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Modelling the twin-cam compound bow

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Academic Dissertation

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Abstract

In this thesis, the statics of the archery twin-cam compound bow is investigated with the help of mathematical models. The simplest case is the model of the round-wheel compound bow, which is then advanced with the Euler-Bernoulli (EB) beam model for the limb deformation. Further, a generalization of the round-wheel compound bow model is examined, and also the twin-cam compound bow with non-round cams is studied.

The force-draw curves of the round-wheel and the twin-cam compound bows are measured and compared to the predictions of the respective models. The extended offset-eccentric model is tested with the help of the round-wheel model, and also by using the twin-cam bow model. The path of the limb tip of the compound bow is checked with both bows by measurements.

The Young's modulus of the round-wheel bow limb is determined, and the effects of the shape of the limb on the force-draw curve are also considered. Furthermore, an approximation on the path of the tip of the straight limb with constant bending stiffness for small deformations is derived.

In the twin-cam bow model, the non-round cams are modelled with the help of cubic splines. In all models of this thesis, the derived non-linear equations are simplified and solved numerically. The consistency and the validity of all the models were also checked in several ways.

Two static quality coefficients for the compound bow are presented. A limit for the force-draw curve of the compound bow is concluded.

As an example, the cams of the compound bow are virtually modified in order to improve the effectiveness of the bow. All the models presented in this thesis may be used for adjusting and designing twin-cam compound bows.

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List of Variables

α	the angular coordinate of the cable cam
β	the angular coordinate of the string cam
A	the ratio between the length of the supposed elastic portion of the limb with respect to the total limb length
D	the draw; the distance between the midpoint of the string and the vertical line that connects the bottoms of the upper and the lower limb
D_0	the value of D in the initial position
D_F	the value of D in the full draw
E	the Young's modulus of the limb material
F	the absolute value of the force acting on the arrow
F_{max}	the peak force; the maximum value of F
h	the distance between the grip (handle) supporting point and the vertical line that connects the bottoms (riser ends) of the upper and the lower limbs
I	the second moment of inertia of the limb
k	the spring constant of the elastic portion of the limb
l	the length coordinate along the limb measured from the bottom of the free limb with length L towards the tip of the limb
L	the length of the free (elastic) limb measured from the point where the limb touches the riser to the axle point along the limb
L_{tot}	the total length of the limb measured from the riser end (the bottom) of the limb to the axle point along the limb
q	the static quality coefficient of the bow
q_F	the quality coefficient of the force-draw curve of the bow
r_c	the radius of the cable cam; the radial distance from the axle point to the edge point of the cable cam
r_s	the radius of the string cam; the radial distance from the axle point to the edge point of the string cam
V	the potential energy of the bow
W	the bending stiffness of the limb
W_0	the constant bending stiffness of the limb

Common Abbreviations

BD	Brent-Dekker
EB	Euler-Bernoulli
FD	force-draw
H	Hickman
LM	Levenberg-Marquardt
RG	Runge-Kutta

Chapter 1

Introduction

In archery, the compound bow is a bow in which a pulley system is used at the end of the limb tips. The pulley system can be designed so that when drawing the bow, the holding force first increases rapidly to its maximum value, keeps there for a while, and then surprisingly decreases to a local minimum value at the aiming position. With a conventional bow, the holding force increases almost linearly with respect to the draw, and the maximum value of the force is reached at the aiming, or full draw, position. Therefore, drawing the compound bow is a harder work when compared to the drawing of the conventional bow with the same initial and full draw and the same maximum holding force. On the other hand, the work done is stored as the potential energy of the limbs, so the potential energy of the compound bow is greater. In shooting process this energy is transferred mainly to the kinetic energy of the arrow. Thus, when these two bows which have been chosen with above-mentioned criteria are compared, the compound bow is both easier to aim and more energetic, and the speed of the arrow that has been shot with the compound bow is higher.

The first compound bow was reportedly built by Claude Lapp in 1938, but his bow never reached the markets [1]. After Allen's patented design 1969 [2], the compound bow became available. Yet, the mechanism of the compound bow has not been investigated until quite recently.

There is no doubt that modelling the compound bow is useful for designing purposes. With accurate compound bow models the manufacturing and testing processes of the bow can be optimized, which saves both time and money. Using bow models it is also possible to adjust the manufactured bow for the specific user.

Nowadays, there are several types of compound bows available. The classification of compound bows is mostly based on the type of pulleys at the tips of the upper and lower limb. The main

compound bow configurations are 1) single-cam, 2) twin-cam, 3) hybrid-cam, and 4) binary-cam [3, 4]. A typical twin-cam compound bow with the upper cam system is presented in Fig. 1.1.

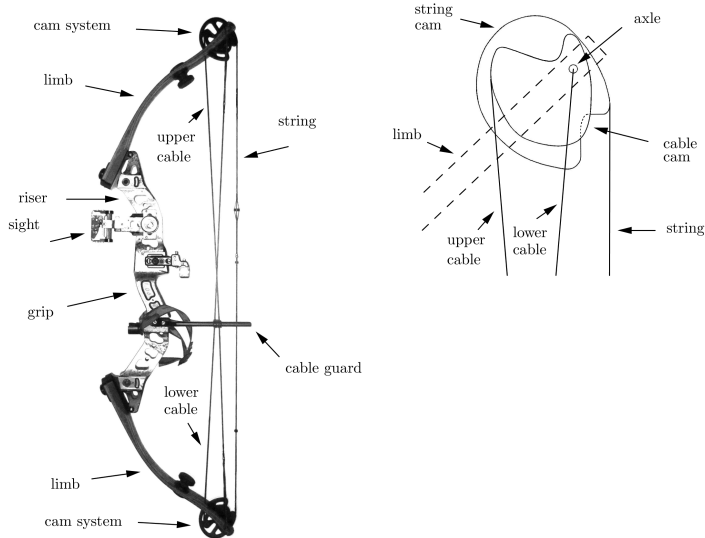


Figure 1.1: A twin-cam compound bow in the initial position and its upper cam system

In all these compound bow configurations, there are three different main parts in the bow: the riser (with the grip), the limbs, and the eccentric/cam systems with the cables and the string at the tips of the limbs. The modelling problem culminates into the bending of limbs and the rotating eccentrics/cams.

In modelling the behaviour of the compound bow, the main aim is to predict the force acting on the arrow with a certain value of draw. This is not a trivial problem at all, even with the simplest round-wheel compound bow. Yet, a solution is certainly needed, as the force-draw relation dominates the drawing, aiming and releasing processes of the bow.

In this thesis, the consideration is restricted only on the statics of the compound bow. Further, we shall examine only the symmetric twin-cam compound bow, while other types of compound bows are beyond the scope of this thesis.

Chapter 2

List of included publications

This thesis consists of an introductory review, followed by four research articles, which are published in international peer-reviewed journals. All the articles are done solely by the author. These articles are referred to in bold Roman numerals in this thesis. All the articles included in this thesis are reprinted under the Creative Commons License.

Article I: An advanced model of the round-wheel compound bow

M. Tiermas, *Meccanica*, 51(5), 1201-1207, (2016). doi: 10.1007/s11012-015-0262-5

Article II: The limb deformation of the compound bow

M. Tiermas, *Meccanica*, 52(6), 1475-1483, (2017). doi: 10.1007/s11012-016-0485-0

Article III: A model of the twin-cam compound bow with cam design options

M. Tiermas, *Meccanica*, 52(1), 421-429, (2017). doi: 10.1007/s11012-016-0395-1

Article IV: The round-wheel compound bow model revisited: a new extension

M. Tiermas, *Sports Engineering*, 20, 155-162, (2017). doi: 10.1007/s12283-017-0225-2

Chapter 3

Previous work

In the background of the research concerning the compound bow, there are important connections between the compound bow models and the earlier studies about the conventional (or traditional) bow. For this reason the following synopsis is presented.

Hickman produced several articles concerning the statics and the dynamics of the conventional long bows [5]. Especially his paper [6] in 1937 presenting the model for the long bow is of great importance and is widely used in the further conventional and compound bow modelling. Klopsteg published a paper [7] in 1943, where he treated among other aspects also the so called archer's paradox: the released arrow should always strike the bow handle and get out of its course. In his paper, Klopsteg explained how the arrow bends during the acceleration time in contact with the string so that it undulates around the bow handle, keeping its straight course. This paradox is more precisely explained by Kooi and Sparenberg in [8] and Kooi in [9].

The working recurve bow is a bow, where the tips of the limbs are initially curved in a sense opposite to the archer. An interesting paper by Schuster in 1969 includes the first model of this bow type [10]. The working recurve bow is more precisely modelled by Kooi and Sparenberg in [11].

A model of the conventional bow with elastic string was introduced by Marlow [12] in 1981. In the appendix of this paper, also the force-draw curve of the compound bow is presented, although without further notes.

The model of the conventional straight bow with elastic string is introduced by Kooi in 1983 [13], and in 1991 he presented a similar model for conventional recurved bow [14]. The Hickman model with Lagrangian mechanics was introduced in 1992 by Tuijn and Kooi [15].

In 2000 Peters compared the conventional bow with the compound bow [16]. While the eccentric model presented in the appendix of his study is quite instructive, it is simple and has only conceptual similarities to the eccentric systems of the compound bow.

The first investigation of the compound bow including a model of the single-cam compound bow was presented by Park in 2009 [3]. In his model, the place of the limb tip is evaluated with the modified Hickman model [6], when the straight limb is supposed to bend at only one hinge point which locates between the tip and the bottom of the undeflected limb. The placement of this hinge point is evaluated by curve fitting. This approximation turned out to be quite accurate. On the other hand, the asymmetric pulley system of the single-cam bow model would lead to a large group of equations, when the problem is eased in [3] by assumption of symmetric bending of the limbs and the simplified treatment of the cams. The visual shape of the cams is then hard to figure out, for the radius with every limb angle are defined as the perpendicular distances between the string/cable line and the axle.

In 2012 Zanevsky introduced an asymmetric model of a special type of compound bow with centric cable eccentric [17]. The focus of this model is on the slight differences of the limb and the initial cam angles caused by the asymmetrical positioning of the grip and the nock point (the point where the arrow meets the string), when due to the complex system also here some approximations are done. The cables are assumed to be exactly vertical all the time. It should be noted that this does not lead to any essential errors with usual limb and cam dimensions. As in [3], also in Zanevsky's model the radius of the cam is expressed in a way that its shape is not easy to find out. The deformation of the limb is assumed similar as in Marlow's paper [12] for the conventional bow, when the limb rotates as a rigid body about the bottom of the limb at the place where it meets the riser. Then the path of the limb tip is not quite realistic.

The previous work left open the question about the exact geometric shape of the cam/eccentric system and its effect on the force-draw (FD) relation of the compound bow. With conventional bows, the shape of the limbs has a crucial effect on the FD relation of the bow, as presented for example in [13], but we don't know if a similar result holds with compound bows. In case of twin-cam compound bows, answers to these questions are presented in the following.

Chapter 4

Modelling the twin-cam compound bow

The serious archer knows that the performance of the bow depends strongly on the shape of the eccentric/cam system. Without question, the action of the bow limbs is also decisive. Unfortunately, modelling the pulleys and their action more specifically leads to a large amount of non-linear equations. If we also simultaneously want to model the deformation of the bow limbs accurately, the problem becomes very laborious.

However, it is possible to divide the problem into two main parts. At first we may concentrate on the pulley system and use a simple approximation for the limbs. Assuming the limb of the bow as a straight rod that has only one hinge point and using the Hooke's law in one dimension for the hinge, the problem is significantly easier to solve. We have only to consider the exact place of this supposed hinge point, which can be found out by measuring the path of the tip of the real limb, and we are ready to focus on the action of the pulley system.

The pulley system with two round eccentrics with the same centre and axle (round-wheel compound bow) is conceptually the easiest case. Let us consider the action of this kind of a pulley system (or shortly wheel) more closely. We shall assume that the wheels at the tips of the upper and the lower limbs are similar, and the bow is in a vertical position with the riser on the left side with respect to the wheels. Note that this configuration is otherwise similar to the one presented in Fig. 1.1, only the shape of the pulleys is different. The eccentrics of the wheel are firmly attached to each other, and the wheel can rotate only around the axle, from which the wheel is connected to the limb.

In Fig. 4.1, the wheel of the upper limb is in the initial position. The string is wrapped around the string eccentric. The upper cable is twisted around the cable eccentric, but on the other side than the string, as seen in Fig. 4.1. The other end of this cable is connected to the axle

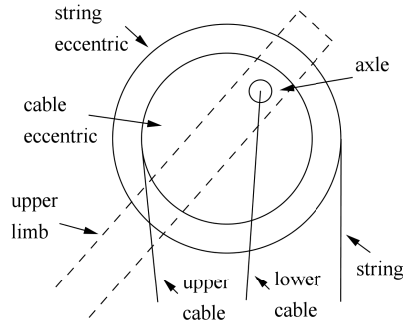


Figure 4.1: The upper wheel of the twin-round-wheel compound bow in the initial position (Article I, the right-hand part of Fig. 1, <https://creativecommons.org/licenses/by/4.0/>)

of the lower wheel, in a similar way as the lower cable is connected to the axle of the upper wheel. The total force acting on the axle, which is the vector sum of the string tension and the two cable tensions, bends the limb. It should be noted that in the initial position the limbs are already deflected, when the tensions of the string and the cables are notable. In balance, the torque caused by the upper cable tension (with respect to the axle) must cancel the torque caused by the string tension. From Fig. 4.1 we further notice that the lever arm of the cable tension, which is the perpendicular distance from the cable line to the axle, is longer than the lever arm of the string tension, so in the initial position the string tension is greater.

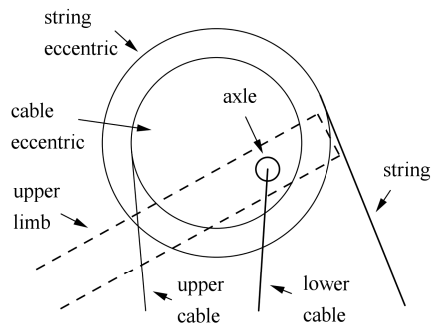


Figure 4.2: The upper wheel of the twin-round-wheel compound bow in drawn position

When the string is drawn from the midpoint of the string, the upper wheel rotates clockwise, as seen in Fig 4.2. An amount of string wraps out from the string eccentric, while an amount of the upper cable wraps in the upper cable eccentric. The cables are then shortened, when the limbs are bent more, so the total force acting on the axle is increased.

The horizontal holding force, which equals the absolute value of the force acting on the arrow, depends not only on the string tension but also on the string angle with respect to the vertical line. Indeed, in Fig. 4.2 the holding force is close to its maximum value, whereas in Fig. 4.1 the holding force is zero. By continuing drawing the bow from situation of Fig. 4.2, the lever arm of the string tension lengthens and the lever arm of the cable tension shortens, when the cables take more load and the string tension decreases.

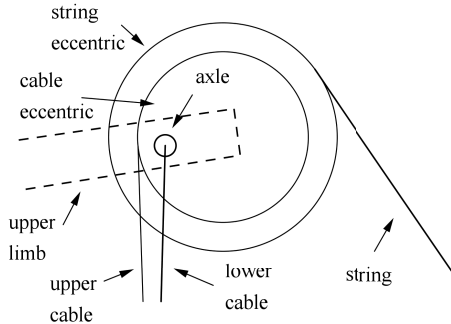


Figure 4.3: The upper wheel of the twin-round-wheel compound bow in full draw

In Fig. 4.3 the bow is in full draw position. The limbs are bent even more, and the total force acting on the axle has its greatest value. However, the lever arm of the string tension is long, and the lever arm of the cable tension is very short. Thus, the cables take most of the load, while the string tension and the holding force are low. It is relatively easy to keep the bow in this (aiming) position.

After writing the relevant equations based on the geometrics and statics of this kind of bow, we also notice that while there are many non-linear equations, they consist only of elementary algebraic and elementary transcendental functions and can be arranged into groups of (non-linear) equations. There is no analytical solution, but as each equation group at a time may also be simplified into one equation of one variable, the model is computable with simple numerical methods.

The conceptual consideration is similar as explained before with the round-wheel bow even if the pulley system includes non-round cams (twin-cam compound bow). However, the quantitative treatment is then more complex. We are interested in the design of the cams, when a promising idea is to choose several edge points of the cams, and using these points to form some interpolating continuous function. As the cam system also rotates, it is natural to use the polar transformation of the coordinates of these edge points with respect to the axle. We also have to remember that with a certain rotation of the cams, the string and the cables do wrap in and

out from the edge of the cams, when the arc length of the edge of the cam must be considered with great care.

Approaching the second part of the modelling problem, the more realistic deformation of the compound bow limb, the Euler-Bernoulli (EB) beam theory is used. Now, the limb is considered as an elastic line of zero thickness, and for every infinitesimal segment of this line a separate value for bending stiffness is affiliated. In this model, the geometrical side profile of the limb may diverge from the straight line, an option which is also used with some compound bows.

With the properly chosen prime variable by using the round-wheel model, it is possible to compute the value of the limb tip y -coordinate without any assumptions about the limbs. Then, also the value of the angle of the total force acting on the limb tip with respect to vertical line can be solved. This reduces the number of variables, when the group of relevant equations concerning the EB deformation of the limb is solvable. The values of x -coordinate of the limb tip and the absolute value of the force acting on the limb tip are then gained, when the rest values needed can be easily calculated with the help of the round-wheel model. Note that in all Articles included in this thesis, the midpoint of the axle of the cam/eccentric system (axle point) and the respective tip point of the limb are defined to coincide.

Chapter 5

Models

5.1 Round-wheel model

In the model of the round-wheel compound bow of Article I, the cables and the string are assumed to be inextensible. The bow is assumed to be symmetric with respect to the line in which the midpoint of the string moves when the bow is drawn or relaxed. The cable guard is ignored, and the cables are supposed to be on the same plane with the string. The limb is assumed to be a rigid straight rod with length L , and it is supposed to bend only on one point whose distance from the tip of the bow limb is AL . The constant A is determined later from the actual limb by measurements. The assumption of the Hooke's law on the bending portion of the deflecting limb is used.

As the eccentric system consists of two round concentric discs, it is convenient to handle the geometrics affiliated to eccentrics using the centre of the discs as origin. Then, a number of equations are derived as based on the geometrics and the statics of the system.

It is possible to simplify the problematic two sets of non-linear equations of the model into two non-linear equations of one variable. There is no analytical solution, and consequently numerical means are called for. For this purpose, the robust Brent-Dekker [18] root finding algorithm is chosen. When solutions of these equations are found, the values of the other variables can be obtained quite straightforwardly, when, for example, with every value of the eccentric rotation angle the respective limb angle, the draw and the force acting on the arrow are obtained.

5.2 Limb deformation model

In the model of Article **II**, the general assumptions relating to the eccentrics, cables and string are the same as in the round-wheel compound bow model. The Hickman model of the limb is replaced with the more realistic Euler-Bernoulli (EB) model for the limb, which is embedded in the round-wheel compound bow model. The limb is represented by an inextensible elastic line with zero thickness. This line is divided into infinitesimal segments, and with every segment a separate bending stiffness is affiliated. In this thesis, this model for the limb deformation is called shortly as EB model.

Now it is possible to consider also the limbs with varying thickness, width, curvature or material. The dominating group of differential equations describing the statics of the system is deduced, and this equation group is united with the equations of the round-wheel model. With every value of the primary variable, the resulting differential equations with the respective geometrical condition equations are solved with trust-region dogleg method [19] Runge-Kutta algorithm included. After the limb tip coordinates are solved, the values of all the other variables (especially the draw and the respective force acting to the arrow) can be concluded with the help of the round-wheel compound bow model.

5.3 Twin-cam model

In the model of Article **III**, the shape of the string cam and the cable cam are modelled using polar coordinates and cubic splines. For the limb, the modified Hickman model as explained before is used. Again, the other assumptions of the model are similar to the assumptions of the round-wheel model. Compared to the round eccentrics, the cams complicate the system a lot. The arc length along the cams must be considered carefully, as well as the tangential lines of the edge of the cams. A total of 29 equations are needed to describe the geometrics and the statics of the system. Still, also here it is possible to simplify the problematic sets of non-linear equations as non-linear equations of one variable. Again the Brent-Dekker method can be used for solving these equations, after which the values of the other variables can be easily calculated with every value of the selected primary variable α (the angular coordinate of the cable cam).

5.4 Extended offset-eccentric model

The round-wheel model is extended to eccentric systems with off-set between the centre of the string and the cable eccentrics in Article **IV**. Then, compared to the round-wheel model, an additional parameter and one extra variable are needed. Consequently, this leads to one additional equation and to some minor modifications of the equations of the round-wheel compound bow model. The other assumptions are quite the same as in case of round-wheel model, including the modified Hickman model for the bow limbs. The equation solving process is also similar, so the off-set eccentric and the round-wheel models are numerically equally robust. As a by-product, also a model for the conventional bow with modified Hickman approximation for the path of the limb tip is introduced.

Chapter 6

Measurements

Two bows were used in testing the models, the round-wheel compound bow "JahtiJakt 55 lbs" and the twin-cam compound bow "Smoke". The data relating to eccentrics/cams of the bows were measured from the enlarged photos of the eccentric/cam systems. The other initial geometrical parameters were measured straightforwardly with a steel ruler from the actual bows. The FD curve of each bow was measured using a bow scale and a steel ruler, with the help of a deflecting rack and a manual winch. Each bow was set on the top of the rack, and the bow was slowly drawn and relaxed with the help of the winch. The readings of the scale and the ruler were recorded with a video camera. The arrangements of the FD curve measurements are presented in Fig. 6.1.

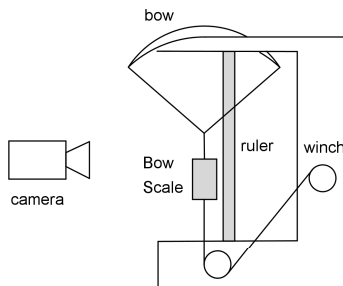


Figure 6.1: The arrangements of the measurement of the force-draw curve (Article I, <https://creativecommons.org/licenses/by/4.0/>)

Before fitting the measured data to the models, it is necessary to evaluate the values of the length of the elastic portion of the bow limbs for each bow, which is determined with the parameter A . The winch has a locking mechanism, so using the deflecting rack each bow was drawn again with several stops, in which the place of the axle point (or the tip of the limb) of

both limbs with respect to the riser and to the other axle point was measured with the ruler. The parameter A was then found by fitting the modified Hickman model into the measured data, using the Levenberg-Marquardt (LM) algorithm [20]. As the bow limbs of the twin-cam bow in question are slightly recurved, the length of the limb of this bow was determined together with parameter A , using LM algorithm.

For each bow, the value of the spring constant k of the elastic portion of the limb was chosen so that the respective model fitted to the measurements as well as possible in the sense of least squares. For continuity reasons, a cubic spline was first fitted to the (D, F) -values of the models, and the least squares method was applied to the values of this spline and the measured data. Both the drawing and relaxing FD data of the bow were included in the fitting.

The bending stiffness $W(l)$ of the limb of the round-wheel bow is needed in the model of Article **II**. First, the width and the thickness of the limb with several length coordinates l was measured with the calipers, when the second moment of inertia $I(l)$ of the limb is known. Then, as $W(l) = EI(l)$ the Young's modulus E of the limb was found by fitting the model of Article **II** into the measured FD data of the round wheel bow using the method of least squares.

Chapter 7

Results

7.1 Model testing

In testing the round-wheel and the twin-cam models, the FD curves of the respective bows were measured while both drawing and relaxing the bow. There is a well known difference between these measurements [21]. Apparently this difference is caused by the friction of the eccentrics/cams and the elastic hysteresis of the bow limbs, string and cables.

The consistency of all the compound bow models was checked by calculating the potential energy of the limbs and comparing it to the respective area of the FD curve. In Article **II**, the bending energy of the limbs was also checked with the help of elliptic integrals in case of straight vertical limbs with constant bending stiffness. Additionally, the twin-cam model was also checked with the help of the round-wheel model in case of round cams. Finally, the offset-eccentric model was cross-checked with the twin-cam model.

It was found that the accuracy of all the models is good. However, in case of strongly precurved limbs, the EB model for the limb deformation was not empirically tested. Neither the offset-eccentric model was tested empirically, for just such bows are not yet manufactured.

7.2 Static quality coefficients

Especially when adjusting and designing the compound bows, it was found useful to define two static quality coefficients for the compound bow.

These measures, the static quality coefficient q and the quality coefficient of the FD curve q_F , are presented in Article **IV**,

$$q = \frac{\int_{D_0}^{D_F} F(D) \, dD}{(D_F - h)F_{max}}, \quad q_F = \frac{\int_{D_0}^{D_F} F(D) \, dD}{(D_F - D_0)F_{max}} \quad (7.1)$$

where F is the absolute value of the force acting on the arrow, D the draw with respect to the line that connects the riser ends of the upper and the lower limbs, D_0 the value of D in the initial position, D_F the full draw (here, the draw with the local minimum value of the force F), h the distance between the grip supporting point and the vertical line that connects the bottoms of the upper and the lower limbs (positive, when the grip supporting point is on the archer's side from this line), and the peak force

$$F_{max} = \max\{F(D)\}, \quad D_0 \leq D \leq D_F \quad (7.2)$$

As an example of making use of these coefficients, let us take three different compound bows. The bow B_1 is "Darton DS-700" by Darton Archery, the bow B_2 is "Bear Escape SD" by Bear Archery, and B_3 is "Smoke" by Hoyt. The bows B_1 and B_3 are twin-cam bows, whereas B_2 is a hybrid-cam bow.

The static characteristics of the bows B_1 , B_2 and B_3 are presented in Table 7.1, and the respective FD curves of the bows are shown in Fig. 7.1. In case of bows B_1 and B_2 , the draws are reported only with respect to the grip, so note these "true draw" values $D_0 - h$ and $D_F - h$ used in Table 7.1, and the axis $D - h$ used in Fig. 7.1. The bow B_3 has $h = 9.5$ cm. The properties and the FD curve of the bow B_3 are presented also in Article **III**, and the respective data of the bows B_1 and B_2 are based on the FD measurements and information presented in [22] and [23]. As in the case of B_3 , also the FD curves of B_1 and B_2 in Fig. 7.1 are gained by curve fitting on both the drawing and the relaxing FD data.

Table 7.1: The bows B_1 , B_2 and B_3 . The values of $D_0 - h$ and $D_F - h$ are expressed in cm, the values of F_{max} and $F(D_F)$ in N, and the value of potential energy $V(D_F)$ in J. Coefficients q_F and q are dimensionless

Bow	$D_0 - h$	$D_F - h$	F_{max}	$F(D_F)$	$V(D_F)$	q_F	q
B_1	18.4	69.3	260.4	45.2	110.7	0.835	0.613
B_2	14.9	61.6	215.4	56.6	78.9	0.784	0.595
B_3	16.3	66.9	259.3	86.9	96.4	0.735	0.556

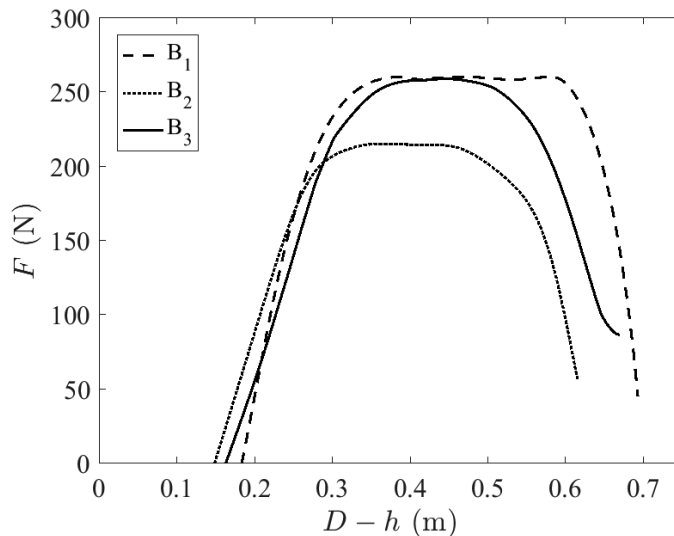


Figure 7.1: The force-draw curves of the bows B_1 , B_2 and B_3

As can be seen in Fig. 7.1, the shapes of the FD curves clearly deviate from each other. From Table 7.1, we notice that without taking into account the peak forces of B_1 and B_3 , there are clear differences between all the other quantities. Usually the bows with such significant differences between the initial draw, full draw or peak force are not comparable to each other. However, with the coefficients q_F and q we may exceed this limitation.

From Table 7.1 we notice that B_1 has the greatest value of both q and q_F , so in this sense it is the best. The bow B_1 has also the greatest value of potential energy $V(D_F)$ in full draw. When comparing the front part of the FD curves in Fig. 7.1, we see that the front part of the FD curve of B_1 is steepest. Also the rear part of the FD curve of B_1 is steepest, whereas the holding force of both B_2 and B_3 decrease significantly slower near the full draw, which reduces the values of q_F and q . Note especially the ending of the FD curve of B_3 at full draw. On the other hand, both the peak force and the potential energy of B_2 are lowest, but when comparing the q -values in Table 7.1, we notice that the static quality of B_2 is clearly better than the static quality of B_3 . The q_F -value of B_2 is about the average of the respective values of B_1 and B_3 , but the q -value of B_2 is closer to the q -value of B_1 . This is due to the small value of D_0 of bow B_2 , which also suggests that it should be possible to improve the static quality of the bows B_1 and B_3 by riser modification.

7.3 Virtual modifications

The cams of the actual twin-cam bow were virtually designed to be more energetic with the help of the twin-cam model. With some work, an energy improvement of 10 % was achieved in Article **III**.

The round-wheel bow can also be designed to be more energetic, especially with the off-set between the centres of the cable and the string eccentrics. For an actual round-wheel bow it was calculated that 18.5 % increment of energy is achievable. However, it should be noted that this result was not gained without modifying both the eccentric systems and the spring constant k of the elastic portion of the limbs.

Using the EB limb deformation model, it was noticed that the bending stiffness $W(l)$ of the limbs affects mainly on the height of the FD curve of the compound bow. On the other hand, the effects of changes of bending stiffness on the shape and on the width of the FD curve are quite minimal. The pre-curvature of the limbs has no significant effect to the general shape of the FD curve of the compound bow; this is quite unlike as with conventional bows, whose FD curve is greatly dependent on the pre-curvature of the limbs [13].

7.4 Some restrictions

When designing the cam radius functions by manipulating cam data, it was found that the resulting geometric shape of the cam may be concave in some points, which leads to serious computational problems. This concerns especially the parts of the cam which are related to the point where the string or the cable contacts the respective cam near initial and full draw, where a strong change in cam radius is usually wanted for maximizing the energy of the FD curve. After each change during the design process of the cams, it is therefore advisable to always check the quality of the cams. This can be done by checking that the respective condition equations for the cam radius functions $r_c(\alpha)$ and $r_s(\beta)$, which are, according to Article **III**,

$$r_c^2 + 2(r'_c)^2 - r_c r''_c > 0, \quad r_s^2 + 2(r'_s)^2 - r_s r''_s > 0 \quad (7.3)$$

are valid in all the domain of the cam radius functions. Note that with offset-eccentric and with round-wheel models this problem does not exist.

Considering the EB limb deformation model, it was noticed that with too small values of the bending stiffness serious computational problems may arise. The same effect is reported also

in [11]. For example, for the bow used in Article **II** the bending stiffness of the limb must be greater than 1.40 Nm^2 everywhere. However, this minimum limit depends also on the other properties of the limbs, such as the initial limb angle and the length of the limb.

There is also a theoretical limit for the FD curve of the compound bow. The front part of the FD curve can not be arbitrary steep. In Article **IV** it was shown that with the fixed force F acting on the arrow, the draw with the same limb and riser parameter values of the respective conventional bow is always the least. This result holds for both the bows with round wheels and non-round cams.

7.5 Links between the models

When comparing the pulley systems of the models presented here, the twin-cam model is the most complex. This model is also numerically most sensitive due to the interpolating cubic splines used for the cam radius functions. We may adjust the shape of both the string cam and the cable cam of the pulley system to be as round eccentrics as possible. Now, selecting the origin of the coordinates suitably the interpolating splines are not needed. The computations are then simplified, and we have the offset-eccentric model. After this, requesting that the offset between the string and the cable eccentric is zero, the model has further simplified into the round-wheel model. Moreover, adjusting the radius of the string cam/eccentric as zero with every value of the angle variable, the string takes all the load and we have a model for the conventional (modified) Hickman bow.

For straight limbs of uniform bending stiffness, the EB limb deformation model and the modified Hickman model have also a connection, which can be used to evaluate the parameters A and k of the modified Hickman model. For small deformations, the path made by the (free) limb of length L can be approximated by an arc of a circle whose radius is $5L/6$ and whose center is located at a distance of $5L/6$ from the tip of the undeflected limb. Using this approximation, the measurement of parameter A is not absolutely necessary, for now we have an estimate, as presented in Article **II**,

$$A = \frac{5}{6} \frac{L}{L_{tot}} \quad (7.4)$$

where L_{tot} is the total length of the limb. Now, an approximation for the spring constant of the elastic portion of the limb can also be evaluated, yielding

$$k = \frac{5W_0}{2L^2} \quad (7.5)$$

where W_0 is the constant bending stiffness of the limb.

Chapter 8

Summary

In this thesis, the statics of the twin-cam compound bow is investigated with the help of mathematical models. The simplest case, a model of the twin-round-wheel compound bow is introduced in Article **I**. In this model, the assumption of the Hooke's law on the bending portion of the deflecting limb proved to be adequate. While the profile of the compound bow limb differs from the traditional Hickman bow limb, a good approximation for the path of the limb tip of the compound bow can be found by supposing that the limb is a rigid straight rod with length L and bends only on one point whose distance from the tip of the bow limb is AL . The model does not take into account the elongation of the string or cables, nevertheless this missing has only a possible minor effect on the FD curve.

In order to study the limb deformation of the compound bow more closely, the EB beam model was inserted to the round-wheel compound bow model in Article **II**. The second moment of inertia of the compound bow limb was measured, and the Young's modulus E of the limb material was determined by fitting the resulting FD curve to the measured data. Both the resulting FD curve and the calculated path of the limb tip differed only very slightly when compared to calculations with the modified H model for the limb deformation. In case of straight (undeflected) compound bow limbs with constant bending stiffness, there is no significant difference whether we use EB model or the modified H model for the deformation of the limb.

An approximation for the path of the limb tip for the straight limb of uniform bending stiffness was derived for small deformations. Then, using the modified H model the measurement of A is no longer an absolute necessity, and k can also be estimated from the bending stiffness of the limb.

The effects of varying the thickness, width and side profile of the limb were considered. It was found that in all cases mainly the height of the FD curve and the full draw changed, while the general shape of the curve remained rather unchanged. Unlike with conventional bows, deviating from the straight side profile of the limb of the compound bow offers no substantial benefits from the viewpoint of statics.

When modelling the compound bow, both the EB model and the modified H model can be used for the deformation of the limb. If the undeflected limbs differ clearly from straight rods of constant width and thickness, the EB model is recommended.

In Article **III**, a model of the twin-cam compound bow with non-round cams is introduced. The shape of the string and the cable cams was modelled with the help of cubic splines. With the help of the model, the cams of an individual bow can be virtually modified for example to get the FD curve of the bow more effective. It turned out that the twin-cam model is particularly sensitive to the cam functions r_c and r_s , so when manipulating the cams it must be checked that both cams remain convex in their domain of operation.

Further, an offset-eccentric model of the compound bow was derived in Article **IV**. It turned out that it is possible to improve the effectiveness of the round-wheel compound bows also by manipulating the eccentric system. Designing the eccentric system of the round-wheel bow such that there is an offset between the centres of the cable and the string eccentric, an improvement of 18.5 % of energy was calculated. However, this result was possible only after modifying also the limbs.

There is a limit for the rising front part of the FD curve of the compound bow. It was shown that with the same force acting on the arrow, the respective traditional bow with the same limb and riser parameter values has always the shortest draw.

The static quality coefficients introduced, q for the compound bow and q_F for the FD curve of the bow, are helpful when designing and adjusting the compound bow. These coefficients may also be used as a part of the bow comparison.

The consistency and the validity of all the models were checked in several ways. The models presented here can be used when adjusting and designing twin-cam compound bows.

Hopefully this thesis will also inspire further research of compound bows. A potential subject for following investigation could deal with the limb deformation using Timoshenko beam theory for example. Another continuation could be to expand the analysis into the dynamics of the twin-cam compound bow.

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