ACADEMIC DISSERTATION

FINDING THE MAIN GAP IN THE BOREL-REDUCIBILITY HIERARCHY

Miguel Moreno

To be presented, with the permission of the Faculty of Science of the University of Helsinki, for public examination in room D122 of Exactum (Gustaf Hällströmin katu 2b), on Monday December 18th 2017, at 12 o'clock noon.

Department of Mathematics and Statistics
Faculty of Science
University of Helsinki
Finland
2017

Supervisor
Tapani Hyttinen, University of Helsinki, Finland

Pre-examiners
Boban Velickovic, University of Paris 7, France
Vera Koponen, Uppsala University, Sweden

Opponent
Andrés Villaveces, Universidad Nacional, Colombia

Custos
Jouko Väänänen, University of Helsinki, Finland

ISBN 978-951-51-3908-5 (paperback)
ISBN 978-951-51-3909-2 (PDF)
http://ethesis.helsinki.fi
Helsinki 2017
Unigrafia Oy

TO MY MOTHER,
MY FIRST MATH TEACHER.

Acknowledgement

I would like to express my deepest gratitude to my supervisor Dr. Tapani Hyttinen for his guidance and support during this work. I am thankful that he always had time for my questions and mathematical discussions. I learned a lot from him during these years, from huge ideas to small details that make a difference. He taught me how to be self-critical and how to do serious mathematical research. He introduced me to the topic of my thesis, taught me the basics, and gave me questions that I would have never considered to answer due to the technical and difficult level of them. I appreciate that he demanded a high level of quality on my work, making me go over my limits and improve myself as a mathematician.

I wish to thank Professor Jouko Väänänen for his support during these years, for acting as the custos at the doctoral defense of this dissertation, and for the opportunities to present my work in many set theory meetings.

I am grateful to my co-author and friend Vadim Kulikov for the mathematical discussions, collaborations, and help in and outside the mathematical environment. I wish to thank Åsa Hirvonen who shared office with me during these years, helped me with question about model theory and about the doctoral program. To the current and former members of the Helsinki Logic Group, apart from the people mentioned above, Mika Hannula, Kaisa Kangas, Juliette Kennedy, Juha Kontinen, Juha Oikkonen, Gianluca Paolini, and Jonni Viertema, for being a community that motivated me during these years to improve myself as a researcher, for the discussions in the Logic Coffee, and for making me feel welcome in a research community. Due to the international connections of the group I had the opportunity to attend to talks given by experts in logic in the world, these talks influenced my work in many useful ways.

I am thankful to the pre-examiners Professor Boban Velickovic and Dr. Vera Koponen for their time in the examination of my thesis, and helpful comments. To my opponent Professor Andrés Villaveces for his time in the examination of my thesis, the fruitful discussions about the connetions between model theory and set theory, and his advises during these years.

I want to thank Philipp Schlicht and Philipp Lücke for the fruitful discussions and their interest in the topic, to my co-author David Asperó for hearing my ideas about reflection principles and his improvements of these ideas.

This research was supported by the DOCTORAL PROGRAMME IN MATHEMATICS AND STATISTICS (Domast), to which I want to express my gratitude. Without their support it would have not been possible to do my research. I wish to thank the administration office of the department for their help with practical issues.

I am thankful to Helen W., Brecht D., Stefan W., Kay S., Petri T., Paolo M., and the math football group for helping me chill with exercise and to the CF Basement for helping me distract with good workouts. I want to thank all my friends for their support during these years, in particular Lila J., Daniel G., Andres G., Jonathan M. K., Jairo D., Cindy E., Giovanno C., Joaquin M., Javier V., Adolfo A., Christian R., Jose A., and Ramon U. for allways having the time for chat no matter the time or distance.

To my friends Klaus I., Rodrigo B., Reinis S., Li L., Katia C., Kristiina S., Diego C., and Saija K. for always been there with a good joke and the right comment when it was needed during an angry, frustrated or sad day. To my former and current flatmates Adrew L., Helen A., and Sarah K. for the dinners, the chats, and the drinks that made me forget about math for a while. To my colleague and close friend Andrés Jaramillo Puentes, thank you for encourage me to pursue a Ph.D. and convinced me not to quit logic, many times during these ten years of friendship.

Finally, to my family. Specially to my aunt Gladys Moreno, thank you for the help to start my master studies. To my aunts Guillermina Wandurraga, and Isabel Wandurraga, and my cousin Lizeth Pardo, thanks for always check if I was doing well. To my aunt Aminta Wandurraga, thank you for those long calls and advises. To my brothers Sergio and Andres, thank you for understand me and push me to pursue my dreams even when I am not sure what am I dreaming.

To my father Leonardo, spanish is supposed to be the best language to express feelings because spanish speakers are full of emotions, even with all that I cannot find the words in spanish to express my gratitude and respect. Thanks for everything, you have done more than what you promised to do or had to do. Gracias.

Prologue

This thesis is constituted by two parts. The first part is divided in four chapters, these are: Introduction, Structure of the thesis, Summary, and Conclusions. The second part contains five research articles.

The first part is intended to give a smooth explanation of each of the five articles of the second part. At the same time, the first part is an attempt to motivate the reader over the second part. It gives a motivation for the reader to study the generalized descriptive set theory, the roll of the author on every chapter of the second part, the motivation and a summary of the results behind each article of the second part, using the least amount of technical language.

Each article of the second part has no modifications from the submitted version of the respective article. The reader is advised that the notation between articles might differ. The articles are presented in the chronological order in which them were produced.

Contents

Chapter 1

Introduction

“ *If it is not difficult, then it is not funny.*

- ”

This thesis is about generalized descriptive set theory. To understand the importance of this area of mathematics I will show some of the motivation behind it. The second chapter is intended to explain the roll of the author on every chapter of the second part. The third chapter allows the reader to understand the motivation and results behind each chapter of the second part, using the least amount of technical language. The fourth chapter summarizes the main results of the second part and is intended to orientate the reader about future researches. All the mathematical proofs presented in this thesis are in the second part.

1.1 Descriptive set theory

Descriptive set theory studies definable sets and functions in Polish spaces. A Polish space is a topological space that is homeomorphic to a separable complete metric space, descriptive set theory is mainly focus on the space ω^ω equipped with the product topology. This space is called the Baire space and it has the property that every Polish space is a continuous image of it. Descriptive set theory is a beautiful area with applications in other areas of mathematics such as analysis, model theory, ergodic theory, and more. It has became one of the main research areas of set theory. The connection between descriptive set theory and model theory (e.g. Scott's and Lopez-Escobar's theorems) comes from the ability of coding countable structures with domain ω, in a countable relational vocabulary, into elements of the

Cantor space, 2^ω. The product topology on the Cantor space coincides with the topology generated by the basic open set of the form $N_\eta = \{\xi \in 2^\omega \mid \eta \subset \xi\}$, where $\eta \in 2^{<\omega}$. For $\mathcal{L} = \{P_m \mid m \in \omega\}$ a relational countable language, the elements of 2^ω code $\mathcal{L}$-structures as follows:

Definition 1.1 *Fix a bijection $\pi\colon \omega^{<\omega} \to \omega$. For every $\eta \in 2^\omega$ define the $\mathcal{L}$-structure $\mathcal{A}_\eta$ with universe ω as follows: For every relation P_m with arity n, every tuple $(a_1, a_2, \ldots, a_n)$ in ω^n satisfies*

$$(a_1, a_2, \ldots, a_n) \in P_m^{\mathcal{A}_\eta} \iff \eta(\pi(m, a_1, a_2, \ldots, a_n)) = 1.$$

Some of the definable sets studied in descriptive set theory are the projective sets ($\Delta^1_1, \Pi^1_1, \Sigma^1_1, \Delta^1_2 \ldots$), some of these sets have interesting properties.

Theorem 1.2 *([16]) Every Σ^1_1 set has the property of Baire.*

Another important class of sets is the Borel class of sets. A set $X \subseteq \omega^\omega$ is Borel if it belongs to the smallest σ-algebra containing the open sets of ω^ω. The class of Borel sets coincide with the class of Δ^1_1 sets, i.e. a subset $X \subset \omega^\omega$ is Borel if and only if X is Π^1_1 and Σ^1_1. The Borel class of subsets of 2^ω is defined in the same way. A function $f : 2^\omega \to 2^\omega$ is a Borel function, if for every open set $X \subset 2^\omega$, $f^{-1}[X]$ is a Borel set in 2^ω. Using Borel functions we can classify equivalence relations on 2^ω by their complexity (the Borel reducibility hierarchy). Suppose E_0 and E_1 equivalence relations on 2^ω. We say that E_0 is Borel reducible to E_1 if there is a Borel function $f\colon 2^\omega \to 2^\omega$ that satisfies $(\eta, \xi) \in E_0 \Leftrightarrow (f(\eta), f(\xi)) \in E_1$. We call f a Borel reduction of E_0 to E_1, and we denoted by $E_0 \ \leq_B \ E_1$. (if f is continuous, then E_0 is continuous reducible to E_1, $E_0 \ \leq_c \ E_1$.) What a reduction tells us about the complexity of two relation is (in this case) that E_0 is as most as complex as E_1. Borel reduction can also be use to classify quassi-orders.

Many results have been obtained in the Borel reducibility hierarchy.

Theorem 1.3 *[21] Let $E \subset 2^\omega \times 2^\omega$ be a Π^1_1 equivalence relation. If E has uncountably many equivalence classes, then $id_{2^\omega} \ \leq_B \ E$.*

As it was explain before, the elements of 2^ω code the structures with domain ω, the isomorphism reltion of model of a first order theory is an equivalence relation. It is natural to think on the isomorphism relation of first order theories as an equivalence relation on the space 2^ω.

Definition 1.4 (The isomorphism relation) *Assume T is a complete first order theory in a countable vocabulary. We define $\cong^\omega_T$ as the relation*

$$\{(\eta, \xi) \in 2^\omega \times 2^\omega \mid (\mathcal{A}_\eta \models T, \mathcal{A}_\xi \models T, \mathcal{A}_\eta \cong \mathcal{A}_\xi) \ or \ (\mathcal{A}_\eta \not\models T, \mathcal{A}_\xi \not\models T)\}.$$

The isomorphism relation with the Borel reducibility give us a notion of complexity for first order theories. We say that a theory T is as most as complex as T' if $\cong^\omega_T \leq_B \cong^\omega_{T'}$. This notion of complexity shows us the connection between Model Theory and Descriptive Set Theory.

In Model Theory, more precisely in Classification Theory there is a notion of complexity for first order theories, this notion is due to Shelah [20]. It is natural to ask if the Borel reducibility notion of complexity and the Classification Theory notion of complexity coincide. In Classification Theory, one of the most important results is the Main Gap Theorem. This theorem tells us that classifiable theories are less complex than non-classifiable ones and their complexities are far apart.

A classifiable theory is a theory with an invariant that determines the structures up to isomorphisms. The theory of a vector space over the field of rational numbers is a classifiable theory, the models are characterized by the dimension.

A theory with no invariant of this kind is a non-classifiable theory. The theory of the order of the rational numbers is a non-classifiable theory.

The Main Gap Theorem tells us that the theory of a vector space over the field of rational number is less complex than the theory of the order of the rational numbers. Unfortunately there is only one model of countable size, up to isomorphisms, of the theory of of the order of the rational numbers and the theory of a vector space over the field of rational number has more than one countable model, up to isomorphisms.

From this we can see that these two complexity notions are not equivalent. Another example of a classifiable theory more complex than the theory of the order of the rational numbers (in the Borel complexity notion) is the one introduced by Koerwien in [13]. He sows the existence of an ω-stable theory T with NDOP, NOTOP, depth 2, and with $\cong^\omega_T$ not Borel.

1.2 Generalized Descriptive set theory

So far the Descriptive Set Theory studies the complexity of a theory by studying the complexity of the countable models. On the other hand in Classification Theory the complexity does not depend on the countable models. Can Descriptive Set Theory study the complexity of the non countable models?

The previous question is about the elements of the set κ^κ, the *generalized Baire space*. To answer it, we will need to define a topology in κ^κ, define the Borel set, and more concepts. This questions were studied in [23] and

[5], for every $\zeta \in \kappa^{<\kappa}$, we call the set

$$[\zeta] = \{\eta \in \kappa^\kappa | \zeta \subset \eta\}$$

a basic open set. The open sets are of the form $\bigcup X$ where X is a collection of basic open sets. Vaught [22], Mekler and Väänänen [17] studied this topology. This topology is called the bounded topology. In Descriptive Set Theory there are three equivalent definitions for the collection of Borel set:

1. The collection of Borel subsets of ω^ω is the smallest set which contains the basic open sets and is closed under union and intersection, both of length ω.

2. $\Delta^1_1 = \Pi^1_1 \cap \Sigma^1_1$.

3. The collection of Borel* subsets of ω^ω is the set of subsets of ω^ω that have a Borel* code.

Each of these definitions can be generalized to a definition in the generalized Baire space. To chose one from the three possible generalization, Friedman, Hyttinen and Kulikov studied them under the assumption $\kappa^{<\kappa} = \kappa$ and try to over come as many difficulties as possible. They show that, under the assumption $\kappa^{<\kappa} = \kappa$, the best candidate for the collection of κ–Borel subsets is:

The collection of κ–Borel subsets of κ^κ is the smallest set which contains the basic open sets and is closed under union and intersection, both of length κ.

The generalization of (1), (2), and (3) have the following property in the space κ^κ under the assumption $\kappa^{<\kappa} = \kappa$

$$Borel \subseteq \Delta^1_1 \subseteq Borel^*,$$

it correspond to the formulas of $L_{\kappa^+\kappa}$, etc. A κ–Borel set is any set in this collection.

The topology of the space and the Borel sets are the basis for Descriptive set theory. Using the this topology and the κ–Borel sets, other notions of Descriptive Set Theory can be generalized to the generalized Baire space. The generalized Cantor space is the subspace 2^κ endowed with the relative subspace topology. The collection of κ–Borel subsets of 2^κ is the smallest set which contains the basic open sets and is closed under union and intersection, both of length κ.

It is easy to see that the generalized Baire space and the generalized Cantor space are very similar, it is possible to use both of them to define a complexity notion, as it was discussed in the classical Baire space, 2^ω. Instead of restricting the study to one of these spaces, we can generalized the complexity notion of the classical case into a notion that involves the generalized Baire space and the generalized Cantor space.

Suppose $X, Y \in \kappa^\kappa$, a function $f : X \to Y$ is a Borel function if for every open set $A \subseteq Y$, $f^{-1}[A]$ is a Borel set in X of κ^κ. Let E_1 and E_2 be equivalence relations on X and Y respectively. If a function $f : X \to Y$ satisfies $E_1(x, y) \Leftrightarrow E_2(f(x), f(y))$, we say that f is a reduction of E_1 to E_2. If there exists a Borel function that is a reduction, we say that E_1 is Borel reducible to E_2 and we denote it by $E_1 \leq_B E_2$.

Let us fix a relational countable language $\mathcal{L} = \{P_n | n < \omega\}$ and a bijection π between $\kappa^{<\omega}$ and κ.

Definition 1.5 *For every $\eta \in \kappa^\kappa$ define the structure $\mathcal{A}_\eta$ with domain κ as follows.*
For every tuple $(a_1, a_2, \ldots, a_n)$ in κ^n

$$(a_1, a_2, \ldots, a_n) \in P_m^{\mathcal{A}_\eta} \Leftrightarrow \text{ the arity of } P_m \text{ is } n \text{ and } \eta(\pi(m, a_1, a_2, \ldots, a_n)) > 0.$$

Definition 1.6 *For every $\eta \in 2^\kappa$ define the structure $\mathcal{A}_\eta$ with domain κ as follows.*
For every tuple $(a_1, a_2, \ldots, a_n)$ in κ^n

$$(a_1, a_2, \ldots, a_n) \in P_m^{\mathcal{A}_\eta} \Leftrightarrow \text{ the arity of } P_m \text{ is } n \text{ and } \eta(\pi(m, a_1, a_2, \ldots, a_n)) = 1.$$

With the structures coded by the elements of 2^κ and κ^κ, it is easy to define the isomorphism relation of structures of size κ in both spaces.

Definition 1.7 (The isomorphism relation) *Assume T is a complete first order theory in a countable vocabulary. We define $\cong_T^\kappa$ as the relation*

$$\{(\eta, \xi) \in \kappa^\kappa \times \kappa^\kappa \mid (\mathcal{A}_\eta \models T, \mathcal{A}_\xi \models T, \mathcal{A}_\eta \cong \mathcal{A}_\xi) \text{ or } (\mathcal{A}_\eta \not\models T, \mathcal{A}_\xi \not\models T)\}.$$

Definition 1.8 *Assume T is a complete first order theory in a countable vocabulary. We define $\cong_T^2$ as the relation*

$$\{(\eta, \xi) \in 2^\kappa \times 2^\kappa \mid (\mathcal{A}_\eta \models T, \mathcal{A}_\xi \models T, \mathcal{A}_\eta \cong \mathcal{A}_\xi) \text{ or } (\mathcal{A}_\eta \not\models T, \mathcal{A}_\xi \not\models T)\}.$$

It is easy to see that the function $\mathcal{F} : \kappa^\kappa \to 2^\kappa$ given by

$$\mathcal{F}(\eta)(\alpha) = \begin{cases} 0 & \text{if } \eta(\alpha) = 0 \\ 1 & \text{otherwise} \end{cases}$$

is a reduction of $\cong^\kappa_T$ to $\cong^2_T$, these two relations are bireducible. With this in mind a notion of complexity for first order complete theories in a countable vocabulary that depends on the complexity of the models of size κ can be define. We say that a theory T is as most as complex as T' if $\cong^\kappa_T \leq_B \cong^\kappa_{T'}$.

The main subject of study in this thesis is the question: *Is it true that for all classifiable theory T and non-classifiable theory T' holds $\cong^\kappa_T \leq_B \cong^\kappa_{T'}$?*

As we saw, the fact that the Borel reducibility measures complexity in a different way than stability theory (in the classical descriptive set theory) was part of the motivation for the generalized descriptive set theory. Anyway, the notions in generalized descriptive set theory were defined in such a way that are not too different to their equivalent in the classical case. This allows the study of many other subjects in the Generalized Baire spaces and similar question to ones asked in the classical case can be asked.

Besides the isomorphism relations, there are other equivalent relations that have been studied in the generalized Baire space, some of those are the relations $(Mod_\lambda(T), \equiv_{\infty,\aleph_0})$. In [15] Laskowski and Shelah studied the Borel reducibility properties of $(Mod_\lambda(T), \equiv_{\infty,\aleph_0})$ for theories T with eni-DOP. The quasi-orders can be studied in the generalized Baire space too, in general, the study of relations in the generalized Baire space is a huge area of studies.

Cardinal characteristics is another example of a subject that carries questions from the classical case to the generalized case. Many cardinal characteristics can be easily generalized, some of them are $\mathfrak{a}(\kappa)$, $\mathfrak{e}(\kappa)$, and $\mathfrak{g}(\kappa)$. Some others need more care to be generalized, like $\mathfrak{p}$. Brooke-Taylor, Fischer, Friedman, and Montoya have studied this in [2].

Generalized descriptive set theory is a growing area in set theory with many applications to other areas. The reader can find more about this subjects and others related to generalized descriptive set theory in [12].

Chapter 2

Structure of the thesis

> “ *Where there is will there is a way.*
>
> - English proverb ”

The main goal of this thesis is to make a contribution to the study of the Borel reducibility hierarchy in the generalized Baire space. Model theory and set theory are two disciplines of mathematical logic which can be used to study the Borel reducibility hierarchy in the generalized Baire space. These two disciplines are connected when the complexity of complete first order theories is studied. Each of these disciplines has its approach to measure the complexity of complete first order theories. The Borel reducibility hierarchy in the generalized Baire space shows us a deep connection between these two approaches, in this thesis I study this connection.

2.1 List of articles

The second part of this thesis consists of the following five articles, the articles are presented in the chronological order of production.

I Tapani Hyttinen and Miguel Moreno, *On the reducibility of isomorphism relations*, Mathematical Logic Quarterly. **63**, 175 – 192 (2017).

II Tapani Hyttinen, Vadim Kulikov, and Miguel Moreno *A generalized Borel-reducibility counterpart of Shelah's main gap theorem*, Archive for Mathematical Logic. **56** no.3, 175 – 185 (2017).

III Miguel Moreno, *The isomorphism relation of theories with S-DOP*. Preprint.

IV David Asperó, Tapani Hyttinen, Vadim Kulikov, and Miguel Moreno *On large cardinals and generalized Baires spaces.* Submited August 2017.

V Vadim Kulikov, and Miguel Moreno *On* Σ_1^1*–completeness in weakly compact cardinals.* Preprint.

The articles are reproduced with the permission of their respective copyright holders. I wish to discuss my honest contribution to each of the articles. The following must be taken with certain precaution, in mathematics is not always easy to determine which part was contributed by whom, in particular when it is the result of many hours of discussion. The five articles were written by me, except for the first part of the introduction of [9], the second paragraph of the introduction of [14] and [[1], Lemma 3.4]. Most of the details of all the articles have been elaborated by me.

The first article, *On the reducibility of isomorphism relations* is a joint work with my supervisor Tapani Hyttinen. The idea to generalize [[4], Lemma 9] to Δ_1^1 equivalent relations was mine [[9], Lemma 2.4]. This generalization gives us a sufficient condition for a Δ_1^1 equivalent relation to be continuous reducible to $E^\kappa_{\lambda\text{-club}}$, for all $\lambda < \kappa$ regular. We realized that for every classifiable theory, the isomorphism relation satisfies this condition given the right Δ_1^1-code. Coding the moves of the Ehrenfeucht-Fraïssé game by ordinals [[9], Definitions 2.3, 2.6] was my idea, this leads to [[9], Lemma 2.7]. From these two results [[9], Theorem 2.8] follows.

The second article, *A generalized Borel-reducibility counterpart of Shelah's main gap theorem* is a joint work with Tapani Hyttinen and Vadim Kulikov. We tried to obtain in the generalized Cantor space a result equivalent to [[9], Theorem 2.8]. Using the same technique of [[9], Theorem 2.8], we realized that the diamond principle implies the result we wanted [[8], Lemma 2]. The details of the proofs have been elaborated by me. Some of these were: show that the forcings needed for Theorem 7 do not destroy the diamond sequence, and show that the preimage of a Borel* set under a Borel function is also a Borel* set.

In the third article, *The isomorphism relation of theories with S-DOP*, I am the only author.

The fourth article, *On large cardinals and generalized Baires spaces* is a joint work with David Asperó, Tapani Hyttinen and Vadim Kulikov. [[1], Theorem 2.11] is due to me, the idea behind is to use λ^+ many times the reduction $E^2_{\lambda\text{-club}} \leq_B E^2_{\lambda^+\text{-club}}$ from [5], and Fodor's lemma. To use the reduction $E^2_{\lambda\text{-club}} \leq_B E^2_{\lambda^+\text{-club}}$ λ^+ many times we needed λ^+ many stationary subsets of $reg(\kappa)$ such that κ $\diamond$–reflect to them. We obtain this by using strongly reflection in L for κ a $\Pi_1^{\lambda^+}$–indescribable. During the preparation of this article I proved that: *If κ is a Π_2^2–indescribable cardinal, then E^κ_{reg} is Borel*–complete.* We improved the technique used in this result to prove [[1], Theorem 3.7].

The fifth article, Σ_1^1*–complete quasiorders on weakly compact cardinals* is a joint work with Vadim Kulikov. This article partially solves an open question posed by Motto Ros. The idea to modify the dual diamond from [1] to solve this question, was mine. The third section is due to me. The consistency of $G_{<\kappa}$–dual diamond is due to me.

2.2 Outline of problems studied in the thesis

In Shelah's stability theory, a classifiable theory is a theory with an invariant that determines the structures up to isomorphisms, a theory with no invariant of this kind is a non-classifiable theory. This tell us that a theory with an invariant of this kind is less complex than a theory with no invariant of this kind. Shelah's stability theory tells us that every countable complete first-order classifiable theory is less complex than all countable complete first-order non-classifiable theories. The subject of study in this thesis was the question: *Are all classifiable theories less complex than all the non-classifiable theories, in the Borel reducibility hierarchy?*. There are two frames where this question can be studied, the generalized Baire space and the generalized Cantor space. It is known that for every theory T, the relations $\cong^2_T$ and $\cong^\kappa_T$ are bireducible. This gives us the freedom to choose in which space we would like to work.

This question was studied in [4],[5], and [6] between other previous works. Some of the results in those works pointed out that the relation equivalence modulo the λ-non-stationary ideal might be one of the keys to understand the reducibility of the isomorphism relation. On the space κ^κ, for every regular $\lambda<\kappa$, we say that $f,g\in\kappa^\kappa$ are $E^\kappa_{\lambda\text{-club}}$ equivalent ($f\ E^\kappa_{\lambda\text{-club}}\ g$) if the set $\{\alpha<\kappa|cf(\alpha)=\lambda\wedge f(\alpha)\neq g(\alpha)\}$ is non-stationary. On the space 2^κ, for every regular $\lambda<\kappa$, we say that $f,g\in\kappa^\kappa$ are $E^2_{\lambda\text{-club}}$ equivalent ($f\ E^2_{\lambda\text{-club}}\ g$) if the set $\{\alpha<\kappa|cf(\alpha)=\lambda\wedge f(\alpha)\neq g(\alpha)\}$ is non-stationary. Some of these results are the following:

Theorem 2.1 *[5, Thm 79] Suppose that $\kappa=\lambda^+=2^\lambda$ and $\lambda^{<\lambda}=\lambda$.*

1. *If T is unstable or superstable with OTOP, then $E^2_{\lambda\text{-}club}\leq_c\ \cong^\kappa_T$.*
2. *If $\lambda\geq 2^\omega$ and T is superstable with DOP, then $E^2_{\lambda\text{-}club}\leq_c\ \cong^\kappa_T$.*

Theorem 2.2 *[5, Thm 86] Suppose that for all $\gamma<\kappa$, $\gamma^\omega<\kappa$ and T is a stable unsuperstable theory. Then $E^2_{\omega\text{-}club}\leq_c\ \cong^\kappa_T$.*

Theorem 2.3 *[4, Cor 14] Suppose T is a countable complete first-order classifiable and shallow theory, then $\cong^\kappa_T \ \leq_B \ E^\kappa_{\lambda\text{-}club}$ holds for all regular $\lambda < \kappa$.*

These results lead to two approaches for the main question,

- *Is it provable in ZFC that $\cong^\kappa_{T'} \ \leq_B \ E^2_{\lambda\text{-}club} \ \leq_B \ \cong^\kappa_T$ holds for all T' classifiable and T non-classifiable?*
- *Is it provable in ZFC that $\cong^\kappa_{T'} \ \leq_B \ E^\kappa_{\lambda\text{-}club} \ \leq_B \ \cong^\kappa_T$ holds for all T' classifiable and T non-classifiable?*

Theorems 2.1 and 2.2 give a partial answer to the second reduction in the first question (above), and Theorem 2.3 give a partial answer to the first reduction to the second question (above). This point out a new possible approach to the main question: *Is it provable in ZFC that $\cong^\kappa_{T'} \ \leq_B \ \cong^\kappa_T$ holds for all T' classifiable and T non-classifiable?*. It can be studied by studying the reducibility between the relations $E^\kappa_{\lambda\text{-club}}$ and $E^2_{\lambda\text{-club}}$, it is clear that $E^2_{\lambda\text{-club}}$ is Borel reducible to $E^\kappa_{\lambda\text{-club}}$. The Borel reducibility of $E^\kappa_{\lambda\text{-club}}$ to $E^2_{\lambda\text{-club}}$ would imply $\cong^\kappa_{T'} \ \leq_B \ \cong^\kappa_T$ for all theories T' classifiable and non-shallow, and T non-classifiable (depending if the former one holds under the cardinal assumptions of Theorems 2.1 and 2.2).

These three are the questions studied in this thesis. The question *Is it provable in ZFC that $\cong^\kappa_{T'} \ \leq_B \ E^\kappa_{\lambda\text{-}club} \ \leq_B \ \cong^\kappa_T$ holds for all T' classifiable and T non-classifiable?* is studied in the fifth and seventh articles. The question *Is it provable in ZFC that $\cong^\kappa_{T'} \ \leq_B \ E^2_{\lambda\text{-}club} \ \leq_B \ \cong^\kappa_T$ holds for all T' classifiable and T non-classifiable?* is studied in the sixth article. The question *Is it provable in ZFC that $E^\kappa_{\lambda\text{-}club} \ \leq_B \ E^2_{\lambda\text{-}club}$?* is studied in the eighth article. The Borel reducibility properties of the relation E^κ_{reg} is studied in the ninth article.

Chapter 3

Summary

“ *Math is not trivial.*

- Jouko Väänänen ”

3.1 On the reducibility of isomorphism relations

3.1.1 Motivation

In this article we studied the Borel-reducibility properties of the relations E^κ_λ. The main motivation is to prove that $\cong^\kappa_{T'} \leq_B E^\kappa_{\lambda\text{-club}} \leq_B \cong^\kappa_T$ holds for all theories T' classifiable and T non-classifiable in ZFC. At the moment this project started, the best result concerning this problem was Theorem 2.3 above [[4], Cor 14]. This result tells us that if T is a classifiable and shallow theory then $\cong^\kappa_T \leq_B E^\kappa_{\lambda\text{-club}}$. This result motivated the study of the reducibility $\cong^\kappa_T \leq_B E^\kappa_{\lambda\text{-club}}$ when T is a classifiable theory.

3.1.2 Results

This article has five sections, the first one is the introduction. The second section is the study of the reduction $\cong^\kappa_T \leq_B E^\kappa_{\lambda\text{-club}}$ when T is a classifiable theory. In this section, Theorem 2.3 is generalized to all classifiable theories [[9], Theorem 2.8], not only to classifiable and shallow.

Theorem 3.1 ([9], Thm 2.8) *Assume T is a classifiable theory and $\lambda < \kappa$ a regular cardinal, the $\cong^\kappa_T$ is continuously reducible to $E^\kappa_{\lambda\text{-club}}$.*

In [4] Theorem 2.3 is obtained as a corollary of a stronger result, this result gives a sufficient condition for a Borel equivalent relation to be Borel reducible to $E^\kappa_{\lambda\text{-club}}$ for all regular $\lambda < \kappa$. In [4] the condition is stated as: *The Borel-code* (t, h) *has club-many good ordinals*; it is a condition over the Borel-code of the relation. In general, this "good condition" can be extended to Δ^1_1-codes and it is sufficient for any Δ^1_1 equivalent relation to be Borel reducible to $E^\kappa_{\lambda\text{-club}}$. The key for Theorem 3.1 was to prove that if T is a classifiable theory, then $\cong^\kappa_T$ satisfies the good condition.

It was already known that if T is a classifiable theory, then $\cong^\kappa_T$ is Δ^1_1 [[5], Theorem 70]. Unfortunately the Δ^1_1-code provided by [[5], Theorem 70] doesn't have club-many good ordinals, this is due to fact that this Δ^1_1-code doesn't use ordinals in the same way as the good condition uses them. [[9], Def 2.3] is a modification of the Δ^1_1-code in [[5], Theorem 70], this modification uses the ordinals in the same way as the good condition and codes the same relation. Using this Δ^1_1-code, Theorem 3.1 is proved in the same way as [[4], Cor 14].

The Third, fourth and fifth sections are the study of the reduction $E^\kappa_{\lambda\text{-club}} \leq_B \cong^\kappa_T$. In the third section we find a theory such that $E^\kappa_{\omega\text{-club}} \leq_B \cong^\kappa_T$ holds under certain cardinal assumptions, this is the first result of this type. Before this result was obtained, it was already known that if T is the theory of dense linear orderings without end points, then $E^\kappa_{\lambda\text{-club}} \leq_B \cong^\kappa_T$ is consistently true [[6], Thm 9].

Lemma 3.2 ([9], Lemma 3.2) *Suppose that for all* $\gamma < \kappa$, $\gamma^\omega < \kappa$ *and* $2^\lambda = \kappa$, *then* $E^\kappa_{\omega\text{-club}} \leq_c \cong^\kappa_{T_\omega}$.

The key for this result was to find the appropriate stable unsuperstable theory such that the reduction of Theorem 2.2 can be extended to the reduction $\mathbf{\Pi}_\lambda E^2_{\omega\text{-club}} \leq_B \cong^\kappa_T$. The result follows from the reduction $E^\kappa_{\omega\text{-club}} \leq_B \mathbf{\Pi}_\lambda E^2_{\omega\text{-club}}$, which holds when $\kappa = 2^\lambda$. In [[6], Thm 7] the authors proved that $E^\kappa_{\omega\text{-club}}$ is Σ^1_1–complete in L, this and Lemma 3.2 imply that $\cong^\kappa_{T_\omega}$ is Σ^1_1–complete in L.

In the fourth section [[9], Definition 4.1] defines coloured trees. In [[9], Definition 4.6] the trees (J_f, c_f) are constructed for all $f \in \kappa^\kappa$ such that, if κ is an inaccessible cardinal and $f, g \in \kappa^\kappa$, then $f\ E^\kappa_{\omega\text{-club}}\ g$ holds if and only if J_f and J_g are isomorphic. These trees were used to prove [[6], Cor 21] mentioned above.

In the fifth section the coloured trees are used to prove that:

Corollary 3.3 ([9], Cor 5.10) *If* T *is a stable theory with the OCP and* κ *is an inaccessible cardinal, then* $E^\kappa_{\omega\text{-club}} \leq_c \cong^\kappa_T$.

The proof is based on Theorem 4 of [10]. Corollary 3.3 implies that $\cong^\kappa_T$ is Σ^1_1–complete in L when κ is inaccessible and T a stable theory with the OCP, it also implies:

Corollary 3.4 ([9], Cor 5.11) *Assume κ is an inaccessible cardinal. If T_1 is a classifiable theory and T_2 is a stable theory with the OCP, then $\cong^\kappa_{T_1} \leq_c \cong^\kappa_{T_2}$.*

3.2 A generalized Borel-reducibility counterpart of Shelah's Main Gap theorem

3.2.1 Motivation

In this article we studied the Borel-reducibility properties of the relations $E^2_{\lambda\text{-club}}$. The main motivation is to prove that $\cong^\kappa_{T'} \leq_B E^2_{\lambda\text{-club}} \leq_B \cong^\kappa_T$ holds for all theories T' classifiable and T non-classifiable, in ZFC. When this project started, the best results concerning this problem were Theorem 2.1 and 2.2 above [[5], Thm 79, Thm86]. These results are about the second reduction ($E^2_{\lambda\text{-club}} \leq_B \cong^\kappa_T$ for T non-classifiable), the other reduction is the one that looked difficult to obtain at that time. This and Theorem 3.1 motivated the study of the reduction $\cong^\kappa_T \leq_B E^2_{\lambda\text{-club}}$ when T is a classifiable theory.

3.2.2 Results

This article has three sections, the first one is the introduction. In the second section, the proof of Theorem 3.1 is modified to obtain

Lemma 3.5 ([8], Lemma 2) *Assume T is a classifiable theory and $\mu < \kappa$ a regular cardinal. If $\diamondsuit_\kappa(X)$ holds, then $\cong^\kappa_T$ is continuously reducible to E_X.*

This result has many important implication, these are presented in the third section. The first of them is that $\cong^\kappa_{T'} \leq_B \cong^\kappa_T$ holds for all theories T' classifiable and T stable unsuperstable, under some cardinality assumptions.

Corollary 3.6 ([8], Cor 2) *Suppose $\kappa = \kappa^{<\kappa} = \lambda^+$ and $\lambda^\omega = \lambda$. If T_1 is classifiable and T_2 is stable unsuperstable, then $\cong^\kappa_{T_1} \leq_c \cong^\kappa_{T_2}$ and $\cong^\kappa_{T_2} \not\leq_B \cong^\kappa_{T_1}$.*

The other implications are related to the consistency of $\cong^\kappa_{T'} \leq_B \cong^\kappa_T$ for all theories T' classifiable and T non-classifiable. Define $H(\kappa)$ as the following property:
If T is classifiable and T' not, then $\cong^\kappa_T \leq_c \cong^\kappa_{T'}$ and $\cong^\kappa_{T'} \not\leq_B \cong^\kappa_T$.

Theorem 3.7 ([8], Thm 6) *Suppose that $\kappa = \kappa^{<\kappa} = \lambda^+$, $2^\lambda > 2^\omega$ and $\lambda^{<\lambda} = \lambda$.*

1. *If $V = L$, then $H(\kappa)$ holds.*

2. *There is a κ-closed forcing notion $\mathbb{P}$ with the κ^+-c.c. which forces $H(\kappa)$.*

Theorem 3.8 ([8], Thm 7) *Suppose that $\kappa = \kappa^{<\kappa} = \lambda^+$, $2^\lambda > 2^\omega$ and $\lambda^{<\lambda} = \lambda$. Then the following statements are consistent.*

1. *If T_1 is classifiable and T_2 is not, then there is an embedding of $(\mathcal{P}(\kappa), \subseteq)$ to $(B^*(T_1, T_2), \leq_B)$, where $B^*(T_1, T_2)$ is the set of all Borel*-equivalence relations strictly between $\cong^\kappa_{T_1}$ and $\cong^\kappa_{T_2}$.*

2. *If T_1 is classifiable and T_2 is unstable, or superstable with OTOP or with DOP, then*

$$\cong^\kappa_{T_1} \leq_c E^2_{\lambda\text{-}club} \leq_c \cong^\kappa_{T_2} \wedge \cong^\kappa_{T_2} \not\leq_B E^2_{\lambda\text{-}club} \wedge E^2_{\lambda\text{-}club} \not\leq_B \cong^\kappa_{T_1} .$$

3.3 The isomorphism relation of theories with S-DOP

3.3.1 Motivation

In this article I study the reduction $E^\kappa_{\lambda\text{-club}} \leq_B \cong^\kappa_T$, where T a non-classifiable theory, I focus on the case when T is a superstable theory with S-DOP. After writing [9] it was clear that the Borel-reducibility properties of $E^\kappa_{\lambda\text{-club}}$ needed to be studied. The results obtained in [9] are very strong, one of them is the Σ^1_1–completeness of theories with OCP in L. These results motivated the study of other kind of non-classifiable theories, Hyttinen recommended me to start by studying the superstable theories with DOP and provided me with some references ([11], [15]).

3.3.2 Results

This article has four sections, the first section is the introduction. In the introduction I study the results obtained by Laskowski and Shelah in [15] about the reducibility of the relations $\equiv^K_{\infty,\aleph_0}$, when $K = Mod_\kappa(T)$. In the second section I constructed the coloured trees that will be needed in the fourth section. These trees are a modification of the trees presented on [5], [6], and [9]. These trees have uncountable height and are very similar to the trees constructed in [11], in [11] the trees were used to construct models of theories with DOP. The construction of these trees in this section was made under the assumption that κ is an inaccessible cardinal, this assumption continued during the rest of the article.

The third section is a discussion of DOP and strong DOP (S-DOP), in this section S-DOP is introduced as a natural strengthening of DOP, and some useful properties of theories with DOP are presented.

In the fourth section I use the coloured trees of the second section and the properties of S-DOP to construct models of a given superstable theory with S-DOP, T. These models are the key to prove the following result:

Corollary 3.9 ([18], Cor 4.15) *Suppose κ is an inaccessible cardinal. Assume T is a superstable theory with S-DOP, then $E^\kappa_{\lambda\text{-}club}$ is continuously reducible to $\cong^\kappa_T$.*

This result has two important implications.

Corollary 3.10 ([18], Cor 4.16) *Suppose κ is an inaccessible cardinal. Assume T_1 is a classifiable theory and T_2 is a superstable theory with S-DOP, then $\cong^\kappa_{T_1} \leq_c \cong^\kappa_{T_2}$.*

Corollary 3.11 ([18], Cor 4.19) *Suppose κ is an inaccessible cardinal. Suppose $V = L$. If T is a superstable theory with S-DOP, then $\cong^\kappa_T$ is Σ^1_1–complete.*

3.4 On large cardinals and generalized Baire spaces

3.4.1 Motivation

In this article we studied the Borel-reducibility properties of the relations $E^2_{\lambda\text{-club}}$ and $E^\kappa_{\lambda\text{-club}}$ between them. The motivation for this article comes from a question asked in [4], *is $E^\kappa_{\lambda\text{-}club}$ Borel reducible to $E^2_{\lambda\text{-}club}$?* As it was mentioned in the previous chapter, an affirmative answer to this question would imply a partial answer for the main question studied during this thesis. In [[8], Cor 2] we obtained a partial answer to this question (Corollary 3.6), now the study is focused on other kind of non-classifiable theories. If $E^\kappa_{\lambda\text{-club}}$ is Borel reducible to $E^2_{\lambda\text{-club}}$, then Theorem 3.1 and Theorem 2.1 would imply that $\cong^\kappa_{T'} \leq_B \cong^\kappa_T$ for all theories T' classifiable and T non-classifiable (under certain cardinality assumptions).

3.4.2 Results

This article has three sections, the first section is the introduction. In the second section the reducibility between different cofinalities is studied. This was studied in previous works, in [[5], Thm 55] it is proved that $E^2_{\lambda\text{-club}} \leq_B E^2_{\lambda^+\text{-club}}$ is consistently true. In this section we study the *strong reflection*, this reflection implies the *good condition* from [4]. The strong reflection holds in the model constructed in [[5], Thm 55], this gives us a model in which $E^2_{\lambda\text{-club}} \leq_B E^2_{\lambda^+\text{-club}}$ and $E^\kappa_{\lambda\text{-club}} \leq_B E^\kappa_{\lambda^+\text{-club}}$ both hold.

Proposition 3.12 ([1], Prop 2.8) *Suppose $\gamma < \lambda$ are regular cardinals. If S^κ_γ strongly reflect to S^κ_λ, then $E^\kappa_{\gamma\text{-club}} \leq_c E^\kappa_{\lambda\text{-club}}$.*

We strengthened [[5], Thm 55] in [[1], Theorem 2.11] by using $\Pi^{\lambda^+}_1$–indescribable cardinals.

Theorem 3.13 ([1], Thm 2.11) *Suppose κ is a $\Pi^{\lambda^+}_1$–indescribable cardinal and that $V = L$. Then there is a forcing extension where κ is collapsed to λ^{++} and $E^{\lambda^{++}}_{\lambda\text{-club}} \leq_c E^2_{\lambda^+\text{-club}}$.*

In the third section we study the Σ^1_1–complete property of the relations E^κ_{reg} and E^2_{reg}. The combinatorial principle *S–Dual Diamond* is introduced in this section, this principle has important implications for the reducibility of these two relations.

Theorem 3.14 ([1], Thm 3.3) *Suppose $S = S^\kappa_\lambda$ for some λ regular cardinal, or $S = reg(\kappa)$ and κ is a weakly compact cardinal. If κ has the S-dual diamond, then $E_S \leq_c E^2_{reg}$, where $E_S = E^\kappa_{\lambda\text{-club}}$ if $S = S^\kappa_\lambda$, or $E_S = E^\kappa_{reg}$ if $S = reg(\kappa)$.*

From [[6], Thm 7] we know that E^κ_{reg} is Σ^1_1–complete in L. This result is improved by showing that in L, E^2_{reg} is Σ^1_1–complete [[1], Cor 3.5]. In [[1] Thm 3.6] we show that if κ is a supercompact cardinal, then $reg(\kappa)$–dual diamond can be forced. This implies the consistency of $E^\kappa_{reg} \leq_B E^2_{reg}$.

After studying the implications of the dual diamond, we proceed to study the implications of κ being Π^1_2–indescribable.

Theorem 3.15 ([1], Thm 3.7) *If κ is a Π^1_2–indescribable cardinal, then E^κ_{reg} is Σ^1_1–complete.*

It is clear that every supercompact cardinal is a Π_2^1–indescribable cardinal, from this result we can conclude that if κ is a supercompact cardinal, then E^κ_{reg} is Σ_1^1–complete can be forced. The last application of Theorem 3.15 is to show that if κ is a Π_2^1–indescribable cardinal, then $\cong^\kappa_{DLO}$ is Σ_1^1–complete.

Theorem 3.16 ([1], Thm 3.9) *Let DLO be the theory of dense linear orderings without end points. If κ is a Π_2^1–indescribable cardinal, then $\cong^\kappa_{DLO}$ is Σ_1^1–complete.*

3.5 Σ_1^1–complete quasiorders on weakly compact cardinals

3.5.1 Motivation

In this article we studied the Borel-reducibility properties of the equivalence relation E^κ_{reg} and the quasi-order $\subset^{reg}$. The motivation for this article comes from a question asked in [[19], Question 11.4], *is the quasi-order $\subseteq^{NS}$ Σ_1^1–complete?* In [[1], Thm 3.5] we proved that if κ is weakly compact and $V = L$, then E^2_{reg} is Σ_1^1–complete. The relation E^2_{reg} is the equivalence relation associated to the quasi-order $\subseteq^{reg}$, it is natural to think that [[1], Thm 3.5] can be extended to the quasi-order $\subseteq^{reg}$ and give a partial answer to [[19], Question 11.4]. At the same time, this improve some of the results obtained in [1].

3.5.2 Results

This article has four sections, the first section is the introduction. In the second section are all the basic definitions. In the third section we focus on the study of quasi-orders. Using the dual diamond introduced in [1] and [[19], Cor 10.24], we prove that it is consistently true that $\subseteq^{reg}$ is Σ_1^1–complete. This result gives us a partial answer to [[19], Question 11.4].

Theorem 3.17 ([14], Thm 3.8) *If κ is weakly compact and $V = L$, then $\subseteq^{NS}$ is Σ_1^1–complete.*

In the fourth section the reducibility of E^κ_{reg} is studied. This was studied in [1], there was proved that if κ is weakly compact and $V = L$, then E^2_{reg} is Σ_1^1–complete. We improve this result by using [[19], Cor 10.24] *if κ is weakly compact, then embeddability of graphs is Σ_1^1–complete.*

Theorem 3.18 ([14], Thm 4.1) *Let DLO be the theory of dense linear orderings without end points. If κ is a weakly compact cardinal, then $\cong^{\kappa}_{DLO}$ is Σ^1_1–complete.*

This shows that the isomorphism of graphs is Σ^1_1–complete, when κ is a weakly compact cardinal.

Chapter 4

Conclusions

> " *As a rule, men worry more about what they can't see than about what they can.*
>
> - Julius Caesar "

So far many results regarding generalized Borel reducibility have been achieved, thanks to them we have an idea of how the generalized Borel hierarchy looks so far and what needs to be studied in the future. In the first part of this chapter, I will give an overview of the generalized Borel hierarchy in different models and point out which assumptions are required. In the second part of this chapter, I will give a list of open problems related to the generalized Borel hierarchy, the answer of any of these questions will give us a better understanding of the generalized Borel hierarchy.

4.1 Maps of the Borel hierarchy

I will give two lists of results concerning Borel-reducibility in the generalized Baire space. The first list contains results previous to this thesis, the second list contains all the main results of this thesis.

4.1.1 Previous results

This list contains all the results that motivated this thesis and are basic results to understand the Borel-reducibility in the generalized Baire space. The results are not listed in chronological order.

- $Borel \subseteq \Delta_1^1 \subseteq Borel^* \subseteq \Sigma_1^1$ [[5], Thm 17].
- 1. $Borel \subsetneq \Delta_1^1$.

2. $\Delta_1^1 \subsetneq \Sigma_1^1$.
3. *If* $V = L$*, then Borel*$^* = \Sigma_1^1$.
4. *If* $V = L$*, then* $\Delta_1^1 \subsetneq$ *Borel** [[5], Thm 18]

- *Assume that* κ *is inaccessible. If the number of equivalence classes of* $\cong_T^\kappa$ *is grater than* κ*, then* $id \ \leq_c \cong_T^\kappa$ [[5], Thm 36].

- *Assume* $\kappa^{<\kappa} = \kappa = \aleph_\alpha > \omega$*,* κ *is not weakly inaccessible and* $\lambda = \mid \alpha + \omega \mid$*. Then the following are equivalent.*

  1. *There is* $\gamma < \omega_1$ *such that* $\beth_\gamma(\lambda) \geq \kappa$*.*
  2. *There is a complete countable theory* T *such that* $id \ \leq_B \cong_T^\kappa$ *and* $\cong_T^\kappa \leq_B \ id$ [[5], Thm 37].

- *Suppose* κ *is a weakly compact cardinal and that* $V = L$*. Then*

  1. $E^2_{\lambda\text{-}club} \ \leq_c \ E^2_{reg}$.
  2. *In a forcing extension* $E^2_{\lambda\text{-}club} \ \leq_c \ E^2_{\lambda^+\text{-}club}$*, in which* $\kappa = \lambda^{++}$ [[5], Thm 55].

- *For a cardinal* κ *which is a successor of a regular cardinal or* κ *inaccessible, there is a cofinality-preserving forcing extension in which for all regular* $\lambda < \kappa$*, the relations* $E^2_{\lambda\text{-}club}$ *are* $\leq_B$*–incomparable with each other* [[5], Thm 56].

- *Assume* $\kappa > 2^\omega$*. If the theory* T *is classifiable and shallow, then* $\cong_T^\kappa$ *is Borel* [[5], Thm 68].

- *If the theory* T *is classifiable, then* $\cong_T^\kappa$ *is* Δ_1^1 [[5], Thm 70].

- 1. *If* T *is unstable, then* $\cong_T^\kappa$ *is not* Δ_1^1
  2. *If the theory* T *is superstable with OTOP, then* $\cong_T^\kappa$ *is not* Δ_1^1*.*
  3. *If the theory* T *is superstable with DOP and* $\kappa > \omega_1$*, then* $\cong_T^\kappa$ *is not* Δ_1^1*.*
  4. *If* T *is stable with DOP and* $\lambda = cf(\lambda) = \lambda(T) + \lambda^{<\kappa(T)} \geq \omega_1$*,* $\kappa > \lambda^+$ *and for all* $\xi < \kappa$*,* $\xi^\lambda < \kappa$*, then then* $\cong_T^\kappa$ *is not* Δ_1^1 [[5], Thm 71].

- *If a first order Theory* T *is classifiable, then for all* $\lambda < \kappa$ *regular, it holds* $E^2_{\lambda\text{-}club} \not\leq_B \cong_T^\kappa$ [[5], Thm 77].

- *Suppose that* $\kappa = \lambda^+ = 2^\lambda$ *and* $\lambda^{<\lambda} = \lambda$*.*

1. *If T is unstable or superstable with OTOP, then* $E^2_{\lambda\text{-club}} \leq_c \cong^\kappa_T$.
2. *If $\lambda \geq 2^\omega$ and T is superstable with DOP, then* $E^2_{\lambda\text{-club}} \leq_c \cong^\kappa_T$ [[5], Thm 79].

- *Suppose that for all $\gamma < \kappa$, $\gamma^\omega < \kappa$ and T is a stable unsuperstable theory. Then* $E^2_{\omega\text{-club}} \leq_c \cong^\kappa_T$ [[5], Thm 86].

- *Suppose T is a countable complete first-order classifiable and shallow theory, then* $\cong^\kappa_T \leq_B E^\kappa_{\lambda\text{-club}}$ *holds for all regular $\lambda < \kappa$.* [[4], Cor 14].

- *($V = L$). Let $\kappa^{<\kappa} = \kappa > \omega$. If $\kappa = \lambda^+$, let $\theta = \lambda$ and if κ is inaccessible, let $\theta = \kappa$. Let $\mu < \kappa$ be a regular cardinal. Then $E^\kappa_{\mu\text{-club}}$ is Σ^1_1–complete* [[6], Thm 7].

- *($V = L$). Suppose $\kappa = \lambda^+$ and λ is regular. The isomorphism relatiopn on the class of dense linear orderings of size κ is Σ^1_1–complete* [[6], Thm 9].

- *($V = L$). Suppose $\kappa = \lambda^+$ and $\lambda = \lambda^\omega$. Then $\cong^\kappa_{T_{\omega+\omega}}$ is Σ^1_1–complete* [[6], Cor 21].

- *It is consistent that $\Delta^1_1 \subsetneq$ Borel$^* \subsetneq \Sigma^1_1$* [[7], Cor 3.2].

- *If κ is weakly compact, then the embeddability of trees is Σ^1_1–complete* [[19], Thm 10.23].

- *If κ is weakly compact, then the embeddability of graphs is Σ^1_1–complete* [[19], Cor 10.24].

4.1.2 Main results in this thesis

This is a list of the main results obtained in this thesis. This thesis contains results concerning model theory or set theory, in this list I will include only the results that are about Borel-reducibility.

- *Assume T is a classifiable theory and $\lambda < \kappa$ a regular cardinal, the $\cong^\kappa_T$ is continuously reducible to $E^\kappa_{\lambda\text{-club}}$* [[9], Thm 2.8].

- *Suppose that for all $\gamma < \kappa$, $\gamma^\omega < \kappa$ and 2^λ, then* $E^\kappa_{\omega\text{-club}} \leq_B \cong^\kappa_{T_\omega}$ [[9], Lemma 3.2].

- *Suppose for all $\gamma < \kappa$, $\gamma^\omega < \kappa$ and $\kappa = 2^\lambda$, $\lambda < \kappa$. If T is a classifiable theory. Then* $\cong^\kappa_T \leq_c \cong^\kappa_{T_\omega}$ [[9], Cor 3.4].

- *If T is a stable theory with the OCP and κ is an inaccessible cardinal, then* $E^\kappa_{\omega\text{-}club} \leq_c \cong^\kappa_T$ [[9], Cor 5.10].

- *Assume κ is an inaccessible cardinal. If T_1 is a classifiable theory and T_2 is a stable theory with the OCP, then* $\cong^\kappa_{T_1} \leq_c \cong^\kappa_{T_2}$ [[9], Cor 5.11].

- *Assume T is a classifiable theory and $\mu < \kappa$ a regular cardinal. If $\diamondsuit_\kappa(X)$ holds, then $\cong^\kappa_T$ is continuously reducible to E_X* [[8], Lemma 2].

- *Assume that $\diamondsuit_\kappa(S^\kappa_\mu)$ holds for all regular $\mu < \kappa$. If a first order theory T is classifiable, then for all regular cardinals $\mu < \kappa$ we have $\cong^\kappa_T \leq_c E^2_{\mu\text{-}club}$ and $E^2_{\mu\text{-}club} \not\leq_B \cong^\kappa_T$* [[8], Cor 1].

- *Suppose $\kappa = \kappa^{<\kappa} = \lambda^+$ and $\lambda^\omega = \lambda$. If T_1 is classifiable and T_2 is stable unsuperstable, then $\cong^\kappa_{T_1} \leq_c \cong^\kappa_{T_2}$ and $\cong^\kappa_{T_2} \not\leq_B \cong^\kappa_{T_1}$* [[8], Cor 2].

- Define $H(\kappa)$ as the following property:
  If T is classifiable and T' not, then $\cong^\kappa_T \leq_c \cong^\kappa_{T'}$ and $\cong^\kappa_{T'} \not\leq_B \cong^\kappa_T$.
  Suppose that $\kappa = \kappa^{<\kappa} = \lambda^+$, $2^\lambda > 2^\omega$ and $\lambda^{<\lambda} = \lambda$.

  1. *If $V = L$, then $H(\kappa)$ holds.*
  2. *There is a κ-closed forcing notion $\mathbb{P}$ with the κ^+-c.c. which forces $H(\kappa)$* [[8], Thm 6] .

- *Suppose that $\kappa = \kappa^{<\kappa} = \lambda^+$, $2^\lambda > 2^\omega$ and $\lambda^{<\lambda} = \lambda$. Then the following statements are consistent.*

  1. *If T_1 is classifiable and T_2 is not, then there is an embedding of $(\mathcal{P}(\kappa), \subseteq)$ to $(B^*(T_1, T_2), \leq_B)$, where $B^*(T_1, T_2)$ is the set of all Borel*-equivalence relations strictly between $\cong^\kappa_{T_1}$ and $\cong^\kappa_{T_2}$.*
  2. *If T_1 is classifiable and T_2 is unstable or superstable with OTOP or with DOP, then*

  $$\cong^\kappa_{T_1} \leq_c E^2_{\lambda\text{-}club} \leq_c \cong^\kappa_{T_2} \wedge \cong^\kappa_{T_2} \not\leq_B E^2_{\lambda\text{-}club} \wedge E^2_{\lambda\text{-}club} \not\leq_B \cong^\kappa_{T_1}$$

  [[8], Thm 7].

- *Assume T is a superstable theory with S-DOP, then $E^\kappa_{\lambda\text{-}club}$ is continuously reducible to $\cong^\kappa_T$* [[18], Cor 4.15].

- *Assume T_1 is a classifiable theory and T_2 is a superstable theory with S-DOP, then $\cong^\kappa_{T_1} \leq_c \cong^\kappa_{T_2}$* [[18], Cor 4.16].

- *Suppose $V = L$. If T is a superstable theory with S-DOP, then $\cong^\kappa_T$ is Σ^1_1–complete* [[18], Cor 4.19].

- *Suppose $\gamma < \lambda$ are regular cardinals. If S^κ_γ strongly reflect to S^κ_λ, then $E^\kappa_{\gamma\text{-}club} \leq_c E^\kappa_{\lambda\text{-}club}$.* [[1], Prop 2.8].

- *Suppose $\lambda < \kappa$ is such that $\lambda^{<\lambda} = \lambda$. If γ is a regular cardinal such that S^κ_γ $\diamond$–reflects to S^κ_λ, then*

  1. $E^2_{\gamma\text{-}club} \leq_c E^2_{\lambda\text{-}club}$.
  2. $E^\kappa_{\gamma\text{-}club} \leq_c E^\kappa_{\lambda\text{-}club}$ [[1], Cor 2.9].

- *Suppose κ is a $\Pi^{\lambda^+}_1$–indescribable cardinal and that $V = L$. Then there is a forcing extension where κ is collapsed to λ^{++} and $E^{\lambda^{++}}_{\lambda\text{-}club} \leq_c E^2_{\lambda^+\text{-}club}$* [[1], Thm 2.11].

- *The following statement is consistent. $E^{\omega_2}_{\omega\text{-}club} \leq_c E^{\omega_2}_{\omega_1\text{-}club}$, and for every $2 < n$ and every $0 \leq k \leq n-3$, $E^{\omega_n}_{\omega_k\text{-}club} \leq_c E^{\omega_n}_{\omega_{n-1}\text{-}club}$* [[1], Cor 2.14].

- *Suppose $S = S^\kappa_\lambda$ for some λ regular cardinal, or $S = reg(\kappa)$ and κ is a weakly compact cardinal. If κ has the S-dual diamond, then $E_S \leq_c E^2_{reg}$, where $E_S = E^\kappa_{\lambda\text{-}club}$ if $S = S^\kappa_\lambda$, or $E_S = E^\kappa_{reg}$ if $S = reg(\kappa)$* [[1], Thm 3.3].

- *Suppose $V = L$ and κ is weakly compact. Then E^2_{reg} is Σ^1_1-complete* [[1], Cor 3.5].

- *Suppose κ is a supercompact cardinal. There is a generic extension $V[G]$ in which $E^\kappa_{reg} \leq_c E^2_{reg}$ holds and κ is still supercompact in the extension* [[1], Thm 3.6].

- *If κ is a Π^1_2–indescribable cardinal, then E^κ_{reg} is Σ^1_1–complete* [[1], Thm 3.7].

- *Suppose κ is a supercompact cardinal. There is a generic extension $V[G]$ in which E^2_{reg} is Σ^1_1–complete* [[1], Cor 3.8].

- *Let DLO be the theory of dense linear orderings without end points. If κ is a Π^1_2–indescribable cardinal, then $\cong^\kappa_{DLO}$ is Σ^1_1–complete* [[1], Thm 3.9].

- *If κ is weakly compact, then $\approx \leq_c E^\kappa_{reg}$ is Σ^1_1–complete.* [[14], Thm 4.2].

- *If κ is a weakly compact cardinal, then $\cong^\kappa_{DLO}$ is Σ_1^1–complete.* [[14], Thm 4.1].
- *If κ is a weakly compact cardinal and has the $G_{<\kappa}$–Dual diamond, then $\subseteq^{reg}$ is Σ_1^1–complete.* [[14], Lemma 3.7].
- *If κ is weakly compact and $V = L$, then $\subseteq^{NS}$ is Σ_1^1–complete.* [[14], Thm 3.8].

4.2 Open questions and further research

Some of the questions listed in here were asked in other articles different from the ones presented in this thesis, those questions have their respective reference. In [12] the reader can find a bigger list of open questions on generalized Baire spaces. The list presented in here contains questions that motivated the articles mentioned above, questions asked during the elaboration of this thesis, and questions that are closely related to the generalized Borel hierarchy but not were mentioned in the previous chapter.

The list is organised according to the four categories

- Generalized Descriptive Set Theory.
- Model Theory and the Main Gap Theorem.
- The reducibility of the isomorphism relation.
- The reducibility of the equivalence modulo the non-stationary ideal.

4.2.1 Generalized Descriptive Set Theory

In [5] a complete study on the generalized descriptive set theory was done, many results were proved. Some of which are considered the basis for the theory. The authors asked the following question, which rises natural from that study.

Question 4.1 (Friedman, Hyttinen, Kulikov; [5]) *How much can be done without the assumption $\kappa^{<\kappa} = \kappa$?*

As it was mentioned before, in the generalized descriptive set theory we know that $Borel \subsetneq \Delta_1^1 \subseteq Borel^* \subseteq \Sigma_1^1$. In [7] it was shown that $Borel^* \neq \Sigma_1^1$, $\Delta_1^1 \neq Borel^*$, and $Borel^* = \Sigma_1^1$ are all consistently true. The consistency of $\Delta_1^1 = Borel^*$ is still open.

Question 4.2 (Friedman, Hyttinen, Kulikov; [5, 7]) *Is it consistent that* $\Delta_1^1 = Borel^*$*?*

In [6] it was proved that in L, that there are *Borel** relations that are Σ_1^1-complete. This implies that *there is a Borel***-complete relation* is consistently true. One of the motivations for [1] was to find a *Borel**-complete relation without the assumption $V = L$. The relation studied in [1] turned out to be Σ_1^1-complete, this rises the next question.

Question 4.3 *Does there exists a Borel***-complete relation that is not* Σ_1^1*-complete?*

The same question can be asked for Δ_1^1, in this case we know that there is no Δ_1^1-complete relation. The question would be about Δ_1^1-hard relations.

Question 4.4 *Does there exists a* Δ_1^1*-hard relation that is not* Σ_1^1*-complete?*

A positive answer in this question could imply a negative answer in Question 4.2. If there is a *Borel** relation in ZFC that is Δ_1^1-hard, then $\Delta_1^1 \neq$ Borel*.

4.2.2 Model Theory and the Main Gap Theorem

In [9] the main theorem tells us that every stable theory with the OCP is more complex than any classifiable theory. Some examples of stable theories with OCP has been found, in [3] the author shows that the theory of the group of p–adic integers is stable and has the OCP. This rises the next question.

Question 4.5 *Does there exists a stable unsuperstable theory that doesn't have OCP?*

In [18] the main theorem tells us that every superstable theory with S-DOP is more complex than any classifiable theory. The following question is a natural question to ask.

Question 4.6 (J. Baldwin) *Does there exists a superstable theory with DOP that doesn't have S-DOP?*

When I asked Shelah this question, his conjecture was that every superstable theory with DOP also has S-DOP.

In [8] it was shown the consistency of: *If* T_1 *is a classifiable theory and* T_2 *is not classifiable, then* T_1 *is as most as complex as* T_2. But it is still open whether it is true in ZFC.

Question 4.7 (Friedman, Hyttinen, Kulikov, Moreno; [4, 8]) *Is there a generalized Borel reducibility counterpart of the Main Gap Theorem in ZFC, i.e. Is it provable in ZFC that $\cong^{\kappa}_{T} \lneq_B \cong^{\kappa}_{T'}$ (note the strict inequality) for all complete first-order theories T and T', T classifiable and T' not? Do we need large cardinal assumptions?*

As it was mentioned before, one approach to this question is to study the reducibility of the relations $E^{\kappa}_{\lambda\text{-club}}$ and $E^{2}_{\lambda\text{-club}}$. A lot have been study on this, but there are three main questions open about the reducibility of these relations.

Question 4.8 *Suppose T is a non-classifiable theory. Does there exists $\lambda < \kappa$ such that $E^{\kappa}_{\lambda\text{-club}} \leq_B \cong^{\kappa}_{T}$? In case it exists, is it the same for all the theories?*

Question 4.9 *Suppose T is a non-classifiable theory. Under which cardinality assumptions on κ, does it hold $E^{2}_{\omega\text{-club}} \leq_B \cong^{\kappa}_{T}$?*

Notice that we know it is consistently true to have $E^{2}_{\omega\text{-club}} \leq_B \cong^{\kappa}_{T}$ for all theory T non-classifiable, this follows from Theorem 55, Theorem 79 and Theorem 86 of [5].

Question 4.10 (Friedman, Hyttinen, Kulikov; [5]) *If $\kappa = \lambda^+$, λ regular and uncountable, does $E^{2}_{\lambda\text{-club}} \leq_B \cong^{\kappa}_{T}$ for all theory T stable unsuperstable?*

4.2.3 The reducibility of the isomorphism relation

The isomorphism relation of every first order complete theory is Σ^1_1, for some theories we know even more. In [5] it was shown the following results

- *If the theory T is classifiable, then $\cong^{\kappa}_{T}$ is Δ^1_1.*
- *If the theory T is classifiable and , and $\kappa > 2^{\omega}$, then $\cong^{\kappa}_{T}$ is Borel.*
- *If the theory T is unstable, then $\cong^{\kappa}_{T}$ is not Δ^1_1.*
- *If the theory T is superstable with OTOP, then $\cong^{\kappa}_{T}$ is not Δ^1_1.*
- *If the theory T is superstable with DOP and $\kappa > \omega_1$, then $\cong^{\kappa}_{T}$ is not Δ^1_1.*

It is still not known if T is a stable unsuperstable theory, then $\cong^{\kappa}_{T}$ is not Δ^1_1.

Question 4.11 (Friedman, Hyttinen, Kulikov; [5]) *Is it consistent that there exists a stable unsuperstable theory T such that $\cong^\kappa_T$ is a Δ^1_1 relation.*

One of the advantages of using the Borel hierarchy to measure the complexity of first order theories, is the possibility of comparing the complexity of two different theories (not necessary classifiable and non-classifiable). Some examples are:

- If $V = L$, then $\cong^\kappa_{DLO}$ is Σ^1_1-complete [6].
- If κ is a Π^1_2–indescribable cardinal, then $\cong^\kappa_{DLO}$ is Σ^1_1-complete [1].
- If $V = L$ and κ is an inaccessible cardinal, then for every stable theory T with OCP, $\cong^\kappa_T$ is Σ^1_1-complete [9].
- If $V = L$ and κ is an inaccessible cardinal, for every superstable theory T with S-DOP, $\cong^\kappa_T$ is Σ^1_1-complete [18].
- If κ is a weakly compact cardinal, then the bi-embeddability of trees is Σ^1_1-complete [19].
- If κ is a weakly compact cardinal, then the bi-embeddability of graphs is Σ^1_1-complete [19].

One natural question to ask is if there are two theories that are not comparable.

Question 4.12 *Do there exist theories T_1 and T_2 such that $\cong^\kappa_{T_1} \not\leq_B \cong^\kappa_{T_2}$ and $\cong^\kappa_{T_2} \not\leq_B \cong^\kappa_{T_1}$.*

As it was mentioned, the different kind of theories have a total or partial characterization using Borel reducibility, e.g. *if $\kappa = \lambda^+ = 2^\lambda > 2^\omega$ where $\lambda^{<\lambda} = \lambda$, then T is classifiable if and only if for al regular $\mu < \kappa$, $E^2_{\mu\text{-}club} \not\leq_B \cong^\kappa_T$.* Notice that there is no total characterization in ZFC for stable theories with OCP or superstable theories with S-DOP.

Question 4.13 *Under which cardinal assumptions there exists a total characterization for all kind of theories (classifiable, stable unsuperstable, superstable with DOP, superstable with OTOP, unstable)?*

Question 4.14 (Friedman, Hyttinen, Kulikov; [5]) *Under which assumptions on κ, does it hold that if the number of equivalence classes of $\cong^\kappa_T$ is grater than κ, then $id \leq_B \cong^\kappa_T$?*

4.2.4 The reducibility of the equivalence modulo the non-stationary ideal

As we saw in the map of the Borel hierarchy, the equivalence modulo the non-stationary ideal is very important when we look for a generalized Borel reducibility counterpart of Shelah's main gap theorem. In [5] it was shown the consistency of: *The relations $E^2_{\lambda\text{-}club}$ are $\leq_B$-incomparable.* The natural question is if this is provable in ZFC.

Question 4.15 (Friedman, Hyttinen, Kulikov; [5]) *Is it consistent that $E^2_{\lambda\text{-}club} \leq_c E^2_{\gamma\text{-}club}$ for $\gamma < \lambda$?*

For the relations $E^\kappa_{\lambda\text{-club}}$ the situation is the oposite, in [6] it was shown the consistency of: *The relations $E^\kappa_{\lambda\text{-}club}$ are Σ^1_1–complete.* In this case the question is about the incomparability of these relations.

Question 4.16 *Is it consistent that $E^\kappa_{\lambda\text{-}club} \not\leq_c E^\kappa_{\gamma\text{-}club}$ holds for all $\gamma \neq \lambda$?*

The Borel-reducibility properties of the equivalence modulo the non-stationary ideal have been studied in the generalized Baire space and in the generalized Cantor space. One natural question is to ask if the equivalence modulo the non-stationary ideal has the same Borel-reducibility properties in the generalized Baire space as in the generalized Cantor space? Some results strengthen the idea that the properties should be the same. An example of this is Corollary 2.9 in [1], in which the combinatorial principle used to get reduction in the generalized Cantor space implies the "same" reduction in the generalized Baire space. This can also be seen with the Borel-reducibility of the isomorphism relation of classifiable theories to the equivalence modulo the λ-non-stationary ideal (in the generalized Baire space this always holds, in the generalized Cantor space we only know that the diamond principle implies this reducibility). A possible approach to some of the previous questions is to study the reducibility of equivalence relations in the generalized Baire space to equivalence relations in the generalized Cantor space. In particular, a positive answer in the following question will imply a partial answer to Question 4.7.

Question 4.17 (Friedman, Hyttinen, Kulikov; [5]) *Is it $E^\kappa_{\lambda\text{-}club}$ Borel reducible to $E^2_{\lambda\text{-}club}$?*

In [1] it was shown that $E^2_{\omega\text{-club}} \leq_c E^{\omega_2}_{\omega\text{-club}} \leq_c E^2_{\omega_1\text{-club}} \leq_c E^{\omega_2}_{\omega_1\text{-club}}$ is consistently true, but it was not shown whether this reductions are strict or not in that model. This leads to the question if it is consistently true that the relations $E^2_{\gamma\text{-club}}$ and $E^\kappa_{\gamma\text{-club}}$ are linearly ordered by $\leq_B$.

Question 4.18 (Aspero, Hyttinen, Kulikov, Moreno; [1]) *Is it consistent that*

$$E^2_{\gamma\text{-}club} \lneq_c E^\kappa_{\gamma\text{-}club} \lneq_c E^2_{\lambda\text{-}club} \lneq_c E^\kappa_{\lambda\text{-}club}$$

holds for all $\gamma, \lambda < \kappa$ *and* $\gamma < \lambda$*?*

References

[1] D. Asperó, T. Hyttinen, V. Kulikov, and M. Moreno, *On large cardinals and generalized Baires spaces*, submitted.

[2] A.D. Brooke-Taylor, V. Fischer, S.D. Friedman, and D. Montoya, *Cardinal characteristics at κ in a small $\mathfrak{u}(\kappa)$ model*, Fundamenta Mathematicae. **203** no.3, 285 – 298 (2015).

[3] F. Calderoni, *The complexity of the embeddability between trosion-free abelian groups of uncountable size*, submited.

[4] S.D. Friedman, T. Hyttinen, and V. Kulikov, *On Borel reducibility in Generalized Baire space*, Fundamenta Mathematicae. **203** no.3, 285 – 298 (2015).

[5] S.D. Friedman, T. Hyttinen, and V. Kulikov, *Generalized descriptive set theory and classification theory*, Memories of the American Mathematical Society. **230** no. 1081, (2014).

[6] T. Hyttinen, and V. Kulikov, *On Σ_1^1-complete equivalence relations on the generalized baire space*, Math. Log. Quart. **61**, 66 – 81 (2015).

[7] T. Hyttinen, and V. Kulikov, *Borel* sets in the generalized Baire space*, in: Sandu, G., van Ditmarsch, H. (eds.) Outstanding Contributions to Logic: Jaakko Hintikka. Springer (in print).

[8] T. Hyttinen, V. Kulikov, and M. Moreno, *A generalized Borel-reducibility counterpart of Shelah's main gap theorem*, Arch. Math. Logic. **56** no.3, 175 – 185 (2017).

[9] T. Hyttinen, and M. Moreno, *On the reducibility of isomorphism relations*, Math Logic Quart. **63**, 175 – 192 (2017).

[10] T. Hyttinen, and S. Shelah, *On the number of elementary submodels of an unuperstable homogeneous structure*, Math. Logic. Quart. Vol. **44**, no. 3, 354 –358 (1998).

[11] T. Hyttinen, and S. Shelah, *Constructing strongly equivalent non-isomorphic models for unsuperstable theories, part C*, The Journal of Symbolic Logic **64** 634 –642, (1999).

[12] Y. Khomskii, G. Laguzzi, B. Löwe, and I. Sharankou, *Questions on generalised Baire spaces*, Math. Log. Quart. **62**, 439 – 456 (2016).

[13] M. Koerwien, *A complicated ω-stable depth 2 theory*, Journal of Symbolic Logic. **75** no.1, 47 – 65 (2011).

[14] V. Kulikov, and M. Moreno *On Σ_1^1–completeness in weakly compact cardinals*, preprint.

[15] M. Laskowski, and S. Shelah, *Borel completeness of some $\aleph_0$-stable theories*, Fundamenta Mathematicae **229**(1) 1–46, (2015).

[16] N. N. Luzin, and W. Sierpiński, *Sur un ensemble non mesurable B*, J. de Math. **9**(2) 53–72, (1923).

[17] A, Mekler, and J. Väänänen, *Trees and Π_1^1 subsets of $\omega_1^{\omega_1}$*, J. Symbolic Logic. **58**(3), 1052–1070, (1993).

[18] M. Moreno, *The isomorphism relation of theories with S-DOP*, preprint.

[19] L. Motto Ros, *The descriptive set-theoretical complexity of the embeddability relation on models of large size*, Ann. Pure Appl. Log. **164** no.12, 1454 – 1492 (2013).

[20] S. Shelah, *Classification theory*, Stud. Logic Found. Math. Vol. **92**, (North-Holland, Amsterdam, 1990).

[21] J. H. Silver, *Counting the number of equivalence classes of Borel an Coanalytic equivalence relations*, Ann. Math. Logic **18** 1–28, (1980).

[22] R. Vaught, *Invariant sets in topology and logic*, Fund. Math. **82**, 269–294, (1974/75).

[23] J. Väänänen, *Games and Trees in infinitary Logic: A Survey*, Quantifiers: Logics, Models and Computation **248**, 105–138, (1995).