and dense. This is a general fact for Hilbert spaces $H_1 \subset H_2$ when there is a continuous and injective map $i : H_1 \to H_2$ with dense range.

---

[1]Often the negative Sobolev spaces are defined to be the duals of the positive Sobolev spaces. Also, in the literature there are many different definitions of the negative Sobolev spaces (which makes it hard to give a proper reference for our claim). We stress that the claim is "clear" in the sense that, for example, using the Laplace-eigenbasis we can write an element of $H_0^1(D)$ as

$$\sum_{n=1}^{\infty} b_n e_n \,,$$

where $\sum_n b_n^2 < \infty$. Then look at L^2 inner products of this function and (2.3.1). For a bit different approach see [1].

[2]To be precise, by the inclusion $L^2(D) \subset H^{-1}(D)$ we mean that we take $f \in L^2(D)$, map it to the corresponding element in $L^2(D)^\star$ by defining $\langle f, g \rangle := \langle f, g \rangle_{L^2}$ and then restrict this functional to $H_0^1(D)$.

Proposition 2.3.4. *Let H_1 and H_2 be Hilbert spaces and assume that there exists a continuous linear injection $\iota : H_1 \to H_2$ with dense range. Denote by $I : H_2 \to H_2^\star$ the isomorphism given by the Riesz–Fréchet representation theorem. Then the map $\iota^\star : H_2^\star \to H_1^\star$ defined by*

$$\langle \iota^\star(\rho), f\rangle = \langle \rho, \iota(f)\rangle$$

is a continuous injection with dense range.

Proof. If $\langle \iota^\star(\rho), f\rangle = \langle \rho, \iota(f)\rangle = 0$ for all $f \in H_1$, then $\langle I^{-1}(\rho), \iota(f)\rangle_{H_2} = 0$ for all $f \in H_1$. Thus $I^{-1}(\rho) \in (\iota(H_1))^\perp$, which implies that $I^{-1}(\rho) = 0$ since $\iota(H_1)$ is dense. Thus $\iota^\star$ is an injection. Continuity follows from the computation

$$\begin{aligned}
\|\iota^\star(\rho)\|_{H_1^\star} &= \sup_{\|f\|_{H_1}=1} |\langle \iota^\star(\rho), f\rangle| = \sup_{\|f\|_{H_1}=1} |\langle \rho, \iota(f)\rangle| \\
&\leqslant \sup_{\|f\|_{H_1}=1} \|\rho\|_{H_2^\star}\|\iota(f)\|_{H_2} \leqslant \sup_{\|f\|_{H_1}=1} \|\rho\|_{H_2^\star} C\|f\|_{H_1} \\
&= C\|\rho\|_{H_2^\star} .
\end{aligned}$$

If $\mathscr{R}(\iota^\star)$ denotes the range of $\iota^\star$, then by Theorem 4.12. in [44] it holds that $\mathscr{R}(\iota^\star)^\perp = \mathscr{N}(\iota)$, where $\mathscr{N}(\iota)$ is the kernel of ι. Since $\mathscr{N}(\iota) = \{0\}$, it follows that the range of $\iota^\star$ is dense. □

Proposition 2.3.5. (The distributional Laplacian) *We define an operator $L : C_c^\infty(D) \to H^{-1}(D)$ by*

$$\langle Lf, g\rangle := -\int_D g(x)\Delta f(x)\, dx\,, \quad (g \in H_0^1(D))\,.$$

This is bounded when we equip $C_c^\infty(D)$ with the Dirichlet norm and $H^{-1}(D)$ with the operator norm. Thus there exists a unique bounded operator $\widetilde{L} : H_0^1(D) \to H^{-1}(D)$, called the distributional Laplacian, that agrees with L on $C_c^\infty(D)$. Explicitly the extension is given by

$$\langle \widetilde{L}f, g\rangle = \int_D \nabla f(x)\cdot\nabla g(x)\, dx = \langle f, g\rangle_{H_0^1}\,, \quad (f, g \in H_0^1(D))\,.$$

Proof. By definitions and Green's formula

$$\begin{aligned}
\|Lf\|_{H^{-1}} &= \sup_{\|g\|_{H_0^1}=1} |\langle Lf, g\rangle| = \sup_{\|g\|_{H_0^1}=1} \left|\int_D (\nabla g)(x)\cdot(\nabla f)(x)\, dx\right| \\
&\leqslant \sup_{\|g\|_{H_0^1}=1} \|g\|_{H_0^1}\|f\|_{H_0^1} = \|f\|_{H_0^1} .
\end{aligned}$$

Thus L is $H^{-1}(D)$-valued and bounded. The existence of a unique bounded extension to $H_0^1(D)$ then follows from the facts that $C_c^\infty(D)$ is dense in $H_0^1(D)$ with respect to the Dirichlet norm and that $H^{-1}(D)$ is a Banach space when equipped with the operator norm. The explicit formula for $\widetilde{L}$ follows from the Green's formula. □

Remark 2.3.6. *From now on we will write $-\Delta f$ instead of $\widetilde{L}f$ even when the Laplacian of $f \in H_0^1(D)$ does not exist in the classical sense.*

Proposition 2.3.7. *The distributional Laplacian $-\Delta$ is an isometric bijection (isomorphism) from $H_0^1(D)$ to $H^{-1}(D)$.*

Proof. If $f \equiv 0$ then clearly $-\Delta f \equiv 0$. For any $f \in H_0^1(D)$ with $\|f\|_{H_0^1} > 0$ Proposition 2.3.5 implies

$$\langle -\Delta f, f/\|f\|_{H_0^1} \rangle = \langle f, f/\|f\|_{H_0^1} \rangle_{H_0^1} = \|f\|_{H_0^1} \,.$$

This implies that $\|-\Delta f\|_{H^{-1}} \geqslant \|f\|_{H_0^1}$ and together with the boundedness established in Proposition 2.3.5 this means that $-\Delta$ is isometric, which implies injectivity.

For surjectivity, fix a dual element $\rho \in H^{-1}(D)$. Then by the Riesz–Fréchet representation theorem there exists a function $f_\rho \in H_0^1(D)$ such that

$$\langle \rho, g \rangle = \langle f_\rho, g \rangle_{H_0^1}$$

for all $g \in H_0^1(D)$. Now by Proposition 2.3.5

$$\langle -\Delta f_\rho, g \rangle = \langle f_\rho, g \rangle_{H_0^1} \,.$$

Thus $-\Delta f_\rho = \rho$, implying surjectivity. □

The fact that $-\Delta : H_0^1(D) \to H^{-1}(D)$ is an isomorphism yields the corollary

Corollary 2.3.8. *The norm induced by the inner product*

$$\langle f, g \rangle_\Delta := \langle -\Delta^{-1} f, -\Delta^{-1} g \rangle_{H_0^1} \,, \quad (f, g \in H^{-1}(D))$$

agrees with operator norm on $H^{-1}(D)$.

Proof. Recall that $\|\cdot\|_{H^{-1}}$ denotes the operator norm. For any $f \in H^{-1}(D)$ we have

$$\begin{aligned} \langle f, f \rangle_\Delta = \langle -\Delta^{-1} f, -\Delta^{-1} f \rangle_{H_0^1} = \langle f, -\Delta^{-1} f \rangle = \left\langle f, \frac{-\Delta^{-1} f}{\|-\Delta^{-1} f\|_{H_0^1}} \right\rangle \|-\Delta^{-1} f\|_{H_0^1} \\ \leqslant \sup_{\|g\|_{H^{-1}}=1} \langle f, -\Delta^{-1} g \rangle \|f\|_\Delta = \sup_{\|g\|_{H_0^1}=1} |\langle f, g \rangle| \|f\|_\Delta = \|f\|_{H^{-1}} \|f\|_\Delta \,. \end{aligned}$$

The second equality follows from the definition of the distributional Laplacian and second to last equality from the isomorphism property of $-\Delta$. Hence $\|f\|_\Delta \leqslant \|f\|_{H^{-1}}$. Also,

$$\|f\|_{H^{-1}} = \sup_{\|g\|_{H_0^1}=1} |\langle f, g \rangle| = \sup_{\|g\|_{H_0^1}=1} |\langle -\Delta^{-1} f, g \rangle_{H_0^1}| \leqslant \sup_{\|g\|_{H_0^1}=1} \|-\Delta^{-1} f\|_{H_0^1} \|g\|_{H_0^1} = \|f\|_\Delta \,.$$

□

In the light of the previous result it is justified to write $\langle f, g\rangle_{H^{-1}}$ instead of $\langle f, g\rangle_\Delta$. Next we list some applications of Sobolev space theory that will be useful for us later.

Proposition 2.3.9. (Elliptic regularity) 1. *If $f \in H_0^1(D)$ and $\langle -\Delta f, g\rangle = 0$ for all $g \in C_c^\infty(D)$, then there exists $\widetilde{f} \in C^\infty(D)$ such that $f(x) = \widetilde{f}(x)$ for almost every $x \in D$ and $-\Delta \widetilde{f}(x) = 0$ for all $x \in D$.*

2. *More generally, if $\Lambda \in (C_c^\infty(D))^\star$ and $\langle \Lambda, -\Delta g\rangle = 0$ for all $g \in C_c^\infty(D)$, then there exists $f_\Lambda \in C^\infty(D)$ such that*

$$\int_D f_\Lambda(x)\, g(x)\, dx = \langle \Lambda, g\rangle$$

for all $g \in C_c^\infty(D)$ and $-\Delta f_\Lambda(x) = 0$ for all $x \in D$.

Proof. This follows for example from Theorem 8.12 and its corollary in [44]. Alternatively, see pages 66-67 in [31]. □

Definition 2.3.10. (Regularity of boundary) *Let $D \subset \mathbb{R}^d$ be open and bounded. We say that the boundary ∂D is C^n if for each point $x_0 \in \partial D$ there exists $r > 0$ and a $C^n(\mathbb{R}^{d-1})$ function γ such that (by possibly relabelling and reorienting the coordinate axes) we have*

$$D \cap B(x_0, r) = \{x \in B(x_0, r) \mid x_d > \gamma(x_1, \ldots, x_{d-1})\}\,.$$

Proposition 2.3.11. (Harmonic extensions) *Let $D \subset \mathbb{R}^d$ be an open and bounded subset with a C^2 boundary and let $g : \partial D \to \mathbb{R}$ be a continuous function. Then the boundary-value problem*

$$\begin{cases} \Delta f(x) = 0\,, & x \in D\,, \\ f(x) = g(x)\,, & x \in \partial D\,. \end{cases}$$

has a unique solution $f \in C^2(D)$.

Proof. See Theorem 2.14 in [18]. □

An important property of Sobolev functions is that while they are defined only modulo sets of Lebesgue measure zero, there still is a canonical way to define the values of a function $f \in H^n(D)$ on the boundary ∂D, even if ∂D has Lebesgue measure zero. It also holds that $f|_{\partial D} \equiv 0$ if and only if $f \in H_0^1(D)$. The next proposition explains precisely what we mean by $f|_{\partial D}$.

Proposition 2.3.12. (Boundary values of Sobolev functions) *Assume that $D \subset \mathbb{R}^d$ is open, bounded and has C^1-boundary. Denote by $L^2(\partial D)$ the space of L^2 functions with respect to the Lebesgue measure on $\partial D \subset D$ [3]. Then there exists a bounded linear operator*

$$T : H^1(D) \to L^2(\partial D)$$

such that

[3] The Lebesgue measure on ∂D can be defined by taking the Lebesgue measure μ on $[0,1)$ and pushing it to ∂D. For example if $D = B(0,1)$, then define $f : [0,1) \to \partial D$ by $f(x) = (\cos x, \sin x)$ and then the Lebesgue measure on ∂D is the push-forward measure $f_*\mu$, see Definition 2.4.11.

1. $Tf = f|_{\partial D}$ *if* $f \in H^1(D) \cap C(\overline{D})$.

2. $\|Tf\|_{L^2(\partial D)} \leqslant C\|f\|_{H^1(D)}$ *for all* $f \in H_0^1(D)$. *The constant* C *depends only on the set* D.

T is called the (Sobolev) trace operator.

Proof. Theorem 1 in section 5.5 of [16]. □

The existence and properties of the Sobolev trace operator will be essential when in Chapter 3 we show that taking a circle-average of a function belonging to $H_0^1(D)$ defines an element of $H^{-1}(D)$.

Remark 2.3.13. (On products of generalized functions) *The dual elements of various test function spaces are usually called generalized functions, or distributions. Examples of test function spaces are* $C_c^\infty(D)$ *and* $H_0^1(D)$. *The classes of generalized functions that we will use are* $(C_c^\infty(D))^\star$ *and* $H^{-1}(D)$.

A common property for both of these classes is that in general one can not multiply elements of them with each other, that is, an object of the form $\Lambda_1\Lambda_2$, *where* $\Lambda_i \in H^{-1}(D)$, *does not make sense necessarily. This leads to problems later when we are trying to define an object of the form* e^Λ *where* Λ *will be the Gaussian free field. We resolve this problem by using the theory of Gaussian multiplicative chaos.*

2.4 Probability theory

Definition 2.4.1. 1. *Let* T *be an arbitrary index set. The cylindrical* σ*-algebra on* $\mathbb{R}^T$ *is the* σ*-algebra generated by the maps* $p_t : \mathbb{R}^T \to \mathbb{R}$, $p_t(f) = f(t)$.

2. *A real-valued stochastic process is a measurable map* $X : (\Omega, \mathcal{F}, \mathbb{P}) \to \mathbb{R}^T$ *where* $\mathbb{R}^T$ *is endowed with the cylindrical* σ*-algebra.*

Definition 2.4.2. (Consistent collections of measures) *Let* T *be an index set and suppose that for each* $t = (t_1, \ldots, t_k) \in T^k$ *we have a probability measure* ν_t *on the product space* $((\mathbb{R}^n)^k, \otimes_{j=1}^k \mathcal{B}(\mathbb{R}^n))$. *The family of probability measures* $(\nu_t)_{t \in T^k, k \in \mathbb{N}}$ *is called* consistent *if the following two properties hold.*

i. *For all permutations* π *of* $\{1, \ldots, k\}$ *we have*

$$\nu_{t_1,\ldots,t_k}(F_1 \times \ldots \times F_k) = \nu_{t_{\pi(1)},\ldots,t_{\pi(k)}}(F_{\pi(1)} \times \ldots \times F_{\pi(k)}),$$

where $F_i \in \mathcal{B}(\mathbb{R}^n)$.

ii. *For all* $F_i \in \mathcal{B}(\mathbb{R}^n)$, $m \in \mathbb{N}$ *we have*

$$\nu_{t_1,\ldots,t_k,t_{k+1},\ldots,t_{k+m}}(F_1 \times \ldots \times F_k \times \mathbb{R}^n \times \ldots \times \mathbb{R}^n) = \nu_{t_1,\ldots,t_k}(F_1 \times \ldots \times F_k).$$

Theorem 2.4.3. (Kolmogorov extension theorem) *Using the notation of the previous definition, if a family of probability measures $(\nu_t)_{t\in T^k,\, k\in\mathbb{N}}$ is consistent, then there exists a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and a stochastic process $X : \Omega \to (\mathbb{R}^n)^T$ such that*

$$\mathbb{P}(X_{t_1} \in F_1, \ldots, X_{t_k} \in F_k) = \nu_{t_1,\ldots,t_k}(F_1 \times \ldots \times F_k)$$

for all $F_i \in \mathcal{B}(\mathbb{R}^n)$.

Proof. See for example [48]. □

Theorem 2.4.4. (Multiparameter Kolmogorov–Čentsov continuity theorem) *Fix positive numbers $t_i \in \mathbb{R}$, $i = 1, \ldots, d$. Let X be a random function $\prod_{i=1}^d [0, t_i] \to \mathbb{R}$. Assume that X satisfies*

$$\mathbb{E}\left[|X(a) - X(b)|^\alpha\right] \leqslant K|a-b|^{d+\beta}$$

for all $a, b \in \prod_{i=1}^d [0, t_i]$ and some fixed positive constants α, β and K. Then there exists a modification of X that is η-Hölder continuous for every $\eta < \beta/\alpha$.

Proof. See Section 2.2.B in [25]. □

Proposition 2.4.5. *Let X be a Gaussian random vector with mean vector μ and covariance matrix V. Its moment generating function is*

$$\mathbb{E}e^{X\cdot\gamma} = e^{\mu\cdot\gamma + \frac{1}{2}\gamma\cdot V\gamma}\,.$$

Proof. We compute the d-dimensional Gaussian integral

$$\mathbb{E}e^{X\cdot\gamma} = \frac{1}{\sqrt{(2\pi)^d \det V}} \int_{\mathbb{R}^d} e^{x\cdot\gamma} e^{-\frac{1}{2}(x-\mu)\cdot V^{-1}(x-\mu)}\, dx\,.$$

We make the change of variables $x = Ay + \mu + V\gamma$, where A is such that $A^{-1}V^{-1}A$ is diagonal (note that A will be symmetric and $\det A = 1$). Denote by λ_i the eigenvalues of V^{-1}. Then

$$\begin{aligned}
\mathbb{E}e^{X\cdot\gamma} &= \frac{1}{\sqrt{(2\pi)^d \det V}} e^{\mu\cdot\gamma + \frac{1}{2}\gamma\cdot V\gamma} \int_{\mathbb{R}^d} e^{-\frac{1}{2}\sum_{i=1}^d \lambda_i y_i^2}\, dy \\
&= \frac{1}{\sqrt{(2\pi)^d \det V}} e^{\mu\cdot\gamma + \frac{1}{2}\gamma\cdot V\gamma} \prod_{i=1}^d \int_{\mathbb{R}} e^{-\frac{\lambda_i y_i^2}{2}}\, dy_i \\
&= \frac{1}{\sqrt{(2\pi)^d \det V}} e^{\mu\cdot\gamma + \frac{1}{2}\gamma\cdot V\gamma} \prod_{i=1}^d \sqrt{\frac{2\pi}{\lambda_i}} \\
&= e^{\mu\cdot\gamma + \frac{1}{2}\gamma\cdot V\gamma}\,.
\end{aligned}$$

□

As a corollary we get the finite-dimensional Girsanov theorem for Gaussian random variables.

Corollary 2.4.6. (Girsanov theorem) *Let X be a centred Gaussian random vector with covariance matrix V. Fix $\alpha \in \mathbb{R}^d$ and define a new probability measure $\mathbb{Q}$ by*

$$\mathbb{Q}(A) = Z^{-1} \int_A e^{\alpha \cdot X} \, d\mathbb{P} \,, \quad (A \in \mathcal{F}) \,,$$

where $Z = \mathbb{E}_{\mathbb{P}} e^{\alpha \cdot X} = e^{\frac{1}{2}\alpha \cdot V\alpha}$. Then under the probability measure $\mathbb{Q}$ the vector X is Gaussian with mean vector $V\alpha$ and covariance matrix V. In other words, if μ is the distribution of X, then

$$\int_{\mathbb{R}^d} F(x) e^{\alpha \cdot x - \frac{1}{2}\alpha \cdot V\alpha} \, \mu(dx) = \int_{\mathbb{R}^d} F(x + V\alpha) \, \mu(dx) \,,$$

where F is such that both of the above integrals are defined.

Proof. The moment generating function of X under $\mathbb{Q}$ is

$$\mathbb{E}_{\mathbb{Q}} e^{\gamma \cdot X} = Z^{-1} \mathbb{E}_{\mathbb{P}} e^{\gamma \cdot X + \alpha \cdot X} = \frac{1}{e^{\frac{1}{2}\alpha \cdot V\alpha}} e^{\frac{1}{2}(\gamma+\alpha)\cdot V(\gamma+\alpha)} = e^{\gamma \cdot V\alpha + \frac{1}{2}\gamma \cdot V\gamma} \,.$$

This is the moment generating function of a Gaussian with mean $V\alpha$ and covariance V. Thus the result follows since the moment generating function determines the distribution of a random vector. □

Proposition 2.4.7. (Probabilistic Wick's formula) *Let $n \in \mathbb{N}$ and assume that $X = (X_1, \dots, X_n)$ is a centered Gaussian vector. Then*

$$\mathbb{E}[X_1 X_2 \dots X_n] = \sum_{p \in P} \prod_{\{i,j\} \in p} \mathbb{E}[X_i X_j] \,,$$

where P denotes the set of all pair-partitions[4] *of the set $\{1, 2, \dots, n\}$.*

Proof. Taylor expanding the characteristic function of $\alpha \cdot X$ for some $\alpha \in \mathbb{R}^d$ yields the result, see Theorem 1.28. in [22]. □

Definition 2.4.8. (Weak convergence of measures) *Let $(X, \mathcal{G})$ be a measurable space that is also a topological space. A family of measures $(\mu_\varepsilon)_{\varepsilon > 0}$ on $(X, \mathcal{G})$ converges to a measure μ on $(X, \mathcal{G})$ weakly if*

$$\lim_{\varepsilon \to 0} \int_X f(x) \, \mu_\varepsilon(dx) = \int_X f(x) \, \mu(dx)$$

for all bounded and continuous functions $f : X \to \mathbb{R}$.

[4] A pair-partition of a set is a partition of the set into unordered pairs. For example the pair-partitions of $\{1, 2, 3, 4\}$ are $\{\{1, 2\}, \{3, 4\}\}$, $\{\{1, 3\}, \{2, 4\}\}$ and $\{\{1, 4\}, \{2, 3\}\}$.

Definition 2.4.9. (Weak convergence of random measures) *Let $(\mu_\varepsilon)_{\varepsilon>0}$ be a family of random Borel measures, that is, μ_ε is defined on a probability space and takes values in the space of all Borel measures on some fixed topological space X.*

We say that $(\mu_\varepsilon)_{\varepsilon>0}$ converges to a measure μ weakly in probability if

$$\lim_{\varepsilon\to 0}\int f(x)\,\mu_\varepsilon(dx) = \int f(x)\,\mu(dx)$$

in probability for every bounded continuous function $f : X \to \mathbb{R}$.

Similarly, we say that $(\mu_\varepsilon)_{\varepsilon>0}$ converges to a measure μ weakly almost surely if the above convergence holds almost surely.

Theorem 2.4.10. (Martingale convergence theorem) *Let $(X_n)_{n=1}^\infty$ be a sequence of real-valued random variables that forms a martingale. Then:*

1. *If $\sup_{n\in\mathbb{N}}\mathbb{E}|X_n| < \infty$, then the limit $\lim_{n\to\infty} X_n =: X$ exists almost surely and $\mathbb{E}|X| < \infty$.*
2. *If $p > 1$ and*

$$\sup_{n\in\mathbb{N}}\mathbb{E}[|X_n|^p] < \infty\,,$$

then there exists a random variable $X \in L^p(\Omega,\mathbb{P})$ such that $X_n \to X$ almost surely and in $L^p(\Omega,\mathbb{P})$.

Proof. Theorems 5.2.8 and 5.4.5 in [15]. □

Definition 2.4.11. *Let $(X,\mathcal{F})$ and $(Y,\mathcal{G})$ be two measurable spaces, $f : X \to Y$ a measurable function and μ a measure on X. We define the push-forward measure $f_*\mu$ of μ under f on Y by*

$$(f_*\mu)(A) := \mu(f^{-1}(A))\,.$$

An important property of push-forward measures is the change of variables formula

$$\int_{f(A)} g(x)\,f_*\mu(dx) = \int_A (g\circ f)(x)\,\mu(dx)\,,$$

where $g \in L^1(f_*\mu)$.

The notion of a cylindrical sigma-algebra can be generalized to an arbitrary Banach space in the following manner.

Definition 2.4.12. *Let X be a Banach space. Then the cylindrical sigma-algebra in X is the sigma-algebra generated by the family of dual elements $(\Lambda)_{\Lambda\in X^\star}$.*

2.5 Gaussian fields on Hilbert spaces

The finite dimensional Gaussian vectors $X : \Omega \to \mathbb{R}^d$ are characterized by the fact that X is Gaussian if and only if $X \cdot \alpha$ is a one-dimensional Gaussian for each deterministic $\alpha \in \mathbb{R}^d$. Furthermore, the finite dimensional standard Gaussian in $\mathbb{R}^d$ can be characterized as

$$X = \sum_{n=1}^{d} X_n e_n \, ,$$

where $(e_n)_{n=1}^d$ is an orthonormal basis of $\mathbb{R}^d$ and $(X_n)_{n=1}^d$ are iid one-dimensional standard Gaussians. We can use this characterization as a starting point for defining analogous Gaussian objects in infinite dimensional Hilbert spaces. However, as is the tradition, in infinite dimensions topological peculiarities make things harder.

Let H be a Hilbert space, $(e_n)_{n\in\mathbb{N}}$ its orthonormal basis and $(X_n)_{n\in\mathbb{N}}$ an iid sequence of standard Gaussians. We investigate the formal series

$$X = \sum_{n=1}^{\infty} X_n e_n \tag{2.5.1}$$

which is our candidate the "standard Gaussian in H". However, it is clear that the series diverges in H almost surely since

$$\|X\|_H^2 = \sum_{n=1}^{\infty} X_n^2$$

and the event $\{\lim_{n\to\infty} X_n = 0\}$ has probability zero. Nevertheless, we can define the "inner products"

$$\sum_{n=1}^{\infty} X_n \langle f, e_n \rangle_H$$

where $f \in H$, and this gives us a family of centered Gaussians indexed by the space H.

Theorem 2.5.1. *Let H be a Hilbert space, $(e_n)_{n\in\mathbb{N}}$ its orthonormal basis and $(X_n)_{n\in\mathbb{N}}$ an iid sequence of standard Gaussians. Then the following statements hold true.*

1. *There exists a stochastic process indexed by H, denoted by $(X(f))_{f\in H}$, such that any vector $(X(f_1), \ldots, X(f_n))$ is a centered Gaussian vector with covariance matrix V given by*

$$V_{ij} = \mathbb{E}\left[X(f_i)X(f_j)\right] = \langle f_i, f_j \rangle_H \, .$$

This process is unique in the sense that any two stochastic processes with the above properties agree in distribution.

2. The series

$$\sum_{n=1}^{\infty} X_n \langle f, e_n \rangle_H$$

converges almost surely and in $L^2(\Omega, \mathbb{P})$ towards a random variable $\mathcal{X}(f)$ which has the distribution $\mathcal{N}(0, \|f\|_H^2)$. Thus the limit agrees in distribution with $X(f)$ defined above.

Proof. 1. In short, this follows from properties of Gaussian vectors and the Kolmogorov extension theorem. Let $f_1, \ldots, f_n \in H$. Since the inner product is symmetric, it suffices to show that if $\alpha_1, \ldots, \alpha_n \in \mathbb{R}$, then

$$\sum_{i,j=1}^{n} \alpha_i \alpha_j \langle f_i, f_j \rangle_H \geqslant 0\,.$$

By linearity, we have

$$\sum_{i,j=1}^{n} \alpha_i \alpha_j \langle f_i, f_j \rangle_H = \left\langle \sum_{i=1}^{n} \alpha_i f_i, \sum_{j=1}^{n} \alpha_j f_j \right\rangle_H = \langle f, f \rangle_H \geqslant 0$$

where $f = \sum_{i=1}^n \alpha_i f_i$. Hence for each choice of $(f_1, \ldots, f_n)$ there exists a centered Gaussian vector $(X(f_1), \ldots, X(f_n))$ with a covariance structure given by $\langle f_i, f_j \rangle_H$ [5]. Uniqueness follows from the fact that the distribution of a Gaussian vector is uniquely determined by its mean and covariance.

Next we want to show the existence of the Gaussian process $(X(f))_{f \in H}$. Above we showed the existence of the finite-dimensional distributions of the process. Now we verify that these finite-dimensional distributions satisfy the assumptions of the Kolmogorov extension theorem.

Let $\nu_{f_1 \cdots f_n}$ be the distribution of the Gaussian vector $(X(f_1), \ldots, X(f_n))$. Now if π is a permutation of $\{1, \ldots, n\}$ and $F_1, \ldots, F_n \in \mathcal{B}(\mathbb{R})$ are Borel sets, our task is to show that

$$\nu_{f_{\pi(1)} \cdots f_{\pi(n)}}(F_{\pi(1)} \times \ldots \times F_{\pi(n)}) = \nu_{f_1 \ldots f_n}(F_1 \times \ldots \times F_n)\,.$$

This follows from a simple change of variables. Define $F_\pi = F_{\pi(1)} \times \ldots \times F_{\pi(n)}$, $V_{ij} = \langle f_i, f_j \rangle_H$ and $(V_\pi)_{ij} = V_{\pi(i)\pi(j)}$. Then by the change of variables $(x_{\pi(1)}, \ldots, x_{\pi(n)}) \mapsto (x_1, \ldots, x_n)$ we

[5] Some matrix algebra shows that such a Gaussian vector is given by $V^{\frac{1}{2}} Y$ where $Y \sim N(0, \mathrm{id}_{\mathbb{R}^n})$ is the standard Gaussian and $V^{\frac{1}{2}}$ is the square root of the matrix $V_{ij} = \langle f_i, f_j \rangle_H$, which is well-defined whenever V is symmetric and positive semidefinite.

have

$$\begin{aligned}\nu_{f_{\pi(1)}\cdots f_{\pi(n)}}(F_{\pi(1)} \times \ldots \times F_{\pi(n)}) &= \int_{F_\pi} e^{-\frac{1}{2}x\cdot(V_\pi)^{-1}x}\, d^n x \\ &= \int_{F_{\pi(1)}} \cdots \int_{F_{\pi(n)}} e^{-\frac{1}{2}\sum_{ij} x_i (V_\pi^{-1})_{ij} x_j}\, dx_n \ldots dx_1 \\ &= \int_{F_1} \cdots \int_{F_n} e^{-\frac{1}{2}\sum_{ij} x_{\pi(i)} (V_\pi^{-1})_{ij} x_{\pi(j)}}\, dx_{\pi(n)} \ldots dx_{\pi(1)} \\ &= \nu_{f_1\ldots f_n}(F_1 \times \ldots \times F_n)\,.\end{aligned}$$

We used the facts that the Jacobian determinant of the transformation equals $(-1)^{\mathrm{sgn}(\pi)}$ at every point and thus has absolute value 1 everywhere, and that

$$\sum_{ij} x_{\pi(i)} (V_\pi^{-1})_{ij} x_{\pi(j)} = \sum_{ij} x_i (V^{-1})_{ij} x_j\,.$$

The second consistency condition follows from the observation that

$$\begin{aligned}\nu_{f_1\ldots f_n f_{n+1}}(F_1 \times \ldots \times F_n \times \mathbb{R}) &= \mathbb{P}(X(f_1) \in F_1, \ldots, X(f_n) \in F_n, X(f_{n+1}) \in \mathbb{R}) \\ &= \mathbb{P}(X(f_1) \in F_1, \ldots, X(f_n) \in F_n) \\ &= \nu_{f_1\ldots f_n}(F_1 \times \ldots \times F_n)\,.\end{aligned}$$

Now the result follows from the Kolmogorov extension theorem 2.4.3.

2. We apply the martingale convergence theorem to the partial sums. Define the filtration $\mathcal{F}_n = \sigma(X_1, \ldots, X_n)$. Now note that

$$\begin{aligned}\mathbb{E}\left(\sum_{n=1}^{k+1} X_n \langle f, e_n\rangle_H \middle| \mathcal{F}_k\right) &= \mathbb{E}\left(X_{k+1}\langle f, e_{k+1}\rangle_H | \mathcal{F}_k\right) + \mathbb{E}\left(\sum_{n=1}^{k} X_n \langle f, e_n\rangle_H \middle| \mathcal{F}_k\right) \\ &= 0 + \sum_{n=1}^{k} X_n \langle f, e_n\rangle_H\,.\end{aligned}$$

Hence the partial sums form a martingale with respect to the filtration $(\mathcal{F}_n)_{n=1}^\infty$. The variance of the partial sum is

$$\begin{aligned}\mathbb{E}\left[\left(\sum_{n=1}^{k} X_n \langle f, e_n\rangle_H\right)^2\right] &= \sum_{n,m=1}^{k} \mathbb{E}[X_n X_m]\langle f, e_n\rangle_H \langle f, e_m\rangle_H \\ &= \sum_{n=1}^{k} \mathbb{E}[X_n^2]\langle f, e_n\rangle_H^2 \\ &= \sum_{n=1}^{k} \langle f, e_n\rangle_H^2\,,\end{aligned}$$

which is bounded by $\|f\|_H^2$. Now by the martingale convergence theorem 2.4.10 there exists a random variable $\mathcal{X}(f)$ such that the convergence

$$\lim_{k\to\infty}\sum_{n=1}^{k} X_n\langle f, e_n\rangle_H = \mathcal{X}(f)$$

holds almost surely and in $L^2(\Omega,\mathbb{P})$. Since the partial sums are centered Gaussians, the limit $\mathcal{X}(f)$ is also a centered Gaussian with variance

$$\mathbb{E}[\mathcal{X}(f)^2] = \lim_{k\to\infty}\sum_{n=1}^{k}\langle f, e_n\rangle_H^2 = \|f\|_H^2\,.$$

□

The convergence of the sums $\sum_n X_n\langle f, e_n\rangle_H$ somewhat justifies the notation (2.5.1) in the sense that X can be thought of as a formal element of H and we can take inner products by $\langle X, f\rangle_H = \mathcal{X}(f)$. We can now give the definition of a Gaussian field on the Hilbert space H.

Definition 2.5.2. *We define the* Gaussian field *on H to be the stochastic process $(X(f))_{f\in H}$ defined by (1) of Theorem 2.5.1.*

Even though the notation might suggest that $f\mapsto\mathcal{X}(f)$ could be a random element of $H^\star$, this is not the case. The map $\mathcal{X}$ is an unbounded functional almost surely:

$$\mathbb{P}\left(\sup_{\|f\|_H=1}|\mathcal{X}(f)|<M\right)\leqslant\mathbb{P}\left(|\mathcal{X}(e_n)|<M\quad\forall n\in\mathbb{N}\right)=\prod_{n\in\mathbb{N}}\mathbb{P}(X_n<M)=0\,.$$

Since M was arbitrary, we get the following result:

Proposition 2.5.3. *Let $\mathcal{X}$ be the Gaussian field on H. Then $\mathbb{P}(\mathcal{X}\in H^\star)=0$.*

The Girsanov theorem generalizes to Gaussian fields on Hilbert spaces.

Proposition 2.5.4. (Girsanov theorem) *For all $f,g\in H$ we have*

$$\mathbb{E}[F(\mathcal{X}(f))e^{\mathcal{X}(g)-\frac{1}{2}\mathbb{E}[\mathcal{X}(g)^2]}] = \mathbb{E}[F(\mathcal{X}(f)+\langle g,f\rangle_H)]\,.$$

Proof. This follows from Theorem 14.1 in [22]. □

2.6 Green functions

In the Introduction we mentioned that the Gaussian free field is a Gaussian field with covariance operator Δ^{-1}, which is naturally interpreted as the integral operator corresponding to the Green function of the Laplacian. In this section we prove some basic properties of the (Dirichlet) Green function of the Laplacian.

Definition 2.6.1. *We define the Green function of the two-dimensional Laplacian* $G_{\mathbb{R}^2} : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R} \cup \{+\infty\}$ *by*

$$G_{\mathbb{R}^2}(x,y) = \frac{1}{2\pi} \log \frac{1}{|x-y|}\,.$$

Definition 2.6.2. *Let* $D \subset \mathbb{R}^2$ *be an open bounded subset with a* C^2 *boundary. The Dirichlet Green function in domain* D *is the function* $G_D : D \times D \to \mathbb{R} \cup \{+\infty\}$ *defined by*

$$G_D(x,y) = G_{\mathbb{R}^2}(x,y) + \xi_D(x,y)\,,$$

where $\xi_D : \overline{D} \times D \to \mathbb{R}$ *is defined as follows. Fix a point* $y \in D$ *and denote by* h_y *the harmonic extension (see Proposition 2.3.11) of the function* $x \mapsto \frac{1}{2\pi} \log \frac{1}{|x-y|} = G_{\mathbb{R}^2}(x,y)$ *from* ∂D *into* D*. Then set* $\xi_D(x,y) = -h_y(x)$*.*

This definition of the Green function will require us to assume C^2-regularity of the boundary for our domains in Chapters 3 and 4. Using probabilistic methods the Green function can also be defined for domains with rough boundary, see [3]. Hence this assumption could be dropped.

Proposition 2.6.3. *The Green function has the following properties.*

1. *For each fixed* $y \in \mathbb{R}^2$*, the function* $x \mapsto G_{\mathbb{R}^2}(x,y)$ *belongs to* $L^1_{\text{loc}}(\mathbb{R}^2)$*. Also, this function is smooth and harmonic in* $\mathbb{R}^2 \setminus \{y\}$*.*

2. *For each* $f \in C_c^\infty(\mathbb{R}^2)$ *and* $y \in \mathbb{R}^2$ *we have*

$$\int_{\mathbb{R}^2} G_{\mathbb{R}^2}(x,y)\Delta f(x)\,dx = -f(y)\,.$$

3. *For each fixed* $y \in D$*, the function* $x \mapsto G_D(x,y)$ *is smooth and harmonic in* $D \setminus \{y\}$ *and approaches* 0 *as* $x \to \partial D$*. Hence we define* $G_D(x,y) = 0$ *for* $x \in \partial D$ *and this extension belongs to* $C^\infty(\overline{D} \setminus \{y\})$*. Furthermore, the function* $x \mapsto G_D(x,y)$ *belongs to* $L^1_{\text{loc}}(\overline{D})$*.*

4. *For each* $f \in C^\infty(\overline{D})$ *and* $y \in D$ *we have*

$$\int_D G_D(x,y)\Delta f(x)\,dx = -f(y)\,.$$

5. *Let* $\mathbb{D}$ *be the unit disk centered at the origin. Then*

$$G_{\mathbb{D}}(x,0) = \frac{1}{2\pi} \log \frac{1}{|x|}\,.$$

Proof. See section 2.2 in [16]. □

Lemma 2.6.4. (Conformal invariance of the Dirichlet energy) *If $\psi : D \to D'$ is conformal mapping between the domains D and D', then for all $f, g \in H_0^1(D')$ we have*

$$\int_{D'} \nabla f(x) \cdot \nabla g(x)\, dx = \int_D \nabla(f \circ \psi)(x) \cdot \nabla(g \circ \psi)(x)\, dx\,.$$

Proof. We first assume that $f, g \in C_c^\infty(D')$. Consider the change of variables $(x_1, x_2) = x = \psi(y) = (\psi_1(y), \psi_2(y))$. Using the Cauchy–Riemann equations we see that the Jacobian determinant of this transformation equals $|\nabla\psi_1(y_1, y_2)|^2$ and then by the properties of the complex derivative, this equals $|\psi'(y)|^2$, where ψ' denotes the complex derivative of ψ. Thus the Jacobian does not vanish at any point. By the chain rule

$$\partial_i(f \circ \psi)(x) = \partial_i\psi_1(x)\partial_1 f(\psi(x)) + \partial_i\psi_2(x)\partial_2 f(\psi(x))\,.$$

Using this and the Cauchy–Riemann equations then yields

$$\begin{aligned}\nabla(f \circ \psi)(x) \cdot \nabla(g \circ \psi)(x) &= |\partial_1\psi_1(x)|^2[\partial_1 f(\psi(x))\partial_1 g(\psi(x)) + \partial_2 f(\psi(x))\partial_2 g(\psi(x))] \\ &\quad + |\partial_2\psi_1(x)|^2[\partial_1 f(\psi(x))\partial_1 g(\psi(x)) + \partial_2 f(\psi(x))\partial_2 g(\psi(x))] \\ &= |\nabla\psi_1(x)|^2[\nabla f(\psi(x)) \cdot \nabla g(\psi(x))]\,.\end{aligned}$$

By this result and the change of variables formula we get

$$\begin{aligned}\int_D \nabla(f \circ \psi)(x) \cdot \nabla(g \circ \psi)(x)\, dx &= \int_D |\nabla\psi_1(x)|^2[\nabla f(\psi(x)) \cdot \nabla g(\psi(x))]\, dx \\ &= \int_{\psi(D)} |\nabla\psi_1(\psi^{-1}(x))|^2[\nabla f(x) \cdot \nabla g(x)] \frac{dx}{|\nabla\psi_1(\psi^{-1}(x))|^2} \\ &= \int_{D'} \nabla f(x) \cdot \nabla g(x)\, dx\,.\end{aligned}$$

The result for $f, g \in H_0^1(D')$ follows by considering sequences of test functions $(f_n), (g_n) \subset C_c^\infty(D')$ converging to f and g in $H_0^1(D')$, respectively, and then passing to the limit in the above integrals. Indeed we have

$$\begin{aligned}\int_{D'} \nabla f(x) \cdot \nabla f(x)\, dx &= \lim_{n\to\infty} \int_{D'} \nabla f_n(x) \cdot \nabla f_n(x)\, dx \\ &= \lim_{n\to\infty} \int_D \nabla(f_n \circ \psi)(x) \cdot \nabla(f_n \circ \psi)(x)\, dx\,.\end{aligned}$$

Since $(f_n \circ \psi)_{n\in\mathbb{N}}$ is Cauchy in $H_0^1(D)$, the above integral has a limit. Thus we can pass to a subsequence $(n_k)_{k\in\mathbb{N}}$ such that $f_{n_k} \to f$ and $\nabla f_{n_k} \to \nabla f$ almost everywhere. To finish the proof we use the fact that now $\nabla(f_{n_k} \circ \psi) \to \nabla(f \circ \psi)$ almost everywhere and the dominated convergence theorem then implies the claim for $f = g$. The case $f \neq g$ follows by the polarization identity

$$\langle f, g\rangle_{H_0^1} = \frac{1}{4}(\|f + g\|_{H_0^1}^2 - \|f - g\|_{H_0^1}^2)\,.$$

□

Lemma 2.6.5. *Assume that $f \in L^1_{\text{loc}}(D')$ and let $\psi : D \to D'$ be a conformal map. Then for all $g \in C_c^\infty(D')$ we have*

$$\int_{D'} f(x)\Delta g(x)\,dx = \int_D (f \circ \psi(x))\Delta(g \circ \psi)(x)\,dx\,.$$

Proof. Take a sequence $(f_n)_{n\in\mathbb{N}} \subset C_c^\infty(D')$ such that $f_n \to f$ in $L^2(D')$. Then Hölder's inequality implies that

$$\lim_{n\to\infty} \int_{D'} f_n(x)\Delta g(x)\,dx = \int_{D'} f(x)\Delta g(x)\,dx\,,$$
$$\lim_{n\to\infty} \int_D (f_n \circ \psi)(x)\Delta(g \circ \psi)(x)\,dx = \int_D (f \circ \psi)(x)\Delta(g \circ \psi)(x)\,dx\,.$$

Then the result follows by using the Green's formula and the previous lemma:

$$\begin{aligned}\int_{D'} f_n(x)\Delta g(x)\,dx &= -\int_{D'} \nabla f_n(x)\cdot\nabla g(x)\,dx \\ &= -\int_D \nabla(f_n\circ\psi)(x)\cdot\nabla(g\circ\psi)(x)\,dx \\ &= \int_D (f_n\circ\psi)(x)\Delta(g\circ\psi)(x)\,dx\,.\end{aligned}$$

□

Definition 2.6.6. *By $R(x, D)$ we denote the conformal radius of D viewed from x, that is, if $\psi : D \to \mathbb{D}$ is a conformal map into the unit disk such that $\psi(x) = 0$, then $R(x, D) = |\psi'(x)|^{-1}$.*

The conformal radius has the following basic properties.

Proposition 2.6.7. *The conformal radius has the following properties.*

1. *Let $\psi : D \to D'$ be a conformal map. Then $R(\psi(x), D') = |\psi'(x)|R(x, D)$.*
2. $\operatorname{dist}(x, \partial D) \leqslant R(x, D) \leqslant 4\operatorname{dist}(x, \partial D)$.

Theorem 2.6.8. *Assume that D and D' are open, simply connected proper subsets of $\mathbb{R}^2$ with C^2 boundary. The Green function has the following properties.*

1. *The Green function is conformally invariant, that is, if $\psi : D' \to D$ is a conformal map, then*

$$G_D(\psi(x), \psi(y)) = G_{D'}(x, y)\,.$$

2. $G_D(x, y) = -\frac{1}{2\pi}\log|x - y| + \frac{1}{2\pi}\log R(y, D) + o(1)$ *as $x \to y$.*

Proof. 1. Assume $f \in C_c^\infty(D')$. Since $x \mapsto G_D(x,y)$ is locally integrable, Lemma 2.6.5 together with Proposition 2.6.3 implies

$$\int_{D'} G_D(\psi(x),\psi(y))\Delta f(x)\,dx = \int_D G_D(x,\psi(y))\Delta f(\psi^{-1}(x))\,dx$$
$$= -f(\psi^{-1}(\psi(x)))$$
$$= -f(x)\,.$$

Now fix $y \in D'$. The above computation shows that if $f \in C_c^\infty(D' \setminus \{y\})$, then

$$\int_{D'} G_D(\psi(x),\psi(y))\Delta f(x)\,dx = 0\,.$$

Now Proposition 2.3.9 implies the existence of a function $h \in C^\infty(D' \setminus \{y\})$ that is also harmonic in $D' \setminus \{y\}$ and has the property

$$\int_{D'} h(x)f(x)\,dx = \int_{D'} G_D(\psi(x),\psi(y))f(x)\,dx$$

for all $f \in C_c^\infty(D' \setminus \{y\})$. Thus $h(x) = G_D(\psi(x),\psi(y))$ for almost every $x \in D' \setminus \{y\}$, and since both of these functions are smooth on that set, we must have $h(x) = G_D(\psi(x),\psi(y))$ for all $x \in D' \setminus \{y\}$.

Now to finish we need to show that $h(x) = G_{D'}(x,y)$ for all $x \in D' \setminus \{y\}$. First, Green's formula implies that for all $f \in C_c^\infty(D' \setminus \{y\})$

$$\int_{D'} h(x)\Delta f(x)\,dx = \int_{D'} \Delta h(x) f(x)\,dx = 0\,.$$

Then since $x \mapsto G_{D'}(x,y)$ has this same property, and the both functions are smooth on $D' \setminus \{y\}$, we must have $h(x,y) = G_{D'}(x,y)$.

2. We use the formula for $G_{\mathbb{D}}(x,0)$ given in Proposition 2.6.3

$$G_{\mathbb{D}}(x,0) = \frac{1}{2\pi}\log\frac{1}{|x|}\,.$$

By definition of the conformal radius $R(0,\mathbb{D}) = 1$. This means that the claim holds in the case $D = \mathbb{D}$ and $y = 0$. Now in the general case choose points $x, y \in D$ and a conformal map $\psi : D \to \mathbb{D}$ such that $\psi(y) = 0$. Now

$$\psi(x) = \psi'(y)(x-y) + \mathcal{O}((x-y)^2)$$

as $x \to y$, so by conformal invariance we have

$$G_D(x,y) = G_{\mathbb{D}}(\psi(x),0) = \frac{1}{2\pi}\log\frac{1}{|\psi(x)|} = -\frac{1}{2\pi}\log\left|\psi'(y)(x-y) + \mathcal{O}((x-y)^2)\right|$$
$$= -\frac{1}{2\pi}\log(|\psi'(y)||x-y|) + o(1) = \frac{1}{2\pi}\log\frac{1}{|x-y|} + \frac{1}{2\pi}\log R(y,D) + o(1)$$

as $x \to y$. The second to last equality holds because

$$\log\left|\psi'(y)(x-y) + \mathcal{O}((x-y)^2)\right| - \log(|\psi'(y)||x-y|) = o(1)$$

as $x \to y$.

□

2.7 Riemannian geometry

In Chapter 5 we will need some basic Riemannian geometry. Here we very briefly summarize the basic terminology and notations. We assume knowledge of basic differential geometry. For a proper treatment see for example [32, 33].

If M is a differentiable manifold (a topological manifold with at least C^1-structure), the tangent space at a point $p \in M$ is denoted by T_pM and the tangent bundle by TM. Similarly T_p^*M and T^*M denote the cotanget space and the cotangent bundle, respectively. If $f : M \to N$ is a map between two differentiable manifolds, then the pullback is denoted by f^*.

Given a manifold M (C^k, smooth, analytic or complex analytic), a Riemannian metric g on M is a symmetric and positive definite 2-tensor field. Symmetry means that for any $p \in M$ and vector fields v, w we have $g(v,w)_p = g(w,v)_p$ and positive definiteness that $g(v,v)_p > 0$ if $v_p \neq 0$. The pair (M, g) is called a Riemannian manifold.

A local frame $(E_1, \ldots, E_n)$ on an n-manifold is an n-tuple of vector fields such that $(E_1, \ldots, E_n)_p$ is a basis for T_pM for every p in some chart. The n-tuple of tensors $(\varphi_1, \ldots, \varphi_n)$ giving the dual bases of the cotangent spaces T_p^*M is called the dual coframe. Let $(E_1, \ldots, E_n)$ be a local frame and $(\varphi_1, \ldots, \varphi_n)$ its dual coframe. In this frame a Riemannian metric can be written as (summation over i and j is implied)

$$g = g_{ij}\varphi^j \otimes \varphi^j \,.$$

The coefficients are $g_{ij} = g(E_i, E_j)$. The symmetry of g implies that $g_{ij} = g_{ji}$. By choosing our local frame to be the coordinate frame, we can write

$$g = g_{ij} dx^i \otimes dx^j \,. \tag{2.7.1}$$

The differential df of a function $f : M \to \mathbb{R}$ is defined to be the covector field which operates as

$$df(v) = v(f)\,, \quad (v \in TM)\,.$$

In a coordinate frame we can write

$$df_p = \frac{\partial f}{\partial x^i}(p)\varphi_p^i\,,$$

where $(\varphi_1, \ldots, \varphi_n)$ is the coframe. Actually, the covector fields φ^i are the differentials of the coordinate vector fields, yielding us the more suggestive notation

$$df_p = \frac{\partial f}{\partial x^i}(p) dx^i_p \, .$$

The gradient $\nabla_g f$ is obtained from df by "raising an index", meaning that $\nabla_g f$ is the vector field satisfying

$$df(v) = g(\nabla_g f, v) \, , \quad (v \in TM) \, .$$

In coordinates this is

$$\nabla_g f = g^{ij} \partial_i f \partial_j \, ,$$

where g^{ij} are the components of the inverse matrix $(g_{ij})^{-1}$. Any oriented Riemannian n-manifold admits a unique n-form λ_g satisfying

$$\lambda_g(e_1, \ldots, e_n) = 1$$

whenever $(e_1, \ldots, e_n)$ is an oriented orthonormal basis for some tangent space $T_p M$, see for example [32]. The n-form λ_g is called the *volume form*. In any oriented local frame the volume form admits the representation

$$\lambda_g = \sqrt{\det g} \, \varphi^1 \wedge \ldots \wedge \varphi^n \, .$$

Here $\det g$ denotes the determinant of the matrix with entries $(g_{ij})_{i,j}$ and $(\varphi^1, \ldots, \varphi^n)$ is the dual coframe.

If $M \subset \mathbb{R}^n$ is an open set, the integral of a compactly supported n-form ω is

$$\int_M \omega = \int_{\mathbb{R}^n} u(x) \, dx$$

where on the right-hand side we have a Lebesgue integral and the function u is such that

$$\omega = u(x) \, dx^1 \wedge \ldots \wedge dx^n$$

in a coordinate frame. Now let M be a general oriented n-manifold and assume that the n-form ω is compactly supported in an open set $U \subset M$. We define the integral of ω to be

$$\int_M \omega = \int_{\varphi(U)} (x^{-1})^* \omega \, ,$$

where (U, x) is a chart. Note that now $(x^{-1})^* \omega$ is an n-form compactly supported in $x(U)$, so that its integral is already defined. Using partitions of unity one can extend the integral to non-compactly supported n-forms, see [33].

With the volume form we can integrate functions on M. Let f be a smooth function $M \to \mathbb{R}$ on an oriented Riemannian manifold (M, g). Now $f \lambda_g$ is an n-form, so we can integrate it. We then define the integral of the *function* f over M to be the integral of the *n-form* $f\lambda_g$ over M.

The Riemann sphere $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$, which we can also think of as the plane $\mathbb{R}^2$ with a point at infinity, admits a complex analytic structure by the two charts $\varphi_1 : \mathbb{C} \to \mathbb{C}$, $\varphi_1(z) = z$ and $\varphi_2 : \widehat{\mathbb{C}} \setminus \{0\} \to \mathbb{C}$, $\varphi_2(z) = \frac{1}{z}$. Alternatively, one can think of these charts as taking values in $\mathbb{R}^2$ ($\mathbb{C} = \mathbb{R}^2$ as sets) so that we have a real analytic manifold. In this interpretation the charts are $\varphi_1(x_1 + ix_2) = (x_1, x_2)$ and $\varphi_2(x_1 + ix_2) = (\frac{x_1}{x_1^2+x_2^2}, -\frac{x_2}{x_1^2+x_2^2})$ with $\varphi_2(\infty) = 0$. A metric g is conformally equivalent to the Euclidean metric if in real coordinates on the $\mathbb{C}$-chart we have

$$g = g(z)(dx^1 \otimes dx^1 + dx^2 \otimes dx^2)\,, \tag{2.7.2}$$

where $z = x_1 + ix_2$, and on the other chart we have

$$g = \frac{g(1/z)}{|z|^4}(dx^1 \otimes dx^1 + dx^2 \otimes dx^2), \tag{2.7.3}$$

where g is assumed to be positive C^2 function that has a finite limit at infinity. We are abusing the notation here a little; on the left-hand side we have the actual Riemannian metric g, which is a 2-tensor field, and on the right-hand side appears the function $z \mapsto g(z)$, which is the density (in the $\mathbb{C}$-chart) of g with respect to the Euclidean metric. This notation is often used in the literature. Comparing to (2.7.1) this means that, for example, on the $\mathbb{C}$-chart we have $g_{ij} = g(z)\delta_{ij}$. Thus if g is conformally equivalent to the Euclidean metric, then

$$(\det g)(z) = g(z)^2\,.$$

The spherical metric $\widehat{g}$ is defined to be the metric for which we have

$$\widehat{g}(z) = \frac{4}{(1 + x_1^2 + x_2^2)^2} = \frac{4}{(1 + |z|^2)^2}$$

on the $\mathbb{C}$-chart. This manifold is isometric to $\mathbb{S}^2$ (the two-dimensional sphere embedded into $\mathbb{R}^3$) via the stereographic projection. Using the volume form λ_g, an integral of a function $f : \widehat{\mathbb{C}} \to \mathbb{R}$ now becomes

$$\int_{\widehat{\mathbb{C}}} f\lambda_g = \int_{\mathbb{R}^2} (f \circ \varphi_1^{-1})(x)\frac{4}{(1 + |x|^2)^2}\,dx$$

where dx is the 2-dimensional Lebesgue measure and $\varphi_1 : \mathbb{C} \to \mathbb{R}^2$ is the chart $x_1 + ix_2 \mapsto (x_1, x_2)$. It is customary to identify f and $f \circ \varphi_1^{-1}$ and to just write

$$\int_{\widehat{\mathbb{C}}} f\lambda_g = \int_{\mathbb{R}^2} f(x)\lambda_g(dx)$$

where $\lambda_g(dx) = \frac{4}{(1+|x|^2)^2}\, dx$. For example for $f \equiv 1$ and $g = \widehat{g}$ we have

$$\int_{\widehat{\mathbb{C}}} \lambda_g = \int_{\mathbb{R}^2} \frac{4}{(1+|x|^2)^2}\, dx = 4\pi\,,$$

yielding the correct area of the sphere. From now on and in Chapter 5 we will adapt this convention: instead of the Riemann sphere $\widehat{\mathbb{C}}$ we will work on $\mathbb{R}^2 \cup \{\infty\}$ with the metric taking the form

$$g = (g \circ \varphi_1^{-1})(x)(dx^1 \otimes dx^1 + dx^2 \otimes dx^2)\,, \quad x = (x_1, x_2)\,,$$

in the chart $\mathbb{R}^2$, and similarly for (2.7.3) in the chart $(\mathbb{R}^2 \cup \{\infty\}) \setminus \{0\}$, and then instead of $(g \circ \varphi_1^{-1})(x)$ we will just write $g(x)$.

We denote the scalar curvature of g by R_g. In our case of metrics conformally equivalent to the Euclidean metric the scalar curvature is given by

$$R_g(x) = -\Delta_g \log \sqrt{(\det g)(x)} = -\Delta_g \log g(x)\,,$$

where Δ_g is the Laplace–Beltrami operator, defined by $\Delta_g = \frac{1}{\sqrt{\det g}}\Delta$. For the spherical metric we have $R_{\widehat{g}}(x) = 2$.

Chapter 3

The Gaussian free field on planar domains

In this chapter we develop basics of the theory of the Gaussian free field (GFF) defined on a planar domain $D \subset \mathbb{R}^2$. We will assume that D is open and bounded with C^2 boundary. First we discuss the definition of the Gaussian free field in detail. Then we proceed to prove some basic properties of the GFF, including conformal invariance, the spatial Markov property and that the circle average process is a Brownian motion. The material is loosely based on [3]. For a slightly different approach see [46].

3.1 Heuristics

In the introduction we considered a statistical field theory defined by a probability distribution heuristically of the form

$$\exp\left(-\frac{1}{2}\int_D |\nabla X(x)|^2\,dx\right)\mathcal{D}X \tag{3.1.1}$$

on the "space of maps $X : D \to \mathbb{R}$". This of course has the form of a Gaussian measure

$$\exp\left(-\frac{1}{2}\langle X, V^{-1}X\rangle\right)\mathcal{D}X$$

and the covariance operator V formally satisfies

$$\langle X, V^{-1}X\rangle = \int_D |\nabla X(x)|^2\,dx\,.$$

Recalling that the distributional Laplacian operates on $f \in H_0^1(D)$ as

$$\langle -\Delta f, f\rangle = \int_D |\nabla f(x)|^2\,dx\,,$$

this suggests that we ought to define V to be $-\Delta^{-1} : H^{-1}(D) \to H_0^1(D)$. By Proposition 2.6.3 we can also write

$$(VX)(y) = (-\Delta^{-1}X)(y) = \int_D G_D(x,y)X(x)\,dx\,. \tag{3.1.2}$$

Heuristically this means that

$$\mathbb{E}[X(x)X(y)] = G_D(x,y)\,.$$

This diverges for $x = y$, and thus $X(x)$ would have to be a centered Gaussian with infinite variance. This strongly suggests that X will not make sense as a pointwise defined object. Thus we want to think of X as a generalized function and sample it with test functions. Then the samplings $X(f)$ are Gaussian random variables and the covariance of two samplings $X(f)$ and $X(g)$ is

$$\begin{aligned}\mathbb{E}\left[X(f)X(g)\right] = \mathbb{E}\left[\int_D X(x)f(x)dx \int_D X(y)g(y)dy\right] &= \iint_{D\times D} \mathbb{E}[X(x)X(y)]f(x)g(y)\,dx\,dy \\ &= \iint_{D\times D} G_D(x,y)f(x)g(y)\,dx\,dy\,.\end{aligned}$$

This way of interpreting V will lead to the zero boundary (or Dirichlet) GFF. Note that one could just as well define V to be a Green function with different boundary condition. For example the Neumann boundary GFF is discussed in Chapter 5 of [3].

To get an idea of what kind of generalized function X might be, we check for which class of functions the covariance of two samplings

$$\mathbb{E}[X(f)X(g)] = \iint_{D\times D} G_D(x,y)f(x)g(y)\,dx\,dy$$

is finite. In terms of the distributional Laplacian and its inverse $-\Delta^{-1} : H^{-1}(D) \to H_0^1(D)$ this means that (recall 2.3.8)

$$\begin{aligned}\iint_{D\times D} G_D(x,y)f(x)f(y)\,dx\,dy &= -\int_D f(x)(\Delta^{-1}f)(x)\,dx \\ &= \|-\Delta^{-1}f\|_{H_0^1}^2 \\ &= \|f\|_{H^{-1}}^2\end{aligned}$$

should be finite too, so we choose f from $H^{-1}(D)$. Hence the object we are looking for is such that $\rho \mapsto X(\rho)$ defines a linear map where $\rho \in H^{-1}(D)$ and $X(\rho)$ is a centered Gaussian with variance

$$\mathbb{E}\left[X(\rho)^2\right] = \|\rho\|_{H^{-1}}^2\,.$$

That is, X is a linear isometry from $H^{-1}(D)$ to $L^2(\Omega, \mathbb{P})$ with $X(\rho) \sim \mathcal{N}(0, \|\rho\|_{H^{-1}}^2)$. This is precisely the Gaussian field on $H^{-1}(D)$, see Definition 2.5.2 and Theorem 2.5.1. This yet does not lead to an actual random generalized function, although we will later see that the GFF can be interpreted as a random element of $H^{-1}(D)$.

3.2 The definition of the Gaussian free field

Definition 3.2.1. *The (zero boundary) Gaussian free field (GFF) on an open, bounded domain $D \subset \mathbb{R}^2$ with C^2 boundary is the Gaussian field on $H^{-1}(D)$ (see Definition 2.5.2), multiplied by $\sqrt{2\pi}$. Thus the GFF is a centered Gaussian process $(X(\rho))_{\rho \in H^{-1}(D)}$ such that the vector $(X(\rho_1), \ldots, X(\rho_n))$ has the covariance matrix*

$$\mathbb{E}[X(\rho_i)X(\rho_j)] = 2\pi \langle \rho_i, \rho_j \rangle_{H^{-1}} .$$

Recall also from Theorem 2.5.1 that if we fix a sequence $(X_n)_{n\in\mathbb{N}}$ of iid standard Gaussians defined on our underlying probability space, $(\Omega, \mathcal{F}, \mathbb{P})$ and if $(e_n)_{n\in\mathbb{N}}$ is an orthonormal basis of $H^{-1}(D)$, then for every $\rho \in H^{-1}(D)$ the series

$$\sqrt{2\pi} \sum_{n=1}^{\infty} X_n \langle \rho, e_n \rangle_{H^{-1}} .$$

converges almost surely and in $L^2(\Omega, \mathbb{P})$ and the limit agrees in distribution with $X(\rho)$.

The multiplication by $\sqrt{2\pi}$ will make some formulas simpler. In the literature there are various slightly different (although equivalent) definitions of the GFF. Our definition resembles the one used in [10].

3.3 Gaussian free field as a random generalized function

In this section we show that the restriction of the GFF to $H_0^1(D)$ has a modification that belongs to $H^{-1}(D)$. Also, we show that this restriction determines the whole GFF. Recall that we think of $H_0^1(D)$ as a subset of $H^{-1}(D)$ by setting for $f \in H_0^1(D)$

$$\langle f, g \rangle := \langle f, g \rangle_{L^2} , \quad (g \in H_0^1(D)) .$$

The Poincaré inequality asserts that defined like this f belongs to $H^{-1}(D)$.

Proposition 3.3.1. *Let $(e_n)_{n\in N}$ be the orthonormal basis of $H^{-1}(D)$ given by $e_n = -\Delta \eta_n$ where $(\eta_n)_{n\in\mathbb{N}}$ is the Laplace-eigenbasis of $H_0^1(D)$ with eigenvalues $(\lambda_n)_{n\in\mathbb{N}}$. Let $(X_n)_{n\in\mathbb{N}}$ be an iid sequence of standard Gaussians. Then the functional*

$$f \mapsto \sqrt{2\pi} \sum_{n=1}^{\infty} X_n \langle f, e_n \rangle_{H^{-1}} , \quad (f \in H_0^1(D)) , \tag{3.3.1}$$

defines an element of $H^{-1}(D)$ almost surely.

Proof. Let $f \in H_0^1(D)$. The sum has the bound

$$\begin{aligned}\left|\sum_{n=1}^{\infty} X_n \langle f, e_n\rangle_{H^{-1}}\right| &\leqslant \sum_{n=1}^{\infty} |X_n||\langle f, e_n\rangle_{H^{-1}}| \\ &\leqslant \left(\sum_{n=1}^{\infty} \frac{X_n^2}{\lambda_n^2}\right)^{1/2} \left(\sum_{n=1}^{\infty} \lambda_n^2 \langle f, e_n\rangle_{H^{-1}}^2\right)^{1/2} \\ &= \left(\sum_{n=1}^{\infty} \frac{X_n^2}{\lambda_n^2}\right)^{1/2} \left(\sum_{n=1}^{\infty} \langle f, \eta_n\rangle_{H_0^1}^2\right)^{1/2} \\ &\leqslant \left(\sum_{n=1}^{\infty} \frac{X_n^2}{\lambda_n^2}\right)^{1/2} \|f\|_{H_0^1} .\end{aligned}$$

We used the fact that

$$\lambda_n^2 \langle f, e_n\rangle_{H^{-1}}^2 = \lambda_n^2 \langle -\Delta^{-1} f, -\Delta^{-1} e_n\rangle_{H_0^1}^2 = \langle -\Delta^{-1} f, e_n\rangle_{H_0^1}^2 = \langle f, \eta_n\rangle_{H_0^1}^2 .$$

We show that the series $\sum_n X_n^2/\lambda_n^2$ is finite almost surely by showing that the expectation is finite. Note that

$$\mathbb{E} \sum_{n=1}^{\infty} \frac{X_n^2}{\lambda_n^2} = \sum_{n=1}^{\infty} \frac{1}{\lambda_n^2} . \tag{3.3.2}$$

By Theorem 2.3.1 we have

$$\lim_{n\to\infty} \frac{\lambda_n^2}{n^2} = \frac{16\pi^2}{|D|^2} .$$

Since the series $\sum_n n^{-2}$ converges, the series $\sum_n \lambda_n^{-2}$ converges too. Now it follows that the expectation (3.3.2) is finite which implies that $\sum_n X_n^2 \lambda_n^{-2}$ is finite almost surely. Hence the functional (3.3.1) belongs to $H^{-1}(D)$ almost surely. □

By Theorem 2.3.1 the Laplace-eigenbasis $(\eta_n)_{n\in\mathbb{N}}$ of $H_0^1(D)$ consists of smooth functions and thus the orthonormal basis $(e_n)_{n\in\mathbb{N}}$ of $H^{-1}(D)$, where $e_n = -\Delta\eta_n$, also consists of smooth functions. Hence we can think of e_n as an element of $L^2(D)$ and we have the following equality for every $f \in H_0^1(D)$

$$\sum_{n=1}^{\infty} X_n \langle f, e_n\rangle_{H^{-1}} = \sum_{n=1}^{\infty} X_n \langle f, \Delta^{-1} e_n\rangle_{L^2} = \sum_{n=1}^{\infty} \frac{X_n}{\lambda_n} \langle f, e_n\rangle_{L^2} .$$

The previous proposition then asserts that the functional

$$f \mapsto \sum_{n=1}^{\infty} \frac{X_n}{\lambda_n} \langle f, e_n\rangle_{L^2}$$

belongs to $H^{-1}(D)$ almost surely. Furthermore, the computation carried out in the proof of the previous proposition implies that for $j > i$ we have

$$\sup_{\|f\|_{H_0^1}=1} \left| \sum_{n=1}^{i} \frac{X_n}{\lambda_n} \langle f, e_n \rangle_{L^2} - \sum_{n=1}^{j} \frac{X_n}{\lambda_n} \langle f, e_n \rangle_{L^2} \right| \leqslant \left(\sum_{k=i+1}^{j} \frac{X_n^2}{\lambda_n^2} \right)^{1/2} .$$

This tends to zero almost surely as $i, j \to \infty$ which means that the series

$$\mathcal{X} := \sqrt{2\pi} \sum_{n=1}^{\infty} \frac{X_n}{\lambda_n} e_n = \sqrt{2\pi} \sum_{n=1}^{\infty} X_n \eta_n \tag{3.3.3}$$

converges in $H^{-1}(D)$[1] almost surely and $\langle \mathcal{X}, f \rangle \stackrel{d}{=} X(f)$ for each $f \in H_0^1(D)$. In fact, we can use any orthonormal basis of $H_0^1(D)$.

Theorem 3.3.2. *Let $(\widetilde{\eta}_n)_{n\in\mathbb{N}}$ be an orthonormal basis of $H_0^1(D)$. Then the series*

$$\sum_{n=1}^{\infty} X_n \widetilde{\eta}_n$$

converges in $H^{-1}(D)$ almost surely. We also have

$$X(f) \stackrel{d}{=} \sqrt{2\pi} \sum_{n=1}^{\infty} X_n \langle f, \widetilde{\eta}_n \rangle_{L^2} , \quad (f \in H_0^1(D)) .$$

Proof. The strategy is to expand $\widetilde{\eta}_n$ using the Laplace-eigenbasis $(\eta_n)_{n\in\mathbb{N}}$ and then use the orthogonality of $(\eta_n)_{n\in\mathbb{N}}$ in $H^{-1}(D)$. We have

$$\begin{aligned}
\mathbb{E} \left\| \sum_{n=1}^{\infty} X_n \widetilde{\eta}_n \right\|_{H^{-1}}^2 &= \mathbb{E} \left\| \sum_{n=1}^{\infty} X_n \sum_{k=1}^{\infty} \langle \widetilde{\eta}_n, \eta_k \rangle_{H_0^1} \eta_k \right\|_{H^{-1}}^2 \\
&= \mathbb{E} \sum_{k=1}^{\infty} \left| \sum_{n=1}^{\infty} X_n \langle \widetilde{\eta}_n, \eta_k \rangle_{H_0^1} \right|^2 \|\eta_k\|_{H^{-1}}^2 \\
&= \sum_{k=1}^{\infty} \sum_{n=1}^{\infty} \langle \widetilde{\eta}_n, \eta_k \rangle_{H_0^1}^2 \|\eta_k\|_{H^{-1}}^2 \\
&= \sum_{k=1}^{\infty} \|\eta_k\|_{H^{-1}}^2 .
\end{aligned}$$

Now observe that

$$\|\eta_k\|_{H^{-1}}^2 = \langle \eta_k, \eta_k \rangle_{H^{-1}} = \langle \Delta^{-1} \eta_k, \Delta^{-1} \eta_k \rangle_{H_0^1} = \frac{1}{\lambda_k^2} .$$

[1]Note that the L^2 inner product $\langle f, e_n \rangle_{L^2}$ coincides with the dual bracket $\langle f, e_n \rangle$ when $f, e_n \in L^2(D)$.

Finally, since $-\Delta : H_0^1(D) \to H^{-1}(D)$ is an isomorphism, the sequence $(\Delta\widetilde{\eta}_n)_{n\in\mathbb{N}}$ forms an orthonormal basis of $H^{-1}(D)$ and thus for $f \in H_0^1(D)$ we have

$$X(f) \overset{d}{=} \sqrt{2\pi}\sum_{n=1}^{\infty} X_n\langle f, \Delta\widetilde{\eta}_n\rangle_{H^{-1}} = \sqrt{2\pi}\sum_{n=1}^{\infty} X_n\langle f, \widetilde{\eta}_n\rangle_{L^2}\,.$$

□

Remark 3.3.3. *A similar computation shows that the series $\sum_n X_n\widetilde{\eta}_n$ converges in $H^{-s}(D)$ almost surely for all $s \in (0,1]$.*

Now if we denote

$$X|_{H_0^1(D)} := \sum_{n=1}^{\infty} X_n\widetilde{\eta}_n$$

for an arbitrary orthonormal basis $(\widetilde{\eta}_n)_{n\in\mathbb{N}}$ of $H_0^1(D)$, then $X|_{H_0^1(D)} \in H^{-1}(D)$ almost surely, meaning that the restriction of the GFF process $(X(\rho))_{\rho\in H^{-1}(D)}$ to $H_0^1(D)$ has a modification that is a random element of $H^{-1}(D)$. Often in the literature this restriction is used as the definition of the GFF. This definition is equivalent to ours since the whole process $(X(\rho))_{\rho\in H^{-1}}$ is determined by the restricted process $(X(\rho))_{\rho\in H_0^1(D)}$ in the sense that if we take an arbitrary $\rho \in H^{-1}(D)$, mollify it with an approximation to the identity $(\theta_\varepsilon)_{\varepsilon>0}$, then $\rho * \theta_\varepsilon$, defined by

$$(\rho * \theta)(x) = \langle \rho, \theta(x - \cdot)\rangle,$$

belongs to $H_0^1(D) \cap C^\infty(D)$ and $X(\rho * \theta_\varepsilon) \to X(\rho)$ in $L^2(\Omega, \mathbb{P})$. We give a proof of this fact.

Proposition 3.3.4. *Assume that $\theta \in C_c^\infty(D)$ is such that $\int_D \theta(x)\,dx = 1$ and define $\theta_\varepsilon(x) = \varepsilon^{-2}\theta(x/\varepsilon)$. Then for all $\rho \in H^{-1}(D)$ we have*

$$X(\rho * \theta_\varepsilon) \to X(\rho)$$

in $L^2(\Omega, \mathbb{P})$ (and in probability) as $\varepsilon \to 0$.

Proof. The convolution $\rho_\varepsilon := \rho * \theta_\varepsilon$ belongs to $C^\infty(D)$ and $\langle \rho_\varepsilon, f\rangle \to \langle \rho, f\rangle$ for all $f \in H_0^1(D)$ as $\varepsilon \to 0$, see [44]. Thus

$$\begin{aligned}\mathbb{E}\left[X(\rho_\varepsilon - \rho)^2\right] &= 2\pi\|\rho_\varepsilon - \rho\|_{H^{-1}}^2 = 2\pi\langle \rho_\varepsilon - \rho, \rho_\varepsilon - \rho\rangle_{H^{-1}} \\ &= 2\pi\big(\langle \rho_\varepsilon, \rho_\varepsilon\rangle_{H^{-1}} - \langle \rho_\varepsilon, \rho\rangle_{H^{-1}} - \langle \rho, \rho_\varepsilon\rangle_{H^{-1}} + \langle \rho, \rho\rangle_{H^{-1}}\big)\,.\end{aligned}$$

Now note that

$$\langle \rho_\varepsilon, \rho\rangle_{H^{-1}} = \langle -\Delta^{-1}\rho_\varepsilon, -\Delta^{-1}\rho\rangle_{H_0^1} = \langle \rho_\varepsilon, -\Delta^{-1}\rho\rangle \to \langle \rho, -\Delta^{-1}\rho\rangle = \langle \rho, \rho\rangle_{H^{-1}}\,.$$

Define $f_\varepsilon = -\Delta^{-1}\rho_\varepsilon$ and $f = -\Delta^{-1}\rho$. The final step is to show that $\langle \rho_\varepsilon, f_\varepsilon\rangle \to \langle \rho, f\rangle$. It holds that $f * \theta_\varepsilon \to f$ and $(\partial^\alpha f) * \theta_\varepsilon \to \partial^\alpha f$, $|\alpha| \leqslant 1$, at Lebesgue points (and especially almost everywhere) and Young's inequality gives $\|(\partial^\alpha f) * \theta_\varepsilon\|_{L^2} \leqslant \|\partial^\alpha f\|_{L^2}\|\theta_\varepsilon\|_{L^1}$. Then Dominated Convergence implies $f * \theta_\varepsilon \to f$ in $H_0^1(D)$. It also holds that $f_\varepsilon = f * \theta_\varepsilon$ since

$$-\Delta(f * \theta_\varepsilon) = -\Delta((-\Delta^{-1}\rho) * \theta_\varepsilon) = \rho * \theta_\varepsilon = -\Delta f_\varepsilon$$

and $-\Delta$ is a bijection. Now we can conclude that

$$\langle \rho_\varepsilon, \rho_\varepsilon \rangle_{H^{-1}} = \langle f_\varepsilon, f_\varepsilon \rangle_{H_0^1} \to \langle f, f \rangle_{H_0^1} = \langle \rho, \rho \rangle_{H^{-1}} .$$

Thus $X(\rho_\varepsilon) \to X(\rho)$ in $L^2(\Omega, \mathbb{P})$. □

In conclusion: if we start with the stochastic process $(X(\rho))_{\rho \in H^{-1}(D)}$, then the restricted process $(X(\rho))_{\rho \in H_0^1(D)}$ has a modification such that $\rho \mapsto X(\rho)$ defines a random element of $H^{-1}(D)$. On the other hand, if we start with the restricted process $(X(\rho))_{\rho \in H_0^1(D)}$, which we can define using the $H^{-1}(D)$-convergent series

$$X|_{H_0^1(D)} = \sum_{n=1}^{\infty} X_n \eta_n ,$$

where $(\eta_n)_{n \in \mathbb{N}}$ is an orthonormal basis of $H_0^1(D)$, then we can retrieve the full process $(X(\rho))_{\rho \in H^{-1}(D)}$ by defining $X(\rho)$ to be the limit in $L^2(\Omega, \mathbb{P})$ of the mollified variables $\langle X|_{H_0^1(D)}, \rho * \theta_\varepsilon \rangle$.

3.4 Properties of the Gaussian free field

Proposition 3.4.1. *For all $f, g \in H_0^1(D)$ we have*

$$\mathbb{E}[X(f)X(g)] = 2\pi \iint_{D \times D} G_D(x, y) f(x) g(x) \, dx \, dy .$$

Proof. We have

$$\mathbb{E}[X(f)X(g)] = 2\pi \langle f, g \rangle_{H^{-1}} = 2\pi \langle -\Delta^{-1} f, -\Delta^{-1} g \rangle_{H_0^1} = 2\pi \langle -\Delta^{-1} f, g \rangle_{L^2} .$$

Now by Proposition 2.6.3 this equals

$$2\pi \langle -\Delta^{-1} f, g \rangle_{L^2} = 2\pi \iint_{D \times D} G_D(x, y) f(x) g(y) \, dx \, dy .$$

□

By the density argument 3.3.4 the above results generalizes for all $\rho, \sigma \in H^{-1}(D)$ in the sense that

$$\mathbb{E}[X(\rho)X(\sigma)] = 2\pi \langle \rho, G^\sigma \rangle , \tag{3.4.1}$$

where $G^\sigma(x) = \langle \sigma, G_D(x, \cdot) \rangle$.

Theorem 3.4.2. (Conformal invariance of the GFF) *Let $\psi : D \to D'$ be a conformal map and X the GFF on D. Define*

$$(X \circ \psi^{-1})(\rho) := X(\rho \circ \psi |\psi'|^{-2}), \quad \rho \in H^{-1}(D'),$$

and similarly $\langle \rho \circ \psi, f \rangle := \langle \rho, f \circ \psi^{-1} |\psi'|^2 \rangle$. Here ψ' denotes the complex derivative of ψ. Then $((X \circ \psi^{-1})(\rho))_{\rho \in H^{-1}(D')}$ is the GFF on D'.

Proof. Let $(X'(\rho))_{\rho \in H^{-1}(D')}$ be the GFF on D' and fix $f, g \in H_0^1(D')$. By change of variables and conformal invariance of the Green function we have

$$\begin{aligned}
\mathbb{E}[X'(f)X'(g)] &= 2\pi \iint_{D' \times D'} G_{D'}(x,y) f(x) g(y) \, dx \, dy \\
&= 2\pi \iint_{D \times D} G_{D'}(\psi(x), \psi(y)) (f \circ \psi)(x) (g \circ \psi)(y) |\psi'(x)|^{-2} |\psi'(y)|^{-2} \, dx \, dy \\
&= 2\pi \iint_{D \times D} G_{D}(x, y) (f \circ \psi)(x) (g \circ \psi)(y) |\psi'(x)|^{-2} |\psi'(y)|^{-2} \, dx \, dy \\
&= \mathbb{E}[X((f \circ \psi)|\psi'|^{-2}) X((g \circ \psi)|\psi'|^{-2})] \\
&= \mathbb{E}[(X \circ \psi^{-1})(f)(X \circ \psi^{-1})(g)] .
\end{aligned}$$

Thus the distributions agree when we restrict X' and $X \circ \psi^{-1}$ to $H_0^1(D')$. Then the density argument laid down in the proof of Proposition 3.3.4 implies that for all $\rho \in H^{-1}(D')$ we have

$$(X \circ \psi^{-1})(\rho) = \lim_{\varepsilon \to 0} (X \circ \psi^{-1})(\rho * \theta_\varepsilon) = \lim_{\varepsilon \to 0} X'(\rho * \theta_\varepsilon) = X'(\rho),$$

where the limits are in $L^2(\Omega, \mathbb{P})$. This implies that the above covariance computation holds for all $f, g \in H^{-1}(D')$. $\square$

Remark 3.4.3. *From the series representation point of view we have*

$$X|_{H_0^1(D)} = \sum_{n=1}^{\infty} X_n \eta_n$$

for an orthonormal basis $(\eta_n)_{n \in \mathbb{N}}$ of $H_0^1(D)$. Then formally

$$X|_{H_0^1(D)} \circ \psi^{-1} = \sum_{n=1}^{\infty} X_n \eta_n \circ \psi^{-1} .$$

Using the fact that the Dirichlet norm is conformally invariant one can show that $(\eta_n \circ \psi^{-1})_{n \in \mathbb{N}}$ is an orthonormal basis of $H_0^1(D')$ and thus from this point of view it is clear that $X|_{H_0^1(D)} \circ \psi^{-1} = X'|_{H_0^1(D')}$.

Theorem 3.4.4. *Assume that $H^{-1}(D) = H_1 \oplus H_2$ for some closed subspaces $H_1, H_2 \subset H^{-1}(D)$. Let $p_1 : H^{-1}(D) \to H_1$ and $p_2 : H^{-1}(D) \to H_2$ be the projection operators onto H_1 and H_2, respectively. Then there exists Gaussian processes $(Y(\rho))_{\rho\in H^{-1}(D)}$ and $(Z(\rho))_{\rho\in H^{-1}(D)}$ such that*

$$X(\rho) = Y(\rho) + Z(\rho)$$

with the following properties

1. *The processes Y and Z are independent.*
2. *For all $\rho \in H^{-1}(D)$ we have $Y(p_2(\rho)) = 0 = Z(p_1(\rho))$.*

Proof. Define $Y(\rho) = X(p_1(\rho))$ and $Z(\rho) = X(p_2(\rho))$. It is clear that $X(\rho) = Y(\rho) + Z(\rho)$ and that $Y(p_2(\rho)) = 0 = Z(p_1(\rho))$.

We have to show the independence of the stochastic processes Y and Z. Since they are Gaussian, it suffices to show that the Gaussian vectors $(Y(\rho_1), \ldots, Y(\rho_n))$ and $(Z(\rho_1) \ldots Z(\rho_n))$ are uncorrelated for any $\rho_i \in H^{-1}(D)$. This in turn reduces to showing that the components are uncorrelated, that is, we have to compute $\mathbb{E}[Y(\rho_i)Z(\rho_j)]$. This is easy because

$$\mathbb{E}[Y(\rho_i)Z(\rho_j)] = 2\pi\langle p_1(\rho_i), p_2(\rho_j)\rangle_{H^{-1}} = 0\,,$$

since $p_1(\rho_i) \in H_1$ and $p_2(\rho_j) \in H_2$ and $H_2 = H_1^{\perp}$. □

Corollary 3.4.5. (Spatial Markov property) *Fix an open subset $U \subset D$. Then we have the decomposition*

$$X(\rho) = X_U(\rho) + \varphi(\rho)\,,$$

where

1. *$(X_U(\rho))_{\rho\in H^{-1}(U)}$ is the GFF on U.*
2. *$(\varphi(\rho))_{\rho\in H^{-1}(D)}$ is a Gaussian process harmonic in U by which we mean that for all $f \in H_0^1(U)$ we have $\varphi(-\Delta f) = 0$. Furthermore, there exists a function $\widetilde{\varphi} \in C^\infty(U)$ that is classically harmonic and $\varphi(f) = \langle \widetilde{\varphi}, f\rangle_{L^2}$ for all $f \in H_0^1(U)$.*
3. *X_U and φ are independent.*

Proof. We will apply the decomposition theorem 3.4.4. To this end, we begin by showing that $H_0^1(D) = H_0^1(U) \oplus \mathrm{Harm}(U)$ where the elements of $H_0^1(U)$ are extended to D by setting them to be 0 on $D \setminus U$, and $\mathrm{Harm}(U)$ is the subset of $H_0^1(D)$ that contains functions which are (classically) harmonic on U. Since $H_0^1(U)$ is a closed subspace of $H_0^1(D)$, it suffices to show that $H_0^1(U)^{\perp} = \mathrm{Harm}(U)$. Then we define $H_1 = -\Delta H_0^1(U)$ and $H_2 = -\Delta\,\mathrm{Harm}(U)$ so that we get the decomposition $H^{-1}(D) = H_1 \oplus H_2$.

If $f \in \mathrm{Harm}(U)$, then for all $g \in H_0^1(U)$ we have

$$\int_D \nabla f(x) \cdot \nabla g(x)\, dx = -\int_U g(x)\Delta f(x)\, dx = 0\,.$$

Thus $g \in H_0^1(U)^\perp$.

If $f \in H_0^1(U)^\perp$, then for all $g \in H_0^1(U)$ the definition of the distributional Laplacian implies

$$0 = \int_D \nabla f(x) \cdot \nabla g(x)\, dx = \langle -\Delta f, g \rangle .$$

This means that the restriction $(-\Delta f)|_{C_c^\infty(U)} \in (C_c^\infty(U))^\star$ is identically zero. Now Proposition 2.3.9 implies that there exists $\widetilde{f} \in C^\infty(U)$ such that $f = \widetilde{f}$ for almost every $x \in U$ and $-\Delta \widetilde{f}(x) = 0$ for all $x \in U$. Thus $f \in \mathrm{Harm}(U)$. Hence $\mathrm{Harm}(U) = H_0^1(U)^\perp$ and we have the decomposition $H_0^1(D) = H_0^1(U) \oplus \mathrm{Harm}(U)$.

Now Theorem 3.4.4 yields the decomposition into two independent terms $X = X_U + \varphi$ where

$$X_U(\rho) := X(\mathrm{pr}_{H_1}(\rho)) , \quad \varphi(\rho) := X(\mathrm{pr}_{H_2}(\rho)) ,$$

where pr_E denotes the projection onto the subspace E and $H_1 = -\Delta H_0^1(U)$, $H_2 = -\Delta\, \mathrm{Harm}(U)$. It is clear that $X(\rho) = X_U(\rho)$ for $\rho \in -\Delta H_0^1(U)$ and $\varphi(-\Delta f) = 0$ for $f \in H_0^1(U)$. Also, the decomposition $H^{-1}(D) = H_1 \oplus H_2$ yields us a natural isometric inclusion map $\iota : H^{-1}(U) \to H^{-1}(D)$ by

$$\langle \iota(\rho), f \rangle := \langle \rho, \mathrm{pr}_{H_0^1(U)}(f) \rangle , \quad (f \in H_0^1(D)) .$$

We will justify this inclusion after we have finished the proof. The important property is that if we think of $H_0^1(U)$ as a subset of $H_0^1(D)$, then $H_1 = -\Delta H_0^1(U) = \iota(H^{-1}(U))$. Now, X_U is the GFF on U since for all $\rho, \sigma \in \iota(H^{-1}(U)) = H_1$ we have

$$\mathbb{E}[X_U(\rho) X_U(\sigma)] = 2\pi \langle \mathrm{pr}_{H_1}(\rho), \mathrm{pr}_{H_1}(\sigma) \rangle_{H^{-1}(D)} = 2\pi \langle \rho, \sigma \rangle_{H^{-1}(D)} = 2\pi \langle \iota^{-1}(\rho), \iota^{-1}(\sigma) \rangle_{H^{-1}(U)} ,$$

where the last equality follows from the isometry property. Thus the covariance agrees with the covariance of the GFF on U.

Since the restrictions $X|_{H_0^1(D)}$ and $X_U|_{H_0^1(U)}$ belong to $H^{-1}(D)$ and $H^{-1}(U)$ almost surely, respectively, it must also hold that $\varphi|_{H_0^1(U)} \in H^{-1}(U)$ almost surely. Also, for every $f \in C_c^\infty(U)$ we have

$$\varphi(-\Delta f) = 0 .$$

By Proposition 2.3.9 there exists $\widetilde{\varphi} \in C^\infty(U)$ such that $-\Delta \widetilde{\varphi}(x) = 0$ for every $x \in U$ and

$$\varphi(f) = \int_U \widetilde{\varphi}(x) f(x)\, dx$$

for every $f \in C_c^\infty(U)$. By density the above expression holds for all $f \in H_0^1(U)$, and thus φ may be identified with $\widetilde{\varphi}$ as an element of $H^{-1}(U)$.

□

Remark 3.4.6. *Now we comment on the inclusion map $\iota : H^{-1}(U) \to H^{-1}(D)$. This remark is somewhat technical and it may be skipped if the reader is convinced that $\rho \in H^{-1}(U)$ defines an element of $H^{-1}(D)$ if we set " $\rho \equiv 0$ on $D \setminus U$"*[2].

Let $-\Delta^D : H_0^1(D) \to H^{-1}(D)$ be the distributional Laplacian and define $-\Delta^U$ similarly. We define the inclusion $\kappa : H_0^1(U) \to H_0^1(D)$ by setting $\kappa(f)$ to be f on U and zero on $D \setminus U$. Now the inclusion $\iota : H^{-1}(U) \to H^{-1}(D)$ should be such that if $\rho \in H^{-1}(U)$, then $\iota(\rho) = -\Delta^D \kappa((-\Delta^U)^{-1}\rho)$. This just means that we want the inclusions $\kappa : H_0^1(U) \to H_0^1(D)$ and $\iota : H^{-1}(U) \to H^{-1}(D)$ to commute with the distributional Laplacians and this will automatically make ι an isometry.

Now if $H^{-1}(D) = H_1 \oplus H_2$ where $H_1 = -\Delta^D \kappa(H_0^1(U))$ and $H_2 = -\Delta^D \operatorname{Harm}(U)$, then for some $f \in \kappa(H_0^1(U))$ we have

$$\langle \operatorname{pr}_{H_1}(\rho), g\rangle = \langle -\Delta^D f, g\rangle = \langle f, g\rangle_{H_0^1(D)} = \int_U \nabla f(x) \cdot \nabla g(x)\, dx\,.$$

Now write $g = \operatorname{pr}_{\kappa(H_0^1(U))}(g) + \operatorname{pr}_{\operatorname{Harm}(U)}(g)$ and we get

$$\langle \operatorname{pr}_{H_1}(\rho), g\rangle = \langle \operatorname{pr}_{H_1}(\rho), \operatorname{pr}_{\kappa(H_0^1(U))}(g)\rangle\,.$$

Also it is easy to see that $\langle \operatorname{pr}_{H_2}(\rho), \operatorname{pr}_{\kappa(H_0^1(U))}(g)\rangle = 0$ and thus $\langle \operatorname{pr}_{H_1}(\rho), g\rangle = \langle \rho, \operatorname{pr}_{\kappa(H_0^1(U))}(g)\rangle$. Similar computation shows that $\langle \operatorname{pr}_{H_2}(\rho), g\rangle = \langle \rho, \operatorname{pr}_{\operatorname{Harm}(U)}(g)\rangle$.

Since $\iota(\rho) := -\Delta^D \kappa((-\Delta^U)^{-1}\rho) \in H_1$, the previous computations imply that

$$\langle \iota(\rho), f\rangle = \langle \iota(\rho), \operatorname{pr}_{\kappa(H_0^1(U))}(f)\rangle$$

for every $f \in H_0^1(D)$. As a composition of three isometries, ι is an isometry. Note that if we think of $H_0^1(U)$ as a subset of $H_0^1(D)$ and $H^{-1}(U)$ as a subset of $H^{-1}(D)$ in the light of these inclusions, then we have $-\Delta H_0^1(U) = H^{-1}(U)$, as we should.

Heuristically the Markov property could be interpreted in the following way. Conditioning on the values of X in $D \setminus U$, the values of X in U are given by $X_U + \varphi$, where X_U is independent of $X|_{D\setminus U}$. Then we could think of the harmonic term φ as the harmonic extension of $X|_{\partial U}$ into U, although we have to remember that it is not clear at all how to interpret this. Thus in some very heuristic sense conditioning on $X|_{\partial U}$ gives the same information about X inside U as conditioning to $X_{D\setminus U}$, hence the name spatial Markov property.

3.5 Circle averages

In this section we define regularized versions of the Gaussian free field X. We do this by taking a circle average of X over an ε-radius circle centered at a point $x \in D$. The circle

[2] One has to be careful with these kind of inclusions, though, since for example $C_c^\infty(U)^\star$ does not make sense as a subspace of $C_c^\infty(D)^\star$. This is essentially because $\rho \in C_c^\infty(U)^\star$ may be too pathological near ∂U to belong to $C_c^\infty(D)^\star$.

averaged Gaussian free field is then a pointwise defined function. In Chapter 4 we will use the circle average to define the exponential of the Gaussian free field.

One could of course use a different regularization method too. However, it turns out that for the sake of defining the exponential $e^{\gamma X}$, all regularization processes (that satisfy some assumptions) provide the same result, see for example [41]. Hence one can choose a regularization according to one's own taste.

Before we can make sense of a circle average of X, we must resolve some technical problems. We want to define the circle average of X around a point x by defining it to be $X(\rho_{x,\varepsilon})$, where $\rho_{x,\varepsilon}$ is a uniform measure on $\partial B(x,\varepsilon)$ with unit mass, and to this end we need to show that $\rho_{x,\varepsilon}$ belongs to $H^{-1}(D)$.

Lemma 3.5.1. *Fix a point $y \in D$ and $\varepsilon > 0$ such that $B(y,\varepsilon) \subset D$. Let $\mu_{y,\varepsilon}$ be the Lebesgue measure on $\partial B(y,\varepsilon)$, normalized so that $\mu_{y,\varepsilon}(\partial B(y,\varepsilon)) = 1$, and let T be the Sobolev trace operator $H^1(B(y,\varepsilon)) \to L^2(\partial B(y,\varepsilon))$ given by Proposition 2.3.12. Then the functional on $H_0^1(D)$, defined by*

$$\rho_{y,\varepsilon}(f) := \int_{\partial B(y,\varepsilon)} (Tf|_{B(y,\varepsilon)})(x)\, \mu_{y,\varepsilon}(dx)\,,$$

is bounded and thus $\rho_{y,\varepsilon} \in H^{-1}(D)$.

Proof. Since T is linear, $\rho_{y,\varepsilon}$ is linear. Boundedness holds because by Proposition 2.3.12

$$\begin{aligned}
|\langle \rho_{y,\varepsilon}, f\rangle| &\leqslant \int_{\partial B(y,\varepsilon)} |(Tf|_{B(y,\varepsilon)})(x)|\, \mu_{y,\varepsilon}(dx) \\
&\leqslant \left(\int_{\partial B(y,\varepsilon)} \mu_{y,\varepsilon}(dx)\right)^{1/2} \left(\int_{\partial B(y,\varepsilon)} |(Tf|_{B(y,\varepsilon)})(x)|^2\, \mu_{y,\varepsilon}(dx)\right)^{1/2} \\
&= \|Tf|_{B(y,\varepsilon)}\|_{L^2(\partial B(y,\varepsilon)} \leqslant C\|f|_{B(y,\varepsilon)}\|_{H^1(B(y,\varepsilon))} \leqslant C'\|f\|_{H_0^1(D)}\,.
\end{aligned}$$

In the last inequality we used the fact that in $H_0^1(D)$ the two norms $\|f\|_{H^1}$ and $\|f\|_{H_0^1}$ are equivalent.

□

Definition 3.5.2. *1. We define the circle average of a Sobolev function $f \in H_0^1(D)$ over a circle $\partial B(x,\varepsilon) \subset D$ to be $\langle \rho_{x,\varepsilon}, f\rangle$, where $\rho_{x,\varepsilon}$ is the functional defined in Lemma 3.5.1.*

2. We define the circle average of the Gaussian free field X over a circle $\partial B(x,\varepsilon) \subset D$ to be $X(\rho_{x,\varepsilon})$, where $\rho_{x,\varepsilon}$ is the functional defined in Lemma 3.5.1. We also introduce the notation $X_\varepsilon(x) = X(\rho_{x,\varepsilon})$.

Recall that we denote by ξ_D the function for which the Green function satisfies

$$G_D(x,y) = \frac{1}{2\pi}\log|x-y| + \xi_D(x,y)\,,$$

see Definition 2.6.2.

Lemma 3.5.3. *Fix $y \in D$ and $\varepsilon > 0$ such that $B(y,\varepsilon) \subset D$. Define $f_{y,\varepsilon} : D \to \mathbb{R}$ by*

$$f_{y,\varepsilon}(x) = \frac{1}{2\pi} \log \frac{1}{\max\{|x-y|, \varepsilon\}} + \xi_D(x,y)\,.$$

Then

$$\int_D |\nabla f_{y,\varepsilon}(x)|^2\, dx = \frac{1}{2\pi} \log \frac{1}{\varepsilon} + \xi_D(y,y) < \infty\,, \tag{3.5.1}$$

and thus $f_{y,\varepsilon} \in H_0^1(D)$. Also $-\Delta f_{y,\varepsilon} = \rho_{y,\varepsilon}$.

Proof. The proof is a straightforward computation, see A.1 in the Appendix. □

Proposition 3.5.4. *The covariance of the circle average satisfies*

$$\mathbb{E}[X_\varepsilon(z)X_\delta(z)] = \log \frac{1}{\max\{\varepsilon,\delta\}} + 2\pi\xi_D(z,z)\,, \tag{3.5.2}$$

and

$$\mathbb{E}[X_\varepsilon(x)X_\delta(y)] = \int_{\partial B(x,\varepsilon)} \log \frac{1}{\max\{|y-z|,\delta\}}\, dS(z) + \xi_D(x,y)\,, \tag{3.5.3}$$

where dS denotes integration with respect to the Lebesgue measure on $\partial B(x,\varepsilon)$.

Proof. Assume $\delta \geqslant \varepsilon$. Now by Lemma 3.5.3

$$\mathbb{E}[X_\varepsilon(y)X_\delta(y)] = \mathbb{E}[X(\rho_{y,\varepsilon})X(\rho_{y,\delta})] = 2\pi\langle f_{y,\varepsilon}, f_{y,\delta}\rangle_{H_0^1}\,.$$

This then simplifies to

$$\langle f_{y,\varepsilon}, f_{y,\delta}\rangle_{H_0^1} = \langle f_{y,\varepsilon}, f_{y,\varepsilon}\rangle_{H_0^1} + \int_{\delta<|x-y|<\varepsilon} \nabla\xi_D(x,y)\cdot\nabla\log\frac{1}{|x-y|}\, dx\,.$$

The latter integral vanishes, which can be seen by using the Green's formula, harmonicity of ξ_D and polar coordinates. Thus the result follows from the formula (3.5.1).

Next we prove the second claim. Observe that

$$\begin{aligned}
\mathbb{E}[X_\varepsilon(x)X_\delta(y)] &= \mathbb{E}[X(\rho_{x,\varepsilon})X(\rho_{y,\delta})] = 2\pi\langle f_{x,\varepsilon}, f_{y,\delta}\rangle_{H_0^1}\\
&= 2\pi\langle -\Delta f_{x,\varepsilon}, f_{y,\delta}\rangle\\
&= 2\pi\int_{\partial B(x,\varepsilon)} f_{y,\delta}(z)\, dS(z)\\
&= \int_{\partial B(x,\varepsilon)} \log\frac{1}{\max\{|y-z|,\delta\}}\, dS(z) + 2\pi\xi_D(x,y)\,.
\end{aligned}$$

The first line follows from definitions and Lemma 3.5.3. The second line is the definition of the distributional Laplacian. Third line follows from Lemma 3.5.3 and the definition of $\rho_{y,\varepsilon}$. The last line follows from the definition of $f_{y,\delta}$. □

Proposition 3.5.5. (Circle average is jointly Hölder continuous) *There exists a modification of X such that $(X_\varepsilon(x), x \in D, 0 < \varepsilon < \operatorname{dist}(x, \partial D))$ is jointly Hölder continuous with exponent $\gamma < 1/2$ on compact subsets of $(x \in D, 0 < \varepsilon < \operatorname{dist}(x, \partial D))$.*

Proof. We use the proof given in [14]. First we show that for each $\varepsilon_0 > 0$ there exists a constant K such that

$$\mathbb{E}\left[|X_\varepsilon(x) - X_\delta(y)|^2\right] \leqslant K\left(|x-y| + |\varepsilon - \delta|\right)$$

for all $x, y \in D$ and $\varepsilon, \delta \in [\varepsilon_0, \infty)$.

Now fix $\varepsilon_0 > 0$. Expanding the square yields

$$\mathbb{E}\left[|X_\varepsilon(x) - X_\delta(y)|^2\right] = \mathbb{E}X_\varepsilon(x)^2 + \mathbb{E}X_\delta(y)^2 - 2\mathbb{E}\left[X_\varepsilon(x)X_\delta(y)\right].$$

Now from (3.5.3) we get

$$\mathbb{E}\left[X_\varepsilon(x)X_\delta(y)\right] = \int_{\partial B(x,\varepsilon)} f_{y,\delta}(z)\, dS(z).$$

Hence we have

$$\begin{aligned}\mathbb{E}\left[|X_\varepsilon(x) - X_\delta(y)|^2\right] &\leqslant \left|\int_{\partial B(x,\varepsilon)} (f_{x,\varepsilon}(z) - f_{y,\delta}(z))\, dS(z)\right| + \left|\int_{\partial B(y,\delta)} (f_{y,\delta}(z) - f_{x,\varepsilon}(z))\, dS(z)\right| \\ &\leqslant K(|x-y| + |\varepsilon - \delta|).\end{aligned}$$

The latter inequality follows from the fact that for a fixed $z \in D$ the function $(x, \varepsilon) \mapsto f_{x,\varepsilon}(z)$ is Lipschitz for $\varepsilon \in [\varepsilon_0, \infty)$ and $x \in D$. To see this, note that the norm of the gradient of $x \mapsto f_{x,\varepsilon}(z)$ is uniformly bounded in D and that $\partial_\varepsilon f_{x,\varepsilon}(z)|_{\varepsilon=\varepsilon'} = -\mathbf{1}(|x-z| = \varepsilon')/\varepsilon'$. Together these imply

$$|f_{x,\varepsilon}(z) - f_{y,\delta}(z)| \leqslant |\nabla_x f_{x,\varepsilon}(z)|_{x=x'}||x-y| + |\partial_\varepsilon f_{x,\varepsilon}(z)|_{\varepsilon=\varepsilon'}||\varepsilon - \delta|$$

for some $\varepsilon' \in (\varepsilon, \delta)$ and x' belonging to the line between x and y. Thus the Lipschitz bound is established for convex D.

For non-convex D, let d be the diameter of D and let D' be a disk with diameter $10d$ such that $\operatorname{dist}(\partial D, \partial D') > 5d$ and $D \subset D'$. Now the Lipschitz bound holds for the GFF on D', and thus it especially holds for $x, y \in D$ and $\varepsilon \in [\varepsilon_0, 5d]$. If we make the domain of the GFF larger, then the variance

$$\mathbb{E}[|X_\varepsilon(x) - X_\delta(y)|^2] = 2\pi\|\rho_{x,\varepsilon} - \rho_{y,\delta}\|^2_{H^{-1}(D)}$$

can only get larger (by (3.5.3) this comes down to using the definition of ξ_D) and thus it follows that the Lipschitz bound holds for the GFF in D when $x, y \in D$, $\varepsilon \in [\varepsilon_0, 5d]$.

Now by using the formula for Gaussian moments we get

$$\mathbb{E}\left[|X_\varepsilon(x) - X_\delta(y)|^\alpha\right] \leqslant C(\alpha)K^{\alpha/2}\left[|x-y| + |\varepsilon - \delta|\right]^{\alpha/2}.$$

To finish, apply the Kolmogorov continuity theorem 2.4.4 with $d = 3$, $\beta = \alpha/2 - 3$ and large α. The $\varepsilon, \delta > \varepsilon_0$ condition leads to the fact that we get continuous modifications only on compact subsets of $(x \in D, 0 < \varepsilon_0 < \operatorname{dist}(x, \partial D))$. □

Theorem 3.5.6. (Brownian motion from the circle average) *For any* $z \in D$ *set* $t_0 = \inf\{t \geqslant 0 : B(z, e^{-t}) \subset D\}$. *Then define* $B_t(z) = X_{e^{-t}}(z)$ *for any* $t \geqslant t_0$. *Then the stochastic process*

$$\mathscr{B}_t(z) := B_{t+t_0}(z) - B_{t_0}(z)\,, \quad (t \geqslant t_0)\,,$$

is the standard Brownian motion.

Proof. We have to show that (i) $(\mathscr{B}_t(z))_{t \geqslant t_0}$ is a Gaussian process, (ii) $\mathbb{E}\mathscr{B}_t(z) = 0$ and $\mathbb{E}[\mathscr{B}_t(z)\mathscr{B}_s(z)] = \min\{t, s\}$, and finally (iii) $t \mapsto \mathscr{B}_t(z)$ is almost surely continuous.

(i). Both $B_{t+t_0}(z)$ and $B_{t_0}(z)$ are Gaussians. In the GFF language they are $X(\rho_{z,e^{-t-t_0}})$ and $X(\rho_{z,e^{-t_0}})$. Thus their difference is just X acting on the generalized function $\rho_{z,e^{-t-t_0}} - \rho_{z,e^{-t_0}}$, which belongs to $H^{-1}(D)$. Thus $\mathscr{B}_t(z) \sim N(0, \|\rho_{z,e^{-t-t_0}} - \rho_{z,e^{-t_0}}\|^2_{H^{-1}})$.

(ii). Clearly $\mathbb{E}\mathscr{B}_t(z) = 0$ holds. The covariance is (recall (3.5.2))

$$\begin{aligned}
\mathbb{E}[\mathscr{B}_t(z)\mathscr{B}_s(z)] &= \mathbb{E}[(X_{e^{-t-t_0}}(z) - X_{e^{-t_0}}(z))(X_{e^{-s-t_0}}(z) - X_{e^{-t_0}}(z))] \\
&= \log \frac{1}{\max\{e^{-t-t_0}, e^{-s-t_0}\}} - \log \frac{1}{\max\{e^{-t-t_0}, e^{-t_0}\}} \\
&\quad - \log \frac{1}{\max\{e^{-t_0}, e^{-s-t_0}\}} + \log \frac{1}{e^{-t_0}} \\
&= \min\{t + t_0, s + t_0\} - \min\{t + t_0, t_0\} - \min\{t_0, s + t_0\} + t_0 \\
&= \min\{t, s\} + t_0 - t_0 - t_0 + t_0 \\
&= \min\{t, s\}\,.
\end{aligned}$$

(iii). This follows from Proposition 3.5.5. □

3.6 Additional remarks on the Gaussian free field

Now we have established all the results related to the GFF that we will apply in Chapters 4 and 5. In this section we review a few properties of the GFF that we strictly speaking do not need later in the thesis, but will be useful for building a more complete understanding of the GFF.

3.6.1 One dimensional Gaussian free field

It is interesting to see how the construction of X behaves when we switch to one-dimensional domains. According to the Introduction the one-dimensional GFF should be a Brownian motion, so this section also works as a consistency check for what we have done so far. The discussion in this section will be semi-heuristic and we will not use any of these results later.

We again start by trying to define a probability measure

$$\exp\left(-\frac{1}{2}\int_0^b |X'(x)|^2\, dx\right) DX$$

on a "space of maps $(0,b) \to \mathbb{R}$". The covariance kernel of X is given by the Green function of the problem

$$\varphi''(x) = -2\pi\rho(x)\,, \quad \varphi(0) = \varphi(b) = 0\,.$$

We obtain the Green function by solving $\varphi''(x) = -2\pi\delta(y-x)$, where δ is the Dirac delta, and the answer is

$$G_y(x) = (y-x)\mathbf{1}_{[0,\infty)}(x-y) + \frac{b-y}{b}x\,.$$

Hence in the case $b=1$ we have

$$\mathbb{E}[X(x)X(y)] = (y-x)\mathbf{1}_{[0,\infty)}(x-y) + (1-y)x = \begin{cases} (1-y)x\,, & x \leqslant y \\ y(1-x)\,, & x > y\,. \end{cases}$$

Thus $\mathbb{E}[X(x)^2] = x(1-x)$ and the covariance is that of a Brownian bridge. In the case $b=\infty$ we have

$$\mathbb{E}[X(x)X(y)] = (y-x)\mathbf{1}_{[0,\infty)}(x-y) + x = \begin{cases} x\,, & x \leqslant y\,, \\ y\,, & x > y\,. \end{cases}$$

This is the same as the covariance of the standard Brownian motion, exactly what we expected.

Note that the pointwise variance of X is well-defined in one dimension. The L^2-normalized eigenfunctions of the Laplacian in one dimension with boundary conditions $\varphi(0) = \varphi(b) = 0$ are

$$\varphi_n(x) = \sqrt{\frac{2}{b}}\sin\left(\frac{2\pi n}{b}x\right)\,.$$

The series representation is the random sine series

$$X(x) = \sqrt{\frac{2}{b}}\sum_{n=1}^{\infty} X_n \frac{b}{2\pi n}\sin\left(\frac{2\pi n}{b}x\right) = \sqrt{\frac{b}{2\pi}}\sum_{n=1}^{\infty}\frac{X_n}{n}\sin\left(\frac{2\pi n}{b}x\right)\,,$$

where $(X_n)_{n\in\mathbb{N}}$ are iid. The L^2-norm is

$$\|X\|^2_{L^2(0,b)} = \frac{b^2}{4\pi^2}\sum_{n=1}^{\infty}\frac{|X_n|^2}{n^2}$$

and since the expectation is finite, the L^2 norm is finite almost surely.

The Markov property states that for any subinterval $I \subset (0,b)$ we have the decomposition to two independent parts

$$X = X_I + \varphi_I$$

where X_I is the GFF on I and φ_I is an affine function in I (in one dimension harmonic functions are affine). This implies that X has independent Gaussian increments.

3.6.2 Gaussian free field from Minlos' theorem

We outline an alternative approach to defining the Gaussian free field. We use the Minlos' theorem.

Theorem 3.6.1. (Minlos' theorem) *Let E be a nuclear space*[3]*. Any continuous positive-definite linear functional $\Lambda : E \to \mathbb{R}$ satisfying $\Lambda(0) = 1$ is the Fourier transform of a probability measure on $E^\star$ defined on the cylindrical sigma-algebra.*

Proof. See [23]. □

We choose our nuclear space to be $H_0^\infty(D) := \bigcap_{n\geqslant 1} H_0^n(D)$ and the function $\Lambda : H_0^\infty(D) \to \mathbb{R}$ by

$$\Lambda(f) := \exp\left(\frac{1}{4\pi}\int_D f(x)\Delta^{-1}f(x)\,dx\right) = \exp\left(-\frac{1}{4\pi}\langle f, f\rangle_{H^{-1}}\right).$$

Note that on $H_0^\infty(D)$ the weak Laplacian Δ is invertible. For $f \in H_0^n(D)$, $n \geqslant 1$, the function $f \mapsto \langle f, f\rangle_{H^{-1}}$ is continuous[4] and thus the Minlos' theorem states that there is a cylinder probability measure ν on $(H_0^\infty(D))^\star \supset H^{-1}(D)$ such that the characteristic function of ν is

$$\int e^{i\langle \rho, f\rangle}\,\nu(d\rho) = e^{\frac{1}{4\pi}\int_D f(x)\Delta f(x)\,dx} = e^{-\frac{1}{4\pi}\|f\|_{H_0^1}}, \quad f \in H_0^\infty(D).$$

This is the characteristic function of a Gaussian measure with covariance operator Δ^{-1} and immediately implies that if X has distribution ν, then $\langle X, \rho\rangle$ has distribution $\mathcal{N}(0, \|\rho\|_{H^{-1}}^2)$ for every $\rho \in H_0^\infty(D)$. Thus ν is a natural interpretation of the measure (3.1.1). However, this is not the approach we take because if X has distribution ν, then $\langle X, \rho\rangle$ does not a priori make sense for all $\rho \in H^{-1}(D)$. We want, for example, the circle average of X to be defined, which is naturally defined by $\langle X, \rho_{x,\varepsilon}\rangle$, where $\rho_{x,\varepsilon}$ is the uniform unit measure on $\partial B(x, \varepsilon)$.

3.6.3 Schwinger functions

The Wick's formula 2.4.7 yields the following result.

Theorem 3.6.2. (Wick's formula for the GFF) *Let $\rho_1, \dots, \rho_n \in H^{-1}(D)$. Then*

$$\mathbb{E}\left[X(\rho_1)X(\rho_2)\dots X(\rho_n)\right] = \sum_{p\in P_n}\prod_{\{\alpha,\beta\}\in p}\mathbb{E}[X(\rho_\alpha)X(\rho_\beta)],$$

where P_n denotes the collection of pair-partitions of the set $\{1, 2, \dots, n\}$.

[3] We will not need the full generality of this theorem and thus one can think of nuclear spaces as spaces where we have a "weaker notion of smoothness" of functions. For example, our application will be in the case $E = \cap_{n\geqslant 1} H_0^n(D)$, which is nuclear.

[4] We take this for granted to avoid the theory of topological vector spaces ($H_0^\infty(D)$ is not a normed space). We will not later use the result given by this remark.

Now we consider in which sense there exists functions $S_k : D^k \to \mathbb{R}$ which satisfy

$$\mathbb{E}\left[X(\rho_1)X(\rho_2)\dots X(\rho_n)\right] = \int_{D^k} S_k(x_1,\dots,x_k)\prod_{i=1}^{n}\rho_i(x)\,dx_i\,.$$

The functions S_k are called the Schwinger functions. We already know that if $\rho_1, \rho_2 \in H^{-1}(D)$ are sufficiently regular pointwise define functions, then

$$\mathbb{E}\left[X(\rho_1)X(\rho_2)\right] = 2\pi\int_{D^2} G_D(x,y)\rho_1(x)\rho_2(y)\,dx\,dy\,.$$

Thus $S_2(x,y) = G_D(x,y)$. Then the Wick's formula implies that

$$\int_{D^k} S_k(x_1,\dots,x_k)\prod_{i=1}^{n}\rho_i(x)\,dx_i = 2\pi\sum_{p\in P_k}\prod_{\{\alpha,\beta\}\in p}\int_{D^2} G_D(x,y)\rho_\alpha(x)\rho_\beta(y)\,dx\,dy$$

and thus

$$S_k(x_1,\dots,x_k) = \begin{cases} 0\,, & k \text{ odd}\,,\\ 2\pi\sum_{p\in P_k}\prod_{\{\alpha,\beta\}\in p} G_D(x_\alpha,x_\beta)\,, & k \text{ even}\,.\end{cases}$$

Chapter 4

Exponentiating the Gaussian free field

In this chapter we exponentiate the Gaussian free field. More precisely we look at a weak limit in probability of the family of measures

$$\varepsilon^{\gamma^2/2} \exp\left(\gamma X_\varepsilon(x)\right) dx\,, \quad (\varepsilon > 0)\,, \tag{4.0.1}$$

as $\varepsilon \to 0$. Here X_ε is a continuous version of the circle average of the Gaussian free field X. The weak limit in probability of (4.0.1) yields a measure which we formally denote by

$$e^{\gamma X(x)}\, dx\,.$$

Even though the notation suggests so, this measure is not absolutely continuous with respect to the Lebesgue measure.

The method we use for proving the convergence is due to Berestycki and was initially published in [4]. In this chapter we present a modified version of the proof, also due to Berestycki [3], complemented with details. Almost sure weak convergence along the sequence $\varepsilon_k = 2^{-k}$ was initially proven in [14]. This would have also followed from the work of Kahane [24][1], who used a different regularization method, together with the results found in [43] which say that the limiting measure is not dependent on the regularization method.

Throughout this chapter $D \subset \mathbb{R}^2$ will be an open and bounded domain with C^2 boundary.

Definition 4.0.3. *We define a family of regularized approximating measures $(M_{\gamma,\varepsilon})_{\varepsilon>0}$ by*

$$M_{\gamma,\varepsilon}(dx) = \varepsilon^{\gamma^2/2} \exp(\gamma X_\varepsilon(x))\, dx\,,$$

where $X_\varepsilon(x)$ is a jointly continuous version of the circle average.

Remark 4.0.4. *The multiplicative factor $\varepsilon^{\gamma^2/2}$ in $M_{\gamma,\varepsilon}$ is essentially due to normalization. Whenever $\partial B(x,\varepsilon) \subset D$, we have (recall formula (3.5.2) and Theorem 2.6.8)*

$$\mathbb{E} e^{\gamma X_\varepsilon(x)} = e^{\frac{\gamma^2}{2}(-\log\varepsilon + \log R(x,D))} = \varepsilon^{-\gamma^2/2} R(x,D)^{\gamma^2/2}\,. \tag{4.0.2}$$

Thus $\mathbb{E}[\varepsilon^{\gamma^2/2} e^{\gamma X_\varepsilon(x)}] = R(x,D)^{\gamma^2/2}$, so the expectation does not depend on ε.

[1] Jean-Pierre Kahane created a general theory of measures of the form $e^{\gamma X(z)}\, dz$ where X is a log-correlated Gaussian field. These measures are called Gaussian multiplicative chaoses. The Gaussian free field is an example of a log-correlated Gaussian field and thus the material in this chapter belongs to the realm of Kahane's theory.

Lemma 4.0.5. *For all Borel sets $A \subset D$ with* $\operatorname{dist}(A, \partial D) > 0$ *we have*

$$\mathbb{E}[M_{\gamma,\varepsilon}(S)] = \int_S R(x,D)^{\gamma^2/2}\, dx \in (0,\infty)$$

whenever $\varepsilon < \operatorname{dist}(A, \partial D)$.

Proof. Using Fubini's theorem and the previous remark

$$\mathbb{E}[M_{\gamma,\varepsilon}(S)] = \int_S \varepsilon^{\gamma^2/2}\mathbb{E}e^{\gamma X_\varepsilon(x)}\, dx = \int_S R(x,D)^{\gamma^2/2}\, dx\,.$$

The fact that this belongs to $(0,\infty)$ follows from properties of the conformal radius (Proposition 2.6.7). □

The circle-average $X_\varepsilon(x)$ can be defined for points x satisfying $\operatorname{dist}(x, \partial D) < \varepsilon$ by defining $X_\varepsilon(x) = X(\rho_{x,\varepsilon})$ where now $\rho_{x,\varepsilon}$ is understood to be the uniform measure on $\partial B(x,\varepsilon) \cap D$ with total mass

$$\rho_{x,\varepsilon}(D) = \frac{|\partial B(x,\varepsilon) \cap D|}{|\partial B(x,\varepsilon)|}\,.$$

However, for such points x it is harder to compute the variance $\mathbb{E}[X(\rho_{x,\varepsilon})^2]$. This is nothing too important since we are interested in the limit $\varepsilon \to 0$. Indeed, if the variables $M_{\gamma,\varepsilon}(S)$ are uniformly integrable, then write

$$S = \bigcup_{n\in\mathbb{N}} S_n$$

where $S_n = \{x \in S : \operatorname{dist}(x, \partial D) > 1/n\}$ and now

$$\lim_{\varepsilon\to 0}\mathbb{E}[M_{\gamma,\varepsilon}(S)] = \lim_{n\to\infty}\mathbb{E}[M_{\gamma,1/n}(S_n)] = \int_S R(x,D)^{\gamma^2/2}\, dx\,.$$

Our main goal is to show the weak convergence in probability of these measures for each $\gamma \in (0,2)$ towards a non-trivial measure. The case $\gamma < \sqrt{2}$, also called the L^2-phase, is easier and will be proven first. After this we give a proof that works for $\gamma \in [\sqrt{2}, 2)$.

Lemma 4.0.6. *For a fixed x, $(\varepsilon^{\gamma^2/2}e^{\gamma X_\varepsilon(x)})_{\varepsilon>0}$ forms a martingale with respect to the filtration $(\mathcal{F}_\varepsilon)_{\varepsilon>0}$, where $\mathcal{F}_\varepsilon = \sigma((X_\delta)_{\delta\geqslant\varepsilon})$.*

Proof. Let $\delta < \varepsilon$. Using the Markov property of X, we get the decomposition

$$X = \widetilde{X} + \varphi\,,$$

where $\widetilde{X}$ is a GFF on $B(x,\varepsilon)$, φ is harmonic on $B(x,\varepsilon)$ and $\widetilde{X}$ and φ are independent of each other. Then for the circle average we have

$$X_\delta(x) = \widetilde{X}_\delta(x) + \varphi(x)\,,$$

where we used the mean value property of harmonic functions. On the other hand,

$$X_\varepsilon(x) = \widetilde{X}_\varepsilon(x) + \varphi(x) = \varphi(x)\,,$$

since $\widetilde{X}$ vanishes on $\partial B(x,\varepsilon)$. Thus by (4.0.2)

$$\begin{aligned}
\mathbb{E}(e^{\gamma X_\delta(x)}|\mathcal{F}_\varepsilon) &= \mathbb{E}(e^{\gamma \widetilde{X}_\delta(x)+\gamma\varphi(x)}|\mathcal{F}_\varepsilon) = \mathbb{E}(e^{\gamma\varphi(x)}|\mathcal{F}_\varepsilon)\mathbb{E}e^{\gamma\widetilde{X}_\delta(x)} \\
&= \mathbb{E}(e^{\gamma X_\varepsilon(x)}|\mathcal{F}_\varepsilon)e^{\frac{\gamma^2}{2}[-\log\delta+\log R(x,B(x,\varepsilon))]} \\
&= e^{\gamma X_\varepsilon(x)}\left(\frac{R(x,B(x,\varepsilon))}{\delta}\right)^{\gamma^2/2}\,.
\end{aligned}$$

The result follows from observing that $R(x,B(x,\varepsilon)) = \varepsilon$. □

4.1 Convergence of the regularized measures

Recall that $(\Omega,\mathcal{F},\mathbb{P})$ denotes our underlying probability space and by $L^2(\Omega,\mathbb{P})$ we denote the set of random variables $\Omega\to\mathbb{R}$ which are square integrable. In the following discussion $S\subset D$ will be open and bounded and $I_\varepsilon := M_{\gamma,\varepsilon}(S)$. We will also adapt the notations

$$\begin{aligned}
\bar{X}_\varepsilon(x) &:= \gamma X_\varepsilon(x) - (\gamma^2/2)\mathbb{E}[X_\varepsilon(x)^2]\,,\\
\sigma(dx) &:= R(x,D)^{\gamma^2/2}\,dx\,.
\end{aligned}$$

4.1.1 The $\gamma < \sqrt{2}$ case

Proposition 4.1.1. *Let $\varepsilon > 0$ and $\delta/\varepsilon = c \in (0,1)$. Then we have the estimate*

$$\mathbb{E}[(I_\varepsilon - I_\delta)^2] \leqslant C\varepsilon^{2-\gamma^2}\,,$$

where C depends only on γ, S and c. Thus if $\gamma\in(0,\sqrt{2})$, then $(I_\varepsilon)_{\varepsilon>0}$ is Cauchy in $L^2(\Omega,\mathbb{P})$ and hence converges in $L^2(\Omega,\mathbb{P})$ (and in probability).

Proof. By Fubini's theorem

$$\begin{aligned}
\mathbb{E}[(I_\varepsilon - I_\delta)^2] &= \mathbb{E}\left[\left(\int_S \left(\varepsilon^{\gamma^2/2}e^{X_\varepsilon(x)} - \delta^{\gamma^2/2}e^{X_\delta(x)}\right)dx\right)^2\right]\\
&= \mathbb{E}\left[\left(\int_S e^{\bar{X}_\varepsilon(x)-\bar{X}_\delta(x)}\,\sigma(dx)\right)^2\right]\\
&= \int_{S\times S}\mathbb{E}\left[(e^{\bar{X}_\varepsilon(x)} - e^{\bar{X}_\delta(x)})(e^{\bar{X}_\varepsilon(y)} - e^{\bar{X}_\delta(y)})\right]\,\sigma(dx)\,\sigma(dy)\,.
\end{aligned}$$

Note that

$$(e^{\bar{X}_\varepsilon(x)} - e^{\bar{X}_\delta(x)})(e^{\bar{X}_\varepsilon(y)} - e^{\bar{X}_\delta(y)}) = e^{\bar{X}_\varepsilon(x)+\bar{X}_\varepsilon(y)}(1-e^{\bar{X}_\delta(x)-\bar{X}_\varepsilon(x)})(1-e^{\bar{X}_\delta(y)-\bar{X}_\varepsilon(y)})\,.$$

Let us now investigate the region where $|x-y| \geqslant 2\varepsilon$ for $(x,y) \in S \times S$. Define $U = B(x,\varepsilon) \cup B(y,\varepsilon)$. The Markov property of the GFF implies the existence of a decomposition into two independent terms $X = X_U + \varphi_U$ where X_U is the GFF on U and φ_U is harmonic in U. Now $X_\varepsilon(x) + X_\varepsilon(y) = \varphi_U(x) + \varphi_U(y)$ depends only on values of φ_U because X_U vanishes on ∂U. Since the balls $B(x,\varepsilon)$ and $B(y,\varepsilon)$ are disjoint, the restriction of X_U to one of these balls is independent of the restriction to the other ball (this is essentially Theorem 3.4.4 for the orthogonal subspaces $H^{-1}(B(x,\varepsilon))$ and $H^{-1}(B(y,\varepsilon))$). Thus

$$X_\delta(x) - X_\varepsilon(x) = \varphi_U(x) + (X_U)_\delta(x) - \varphi_U(x) = (X_U)_\delta(x)$$

and

$$X_\delta(y) - X_\varepsilon(y) = \varphi_U(x) + (X_U)_\delta(y) - \varphi_U(x) = (X_U)_\delta(y)$$

are independent of each other when $|x-y| \geqslant 2\varepsilon$. They are also independent of φ_U and thus independent of $X_\varepsilon(x) + X_\varepsilon(y)$. In the $|x-y| \geqslant 2\varepsilon$ region this independence implies that

$$\begin{aligned}&\mathbb{E}\left[e^{\bar{X}_\varepsilon(x)+\bar{X}_\varepsilon(y)}(1-e^{\bar{X}_\delta(x)-\bar{X}_\varepsilon(x)})(1-e^{\bar{X}_\delta(y)-\bar{X}_\varepsilon(y)})\right] \\ &= \mathbb{E}[e^{\bar{X}_\varepsilon(x)+\bar{X}_\varepsilon(y)}]\mathbb{E}[1-e^{\bar{X}_\delta(x)-\bar{X}_\varepsilon(x)}]\mathbb{E}[1-e^{\bar{X}_\delta(y)-\bar{X}_\varepsilon(y)}]\,.\end{aligned}$$

The latter two expectations vanish. This follows from a simple computation, or by applying the Markov property as in the proof of Lemma 4.0.6. This implies that the only non-zero contribution to the integral comes from the subset of $S \times S$ where $|x-y| < 2\varepsilon$.

Combining this result with Cauchy–Schwartz gives

$$\begin{aligned}\mathbb{E}[(I_\varepsilon - I_\delta)^2] &= \int_{|x-y|<2\varepsilon} \mathbb{E}\left[(e^{\bar{X}_\varepsilon(x)} - e^{\bar{X}_\delta(x)})(e^{\bar{X}_\varepsilon(y)} - e^{\bar{X}_\delta(y)})\right]\,\sigma(dx)\,\sigma(dy) \\ &\leqslant \int_{|x-y|<2\varepsilon} \sqrt{\mathbb{E}[(e^{\bar{X}_\varepsilon(x)} - e^{\bar{X}_\delta(x)})^2]\mathbb{E}[(e^{\bar{X}_\varepsilon(y)} - e^{\bar{X}_\delta(y)})^2]}\,\sigma(dx)\,\sigma(dy)\,.\end{aligned}$$

The $L^2(\Omega,\mathbb{P})$-norms in the integral can be estimated as

$$\begin{aligned}\sqrt{\mathbb{E}[(e^{\bar{X}_\varepsilon(x)} - e^{\bar{X}_\delta(x)})^2]} &= \|e^{\bar{X}_\varepsilon(x)} - e^{\bar{X}_\delta(x)}\|_{L^2(\Omega,\mathbb{P})} \leqslant \|e^{\bar{X}_\varepsilon(x)}\|_{L^2(\Omega,\mathbb{P})} + \|e^{\bar{X}_\delta(x)}\|_{L^2(\Omega,\mathbb{P})} \\ &= \sqrt{\mathbb{E}\left[e^{2\bar{X}_\varepsilon(x)}\right]} + \sqrt{\mathbb{E}\left[e^{2\bar{X}_\delta(x)}\right]} \\ &= \sqrt{\varepsilon^{-\gamma^2}R(x,D)^{\gamma^2/2}} + \sqrt{\delta^{-\gamma^2}R(x,D)^{\gamma^2/2}} \\ &= \sqrt{\varepsilon^{-\gamma^2}R(x,D)^{\gamma^2/2}} + \sqrt{(c\varepsilon)^{-\gamma^2}R(x,D)^{\gamma^2/2}} \\ &= (1+c^{-\gamma^2/2})\varepsilon^{-\gamma^2/2}R(x,D)^{\gamma^2/4}\,,\end{aligned}$$

where $c = \delta/\varepsilon$. Since the conformal radii remain bounded on D (see Proposition 2.6.7), we may further estimate

$$\begin{aligned}\mathbb{E}[(I_\varepsilon - I_\delta)^2] &\leqslant (1+c^{-\gamma^2/2})^2\varepsilon^{-\gamma^2}\int_{|x-y|<2\varepsilon} R(x,D)^{\gamma^2/4}R(y,D)^{\gamma^2/4}\,\sigma(dx)\,\sigma(dy) \\ &\leqslant (1+c^{-\gamma^2/2})^2\varepsilon^{-\gamma^2}C(S,\gamma)\varepsilon^2 = C(S,\gamma)(1+c^{-\gamma^2/2})^2\varepsilon^{2-\gamma^2}\,,\end{aligned}$$

where $C(S,\gamma)$ is some constant depending on S and γ. This estimate implies that $(I_\varepsilon)_{\varepsilon>0}$ is Cauchy in $L^2(\Omega,\mathbb{P})$ whenever $\gamma\in(0,\sqrt{2})$. □

Proposition 4.1.2. *Let* $\varepsilon_k=2^{-k}$ *and* $\gamma\in(0,\sqrt{2})$. *The sequence* $(I_{\varepsilon_k})_{k\in\mathbb{N}}$ *converges almost surely.*

Proof. The previous proposition implies the bound

$$\mathbb{E}[(I_{\varepsilon_k}-I_{\varepsilon_{k+1}})^2]\leqslant C(2^{-k})^{2-\gamma^2}=\frac{C}{2^{k(2-\gamma^2)}}\,.$$

Chebysev's inequality yields

$$\mathbb{P}(\{|I_{\varepsilon_k}-I_{\varepsilon_{k+1}}|\geqslant\alpha_k\})\leqslant\frac{C}{\alpha_k^2 2^{k(2-\gamma^2)}}\,.$$

We would like to apply the Borel–Cantelli lemma, so we want to choose the sequence $(\alpha_k)_{k\in\mathbb{N}}$ in such a way that the series

$$\sum_{k=1}^{\infty}\frac{1}{\alpha_k^2 2^{k(2-\gamma^2)}}$$

converges, while also $\sum_k\alpha_k<\infty$. We may choose α_k to be $\beta^{k/2}$ where $\beta\in(1/2^{2-\gamma^2},1)\neq\emptyset$ because now

$$\sum_{k=1}^{\infty}\frac{1}{\alpha_k^2 2^{k(2-\gamma^2)}}=\sum_{k=1}^{\infty}\left(\frac{1}{2^{2-\gamma^2}\beta}\right)^k<\infty\,.$$

Also $\beta^{k/2}\to0$ as $k\to\infty$ because $\beta<1$. Now by Borel–Cantelli

$$\mathbb{P}\left(\limsup_{k\to\infty}\{|I_{\varepsilon_k}-I_{\varepsilon_{k+1}}|\geqslant\alpha_k\}\right)=0\,,$$

which means that almost surely there exists n such that for all $k\geqslant n$ we have

$$|I_{\varepsilon_k}-I_{\varepsilon_{k+1}}|<\alpha_k\,.$$

Then for any two indices $j>i\geqslant n$ we almost surely have

$$|I_{\varepsilon_i}-I_{\varepsilon_j}|\leqslant\sum_{k=i}^{j-1}|I_{\varepsilon_k}-I_{\varepsilon_{k+1}}|<\sum_{k=1}^{j-1}\alpha_k\leqslant\sum_{k=i}^{\infty}\alpha_k\,.$$

The upper bound goes to 0 as $i\to\infty$, which means that

$$\lim_{i,j\to\infty}|I_{\varepsilon_i}-I_{\varepsilon_j}|=0$$

almost surely. Hence there exists an almost sure limit of $(I_{\varepsilon_k})_{k\in\mathbb{N}}$. □

Theorem 4.1.3. *For $\gamma \in (0, \sqrt{2})$ the family of measures $(M_{\gamma,2^{-k}})_{k\in\mathbb{N}}$ converges weakly almost surely as $\varepsilon \to 0$.*

Proof. For every open set $S \subset D$ we have almost sure convergence of $M_{\gamma,2^{-k}}(S)$. Especially $(M_{\gamma,2^{-k}}(D))_{k\in\mathbb{N}}$ is a bounded sequence almost surely, which implies that

$$\left|\int_D f(x) M_{\gamma,2^{-k}}(dx)\right| \leqslant C\|f\|_{L^\infty}\,, \quad f \in C_c(D)\,,$$

almost surely. Conditioning on this almost sure event $(M_{\gamma,2^{-k}})_{k\in\mathbb{N}}$ forms a bounded sequence with respect to the operator norm in $C_c(D)^\star$. The Banach–Alaoglu Theorem 2.2.5 then implies that there exists a weak-$\star$ convergent subsequence of $(M_{\gamma,2^{-k}})_{k\in\mathbb{N}}$, which we denote by $(M_{\gamma,x_k})_{k\in\mathbb{N}}$. Now we define $M_\gamma \in C_c(D)^\star$ to be the limit of this weak-$\star$ convergent subsequence. By the Riesz representation theorem 2.2.7 the measure M_γ is a positive Radon measure. What we have established is that almost surely for all $f \in C_c(D)$

$$\lim_{k\to\infty} \int_D f(x) M_{\gamma,x_k}(dx) = \int_D f(x) M_\gamma(dx)\,.$$

The above convergence can be extended to all $C_b(D)$-functions (continuous and bounded)[2]. Hence we conclude that $M_{\gamma,x_k} \to M_\gamma$ weakly almost surely.

Next we show that the whole sequence $(M_{\gamma,2^{-k}})_{k\in\mathbb{N}}$ converges to M_γ. Fix a function $f \in C_b(D)$ and a sequence $(f_n)_{n\in\mathbb{N}}$ of functions such that

$$\lim_{n\to\infty} \sup_{x\in D} |f(x) - f_n(x)| = 0\,,$$

where the functions f_n are of the form

$$f_n(x) = \sum_{i=1}^{k(n)} c_{n,i} \mathbf{1}(x \in S_{n,i})\,,$$

and $S_{n,i} \subset D$ are open sets. Then

$$\begin{aligned}\left|\int_D f(x)\, M_{\gamma,2^{-k}}(dx) - \int_D f(x)\, M_\gamma(dx)\right| \leqslant &\int_D |f(x) - f_n(x)| M_{\gamma,2^{-k}}(dx) \\ &+ \left|\int_D f_n(x)(M_{\gamma,2^{-k}}(dx) - M_\gamma(dx))\right| \\ &+ \int_D |f(x) - f_n(x)| M_\gamma(dx)\,.\end{aligned}$$

[2]Since $M_{\gamma,x_k} \to M_\gamma$ weakly, the sequence of measures $(M_{\gamma,x_k})_{k\in\mathbb{N}}$ is tight, that is, for each $\delta > 0$ there exists a compact set $K_\delta \subset D$ such that $M_{\gamma,x_k}(D \setminus K_\delta) < \delta$ for all k. This implies that $C_c(D)$ is dense in $C_b(D)$ with respect to the $L^1(D, M_{\gamma,x_k})$-topology for all k. The claim follows.

The first and the last integrals vanish in the limit $n \to \infty$. The second term can be written in the form

$$\left| \int_D f_n(x)(M_{\gamma,2^{-k}}(dx) - M_\gamma(dx)) \right| = \left| \sum_{i=1}^{k(n)} c_{n,i} \big(M_{\gamma,2^{-k}}(S_{n,i}) - M_\gamma(S_{n,i}) \big) \right|$$
$$\leqslant \sum_{i=1}^{k(n)} |c_{n,i}| \left| M_{\gamma,2^{-k}}(S_{n,i}) - M_\gamma(S_{n,i}) \right| .$$

We can assume that the sets $S_{n,i}$ are taken from some countable collection of open sets that generates the topology of D, say rectangles of the form $I_1 \times I_2$ where the intervals I_i have rational endpoints. We know that a subsequence of $(M_{\gamma,x_k}|_S)_{k\in\mathbb{N}}$ converges weakly to some measure $\widetilde{M}_\gamma$ on S (same argument as above) and we also know that for all $f \in C_c(S)$

$$\lim_{k\to\infty} \int_S f(x) M_{\gamma,x_k}|_S(dx) = \int_S f(x) M_\gamma(dx)$$

so we also have $M_{\gamma,x_k}|_S \to M_\gamma|_S$ weakly. Thus $\widetilde{M}_\gamma = M_\gamma|_S$ and especially $M_{\gamma,x_k}(S) \to M_\gamma(S)$ almost surely. Since the whole sequence $(M_{\gamma,2^{-k}}(S))_{k\in\mathbb{N}}$ converges almost surely, we must have $M_{\gamma,2^{-k}}(S) \to M_\gamma(S)$ almost surely. Then since $M_{\gamma,2^{-k}}(I_1 \times I_2) \to M_\gamma(I_1 \times I_2)$ almost surely, we can assume that this almost sure convergence occurs simultaneously for all our (countable amount of) rational rectangles.

In conclusion, for all $\delta > 0$ there exist a threshold n_δ such that for $n > n_\delta$ we have

$$\left| \int_D f(x)\, M_{\gamma,2^{-k}}(dx) - \int_D f(x)\, M_\gamma(dx) \right| \leqslant \frac{\delta}{3} + \sum_{i=1}^{k(n)} |c_{n,i}| \left| M_{\gamma,2^{-k}}(S_{n,i}) - M_\gamma(S_{n,i}) \right| + \frac{\delta}{3}$$

and there also exists a threshold k_δ such that for $k > k_\delta$ we have

$$\sum_{i=1}^{k(n)} |c_{n,i}| \left| M_{\gamma,2^{-k}}(S_{n,i}) - M_\gamma(S_{n,i}) \right| < \frac{\delta}{3} .$$

Thus $M_{\gamma,2^{-k}} \to M_\gamma$ weakly almost surely. □

Theorem 4.1.4. *The family of measures $(M_{\gamma,\varepsilon})_{\varepsilon>0}$ converges to M_γ weakly in probability as $\varepsilon \to 0$.*

Proof. Fix a function $f \in C_b(D)$ and a sequence $(\varepsilon_n)_{n\in\mathbb{N}}$ such that $\varepsilon_n \searrow 0$ as $n \to \infty$. We define the random variables

$$Y(n) = \int_D f(x)\, M_{\gamma,\varepsilon_n}(dx) .$$

Showing that $Y(n) \to \int f\, dM_\gamma$ in probability is equivalent to showing that for every subsequence $(\varepsilon_{n_i})_{i\in\mathbb{N}}$ of $(\varepsilon_n)_{n\in\mathbb{N}}$ there exists a further subsequence $(\varepsilon_{n_{i_j}})_{j\in\mathbb{N}}$ for which

$$\lim_{j\to\infty} Y(n_{i_j}) = \int_D f(x)\, M_\gamma(dx) \tag{4.1.1}$$

almost surely.

By Proposition 4.1.1 we have that $M_{\gamma,\varepsilon_n}(D) \to M_\gamma(D)$ in probability. Thus there exists a subsequence $(M_{\gamma,\varepsilon_{n_i}}(D))_{i\in\mathbb{N}}$ that converges to $M_\gamma(D)$ almost surely. Then the Banach–Alaoglu theorem implies (see the beginning of the proof of Proposition 4.1.3) that there exists a further subsequence $(\varepsilon_{n_{i_j}})_{j\in\mathbb{N}}$ such that along this sequence the measures $M_{\gamma,\varepsilon}$ converge weakly to M_γ almost surely. Thus each subsequence of $(\varepsilon_n)_{n\in\mathbb{N}}$ has a further subsequence for which (4.1.1) holds almost surely. This is equivalent with $Y(n)$ converging to $\int_D f\, dM_\gamma$ in probability, that is, M_{γ,ε_n} to M_γ weakly in probability. Since this holds for all sequences $\varepsilon_n \searrow 0$, it follows that $(M_{\gamma,\varepsilon})_{\varepsilon>0}$ converges to M_γ weakly in probability. □

4.1.2 The $\gamma \geqslant \sqrt{2}$ case

Now we start working towards the proof of convergence in the $\gamma \in [\sqrt{2}, 2)$ phase.

Definition 4.1.5. *We say a point $x \in D$ is α-thick if*

$$\liminf_{\varepsilon\to 0} \frac{X_\varepsilon(x)}{\log(1/\varepsilon)} = \alpha\,.$$

We also define

$$G_\varepsilon^\alpha(x) = \{X_\varepsilon(x) \leqslant \alpha \log(1/\varepsilon)\}\,,$$

which is the event that the point x is not "too thick" at scale ε. The problem with the $\gamma \geqslant \sqrt{2}$ case is that some very thick points make the second moment of I_ε grow too large. The next lemma provides an estimate for the expectation of $e^{\bar{X}_\varepsilon(x)}$ when we remove these thick points.

Lemma 4.1.6. *For $\alpha > \gamma$ we have*

$$\mathbb{E}[e^{\bar{X}_\varepsilon(x)} \mathbf{1}_{G_\varepsilon^\alpha(x)}] \geqslant 1 - p_\alpha(\varepsilon)\,,$$

where $p_\alpha(\varepsilon) \to 0$ polynomially fast as $\varepsilon \to 0$. The same estimate holds if $\bar{X}_\varepsilon(x)$ is replaced with $\bar{X}_{\varepsilon/2}(x)$.

Proof. Define a new probability measure $\mathbb{Q}$ on Ω by

$$d\mathbb{Q} = e^{\bar{X}_\varepsilon(x)}\, d\mathbb{P}\,.$$

Define $G^\sigma(x) = \langle \sigma, G_D(x,\cdot)\rangle$ where $\sigma \in H^{-1}(D)$. Now by Girsanov's theorem for Gaussian fields 2.5.4

$$\begin{aligned}\mathbb{E}[e^{\bar{X}_\varepsilon(x)} \mathbf{1}_{G_\varepsilon^\alpha(x)}] &= \mathbb{E}[\mathbf{1}(X(\rho_{x,\varepsilon}) \leqslant \alpha \log(1/\varepsilon) - \gamma\langle \rho_{x,\varepsilon}, G^{\rho_{x,\varepsilon}}\rangle)] \\ &= \mathbb{E}[\mathbf{1}(X_\varepsilon(x) \leqslant \alpha \log(1/\varepsilon) - \gamma 2\pi \|\rho_{x,\varepsilon}\|_{H^{-1}}^2)]\,.\end{aligned}$$

Then by 3.5.3 we have $2\pi\|\rho_{x,\varepsilon}\|_{H^{-1}}^2 = \log(1/\varepsilon) + \log R(x,D)$. Thus by the standard estimate for Gaussian variables $Y \sim \mathcal{N}(0,\sigma^2)$

$$\mathbb{P}(Y \leqslant t) \geqslant 1 - \frac{1}{\sqrt{2\pi}} \frac{\sigma}{t} e^{-\frac{t^2}{2\sigma^2}},$$

where $t > 0$, we get

$$\begin{aligned}\mathbb{E}[e^{\bar{X}_\varepsilon(x)} \mathbf{1}_{G_\varepsilon^\alpha(x)}] &= \mathbb{P}(X_\varepsilon(x) \leqslant (\alpha-\gamma)\log(1/\varepsilon) - \gamma \log R(x,D)) \\ &\geqslant 1 - \frac{1}{\sqrt{2\pi}} \frac{\sqrt{\mathbb{E}[X_\varepsilon(x)^2]}}{t(\varepsilon)} \exp\left(-\frac{t(\varepsilon)^2}{2\mathbb{E}[X_\varepsilon(x)^2]}\right),\end{aligned}$$

where $\mathbb{E}[X_\varepsilon(x)^2] = \log(1/\varepsilon) + \log R(x,D)$ and $t(\varepsilon) = (\alpha-\gamma)\log(1/\varepsilon) - \gamma \log R(x,D)$. The claim follows. When $\bar{X}_\varepsilon(x)$ is replaced by $\bar{X}_{\varepsilon/2}(x)$, the same argument works. □

Now we see that the thick points do not contribute significantly to I_ε. We can therefore safely remove them. Fix $\alpha > \gamma$ and define

$$J_\varepsilon = \int_S e^{\bar{X}_\varepsilon(x)} \mathbf{1}_{G_\varepsilon^\alpha(x)}\, \sigma(dx), \quad J'_{\varepsilon/2} = \int_S e^{\bar{X}_{\varepsilon/2}(x)} \mathbf{1}_{G_\varepsilon^\alpha(x)}\, \sigma(dx).$$

The reason for these definitions is that Lemma 4.1.6 implies

$$\begin{aligned}\mathbb{E}|I_\varepsilon - J_\varepsilon| &= \int_S \mathbb{E}\left[e^{\bar{X}_\varepsilon(x)}(1 - \mathbf{1}_{G_\varepsilon^\alpha(x)})\right] \sigma(dx) \leqslant \int_S \sigma(dx) - \int_S (1 - p_\alpha(\varepsilon))\, \sigma(dx) \\ &= p_\alpha(\varepsilon)\sigma(S),\end{aligned}$$

which tends to 0 in the limit $\varepsilon \to 0$. The same holds for $\mathbb{E}|I_{\varepsilon/2} - J'_{\varepsilon/2}|$.

Proposition 4.1.7. *We have the estimate $\mathbb{E}[(J_\varepsilon - J'_{\varepsilon/2})^2] < \varepsilon^r$ for some $r > 0$. It follows that $(I_\varepsilon)_{\varepsilon>0}$ converges in $L^1(\Omega,\mathbb{P})$ (and in probability). Along the sequence $\varepsilon_k = 2^{-k}$ the convergence occurs almost surely.*

Proof. Define $\delta = \varepsilon/2$. Similarly to the $\gamma < \sqrt{2}$ case, we obtain

$$\mathbb{E}[(J_\varepsilon - J'_{\varepsilon/2})^2] \leqslant \int_{|x-y|<2\varepsilon} \sqrt{\mathbb{E}[((e^{\bar{X}_\varepsilon(x)} - e^{\bar{X}_\delta(x)})\mathbf{1}_{G_\varepsilon^\alpha(x)})^2]\mathbb{E}[((e^{\bar{X}_\varepsilon(y)} - e^{\bar{X}_\delta(y)})\mathbf{1}_{G_\varepsilon^\alpha(y)})^2]}\, \sigma(dx)\, \sigma(dy).$$

Then we estimate

$$\begin{aligned}\sqrt{\mathbb{E}[((e^{\bar{X}_\varepsilon(x)} - e^{\bar{X}_\delta(x)})\mathbf{1}_{G_\varepsilon^\alpha(x)})^2]} &= \|(e^{\bar{X}_\varepsilon(x)} - e^{\bar{X}_\delta(x)})\mathbf{1}_{G_\varepsilon^\alpha(x)}\|_{L^2(\Omega,\mathbb{P})} \\ &\leqslant \|e^{\bar{X}_\varepsilon(x)}\mathbf{1}_{G_\varepsilon^\alpha(x)}\|_{L^2(\Omega,\mathbb{P})} + \|e^{\bar{X}_\delta(x)}\mathbf{1}_{G_\varepsilon^\alpha(x)}\|_{L^2(\Omega,\mathbb{P})} \\ &= \sqrt{\mathbb{E}[e^{2\bar{X}_\varepsilon(x)}\mathbf{1}_{G_\varepsilon^\alpha(x)}]} + \sqrt{\mathbb{E}[e^{2\bar{X}_\delta(x)}\mathbf{1}_{G_\varepsilon^\alpha(x)}]}.\end{aligned}$$

By Girsanov's theorem

$$\begin{aligned}
\mathbb{E}[e^{2\bar{X}_\varepsilon(x)}\mathbf{1}_{G_\varepsilon^\alpha(x)}] &= \varepsilon^{-\gamma^2}R(x,D)^{\gamma^2}\mathbb{P}\big(X_\varepsilon(x)+2\gamma 2\pi\|\rho_{x,\varepsilon}\|_{H^{-1}}^2 \leqslant \alpha\log(1/\varepsilon)\big)\\
&= \varepsilon^{-\gamma^2}R(x,D)^{\gamma^2}\mathbb{P}\big(X_\varepsilon(x)\leqslant(\alpha-2\gamma)\log(1/\varepsilon)-2\gamma\log R(x,D)\big)\,,\\
\mathbb{E}[e^{2\bar{X}_\delta(x)}\mathbf{1}_{G_\varepsilon^\alpha(x)}] &= \delta^{-\gamma^2}R(x,D)^{\gamma^2}\mathbb{P}\big(X_\varepsilon(x)+2\gamma 2\pi\|\rho_{x,\delta}\|_{H^{-1}}^2 \leqslant \alpha\log(1/\varepsilon)\big)\\
&= \delta^{-\gamma^2}R(x,D)^{\gamma^2}\mathbb{P}\big(X_\varepsilon(x)\leqslant(\alpha-2\gamma)\log(1/\varepsilon)-2\gamma(\log R(x,D)+\log 2)\big)
\end{aligned}$$

These can be estimated by using the standard estimate for a Gaussian variable $Y\sim\mathcal{N}(0,\sigma^2)$ (we assume that $\alpha<2\gamma$ and will later choose α close to γ. Because of this, we will assume that $t<0$ in the following estimate)

$$\begin{aligned}
\mathbb{P}(Y\leqslant t) &= \frac{1}{\sqrt{2\pi\sigma^2}}\int_{-\infty}^{t}e^{-\frac{x^2}{2\sigma^2}}\,dx = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\frac{t}{\sigma}}e^{-\frac{x^2}{2}}\,dx = \frac{1}{\sqrt{2\pi}}\int_{-\frac{t}{\sigma}}^{\infty}e^{-\frac{x^2}{2}}\,dx\\
&\leqslant \frac{1}{\sqrt{2\pi}}\frac{\sigma}{-t}e^{-\frac{t^2}{2\sigma^2}}\,.
\end{aligned}$$

Then a straightforward computation shows that

$$\begin{aligned}
\mathbb{P}(X_\varepsilon(x)\leqslant(\alpha-2\gamma)\log(1/\varepsilon)-2\gamma\log R(x,D)) &\leqslant \mathcal{O}(1)\exp\left(-\frac{(2\gamma-\alpha)^2\log 1/\varepsilon}{2}\right)\\
&= \mathcal{O}(1)\varepsilon^{\frac{(2\gamma-\alpha)^2}{2}}
\end{aligned}$$

and similarly for the other term. Hence we obtain

$$\sqrt{\mathbb{E}[((e^{\bar{X}_\varepsilon(x)}-e^{\bar{X}_\delta(x)})\mathbf{1}_{G_\varepsilon^\alpha(x)})^2]\mathbb{E}[((e^{\bar{X}_\varepsilon(y)}-e^{\bar{X}_\delta(y)})\mathbf{1}_{G_\varepsilon^\alpha(y)})^2]}\leqslant\mathcal{O}(1)\varepsilon^{\frac{(2\gamma-\alpha)^2}{2}-\gamma^2}$$

so that

$$\begin{aligned}
\mathbb{E}[(J_\varepsilon-J'_{\varepsilon/2})^2] &\leqslant \mathcal{O}(1)\varepsilon^{\frac{(2\gamma-\alpha)^2}{2}-\gamma^2}\int_{|x-y|<2\varepsilon}\sigma(dx)\,\sigma(dy)\\
&= \mathcal{O}(1)\varepsilon^{2-\gamma^2+\frac{(2\gamma-\alpha)^2}{2}}\,.
\end{aligned}$$

We have already assumed that $\alpha<2\gamma$ and will later need the assumption that $\alpha>\gamma$. For $\gamma\in[\sqrt{2},2)$ and $\alpha>\gamma$ close enough to γ the exponent of ε can be made positive[3]. Indeed, the zeros of $f(\alpha)=2-\gamma^2+\frac{(2\gamma-\alpha)^2}{2}$ are

$$\alpha_\pm = 2\gamma\pm\sqrt{2}\sqrt{\gamma^2-2}\,.$$

The inequality $\alpha_->\gamma$ holds for $\gamma\in[\sqrt{2},2)$ and thus whenever $\gamma\in[\sqrt{2},2)$, we can choose $\alpha\in(\gamma,\alpha_-)$ and for this α the exponent of ε is positive.

[3]For $\gamma=2$ this fails since now $\alpha>2$ and $2-2^2+(4-\alpha)^2/2<-2+2=0$.

Now recalling Lemma 4.1.6 we can estimate

$$\begin{aligned}\mathbb{E}|I_\varepsilon - I_{\varepsilon/2}| &\leqslant \mathbb{E}|I_\varepsilon - J_\varepsilon| + \mathbb{E}|J_\varepsilon - J'_{\varepsilon/2}| + \mathbb{E}|J'_{\varepsilon/2} - I_{\varepsilon/2}| \\ &\leqslant 2p_\alpha(\varepsilon)\sigma(S) + \sqrt{\mathbb{E}|J_\varepsilon - J'_{\varepsilon/2}|^2} \\ &\leqslant 2p_\alpha(\varepsilon)\sigma(S) + \varepsilon^{r/2},\end{aligned}$$

where $r = 2 - \gamma^2 + \frac{(2\gamma-\alpha)^2}{2} > 0$ for $\alpha \in (\gamma, 2\gamma)$ sufficiently close to γ. This implies that $(I_\varepsilon)_{\varepsilon>0}$ is Cauchy in $L^1(\Omega)$.

Next we prove the almost sure convergence along negative powers of two. Chebysev's inequality gives

$$\mathbb{P}(\{|I_{2^{-k}} - I_{2^{-k-1}}| \geqslant \alpha_k\}) \leqslant \frac{1}{\alpha_k}\mathbb{E}|I_{2^{-k}} - I_{2^{-k-1}}| \leqslant C\sum_{k=1}^{\infty}\frac{p_\alpha(2^{-k})}{\alpha_k} + C\sum_{k=1}^{\infty}\frac{1}{\alpha_k}2^{-k\frac{2-\gamma^2}{2} - k\frac{(2\gamma-\alpha)^2}{4}}.$$

Set $\alpha_k = \beta^k$. We want to choose $\beta < 1$ such that both the series above are finite. Recall that $p_\alpha(\varepsilon) = \mathcal{O}(\varepsilon^n)$ for some n. Thus for the first series to be finite it suffices have $\beta > 1/2$. For the second we must have

$$2^{\frac{\gamma^2-2}{2} - \frac{(2\gamma-\alpha)^2}{4}} < \beta.$$

Thus our choice of β will have to belong to both $(\frac{1}{2}, 1)$ and $(2^{\frac{\gamma^2-2}{2} - \frac{(2\gamma-\alpha)^2}{4}}, 1)$. For the latter interval to be non-empty we must have

$$\frac{\gamma^2-2}{2} - \frac{(2\gamma-\alpha)^2}{4} < 0.$$

This holds for $\alpha < 2\gamma - \sqrt{2\gamma^2-4}$ and $\alpha > 2\gamma + \sqrt{2\gamma^2-4}$. We have to assume the first bound since we have previously assumed that $\alpha < 2\gamma$. Note that for $\gamma \in [\sqrt{2}, 2)$ we have $2\gamma - \sqrt{2\gamma^2-4} > \gamma$. Hence we can choose $\alpha \in (\gamma, 2\gamma - \sqrt{2\gamma^2-4})$ and then $(2^{\frac{\gamma^2-2}{2} - \frac{(2\gamma-\alpha)^2}{4}}, 1) \neq \emptyset$. Thus we conclude that choosing

$$\alpha \in (\gamma, 2\gamma - \sqrt{2}\sqrt{\gamma^2-2})$$

makes it possible to choose $\beta \in (\frac{1}{2}, 1) \cap (2^{\frac{\gamma^2-2}{2} - \frac{(2\gamma-\alpha)^2}{4}}, 1)$. Now the rest of the claim follows as in the $\gamma < \sqrt{2}$ case. □

Theorem 4.1.8. *For $\gamma \in (0, 2)$, the sequence of measures $(M_{\gamma,\varepsilon})_{\varepsilon>0}$ converges weakly in probability. Along the sequence $\varepsilon_k = 2^{-k}$ the convergence occurs weakly almost surely.*

Proof. This follows from combining the above results with the proofs of Theorems 4.1.3 and 4.1.4. □

The limit measure can not be trivial with probability 1. This follows from the previous result (4.0.5) which says that $\mathbb{E}[M_{\gamma,\varepsilon}(S)] = \int_S \sigma(dx)$, since now

$$\mathbb{E}[M_\gamma(S)] = \lim_{\varepsilon\to 0} \mathbb{E}[M_{\gamma,\varepsilon}(S)] = \int_S \sigma(dx) > 0\,.$$

The limit can be taken by Proposition 4.1.1 in the $\gamma \in (0,\sqrt{2})$ phase and by Proposition 4.1.7 in the $\gamma \in [\sqrt{2},2)$ phase. In the next section we show that $M_\gamma(S) > 0$ almost surely for all open sets $S \subset D$.

4.2 Properties of the limit measure

The circle average approximations $(M_{\gamma,\varepsilon}(S))_{\varepsilon>0}$ do not form a martingale and thus we could not use the martingale convergence theorem to establish convergence as $\varepsilon \to 0$. Next we introduce a sequence of approximating measures to M_γ which form a martingale.

Proposition 4.2.1. *Let $(\eta_n)_{n\in\mathbb{N}}$ be the Laplace-eigenbasis (or any other basis consisting of smooth functions) of $H_0^1(D)$ and $(X_n)_{n\in\mathbb{N}}$ an iid sequence of standard Gaussians. Define*

$$X^n(x) = \sqrt{2\pi}\sum_{i=1}^n X_i\eta_i(x)$$

and

$$M_\gamma^n(S) = \int_S \exp\left(\gamma X^n(x) - \frac{\gamma^2}{2}\mathbb{E}[X^n(x)^2]\right)\sigma(dx)$$

for all Borel sets $S \subset D$. Then almost surely $\lim_{n\to\infty} M_\gamma^n(S) = M_\gamma(S)$.

Proof. Recall that the functions η_n are smooth and hence the pointwise values $\eta_n(x)$ are well-defined. The sequence $(M_\gamma^n(S))_{n\in\mathbb{N}}$ forms a positive martingale with respect to the filtration $(\mathcal{F}_n)_{n\in\mathbb{N}}$ generated by $(X_n)_{n\in\mathbb{N}}$. Indeed, for $n > k$ we have

$$\begin{aligned}\mathbb{E}(M_\gamma^n(S) \mid \mathcal{F}_k) &= \int_S e^{\gamma X^k(x) - \frac{\gamma^2}{2}\mathbb{E}[X^k(x)^2]}\mathbb{E}\left(e^{\gamma\sqrt{2\pi}\sum_{i=k+1}^n X_i\eta_i(x) - \frac{\gamma^2}{2}\sqrt{2\pi}\sum_{i=k+1}^n \mathbb{E}[X_i\eta_i(x)^2]}\right)\sigma(dx)\\ &= M_\gamma^k(S)\,.\end{aligned}$$

Now by the martingale convergence theorem there exists an almost sure limit $M_\gamma^*(S)$. It holds that $M_\gamma(S) = M_\gamma^*(S)$ almost surely, which we will prove next.

We will show that (i) $M_\gamma^*(S) \geqslant M_\gamma(S)$ almost surely and (ii) $\mathbb{E}M_\gamma(S) \geqslant \mathbb{E}M_\gamma^*(S)$. Then the claim follows.

(i). Define

$$X^{>n} := \sqrt{2\pi}\sum_{i=n+1}^\infty X_i\eta_i\,.$$

By Theorem 3.3.2 the series converges in $H^{-1}(D)$. Now for the circle averages we have

$$X_\varepsilon(x) = X_\varepsilon^n(x) + X_\varepsilon^{>n}(x)\,,$$

where we defined

$$X_\varepsilon^{>n}(x) := \sum_{i=n+1}^{\infty} X_i \langle \rho_{x,\varepsilon}, \eta_i \rangle\,.$$

Recall that $\rho_{x,\varepsilon}$ denotes the unit measure on $\partial B(x,\varepsilon)$ and the series converges in $L^2(\Omega,\mathbb{P})$ and almost surely by the martingale convergence theorem (same argument as in the proof of Theorem 2.5.1).

Hence

$$\varepsilon^{\gamma^2/2} e^{\gamma X_\varepsilon(x)} = \varepsilon^{\gamma^2/2} e^{\gamma X_\varepsilon^n(x)} e^{\gamma X_\varepsilon^{>n}(x)}\,.$$

Integrating and conditioning gives

$$\begin{aligned}\mathbb{E}(M_{\gamma,\varepsilon}(S) \mid \mathcal{F}_n) &= \int_S \varepsilon^{\gamma^2/2} e^{\gamma X_\varepsilon^n(x)} \mathbb{E} e^{\gamma X_\varepsilon^{>n}(x)}\, dx = \int_S \varepsilon^{\gamma^2/2} e^{\gamma X_\varepsilon^n(x)} e^{\frac{\gamma^2}{2}(\mathbb{E}[X_\varepsilon(x)^2] - \mathbb{E}[X_\varepsilon^n(x)^2])}\, dx \\ &= \int_S e^{\gamma X_\varepsilon^n(x) - \frac{\gamma^2}{2}\mathbb{E}[X_\varepsilon^n(x)^2]}\, \sigma(dx)\,.\end{aligned}$$

Now in the limit $\varepsilon \to 0$ the right-hand side converges to $M_\gamma^n(S)$ since X^n is smooth in $\overline{D}$. Thus $M_\gamma^n(S) = \lim_{\varepsilon\to 0} \mathbb{E}(M_{\gamma,\varepsilon}(S) \mid \mathcal{F}_n)$. Fatou's lemma now implies that

$$\begin{aligned}M_\gamma^n(S) &= \lim_{k\to\infty} \mathbb{E}(M_{\gamma,2^{-k}}(S) \mid \mathcal{F}_n) \\ &\geqslant \mathbb{E}(\lim_{k\to\infty} M_{\gamma,2^{-k}}(S) \mid \mathcal{F}_n) \\ &= \mathbb{E}(M_\gamma(S) \mid \mathcal{F}_n)\end{aligned}$$

Lévy's zero-one law implies that the convergence $\mathbb{E}(M_\gamma(S)|\mathcal{F}_n) \to \mathbb{E}(M_\gamma(S)|\mathcal{F}_\infty)$, where $\mathcal{F}_\infty = \sigma(\cup_{n=1}^\infty \mathcal{F}_n)$, holds almost surely and in $L^1(\Omega,\mathbb{P})$. This and the fact that $M_\gamma(S)$ is $\mathcal{F}_\infty$-measurable implies

$$M_\gamma^*(S) \geqslant M_\gamma(S)\,.$$

(ii). By Fatou's lemma

$$\mathbb{E} M_\gamma^*(S) \leqslant \lim_{n\to\infty} \mathbb{E} M_\gamma^n(S) = \lim_{n\to\infty} \int_S \sigma(dx) = \int_S \sigma(dx)\,.$$

On the other hand, we have previously shown that $\mathbb{E} M_{\gamma,\varepsilon}(S) = \int_S \sigma(dx)$. Now by the L^1-convergence of $(M_{\gamma,\varepsilon}(S))_{\varepsilon>0}$ we have $\int_S \sigma(dx) = \lim_{\varepsilon\to 0} \mathbb{E} M_{\gamma,\varepsilon}(S) = \mathbb{E} M_\gamma(S)$. This proves the claim. □

Theorem 4.2.2. *If $S \subset D$ is open and $\gamma \in (0,2)$, then $M_\gamma(S) > 0$ almost surely.*

Proof. We have

$$M_\gamma(S) = \lim_{n\to\infty} M_\gamma^n(S) = \lim_{n\to\infty} \int_S e^{\gamma X^n(x) - \frac{\gamma^2}{2}\mathbb{E}[X^n(x)^2]}\,\sigma(dx)$$
$$\leqslant \sup_{x\in S} e^{\gamma X^k(x) - \frac{\gamma^2}{2}\mathbb{E}[X^k(x)^2]} \lim_{n\to\infty} \int_S e^{\gamma(X^n(x)-X^k(x)) - \frac{\gamma^2}{2}\mathbb{E}[(X^n(x)-X^k(x))^2]}\,\sigma(dx)\,.$$

Similarly,

$$M_\gamma(S) \geqslant \inf_{x\in S} e^{\gamma X^k(x) - \frac{\gamma^2}{2}\mathbb{E}[X^k(x)^2]} \lim_{n\to\infty} \int_S e^{\gamma(X^n(x)-X^k(x)) - \frac{\gamma^2}{2}\mathbb{E}[(X^n(x)-X^k(x))^2]}\,\sigma(dx)\,.$$

Note that $X^n - X^k = \sqrt{2\pi}\sum_{i=k+1}^n X_i\eta_i$ so that $X^n - X^k$ is independent of $X_1, \ldots, X_k$. Also, the above two inequalities entail that

$$\{M_\gamma(S) > 0\} = \left\{ \lim_{n\to\infty} \int_S e^{\gamma(X^n(x)-X^k(x)) - \frac{\gamma^2}{2}\mathbb{E}[(X^n(x)-X^k(x))^2]}\,\sigma(dx) > 0 \right\}\,.$$

The event on the right-hand side is independent of $X_1, \ldots, X_k$ and k was arbitrary. Hence it is a tail event and so is $\{M_\gamma(S) > 0\}$. By Kolmogorov's zero-one law we must have $\mathbb{P}(\{M_\gamma(S) > 0\}) = 1$ since

$$\mathbb{E}[M_\gamma(S)] = \int_S \sigma(dx) > 0$$

implies that $\mathbb{P}(\{M_\gamma(S) > 0\}) > 0$. □

It also holds that M_γ is atomless almost surely for $\gamma \in (0,2)$. The limit measure is supported on γ-thick points in the sense that if we sample a point $x \in D$ according to the measure M_γ, normalized to be probability measure, then

$$\lim_{\varepsilon\to 0} \frac{X_\varepsilon(x)}{\log(1/\varepsilon)} = \gamma$$

almost surely. For a proof see [3]. The set of γ-thick points $\mathcal{T}_\gamma$ has Hausdorff dimension $\max\{2-\gamma^2/2, 0\}$ and $\mathcal{T}_\gamma$ is empty if $\gamma > 2$. One could say that for $\gamma < 2$ the measure M_γ is non-trivial since there are γ-thick points on which the measure is concentrated and if $\gamma > 2$ then the absence of γ-thick points degenerates the measure to the zero measure. For $\gamma = 2$ the set $\mathcal{T}_\gamma$ is non-empty and one can construct a non-trivial measure in this case, called the critical Gaussian multiplicative chaos [11, 12].

From the general theory of Gaussian multiplicative chaos it follows that the moments $\mathbb{E}[M_\gamma(B)^q]$ are finite where $B \subset D$ is an open ball and $q \in (-\infty, 4/\gamma^2)$. The measure also exhibits the multifractal behaviour

$$\mathbb{E}[M_\gamma(B(x,r))^q] \sim C_x r^{\zeta(q)}$$

as $r \to 0$. Here $\zeta(q) = (2+\gamma^2/2)q - \gamma^2 q^2/2$, $q \in (-\infty, 4/\gamma^2)$ and $C_x > 0$ is a constant depending on x. For a more detailed discussion see the review [41].

Theorem 4.2.3. (Conformal covariance) *Let $f : D \to D'$ be a conformal map and X and X' the Gaussian free fields on D and D', respectively. Then we have*

$$f_* M_\gamma \stackrel{d}{=} e^{\gamma Q \log |(f^{-1})'|} M'_\gamma \,,$$

where $Q = \frac{2}{\gamma} + \frac{\gamma}{2}$, M_γ is the exponential of X, M'_γ is the exponential of X' and $f_* M_\gamma$ is the push-forward measure of M_γ under f. In other words

$$\int_S (g \circ f)(x) \, M_\gamma(dx) \stackrel{d}{=} \int_{f(S)} g(x) e^{\gamma Q \log |(f^{-1})'(x)|} \, M'_\gamma(dx)$$

whenever $g \circ f \in L^1(M_\gamma)$.

Proof. Recall that by conformal invariance of the GFF we have $X' \stackrel{d}{=} X \circ f^{-1}$. Let $(\eta_n)_{n\in\mathbb{N}}$ be the Laplace-eigenbasis of $H_0^1(D)$ and define

$$X^n = \sqrt{2\pi} \sum_{i=1}^n X_i \eta_i \,,$$
$$X'^{\,n} = \sqrt{2\pi} \sum_{i=1}^n X'_i (\eta_i \circ f^{-1}) \,,$$

where $(X_i)_{i\in\mathbb{N}}$ and $(X'_i)_{i\in\mathbb{N}}$ are sequences of iid standard Gaussians. Let $S \subset D$ be a Borel set. We define

$$M_\gamma^n(S) = \int_S \exp\left(\gamma X^n(x) - \frac{\gamma^2}{2} \mathbb{E}[X^n(x)^2]\right) \sigma(dx) \,,$$
$$M'^{\,n}_\gamma(S) = \int_S \exp\left(\gamma X'^{\,n}(x) - \frac{\gamma^2}{2} \mathbb{E}[X'^{\,n}(x)^2]\right) \sigma(dx) \,.$$

By Proposition 4.2.1 we have $\lim_{n\to\infty} M_\gamma^n(S) = M_\gamma(S)$ almost surely.

By conformal invariance of the Dirichlet energy $(\eta_n \circ f^{-1})_{n\in\mathbb{N}}$ is an orthonormal basis of $H_0^1(D')$ consisting of smooth functions. Thus the limit $\lim_{n\to\infty} X'^{\,n}$ is the GFF on D'. Now for Borel sets $S \subset D'$ we have

$$\begin{aligned}
f_* M_\gamma^n(S) &= \int_{f^{-1}(S)} e^{\gamma X^n(x) - \frac{\gamma^2}{2}\mathbb{E}[X^n(x)^2]} R(x, D)^{\gamma^2/2} \, dx \\
&= \int_S e^{\gamma X^n(f^{-1}(x)) - \frac{\gamma^2}{2}\mathbb{E}[X^n(f^{-1}(x))^2]} R(f^{-1}(x), D)^{\gamma^2/2} \frac{dx}{|f'(f^{-1}(x))|^2} \\
&\stackrel{d}{=} \int_S e^{\gamma X'^{n}(x) - \frac{\gamma^2}{2}\mathbb{E}[X'^{n}(x)^2]} R(f^{-1}(x), D)^{\gamma^2/2} \frac{dx}{|f'(f^{-1}(x))|^2} \\
&= \int_S e^{\gamma X'^{n}(x) - \frac{\gamma^2}{2}\mathbb{E}[X'^{n}(x)^2]} \frac{R(x, D')^{\gamma^2/2}}{|f'(f^{-1}(x))|^{\gamma^2/2}} \frac{dx}{|f'(f^{-1}(x))|^2} \\
&= \int_S |(f^{-1})'(x)|^{2+\gamma^2/2} e^{\gamma X'^{n}(x) - \frac{\gamma^2}{2}\mathbb{E}[X'^{n}(x)^2]} \, \sigma(dx) \\
&= \int_S e^{\gamma Q \log |(f^{-1})'(x)|} \, M'^{\,n}_\gamma(dx) \,.
\end{aligned}$$

Letting $n \to \infty$ yields

$$f_* M_\gamma^*(S) \overset{d}{=} \int_S e^{\gamma Q \log |(f^{-1})'(z)|} \, M_\gamma'(dz)$$

by Proposition 4.2.1. The result follows since $M_\gamma^*(S) = M_\gamma(S)$ almost surely. □

Chapter 5

Probabilistic Liouville field theory on the sphere

In this chapter we construct the probabilistic Liouville field theory and the corresponding Liouville quantum gravity measure on the two-dimensional sphere. Liouville quantum field theory was introduced by Polyakov [40] in his work on string theory as a building block of Liouville quantum gravity.

In the probabilistic approach to constructing a quantum field theory described by an action functional S we want to make sense of the partition function

$$Z(g) = \int_{\mathscr{F}} e^{-S(X,g)} \, \mathcal{D}X \, ,$$

or more generally the functional integral

$$Z(g,F) = \int_{\mathscr{F}} F(X) e^{-S(X,g)} \, \mathcal{D}X \, ,$$

where $\mathscr{F}$ is some space of fields $M \to \mathbb{R}$, $F : \mathscr{F} \to \mathbb{R}$ a continuous and bounded function and M is a manifold with Riemannian metric g. The mathematical problem is then to define the measure $e^{-S(X,g)} \, \mathcal{D}X$ and showing the convergence of the above integrals. Central objects of the theory are the correlation functions (or exponential moments)

$$Z(g; (z_1, \alpha_1), \ldots, (z_n, \alpha_n)) := \int_{\mathscr{F}} e^{\sum_{i=1}^{n} \alpha_i X(z_i)} e^{-S(X,g)} \, \mathcal{D}X \, , \tag{5.0.1}$$

where $z_1, \ldots, z_n \in M$ and $\alpha_1, \ldots, \alpha_n \in \mathbb{R}$. Of course if the space $\mathscr{F}$ consists of generalized functions, then some regularization is needed. See Section 1.1.3 for a brief discussion, or [19] for a complete discussion, on how this is related to the Hilbert space and operator valued fields of an actual quantum field theory.

Following [20], we call the theory a conformal field theory (CFT) if it possesses certain conformal symmetries[1]. First requirement is that the partition function is invariant under

[1] Note that the article in question focuses on surfaces with genus 2 or higher. The following discussion is not supposed to be specific for the sphere (or the Liouville theory). The point is to give some general idea of the properties of CFTs. In the spherical Liouville theory the picture turns out to be a bit different; for example the partition function diverges.

diffeomorphisms[2], meaning that $Z(\psi^* g) = Z(g)$, where ψ is a diffeomorphism $\psi : M \to M$. Furthermore, we want that for all $\varphi \in C^\infty(M)$ we have

$$Z(e^\varphi g) = Z(g) \exp\left(\frac{c}{96\pi}\int_M (|\nabla_g \varphi(x)|^2 + 2R_g(x)\varphi(x))\sqrt{\det g(x)}\,dx\right).$$

This relation is called Weyl anomaly, or conformal anomaly. The constant c is called the central charge of the theory. In a CFT there should be objects called primary fields $(\theta_i)_{i\in I}$ and we denote their correlation functions by

$$Z(g, \theta_{i_1}(z_1), \ldots, \theta_{i_n}(z_n)) = \int_{\mathscr{F}} \prod_{k=1}^{n} \theta_{i_k}(z_k) e^{-S(X,g)}\,\mathcal{D}X.$$

A CFT is described by its partition function and the set of correlation functions of the primary fields. The correlation functions should satisfy similar conditions as the partition function; the invariance under diffeomorphisms

$$Z(g, \theta_{i_1}(\psi(z_1)), \ldots, \theta_{i_n}(\psi(z_n))) = Z(\psi^* g, \theta_{i_1}(z_1), \ldots, \theta_{i_n}(z_n))$$

and the conformal anomaly

$$\frac{Z(e^\varphi g, \theta_{i_1}(z_1), \ldots, \theta_{i_n}(z_n))}{Z(g, \theta_{i_1}(z_1), \ldots, \theta_{i_n}(z_n))} = e^{-\sum_{k=1}^n \Delta_{i_k}\varphi(x_k) + \frac{c}{96\pi}\int_M (|\nabla_g\varphi(x)|^2 + 2R_g(x)\varphi(x))\sqrt{\det g(x)}\,dx}.$$

Here c is again the central charge and the numbers Δ_i are called the conformal weights of the primary fields.

In Liouville field theory we choose the action functional to be the Liouville action

$$S_L(X, g) = \frac{1}{4\pi}\int_M (|\nabla_g X(x)|^2 + QR_g(x)X(x) + 4\pi\mu e^{\gamma X(x)})\sqrt{\det g(x)}\,dx, \tag{5.0.2}$$

where $\gamma \in (0, 2]$ (we will not cover the case $\gamma = 2$) and $\mu > 0$ are parameters, and $Q = \gamma/2 + 2/\gamma$. This action describes an interacting quantum field theory with the interaction term

$$\mu \int_M e^{\gamma X(x)}\sqrt{\det g(x)}\,dx.$$

The task is now to define the functional integral

$$\mathbb{E}[F(X)] = \int_{\mathscr{F}} F(X) e^{-S_L(X,g)}\,\mathcal{D}X, \tag{5.0.3}$$

where $\mathscr{F}$ is some space of fields $M \to \mathbb{R}$ ($\mathscr{F}$ will actually be a space of generalized functions) and F some function $F : \mathscr{F} \to \mathbb{R}$. Then a suitable choices of F will correspond to the partition function and the correlation functions.

[2]For example if one wants to think of the Riemann sphere as a complex analytic manifold, then the diffeomorphisms are the Möbius transformations.

Physicists have known for a long time that the Liouville theory is a CFT with the central charge $c = 1 + 6Q^2$. Letting γ vary in the interval $(0, 2]$ this means that c can take any value in $[25, \infty)$. The actual rigorous formulation of the theory has been given recently, initially on the sphere in [7], on the disk in [21], on the torus in [9] and on surfaces with higher genus in [20]. Other aspects of the theory are discussed in [8, 29], see also the lecture notes [28, 42]. There is also the non-probabilistic, perturbative approach [49]. A review of the physics literature is found in [39]. Many of the articles mentioned contain expositions of the wider context and the history of the theory. Section 2 of the lecture notes [28] summarize the construction of the actual quantum field theory using the probabilistic theory. Section 6 of [9] contains an explanation of the relation between LQFT and LQG in the genus 1 case. Much more complete discussion (in genus 2) of this relation can be found in [20], which contains a quite extensive exposition of Polyakov's work, directed towards mathematicians. In that paper it is also shown that the Liouville field theory is a CFT (in the sense we described above). The situation is a bit different on the sphere since, for example, the partition function diverges. The theory still has many conformal symmetries, for example the functional integral exhibits the Weyl anomaly and the correlation functions have some conformal symmetries too, allowing us to call the LQFT on the sphere a CFT.

We will construct the theory on the two-dimensional sphere, which is probably the simplest case. We will need some results concerning the Gaussian free field and its exponential on the sphere. Most of the results established in Chapters 3 and 4 hold also in the spherical case, often with the same proofs. Sometimes we choose to re-prove a result, possibly with different arguments, and sometimes we refer to the proofs given in the planar domain setting.

5.1 Notation and a lemma

We define $\overline{C}(\mathbb{R}^2)$ to be the set of continuous functions on $\mathbb{R}^2$ that admit a finite limit at infinity. Then we naturally define $\overline{C}^k(\mathbb{R}^2)$ to be the set of functions for which all derivatives up to order k belong to $\overline{C}(\mathbb{R}^2)$.

Recall the conventions and notations from Section 2.7, especially the fact that we will work on the plane $\mathbb{R}^2$ instead of the Riemann sphere $\widehat{\mathbb{C}}$. In whole of Chapter 5 g will denote a Riemannian metric on $\mathbb{R}^2$, conformally equivalent to the Euclidean metric (see Section 2.7 for details). We denote by ∇_g the gradient, by $\Delta_g = \frac{1}{\sqrt{\det g}}\Delta$ the Laplace–Beltrami operator, by $R_g(x) := -\Delta_g \log \sqrt{(\det g)(x)}$ the scalar curvature and by λ_g the volume form corresponding to g, which means that

$$\int_{\mathbb{R}^2} \lambda_g(dx) = \int_{\mathbb{R}^2} \sqrt{(\det g)(x)}\, dx = \int_{\mathbb{R}^2} g(x)\, dx\,.$$

The same symbols without indices $(\nabla, \Delta, \lambda)$ correspond to the Euclidean case, that is, $g(x) = 1$.

Recall that for the spherical metric the density with respect to the Euclidean metric is

$$\widehat{g}(x) = \frac{4}{(1+|x|^2)^2} \,.$$

The scalar curvature is $R_{\widehat{g}} = 2$ and the total area is $\int_{\mathbb{R}^2} \lambda_{\widehat{g}} = 4\pi$.

Two metrics g' and g are said to be conformally equivalent to each other if there exists a function $\varphi \in \overline{C}^2(\mathbb{R}^2)$ such that the densities with respect to the Euclidean metric satisfy

$$g'(x) = e^{\varphi(x)} g(x)$$

and $\int_{\mathbb{R}^2} |\nabla \varphi|^2 \, d\lambda < \infty$. A quick computation then shows that the scalar curvature $R_{g'}$ is

$$R_{g'} = e^{-\varphi}(R_g - \Delta_g \varphi) \,. \tag{5.1.1}$$

We define the g-mean of a function h by

$$m_g(h) := \frac{1}{\int_{\mathbb{R}^2} g(x)\, dx} \int_{\mathbb{R}^2} h(x) g(x) \, dx = \frac{1}{\lambda_g(\mathbb{R}^2)} \int_{\mathbb{R}^2} h \, d\lambda_g \,.$$

By $L^2(\mathbb{R}^2, g)$ we denote the space of functions f for which $\int_{\mathbb{R}^2} |f|^2 \, d\lambda_g < \infty$. We equip this space with the inner product

$$\langle f, h \rangle_{L^2(\mathbb{R}^2,g)} := \langle f, h \rangle_g := \int_{\mathbb{R}^2} f(x) h(x) \, \lambda_g(dx) \,.$$

We define $H^1(\mathbb{R}^2, g)$ to be the closure of $\overline{C}^\infty(\mathbb{R}^2)$ with respect to the Sobolev-norm

$$\|h\|^2_{H^1(\mathbb{R}^2,g)} = \int_{\mathbb{R}^2} |h(x)|^2 \, \lambda_g(dx) + \int_{\mathbb{R}^2} |\nabla_g h(x)|^2 \, \lambda_g(dx) \,.$$

Note that $H^1(\mathbb{R}^2, g)$ is a Hilbert space. In the above expression the latter integral is known as the Dirichlet energy, and it is independent of the metric since by Green's formula

$$\int_{\mathbb{R}^2} |\nabla_g h|^2 \, d\lambda_g = -\int_{\mathbb{R}^2} h \Delta_g h \, d\lambda_g = -\int_{\mathbb{R}^2} h \Delta h \, d\lambda = \int_{\mathbb{R}^2} |\nabla h|^2 \, d\lambda \,. \tag{5.1.2}$$

If g' is conformally equivalent to g, then the norms $\|h\|_{H^1(\mathbb{R}^2,g)}$ and $\|h\|_{H^1(\mathbb{R}^2,g')}$ are equivalent. This follows from the fact that $e^{\varphi(x)} \in [m, M]$ for a sufficiently large $M > 0$ and sufficiently small $m > 0$ since $\varphi \in \overline{C}^2(\mathbb{R}^2)$.

By $H^{-1}(\mathbb{R}^2, g)$ we denote the dual of $H^1(\mathbb{R}^2, g)$. For $\rho \in H^{-1}(\mathbb{R}^2, g)$ and $f \in H^1(\mathbb{R}^2, g)$ we define $\langle \rho, f \rangle_g := \rho(f)$. Formally $\langle \rho, f \rangle_g = \int_{\mathbb{R}^2} \rho(x) f(x) \, \lambda_g(dx)$.

Since we are working on the plane $\mathbb{R}^2$, given a Möbius transformation $\psi : \mathbb{C} \to \mathbb{C}$, we "should" define it to be a map $\mathbb{R}^2 \to \mathbb{R}^2$ given by $(x, y) \mapsto (\operatorname{Re} \psi(x + iy), \operatorname{Im} \psi(x + iy))$. However, for convenience, we will still use the complex analytic notation $\psi(z) = \frac{az+b}{cz+d}$ and we will understand this as a map on the plane.

Lemma 5.1.1. *Let $\psi(z) = \frac{az+b}{cz+d}$ be a Möbius transformation. Then the following equalities hold true*

$$\int_{\mathbb{R}^2} \log|x-y|\, \lambda_{\widehat{g}}(dy) = 2\pi \log(1+|x|^2)\,, \tag{5.1.3}$$

$$m_{\widehat{g}}\left(\log\frac{1}{|x-\cdot|}\right) = \frac{1}{4}\log\widehat{g}(x) - \frac{1}{2}\log 2\,, \tag{5.1.4}$$

$$m_{g_\psi}\left(\log\frac{1}{|x-\cdot|}\right) - \frac{1}{2}\theta_{g_\psi} = \frac{1}{4}\log\widehat{g}(\psi(x)) + \frac{1}{2}\log|\psi'(x)| - \frac{1}{2}\theta_{\widehat{g}} - \frac{1}{2}\log 2\,. \tag{5.1.5}$$

where $g_\psi = |\psi'|^2 \widehat{g}\circ\psi$, ψ' is the complex derivative of ψ and

$$\theta_g = \frac{1}{\lambda_g(\mathbb{R}^2)^2}\iint_{\mathbb{R}^2\times\mathbb{R}^2} \log\frac{1}{|x-y|}\, \lambda_g(dx)\,\lambda_g(dy)\,.$$

Proof. 1.

$$\begin{aligned}
\int_{\mathbb{R}^2} \log|x-y|\, \lambda_{\widehat{g}}(dy) &= 4\int_{\mathbb{R}^2} \frac{\log|x-y|}{(1+|y|^2)^2}\, dy \\
&= 2\int_0^\infty \int_0^{2\pi} \frac{\log(|x|^2 - 2|x|r\cos\varphi + r^2)}{(1+r^2)^2} r\, dr\, d\varphi \\
&= 2\int_0^\infty \frac{\log\max\{|x|, r\}}{(1+r^2)^2} r\, dr \\
&= 2\pi\log(1+|x|^2)\,.
\end{aligned}$$

2. The first identity implies

$$\begin{aligned}
-2m_{\widehat{g}}\left(\log\frac{1}{|x-\cdot|}\right) &= 2\frac{1}{4\pi}\int_{\mathbb{R}^2} \log|x-y|\, \lambda_{\widehat{g}}(dy) = \log(|x|^2+|1|^2) - \log|1|^2 \\
&= -\log\sqrt{\frac{1}{4}\frac{4}{(1+|x|^2)^2}} = -\frac{1}{2}\log\widehat{g}(x) + \log 2\,.
\end{aligned}$$

3. By a change of variables

$$\begin{aligned}
-2m_{g_\psi}\left(\log\frac{1}{|x-\cdot|}\right) + \theta_{g_\psi} &= \frac{1}{(4\pi)^2}\iint_{\mathbb{R}^2\times\mathbb{R}^2} \log\frac{|x-z||x-z'|}{|z-z'|}\, \lambda_{g_\psi}(dz)\,\lambda_{g_\psi}(dz') \\
&= \frac{1}{(4\pi)^2}\iint_{\mathbb{R}^2\times\mathbb{R}^2} \log\frac{|x-\psi^{-1}(z)||x-\psi^{-1}(z')|}{|\psi^{-1}(z)-\psi^{-1}(z')|}\, \lambda_{\widehat{g}}(dz)\,\lambda_{\widehat{g}}(dz')\,.
\end{aligned}$$

Using continuity of the function

$$(x,y) \mapsto \frac{1}{(4\pi)^2}\iint_{\mathbb{R}^2\times\mathbb{R}^2} \log\frac{|x-\psi^{-1}(z)||y-\psi^{-1}(z')|}{|\psi^{-1}(z)-\psi^{-1}(z')|}\, \lambda_{\widehat{g}}(dz)\,\lambda_{\widehat{g}}(dz')$$

we can write

$$
\begin{aligned}
&-2m_{g_\psi}\left(\log\frac{1}{|x-\cdot|}\right)+\theta_{g_\psi}\\
&=\lim_{y\to x}\frac{1}{(4\pi)^2}\iint_{\mathbb{R}^2\times\mathbb{R}^2}\log\frac{|x-\psi^{-1}(z)||y-\psi^{-1}(z')|}{|\psi^{-1}(z)-\psi^{-1}(z')|}\,\lambda_{\widehat{g}}(dz)\,\lambda_{\widehat{g}}(dz')\\
&=\lim_{y\to x}\left(\frac{1}{(4\pi)^2}\iint_{\mathbb{R}^2\times\mathbb{R}^2}\log\frac{|x-\psi^{-1}(z)||y-\psi^{-1}(z')|}{|x-y||\psi^{-1}(z)-\psi^{-1}(z')|}\,\lambda_{\widehat{g}}(dz)\,\lambda_{\widehat{g}}(dz')+\log|x-y|\right)\\
&=\lim_{y\to x}\left(\frac{1}{(4\pi)^2}\iint_{\mathbb{R}^2\times\mathbb{R}^2}\log\frac{|\psi(x)-z||\psi(y)-z'|}{|\psi(x)-\psi(y)||z-z'|}\,\lambda_{\widehat{g}}(dz)\,\lambda_{\widehat{g}}(dz')+\log|x-y|\right)\\
&=\frac{1}{(4\pi)^2}\iint_{\mathbb{R}^2\times\mathbb{R}^2}\log\frac{|\psi(x)-z||\psi(y)-z'|}{|z-z'|}\,\lambda_{\widehat{g}}(dz)\,\lambda_{\widehat{g}}(dz')-\lim_{y\to x}\log\frac{|\psi(x)-\psi(y)|}{|x-y|}\,.
\end{aligned}
$$

This is equal to $-2m_{\widehat{g}}(-\log|\psi(x)-\cdot|)+\theta_{\widehat{g}}-\log|\psi'(x)|$. Now the claim follows from the previous identity. □

5.2 The Gaussian free field on the two-dimensional sphere

Now we want to define the measure

$$\exp\left(-\frac{1}{2}\int_{\mathbb{R}^2}|\nabla_g X|^2\,d\lambda_g\right)\mathcal{D}X \tag{5.2.1}$$

on the sphere. We again choose to interpret this measure as the distribution of the GFF. To fix an arbitrary constant related to the definition of the GFF[3], we will impose the zero mean condition, which formally means

$$\int X_g\,d\lambda_g=0\,. \tag{5.2.2}$$

This leads to the *zero mean Gaussian Free Field*. To force this condition we could guess that the covariance kernel will be

$$G_g(x,y):=\log\frac{1}{|x-y|}-m_g\left(\log\frac{1}{|x-\cdot|}\right)-m_g\left(\log\frac{1}{|\cdot-y|}\right)+\theta_g\,, \tag{5.2.3}$$

where

$$\theta_g=\frac{1}{\lambda_g(\mathbb{R}^2)^2}\iint_{\mathbb{R}^2\times\mathbb{R}^2}\log\frac{1}{|x-y|}\,\lambda_g(dx)\,\lambda_g(dy)\,.$$

[3]In the planar domain case this constant was fixed by requiring that the GFF vanishes on the boundary. Now on the sphere we have no boundary.

This definition forces the zero mean condition

$$\iint_{\mathbb{R}^2\times\mathbb{R}^2} G_g(x,y)\,\lambda_g(dx)\,\lambda_g(dy) = 0\,.$$

The integral kernel G_g is actually the Green function of the problem

$$-\Delta_g u = 2\pi f\,, \quad \int_{\mathbb{R}^2} u(x)\,\lambda_g(dx) = 0\,. \tag{5.2.4}$$

Now we give the precise definition of the GFF on the two-dimensional sphere. The situation is not completely analogous to the planar domain case. Reason for this is that the (distributional) Laplace–Beltrami operator Δ_g is not invertible as an operator acting on $H^1(\mathbb{R}^2, g)$ while the (distributional) Laplacian Δ is invertible on $H_0^1(D)$, $D \subset \mathbb{R}^2$.

Definition 5.2.1. *Let $(e_n)_{n=0}^{\infty}$ be the orthonormal basis of $L^2(\mathbb{R}^2, g)$ consisting of eigenfunctions of Δ_g with eigenvalues $(c_n)_{n=0}^{\infty}$, ordered so that $c_0 < c_1 < c_2 \dots$. We define the (zero mean) Gaussian free field on the two-dimensional sphere by*

$$X_g := \sqrt{2\pi}\sum_{n=1}^{\infty}\frac{X_n}{\sqrt{c_n}}e_n\,, \quad \langle X_g, \rho\rangle_g := \sqrt{2\pi}\sum_{n=1}^{\infty}\frac{X_n}{\sqrt{c_n}}\langle \rho, e_n\rangle_g$$

where $(X_n)_{n=1}^{\infty}$ is an iid sequence of standard Gaussians and $\rho \in H^{-1}(\mathbb{R}^2, g)$.

As in the planar domain case, the series defining X_g converges in $H^{-1}(\mathbb{R}^2, g)$ and the series defining $\langle X_g, \rho\rangle_g$ converges almost surely and in $L^2(\Omega, \mathbb{P})$ for every $\rho \in H^{-1}(\mathbb{R}^2, g)$.

Note that we left out the eigenfunction e_0 from the definition. It holds that $c_0 = 0$ and all the other eigenvalues are positive. The first eigenfunction is the constant $e_0 = 1/\lambda_g(\mathbb{R}^2)$. This means that leaving out the constant mode e_0 ensures the zero mean condition. This is also the reason why we picked the eigenbasis $(e_n/\sqrt{c_n})_{n=0}^{\infty}$ of $H^1(\mathbb{R}^2, g)$; it is particularly simple to enforce the zero mean condition with this basis.

Theorem 5.2.2. *We have*

$$\mathbb{E}[\langle X_g, f\rangle_g \langle X_g, h\rangle_g] = \int_{\mathbb{R}^2\times\mathbb{R}^2} G_g(x,y) f(x) h(y)\,\lambda_g(dx)\,\lambda_g(dy)\,,$$

for all $f, h \in H^1(\mathbb{R}^2, g)$, where G_g is defined by (5.2.3).

Proof. Let $f, h \in H^1(\mathbb{R}^2, g)$ and write $f = \sum_{n\geqslant 1} f_n e_n + \lambda_g(f) e_0$ where $f_n = \langle f, e_n\rangle_g$ and similarly for h. Then the covariance structure is

$$\begin{aligned}\mathbb{E}\left[\langle X_g, f\rangle_g \langle X_g, h\rangle_g\right] &= 2\pi\sum_{n,m=1}^{\infty}\frac{f_n}{\sqrt{c_n}}\frac{h_m}{\sqrt{c_m}}\mathbb{E}[X_n X_m] = 2\pi\sum_{n=1}^{\infty}\frac{f_n h_n}{c_n}\\ &=: \int_{\mathbb{R}^2\times\mathbb{R}^2} C_g(x,y) f(x) h(y)\,\lambda_g(dx)\lambda_g(dy)\,.\end{aligned}$$

It follows that for $f \in \overline{C}^2(\mathbb{R}^2)$ we have

$$\mathbb{E}\left[\langle X_g, -\Delta_g f\rangle_g \langle X_g, h\rangle_g\right] = 2\pi \sum_{n=1}^{\infty} \frac{(-\Delta_g f)_n g_n}{c_n} = 2\pi \sum_{n=1}^{\infty} f_n h_n = 2\pi(\langle f, h\rangle_g - m_g(f) m_g(h))\,,$$

because $(-\Delta_g f)_n = \langle -\Delta_g f, e_n\rangle_g = \langle f, -\Delta_g e_n\rangle_g = c_n f_n$. Thus we obtain the distributional identity

$$\langle \Delta_g C_g(\cdot, y), f\rangle_g = \langle C_g(\cdot, y), \Delta_g f\rangle_g = \int_{\mathbb{R}^2} C_g(x,y) \Delta_g f(x)\, \lambda_g(dx) = 2\pi \left(f(y) - \frac{m_g(f)}{\lambda_g(\mathbb{R}^2)} \right)\,,$$

which then by density extends to all $f \in H^1(\mathbb{R}^2, g)$. This implies that C_g is the Green function of the problem (5.2.4) and thus

$$C_g(x,y) = \log \frac{1}{|x-y|} - \xi_g(x,y)\,,$$

where function ξ_g is such that

$$\int_{\mathbb{R}^2 \times \mathbb{R}^2} C_g(x,y)\, \lambda_g(dx) \lambda_g(dy) = 0\,,$$

or explicitly

$$\xi(x,y) = -m_g \left(\log \frac{1}{|x - \cdot|} \right) - m_g \left(\log \frac{1}{|\cdot - y|} \right) + \frac{1}{\lambda_g(\mathbb{R}^2)^2} \int_{\mathbb{R}^2 \times \mathbb{R}^2} \log \frac{1}{|z - z'|}\, \lambda_g(dz) \lambda_g(dz')\,.$$

Hence C_g agrees with the function G_g defined in (5.2.3). □

The density argument laid down in Proposition 3.3.4 would yield us the covariance $\mathbb{E}[\langle X_g, \rho\rangle_g \langle X, \sigma\rangle_g]$ for arbitrary $\rho, \sigma \in H^{-1}(\mathbb{R}^2, g)$, but we will mostly work with the restriction $X_g|_{H^1(\mathbb{R}^2, g)}$ so the above result is sufficient for our purposes.

In Chapter 2 we showed that the planar Green functions are invariant under conformal maps. The conformal maps on the sphere are the Möbius transformations and the spherical Green functions have the same invariance property.

Proposition 5.2.3. *Let ψ be a Möbius transform of the sphere. Define a metric $g_\psi(x) = |\psi'(x)|^2 g(\psi(x))$. Then*

$$G_{g_\psi}(x,y) = G_g(\psi(x), \psi(y))\,.$$

Proof. We can write (5.2.3) as

$$G_{g_\psi}(x,y) = \frac{1}{(4\pi)^2} \iint_{\mathbb{R}^2 \times \mathbb{R}2} \log \frac{|x-z||y-z'|}{|x-y||z-z'|}\, \lambda_{g_\psi}(dz)\, \lambda_{g_\psi}(dz')\,.$$

Making a change of variables and using the fact that ψ preserves cross ratios yields

$$\begin{aligned} G_{g_\psi}(x,y) &= \frac{1}{(4\pi)^2}\iint_{\mathbb{R}^2\times\mathbb{R}^2} \log\frac{|x-\psi^{-1}(z)||y-\psi^{-1}(z')|}{|x-y||\psi^{-1}(z)-\psi^{-1}(z')|}\,\lambda_{\hat{g}}(dz)\,\lambda_{\hat{g}}(dz') \\ &= \frac{1}{(4\pi)^2}\iint_{\mathbb{R}^2\times\mathbb{R}^2} \log\frac{|\psi(x)-z||\psi(y)-z'|}{|\psi(x)-\psi(y)||z-z'|}\,\lambda_{\hat{g}}(dz)\,\lambda_{\hat{g}}(dz') \\ &= G_g(\psi(x),\psi(y))\,. \end{aligned}$$

□

Corollary 5.2.4. (Conformal covariance) *Let ψ be a Möbius transformation, set $g_\psi = |\psi'|^2 g\circ\psi$ and define*

$$\langle X_g\circ\psi, f\rangle_{g_\psi} := \langle X_g, f\circ\psi^{-1}\rangle_g\,.$$

Then $X_g\circ\psi \overset{d}{=} X_{g_\psi}$.

Proof.

$$\begin{aligned} \mathbb{E}\left[\langle X_{g_\psi}, f\rangle_{g_\psi}\langle X_{g_\psi}, h\rangle_{g_\psi}\right] &= \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} G_{g_\psi}(x,y)f(x)h(y)\,\lambda_{g_\psi}(dx)\,\lambda_{g_\psi}(dy) \\ &= \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} G_g(\psi(x),\psi(y))f(x)h(y)\,\lambda_{g_\psi}(dx)\,\lambda_{g_\psi}(dy) \\ &= \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} G_g(x,y)(f\circ\psi^{-1})(x)(h\circ\psi^{-1})(y)\,\lambda_g(dx)\,\lambda_g(dy) \\ &= \mathbb{E}\left[\langle X_g, f\circ\psi^{-1}\rangle_g\langle X_g, h\circ\psi^{-1}\rangle_g\right]\,. \end{aligned}$$

This proves the claim. □

Given conformally equivalent metrics g and g' with $g' = e^\varphi g$ we could imagine computing the mean of X_g with respect to g' formally by

$$\frac{1}{\lambda_{g'}(\mathbb{R}^2)}\int_{\mathbb{R}^2} X_g(x)\,\lambda_{g'}(dx) = \frac{1}{\lambda_{g'}(\mathbb{R}^2)}\int_{\mathbb{R}^2} X_g(x)\frac{g'(x)}{g(x)}\,\lambda_g(dx)\,.$$

Since $g'/g = e^\varphi$, we define the mean of X_g with respect to g' by

$$m_{g'}(X_g) := \frac{1}{\lambda_{g'}(\mathbb{R}^2)}\langle X_g, e^\varphi\rangle_g\,.$$

Proposition 5.2.5. (Rule for changing metrics) *Let $g' = e^\varphi g$ be a metric that is conformally equivalent to g. Let X_g and $X_{g'}$ be the corresponding Gaussian free fields. Then*

$$\langle X_{g'}, f\rangle_{g'} \overset{d}{=} \langle X_g - m_{g'}(X_g), fe^\varphi\rangle_g\,.$$

Proof. See A.2. □

Remark 5.2.6. *Assume that $g' = e^{\varphi} g$ is conformally equivalent to g. Since the norms of $H^1(\mathbb{R}^2, g')$ and $H^1(\mathbb{R}^2, g)$ are equivalent, the duals $H^{-1}(\mathbb{R}^2, g')$ and $H^{-1}(\mathbb{R}^2, g)$ are actually the same set of functionals. It holds that $X_{g'} \in H^{-1}(\mathbb{R}^2, g)$ when $X_{g'}$ is defined to be the functional*

$$\langle X_{g'}, f \rangle_g := \langle X_{g'}, f e^{-\varphi} \rangle_{g'}, \quad (f \in H^1(\mathbb{R}^2, g)) .$$

Now the previous proposition says that with this definition we have $X_{g'} - m_g(X_{g'}) \overset{d}{=} X_g$ for all g' conformally equivalent to g.

Now we see that the GFF is not invariant under conformal change of metrics. If we proceeded to the construction of the Liouville field theory with this definition of the GFF, the theory would not be symmetric enough to be a CFT. But since the probability distributions of the GFFs corresponding to two different metrics only differ by an additive random constant (and the factor e^{φ}, but this really is just a term that switches the metric from g to g' so its presence is expected), we can modify the distribution of the GFF in such a way that all additive constants are omitted. We now describe this process.

Definition 5.2.7. *We denote the distribution of $X_g|_{H^1(\mathbb{R}^2,g)}$ by $\mathbb{P}_g$. Since the series defining $X_g|_{H^1(\mathbb{R}^2,g)}$ converges in $H^{-1}(\mathbb{R}^2, g)$, $\mathbb{P}_g$ defines a probability measure on $H^{-1}(\mathbb{R}^2, g)$. Note that if $g' = e^{\varphi} g$, then the measure $\mathbb{P}_{g'}$ is defined on the same space as $\mathbb{P}_g$.*

We will work in the conformal class of the spherical metric $\widehat{g}$. Let $g = e^{\varphi}\widehat{g}$. Since the restriction $X_g|_{H^1(\mathbb{R}^2,g)}$ is a random element of $H^{-1}(\mathbb{R}^2, \widehat{g})$, it defines a probability measure $\mathbb{P}_g$ on $H^{-1}(\mathbb{R}^2, \widehat{g})$. Define a map $f : H^{-1}(\mathbb{R}^2, \widehat{g}) \times \mathbb{R} \to H^{-1}(\mathbb{R}^2, \widehat{g})$ by $f(X, c) = X + c$ and let m be the Lebesgue measure on $\mathbb{R}$. Consider the push-forward measure

$$\nu_{\text{GFF}} := f_*(\mathbb{P}_g \otimes m)$$

on $H^{-1}(\mathbb{R}^2, \widehat{g})$. Now for all cylindrical sets $A \subset H^{-1}(\mathbb{R}^2, \widehat{g})$ we have

$$\nu_{\text{GFF}}(A) = (\mathbb{P}_g \otimes m)(\{(X, c) \in H^{-1}(\mathbb{R}^2, \widehat{g}) \times \mathbb{R} : X + c \in A\}) .$$

Remember that c is a uniformly distributed "random" constant. Thus the measure ν_{GFF} behaves similarly to $\mathbb{P}_g$ except that it omits additive constants.

Note that $\int \nu_{\text{GFF}}(DX) = \infty$, meaning that ν_{GFF} is not a probability measure, but ν_{GFF} is invariant under conformal change of metrics

$$\begin{aligned} f_*(\mathbb{P}_g \otimes m)(A) &= (\mathbb{P}_g \otimes m)(\{(X, c) \in H^{-1}(\mathbb{R}^2, \widehat{g}) \times \mathbb{R} : X + c \in A\}) \\ &= (\mathbb{P}_{\widehat{g}} \otimes m)(\{(X, c) \in H^{-1}(\mathbb{R}^2, \widehat{g}) \times \mathbb{R} : X - m_g(X) + c \in A\}) \\ &= (\mathbb{P}_{\widehat{g}} \otimes m)(\{(X, c) \in H^{-1}(\mathbb{R}^2, \widehat{g}) \times \mathbb{R} : X + c \in A\}) \\ &= f_*(\mathbb{P}_{\widehat{g}} \otimes m)(A) . \end{aligned}$$

In the second equality we used Remark 5.2.6 and in the third equality we used translation invariance of m. Thus for $F \in L^1(\nu_{\text{GFF}})$ and g conformally equivalent to $\widehat{g}$ we define

$$\int_{H^{-1}(\mathbb{R}^2,\widehat{g})} F(X) e^{-\frac{1}{4\pi}\int_{\mathbb{R}^2} |\nabla_g X|^2 \lambda_g} \mathcal{D}X := \int_{H^{-1}(\mathbb{R}^2,\widehat{g})} F(X)\, \nu_{\text{GFF}}(DX) .$$

We can also write

$$\begin{aligned}\int_{H^{-1}(\mathbb{R}^2,\widehat{g})} F(X)\,\nu_{\mathrm{GFF}}(DX) &= \int_{H^{-1}(\mathbb{R}^2,\widehat{g})} F(X)\, f_*(\mathbb{P}_g\otimes m)(DX)\\ &= \int_{H^{-1}(\mathbb{R}^2,\widehat{g})\times\mathbb{R}} F(f(X))\,(\mathbb{P}_g\otimes m)(DX)\\ &= \int_{\mathbb{R}}\int_{H^{-1}(\mathbb{R}^2,\widehat{g})} F(c+X)\mathbb{P}_g(DX)\,dc\\ &= \int_{\mathbb{R}} \mathbb{E}_g[F(c+X_g)]\,dc\,,\end{aligned}$$

where $\mathbb{E}_g$ is the expectation corresponding to the probability measure $\mathbb{P}_g$.

5.3 Exponentiating the Gaussian free field

Next we aim to define a measure of the form

$$\nu_g(DX) = e^{-\frac{1}{4\pi}\int_{\mathbb{R}^2}(QR_gXg+4\pi\mu e^{\phi_g})\,dx}\nu_{\mathrm{GFF}}(DX)\,,$$

where $\phi_g = X + \frac{Q}{2}\log g$[4]. We again define the circle average

$$X_{g,\varepsilon}(x) := \langle X_g, \rho_{x,\varepsilon}\rangle_g\,,$$

where $\rho_{x,\varepsilon}$ is a uniform probability measure on $\partial B(x,\varepsilon)$ (these belong to $H^{-1}(\mathbb{R}^2, g)$, see for example the proofs in the planar domain case). Next we compute the variance of the circle average.

Proposition 5.3.1. 1. *One has*

$$\lim_{\varepsilon\to 0}\mathbb{E}[X_{\widehat{g},\varepsilon}(x)^2] + \log\varepsilon + \frac{1}{2}\log\widehat{g}(x) = \theta_{\widehat{g}} + \log 2$$

uniformly on $\mathbb{R}^2$.

2. *Let* ψ *be a Möbius transformation. Then*

$$\lim_{\varepsilon\to 0}\mathbb{E}[(X_{\widehat{g}}\circ\psi)_\varepsilon(x)^2] + \log\varepsilon + \frac{1}{2}\log\widehat{g}(\psi(x)) + \log|\psi'(x)| = \theta_{\widehat{g}} + \log 2$$

uniformly on $\mathbb{R}^2$.

[4]The term $\frac{Q}{2}\log g$ comes from the fact that we think of the field as an object defined on the plane. See Section 3.4 of [42] for the justification.

Proof. *1.* This is similar to the planar domain case, see (3.5.2). The covariance result 5.2.2 generalizes to all $\rho \in H^{-1}(\mathbb{R}^2, g)$ by using the density argument 3.3.4. Thus using the formula for $G_{\widehat{g}}$ we get

$$\begin{aligned}\mathbb{E}[X_{\widehat{g},\varepsilon}(x)^2] &= \frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi} G_{\widehat{g}}(x+\varepsilon e^{i\theta}, x+\varepsilon e^{i\theta'})\,d\theta\,d\theta' \\ &= \log\frac{1}{\varepsilon} - \frac{2}{2\pi}\int_0^{2\pi} m_{\widehat{g}}\left(\log\frac{1}{|x+\varepsilon e^{i\theta}-\cdot|}\right)d\theta + \theta_{\widehat{g}}\,.\end{aligned}$$

We apply Lemma 5.1.1 to the middle term to get

$$\begin{aligned}-\frac{2}{2\pi}\int_0^{2\pi} m_{\widehat{g}}\left(\log\frac{1}{|x+\varepsilon e^{i\theta}-\cdot|}\right)d\theta &= \frac{1}{2\pi}\int_0^{2\pi}(-\frac{1}{2}\log\widehat{g}(x+\varepsilon e^{i\theta}) + \log 2)\,d\theta \\ &= \log 2 - \frac{1}{2}[\log\widehat{g}(x)]_\varepsilon\,.\end{aligned}$$

Thus the variance of the circle average is

$$\mathbb{E}[X_{\widehat{g},\varepsilon}(x)^2] = \log\frac{1}{\varepsilon} - \frac{1}{2}[\log\widehat{g}(x)]_\varepsilon + \log 2 + \theta_{\widehat{g}}\,.$$

Also,

$$\begin{aligned}\left|\mathbb{E}[X_{\widehat{g},\varepsilon}(x)^2] + \log\varepsilon + \frac{1}{2}\log\widehat{g}(x) - (\theta_{\widehat{g}} + \log 2)\right| &= |\log\widehat{g}(x) - [\log\widehat{g}(x)]_\varepsilon| \\ &\leqslant \frac{1}{2\pi}\int_0^{2\pi}\left|\log\frac{\widehat{g}(x)}{\widehat{g}(x+\varepsilon e^{i\theta})}\right|d\theta \\ &\leqslant -\log\left(1 - \frac{\varepsilon^2}{1+|x|^2}\right) \\ &\leqslant -\log(1-\varepsilon^2)\,.\end{aligned}$$

Thus the convergence is uniform on $\mathbb{R}^2$.

2. $X_{\widehat{g}_\psi} \overset{d}{=} X_{\widehat{g}} \circ \psi$ by 5.2.4. Also again by applying Lemma 5.1.1 we get

$$\begin{aligned}\mathbb{E}[X_{\widehat{g}_\psi,\varepsilon}(x)^2] &= \log\frac{1}{\varepsilon} - \frac{2}{2\pi}\int_0^{2\pi} m_{\widehat{g}_\psi}\left(\log\frac{1}{|x+\varepsilon e^{i\theta}-\cdot|}\right)d\theta + \theta_{\widehat{g}_\psi} \\ &= \log\frac{1}{\varepsilon} - \frac{1}{2}[\log\widehat{g}\circ\psi(x)]_\varepsilon - \frac{1}{2}[\log|\psi'(x)|^2]_\varepsilon + \log 2 + \theta_{\widehat{g}} - \theta_{\widehat{g}_\psi} + \theta_{\widehat{g}_\psi}\,.\end{aligned}$$

Then the uniform convergence clearly follows. □

Definition 5.3.2. *We define the regularized measures $M_{g,\gamma,\varepsilon}$ on $\mathbb{R}^2$ by*

$$M_{g,\gamma,\varepsilon}(dx) := \varepsilon^{\gamma^2/2} e^{\gamma(X_{g,\varepsilon}(x) + \frac{Q}{2}\log g(x))}\,\lambda(dx)\,,$$

where $Q = \gamma/2 + 2/\gamma$.

Proposition 5.3.3. *For $\gamma \in (0,2)$ the limit*

$$M_{g,\gamma}(dx) := \lim_{\varepsilon\to 0} M_{g,\gamma,\varepsilon}(dx) = e^{\frac{\gamma^2}{2}(\theta_g+\log 2)} \lim_{\varepsilon\to 0} e^{\gamma X_{g,\varepsilon}-\frac{\gamma^2}{2}\mathbb{E}[X_{g,\varepsilon}(x)^2]}\, \lambda_g(dx)$$

exists in the sense of weak convergence in probability. The limiting measure is non-trivial and is up to a multiplicative constant the Gaussian multiplicative chaos of the field X_g with respect to the measure λ_g.

Proof. Follows from the results in [41]. See also Theorem 4.1.8. □

Theorem 5.3.4. (Möbius covariance) *Let ψ a Möbius transform of the sphere and $f \in \overline{C}^2(\mathbb{R}^2)$. Then*

$$\int_{\mathbb{R}^2} f(x)\, M_{\widehat{g},\gamma}(dx) \stackrel{d}{=} \int_{\mathbb{R}^2} (f\circ\psi)(x)\, M_{\widehat{g}_\psi,\gamma}(dx)\,,$$

where $\widehat{g}_\psi = |\psi'|^2 \widehat{g}\circ\psi$.

Proof. We could use the proof of Theorem 4.2.3 but instead we use the argument given in [7]. Using the change of variables $x \mapsto \psi(x)$ we get

$$\begin{aligned}
\int_{\mathbb{R}^2} f(x)\, M_{g,\gamma,\varepsilon}(dx) &= \int_{\mathbb{R}^2} f(x)\varepsilon^{\gamma^2/2} e^{\gamma(X_{\widehat{g},\varepsilon}(x)+\frac{Q}{2}\log\widehat{g}(x))}\,\lambda(dx) \\
&= \int_{\mathbb{R}^2} f(\psi(x))\varepsilon^{\gamma^2/2} e^{\gamma\left(X_{\widehat{g},\varepsilon}(\psi(x))+\frac{Q}{2}\log\widehat{g}(\psi(x))\right)}|\psi'(x)|^2\,\lambda(dx) \\
&= \int_{\mathbb{R}^2} (f\circ\psi)(x)\left(\frac{\varepsilon}{|\psi'(x)|}\right)^{\gamma^2/2} e^{\gamma(X_{\widehat{g},\varepsilon}(\psi(x))+\frac{Q}{2}\log\widehat{g})} e^{\gamma\frac{Q}{2}\varphi(x)}\,\lambda(dx)\,,
\end{aligned}$$

where $e^{\varphi} = \widehat{g}_\psi/\widehat{g}$. Proposition 5.3.1 implies

$$\lim_{\varepsilon\to 0}\left(\mathbb{E}[X_{\widehat{g},\varepsilon}(\psi(z))^2] - \mathbb{E}[(X_{\widehat{g}}\circ\psi)_{\epsilon/|\psi'(z)|}(z)^2]\right) = 0$$

on the set $A_\eta := B(0,\frac{1}{\eta})\setminus B(-\frac{d}{c},\eta)$, where $\eta > 0$ (we cut out the singularity of ψ and ψ'). Then multiplicative chaos theory[5] implies that the measures

$$\left(\frac{\varepsilon}{|\psi'|}\right)^{\gamma^2/2} e^{\gamma(X_{\widehat{g},\varepsilon}\circ\psi+\frac{Q}{2}\log\widehat{g})}\,d\lambda \quad \text{and} \quad \varepsilon^{\gamma^2/2}e^{\gamma((X_{\widehat{g}}\circ\psi)_\varepsilon+\frac{Q}{2}\log\widehat{g})}\,d\lambda$$

[5] This is uniqueness of the chaos measure, see for example [41].

converge to the same measure on A_η. Again applying Proposition 5.3.1 yields

$$\begin{aligned}\mathbb{E}\left[\int_{A_\eta^c}\left(\frac{\varepsilon}{|\psi'|}\right)^{\frac{\gamma^2}{2}} e^{\gamma(X_{\widehat{g},\varepsilon}\circ\psi+\frac{Q}{2}\log\widehat{g})}\,d\lambda\right] &= \int_{A_\eta^c}\left(\frac{\varepsilon}{|\psi'|}\right)^{\frac{\gamma^2}{2}} e^{\frac{\gamma^2}{2}(\log\frac{1}{\varepsilon}-\frac{1}{2}[\log\widehat{g}\circ\psi]_\varepsilon+\log 2+\theta_{\widehat{g}})}e^{\gamma\frac{Q}{2}\log\widehat{g}}\,d\lambda\\ &\leqslant C\int_{A_\eta^c}|\psi'|^{-\gamma^2/2}e^{\frac{\gamma^2}{2}(-\frac{1}{2}\log\widehat{g}\circ\psi+\log 2+\theta_{\widehat{g}})}e^{\log\widehat{g}+\frac{\gamma^2}{4}\log\widehat{g}}\,d\lambda\\ &= C\int_{A_\eta^c}\left(\frac{\widehat{g}}{\widehat{g}_\psi}\right)^{\gamma^2/4}\,d\lambda_{\widehat{g}}\\ &= C\int_{A_\eta^c}e^{-\varphi\frac{\gamma^2}{4}}\,d\lambda_{\widehat{g}}\,,\end{aligned}$$

which tends to 0 as $\eta\to 0$. We conclude that

$$\begin{aligned}\lim_{\varepsilon\to 0}\int_{\mathbb{R}^2}f(x)\,M_{\widehat{g},\gamma,\varepsilon}(x) &= \lim_{\varepsilon\to 0}\varepsilon^{\gamma^2/2}\int_{\mathbb{R}^2}(f\circ\psi)(x)e^{\gamma\frac{Q}{2}\varphi}e^{\gamma((X_{\widehat{g}}\circ\psi)_\varepsilon(x)+\frac{Q}{2}\log\widehat{g}(x))}\,\lambda(dx)\\ &\overset{d}{=}\lim_{\varepsilon\to 0}\varepsilon^{\gamma^2/2}\int_{\mathbb{R}^2}(f\circ\psi)(x)e^{\gamma((X_{\widehat{g}_\psi})_\varepsilon(x)+\frac{Q}{2}\log\widehat{g}_\psi(x))}\,\lambda(dx)\\ &= \int_{\mathbb{R}^2}(f\circ\psi)(x)\,M_{\widehat{g}_\psi,\gamma}(dx)\,,\end{aligned}$$

where the equality in distribution holds by the conformal invariance of the GFF. □

Now we are ready to define the measure corresponding to the Liouville action.

Definition 5.3.5. *For g conformally equivalent to $\widehat{g}$ we define the measure ν_g on $H^{-1}(\mathbb{R}^2,\widehat{g})$ by setting*

$$\begin{aligned}\nu_g(DX) &:= e^{-\frac{1}{4\pi}\int_{\mathbb{R}^2}QR_g(c+X_g)\,d\lambda_g}e^{-\mu e^{\gamma c}M_{g,\gamma}(\mathbb{R}^2)}\mathbb{P}_g(DX_g)\,dc\\ &= e^{-\frac{1}{4\pi}\int_{\mathbb{R}^2}QR_gX\,d\lambda_g}e^{-\mu\zeta(X)}\,\nu_{GFF}(DX)\,,\end{aligned}$$

where $M_{g,\gamma}(\mathbb{R}^2) := \int_{\mathbb{R}^2}\,dM_{g,\gamma}$, $\int_{\mathbb{R}^2}R_gX_g\,d\lambda_g := \langle X_g,R_g\rangle_g$ and $\zeta(X_g+c) = e^{\gamma c}M_{g,\gamma}(\mathbb{R}^2)$. This is a natural interpretation of the measure $e^{-S_L(X,g)}\,\mathcal{D}X$ discussed before.

5.4 Weyl anomaly and Möbius covariance of the functional integral

In this section we prove covariance of the functional integral $\int F(X+\frac{Q}{2}\log g)\,\nu_g(DX)$ under conformal change of metrics (or Weyl transformations) and under Möbius transformations. Both of these symmetries are already apparent in the classical field theory described by the Liouville action (5.0.2). If $g' = e^\varphi g$ then we have

$$S_L(X,e^\varphi g) = S_L\left(X+\frac{Q}{2}\varphi,g\right) - \frac{Q^2}{16\pi}\int_{\mathbb{R}^2}\left(|\nabla_g\varphi(x)|^2+2R_g(x)\varphi(x)\right)\lambda_g(dx)\,,$$

so that the Weyl transformation produces a shift $X \mapsto X + (Q/2)\varphi$ in the field and also adds a constant to the action (which does not affect the stationary values of the action).

The conformal bijections $\mathbb{C} \cup \{\infty\} \to \mathbb{C} \cup \{\infty\}$ are given by the Möbius transformations $\psi(z) = \frac{az+b}{cz+d}$. Change of variables computation yields

$$S_L(X \circ \psi^{-1}, g) = S_L\left(X + \frac{Q}{2}\varphi, g\right),$$

where $e^{\varphi} g = |\psi'|^2 g \circ \psi$. Thus a Möbius transformation also produces a shift in the field.

Lemma 5.4.1. *If $g = e^{\varphi}\widehat{g}$, then for any $F \in L^1(\mathbb{P}_g \otimes m)$ we have*

$$\begin{aligned}
&\int_{\mathbb{R}} \int_{H^{-1}(\mathbb{R}^2, \widehat{g})} F(c + X) \exp\left(-\mu e^{\gamma c} M_{g,\gamma}(\mathbb{R}^2)\right) \mathbb{P}_g(DX)\, dc \\
&= \int_{\mathbb{R}} \int_{H^{-1}(\mathbb{R}^2, \widehat{g})} F(c + X) \exp\left(-\mu e^{\gamma c} M_{\widehat{g},\gamma}(e^{\gamma \frac{Q}{2}\varphi})\right) \mathbb{P}_{\widehat{g}}(DX)\, dc\,.
\end{aligned}$$

Proof. Recall that

$$M_{g,\gamma}(\mathbb{R}^2) = \lim_{\varepsilon \to 0} \varepsilon^{\gamma^2/2} \int_{\mathbb{R}^2} e^{\gamma X_{g,\varepsilon}(x)} e^{\gamma \frac{Q}{2} \log g(x)}\, \lambda(dx)\,,$$

where the convergence holds in probability. The circle average $X_{g,\varepsilon}(x)$ is clearly $\sigma(X_g)$-measurable. From this it follows that $e^{\gamma c} M_{g,\gamma,\varepsilon}(\mathbb{R}^2)$ is $\sigma(c + X_g)$-measurable and hence, being the limit in probability, $e^{\gamma c} M_{g,\gamma}(\mathbb{R}^2)$ is also $\sigma(c + X_g)$-measurable. Therefore, we can write $e^{\gamma c} M_{g,\gamma}(\mathbb{R}^2) = \zeta(c + X_g)$ for some map $\zeta : H^{-1}(\mathbb{R}^2, \widehat{g}) \to \mathbb{R}$. Actually ζ is defined by

$$\zeta(c + X_g) = e^{\gamma c} \lim_{\varepsilon \to 0} \varepsilon^{\gamma^2/2} \int_{\mathbb{R}^2} e^{\gamma \langle X_g, \rho_{x,\varepsilon}\rangle_g + \gamma \frac{Q}{2} \log g(x)}\, \lambda(dx)$$

where the limit is in probability and thus

$$\begin{aligned}
\zeta(c + X_{\widehat{g}}) &= e^{\gamma c} \lim_{\varepsilon \to 0} \varepsilon^{\gamma^2/2} \int_{\mathbb{R}^2} e^{\gamma X_{\widehat{g},\varepsilon}(x) + \gamma \frac{Q}{2} \log g(x)}\, \lambda(dx) \\
&= e^{\gamma c} \lim_{\varepsilon \to 0} \varepsilon^{\gamma^2/2} \int_{\mathbb{R}^2} e^{\gamma X_{\widehat{g},\varepsilon}(x) + \gamma \frac{Q}{2} \log \widehat{g}(x)} e^{\gamma \frac{Q}{2}\varphi(x)}\, \lambda(dx) \\
&= e^{\gamma c} M_{\widehat{g}}(e^{\gamma \frac{Q}{2}\varphi})\,.
\end{aligned}$$

We used the fact that $e^{\gamma \frac{Q}{2}\varphi}$ is continuous and bounded which implies that $M_{\widehat{g},\varepsilon}(e^{\gamma \frac{Q}{2}\varphi}) \to M_{\widehat{g}}(e^{\gamma \frac{Q}{2}\varphi})$ in probability.

Now in order to prove the claim we invoke the previously established metric independence of the measure $\nu_{\mathrm{GFF}} = f_*(\mathbb{P}_{\widehat{g}} \otimes m) = f_*(\mathbb{P}_g \otimes m)$ to get

$$\begin{aligned}
\int_{\mathbb{R}} \int_{H^{-1}(\mathbb{R}^2, \widehat{g})} F(c + X) e^{-\mu \zeta(c+X)}\, \mathbb{P}_{\widehat{g}}(DX)\, dc &= \int_{H^{-1}(\mathbb{R}^2, \widehat{g})} F(X) e^{-\mu \zeta(X)}\, f_*(\mathbb{P}_{\widehat{g}} \otimes m)(DX) \\
&= \int_{H^{-1}(\mathbb{R}^2, \widehat{g})} F(X) e^{-\mu \zeta(X)}\, f_*(\mathbb{P}_g \otimes m)(DX) \\
&= \int_{\mathbb{R}} \int_{H^{-1}(\mathbb{R}^2, \widehat{g})} F(c + X) e^{-\mu \zeta(c+X)}\, \mathbb{P}_g(DX)\, dc
\end{aligned}$$

Now the claim follows from the discussion above. □

For the GFF on the sphere the Girsanov theorem 2.5.4 takes the form

Proposition 5.4.2. *For any $\rho \in H^1(\mathbb{R}^2, g)$ we have*

$$\int_{H^{-1}(\mathbb{R}^2,\widehat{g})} F(X) e^{\langle X,\rho\rangle_g - \frac{1}{2}\mathbb{E}[\langle X,\rho\rangle_g^2]}\, \mathbb{P}_g(DX) = \int_{H^{-1}(\mathbb{R}^2,\widehat{g})} F(X + \langle G_g(\cdot,y),\rho\rangle_g)\, \mathbb{P}_g(DX)\,,$$

where

$$\langle G_g(\cdot,y),\rho\rangle_g = \int_{\mathbb{R}^2} G_g(x,y)\rho(x)\, \lambda_g(dx) \in H^{-1}(\mathbb{R}^2,g)\,.$$

Theorem 5.4.3. (Weyl anomaly) *Let $F \in L^1(\nu_{\widehat{g}})$ and $g = e^{\varphi}\widehat{g}$ be conformally equivalent to $\widehat{g}$. Then*

$$\int_{H^{-1}(\mathbb{R}^2,\widehat{g})} F(\phi_g(X))\, \nu_g(DX) = e^{\frac{Q^2}{16\pi}\int_{\mathbb{R}^2}(|\nabla_{\widehat{g}}\varphi|^2 + 2R_{\widehat{g}}\varphi)\, d\lambda_{\widehat{g}}} \int_{H^{-1}(\mathbb{R}^2,\widehat{g})} F(\phi_{\widehat{g}}(X))\, \nu_{\widehat{g}}(DX)\,,$$

where $\phi_g(X) = X + \frac{Q}{2}\log g$. Thus the functional integral is invariant under a conformal change of metric up to some anomaly term $A(\varphi,g) = e^{\frac{Q^2}{16\pi}\int_{\mathbb{R}^2}(|\nabla_{\widehat{g}}\varphi|^2+2R_{\widehat{g}}\varphi)\, d\lambda_{\widehat{g}}}$.

Proof. Applying Lemma 5.4.1 yields

$$\int F(\phi_g)\, d\nu_g = \int F\left(\phi_{\widehat{g}} + \frac{Q}{2}\varphi\right) e^{-\frac{Q}{4\pi}\int R_g(c+X_{\widehat{g}})\, \lambda_g - \mu e^{\gamma c} M_{\widehat{g},\gamma}(e^{\gamma\frac{Q}{2}\varphi})}\, d\mathbb{P}_{\widehat{g}}\, dc\,.$$

Next we simplify the curvature integral in the exponent. By the Gauss–Bonnet theorem[6] $\int_{\mathbb{R}^2} R_g\, d\lambda_g = \int_{\mathbb{R}^2} R_{\widehat{g}}\, d\lambda_{\widehat{g}}$. Also since $R_g = e^{-\varphi}(R_{\widehat{g}} - \Delta_{\widehat{g}}\varphi)$, we get

$$\int_{\mathbb{R}^2} R_g X_{\widehat{g}}\, d\lambda_g = \int_{\mathbb{R}^2} (R_{\widehat{g}} - \Delta_{\widehat{g}}\varphi) X_{\widehat{g}}\, d\lambda_{\widehat{g}} = \int_{\mathbb{R}^2} R_{\widehat{g}} X_{\widehat{g}}\, d\lambda_{\widehat{g}} - \langle X_{\widehat{g}}, \Delta_{\widehat{g}}\varphi\rangle_{\widehat{g}}\,.$$

Hence

$$\int F(\phi_g)\, d\nu_g = \int F(\phi_{\widehat{g}} + \frac{Q}{2}\varphi) e^{-\frac{1}{4\pi}\int_{\mathbb{R}^2} QR_{\widehat{g}}(c+X_{\widehat{g}})\, d\lambda_{\widehat{g}}} e^{\frac{Q}{4\pi}\langle X_{\widehat{g}},\Delta_{\widehat{g}}\varphi\rangle_{\widehat{g}}} e^{-\mu e^{\gamma c} M_{\widehat{g},\gamma}(e^{\gamma\frac{Q}{2}\varphi})}\, d\mathbb{P}_{\widehat{g}}\, dc\,.$$

Now we want to apply the Girsanov theorem to the term $e^{\frac{Q}{4\pi}\langle X_{\widehat{g}},\Delta_{\widehat{g}}\varphi\rangle_{\widehat{g}}}$. This has the effect of shifting $X_{\widehat{g}}$ by

$$\frac{Q}{4\pi}\int_{\mathbb{R}^2} G_{\widehat{g}}(x,y)\Delta_{\widehat{g}}\varphi(y)\, \lambda_{\widehat{g}}(dy) = -\frac{Q}{2}(\varphi(x) - m_{\widehat{g}}(\varphi))\,,$$

[6] The Gauss–Bonnet theorem states that the integral of the curvature is $\int R_g\, d\lambda_g = 8\pi(1-G)$ where G is the genus of the manifold (zero for the sphere). Thus the integral agrees for different metrics.

where we used the fact that $G_{\widehat{g}}$ is the Green kernel of the problem $-\Delta_{\widehat{g}} f = 2\pi\rho$ with $\int f\, d\lambda_{\widehat{g}} = 0$. We also must multiply the whole integral by the normalization constant

$$\mathbb{E} e^{\frac{Q}{4\pi}\langle X_{\widehat{g}}, \Delta_{\widehat{g}}\varphi\rangle_{\widehat{g}}} = e^{\frac{Q^2}{32\pi^2}\mathbb{E}[\langle X_{\widehat{g}}, \Delta_{\widehat{g}}\varphi\rangle_{\widehat{g}}^2]},$$

where

$$\begin{aligned}
\mathbb{E}[\langle X_{\widehat{g}}, \Delta_{\widehat{g}}\varphi\rangle_{\widehat{g}}^2] &= \iint_{\mathbb{R}^2\times\mathbb{R}^2} G_{\widehat{g}}(x,y)\Delta_{\widehat{g}}\varphi(x)\Delta_{\widehat{g}}\varphi(y)\, \lambda_{\widehat{g}}(dx)\, \lambda_{\widehat{g}}(dy) \\
&= -2\pi \int_{\mathbb{R}^2} (\varphi(x) - m_{\widehat{g}}(\varphi))\Delta_{\widehat{g}}\varphi(x)\, \lambda_{\widehat{g}}(dx) \\
&= 2\pi \int_{\mathbb{R}^2} |\nabla_{\widehat{g}}\varphi|^2\, d\lambda_{\widehat{g}}\,.
\end{aligned}$$

Thus after applying the Girsanov theorem and shifting the c-integral by $\frac{Q}{2} m_{\widehat{g}}(\varphi)$ we have

$$\begin{aligned}
&\int F(\phi_g)\, d\nu_g \\
&= e^{\frac{Q^2}{16\pi}\int_{\mathbb{R}^2}|\nabla_{\widehat{g}}\varphi|^2\, d\lambda_{\widehat{g}}} \\
&\quad \times \int F(\phi_{\widehat{g}} + \frac{Q}{2} m_{\widehat{g}}(\varphi)) e^{-\frac{1}{4\pi}\int_{\mathbb{R}^2} Q R_{\widehat{g}}(c + X_{\widehat{g}} - \frac{Q}{2}\varphi + \frac{Q}{2} m_{\widehat{g}}(\varphi))\, d\lambda_{\widehat{g}}} e^{-\mu e^{\gamma c}\zeta(X_{\widehat{g}} - \frac{Q}{2}(\varphi - m_{\widehat{g}}(\varphi))}\, d\mathbb{P}_{\widehat{g}}\, dc \\
&= e^{\frac{Q^2}{16\pi}\int_{\mathbb{R}^2}(|\nabla_{\widehat{g}}\varphi|^2 + 2\int_{\mathbb{R}^2} R_{\widehat{g}}\varphi)\, \lambda_{\widehat{g}}} \int F(\phi_{\widehat{g}})\, d\nu_{\widehat{g}}\,.
\end{aligned}$$

We used the fact that $\zeta(X_{\widehat{g}} - (Q/2)\varphi) = M_{\widehat{g},\gamma}(\mathbb{R}^2)$ which follows from the definition of ζ, discussed in the proof of Lemma 5.4.1. □

Remark 5.4.4. *The Weyl anomaly now tells us the central charge of the LQFT. In the beginning of this chapter we mentioned that in the physics literature it is established that the central charge is $1 + 6Q^2$ while our computation would imply that it is $6Q^2$. The discrepancy comes from the fact that we have used the normalized distribution of the GFF instead of the unnormalized one.*

As a corollary of the Weyl anomaly relation we get covariance of the functional integral under Möbius transformations.

Corollary 5.4.5. (Möbius covariance) *Let ψ be a Möbius transform of the sphere. Then*

$$\int_{H^{-1}(\mathbb{R}^2,\widehat{g})} F\left(\phi_g \circ \psi\right) \nu_g(DX) = \int_{H^{-1}(\mathbb{R}^2,\widehat{g})} F\left(\phi_g - Q\log|\psi'|\right) \nu_g(DX)\,,$$

where $\phi_g = X + \frac{Q}{2}\log g$ and $\psi' : \mathbb{R}^2 \to \mathbb{R}^2$ denotes the complex derivative of ψ.

Proof. Our goal is to apply the Weyl anomaly. Hence we want to write the left-hand side as $\int F(f(\phi_{g_\psi}))\, d\nu_{g_\psi}$, where $g_\psi = |\psi'|^2 g\circ\psi$. By definition

$$\begin{aligned}
&\int F(\phi_g\circ\psi)\, d\nu_g \\
&= \int F\left(c + X\circ\psi + \frac{Q}{2}\log(g\circ\psi)\right) e^{-\frac{1}{4\pi}\int_{\mathbb{R}^2} QR_g(c+X)\, d\lambda_g} e^{-\mu e^{\gamma c} M_{g,\gamma}(\mathbb{R}^2)}\, \mathbb{P}_g(DX)\, dc\,,
\end{aligned}$$

and recall that $X_g\circ\psi \in H^{-1}(\mathbb{R}^2, g)$ is defined by

$$\langle X_g\circ\psi, f\rangle_g := \langle X_g\circ\psi, fe^{-\varphi}\rangle_{g_\psi} \overset{d}{=} \langle X_{g_\psi}, fe^{-\varphi}\rangle_{g_\psi}\,,$$

where $e^{-\varphi} = g/g_\psi$. By Möbius covariance of the chaos measure 5.3.4 we have the equality in distribution $M_{g,\gamma}(\mathbb{R}^2) \overset{d}{=} M_{g_\psi,\gamma}(\mathbb{R}^2)$. Recall the definition $R_g = -\Delta_g \log g$ and that $\Delta_g = g^{-1}\Delta$. Note that

$$\begin{aligned}
\mathbb{E}\left[\langle X_{g_\psi}, R_{g_\psi}\rangle_{g_\psi}^2\right] &= \iint_{\mathbb{R}^2\times\mathbb{R}^2} G_{g_\psi}(x,y)\frac{\Delta\log g_\psi(x)}{|\psi'(x)|^2 g\circ\psi(x)}\frac{\Delta\log g_\psi(y)}{|\psi'(y)|^2 g\circ\psi(y)}\,\lambda_{g_\psi}(dx)\,\lambda_{g_\psi}(dy) \\
&= \iint_{\mathbb{R}^2\times\mathbb{R}^2} G_g(x,y) R_g(x) R_g(y)\,\lambda_g(dx)\,\lambda_g(dy) \\
&= \mathbb{E}[\langle X_g, R_g\rangle_g^2]\,.
\end{aligned}$$

Hence $\int X_g R_g\, d\lambda_g \overset{d}{=} \int X_{g_\psi} R_{g_\psi}\, d\lambda_{g_\psi}$. By Gauss–Bonnet we have

$$\int_{\mathbb{R}^2} QR_g c\,\lambda_g(dx) = \int_{\mathbb{R}^2} QR_{g_\psi} c\,\lambda_{g_\psi}(dx)\,.$$

These observations and $X_g\circ\psi \overset{d}{=} X_{g_\psi}$ together yield

$$\begin{aligned}
&\int F(\phi_g\circ\psi)\, d\nu_g \\
&= \int F\left(c + X + \frac{Q}{2}\log(g\circ\psi)\right) e^{-\frac{1}{4\pi}\int_{\mathbb{R}^2} QR_{g_\psi}(c+X)\, d\lambda_{g_\psi}} e^{-\mu e^{\gamma c} M_{g_\psi,\gamma}(\mathbb{R}^2)}\, \mathbb{P}_{g_\psi}(DX)\, dc \\
&= \int F\left(\phi_{g_\psi} - Q\log|\psi'|^2\right)\, \nu_{g_\psi}(DX)\,.
\end{aligned}$$

Now by Weyl anomaly 5.4.3 this is equal to

$$e^{A(\varphi,g)}\int F(\phi_g - Q\log|\psi'|)\, d\nu_g\,,$$

where

$$A(\varphi,g) = \frac{Q^2}{16\pi}\int_{\mathbb{R}^2}(|\nabla_g\varphi|^2 + 2R_g\varphi)\, d\lambda_g\,,\quad \varphi = \log\left(|\psi'|^2\frac{g\circ\psi}{g}\right)\,.$$

It holds that $A(\varphi, g) = 0$. To see this first observe that

$$\begin{aligned}
&\int_{\mathbb{R}^2} |\nabla_g \varphi(x)|^2 \,\lambda_g(dx) + 2\int_{\mathbb{R}^2} R_g(x)\varphi(x)\,\lambda_g(dx) \\
&= -\int_{\mathbb{R}^2} \varphi(x)\Delta_g\varphi(x)\,\lambda_g(dx) + 2\int_{\mathbb{R}^2} R_g(x)\varphi(x)\,\lambda_g(dx)\,.
\end{aligned}$$

We have $\varphi = \log|\psi'(x)|^2 + \log(g\circ\psi)(x) - \log g(x)$. It holds that $\Delta_g \log|\psi'(x)|^2 = 0$ and thus all the integrals with the term $\log|\psi'(x)|^2$ vanish since we can always use integration by parts to get a term $\Delta_g \log|\psi'(x)|^2$ (recall that $R_g = -\Delta_g \log g$). Define $h = \log g$ and now

$$\begin{aligned}
&\int_{\mathbb{R}^2} |\nabla_g \varphi(x)|^2 \,\lambda_g(dx) + 2\int_{\mathbb{R}^2} R_g(x)\varphi(x)\,\lambda_g(dx) \\
&= \int_{\mathbb{R}^2} |\nabla_g (h\circ\psi(x) - h(x)|^2\,\lambda_g(dx) + 2\int_{\mathbb{R}^2} \nabla_g((h\circ\psi)(x) - h(x))\cdot\nabla_g h(x)\,\lambda_g(dx) \\
&= \int_{\mathbb{R}^2} |\nabla_g (h\circ\psi)(x)|^2\,\lambda_g(dx) - \int_{\mathbb{R}^2} |\nabla_g h(x)|^2\,\lambda_g(dx)\,.
\end{aligned}$$

This vanishes by the conformal invariance of the Dirichlet energy. □

5.5 Vertex operators and Seiberg bounds

The covariance of ν_g under Möbius transformations suggests that ν_g is "too symmetric" to be a finite measure. Indeed, for the spherical metric $\widehat{g}$ we have

$$\int_{H^{-1}(\mathbb{R}^2,\widehat{g})} \nu_{\widehat{g}}(DX) = \int_{\mathbb{R}} e^{-2Qc} \int_{H^{-1}(\mathbb{R}^2,\widehat{g})} e^{-\mu e^{\gamma c} M_{\widehat{g},\gamma}(\mathbb{R}^2)}\,\mathbb{P}_{\widehat{g}}(DX_{\widehat{g}})\;dc\,.$$

This diverges due to the integration over negative c. Weyl anomaly then implies that the integral diverges for all metrics conformally equivalent to $\widehat{g}$. Since we decide to use the interpretation

$$\nu_g(DX) = e^{-S_L(X,g)}\,\mathcal{D}X\,,$$

this means that the partition function of the Liouville field theory on the sphere

$$Z_L(g) = \int_{H^{-1}(\mathbb{R}^2,\widehat{g})} e^{-S_L(X,g)}\,\mathcal{D}X$$

diverges. On surfaces with genus 2 or higher the partition function converges, see Proposition 4.1 in [20].

However, in this section we prove that the correlation functions (or the exponential moments)

$$Z(g;(z_1,\alpha_1),\ldots,(z_n,\alpha_n)) = \int e^{\sum_{i=1}^n \alpha_i X(z_i)} e^{-S_L(X,g)}\,\mathcal{D}X = \int e^{\sum_{i=1}^n \alpha_i X(z_i)} \nu_g(DX) \quad (5.5.1)$$

have a non-trivial limit under the necessary and sufficient conditions

$$\forall i \quad \alpha_i < Q \quad \text{and} \quad \sum_{i=1}^{n} \alpha_i > 2Q\,.$$

These are called the Seiberg bounds, and they immediately imply that we need at least three points z_i for convergence.

Since the measure ν_g lives in a space of generalized functions, we have to use regularization to make sense of the correlations (5.5.1). Let $g = e^{\varphi}\widehat{g}$. We define the regularized vertex operators by

$$V_{g,\alpha,\varepsilon}(x) := \varepsilon^{\alpha^2/2} e^{\alpha(c+X_{g,\varepsilon}(x)+\frac{Q}{2}\log g(x))}$$

and the correlation function of the regularized vertex operators

$$\begin{aligned}&\Pi_{\gamma,\mu}^{(z_i,\alpha_i)_i}(g,F,\varepsilon)\\&:= e^{\frac{1}{24\pi}\int_{\mathbb{R}^2}|\nabla_{\widehat{g}}\varphi|^2+2R_{\widehat{g}}\varphi)\lambda_{\widehat{g}}}\int F(\phi_g)\prod_i V_{g,\alpha_i,\varepsilon}(z_i)e^{-\frac{Q}{4\pi}\int_{\mathbb{R}^2}R_g(c+X_g)\,d\lambda_g-\mu e^{\gamma c}M_{g,\gamma,\varepsilon}(\mathbb{R}^2)}\,\mathbb{P}_g(DX)\,dc\,.\end{aligned}$$

Recall that $\phi_g = c + X_g + \frac{Q}{2}\log g$. Note that, when $F \equiv 1$, this is a regularized version of (5.5.1). We would like to pass to the limit to obtain

$$\lim_{\varepsilon\to 0}\Pi_{\gamma,\mu}^{(z_i,\alpha_i)_i}(g,F,\varepsilon) = \int_{H^{-1}(\mathbb{R}^2,\widehat{g})} F(\phi_g)\prod_i e^{\alpha_i\phi_g(z_i)}\,\nu_g(DX)\,.$$

The formal terms $e^{\alpha_i\phi_g(z_i)}$ are called vertex operators and the integral with $F \equiv 1$ is called the correlation function of the vertex operators. These functions give a precise mathematical meaning for the correlations (5.5.1).

Theorem 5.5.1. *Let $\sum_i \alpha_i > 2Q$. Then the limit*

$$\lim_{\varepsilon\to 0}\Pi_{\gamma,\mu}^{(z_i,\alpha_i)_i}(\widehat{g},F,\varepsilon) =: \Pi_{\gamma,\mu}^{(z_i,\alpha_i)_i}(\widehat{g},F)$$

exists for bounded and continuous F. The limit is nonzero if $\alpha_i < Q$ for all i. If $\alpha_i \geqslant Q$ for some i then the limit vanishes identically.

Proof. We begin by simplifying $\Pi_{\gamma,\mu}^{(z_i,\alpha_i)_i}(\widehat{g},F,\varepsilon)$. For the spherical metric $\widehat{g}$ we have $R_{\widehat{g}} = 2$. Since $X_{\widehat{g}}$ has a vanishing $\lambda_{\widehat{g}}$-mean, we get

$$\int_{\mathbb{R}^2} R_{\widehat{g}}(c + X_{\widehat{g}})\,d\lambda_{\widehat{g}} = 8\pi c\,.$$

Proposition 5.3.1 implies that

$$\mathbb{E}X_{\widehat{g},\varepsilon}(x)^2 = \log\frac{1}{\varepsilon} - \frac{1}{2}\log\widehat{g}(x) + \theta_{\widehat{g}} + \log 2 + o(1)\,.$$

as $\varepsilon \to 0$. Thus

$$\varepsilon^{\frac{\alpha_i^2}{2}} e^{\alpha_i X_{\widehat{g},\varepsilon}(x)} = e^{\frac{\alpha_i^2}{2}(\theta_{\widehat{g}}+\log 2)} \widehat{g}(z_i)^{-\frac{\alpha_i^2}{4}} e^{\alpha_i X_{\widehat{g},\varepsilon}(x) - \frac{\alpha_i^2}{2}\mathbb{E}[X_{\widehat{g},\varepsilon}(x)^2]}(1+o(1)) \,.$$

Let $\mathbb{E}_{\widehat{g}}$ be the expectation with respect to $\mathbb{P}_{\widehat{g}}$. Now we have

$$\begin{aligned}
&\Pi_{\gamma,\mu}^{(z_i,\alpha_i)_i}(\widehat{g}, F, \varepsilon) \\
&= \int_{\mathbb{R}} e^{-2Qc} \mathbb{E}_{\widehat{g}}\Big[F(\phi_{\widehat{g}}) \prod_i e^{\frac{\alpha_i^2}{2}(\theta_{\widehat{g}}+\log 2)} \widehat{g}(z_i)^{-\frac{\alpha_i^2}{4}} e^{\alpha_i(c + X_{\widehat{g},\varepsilon}(z_i) + \frac{Q}{2}\log \widehat{g}(z_i)) - \frac{\alpha_i^2}{2}\mathbb{E}[X_{\widehat{g},\varepsilon}(z_i)^2]}(1+o(1)) \\
&\quad \exp\Big(-\mu e^{\gamma c} M_{\widehat{g},\gamma,\varepsilon}(\mathbb{R}^2)\Big)\Big]\, dc \\
&= e^D \left(\prod_i \widehat{g}(z_i)^{-\frac{\alpha_i^2}{4} + \frac{Q}{2}\alpha_i}\right) \int_{\mathbb{R}} e^{\left(\sum_i \alpha_i - 2Q\right)c} \mathbb{E}_{\widehat{g}}\Big[F(\phi_{\widehat{g}}) e^{\sum_i (\alpha_i X_{\widehat{g},\varepsilon}(z_i) - \frac{\alpha_i^2}{2}\mathbb{E}[X_{\widehat{g},\varepsilon}(z_i)^2])}(1+o(1)) \\
&\quad \exp\Big(-\mu e^{\gamma c} M_{\widehat{g},\gamma,\varepsilon}(\mathbb{R}^2)\Big)\Big]\, dc \,,
\end{aligned}$$

where $D = \sum_i \frac{\theta_{\widehat{g}}+\log 2}{2}\alpha_i^2$. Note that

$$\mathbb{E}\left[\sum_i \alpha_i X_{\widehat{g},\varepsilon}(z_i)\right]^2 = \sum_{i=1}^n \alpha_i^2 \mathbb{E}\left[X_{\widehat{g},\varepsilon}(z_i)^2\right] + \sum_{i\neq j} \alpha_i\alpha_j \mathbb{E}\left[X_{\widehat{g},\varepsilon}(z_i) X_{\widehat{g},\varepsilon}(z_j)\right] \,.$$

We define the circle average

$$H_{\widehat{g},\varepsilon}(x) = \sum_i \alpha_i \langle G(\cdot, x), \rho_{z_i,\varepsilon}\rangle_{\widehat{g}} \,.$$

Applying Girsanov theorem on the term $e^{\sum_i (\alpha_i X_{\widehat{g},\varepsilon}(z_i) - \frac{\alpha_i^2}{2}\mathbb{E}[X_{\widehat{g},\varepsilon}(z_i)^2])}$ yields

$$\begin{aligned}
\Pi_{\gamma,\mu}^{(z_i,\alpha_i)_i}(\widehat{g}, F, \varepsilon) = e^{C_\varepsilon(z)} &\left(\prod_i \widehat{g}(z_i)^{-\frac{\alpha_i^2}{4} + \frac{Q}{2}\alpha_i}\right) \int_{\mathbb{R}} e^{\left(\sum_i \alpha_i - 2Q\right)c} \mathbb{E}_{\widehat{g}}\Big[F(\phi_g + H_{\widehat{g},\varepsilon})(1+o(1)) \\
&\exp\Big(-\mu e^{\gamma c} M_{\widehat{g},\gamma,\varepsilon}(e^{\gamma H_{\widehat{g},\varepsilon}})\Big)\Big]\, dc \,,
\end{aligned}$$

where

$$C_\varepsilon(z) = \frac{1}{2}\sum_{i\neq j} \alpha_i\alpha_j \mathbb{E}[X_{\widehat{g},\varepsilon}(z_i) X_{\widehat{g},\varepsilon}(z_j)] + D \overset{\varepsilon\to 0}{\longrightarrow} \frac{1}{2}\sum_{i\neq j}\alpha_i\alpha_j G_{\widehat{g}}(z_i,z_j) + D \,.$$

We also define $C(z) = \lim_{\varepsilon\to 0} C_\varepsilon(z)$. It holds that $H_{\widehat{g},\varepsilon}(x) \to H_{\widehat{g}}(x) := \sum_i \alpha_i G_{\widehat{g}}(z_i, x)$ in $H^{-1}(\mathbb{R}^2, \widehat{g})$[7]. This implies that it suffices to consider the convergence in the case $F \equiv 1$.

[7] This follows from the fact that for all $\varphi \in H^1(\mathbb{R}^2, \widehat{g})$ we have

$$\begin{aligned}
|\langle G_{\widehat{g}}(z_i,\cdot), \varphi\rangle - \langle G_{\widehat{g},\varepsilon}(z_i,\cdot),\varphi\rangle| &= \left|\int_{\mathbb{R}^2} \varphi(x) \log\frac{1}{|x-z_i|}\,\lambda_{\widehat{g}}(dx) - \int_{\mathbb{R}^2}\varphi(x)\log\frac{1}{\max\{\varepsilon, |x-z_i|\}}\,\lambda_{\widehat{g}}(dx)\right|^2 \\
&\leqslant \|\varphi\|^2_{L^2(\mathbb{R},\widehat{g})} \left|\int_{B(z_i,\varepsilon)} \left(\log\frac{\varepsilon}{|x-z_i|}\right)^2 \lambda_{\widehat{g}}(dx)\right| \,.
\end{aligned}$$

This converges to 0 as $\varepsilon \to 0$.

Define

$$Z_\varepsilon = M_{\widehat{g},\gamma,\varepsilon}(e^{\gamma H_{\widehat{g},\varepsilon}}) = \varepsilon^{\gamma^2/2} \int_{\mathbb{R}^2} e^{\gamma(X_{\widehat{g},\varepsilon} + H_{\widehat{g},\varepsilon} + \frac{Q}{2}\log \widehat{g})}\, d\lambda\,.$$

We have $|H_{\widehat{g},\varepsilon}(x)| \leqslant C_\varepsilon$ because $|G(z_i,x)|$ tends to a constant as $|x| \to \infty$. Now $\mathbb{E}(Z_\varepsilon) < \infty$ by Proposition 5.3.3 and thus $Z_\varepsilon < \infty$ $\mathbb{P}$-almost surely. This implies that there exists $A > 0$ such that $\mathbb{P}(Z_\varepsilon \leqslant A) > 0$ and hence we may estimate (with $\mathbb{E}e^{-Z_\varepsilon} \geqslant e^{-\mathbb{E}Z_\varepsilon}$)

$$\begin{aligned}
\Pi_{\gamma,\mu}^{(z_i,\alpha_i)_i}(\widehat{g},1,\varepsilon) &\geqslant e^{C_\varepsilon(z)} \left(\prod_i \widehat{g}(z_i)^{-\frac{\alpha_i^2}{4}+\frac{Q}{2}\alpha_i}\right) \int_{-\infty}^0 e^{\left(\sum_i \alpha_i - 2Q\right)c}(1+o(1))\mathbb{E}e^{-\mu e^{\gamma c} Z_\varepsilon}\, dc \\
&\geqslant e^{C_\varepsilon(z)} \left(\prod_i \widehat{g}(z_i)^{-\frac{\alpha_i^2}{4}+\frac{Q}{2}\alpha_i}\right) \int_{-\infty}^0 e^{\left(\sum_i \alpha_i - 2Q\right)c}(1+o(1))e^{-\mu e^{\gamma c}\mathbb{E} Z_\varepsilon}\, dc \\
&\geqslant e^{C_\varepsilon(z)} \left(\prod_i \widehat{g}(z_i)^{-\frac{\alpha_i^2}{4}+\frac{Q}{2}\alpha_i}\right) \int_{\mathbb{R}} e^{\left(\sum_i \alpha_i - 2Q\right)c}(1+o(1))e^{-\mu e^{\gamma c} A\mathbb{P}(Z_\varepsilon \leqslant A)}\, dc \\
&= +\infty
\end{aligned}$$

if $\sum_i \alpha_i \leqslant 2Q$. This leads us to the first Seiberg bound $\sum_i \alpha_i > 2Q$.

To finish, we make the substitution $u = \mu e^{\gamma c} Z_\varepsilon \iff c = \gamma^{-1}\log \frac{u}{\mu Z_\varepsilon}$ which yields

$$\begin{aligned}
\int_{\mathbb{R}} \mathbb{E}_{\widehat{g}}\left[e^{c\left(\sum_i \alpha_i - 2Q\right)} e^{-\mu e^{\gamma c} Z_\varepsilon}\right] dc &= \mathbb{E}_{\widehat{g}}\left[\int_0^\infty e^{\frac{1}{\gamma}\log\frac{u}{\mu Z_\varepsilon}\left(\sum_i \alpha_i - 2Q\right)} e^{-u} \frac{du}{\gamma u}\right] \\
&= \frac{1}{\gamma}\mathbb{E}_{\widehat{g}}\left[(\mu Z_\varepsilon)^{-\frac{1}{\gamma}\left(\sum_i \alpha_i - 2Q\right)}\right] \int_0^\infty u^{\frac{1}{\gamma}\left(\sum_i \alpha_i - 2Q\right)-1} e^{-u}\, du \\
&= \frac{\mu^{-\frac{1}{\gamma}\left(\sum_i \alpha_i - 2Q\right)}}{\gamma} \Gamma\left(\frac{1}{\gamma}(\sum_i \alpha_i - 2Q)\right) \mathbb{E}_{\widehat{g}}\left[Z_\varepsilon^{-\frac{1}{\gamma}\left(\sum_i \alpha_i - 2Q\right)}\right],
\end{aligned}$$

where Γ denotes the Gamma function. In conclusion, we have shown that

$$\begin{aligned}
&\Pi_{\gamma,\mu}^{(z_i,\alpha_i)_i}(\widehat{g},F,\varepsilon) \\
&= e^{C_\varepsilon(z)} \left(\prod_i \widehat{g}(z_i)^{\Delta_\alpha}\right) \int_{\mathbb{R}} e^{sc}\mathbb{E}_{\widehat{g}}\Big[F(\phi_g + H_{\widehat{g},\varepsilon}) \exp\Big(-\mu e^{\gamma c} M_{\widehat{g},\gamma,\varepsilon}(e^{\gamma H_{\widehat{g},\varepsilon}})\Big)\Big]\, dc\,,
\end{aligned}$$

where $\Delta_\alpha = \frac{\alpha}{2}(Q - \frac{\alpha}{2})$ and $s = \big(\sum_i \alpha_i - 2Q\big) > 0$. Also,

$$\Pi_{\gamma,\mu}^{(z_i,\alpha_i)_i}(\widehat{g},1,\varepsilon) = e^{C_\varepsilon(z)} \left(\prod_i \widehat{g}(z_i)^{\Delta_\alpha}\right) \frac{\mu^{-\frac{s}{\gamma}}}{\gamma}\Gamma\left(\frac{s}{\gamma}\right) \mathbb{E}_{\widehat{g}}\left[Z_\varepsilon^{-\frac{s}{\gamma}}\right]$$

Now the claim follows from the following lemma. □

Lemma 5.5.2. *Let $s < 0$. If $\alpha_i < Q$ for all i then*

$$\lim_{\varepsilon \to 0} \mathbb{E}_{\widehat{g}}[Z_\varepsilon^s] = \mathbb{E}_{\widehat{g}}[Z_0^s] \,,$$

where

$$Z_0 = \int_{\mathbb{R}^2} e^{\gamma H_{\widehat{g}}(x)} M_{\widehat{g},\gamma}(dx)$$

and $0 < \mathbb{E}_{\widehat{g}}[Z_0^s] < \infty$.

If $\alpha_i \geqslant Q$ for some i, then the limit is 0.

Proof. The proof is quite involved and can be found in [7] (Lemma 3.3). □

As a corollary of the previous computation, we obtain the following result.

Theorem 5.5.3. (KPZ scaling laws) *We have the following scaling laws for the insertions $(z_i, \alpha_i)_i$*

$$\Pi_{\gamma,\mu}^{(z_i,\alpha_i)_i}(\widehat{g}, 1) = \mu^{\frac{2Q - \sum_i \alpha_i}{\gamma}} \Pi_{\gamma,1}^{(z_i,\alpha_i)_i}(\widehat{g}, 1)$$

where

$$\Pi_{\gamma,1}^{(z_i,\alpha_i)_i}(\widehat{g}, 1) = e^{C(z)} \left(\prod_i \widehat{g}(z_i)^{\Delta_{\alpha_i}} \right) \frac{1}{\gamma} \Gamma \left(\frac{1}{\gamma} \big(\sum_i \alpha_i - 2Q \big) \right) \mathbb{E}_{\widehat{g}} \left[Z_0^{-\frac{1}{\gamma} \left(\sum_i \alpha_i - 2Q \right)} \right]$$

and we defined

$$\Delta_\alpha = -\frac{\alpha^2}{4} + \frac{Q}{2}\alpha = \frac{\alpha}{2} \left(Q - \frac{\alpha}{2} \right) \,.$$

Also,

$$\Pi_{\gamma,\mu}^{(z_i,\alpha_i)_i}(\widehat{g}, F) = e^{C(z)} \left(\prod_i \widehat{g}(z_i)^{\Delta_{\alpha_i}} \right) \int_{\mathbb{R}} e^{\left(\sum_i \alpha_i - 2Q \right) c} \mathbb{E}_{\widehat{g}} \Big[F(\phi_g + H_{\widehat{g}})) e^{-\mu e^{\gamma c} Z_0} \Big] \, dc \,.$$

Using the Möbius covariance of ν_g we see that the vertex operators are the primary fields of Liouville field theory. First we define the short hand notation

$$\nu_{g,\varepsilon}(DX, dc) := e^{-\frac{Q}{4\pi} \int_{\mathbb{R}^2} R_g (c+X) \, d\lambda_g - \mu e^{\gamma c} M_{g,\gamma,\varepsilon}(\mathbb{R}^2)} \, \mathbb{P}_g(DX) \, dc \,.$$

Let $g = e^\varphi \widehat{g}$. Now

$$\begin{aligned} \Pi_{\gamma,\mu}^{(\psi(z_i),\alpha_i)_i}(g, 1, \varepsilon) &= e^{\frac{1}{24\pi} \int_{\mathbb{R}^2} |\nabla_{\widehat{g}} \varphi|^2 + 2R_{\widehat{g}} \varphi) \lambda_{\widehat{g}}} \int \prod_i V_{g,\alpha_i,\varepsilon}(\psi(z_i)) \nu_{g,\varepsilon}(DX, dc) \\ &= e^{\frac{1}{24\pi} \int_{\mathbb{R}^2} |\nabla_{\widehat{g}} \varphi|^2 + 2R_{\widehat{g}} \varphi) \lambda_{\widehat{g}}} \int \prod_i |\varepsilon \psi'(z_i)|^{\frac{\alpha_i^2}{2}} e^{\alpha_i (\phi_g \circ \psi)_\varepsilon (z_i)} \nu_{g,\varepsilon}(DX, dc) \,. \end{aligned}$$

In the limit $\varepsilon \to 0$ the Möbius covariance yields

$$\Pi_{\gamma,\mu}^{(\psi(z_i),\alpha_i)_i}(g,1) = \lim_{\varepsilon\to 0} \Pi_{\gamma,\mu}^{(\psi(z_i),\alpha_i)_i}(g,1,\varepsilon) = \prod_i |\psi'(z_i)|^{\frac{\alpha_i^2}{2}} e^{-\alpha_i Q \log|\psi'(z_i)|} \Pi_{\gamma,\mu}^{(z_i,\alpha_i)_i}(\widehat{g},1)$$
$$= \prod_i |\psi'(z_i)|^{-2\Delta_{\alpha_i}} \Pi_{\gamma,\mu}^{(z_i,\alpha_i)_i}(g,1)\,.$$

Here $\Delta_{\alpha_i} = \frac{\alpha_i}{2}(Q - \frac{\alpha_i}{2})$ is the conformal weight of the primary field (vertex operator) $e^{\alpha_i \phi_g}$.

5.6 The Liouville quantum gravity measure

We are finally ready to define the (non-critical) Liouville quantum gravity measure $\rho_{g,\gamma,\mu}$ mentioned in the Introduction. Set $\mathbf{z} = (z_1, \ldots, z_n)$, $\alpha = (\alpha_1, \ldots, \alpha_n)$ and we will assume X to be distributed according to the measure

$$d\nu_g^{(\mathbf{z},\alpha)} := Z^{-1} \prod_{i=1}^n V_{g,\alpha_i}(z_i)\, d\nu_g\,,$$

where

$$Z = \int_{H^{-1}(\mathbb{R}^2,g)} \prod_{i=1}^n V_{g,\alpha_i}(z_i)\, d\nu_g$$

and g is some fixed background metric, conformally equivalent to the Euclidean metric. We assume that the numbers α_i satisfy the Seiberg bounds. Then we define $\rho_{g,\gamma,\mu}$ to be the exponential of $\phi_g = X + \frac{Q}{2}\log g$. More precisely the definition is

Definition 5.6.1. (Liouville quantum gravity measure) *We define the measure $\rho_{g,\gamma,\mu}$ as the exponential of ϕ_g with respect to the distribution $\nu_g^{(z,\alpha)}$, that is, if $F : \mathbb{R}^k \to \mathbb{R}$ and $A_i \in \mathcal{B}(\mathbb{R}^2)$ are Borel sets, then*

$$\mathbb{E}[F(\rho_{g,\gamma.\mu}(A_1), \ldots, \rho_{g,\gamma,\mu}(A_k))]$$
$$:= Z^{-1} \lim_{\varepsilon\to 0} \int F\left(e^{\gamma c} M_{g,\gamma,\varepsilon}(A_1), \ldots, e^{\gamma c} M_{g,\gamma,\varepsilon}(A_k)\right) \prod_{i=1}^n V_{g,\alpha_i,\varepsilon}(z_i)\, \nu_{g,\varepsilon}(DX, dc)\,,$$

where

$$\nu_{g,\varepsilon}(DX, dc) = e^{-\frac{Q}{4\pi}\int_{\mathbb{R}^2} R_g(c+X)\, d\lambda_g - \mu e^{\gamma c} M_{g,\gamma,\varepsilon}(\mathbb{R}^2)}\, \mathbb{P}_g(DX)\, dc\,,$$

and Z is as above.

From the random surface point of view (see Section 1.2) the motivation to construct the random measure $\rho_{g,\gamma,\mu}$ was the conjectured relationship with the scaling limit of random planar maps (possibly coupled with statistical physics models) conformally embedded to the sphere. Next we check that the total volumes of $\rho_{g,\gamma,\mu}$ and the scaling limit agree in distribution. Recall that the total volume of the scaling limit measure has a gamma distribution with parameters $2 - 4/\gamma^2$ and μ. For the Liouville measure with general insertions we have the following result.

Theorem 5.6.2. *The volume of the space $\rho_{\widehat{g},\gamma,\mu}(\mathbb{R}^2)$ is gamma distributed with parameters $\frac{\sum_i \alpha_i - 2Q}{\gamma}$ and μ.*

Proof. Define $s = \big(\sum_i \alpha_i - 2Q\big)$. Using Theorem 5.5.3 and making the change of variables $y = e^{\gamma c} Z_0$ yields

$$\begin{aligned}
&\mathbb{E}[F(\rho_{\widehat{g},\gamma,\mu}(\mathbb{R}^2))] \\
&= Z^{-1} \int \int_{\mathbb{R}} F(e^{\gamma c} M_{\widehat{g},\gamma}(\mathbb{R}^2)) \prod_{i=1}^{n} V_{\widehat{g},\alpha_i}(z_i)\, e^{-\frac{1}{4\pi}\int_{\mathbb{R}^2} Q R_{\widehat{g}}(c+X)\, d\lambda_{\widehat{g}}} e^{-\mu e^{\gamma c} M_{\widehat{g},\gamma}(\mathbb{R}^2)} \mathbb{P}_{\widehat{g}}(DX)\, dc \\
&= Z^{-1} e^{C(\mathbf{z})} \left(\prod_i \widehat{g}(z_i)^{\Delta_{\alpha_i}} \right) \int_{\mathbb{R}} e^{sc} \mathbb{E}_{\widehat{g}} \Big[F(e^{\gamma c} Z_0) e^{-\mu e^{\gamma c} Z_0} \Big]\, dc\,. \\
&= Z^{-1} e^{C(\mathbf{z})} \left(\prod_i \widehat{g}(z_i)^{\Delta_{\alpha_i}} \right) \mathbb{E}_{\widehat{g}} \left[Z_0^{-\frac{s}{\gamma}} \right] \int_0^\infty F(y) y^{\frac{s}{\gamma}} \frac{e^{-\mu y}}{\gamma} \frac{dy}{y} \\
&= \frac{1}{\Gamma(\frac{s}{\gamma})} \int_0^\infty F(y) \mu^{\frac{s}{\gamma}} y^{\frac{s}{\gamma}-1} e^{-\mu y}\, dy\,.
\end{aligned}$$

And we are done. We used the fact that

$$Z = \mu^{-\frac{s}{\gamma}} e^{C(z)} \left(\prod_i \widehat{g}(z_i)^{\Delta_{\alpha_i}} \right) \frac{1}{\gamma} \Gamma\left(\frac{s}{\gamma}\right) \mathbb{E}_{\widehat{g}} \left[Z_0^{-\frac{s}{\gamma}} \right]\,.$$

□

Remark 5.6.3. *With three insertions z_i and setting $\alpha_i = \gamma$ for each i we see that $\rho_{\widehat{g},\gamma,\mu}(\mathbb{R}^2)$ has a gamma distribution with parameters $2 - \frac{4}{\gamma^2}$ and μ. This agrees with the scaling limit considered in Section 1.2.3. The three insertions are analogous to the three marked faces in the discrete models, which made the embedding into the sphere unique.*

Appendix A

Computations

Recall that we denote by ξ_D the function for which the Green function satisfies

$$G_D(x,y) = \frac{1}{2\pi} \log|x-y| + \xi_D(x,y)\,,$$

see Definition 2.6.2.

Lemma A.1. Fix $y \in D$ and $\varepsilon > 0$ such that $B(y,\varepsilon) \subset D$. Define $f_{y,\varepsilon} : D \to \mathbb{R}$ by

$$f_{y,\varepsilon}(x) = \frac{1}{2\pi} \log \frac{1}{\max\{|x-y|,\varepsilon\}} + \xi_D(x,y)\,.$$

Then $f_{y,\varepsilon} \in H_0^1(D)$ and $-\Delta f_{y,\varepsilon} = \rho_{y,\varepsilon}$.

Proof. The function $f_{y,\varepsilon}$ is continuous in D and the definition of ξ_D implies that $f_{y,\varepsilon} \to 0$ as $x \to \partial D$. Also, the weak gradient satisfies

$$\begin{aligned}
&\int_D |\nabla f_{y,\varepsilon}(x)|^2\,dx \\
&\quad= \int_D |\nabla \xi_D(x,y)|^2\,dx + \frac{1}{4\pi^2}\int_{D\setminus \overline{B}(y,\varepsilon)} \left|\nabla \log\frac{1}{|x-y|}\right|^2 dx \\
&\qquad+ \frac{1}{\pi}\int_{D\setminus \overline{B}(y,\varepsilon)} \nabla\xi_D(x,y)\cdot\nabla\log\frac{1}{|x-y|}\,dx \\
&\quad= \int_{\partial D} \xi_D(x,y)\nabla\xi_D(x,y)\cdot\hat{n}(x)\,dS(x) + \frac{1}{4\pi^2}\int_{\partial D}\log\frac{1}{|x-y|}\nabla\log\frac{1}{|x-y|}\cdot\hat{n}(x)\,dS(x) \\
&\qquad+ \frac{1}{4\pi^2}\log\frac{1}{\varepsilon}\int_{\partial B(y,\varepsilon)}\nabla\log\frac{1}{|x-y|}\cdot\hat{n}\,dS(x) + \frac{1}{\pi}\int_{\partial D}\xi_D(x,y)\nabla\log\frac{1}{|x-y|}\,dx \\
&\qquad+ \frac{1}{\pi}\int_{\partial B(y,\varepsilon)}\xi_D(x,y)\nabla\log\frac{1}{|x-y|}\cdot\hat{n}(x)\,dS(x)\,.
\end{aligned}$$

For example by using polar coordinates we get

$$\int_{\partial B(y,\varepsilon)} (1+\xi_D(x,y))\nabla\log\frac{1}{|x-y|}\cdot\hat{n}(x)\,dS(x) = 2\pi(1+\xi_D(x,y))\,,$$

where we used the mean value property of harmonic functions. Using the definition of ξ_D yields

$$\frac{1}{4\pi^2}\int_{\partial D}\log\frac{1}{|x-y|}\nabla\log\frac{1}{|x-y|}\cdot\hat{n}(x)\,dS(x)=-\frac{1}{2\pi}\int_{\partial D}\xi_D(x,y)\nabla\log\frac{1}{|x-y|}\cdot\hat{n}(x)\,dS(x)\,.$$

Note also that repetitively using Green's identity gives the result

$$\begin{aligned}\int_{\partial D}\xi_D(x,y)\nabla\xi_D(x,y)\cdot\hat{n}(x)\,dS(x)&=-\frac{1}{2\pi}\int_{\partial D}\log\frac{1}{|x-y|}\nabla\xi_D(x,y)\cdot\hat{n}(x)\,dS(x)\\&=-\frac{1}{2\pi}\int_{D}\nabla\log\frac{1}{|x-y|}\cdot\nabla\xi_D(x,y)\,dx\\&=-\frac{1}{2\pi}\int_{\partial D}\xi_D(x,y)\nabla\log\frac{1}{|x-y|}\cdot\hat{n}(x)\,dS(x)\\&\quad-\frac{1}{2\pi}\int_{\partial B(y,\varepsilon)}\xi_D(x,y)\nabla\log\frac{1}{|x-y|}\cdot\hat{n}(x)\,dS(x)\,.\end{aligned}$$

Plugging in all these results then yields the final result

$$\int_D|\nabla f_{y,\varepsilon}(x)|^2\,dx=\frac{1}{2\pi}\log\frac{1}{\varepsilon}+\xi_D(y,y)<\infty\,. \tag{A.0.1}$$

The conclusion is that $f_{y,\varepsilon}\in H_0^1(D)$.

Fix $g\in C_c^\infty(D)$. We first compute

$$\begin{aligned}\langle-\Delta f_{y,\varepsilon},g\rangle&=\int_D\nabla f_{y,\varepsilon}(x)\cdot\nabla g(x)\,dx\\&=\frac{1}{2\pi}\int_{D\setminus\overline{B}(y,\varepsilon)}\nabla\log\frac{1}{|x-y|}\cdot\nabla g(x)\,dx+\int_D\nabla\xi_D(x,y)\cdot\nabla g(x)\,dx\,.\end{aligned}$$

Since $x\mapsto\xi_D(x,y)$ is harmonic and g vanishes on ∂D, the latter integral vanishes due to Green's formula. Then applying Green's formula to the first integral yields

$$\begin{aligned}\langle-\Delta f_{y,\varepsilon},g\rangle&=-\frac{1}{2\pi}\int_{D\setminus\overline{B}(y,\varepsilon)}g(x)\Delta\log\frac{1}{|x-y|}\,dx+\frac{1}{2\pi}\int_{\partial B(y,\varepsilon)}g(x)\nabla\log\frac{1}{|x-y|}\cdot\hat{n}(x)\,dS(x)\\&\quad+\frac{1}{2\pi}\int_{\partial D}g(x)\nabla\log\frac{1}{|x-y|}\cdot\hat{n}(x)\,dS(x)\\&=0+\frac{1}{2\pi}\int_{\partial B(y,\varepsilon)}g(x)\,dS(x)+0\\&=\langle\rho_{y,\varepsilon},g\rangle\,.\end{aligned}$$

Now $-\Delta f_{y,\varepsilon}$ agrees with $\rho_{y,\varepsilon}$ on $C_c^\infty(D)$ which is dense in $C_c^\infty(D)$. Thus $-\Delta f_{y,\varepsilon}=\rho_{y,\varepsilon}$. □

Proposition A.2. Let $g=e^\varphi\widehat{g}$ be a metric that is conformally equivalent to the spherical metric $\widehat{g}$. Let $X_{\widehat{g}}$ and X_g be the corresponding Gaussian free fields. Then

$$\langle X_g,f\rangle_g\overset{d}{=}\langle X_{\widehat{g}}-m_g(X_{\widehat{g}}),fe^\varphi\rangle_{\widehat{g}}\,.$$

Proof. We show that the covariance structures agree. For the left-hand side we get

$$\begin{aligned}\mathbb{E}\left[\langle e^{\varphi}(X_{\widehat{g}} - m_g(X_{\widehat{g}})), f\rangle_{\widehat{g}}\langle e^{\varphi}(X_{\widehat{g}} - m_g(X_{\widehat{g}})), h\rangle_{\widehat{g}}\right] = \mathbb{E}[\langle X_{\widehat{g}}, e^{\varphi} f\rangle_{\widehat{g}}\langle X_{\widehat{g}}, e^{\varphi} h\rangle_{\widehat{g}}] \\ - m_g(h)\mathbb{E}[\langle X_{\widehat{g}}, e^{\varphi} f\rangle_{\widehat{g}}\langle X_{\widehat{g}}, e^{\varphi}\rangle_{\widehat{g}}] \\ - m_g(f)\mathbb{E}[\langle X_{\widehat{g}}, e^{\varphi}\rangle_{\widehat{g}}\langle X_{\widehat{g}}, e^{\varphi} h\rangle_{\widehat{g}}] \\ + m_g(f)m_g(h)\mathbb{E}[\langle X_{\widehat{g}}, e^{\varphi}\rangle_{\widehat{g}}\langle X_{\widehat{g}}, e^{\varphi}\rangle_{\widehat{g}}]\,.\end{aligned}$$

Now since the covariance kernel of $X_{\widehat{g}}$ is the Green function (5.2.3), we get

$$\begin{aligned}\mathbb{E}\left[\langle X_{\widehat{g}} - m_g(X_{\widehat{g}}), fe^{\varphi}\rangle_{\widehat{g}}\langle X_{\widehat{g}} - m_g(X_{\widehat{g}}), he^{\varphi}\rangle_{\widehat{g}}\right] &= \iint_{\mathbb{R}^2\times\mathbb{R}^2} G_{\widehat{g}}(x,y)f(x)h(y)\,\lambda_g(dx)\,\lambda_g(dy) \\ &- m_g(h)\iint_{\mathbb{R}^2\times\mathbb{R}^2} G_{\widehat{g}}(x,y)f(x)\,\lambda_g(dx)\,\lambda_g(dy) \\ &- m_g(f)\iint_{\mathbb{R}^2\times\mathbb{R}^2} G_{\widehat{g}}(x,y)h(y)\,\lambda_g(dx)\,\lambda_g(dy) \\ &+ m_g(f)m_g(h)\iint_{\mathbb{R}^2\times\mathbb{R}^2} G_{\widehat{g}}(x,y)\,\lambda_g(dx)\,\lambda_g(dy)\end{aligned}$$

$$\begin{aligned}\iint_{\mathbb{R}^2\times\mathbb{R}^2} G_{\widehat{g}}(x,y)f(x)h(y)\,\lambda_g(dx)\,\lambda_g(dy) &= \iint_{\mathbb{R}^2\times\mathbb{R}^2} \log\frac{1}{|x-y|}f(x)h(y)\,\lambda_g(dx)\,\lambda_g(dy) \\ &- \frac{\lambda_g(h)}{\lambda_{\widehat{g}}(\mathbb{R}^2)}\iint_{\mathbb{R}^2\times\mathbb{R}^2} \log\frac{1}{|x-z|}f(x)\,\lambda_g(dx)\,\lambda_{\widehat{g}}(dz) \\ &- \frac{\lambda_g(f)}{\lambda_{\widehat{g}}(\mathbb{R}^2)}\iint_{\mathbb{R}^2\times\mathbb{R}^2} \log\frac{1}{|y-z|}h(y)\,\lambda_g(dy)\,\lambda_{\widehat{g}}(dz) \\ &+ \lambda_g(f)\lambda_g(h)\theta_{\widehat{g}}\,.\end{aligned}$$

$$\begin{aligned}m_g(h)\iint_{\mathbb{R}^2\times\mathbb{R}^2} G_{\widehat{g}}(x,y)f(x)\,\lambda_g(dx)\,\lambda_g(dy) &= m_g(h)\iint_{\mathbb{R}^2\times\mathbb{R}^2} \log\frac{1}{|x-y|}f(x)\,\lambda_g(dx)\,\lambda_g(dy) \\ &- \frac{\lambda_g(h)}{\lambda_{\widehat{g}}(\mathbb{R}^2)}\iint_{\mathbb{R}^2\times\mathbb{R}^2} \log\frac{1}{|x-z|}f(x)\,\lambda_g(dx)\,\lambda_{\widehat{g}}(dz) \\ &- m_g(h)\frac{\lambda_g(f)}{\lambda_{\widehat{g}}(\mathbb{R}^2)}\iint_{\mathbb{R}^2\times\mathbb{R}^2} \log\frac{1}{|z-y|}\,\lambda_g(dy)\,\lambda_{\widehat{g}}(dz) \\ &+ \lambda_g(h)\lambda_g(f)\theta_{\widehat{g}}\,.\end{aligned}$$

$$\begin{aligned}m_g(f)\iint_{\mathbb{R}^2\times\mathbb{R}^2} G_{\widehat{g}}(x,y)\,\lambda_g(dx)\,\lambda_g(dy) &= m_g(f)\iint_{\mathbb{R}^2\times\mathbb{R}^2} \log\frac{1}{|x-y|}h(y)\,\lambda_g(dx)\,\lambda_g(dy) \\ &- m_g(f)\frac{\lambda_g(h)}{\lambda_{\widehat{g}}(\mathbb{R}^2)}\iint_{\mathbb{R}^2\times\mathbb{R}^2} \log\frac{1}{|x-z|}\,\lambda_g(dx)\,\lambda_{\widehat{g}}(dz) \\ &- \frac{\lambda_g(f)}{\lambda_{\widehat{g}}(\mathbb{R}^2)}\iint_{\mathbb{R}^2\times\mathbb{R}^2} \log\frac{1}{|z-y|}h(y)\,\lambda_{\widehat{g}}(dy)\,\lambda_{\widehat{g}}(dz) \\ &+ \lambda_g(f)\lambda_g(h)\theta_{\widehat{g}}\,.\end{aligned}$$

$$
\begin{aligned}
m_g(f)m_g(h)\iint_{\mathbb{R}^2\times\mathbb{R}^2} G_{\widehat{g}}(x,y)\,\lambda_g(dx)\,\lambda_g(dy) &= m_g(f)m_g(h)\iint_{\mathbb{R}^2\times\mathbb{R}^2}\log\frac{1}{|x-y|}\,\lambda_g(dx)\,\lambda_g(dy)\\
&\quad - m_g(f)\frac{\lambda_g(h)}{\lambda_{\widehat{g}}(\mathbb{R}^2)}\iint_{\mathbb{R}^2\times\mathbb{R}^2}\log\frac{1}{|x-z|}\,\lambda_g(dx)\,\lambda_{\widehat{g}}(dz)\\
&\quad - m_g(h)\frac{\lambda_g(f)}{\lambda_{\widehat{g}}(\mathbb{R}^2)}\iint_{\mathbb{R}^2\times\mathbb{R}^2}\log\frac{1}{|y-z|}\,\lambda_g(dy)\,\lambda_{\widehat{g}}(dz)\\
&\quad + \lambda_g(f)\lambda_g(h)\theta_{\widehat{g}}\,.
\end{aligned}
$$

Most terms cancel out and we are left with

$$
\begin{aligned}
\mathbb{E}\left[X_{\widehat{g}} - m_g(X_{\widehat{g}}), fe^{\varphi}\rangle_{\widehat{g}}\langle X_{\widehat{g}} - m_g(X_{\widehat{g}}), he^{\varphi}\rangle_{\widehat{g}}\right] &= \iint_{\mathbb{R}^2\times\mathbb{R}^2} G_g(x,y)f(x)h(y)\,\lambda_g(dx)\,\lambda_g(dy)\\
&= \mathbb{E}[\langle X_g, f\rangle_g\langle X_g, h\rangle_g]\,.
\end{aligned}
$$

Now we are done. □

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