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The full set of solutions of linear rational expectations models

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HIGHLIGHTS

- Characterisation of the extent of non-uniqueness in linear rational expectations models.
- Impulse response function for all solutions of linear rational expectations models.
- Some equilibria characterised as indeterminate in Lubik and Schorfheide (2003) turn out to be determinate.

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ABSTRACT

This article characterises the dimension of indeterminacy of linear rational expectations (LRE) models and derives their full set of solutions. It extends the analysis of indeterminate equilibria in Lubik and Schorfheide (2003) where some equilibria are incorrectly classified as indeterminate even though they entail the same observable outcome.

1. Introduction

In this article, the uniqueness condition (equation (42) in Sims, 2001) is analysed in order to characterise the dimension of indeterminacy and to derive the full set of solutions of LRE models. An example of an LRE model with a unique solution but whose existence condition is satisfied by multiple candidates is presented. This example is characterised as having indeterminate equilibria by Lubik and Schorfheide (2003).

The treatment of indeterminate equilibria in Lubik and Schorfheide (2003) analyses Sims (2001)'s existence condition, a necessary and sufficient condition for the existence of a stable and causal solution of an LRE model. For an LRE model to have multiple solutions, it is necessary that a certain equation system involved in the existence condition has multiple solutions; however, it is not sufficient for non-uniqueness.

In Section 2, the model of Lubik and Schorfheide (2003) is introduced. In Section 3, we analyse all solutions of the existence condition which correspond to different equilibria on which the economic agents may coordinate and which correspond to stable and causal solutions of the LRE model. Subsequently, we examine under which condition the solutions of the existence condition entail the same observational outcome and characterise the extent of non-uniqueness. Finally, we discuss a New Keynesian (NK) monetary model whose solution is unique even though the existence condition is satisfied by multiple candidates. In Section 4, the dimensionality of the problem as well as other approaches for solving LRE models are discussed.

2. Model

We consider the system

$$\Gamma_0 y_t = \Gamma_1 y_{t-1} + \Psi \varepsilon_t + \Pi \eta_t, \quad t \in \mathbb{Z}, \tag{1}$$

where the vector of endogenous variables y_t is n-dimensional, and the l-dimensional vector ε_t and the k-dimensional vector η_t comprise the exogenous shocks and the endogenous forecast errors, respectively. Furthermore, we assume that there are no redundant equations. The process (ε_t) is a martingale difference sequence, i.e. it satisfies $\mathbb{E}_t(\varepsilon_{t+1}) = 0$ where $\mathbb{E}_t(\cdot)$ denotes the conditional expectation, conditional on the information set $H_{\varepsilon,\zeta}(t)$ which is the closure of the span of present and past components of (ε_t) and (ζ_t), the p-dimensional martingale difference sequence of sunspot

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shocks orthogonal to (ε_t). Moreover, the stability condition (equation (9) in Sims, 2001) is assumed to apply to all components of the solution (y_t) of (1), i.e.

$$\mathbb{E}_t\left(\xi^{-h}y_{t+h}\right) \xrightarrow{h\to\infty} 0,\ \xi > 1, \tag{2}$$

holds. We call a process (y_t) which satisfies (1) and (2), and which is contained in $H_{\varepsilon,\zeta}(t)$ at time t a *stable and causal solution* of the LRE model.

In order to analyse the existence and uniqueness of stable and causal solutions, we transform and partition system (1) into a "stable" and an "unstable" block of n_S and n_U rows respectively, where $n_S + n_U = n$. To this end, we apply the so-called QZ-decomposition to (Γ_0, Γ_1), i.e. the matrices Γ_0 and Γ_1 are decomposed as $\Gamma_0 = Q'\Lambda Z'$, $\Gamma_1 = Q'\Omega Z'$, where $QQ' = I_n = ZZ'$, and Λ and Ω are upper-triangular. Left-multiplying (1) by Q and defining $w_t = Z'y_t$, we obtain

$$\begin{pmatrix}\Lambda_{SS} & \Lambda_{SU}\\ 0 & \Lambda_{UU}\end{pmatrix}\begin{pmatrix}w_t^S\\ w_t^U\end{pmatrix} = \begin{pmatrix}\Omega_{SS} & \Omega_{SU}\\ 0 & \Omega_{UU}\end{pmatrix}\begin{pmatrix}w_{t-1}^S\\ w_{t-1}^U\end{pmatrix} + \begin{pmatrix}Q_{S\bullet}\\ Q_{U\bullet}\end{pmatrix}\Psi\varepsilon_t + \begin{pmatrix}Q_{S\bullet}\\ Q_{U\bullet}\end{pmatrix}\Pi\eta_t \tag{3}$$

where the stable and unstable block, indexed with "S" and "U" respectively, pertain to whether the absolute values of the ratios[1] ω_{ii}/λ_{ii} of diagonal elements of Λ and Ω are smaller than ξ. Consequently, the vectors w_t^S and w_t^U are of dimensions n_S and n_U, and $Q_{S\bullet}$ and $Q_{U\bullet}$ are of dimensions $(n_S \times n)$ and $(n_U \times n)$ respectively.

3. Theory and results

3.1. Analysis of the existence condition

Here, we derive the condition for existence of a stable and causal solution of (1). Assuming the existence condition holds, we subsequently characterise its full set of solutions.

To derive the existence condition, we concentrate on the unstable block of (3), i.e.

$$\Lambda_{UU}w_t^U = \Omega_{UU}w_{t-1}^U + Q_{U\bullet}\Psi\varepsilon_t + Q_{U\bullet}\Pi\eta_t, \tag{4}$$

rewritten as $w_{t-1}^U = \Omega_{UU}^{-1}\Lambda_{UU}w_t^U - (Q_{U\bullet}\Psi\varepsilon_t + Q_{U\bullet}\Pi\eta_t)$. The only solution $\left(w_t^U\right)$ of (4) satisfying (2) is the one depending on future values of ε_t and η_t, i.e. $w_{t-1}^U = -\sum_{j=0}^{\infty}\left(\Omega_{UU}^{-1}\Lambda_{UU}\right)^j\left(Q_{U\bullet}\Psi\varepsilon_{t+j} + Q_{U\bullet}\Pi\eta_{t+j}\right)$. From the causality condition that y_t be contained in the conditioning set $H_{\varepsilon,\zeta}(t)$, i.e. $\mathbb{E}_t(y_t) = y_t$ and thus $\mathbb{E}_t\left(w_t^U\right) = w_t^U$, it follows that for any stable and causal solution $\left(w_t^U\right)$ of (4) the equation

$$Q_{U\bullet}\Psi\varepsilon_t + Q_{U\bullet}\Pi\eta_t = 0 \tag{5}$$

must hold. One can find for any given exogenous shock ε_t endogenous forecast errors η_t^* such that Eq. (5) holds (and consequently a stable and causal solution (y_t) exists) if and only if the existence condition (equation (40) in Sims, 2001)

$$span_{col}\left(Q_{U\bullet}\Psi\right) \subseteq span_{col}\left(Q_{U\bullet}\Pi\right) \tag{6}$$

holds, i.e. the space spanned by the columns of $Q_{U\bullet}\Psi$ must be contained in the space spanned by the columns of $Q_{U\bullet}\Pi$. Henceforth, we will assume that (6) holds.

One may represent the set of all solutions η_t^* of (5) for given exogenous shock ε_t by using the singular value decomposition (SVD) of the $(n_U \times k)$-dimensional matrix $Q_{U\bullet}\Pi$, i.e.

$$Q_{U\bullet}\Pi = \begin{pmatrix}U_{\bullet 1} & U_{\bullet 2}\end{pmatrix}\begin{pmatrix}D_{11} & 0\\ 0 & 0\end{pmatrix}\begin{pmatrix}V_{\bullet 1}'\\ V_{\bullet 2}'\end{pmatrix} = U_{\bullet 1}D_{11}V_{\bullet 1}'$$

where D_{11} is an $(r \times r)$-dimensional diagonal matrix with positive diagonal elements, and $U = \begin{pmatrix}U_{\bullet 1} & U_{\bullet 2}\end{pmatrix}$ and $V = \begin{pmatrix}V_{\bullet 1}'\\ V_{\bullet 2}'\end{pmatrix}$ are orthogonal matrices. For orthogonal processes (ε_t) and (ζ_t), the set of all solutions of (5) is $\eta_t^* = -V_{\bullet 1}D_{11}^{-1}U_{\bullet 1}'Q_{U\bullet}\Psi\varepsilon_t + V_{\bullet 2}(M_1\varepsilon_t + M_2\zeta_t)$, where $M_1 \in \mathbb{R}^{(k-r)\times l}$ and $M_2 \in \mathbb{R}^{(k-r)\times p}$ parametrise the kernel of $Q_{U\bullet}\Pi$, i.e. the linear combinations of endogenous forecast errors which do not influence the stable and causal solution $\left(w_t^U\right)$ of (4).

3.2. Analysis of the uniqueness condition

In this section, we quantify the extent of non-uniqueness of LRE models. Lubik and Schorfheide (2003) consider an equilibrium to be indeterminate whenever there is more than one solution η_t^* of (5) which entails a stable and causal solution $\left(w_t^U\right)$ of (4). However, multiple solutions η_t^* of (5) are necessary but not sufficient for different stable and causal solutions $\left(y_t^{(1)}\right)$ and $\left(y_t^{(2)}\right)$ of (1) to exist. Thus, certain determinate equilibria (in the sense that they entail the same observational outcome) would be classified as indeterminate by Lubik and Schorfheide (2003). A necessary and sufficient condition for the uniqueness of a stable and causal solution of (1) is equation (42) in Sims (2001) which requires that the space spanned by the rows of $Q_{S\bullet}\Pi$ be contained in the space spanned by the rows of $Q_{U\bullet}\Pi$, i.e.

$$span_{row}\left(Q_{S\bullet}\Pi\right) \subseteq span_{row}\left(Q_{U\bullet}\Pi\right). \tag{7}$$

Intuitively, the solution (y_t) of (1) is unique if and only if all linear combinations of endogenous forecast errors η_t which have an observable effect on the stable and causal solution $\left(w_t^S\right)$ of the stable block in (3) also have an observable effect on the stable and causal solution $\left(w_t^U\right)$ of the unstable block (4). In this case, the endogenous forecast errors which could have an observable effect on (y_t) are already pinned down uniquely by the existence condition.

As an example, consider $Q = I_3$, $\Psi = \left(\begin{array}{c|cc}1 & 1 & 0\end{array}\right)'$, and $\Pi = \begin{pmatrix}1 & 1 & 1\\ \hline 1 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}$ and assume that there are one stable and two unstable eigenvalues. Condition (6) is satisfied because $Q_{U\bullet}\Psi = \begin{pmatrix}1 & 0\end{pmatrix}'$ is contained in the space spanned by the columns of $Q_{U\bullet}\Pi = \begin{pmatrix}1 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}$. Since the kernel of $\begin{pmatrix}1 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}$ is non-trivial, Lubik and Schorfheide (2003) would conclude that the equilibrium is indeterminate. However, (7) is satisfied as well because the first row of Π is contained in the space spanned by the second and third row of Π. Hence, there is only one solution (y_t) of (1).

When the uniqueness condition (7) is not satisfied, it is necessary to characterise the extent of non-uniqueness by using the following definition.

Definition 1. The *dimension of indeterminacy* of the LRE model (1) is equal to the rank of the projection of the row space of $Q_{S\bullet}\Pi$ on the orthogonal complement of the row space of $Q_{U\bullet}\Pi$.

The dimension of indeterminacy denotes the number of non-trivial linear combinations of endogenous forecast errors η_t which satisfy the following two conditions. Firstly, they should have no observable effect on the stable and causal solution $\left(w_t^U\right)$ of (4) and are thus not pinned down by (5). Secondly, they should have an observable effect on the stable and causal solution $\left(w_t^S\right)$ of the stable block of (3).

In order to relate the dimension of indeterminacy to functions of the parameter matrices $(\Gamma_0, \Gamma_1, \Psi, \Pi)$ in (1), we introduce the SVD of $Q_{S\bullet}\Pi$ as

$$Q_{S\bullet}\Pi = \begin{pmatrix}\tilde{U}_{\bullet 1} & \tilde{U}_{\bullet 2}\end{pmatrix}\begin{pmatrix}\tilde{D}_{11} & 0\\ 0 & 0\end{pmatrix}\begin{pmatrix}\tilde{V}_{\bullet 1}'\\ \tilde{V}_{\bullet 2}'\end{pmatrix} = \tilde{U}_{\bullet 1}\tilde{D}_{11}\tilde{V}_{\bullet 1}'$$

[1] The ratios are ordered ascendingly with respect to absolute value. If $\lambda_{ii} = 0$, the ratio is considered to be infinite.

where $\tilde{D}_{11}$ is an $(s \times s)$-dimensional diagonal matrix with positive diagonal elements, and $U = \begin{pmatrix} \tilde{U}_{\bullet 1} & \tilde{U}_{\bullet 2} \end{pmatrix}$ and $V = \begin{pmatrix} \tilde{V}'_{\bullet 1} \\ \tilde{V}'_{\bullet 2} \end{pmatrix}$ are orthogonal matrices.

Theorem 1. *The dimension of indeterminacy is equal to rank* $\left(\tilde{V}'_{\bullet 1} V_{\bullet 2}\right)$.

Proof. It follows from the definition of the dimension of indeterminacy that

$$\begin{aligned} &rank\left(Q_{S\bullet}\Pi - Proj\left(Q_{S\bullet}\Pi \mid Q_{U\bullet}\Pi\right)\right) \\ &\quad = rank\left\{Q_{S\bullet}\Pi\left[I_k - \left((Q_{U\bullet}\Pi)^{\dagger} Q_{U\bullet}\Pi\right)\right]\right\} \\ &\quad = rank\left\{\tilde{U}_{\bullet 1}\tilde{D}_{11}\tilde{V}'_{\bullet 1}\left[I_k - \left(V_{\bullet 1}D_{11}^{-1}U'_{\bullet 1}U_{\bullet 1}D_{11}V'_{\bullet 1}\right)\right]\right\} \\ &\quad = rank\left[\tilde{V}'_{\bullet 1}\left(V_{\bullet 2}V'_{\bullet 2}\right)\right] \\ &\quad = rank\left(\tilde{V}'_{\bullet 1}V_{\bullet 2}\right), \end{aligned}$$

where $Proj\,(A \mid B)$ denotes the projection of the row space of A on the row space of B. The fourth line follows from the third line because $I_k - V_{\bullet 1}V'_{\bullet 1} = V_{\bullet 2}V'_{\bullet 2}$. □

Note that Lubik and Schorfheide (2003) would characterise the dimension of indeterminacy as $rank\,(V_{\bullet 2})$, i.e. the number of non-trivial linearly independent linear combinations of endogenous forecast errors which do not have an observable effect on the causal and stable solution $\left(w_t^U\right)$ of (4).

3.3. Analysis of the impulse response function

In this section, we derive the impulse response function of the solutions (y_t) of (1) with respect to the exogenous shocks ε_t and the reduced sunspot shocks $\zeta_t^* = M_2\zeta_t$. We proceed by solving system (3) stepwise such that we first obtain a solution for $\left(w_t^U\right)$ and then for $\left(w_t^S\right)$. Subsequently, this solution is transformed back to the original variables $y_t = Zw_t$.

First, we obtain a stable and causal solution $\left(w_t^U\right)$ of system (3) which is necessarily identically zero. We next consider the stable block of system (3), i.e.

$$\Lambda_{SS}w_t^S = \Omega_{SS}w_{t-1}^S + Q_{S\bullet}\Psi\varepsilon_t + Q_{S\bullet}\Pi\eta_t.$$

Substituting $\eta_t = -V_{\bullet 1}D_{11}^{-1}U'_{\bullet 1}Q_{U\bullet}\Psi\varepsilon_t + V_{\bullet 2}\left(M_1\varepsilon_t + \zeta_t^*\right)$, premulitplying the inverse of Λ_{SS}, and solving the system such that the solution is stable and causal, we obtain

$$\begin{aligned} w_t^S = &\sum_{j=0}^{\infty}\left(\Lambda_{SS}^{-1}\Omega_{SS}\right)^j\Lambda_{SS}^{-1}\left(\, I \;\middle|\; -Q_{S\bullet}\Pi\left(V_{\bullet 1}D_{11}^{-1}U'_{\bullet 1}\right)\,\right) \\ &\times\begin{pmatrix} Q_{S\bullet} \\ Q_{U\bullet}\end{pmatrix}\Psi\varepsilon_{t-j} + \cdots\cdots \\ &+\sum_{j=0}^{\infty}\left(\Lambda_{SS}^{-1}\Omega_{SS}\right)^j\Lambda_{SS}^{-1}\left(Q_{S\bullet}\Pi\right)V_{\bullet 2}\left(M_1\varepsilon_{t-j} + \zeta_{t-j}^*\right). \end{aligned}$$

Finally, the effects of the reduced sunspots ζ_t^* and the exogenous shocks ε_t on the solution $y_t = Z_{\bullet S}w_t^S$, where $Z_{\bullet S}$ are the first n_S columns of Z, are

$$\frac{\partial y_t}{\partial\zeta_t^*} = Z_{\bullet S}\Lambda_{SS}^{-1}\left(Q_{S\bullet}\Pi\right)V_{\bullet 2} = Z_{\bullet S}\Lambda_{SS}^{-1}\left(\tilde{U}_{\bullet 1}\tilde{D}_{11}\tilde{V}'_{\bullet 1}\right)V_{\bullet 2}$$

and

$$\begin{aligned}\frac{\partial y_t}{\partial\varepsilon_t} &= Z_{\bullet S}\Lambda_{SS}^{-1}\left(\, I \;\middle|\; -\left(\tilde{U}_{\bullet 1}\tilde{D}_{11}\tilde{V}'_{\bullet 1}\right)\left(V_{\bullet 1}D_{11}^{-1}U'_{\bullet 1}\right)\,\right)Q\Psi \\ &\quad + \cdots\cdots + Z_{\bullet S}\Lambda_{SS}^{-1}\left(\tilde{U}_{\bullet 1}\tilde{D}_{11}\tilde{V}'_{\bullet 1}\right)V_{\bullet 2}M_1. \end{aligned} \tag{8}$$

If (7) holds, it follows that $\tilde{V}'_{\bullet 1}V_{\bullet 2} = 0$ and thus $\partial y_t/\partial\zeta_t^* = 0$ and the second summand in (8) is zero. Lubik and Schorfheide (2003) analyse the derivatives of $\Gamma_0 y_t$ (instead of y_t). Implications of the violation of (7) are not reflected in their analysis.

3.4. An economic example

We illustrate by means of an economic example the possibility that Sims' uniqueness condition is satisfied when there are multiple equilibria satisfying the existence condition.

Consider the NK model

$$\begin{aligned} \mathbb{E}_t\left(y_{t+1}\right) + \sigma\pi_t &= y_t + \sigma R_t \\ \beta\mathbb{E}_t\left(\pi_{t+1}\right) &= \pi_t - \kappa y_t \\ R_t &= \phi_R R_{t-1} + \phi_\pi\pi_t + \phi_y y_t + \varepsilon_t^R \end{aligned}$$

where y_t, π_t, and R_t are log-deviations from the steady state of output, inflation and the interest rate. Furthermore, σ denotes the intertemporal substitution elasticity, β the time preference rate, κ^{-1} measures the elasticity of aggregate supply with respect to inflation, ϕ_R denotes the interest rate smoothing coefficient, ϕ_π denotes the elasticity of the interest rate response with respect to inflation, and ϕ_y denotes the elasticity of the interest rate response with respect to output. Finally, ε_t^R denotes an unanticipated policy implementation shock. A similar monetary rule can be found in equation (26) of Lubik and Marzo (2007).

We transform the equations above, using the methods described in Sims (2001) and Lubik and Schorfheide (2003), by replacing the conditional expectations with new variables $\xi_t^y = \mathbb{E}_t\left(y_{t+1}\right)$ and $\xi_t^\pi = \mathbb{E}_t\left(\pi_{t+1}\right)$, and adding the respective equations to obtain

$$\begin{aligned}\begin{pmatrix} 1 & 0 & -\sigma \\ 0 & \beta & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix}\xi_t^y \\ \xi_t^\pi \\ R_t\end{pmatrix} &= \begin{pmatrix} 1 & 0 & 0 \\ -\kappa & 1 & 0 \\ 0 & 0 & \phi_R\end{pmatrix}\begin{pmatrix}\xi_{t-1}^y \\ \xi_{t-1}^\pi \\ R_{t-1}\end{pmatrix} \\ &\quad + \begin{pmatrix}0 \\ 0 \\ 1\end{pmatrix}\varepsilon_t^R + \begin{pmatrix} 1 & -\sigma \\ -\kappa & 1 \\ \phi_y & \phi_\pi\end{pmatrix}\begin{pmatrix}\eta_t^y \\ \eta_t^\pi\end{pmatrix}.\end{aligned}$$

It can be verified that for $\kappa = \frac{1}{\sigma}$ and $\phi_y = -\phi_\pi\kappa$ and, e.g., $\beta = 0.95$ and $\phi_R = 0.6$, the existence condition has a non-trivial kernel but the uniqueness condition is satisfied.[2] Lubik and Schorfheide (2003) classify a model like the one above as indeterminate even though its solution (y_t) is unique.

4. Discussion

In this section, we start by discussing the dimensionality, including some general remarks regarding the rank of matrices depending on a low-dimensional parameter vector, of the problem at hand. Subsequently, the characterisation of the dimension of indeterminacy is related to the notion of indeterminacy used in Lubik and Schorfheide (2003) and Farmer et al. (2015). Finally, we mention how other articles on solution methods for LRE models treat indeterminate equilibria.

[2] The software package `indeterminateR`, written in the R computing environment R-Core-Team (2017), contains more detail on the derivation of the NK model and checks the existence and uniqueness condition as described above. It can be installed with the command `devtools::install_github(''bfunovits/indeterminateR'')`.

4.1. On the rank of matrices depending on deep structural parameters

In order to discuss the dimensionality of the problem, let us consider the following. If the number k of endogenous forecast errors is strictly larger[3] than the number of equations n, it follows that the maximal dimension of indeterminacy n_S is strictly smaller than the maximal dimension $(k - n_U) > (n - n_U) = n_S$ of the kernel of the existence condition (5), i.e. the dimension of indeterminacy according to Lubik and Schorfheide (2003).

Moreover, matrices depending on a low-dimensional vector of deep structural parameters are not necessarily of full rank. As an example consider the NK model analysed in Komunjer and Ng (2011) which involves a so-called controllability matrix of dimension (8×24) whose rank is equal to three. Taking the possible rank deficiencies in $Q_{U\bullet}\Pi$ and $Q_{S\bullet}\Pi$ into account is maybe the most important one among the major achievements of the paper by Sims (2001). The respective SVDs of $Q_{U\bullet}\Pi$ and $Q_{S\bullet}\Pi$ are a useful tool for analysing them. On the one hand, the rows of $Q_{U\bullet}\Pi$ might be linearly dependent which implies a relatively higher dimensional kernel. On the other hand, linearly dependent rows in $Q_{S\bullet}\Pi$ imply a relatively smaller dimension of indeterminacy.

In the applications described in Lubik and Schorfheide (2003) and Farmer et al. (2015), these rank deficiencies do not occur firstly because of the small number of deep structural parameters relative to the dimension of their models, and secondly because of implicit and explicit rank restrictions on certain matrices. The first point is exemplified by the determinate solution in Lubik and Schorfheide (2003) with two endogenous variables and one shock which is parametrised by three parameters. As an example of an implicit rank assumption, note that both Lubik and Schorfheide (2003) and Farmer et al. (2015) invert the matrix Λ_{UU}, see equation (12) in Farmer et al. (2015) and Eq. (7) in Lubik and Schorfheide (2003). Assuming invertibility of this matrix is equivalent to assuming invertibility of Γ_0 in Sims' canonical form. An explicit rank assumption in Farmer et al. (2015) is the definition of a regular indeterminate equilibrium on page 21 in Farmer et al. (2015) which requires that each selection of n_U columns of the $(n_U \times k)$-dimensional matrix $Q_{U\bullet}\Pi$ is non-singular.

4.2. Internal and external dimension of indeterminacy

Despite making stronger (implicit or explicit) rank assumptions, the contributions of Lubik and Schorfheide (2003) and Farmer et al. (2015)[4] are important for the analysis and understanding of indeterminate equilibria. This article can be interpreted as an extension in the following sense. On the one hand, the papers mentioned above focus on indeterminacy at the causal and stable modelling stage. They are concerned with the internal characteristics[5] of an LRE model and thus analyse internal indeterminacy. This article, on the other hand, focuses on the observable consequences of indeterminacy, i.e. on the external characteristics of an LRE model and in particular on the observable consequences of indeterminacy.

[3] Note that Sims' canonical form allows for this case; it covers more general models than LRE or DSGE models.

[4] Note that Farmer et al. (2015) mention in their equation (17) that a solution might still be unique even when the existence condition is satisfied by multiple candidates.

[5] For precise definitions of internal and external characteristics of an econometric model see Deistler and Seifert (1978).

4.3. Other approaches and algorithmic implementation

In order to broaden the perspective of this article, it is mentioned here that while there are many alternative algorithms that work in case of a unique solution, there is considerable variation in practice as to what to do under indeterminacy.[6]

The article King and Watson (1998), for example, imposes minimal conditions (equivalent to the assumption by Sims (2001) that there be no coincident zeros in the QZ-decomposition) on the rank of the matrices involved in their canonical form. In contrast to Sims (2001), however, the condition for existence and uniqueness is neither stated as a theorem nor is it minimal: King and Watson (1998) require that a certain (sub-)matrix be square and invertible. Inspection of the associated algorithmic implementation[7] in MATLAB shows that an exception is thrown when the equilibrium is indeterminate.

The model in Al-Sadoon (2017) contains, in the same way as Anderson and Moore (1985), a finite number of leads and lags of endogenous variables. In contrast to the latter paper, however, the former one also includes conditional expectations and treats the impact of different conditioning sets in the conditional expectations. Even though Al-Sadoon (2017) characterises existence, uniqueness, and non-uniqueness elegantly with partial indices of certain polynomial matrices, there is no algorithmic implementation as in Farmer et al. (2015), Lubik and Schorfheide (2003), or Sims (2001). Moreover, Sims' canonical form allows, e.g., for perception variables, and is more general than the one in Al-Sadoon (2017).

Under the condition that the dimension of the kernel of the existence condition coincides with the dimension of indeterminacy, the Dynare, see Adjemian et al. (2017), implementation by Farmer et al. (2015) provides a convenient way to parametrise solutions pertaining to indeterminate equilibria.

5. Conclusion

This article characterises the observable extent of non-uniqueness of stable and causal solutions of the LRE model (1). Moreover, it shows that some LRE models are classified as indeterminate by Lubik and Schorfheide (2003) even though they entail the same observable outcome. Thus, the correct region of indeterminacy is sometimes smaller than the one obtained by Lubik and Schorfheide (2003).

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Appendix A. Supplementary data

Supplementary material related to this article can be found online at http://dx.doi.org/10.1016/j.econlet.2017.09.021.

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[6] I am grateful to a referee for pointing this out.

[7] It can be downloaded from http://people.bu.edu/rking/REmodels/KWRE.zip.

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