ACADEMIC DISSERTATION

FINDING THE MAIN GAP IN THE
BOREL-REDUCIBILITY HIERARCHY

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Unigrafia Oy
TO MY MOTHER,
MY FIRST MATH TEACHER.
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Prologue

This thesis is constituted by two parts. The first part is divided in four chapters, these are: Introduction, Structure of the thesis, Summary, and Conclusions. The second part contains five research articles.

The first part is intended to give a smooth explanation of each of the five articles of the second part. At the same time, the first part is an attempt to motivate the reader over the second part. It gives a motivation for the reader to study the generalized descriptive set theory, the roll of the author on every chapter of the second part, the motivation and a summary of the results behind each article of the second part, using the least amount of technical language.

Each article of the second part has no modifications from the submitted version of the respective article. The reader is advised that the notation between articles might differ. The articles are presented in the chronological order in which them were produced.
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Chapter 1

Introduction

“If it is not difficult, then it is not funny.

This thesis is about generalized descriptive set theory. To understand the importance of this area of mathematics I will show some of the motivation behind it. The second chapter is intended to explain the roll of the author on every chapter of the second part. The third chapter allows the reader to understand the motivation and results behind each chapter of the second part, using the least amount of technical language. The fourth chapter summarizes the main results of the second part and is intended to orientate the reader about future researches. All the mathematical proofs presented in this thesis are in the second part.

1.1 Descriptive set theory

Descriptive set theory studies definable sets and functions in Polish spaces. A Polish space is a topological space that is homeomorphic to a separable complete metric space, descriptive set theory is mainly focus on the space $\omega^\omega$ equipped with the product topology. This space is called the Baire space and it has the property that every Polish space is a continuous image of it. Descriptive set theory is a beautiful area with applications in other areas of mathematics such as analysis, model theory, ergodic theory, and more. It has became one of the main research areas of set theory. The connection between descriptive set theory and model theory (e.g. Scott’s and Lopez-Escobar’s theorems) comes from the ability of coding countable structures with domain $\omega$, in a countable relational vocabulary, into elements of the
Cantor space, $2^\omega$. The product topology on the Cantor space coincides with the topology generated by the basic open set of the form $N_\eta = \{ \xi \in 2^\omega \mid \eta \subset \xi \}$, where $\eta \in 2^{<\omega}$. For $L = \{ P_m \mid m \in \omega \}$ a relational countable language, the elements of $2^\omega$ code $L$-structures as follows:

**Definition 1.1** Fix a bijection $\pi: \omega^{<\omega} \to \omega$. For every $\eta \in 2^\omega$ define the $L$-structure $A_\eta$ with universe $\omega$ as follows: For every relation $P_m$ with arity $n$, every tuple $(a_1, a_2, \ldots, a_n)$ in $\omega^n$ satisfies

$$(a_1, a_2, \ldots, a_n) \in P_{m_\eta} \iff \eta(\pi(m, a_1, a_2, \ldots, a_n)) = 1.$$ 

Some of the definable sets studied in descriptive set theory are the projective sets ($\Delta^1_1, \Pi^1_1, \Sigma^1_1, \Delta^1_2, \ldots$), some of these sets have interesting properties.

**Theorem 1.2** ([16]) Every $\Sigma^1_1$ set has the property of Baire.

Another important class of sets is the Borel class of sets. A set $X \subseteq \omega^\omega$ is Borel if it belongs to the smallest $\sigma$-algebra containing the open sets of $\omega^\omega$. The class of Borel sets coincide with the class of $\Delta^1_1$ sets, i.e. a subset $X \subseteq \omega^\omega$ is Borel if and only if $X$ is $\Pi^1_1$ and $\Sigma^1_1$. The Borel class of subsets of $2^\omega$ is defined in the same way. A function $f: 2^\omega \to 2^\omega$ is a Borel function, if for every open set $X \subseteq 2^\omega$, $f^{-1}[X]$ is a Borel set in $2^\omega$. Using Borel functions we can classify equivalence relations on $2^\omega$ by their complexity (the Borel reducibility hierarchy). Suppose $E_0$ and $E_1$ equivalence relations on $2^\omega$. We say that $E_0$ is Borel reducible to $E_1$ if there is a Borel function $f: 2^\omega \to 2^\omega$ that satisfies $(\eta, \xi) \in E_0 \iff (f(\eta), f(\xi)) \in E_1$. We call $f$ a Borel reduction of $E_0$ to $E_1$, and we denoted by $E_0 \leq_B E_1$. (If $f$ is continuous, then $E_0$ is continuous reducible to $E_1$, $E_0 \leq_c E_1$.) What a reduction tells us about the complexity of two relation is (in this case) that $E_0$ is as most as complex as $E_1$. Borel reduction can also be use to classify quasi-orders.

Many results have been obtained in the Borel reducibility hierarchy.

**Theorem 1.3** [21] Let $E \subseteq 2^\omega \times 2^\omega$ be a $\Pi^1_1$ equivalence relation. If $E$ has uncountably many equivalence classes, then $id_{2^\omega} \leq_B E$.

As it was explain before, the elements of $2^\omega$ code the structures with domain $\omega$, the isomorphism relation of model of a first order theory is an equivalence relation. It is natural to think on the isomorphism relation of first order theories as an equivalence relation on the space $2^\omega$.

**Definition 1.4** (The isomorphism relation) Assume $T$ is a complete first order theory in a countable vocabulary. We define $\cong_T^\omega$ as the relation

$$\{(\eta, \xi) \in 2^\omega \times 2^\omega \mid (A_\eta \models T, A_\xi \models T, A_\eta \cong A_\xi) \text{ or } (A_\eta \not\models T, A_\xi \not\models T)\}.$$
The isomorphism relation with the Borel reducibility give us a notion of complexity for first order theories. We say that a theory $T$ is as most as complex as $T'$ if $\cong_T \leq_{B} \cong_{T'}$. This notion of complexity shows us the connection between Model Theory and Descriptive Set Theory.

In Model Theory, more precisely in Classification Theory there is a notion of complexity for first order theories, this notion is due to Shelah [20]. It is natural to ask if the Borel reducibility notion of complexity and the Classification Theory notion of complexity coincide. In Classification Theory, one of the most important results is the Main Gap Theorem. This theorem tells us that classifiable theories are less complex than non-classifiable ones and their complexities are far apart.

A classifiable theory is a theory with an invariant that determines the structures up to isomorphisms. The theory of a vector space over the field of rational numbers is a classifiable theory, the models are characterized by the dimension.

A theory with no invariant of this kind is a non-classifiable theory. The theory of the order of the rational numbers is a non-classifiable theory.

The Main Gap Theorem tells us that the theory of a vector space over the field of rational number is less complex than the theory of the order of the rational numbers. Unfortunately there is only one model of countable size, up to isomorphisms, of the theory of of the order of the rational numbers and the theory of a vector space over the field of rational number has more than one countable model, up to isomorphisms.

From this we can see that these two complexity notions are not equivalent. Another example of a classifiable theory more complex than the theory of the order of the rational numbers (in the Borel complexity notion) is the one introduced by Koerwien in [13]. He sows the existence of an $\omega$-stable theory $T$ with NDOP, NOTOP, depth 2, and with $\cong_T$ not Borel.

1.2 Generalized Descriptive set theory

So far the Descriptive Set Theory studies the complexity of a theory by studying the complexity of the countable models. On the other hand in Classification Theory the complexity does not depend on the countable models. Can Descriptive Set Theory study the complexity of the non countable models?

The previous question is about the elements of the set $\kappa^\kappa$, the *generalized Baire space*. To answer it, we will need to define a topology in $\kappa^\kappa$, define the Borel set, and more concepts. This questions were studied in [23] and
for every $\zeta \in \kappa^{<\kappa}$, we call the set

$$[\zeta] = \{\eta \in \kappa^\kappa | \zeta \subset \eta\}$$

a basic open set. The open sets are of the form $\bigcup X$ where $X$ is a collection of basic open sets. Vaught [22], Mekler and Väänänen [17] studied this topology. This topology is called the bounded topology. In Descriptive Set Theory there are three equivalent definitions for the collection of Borel set:

1. The collection of Borel subsets of $\omega^\omega$ is the smallest set which contains the basic open sets and is closed under union and intersection, both of length $\omega$.

2. $\Delta^1_1 = \Pi^1_1 \cap \Sigma^1_1$.

3. The collection of Borel* subsets of $\omega^\omega$ is the set of subsets of $\omega^\omega$ that have a Borel* code.

Each of these definitions can be generalized to a definition in the generalized Baire space. To chose one from the three possible generalization, Friedman, Hyttinen and Kulikov studied them under the assumption $\kappa^{<\kappa} = \kappa$ and try to over come as many difficulties as possible. They show that, under the assumption $\kappa^{<\kappa} = \kappa$, the best candidate for the collection of $\kappa$–Borel subsets is:

*The collection of $\kappa$–Borel subsets of $\kappa^\kappa$ is the smallest set which contains the basic open sets and is closed under union and intersection, both of length $\kappa$.*

The generalization of (1), (2), and (3) have the following property in the space $\kappa^\kappa$ under the assumption $\kappa^{<\kappa} = \kappa$

$$\text{Borel} \subseteq \Delta^1_1 \subseteq \text{Borel}^*,$$

it correspond to the formulas of $L_{\kappa^{+\kappa}}$, etc. A $\kappa$–Borel set is any set in this collection.

The topology of the space and the Borel sets are the basis for Descriptive set theory. Using the this topology and the $\kappa$–Borel sets, other notions of Descriptive Set Theory can be generalized to the generalized Baire space. The generalized Cantor space is the subspace $2^\kappa$ endowed with the relative subspace topology. The collection of $\kappa$–Borel subsets of $2^\kappa$ is the smallest set which contains the basic open sets and is closed under union and intersection, both of length $\kappa$.

It is easy to see that the generalized Baire space and the generalized Cantor space are very similar, it is possible to use both of them to define
1.2 Generalized Descriptive set theory

a complexity notion, as it was discussed in the classical Baire space, $2^\omega$. Instead of restricting the study to one of these spaces, we can generalize the complexity notion of the classical case into a notion that involves the generalized Baire space and the generalized Cantor space.

Suppose $X, Y \in \kappa$, a function $f : X \to Y$ is a Borel function if for every open set $A \subseteq Y$, $f^{-1}[A]$ is a Borel set in $X$ of $\kappa$. Let $E_1$ and $E_2$ be equivalence relations on $X$ and $Y$ respectively. If a function $f : X \to Y$ satisfies $E_1(x, y) \Leftrightarrow E_2(f(x), f(y))$, we say that $f$ is a reduction of $E_1$ to $E_2$. If there exists a Borel function that is a reduction, we say that $E_1$ is Borel reducible to $E_2$ and we denote it by $E_1 \leq_B E_2$.

Let us fix a relational countable language $L = \{ P_n \mid n < \omega \}$ and a bijection $\pi$ between $\kappa < \omega$ and $\kappa$.

**Definition 1.5** For every $\eta \in \kappa$, define the structure $A_\eta$ with domain $\kappa$ as follows. For every tuple $(a_1, a_2, \ldots, a_n)$ in $\kappa^n$

$$(a_1, a_2, \ldots, a_n) \in P^A_m \Leftrightarrow \text{the arity of } P_m \text{ is } n \text{ and } \eta(\pi(m, a_1, a_2, \ldots, a_n)) > 0.$$  

**Definition 1.6** For every $\eta \in 2^\kappa$, define the structure $A_\eta$ with domain $\kappa$ as follows. For every tuple $(a_1, a_2, \ldots, a_n)$ in $\kappa^n$

$$(a_1, a_2, \ldots, a_n) \in P^A_m \Leftrightarrow \text{the arity of } P_m \text{ is } n \text{ and } \eta(\pi(m, a_1, a_2, \ldots, a_n)) = 1.$$  

With the structures coded by the elements of $2^\kappa$ and $\kappa$, it is easy to define the isomorphism relation of structures of size $\kappa$ in both spaces.

**Definition 1.7 (The isomorphism relation)** Assume $T$ is a complete first order theory in a countable vocabulary. We define $\cong^T_\kappa$ as the relation

$$\{ (\eta, \xi) \in \kappa \times \kappa \mid (A_\eta \models T, A_\xi \models T, A_\eta \cong A_\xi) \text{ or } (A_\eta \not\models T, A_\xi \not\models T) \}.$$  

**Definition 1.8** Assume $T$ is a complete first order theory in a countable vocabulary. We define $\cong^T_2$ as the relation

$$\{ (\eta, \xi) \in 2^\kappa \times 2^\kappa \mid (A_\eta \models T, A_\xi \models T, A_\eta \cong A_\xi) \text{ or } (A_\eta \not\models T, A_\xi \not\models T) \}.$$  

It is easy to see that the function $F : \kappa \to 2^\kappa$ given by

$$F(\eta)(\alpha) = \begin{cases} 0 & \text{if } \eta(\alpha) = 0 \\ 1 & \text{otherwise} \end{cases}$$
is a reduction of $\cong_T^\kappa$ to $\cong_T^2$, these two relations are bireducible. With this in mind a notion of complexity for first order complete theories in a countable vocabulary that depends on the complexity of the models of size $\kappa$ can be define. We say that a theory $T$ is as most as complex as $T'$ if $\cong_T^\kappa \leq_B \cong_{T'}^\kappa$.

The main subject of study in this thesis is the question: *Is it true that for all classifiable theory $T$ and non-classifiable theory $T'$ holds $\cong_T^\kappa \leq_B \cong_{T'}^\kappa$?*

As we saw, the fact that the Borel reducibility measures complexity in a different way than stability theory (in the classical descriptive set theory) was part of the motivation for the generalized descriptive set theory. Anyway, the notions in generalized descriptive set theory were defined in such a way that are not too different to their equivalent in the classical case. This allows the study of many other subjects in the Generalized Baire spaces and similar question to ones asked in the classical case can be asked.

Besides the isomorphism relations, there are other equivalent relations that have been studied in the generalized Baire space, some of those are the relations $(\text{Mod}_\lambda(T), \equiv_\infty, \aleph_0)$. In [15] Laskowski and Shelah studied the Borel reducibility properties of $(\text{Mod}_\lambda(T), \equiv_\infty, \aleph_0)$ for theories $T$ with eni-DOP. The quasi-orders can be studied in the generalized Baire space too, in general, the study of relations in the generalized Baire space is a huge area of studies.

Cardinal characteristics is another example of a subject that carries questions from the classical case to the generalized case. Many cardinal characteristics can be easily generalized, some of them are $a(\kappa)$, $\mathfrak{c}(\kappa)$, and $\mathfrak{g}(\kappa)$. Some others need more care to be generalized, like $\mathfrak{p}$. Brooke-Taylor, Fischer, Friedman, and Montoya have studied this in [2].

Generalized descriptive set theory is a growing area in set theory with many applications to other areas. The reader can find more about this subjects and others related to generalized descriptive set theory in [12].
Chapter 2

Structure of the thesis

“Where there is will there is a way.”

- English proverb

The main goal of this thesis is to make a contribution to the study of the Borel reducibility hierarchy in the generalized Baire space. Model theory and set theory are two disciplines of mathematical logic which can be used to study the Borel reducibility hierarchy in the generalized Baire space. These two disciplines are connected when the complexity of complete first order theories is studied. Each of these disciplines has its approach to measure the complexity of complete first order theories. The Borel reducibility hierarchy in the generalized Baire space shows us a deep connection between these two approaches, in this thesis I study this connection.

2.1 List of articles

The second part of this thesis consists of the following five articles, the articles are presented in the chronological order of production.


IV David Asperó, Tapani Hyttinen, Vadim Kulikov, and Miguel Moreno


V Vadim Kulikov, and Miguel Moreno *On $\Sigma_1^1$-completeness in weakly compact cardinals.* Preprint.

The articles are reproduced with the permission of their respective copyright holders. I wish to discuss my honest contribution to each of the articles. The following must be taken with certain precaution, in mathematics is not always easy to determine which part was contributed by whom, in particular when it is the result of many hours of discussion. The five articles were written by me, except for the first part of the introduction of [9], the second paragraph of the introduction of [14] and [[1], Lemma 3.4]. Most of the details of all the articles have been elaborated by me.

The first article, *On the reducibility of isomorphism relations* is a joint work with my supervisor Tapani Hyttinen. The idea to generalize [[4], Lemma 9] to $\Delta_1^1$ equivalent relations was mine [[9], Lemma 2.4]. This generalization gives us a sufficient condition for a $\Delta_1^1$ equivalent relation to be continuous reducible to $E^\kappa_\text{club}$, for all $\lambda < \kappa$ regular. We realized that for every classifiable theory, the isomorphism relation satisfies this condition given the right $\Delta_1^1$-code. Coding the moves of the Ehrenfeucht-Fraïssé game by ordinals [[9], Definitions 2.3, 2.6] was my idea, this leads to [[9], Lemma 2.7]. From these two results [[9], Theorem 2.8] follows.

The second article, *A generalized Borel-reducibility counterpart of Shelah's main gap theorem* is a joint work with Tapani Hyttinen and Vadim Kulikov. We tried to obtain in the generalized Cantor space a result equivalent to [[9], Theorem 2.8]. Using the same technique of [[9], Theorem 2.8], we realized that the diamond principle implies the result we wanted [[8], Lemma 2]. The details of the proofs have been elaborated by me. Some of these were: show that the forcings needed for Theorem 7 do not destroy the diamond sequence, and show that the preimage of a Borel* set under a Borel function is also a Borel* set.

In the third article, *The isomorphism relation of theories with S-DOP,* I am the only author.

The fourth article, *On large cardinals and generalized Baires spaces* is a joint work with David Asperó, Tapani Hyttinen and Vadim Kulikov. [[1], Theorem 2.11] is due to me, the idea behind is to use $\lambda^+$ many times the reduction $E^\lambda_\text{club} \leq_B E^{\lambda^+}_\text{club}$ from [5], and Fodor’s lemma. To use the reduction $E^\lambda_\text{club} \leq_B E^{\lambda^+}_\text{club}$ $\lambda^+$ many times we needed $\lambda^+$ many stationary subsets of $\text{reg}(\kappa)$ such that $\kappa \triangleleft \text{reflect}$ to them. We obtain this by using strongly reflection in $L$ for $\kappa$ a $\Pi^\lambda_1$-indescribable.
the preparation of this article I proved that: If $\kappa$ is a $\Pi^2_2$–indescribable cardinal, then $E^\kappa_{\text{reg}}$ is Borel$^\ast$–complete. We improved the technique used in this result to prove [[1], Theorem 3.7].

The fifth article, $\Sigma^1_1$–complete quasiorders on weakly compact cardinals is a joint work with Vadim Kulikov. This article partially solves an open question posed by Motto Ros. The idea to modify the dual diamond from [1] to solve this question, was mine. The third section is due to me. The consistency of $G_{<\kappa}$–dual diamond is due to me.

2.2 Outline of problems studied in the thesis

In Shelah’s stability theory, a classifiable theory is a theory with an invariant that determines the structures up to isomorphisms, a theory with no invariant of this kind is a non-classifiable theory. This tell us that a theory with an invariant of this kind is less complex than a theory with no invariant of this kind. Shelah’s stability theory tells us that every countable complete first-order classifiable theory is less complex than all countable complete first-order non-classifiable theories. The subject of study in this thesis was the question: Are all classifiable theories less complex than all the non-classifiable theories, in the Borel reducibility hierarchy?. There are two frames where this question can be studied, the generalized Baire space and the generalized Cantor space. It is known that for every theory $T$, the relations $\sim_\lambda^\kappa$ and $\sim_\kappa^\kappa$ are bireducible. This gives us the freedom to choose in which space we would like to work.

This question was studied in [4],[5], and [6] between other previous works. Some of the results in those works pointed out that the relation equivalence modulo the $\lambda$-non-stationary ideal might be one of the keys to understand the reducibility of the isomorphism relation. On the space $\kappa^\kappa$, for every regular $\lambda < \kappa$, we say that $f, g \in \kappa^\kappa$ are $E^\kappa_{\lambda,\text{club}}$ equivalent ($f \sim_{E^\kappa_{\lambda,\text{club}}} g$) if the set $\{\alpha < \kappa | cf(\alpha) = \lambda \land f(\alpha) \neq g(\alpha)\}$ is non-stationary. On the space $2^\kappa$, for every regular $\lambda < \kappa$, we say that $f, g \in \kappa^\kappa$ are $E^2_{\lambda,\text{club}}$ equivalent ($f \sim_{E^2_{\lambda,\text{club}}} g$) if the set $\{\alpha < \kappa | cf(\alpha) = \lambda \land f(\alpha) \neq g(\alpha)\}$ is non-stationary. Some of these results are the following:

Theorem 2.1 [5, Thm 79] Suppose that $\kappa = \lambda^+ = 2^\lambda$ and $\lambda^{<\lambda} = \lambda$.

1. If $T$ is unstable or superstable with OTOP, then $E^2_{\lambda,\text{club}} \leq_c \equiv^\kappa_T$.

2. If $\lambda \geq 2^\omega$ and $T$ is superstable with DOP, then $E^2_{\lambda,\text{club}} \leq_c \equiv^\kappa_T$.

Theorem 2.2 [5, Thm 86] Suppose that for all $\gamma < \kappa$, $\gamma^\omega < \kappa$ and $T$ is a stable unsuperstable theory. Then $E^2_{\omega,\text{club}} \leq_c \equiv^\kappa_T$. 


Theorem 2.3 [4, Cor 14] Suppose $T$ is a countable complete first-order classifiable and shallow theory, then $\cong_{T}^{\kappa} \leq_{B} E_{\lambda\text{-club}}^{\kappa}$ holds for all regular $\lambda < \kappa$.

These results lead to two approaches for the main question,

- Is it provable in ZFC that $\cong_{T'}^{\kappa} \leq_{B} E_{\lambda\text{-club}}^{2} \leq_{B} \cong_{T}^{\kappa}$ holds for all $T'$ classifiable and $T$ non-classifiable?

- Is it provable in ZFC that $\cong_{T'}^{\kappa} \leq_{B} E_{\lambda\text{-club}}^{\kappa} \leq_{B} \cong_{T}^{\kappa}$ holds for all $T'$ classifiable and $T$ non-classifiable?

Theorems 2.1 and 2.2 give a partial answer to the second reduction in the first question (above), and Theorem 2.3 gives a partial answer to the first reduction to the second question (above). This point out a new possible approach to the main question: Is it provable in ZFC that $\cong_{T'}^{\kappa} \leq_{B} \cong_{T}^{\kappa}$ holds for all $T'$ classifiable and $T$ non-classifiable?. It can be studied by studying the reducibility between the relations $E_{\lambda\text{-club}}^{\kappa}$ and $E_{\lambda\text{-club}}^{2}$, it is clear that $E_{\lambda\text{-club}}^{2}$ is Borel reducible to $E_{\lambda\text{-club}}^{\kappa}$. The Borel reducibility of $E_{\lambda\text{-club}}^{\kappa}$ to $E_{\lambda\text{-club}}^{2}$ would imply $\cong_{T'}^{\kappa} \leq_{B} \cong_{T}^{\kappa}$ for all theories $T'$ classifiable and non-shallow, and $T$ non-classifiable (depending if the former one holds under the cardinal assumptions of Theorems 2.1 and 2.2).

These three are the questions studied in this thesis. The question Is it provable in ZFC that $\cong_{T'}^{\kappa} \leq_{B} E_{\lambda\text{-club}}^{\kappa} \leq_{B} \cong_{T}^{\kappa}$ holds for all $T'$ classifiable and $T$ non-classifiable? is studied in the fifth and seventh articles. The question Is it provable in ZFC that $\cong_{T'}^{\kappa} \leq_{B} E_{\lambda\text{-club}}^{2} \leq_{B} \cong_{T}^{\kappa}$ holds for all $T'$ classifiable and $T$ non-classifiable? is studied in the sixth article. The question Is it provable in ZFC that $E_{\lambda\text{-club}}^{\kappa} \leq_{B} E_{\lambda\text{-club}}^{2}$? is studied in the eighth article. The Borel reducibility properties of the relation $E_{\text{reg}}^{\kappa}$ is studied in the ninth article.
Chapter 3

Summary

“Math is not trivial.
- Jouko Väänänen”

3.1 On the reducibility of isomorphism relations

3.1.1 Motivation

In this article we studied the Borel-reducibility properties of the relations $E^\kappa_\lambda$. The main motivation is to prove that $\cong^\kappa_T \leq_B E^\kappa_\lambda \leq_B \cong^\kappa_T$ holds for all theories $T'$ classifiable and $T$ non-classifiable in ZFC. At the moment this project started, the best result concerning this problem was Theorem 2.3 above [[4], Cor 14]. This result tells us that if $T$ is a classifiable and shallow theory then $\cong^\kappa_T \leq_B E^\kappa_\lambda$. This result motivated the study of the reducibility $\cong^\kappa_T \leq_B E^\kappa_\lambda$ when $T$ is a classifiable theory.

3.1.2 Results

This article has five sections, the first one is the introduction. The second section is the study of the reduction $\cong^\kappa_T \leq_B E^\kappa_\lambda$ when $T$ is a classifiable theory. In this section, Theorem 2.3 is generalized to all classifiable theories [[9], Theorem 2.8], not only to classifiable and shallow.

Theorem 3.1 ([9], Thm 2.8) Assume $T$ is a classifiable theory and $\lambda < \kappa$ a regular cardinal, the $\cong^\kappa_T$ is continuously reducible to $E^\kappa_\lambda$.

In [4] Theorem 2.3 is obtained as a corollary of a stronger result, this result gives a sufficient condition for a Borel equivalent relation to be Borel
reducible to $E_{\lambda}^{\kappa}$-club for all regular $\lambda < \kappa$. In [4] the condition is stated as: 
*The Borel-code $(t,h)$ has club-many good ordinals*; it is a condition over the Borel-code of the relation. In general, this “good condition” can be extended to $\Delta_1^1$-codes and it is sufficient for any $\Delta_1^1$ equivalent relation to be Borel reducible to $E_{\lambda}^{\kappa}$-club. The key for Theorem 3.1 was to prove that if $T$ is a classifiable theory, then $\cong^T_T$ satisfies the good condition.

It was already known that if $T$ is a classifiable theory, then $\cong^T_T$ is $\Delta_1^1$ [5], Theorem 70]. Unfortunately the $\Delta_1^1$-code provided by [5], Theorem 70 doesn’t have club-many good ordinals, this is due to fact that this $\Delta_1^1$-code doesn’t use ordinals in the same way as the good condition uses them. [9], Def 2.3 is a modification of the $\Delta_1^1$-code in [5], Theorem 70], this modification uses the ordinals in the same way as the good condition and codes the same relation. Using this $\Delta_1^1$-code, Theorem 3.1 is proved in the same way as [4], Cor 14.

The Third, fourth and fifth sections are the study of the reduction $E_{\lambda}^{\kappa}$-club $\leq B \cong^T_T$. In the third section we find a theory such that $E_{\omega}^{\kappa}$-club $\leq B \cong^T_T$ holds under certain cardinal assumptions, this is the first result of this type. Before this result was obtained, it was already known that if $T$ is the theory of dense linear orderings without end points, then $E_{\lambda}^{\kappa}$-club $\leq B \cong^T_T$ is consistently true [6], Thm 9].

**Lemma 3.2 ([9], Lemma 3.2)*** Suppose that for all $\gamma < \kappa$, $\gamma^\omega < \kappa$ and $2^\lambda = \kappa$, then $E_{\omega}^{\kappa}$-club $\leq c \cong^T_T$.

The key for this result was to find the appropriate stable unsuperstable theory such that the reduction of Theorem 2.2 can be extended to the reduction $\Pi_\lambda E_{\omega}^{2}$-club $\leq B \cong^T_T$. The result follows from the reduction $E_{\omega}^{\kappa}$-club $\leq B \Pi_\lambda E_{\omega}^{\kappa}$-club, which holds when $\kappa = 2^\lambda$. In [6], Thm 7] the authors proved that $E_{\omega}$-club is $\Sigma_1^1$-complete in $L$, this and Lemma 3.2 imply that $\cong^T_T$ is $\Sigma_1^1$-complete in $L$.

In the fourth section [9], Definition 4.1] defines coloured trees. In [9], Definition 4.6] the trees $(J_f, cf)$ are constructed for all $f \in \kappa^\kappa$ such that, if $\kappa$ is an inaccessible cardinal and $f, g \in \kappa^\kappa$, then $f E_{\omega}$-club $g$ holds if and only if $J_f$ and $J_g$ are isomorphic. These trees were used to prove [6], Cor 21] mentioned above.

In the fifth section the coloured trees are used to prove that:

**Corollary 3.3 ([9], Cor 5.10)*** If $T$ is a stable theory with the OCP and $\kappa$ is an inaccessible cardinal, then $E_{\omega}$-club $\leq c \cong^T_T$.

The proof is based on Theorem 4 of [10]. Corollary 3.3 implies that $\cong^T_T$ is $\Sigma_1^1$-complete in $L$ when $\kappa$ is inaccessible and $T$ a stable theory with the OCP, it also implies:
3.2 A generalized Borel-reducibility counterpart of Shelah’s Main Gap theorem

**Corollary 3.4** ([9], Cor 5.11) Assume \( \kappa \) is an inaccessible cardinal. If \( T_1 \) is a classifiable theory and \( T_2 \) is a stable theory with the OCP, then 
\[
\preccurlyeq_{T_1} \leq c \preccurlyeq_{T_2}.
\]

### 3.2.1 Motivation

In this article we studied the Borel-reducibility properties of the relations \( E^2_{\lambda\text{-club}} \). The main motivation is to prove that 
\[
\preccurlyeq_{T'} \leq_B E^2_{\lambda\text{-club}} \leq_B \preccurlyeq_T
\]
holds for all theories \( T' \) classifiable and \( T \) non-classifiable, in ZFC. When this project started, the best results concerning this problem were Theorem 2.1 and 2.2 above [[5], Thm 79, Thm86]. These results are about the second reduction \( (E^2_{\lambda\text{-club}} \leq_B \preccurlyeq_T \) for \( T \) non-classifiable), the other reduction is the one that looked difficult to obtain at that time. This and Theorem 3.1 motivated the study of the reduction \( \preccurlyeq_{T'} \leq_B E^2_{\lambda\text{-club}} \) when \( T \) is a classifiable theory.

### 3.2.2 Results

This article has three sections, the first one is the introduction. In the second section, the proof of Theorem 3.1 is modified to obtain

**Lemma 3.5** ([8], Lemma 2) Assume \( T \) is a classifiable theory and \( \mu < \kappa \) a regular cardinal. If \( \diamondsuit_{\kappa}(X) \) holds, then \( \preccurlyeq_{T'} \) is continuously reducible to \( E_X \).

This result has many important implication, these are presented in the third section. The first of them is that \( \preccurlyeq_{T'} \leq_B \preccurlyeq_T \) holds for all theories \( T' \) classifiable and \( T \) stable unsuperstable, under some cardinality assumptions.

**Corollary 3.6** ([8], Cor 2) Suppose \( \kappa = \kappa^{<\kappa} = \lambda^+ \) and \( \lambda^{<\lambda} = \lambda \). If \( T_1 \) is classifiable and \( T_2 \) is stable unsuperstable, then 
\[
\preccurlyeq_{T_1} \leq c \preccurlyeq_{T_2} \text{ and } \preccurlyeq_{T_2} \not\leq_B \preccurlyeq_{T_1}.
\]

The other implications are related to the consistency of \( \preccurlyeq_{T'} \leq_B \preccurlyeq_T \) for all theories \( T' \) classifiable and \( T \) non-classifiable. Define \( H(\kappa) \) as the following property:

If \( T \) is classifiable and \( T' \) not, then \( \preccurlyeq_T \leq c \preccurlyeq_{T'} \), and \( \preccurlyeq_{T'} \not\leq_B \preccurlyeq_T \).
Theorem 3.7 ([8], Thm 6) Suppose that $\kappa = \kappa^+ = \lambda^+$, $2^\lambda > 2^\omega$ and $\lambda^{<\lambda} = \lambda$.

1. If $V = L$, then $H(\kappa)$ holds.

2. There is a $\kappa$-closed forcing notion $\mathbb{P}$ with the $\kappa^+$-c.c. which forces $H(\kappa)$.

Theorem 3.8 ([8], Thm 7) Suppose that $\kappa = \kappa^+ = \lambda^+$, $2^\lambda > 2^\omega$ and $\lambda^{<\lambda} = \lambda$. Then the following statements are consistent.

1. If $T_1$ is classifiable and $T_2$ is not, then there is an embedding of $(\mathcal{P}(\kappa), \subseteq)$ to $(B^*(T_1, T_2), \leq_B)$, where $B^*(T_1, T_2)$ is the set of all Borel*-equivalence relations strictly between $\equiv_{T_1}^\kappa$ and $\equiv_{T_2}^\kappa$.

2. If $T_1$ is classifiable and $T_2$ is unstable, or superstable with OTOP or with DOP, then

$$\equiv_{T_1}^\kappa \leq c \bigwedge_{\lambda, \text{club}} E^{2}_{\lambda, \text{club}} \leq c \bigwedge_{T_2} \leq_B \bigwedge_{\lambda, \text{club}} E^{2}_{\lambda, \text{club}} \leq_B \equiv_{T_1}^\kappa.$$ 

3.3 The isomorphism relation of theories with S-DOP

3.3.1 Motivation

In this article I study the reduction $E^\kappa_{\lambda, \text{club}} \leq_B \equiv_T^\kappa$, where $T$ a non-classifiable theory, I focus on the case when $T$ is a superstable theory with S-DOP. After writing [9] it was clear that the Borel-reducibility properties of $E^\kappa_{\lambda, \text{club}}$ needed to be studied. The results obtained in [9] are very strong, one of them is the $\Sigma^1_1$-completeness of theories with OCP in $L$. These results motivated the study of other kind of non-classifiable theories, Hyttinen recommended me to start by studying the superstable theories with DOP and provided me with some references ([11], [15]).

3.3.2 Results

This article has four sections, the first section is the introduction. In the introduction I study the results obtained by Laskowski and Shelah in [15] about the reducibility of the relations $\equiv^K_{\infty, \aleph_0}$, when $K = \text{Mod}_\kappa(T)$. In the second section I constructed the coloured trees that will be needed in the fourth section. These trees are a modification of the trees presented on [5], [6], and [9]. These trees have uncountable height and are very similar to the trees constructed in [11], in [11] the trees were used to construct models of
theories with DOP. The construction of these trees in this section was made under the assumption that $\kappa$ is an inaccessible cardinal, this assumption continued during the rest of the article.

The third section is a discussion of DOP and strong DOP (S-DOP), in this section S-DOP is introduced as a natural strengthening of DOP, and some useful properties of theories with DOP are presented.

In the fourth section I use the coloured trees of the second section and the properties of S-DOP to construct models of a given superstable theory with S-DOP, $T$. These models are the key to prove the following result:

**Corollary 3.9** ([18], Cor 4.15) Suppose $\kappa$ is an inaccessible cardinal. Assume $T$ is a superstable theory with S-DOP, then $E_{\lambda\text{-club}}^\kappa$ is continuously reducible to $\cong_T^\kappa$.

This result has two important implications.

**Corollary 3.10** ([18], Cor 4.16) Suppose $\kappa$ is an inaccessible cardinal. Assume $T_1$ is a classifiable theory and $T_2$ is a superstable theory with S-DOP, then $\cong_{T_1}^\kappa \leq_c \cong_{T_2}^\kappa$.

**Corollary 3.11** ([18], Cor 4.19) Suppose $\kappa$ is an inaccessible cardinal. Suppose $V = L$. If $T$ is a superstable theory with S-DOP, then $\cong_T^\kappa$ is $\Sigma_1^1$-complete.

### 3.4 On large cardinals and generalized Baire spaces

#### 3.4.1 Motivation

In this article we studied the Borel-reducibility properties of the relations $E_{\lambda\text{-club}}^2$ and $E_{\lambda\text{-club}}^\kappa$ between them. The motivation for this article comes from a question asked in [4], is $E_{\lambda\text{-club}}^\kappa$ Borel reducible to $E_{\lambda\text{-club}}^2$? As it was mentioned in the previous chapter, an affirmative answer to this question would imply a partial answer for the main question studied during this thesis. In [[8], Cor 2] we obtained a partial answer to this question (Corollary 3.6), now the study is focused on other kind of non-classifiable theories. If $E_{\lambda\text{-club}}^\kappa$ is Borel reducible to $E_{\lambda\text{-club}}^2$, then Theorem 3.1 and Theorem 2.1 would imply that $\cong_{T'}^\kappa \leq_B \cong_T^\kappa$ for all theories $T'$ classifiable and $T$ non-classifiable (under certain cardinality assumptions).
3.4.2 Results

This article has three sections, the first section is the introduction. In the second section the reducibility between different cofinalities is studied. This was studied in previous works, in [[5], Thm 55] it is proved that $E^2_{\lambda\text{-club}} \leq_B E^2_{\lambda^+\text{-club}}$ is consistently true. In this section we study the strong reflection, this reflection implies the good condition from [4]. The strong reflection holds in the model constructed in [[5], Thm 55], this gives us a model in which $E^2_{\lambda\text{-club}} \leq_B E^2_{\lambda^+\text{-club}}$ and $E^\kappa_{\lambda\text{-club}} \leq_B E^\kappa_{\lambda^+\text{-club}}$ both hold.

**Proposition 3.12 ([1], Prop 2.8)** Suppose $\gamma < \lambda$ are regular cardinals. If $S^\kappa_\gamma$ strongly reflect to $S^\kappa_\lambda$, then $E^\kappa_{\gamma\text{-club}} \leq c E^\kappa_{\lambda\text{-club}}$.

We strengthened [[5], Thm 55] in [[1], Theorem 2.11] by using $\Pi^+_1$–indescribable cardinals.

**Theorem 3.13 ([1], Thm 2.11)** Suppose $\kappa$ is a $\Pi^+_1$–indescribable cardinal and that $V = L$. Then there is a forcing extension where $\kappa$ is collapsed to $\lambda^{++}$ and $E^\kappa_{\lambda\text{-club}} \leq c E^2_{\lambda^+\text{-club}}$.

In the third section we study the $\Sigma^1_1$–complete property of the relations $E^\kappa_{\text{reg}}$ and $E^2_{\text{reg}}$. The combinatorial principle $S$–Dual Diamond is introduced in this section, this principle has important implications for the reducibility of these two relations.

**Theorem 3.14 ([1], Thm 3.3)** Suppose $S = S^\kappa_\lambda$ for some $\lambda$ regular cardinal, or $S = \text{reg}(\kappa)$ and $\kappa$ is a weakly compact cardinal. If $\kappa$ has the $S$-dual diamond, then $E_S \leq c E^2_{\text{reg}}$, where $E_S = E^\kappa_{\lambda\text{-club}}$ if $S = S^\kappa_\lambda$, or $E_S = E^\kappa_{\text{reg}}$ if $S = \text{reg}(\kappa)$.

From [[6], Thm 7] we know that $E^\kappa_{\text{reg}}$ is $\Sigma^1_1$–complete in $L$. This result is improved by showing that in $L$, $E^2_{\text{reg}}$ is $\Sigma^1_1$–complete [[1], Cor 3.5]. In [[1] Thm 3.6] we show that if $\kappa$ is a supercompact cardinal, then $\text{reg}(\kappa)$–dual diamond can be forced. This implies the consistency of $E^\kappa_{\text{reg}} \leq_B E^2_{\text{reg}}$.

After studying the implications of the dual diamond, we proceed to study the implications of $\kappa$ being $\Pi^+_2$–indescribable.

**Theorem 3.15 ([1], Thm 3.7)** If $\kappa$ is a $\Pi^+_2$–indescribable cardinal, then $E^\kappa_{\text{reg}}$ is $\Sigma^1_1$–complete.
3.5 $\Sigma^1_1$–complete quasiorders on weakly compact cardinals

It is clear that every supercompact cardinal is a $\Pi^1_2$–indescribable cardinal, from this result we can conclude that if $\kappa$ is a supercompact cardinal, then $E^\kappa_{\text{reg}}$ is $\Sigma^1_1$–complete can be forced. The last application of Theorem 3.15 is to show that if $\kappa$ is a $\Pi^1_2$–indescribable cardinal, then $\cong^\kappa_{\text{DLO}}$ is $\Sigma^1_1$–complete.

**Theorem 3.16 ([1], Thm 3.9)** Let $\text{DLO}$ be the theory of dense linear orderings without end points. If $\kappa$ is a $\Pi^1_2$–indescribable cardinal, then $\cong^\kappa_{\text{DLO}}$ is $\Sigma^1_1$–complete.

3.5 $\Sigma^1_1$–complete quasiorders on weakly compact cardinals

3.5.1 Motivation

In this article we studied the Borel-reducibility properties of the equivalence relation $E^\kappa_{\text{reg}}$ and the quasi-order $\subseteq^\text{reg}$. The motivation for this article comes from a question asked in [[19], Question 11.4], is the quasi-order $\subseteq^{\text{NS}} \Sigma^1_1$–complete? In [[1], Thm 3.5] we proved that if $\kappa$ is weakly compact and $V = L$, then $E^2_{\text{reg}}$ is $\Sigma^1_1$–complete. The relation $E^2_{\text{reg}}$ is the equivalence relation associated to the quasi-order $\subseteq^{\\text{reg}}$, it is natural to think that [[1], Thm 3.5] can be extended to the quasi-order $\subseteq^{\text{reg}}$ and give a partial answer to [[19], Question 11.4]. At the same time, this improve some of the results obtained in [1].

3.5.2 Results

This article has four sections, the first section is the introduction. In the second section are all the basic definitions. In the third section we focus on the study of quasi-orders. Using the dual diamond introduced in [1] and [[19], Cor 10.24], we prove that it is consistently true that $\subseteq^{\text{reg}}$ is $\Sigma^1_1$–complete. This result gives us a partial answer to [[19], Question 11.4].

**Theorem 3.17 ([14], Thm 3.8)** If $\kappa$ is weakly compact and $V = L$, then $\subseteq^{\text{NS}}$ is $\Sigma^1_1$–complete.

In the fourth section the reducibility of $E^\kappa_{\text{reg}}$ is studied. This was studied in [1], there was proved that if $\kappa$ is weakly compact and $V = L$, then $E^2_{\text{reg}}$ is $\Sigma^1_1$–complete. We improve this result by using [[19], Cor 10.24] if $\kappa$ is weakly compact, then embeddability of graphs is $\Sigma^1_1$–complete.
Theorem 3.18 ([14], Thm 4.1) Let $DLO$ be the theory of dense linear orderings without end points. If $\kappa$ is a weakly compact cardinal, then $\cong^\kappa_{DLO}$ is $\Sigma^1_1$-complete.

This shows that the isomorphism of graphs is $\Sigma^1_1$-complete, when $\kappa$ is a weakly compact cardinal.
Chapter 4

Conclusions

“As a rule, men worry more about what they can’t see than about what they can.”

- Julius Caesar

So far many results regarding generalized Borel reducibility have been achieved, thanks to them we have an idea of how the generalized Borel hierarchy looks so far and what needs to be studied in the future. In the first part of this chapter, I will give an overview of the generalized Borel hierarchy in different models and point out which assumptions are required. In the second part of this chapter, I will give a list of open problems related to the generalized Borel hierarchy, the answer of any of these questions will give us a better understanding of the generalized Borel hierarchy.

4.1 Maps of the Borel hierarchy

I will give two lists of results concerning Borel-reducibility in the generalized Baire space. The first list contains results previous to this thesis, the second list contains all the main results of this thesis.

4.1.1 Previous results

This list contains all the results that motivated this thesis and are basic results to understand the Borel-reducibility in the generalized Baire space. The results are not listed in chronological order.

- $\text{Borel} \subseteq \Delta^1_1 \subseteq \text{Borel}^* \subseteq \Sigma^1_1$ [[5], Thm 17].
- 1. $\text{Borel} \nsubseteq \Delta^1_1$.
2. \( \Delta^1 \subset \Sigma^1 \).

3. If \( V = L \), then \( \text{Borel}^* = \Sigma^1 \).

4. If \( V = L \), then \( \Delta^1 \subset \text{Borel}^* \) \([5], \text{Thm 18}\).

- Assume that \( \kappa \) is inaccessible. If the number of equivalence classes of \( \equiv^\kappa_T \) is greater than \( \kappa \), then \( \text{id} \leq c \equiv^\kappa_T \) \([5], \text{Thm 36}\).

- Assume \( \kappa < \kappa = \aleph_\alpha > \omega \), \( \kappa \) is not weakly inaccessible and \( \lambda = |\alpha + \omega| \). Then the following are equivalent.
  1. There is \( \gamma < \omega_1 \) such that \( \beth_\gamma(\lambda) \geq \kappa \).
  2. There is a complete countable theory \( T \) such that \( \text{id} \leq_B \equiv^\kappa_T \) and \( \equiv^\kappa_T \leq_B \text{id} \) \([5], \text{Thm 37}\).

- Suppose \( \kappa \) is a weakly compact cardinal and that \( V = L \). Then
  1. \( E^2_{\lambda \text{-club}} \leq c E^2_{\lambda \text{-reg}} \).
  2. In a forcing extension \( E^2_{\lambda \text{-club}} \leq c E^2_{\lambda^+ \text{-club}} \), in which \( \kappa = \lambda^+ \) \([5], \text{Thm 55}\). 

- For a cardinal \( \kappa \) which is a successor of a regular cardinal or \( \kappa \) inaccessible, there is a cofinality-preserving forcing extension in which for all regular \( \lambda < \kappa \), the relations \( E^2_{\lambda \text{-club}} \) are \( \leq_B \)-incomparable with each other \([5], \text{Thm 56}\). 

- Assume \( \kappa > 2^\omega \). If the theory \( T \) is classifiable and shallow, then \( \equiv^\kappa_T \) is Borel \([5], \text{Thm 68}\).

- If the theory \( T \) is classifiable, then \( \equiv^\kappa_T \) is \( \Delta^1 \) \([5], \text{Thm 70}\).

- 1. If \( T \) is unstable, then \( \equiv^\kappa_T \) is not \( \Delta^1 \)
  2. If the theory \( T \) is superstable with OTOP, then \( \equiv^\kappa_T \) is not \( \Delta^1 \).
  3. If the theory \( T \) is superstable with DOP and \( \kappa > \omega_1 \), then \( \equiv^\kappa_T \) is not \( \Delta^1 \).
  4. If \( T \) is stable with DOP and \( \lambda = \text{cf}(\lambda) = \lambda(T) + \lambda^{<\kappa(T)} \geq \omega_1 \), \( \kappa > \lambda^+ \) and for all \( \xi < \kappa \), \( \xi^\lambda < \kappa \), then then \( \equiv^\kappa_T \) is not \( \Delta^1 \) \([5], \text{Thm 71}\). 

- If a first order Theory \( T \) is classifiable, then for all \( \lambda < \kappa \) regular, it holds \( E^2_{\lambda \text{-club}} \not\leq_B \equiv^\kappa_T \) \([5], \text{Thm 77}\).

- Suppose that \( \kappa = \lambda^+ = 2^\lambda \) and \( \lambda^{<\lambda} = \lambda \).
1. If $T$ is unstable or superstable with OTOP, then $E^2_{\lambda\text{-}club} \leq_c \cong T$.
2. If $\lambda \geq 2^\omega$ and $T$ is superstable with DOP, then $E^2_{\lambda\text{-}club} \leq_c \cong T$ \cite{5}, Thm 79.

- Suppose that for all $\gamma < \kappa$, $\gamma^\omega < \kappa$ and $T$ is a stable unsuperstable theory. Then $E^2_{\omega\text{-}club} \leq_c \cong T$ \cite{5}, Thm 86.

- Suppose $T$ is a countable complete first-order classifiable and shallow theory, then $\cong_T \leq_B E^\kappa_{\lambda\text{-}club}$ holds for all regular $\lambda < \kappa$. \cite{4}, Cor 14.

- ($V = L$). Let $\kappa < \kappa = \kappa > \omega$. If $\kappa = \lambda^+$, let $\theta = \lambda$ and if $\kappa$ is inaccessible, let $\theta = \kappa$. Let $\mu < \kappa$ be a regular cardinal. Then $E^\kappa_{\mu\text{-}club}$ is $\Sigma^1_1$-complete \cite{6}, Thm 7.

- ($V = L$). Suppose $\kappa = \lambda^+$ and $\lambda$ is regular. The isomorphism relation on the class of dense linear orderings of size $\kappa$ is $\Sigma^1_1$-complete \cite{6}, Thm 9.

- ($V = L$). Suppose $\kappa = \lambda^+$ and $\lambda = \lambda^\omega$. Then $\cong_{T\omega+\omega}^\kappa$ is $\Sigma^1_1$-complete \cite{6}, Cor 21.

- It is consistent that $\Delta^1_1 \subsetneq \text{Borel}^\kappa \subsetneq \Sigma^1_1$ \cite{7}, Cor 3.2.

- If $\kappa$ is weakly compact, then the embeddability of trees is $\Sigma^1_1$-complete \cite{19}, Thm 10.23.

- If $\kappa$ is weakly compact, then the embeddability of graphs is $\Sigma^1_1$-complete \cite{19}, Cor 10.24.

### 4.1.2 Main results in this thesis

This is a list of the main results obtained in this thesis. This thesis contains results concerning model theory or set theory, in this list I will include only the results that are about Borel-reducibility.

- Assume $T$ is a classifiable theory and $\lambda < \kappa$ a regular cardinal, the $\cong_T^\kappa$ is continuously reducible to $E^\kappa_{\lambda\text{-}club}$ \cite{9}, Thm 2.8.

- Suppose that for all $\gamma < \kappa$, $\gamma^\omega < \kappa$ and $2^\lambda$, then $E^\kappa_{\omega\text{-}club} \leq_B \cong_{T\omega}^\kappa$ \cite{9}, Lemma 3.2.

- Suppose for all $\gamma < \kappa$, $\gamma^\omega < \kappa$ and $2^\lambda$, $\lambda < \kappa$. If $T$ is a classifiable theory. Then $\cong_T^\kappa \leq_c \cong_{T\omega}^\kappa$ \cite{9}, Cor 3.4.
• If $T$ is a stable theory with the OCP and $\kappa$ is an inaccessible cardinal, then $E^\kappa_{\omega\text{-club}} \leq_c \cong^\kappa_T$ [[9], Cor 5.10].

• Assume $\kappa$ is an inaccessible cardinal. If $T_1$ is a classifiable theory and $T_2$ is a stable theory with the OCP, then $\cong^\kappa_{T_1} \leq_c \cong^\kappa_{T_2}$ [[9], Cor 5.11].

• Assume $T$ is a classifiable theory and $\mu < \kappa$ a regular cardinal. If $\diamondsuit_{\kappa}(X)$ holds, then $\cong^\kappa_T$ is continuously reducible to $E_X$ [[8], Lemma 2].

• Assume that $\diamondsuit_{\kappa}(S^\kappa_\mu)$ holds for all regular $\mu < \kappa$. If a first order theory $T$ is classifiable, then for all regular cardinals $\mu < \kappa$ we have $\cong^\kappa_T \leq_c E^2_{\mu\text{-club}}$ and $E^2_{\mu\text{-club}} \not\leq_B \cong^\kappa_T$ [[8], Cor 1].

• Assume that $\diamondsuit_{\kappa}(S^\kappa_\mu)$ holds for all regular $\mu < \kappa$. If a first order theory $T$ is classifiable, then for all regular cardinals $\mu < \kappa$ we have $\cong^\kappa_T \leq_c E^2_{\mu\text{-club}}$ and $E^2_{\mu\text{-club}} \not\leq_B \cong^\kappa_T$ [[8], Cor 1].

• Suppose $\kappa = \kappa^{<\kappa} = \lambda^+$ and $\lambda^\omega = \lambda$. If $T_1$ is classifiable and $T_2$ is stable unsuperstable, then $\cong^\kappa_{T_1} \leq_c \cong^\kappa_{T_2}$ and $\cong^\kappa_{T_2} \not\leq_B \cong^\kappa_{T_1}$ [[8], Cor 2].

• Define $H(\kappa)$ as the following property:
  If $T$ is classifiable and $T'$ not, then $\cong^\kappa_{T'} \leq_c \cong^\kappa_T$, and $\cong^\kappa_{T'}, \not\leq_B \cong^\kappa_{T'}$. Suppose that $\kappa = \kappa^{<\kappa} = \lambda^+$, $2^\lambda > 2^\omega$ and $\lambda^{<\lambda} = \lambda$.

  1. If $V = L$, then $H(\kappa)$ holds.
  2. There is a $\kappa$-closed forcing notion $\mathbb{P}$ with the $\kappa^+$-c.c. which forces $H(\kappa)$ [[8], Thm 6].

• Suppose that $\kappa = \kappa^{<\kappa} = \lambda^+$, $2^\lambda > 2^\omega$ and $\lambda^{<\lambda} = \lambda$. Then the following statements are consistent.

  1. If $T_1$ is classifiable and $T_2$ is not, then there is an embedding of $(\mathcal{P}(\kappa), \subseteq)$ to $(B^*(T_1, T_2), \leq_B)$, where $B^*(T_1, T_2)$ is the set of all Borel*-equivalence relations strictly between $\cong^\kappa_{T_1}$ and $\cong^\kappa_{T_2}$.
  2. If $T_1$ is classifiable and $T_2$ is unstable or superstable with OTOP or with DOP, then

\[
\cong^\kappa_{T_1} \leq_c E^2_{\lambda\text{-club}} \leq_c \cong^\kappa_{T_2} \land \cong^\kappa_{T_2} \not\leq_B E^2_{\lambda\text{-club}} \land E^2_{\lambda\text{-club}} \not\leq_B \cong^\kappa_{T_1}
\]

[[8], Thm 7].

• Assume $T$ is a superstable theory with $S$-DOP, then $E^\kappa_{\lambda\text{-club}}$ is continuously reducible to $\cong^\kappa_T$ [[18], Cor 4.15].

• Assume $T_1$ is a classifiable theory and $T_2$ is a superstable theory with $S$-DOP, then $\cong^\kappa_{T_1} \leq_c \cong^\kappa_{T_2}$ [[18], Cor 4.16].
4.1 Maps of the Borel hierarchy

• Suppose $V = L$. If $T$ is a superstable theory with $S$-DOP, then $\cong_T$ is $\Sigma_1^1$-complete [[18], Cor 4.19].

• Suppose $\gamma < \lambda$ are regular cardinals. If $S^\kappa_\gamma$ strongly reflect to $S^\kappa_\lambda$, then $E^\kappa_\gamma_{\text{club}} \leq_c E^\kappa_\lambda_{\text{club}}$. [[1], Prop 2.8].

• Suppose $\lambda < \kappa$ is such that $\lambda < \lambda = \lambda$. If $\gamma$ is a regular cardinal such that $S^\kappa_\gamma \leftrightharpoons$ reflects to $S^\kappa_\lambda$, then
  1. $E^2_\gamma_{\text{club}} \leq_c E^2_\lambda_{\text{club}}$.
  2. $E^\kappa_\gamma_{\text{club}} \leq_c E^\kappa_\lambda_{\text{club}}$ [[1], Cor 2.9].

• Suppose $\kappa$ is a $\Pi^+_1$–indescribable cardinal and that $V = L$. Then there is a forcing extension where $\kappa$ is collapsed to $\lambda^{++}$ and $E^\lambda_{\kappa+} \leq_c E^\lambda_2_{\text{club}}$ [[1], Thm 2.11].

• The following statement is consistent. $E^{w^2_{\omega}}_{\kappa-\text{club}} \leq_c E^{w^2_{\omega_1}}_{\kappa-\text{club}}$, and for every $2 < n$ and every $0 \leq k \leq n - 3$, $E^{w^2_{\omega_0}}_{\kappa-\text{club}} \leq_c E^{w^2_{\omega_{k-1}}}_{\kappa-\text{club}}$ [[1], Cor 2.14].

• Suppose $S = S^\kappa_\lambda$ for some $\lambda$ regular cardinal, or $S = \text{reg}(\kappa)$ and $\kappa$ is a weakly compact cardinal. If $\kappa$ has the $S$-dual diamond, then $E_S \leq_c E^\kappa_{\text{reg}}$, where $E_S = E^\kappa_{\lambda-\text{club}}$ if $S = S^\kappa_\lambda$, or $E_S = E^\kappa_{\text{reg}}$ if $S = \text{reg}(\kappa)$ [[1], Thm 3.3].

• Suppose $V = L$ and $\kappa$ is weakly compact. Then $E^2_{\text{reg}}$ is $\Sigma_1^1$-complete [[1], Cor 3.5].

• Suppose $\kappa$ is a supercompact cardinal. There is a generic extension $V[G]$ in which $E^\kappa_{\text{reg}} \leq_c E^2_{\text{reg}}$ holds and $\kappa$ is still supercompact in the extension [[1], Thm 3.6].

• If $\kappa$ is a $\Pi^1_2$–indescribable cardinal, then $E^\kappa_{\text{reg}}$ is $\Sigma_1^1$-complete [[1], Thm 3.7].

• Suppose $\kappa$ is a supercompact cardinal. There is a generic extension $V[G]$ in which $E^2_{\text{reg}}$ is $\Sigma_1^1$-complete [[1], Cor 3.8].

• Let DLO be the theory of dense linear orderings without end points. If $\kappa$ is a $\Pi^1_2$–indescribable cardinal, then $\cong_{\text{DLO}}$ is $\Sigma_1^1$-complete [[1], Thm 3.9].

• If $\kappa$ is weakly compact, then $\sim \leq_c E^\kappa_{\text{reg}}$ is $\Sigma_1^1$-complete. [[14], Thm 4.2].
• If $\kappa$ is a weakly compact cardinal, then $\cong^{\kappa}_{DLO}$ is $\Sigma^1_1$-complete. [[14], Thm 4.1].

• If $\kappa$ is a weakly compact cardinal and has the $G_{<\kappa}$-Dual diamond, then $\subseteq^{\text{reg}}$ is $\Sigma^1_1$-complete. [[14], Lemma 3.7].

• If $\kappa$ is weakly compact and $V = L$, then $\subseteq^{\text{NS}}$ is $\Sigma^1_1$-complete. [[14], Thm 3.8].

4.2 Open questions and further research

Some of the questions listed in here were asked in other articles different from the ones presented in this thesis, those questions have their respective reference. In [12] the reader can find a bigger list of open questions on generalized Baire spaces. The list presented in here contains questions that motivated the articles mentioned above, questions asked during the elaboration of this thesis, and questions that are closely related to the generalized Borel hierarchy but not were mentioned in the previous chapter.

The list is organised according to the four categories

• Generalized Descriptive Set Theory.

• Model Theory and the Main Gap Theorem.

• The reducibility of the isomorphism relation.

• The reducibility of the equivalence modulo the non-stationary ideal.

4.2.1 Generalized Descriptive Set Theory

In [5] a complete study on the generalized descriptive set theory was done, many results were proved. Some of which are considered the basis for the theory. The authors asked the following question, which rises natural from that study.

**Question 4.1 (Friedman, Hyttinen, Kulikov; [5])** How much can be done without the assumption $\kappa^{<\kappa} = \kappa$?

As it was mentioned before, in the generalized descriptive set theory we know that $\text{Borel} \subsetneq \Delta^1_1 \subsetneq \text{Borel}^* \subseteq \Sigma^1_1$. In [7] it was shown that $\text{Borel}^* \neq \Sigma^1_1$, $\Delta^1_1 \neq \text{Borel}^*$, and $\text{Borel}^* = \Sigma^1_1$ are all consistently true. The consistency of $\Delta^1_1 = \text{Borel}^*$ is still open.
4.2 Open questions and further research

Question 4.2 (Friedman, Hyttinen, Kulikov; [5, 7]) Is it consistent that $\Delta_1^1 = \text{Borel}^*$?

In [6] it was proved that in $L$, that there are $\text{Borel}^*$ relations that are $\Sigma_1^1$-complete. This implies that there is a $\text{Borel}^*$-complete relation is consistently true. One of the motivations for [1] was to find a $\text{Borel}^*$-complete relation without the assumption $V = L$. The relation studied in [1] turned out to be $\Sigma_1^1$-complete, this rises the next question.

Question 4.3 Does there exists a $\text{Borel}^*$-complete relation that is not $\Sigma_1^1$-complete?

The same question can be asked for $\Delta_1^1$, in this case we know that there is no $\Delta_1^1$-complete relation. The question would be about $\Delta_1^1$-hard relations.

Question 4.4 Does there exists a $\Delta_1^1$-hard relation that is not $\Sigma_1^1$-complete?

A positive answer in this question could imply a negative answer in Question 4.2. If there is a $\text{Borel}^*$ relation in ZFC that is $\Delta_1^1$-hard, then $\Delta_1^1 \neq \text{Borel}^*$.

4.2.2 Model Theory and the Main Gap Theorem

In [9] the main theorem tells us that every stable theory with the OCP is more complex than any classifiable theory. Some examples of stable theories with OCP has been found, in [3] the author shows that the theory of the group of $p$-adic integers is stable and has the OCP. This rises the next question.

Question 4.5 Does there exists a stable unsuperstable theory that doesn’t have OCP?

In [18] the main theorem tells us that every superstable theory with S-DOP is more complex than any classifiable theory. The following question is a natural question to ask.

Question 4.6 (J. Baldwin) Does there exists a superstable theory with DOP that doesn’t have S-DOP?

When I asked Shelah this question, his conjecture was that every superstable theory with DOP also has S-DOP.

In [8] it was shown the consistency of: If $T_1$ is a classifiable theory and $T_2$ is not classifiable, then $T_1$ is as most as complex as $T_2$. But it is still open whether it is true in ZFC.
Question 4.7 (Friedman, Hyttinen, Kulikov, Moreno; [4, 8]) Is there a generalized Borel reducibility counterpart of the Main Gap Theorem in ZFC, i.e. Is it provable in ZFC that \( \preceq_T^\kappa \preceq_B \preceq_T^{\kappa} \), (note the strict inequality) for all complete first-order theories \( T \) and \( T' \), \( T \) classifiable and \( T' \) not? Do we need large cardinal assumptions?

As it was mentioned before, one approach to this question is to study the reducibility of the relations \( E^\kappa_\lambda \)-club and \( E^2_\lambda \)-club. A lot have been study on this, but there are three main questions open about the reducibility of these relations.

Question 4.8 Suppose \( T \) is a non-classifiable theory. Does there exists \( \lambda < \kappa \) such that \( E^\kappa_\lambda \)-club \( \preceq_B \preceq_T^{\kappa} \)? In case it exists, is it the same for all the theories?

Question 4.9 Suppose \( T \) is a non-classifiable theory. Under which cardinality assumptions on \( \kappa \), does it hold \( E^2_\omega \)-club \( \preceq_B \preceq_T^{\kappa} \)?

Notice that we know it is consistently true to have \( E^2_\omega \)-club \( \preceq_B \preceq_T^{\kappa} \) for all theory \( T \) non-classifiable, this follows from Theorem 55, Theorem 79 and Theorem 86 of [5].

Question 4.10 (Friedman, Hyttinen, Kulikov; [5]) If \( \kappa = \lambda^+ \), \( \lambda \) regular and uncountable, does \( E^2_\lambda \)-club \( \preceq_B \preceq_T^{\kappa} \) for all theory \( T \) stable unsuperstable?

4.2.3 The reducibility of the isomorphism relation

The isomorphism relation of every first order complete theory is \( \Sigma^1_1 \), for some theories we know even more. In [5] it was shown the following results

- If the theory \( T \) is classifiable, then \( \preceq_T^{\kappa} \) is \( \Delta^1_1 \).
- If the theory \( T \) is classifiable and , and \( \kappa > 2^\omega \), then \( \preceq_T^{\kappa} \) is Borel.
- If the theory \( T \) is unstable, then \( \preceq_T^{\kappa} \) is not \( \Delta^1_1 \).
- If the theory \( T \) is superstable with OTOP, then \( \preceq_T^{\kappa} \) is not \( \Delta^1_1 \).
- If the theory \( T \) is superstable with DOP and \( \kappa > \omega_1 \), then \( \preceq_T^{\kappa} \) is not \( \Delta^1_1 \).

It is still not known if \( T \) is a stable unsuperstable theory, then \( \preceq_T^{\kappa} \) is not \( \Delta^1_1 \).
Question 4.11 (Friedman, Hyttinen, Kulikov; [5]) Is it consistent that there exists a stable unsuperstable theory $T$ such that $\cong^\kappa_T$ is a $\Delta^1_1$ relation.

One of the advantages of using the Borel hierarchy to measure the complexity of first order theories, is the possibility of comparing the complexity of two different theories (not necessary classifiable and non-classifiable). Some examples are:

- If $V = L$, then $\cong^\kappa_{DLO}$ is $\Sigma^1_1$-complete [6].
- If $\kappa$ is a $\Pi^1_2$-indescribable cardinal, then $\cong^\kappa_{DLO}$ is $\Sigma^1_1$-complete [1].
- If $V = L$ and $\kappa$ is an inaccessible cardinal, then for every stable theory $T$ with OCP, $\cong^\kappa_T$ is $\Sigma^1_1$-complete [9].
- If $V = L$ and $\kappa$ is an inaccessible cardinal, for every superstable theory $T$ with S-DOP, $\cong^\kappa_T$ is $\Sigma^1_1$-complete [18].
- If $\kappa$ is a weakly compact cardinal, then the bi-embeddability of trees is $\Sigma^1_1$-complete [19].
- If $\kappa$ is a weakly compact cardinal, then the bi-embeddability of graphs is $\Sigma^1_1$-complete [19].

One natural question to ask is if there are two theories that are not comparable.

Question 4.12 Do there exist theories $T_1$ and $T_2$ such that $\cong^\kappa_{T_1} \not\leq_B \cong^\kappa_{T_2}$ and $\cong^\kappa_{T_2} \not\leq_B \cong^\kappa_{T_1}$.

As it was mentioned, the different kind of theories have a total or partial characterization using Borel reducibility, e.g. if $\kappa = \lambda^+ = 2^\lambda > 2^\omega$ where $\lambda^{<\lambda} = \lambda$, then $T$ is classifiable if and only if for all regular $\mu < \kappa$, $E^2_{\mu-\text{club}} \not\leq_B \cong^\kappa_T$. Notice that there is no total characterization in ZFC for stable theories with OCP or superstable theories with S-DOP.

Question 4.13 Under which cardinal assumptions there exists a total characterization for all kind of theories (classifiable, stable unsuperstable, superstable with DOP, superstable with OTOP, unstable)?

Question 4.14 (Friedman, Hyttinen, Kulikov; [5]) Under which assumptions on $\kappa$, does it hold that if the number of equivalence classes of $\cong^\kappa_T$ is greater than $\kappa$, then $\text{id} \leq_B \cong^\kappa_T$?
4.2.4 The reducibility of the equivalence modulo the non-stationary ideal

As we saw in the map of the Borel hierarchy, the equivalence modulo the non-stationary ideal is very important when we look for a generalized Borel reducibility counterpart of Shelah’s main gap theorem. In [5] it was shown the consistency of: The relations $E^2_{\lambda,\text{club}}$ are $\leq_B$-incomparable. The natural question is if this is provable in ZFC.

**Question 4.15** (Friedman, Hyttinen, Kulikov; [5]) *Is it consistent that $E^2_{\lambda,\text{club}} \leq_c E^2_{\gamma,\text{club}}$ for $\gamma < \lambda$?*

For the relations $E^\kappa_{\lambda,\text{club}}$ the situation is the oposite, in [6] it was shown the consistency of: The relations $E^\kappa_{\lambda,\text{club}}$ are $\Sigma^1_1$-complete. In this case the question is about the incomparability of these relations.

**Question 4.16** *Is it consistent that $E^\kappa_{\lambda,\text{club}} \not\leq_c E^\kappa_{\gamma,\text{club}}$ holds for all $\gamma \neq \lambda$?*

The Borel-reducibility properties of the equivalence modulo the non-stationary ideal have been studied in the generalized Baire space and in the generalized Cantor space. One natural question is to ask if the equivalence modulo the non-stationary ideal has the same Borel-reducibility properties in the generalized Baire space as in the generalized Cantor space? Some results strengthen the idea that the properties should be the same. An example of this is Corollary 2.9 in [1], in which the combinatorial principle used to get reduction in the generalized Cantor space implies the “same” reduction in the generalized Baire space. This can also be seen with the Borel-reducibility of the isomorphism relation of classifiable theories to the equivalence modulo the $\lambda$-non-stationary ideal (in the generalized Baire space this always holds, in the generalized Cantor space we only know that the diamond principle implies this reducibility). A possible approach to some of the previous questions is to study the reducibility of equivalence relations in the generalized Baire space to equivalence relations in the generalized Cantor space. In particular, a positive answer in the following question will imply a partial answer to Question 4.7.

**Question 4.17** (Friedman, Hyttinen, Kulikov; [5]) *Is it $E^\kappa_{\lambda,\text{club}}$ Borel reducible to $E^2_{\lambda,\text{club}}$?*

In [1] it was shown that $E^2_{\omega,\text{club}} \leq_c E^{\omega_2}_{\omega,\text{club}} \leq_c E^2_{\omega_1,\text{club}} \leq_c E^{\omega_2}_{\omega_1,\text{club}}$ is consistently true, but it was not shown whether this reductions are strict or not in that model. This leads to the question if it is consistently true that the relations $E^2_{\gamma,\text{club}}$ and $E^\kappa_{\gamma,\text{club}}$ are linearly ordered by $\leq_B$. 
Question 4.18 (Aspero, Hyttinen, Kulikov, Moreno; [1]) Is it consistent that

\[ E_\gamma^{\kappa \text{-club}} \leq_c E_{\kappa \text{-club}} \leq_c E_\lambda^{\kappa \text{-club}} \leq_c E_{\lambda \text{-club}} \]

holds for all \( \gamma, \lambda < \kappa \) and \( \gamma < \lambda \)?
References


[14] V. Kulikov, and M. Moreno *On \(\Sigma^1_1\)-completeness in weakly compact cardinals*, preprint.


