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Daoxiang, Zhang

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ON THE HARDY–CARLEMAN INEQUALITY FOR A NEGATIVE EXPONENT

ZHANG DAOXIANG AND PING YAN

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Abstract. In this paper we settle an open problem raised by B. Yang (2005, Taiwanese Journal of Mathematics 9, 469-475), by using Hölder’s and Bernoulli’s inequalities. We give a strengthened Hardy-Carleman inequality for a negative exponent.

1. Introduction

The following inequality of Hardy’s is well known [2, Chap. 9.12]:

\[ \sum_{n=1}^{\infty} \left( \frac{1}{n} \sum_{k=1}^{n} a_k^p \right)^{1/p} < \left( \frac{p}{p-1} \right)^{p} \sum_{n=1}^{\infty} a_n. \] (1)

Here \( p > 1 \), \( a_n \geq 0 \) (\( n \in \mathbb{N} \)) and \( 0 < \sum_{n=1}^{\infty} a_n < \infty \).

The constant \( \left( \frac{p}{p-1} \right)^{p} \) in (1) is the best possible. As \( p \) tends to infinity the inequality (1) reduces to Carleman’s inequality

\[ \sum_{n=1}^{\infty} (a_1 a_2 \cdots a_n)^{1/n} < e \sum_{n=1}^{\infty} a_n, \] (2)

where the constant \( e \) in (2) is still the best possible [2, Chap. 9.12]. The inequalities (1) and (2) are important in analysis and its applications [3].

In [8], we proved the following strengthened version of (2).

\[ \sum_{n=1}^{\infty} (a_1 a_2 \cdots a_n)^{1/n} < e \sum_{n=1}^{\infty} \left( 1 + \frac{1}{n + \frac{1}{5}} \right)^{-\frac{1}{2}} a_n. \] (3)

Some other strengthened versions of (2) and related results can be found in [1, 7, 8, 9, 11].


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If we set $p = \frac{1}{r}$ in (1), then we have $0 < r < 1$, and (1) is equivalent to the inequality

$$\sum_{n=1}^{\infty} \left( \frac{1}{n} \sum_{k=1}^{n} a_k^r \right)^{1/r} < \left( \frac{1}{1-r} \right)^{1/r} \sum_{n=1}^{\infty} a_n,$$

where the constant $\left( \frac{1}{1-r} \right)^{1/r}$ is the best possible.

Thanh et al. [6] discussed (4) for $r \in (-\infty, 0)$, and proved the following result: If $a_n \geq 0$ for $n \in \mathbb{N}$ and $0 < \sum_{n=1}^{\infty} a_n < \infty$, then

$$\sum_{n=1}^{\infty} \left( \frac{1}{n} \sum_{k=1}^{n} a_k^r \right)^{1/r} < \left( \frac{1}{1-r} \right)^{1/r} \sum_{n=1}^{\infty} a_n$$

if $-1 \leq r < 0$ and

$$\sum_{n=1}^{\infty} \left( \frac{1}{n} \sum_{k=1}^{n} a_k^r \right)^{1/r} < \frac{1}{r-1} \sum_{n=1}^{\infty} a_n$$

if $r < -1$.

If we replace $r$ by $-r$, in (5) and (6) we obtain

$$\sum_{n=1}^{\infty} \left( \frac{n}{\sum_{k=1}^{n} a_k^r} \right)^{1/r} < (1+r)^{1/r} \sum_{n=1}^{\infty} a_n$$

if $0 < r \leq 1$ and

$$\sum_{n=1}^{\infty} \left( \frac{n}{\sum_{k=1}^{n} a_k^r} \right)^{1/r} < \frac{r}{1+r} 2^{\frac{1-r}{r}} \sum_{n=1}^{\infty} a_n$$

if $1 < r < \infty$.

Recently, Yang [10] proved that the constant $(1+r)^{1/r}$ in (7) is the best possible for $0 < r \leq 1$. At the end of paper [10], Yang posed the question:

Is the constant factor $\frac{r}{1+r} 2^{\frac{1+r}{r}}$ in (8) the best possible or not for $1 < r < \infty$?

In this paper we solve this problem. We give a strengthened Hardy-Carleman inequality for a negative exponent.

2. Main results

In this section, we prove the following theorem.

**Theorem 2.1.** Let $1 < r < \infty$, $a_n \geq 0$ ($n \in \mathbb{N}$) and $0 < \sum_{n=1}^{\infty} a_n < \infty$. Then

$$\sum_{n=1}^{\infty} \left( \frac{n}{\sum_{k=1}^{n} a_k^r} \right)^{1/r} < \frac{1}{r} (1+r)^{\frac{1-r}{r}} \sum_{n=1}^{\infty} a_n.$$
To prove Theorem 2.1, we use Hölder’s inequality (with negative exponent $p$) (see [4, page 29]) and Bernoulli’s inequality. For the convenience of the reader we start by recalling these results.

**Lemma 2.1.** (Hölder’s inequality) Suppose that $p < 0$, $\frac{1}{p} + \frac{1}{q} = 1$, $f(x)$, $g(x) \geq 0$ for $x \in [a, b]$, and $f \in L^p[a, b]$, $g \in L^q[a, b]$. Then

$$\int_a^b f(t)g(t)dt \geq \left( \int_a^b f^p(t)dt \right)^{1/p} \left( \int_a^b g^q(t)dt \right)^{1/q},$$

where equality holds only if there exist real numbers $\alpha$ and $\beta$, such that $\alpha^2 + \beta^2 > 0$, and $\alpha f^p(x) = \beta g^q(x)$, a.e. in $[a, b]$.

**Lemma 2.2.** (Bernoulli inequality) Suppose that $x \geq -1$ and $0 < \alpha < 1$. Then

$$(1 + x)^{\alpha} \leq 1 + \alpha x,$$

where equality holds if and only if $x = 0$.

We also need the following lemmas.

**Lemma 2.3.** Suppose that $0 < \alpha < 1$ and $x > 0$. Then

$$1 + \frac{\alpha x}{1 + (1 - \alpha)x} < (1 + x)^{\alpha}.$$  

*Proof.* We rewrite this inequality as

$$1 + x + \alpha x (1 + x)^{\alpha} < (1 + x)^{1+\alpha}.$$  

We define

$$\varphi(x) = (1 + x)^{1+\alpha} - \alpha x (1 + x)^{\alpha} - x - 1 \quad \text{for} \quad x \geq 0.$$  

Simple computation yields

$$\varphi'(x) = (1 + x)^{\alpha} - \alpha^2 x (1 + x)^{\alpha-1} - 1,$$

$$\varphi''(x) = (\alpha - \alpha^2) (1 + x)^{\alpha-1} - \alpha^2 (\alpha - 1) x (1 + x)^{\alpha-2}.$$  

It follows that $\varphi''(x) > 0$ for $x > 0$ and $0 < \alpha < 1$, $\varphi'(0) = 0$ and $\varphi(0) = 0$. Thus, $\varphi(x)$ is strictly increasing and $\varphi(x) > 0$ for $x > 0$. This completes the proof of Lemma 2.3. \qed

**Lemma 2.4.** Suppose that $r > 1$ and $x \geq 1$. Then

$$(1 + x)^{\frac{1+r}{r}} - x^{\frac{1+r}{r}} > \frac{1 + r}{r} x^{\frac{1}{r}}.$$
Proof. We rewrite this inequality as
\[
(1 + x) \left( 1 + \frac{1}{x} \right)^{\frac{1}{r}} - x > \frac{1 + r}{r}.
\]
This is true. Since by Lemma 2.3, we have
\[
\left( 1 + \frac{1}{x} \right)^{\frac{1}{r}} > 1 + \frac{\frac{1}{r}x}{1 + (1 - \frac{1}{r})x} > 1 + \frac{1}{r(1 + x)}.
\]
This completes the proof of Lemma 2.4. □

Lemma 2.5. We have
(i) \(2^\frac{1}{r} < \left( 1 + \frac{1}{r} \right)^{\frac{1}{r}}\) for \(r > 1\).
(ii) \(\frac{r^2}{1 + r} 2\left( 1 + \frac{1}{r} \right)^{\frac{1}{r}} > \frac{1}{r} (1 + r)^{\frac{1}{r}}\) for \(r > \frac{26}{5}\).

Proof. (i) By Bernoulli’s inequality we have
\[
2^\frac{1}{r} = (1 + 1)^{\frac{1}{r}} < \frac{1 + r}{r}.
\]
(ii) We rewrite this inequality as
\[
\frac{r^2}{1 + r} > \left( \frac{1 + r}{2} \right)^{\frac{1}{r}}.
\]
By Bernoulli’s inequality, it follows
\[
\left( \frac{1 + r}{2} \right)^{\frac{1}{r}} = \frac{1 + r}{2} \left( \frac{1 + r - 1}{2} \right)^{\frac{1}{r}} < \frac{1 + r}{2} \left( 1 + \frac{r - 1}{2r} \right) = \frac{r^2(r - 5) - r + 1}{4r(1 + r)} + \frac{r^2}{1 + r}
\]
for \(r > \frac{26}{5}\).
This completes the proof of Lemma 2.5. □

Proof of Theorem 2.1. Let \(r > 1\) and set \(p = -\frac{1}{r}\), \(a = 1\), \(b = x > 1\), \(f(x) = a_n\), 
\(g_n(x) = (x - 1)^{\frac{1}{1 + r}}\) for \(x \in [n, n + 1]\) and \(n \in \mathbb{N}\). Hölder’s inequality then yields
\[
\left( \int_1^x f(t)g(t)dt \right)^{-\frac{1}{r}} \leq \left( \int_1^x f^{-\frac{1}{r}}(t)dt \right) \left( \int_1^x g^{\frac{1}{1 + r}}(t)dt \right)^{\frac{1}{1 + r}}.
\]
It follows that

\[
\left( \int_1^x f^{-r}(t) \, dt \right)^{-\frac{1}{r}} = \left( \int_1^x ((t-1)^{1+r} f(t))^{-r} ((t-1)^{1+r}) \, dt \right)^{-\frac{1}{r}} \\
< \left( \int_1^x (t-1)^{1+r} f(t) \, dt \right) \left( \int_1^x (t-1)^{r} \, dt \right)^{-1+r} \\
< (1+r)^{\frac{1+r}{r}} (x-1) \frac{1+2r}{r} \int_1^x (t-1)^{1+r} f(t) \, dt.
\]

Then we have

\[
\left( \frac{x-1}{\int_1^x f^{-r}(t) \, dt} \right)^{1/r} < (1+r)^{\frac{1+r}{r}} (x-1)^{-r-2} \int_1^x (t-1)^{1+r} f(t) \, dt.
\]

Then we obtain

\[
\int_1^\infty \left( \frac{x-1}{\int_1^x f^{-r}(t) \, dt} \right)^{1/r} \, dx < (1+r)^{\frac{1+r}{r}} \int_1^\infty (x-1)^{-r-2} \int_1^x (t-1)^{1+r} f(t) \, dt \, dx
\]

\[
= (1+r)^{\frac{1+r}{r}} \int_1^\infty \left( \int_t^\infty (x-1)^{-r-2} \, dx \right) (t-1)^{1+r} f(t) \, dt
\]

\[
= (1+r)^{\frac{1}{r}} \int_1^\infty f(t) \, dt
\]

\[
= (1+r)^{\frac{1}{r}} \sum_{n=1}^\infty a_n.
\]

By the definition of \( f(x) \), Lemmas 2.3, 2.4 and 2.5, we have

\[
\int_1^\infty \left( \frac{x-1}{\int_1^x f^{-r}(t) \, dt} \right)^{1/r} \, dx > \int_1^2 \left( \frac{x-1}{\int_1^x f^{-r}(t) \, dt} \right)^{1/r} \, dx + \int_2^\infty \left( \frac{x-1}{\int_1^x f^{-r}(t) \, dt} \right)^{1/r} \, dx
\]

\[
> \int_2^\infty (x-1)^{1/r} \, dx \sum_{n=2}^\infty \int_1^{n+1} \frac{x-1}{\int_1^x f^{-r}(t) \, dt} \, dx
\]

\[
= \frac{r}{a_1^{-1}} + \sum_{n=2}^\infty \frac{r}{1+r} (\frac{n^{1/r} - (n-1)^{1+r}}{(\sum_{k=1}^n a_k^{-r})^{1/r}})
\]

\[
> \frac{r}{a_1^{-1}} + \sum_{n=2}^\infty (\frac{n-1}{\sum_{k=1}^n a_k^{-r}})^{1/r}
\]

\[
> \frac{r}{a_1^{-1}} + \left( \frac{\frac{1}{2}}{\sum_{k=1}^n a_k^{-r}} \right)^{1/r} \sum_{n=2}^\infty \frac{n^{-1/r}}{(\sum_{k=1}^n a_k^{-r})^{1/r}}
\]

\[
> \frac{r}{1+r} \sum_{n=1}^\infty \frac{n}{(\sum_{k=1}^n a_k^{-r})^{1/r}}.
\]

This completes the proof of Theorem 2.1. \( \square \)

**Remark 1.** By Theorem 2.1 and Lemma 2.5 (ii), we know that the constant factor \( \frac{r}{1+r} \frac{1+2r}{r} \) in (8) is not the best possible for \( r > \frac{26}{5} \). We give a strengthened Hardy-Carleman inequality (9) for a negative exponent.
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Zhang Daoxiang
Mathematics and Computer Science
Anhui Normal University
Wuhu, Anhui 241002, P. R. China
and
Department of Mathematics and Statistics
University of Helsinki
P. O. Box 68, FIN-00014 Helsinki, Finland

Ping Yan
School of Sciences
Zhejiang A & F University
Hangzhou 311300, P. R. China
and
Department of Mathematics and Statistics
University of Helsinki
P. O. Box 68, FIN-00014 Helsinki, Finland
e-mail: ping.yan@helsinki.fi