On the multifractal spectra of mappings of finite distortion

Lauri Hitruhin

Academic dissertation

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List of original articles

This thesis is based on the following articles:

[A] Lauri Hitruhin
On multifractal spectrum of quasiconformal mappings.
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[B] Lauri Hitruhin
Pointwise rotation for mappings with exponentially integrable distortion.
Proc. Amer. Math. Soc. 144 (2016), 5183-5195.

[C] Lauri Hitruhin
Rotational properties of homeomorphisms with integrable distortion.
Manuscript

[D] Lauri Hitruhin
Joint rotational and stretching multifractal spectra of mappings with integrable distortion.
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Contents

1. Introduction

This thesis considers planar quasiconformal mappings and certain subfamilies of mappings of finite distortion, concentrating especially on their stretching and rotational properties. This research adds to the long history that the study of quasiconformal mappings has in the Finnish mathematical landscape, starting with the classical works of Ahlfors, Lehto, Martio, Rickman and Väisälä.

Given a domain $\Omega \subset \mathbb{C}$ we say that a sense-preserving homeomorphism $f : \Omega \to \mathbb{C}$ is K-quasiconformal, where $1 \leq K < \infty$, if and only if

- $f \in W^{1,2}_{\mathrm{loc}}(\Omega)$ and
- $|Df(z)|^2 \leq K J_f(z)$ almost everywhere in Ω.

Here $|Df(z)|$ is the operator norm of the differential, $J_f(z)$ its Jacobian and K is the constant that controls the distortion of the mapping.

This definition for quasiconformal mappings is called the *analytic* definition, and it is the one that is typically used these days. However, there are also two other classical definitions for these mappings.

The *metric* definition is based on the idea that infinitesimal balls are mapped to infinitesimal ellipses with bounded eccentricity, that is

$$\limsup_{r \to 0} \frac{\sup_{|h|=r} |f(z) - f(z+h)|}{\inf_{|h|=r} |f(z) - f(z+h)|} \leq H$$

for some constant $H \geq 1$. Note, that the metric definition makes sense in an arbitrary metric space and that the constants K and H play similar roles in these definitions.

The final definition is *geometric*, which defines quasiconformal mappings using the modulus of path families, which we denote by $M(\Gamma)$ and define later, by demanding that

$$\frac{1}{K} M(f(\Gamma)) \leq M(\Gamma) \leq K M(f(\Gamma))$$

for every path family Γ in Ω.

All of these definitions highlight in their own way how quasiconformal mappings naturally generalize conformal mappings, the analytic via the constant K, the metric via the constant H, and the geometric via the conformal invariance of the modulus of path families.

The equivalence of these definitions in the Euclidean setting was a longstanding open problem and many mathematicians, Ahlfors, Bers, Gehring, Lehto and Mori to name a few, contributed towards its solution during the 50s. Finally the articles by Bers and Gehring in the late 50s and early 60s conclusively established the equivalence, see [8] and [13].

These three definitions for quasiconformality date from different times and arose from different points of view taken during the development of the theory. The interplay between these definitions, and thus between the different points of view of the theory, was central for quasiconformal mappings to achieve the interest and importance that the field nowadays has. The fact that the quasiconformal mappings can be approached in such different ways partly explains why they play a role in so many seemingly unrelated mathematical fields.

The history of quasiconformal mappings can be dated back to Grötzsch and his paper [21] in 1928. He asked to find *the most nearly conformal* *mapping* that maps a given square S to a given rectangle R, while mapping vertices to vertices. To answer this question, one first has to consider what it means to be nearly conformal, which started the process towards generalizing the family of conformal mappings to what would later become the family of quasiconformal mappings.

Not long after this first step by Grötzsch, the importance of quasiconformal mappings in complex analysis was discovered by Ahlfors and Teichmüller in the 30s, see [1] and [41]. From these early stages of the theory we would also like to emphasize the contribution by Morrey, who proved his fundamental existence theorem for quasiconformal mappings in [39]. Later on, Ahlfors in his article [2] studied various definitions for quasiconformal mappings in the correct generality, that is, without any a priori assumptions on the smoothness of mappings. His article drastically changed the field, as previously quasiconformality was often studied inside the family of diffeomorphisms. One could argue, that this paper marks the start of the modern theory of quasiconformal mappings.

Interest towards quasiconformality soared after these results and there was a notable increase in the number of mathematicians working on the field. Especially influential was the systematic study of quasiconformal mappings in space by Gehring and Väisälä that started in the late 50s, see, for example, [14], [15] and [16] by Gehring, [43] by Väisälä, and [18] by Gehring and Väisälä. They were an integral part in establishing the foundations of the modern theory of quasiconformal mappings together with the so-called *Russian school*, which consisted of mathematicians like Bojarski, Lavrentiev and Reshetnyak, that was also actively studying quasiconformal mappings, see, for example, [32], [33], [9] and [10].

During these times also the so-called *Finnish school* of quasiconformal mappings, which had close connections to both Gehring and the Russian school, started to form around Lehto, Martio, Rickman and Väisälä. The collaboration between the Finnish school and Gehring can be seen, in addition to the joint work of Gehring and Väisälä, in the founding of the famous

Gehring Lehto theorem, see [17], which ensures that every planar homeomorphic mapping of finite distortion is differentiable almost everywhere. On the other hand, Martio, Rickman and Väisälä, among other things, continued the work of Reshetnyak and studied quasiregular mappings, which generalize quasiconformal mappings by dropping the assumption that the mapping has to be a homeomorphism, in their articles [34] and [35].

Quasiconformal mappings have since the early 60s had a special place in the Finnish mathematical scene, and the quasiconformal torch has then been passed along by Astala, Koskela, Pankka and Vuorinen, just to name few.

1.1 Mappings of finite distortion

Nowadays quasiconformal mappings play an integral role in a myriad of mathematical fields, including non-linear PDEs, conformal and holomorphic dynamics, conformal geometry, holomorphic motions, fluid dynamics and calculus of variations. In many of these fields the need to extend quasiconformal notions further to a degenerate setting, where the distortion K is not bounded, arises naturally.

This has led to the development of the theory of mappings of finite distortion, which are a natural generalization of quasiconformal mappings. David was the first to consider such mappings in his article [12], where he extended the classical existence theorem of Morrey for mappings with exponentially integrable distortion, which we shall define shortly. Another significant early paper on this subject was [28] by Iwaniec and Šverák, where a Stoilow type factorization was proved under weak assumptions.

The modern definition for mappings of finite distortion resembles the analytic definition of quasiconformal mappings, but relaxes the conditions on the distortion and Sobolev regularity.

Let $\Omega \subset \mathbb{C}$ be a domain. We say that a mapping $f : \Omega \to \mathbb{C}$ has finite distortion if the following conditions hold:

- $f \in W^{1,1}_{\text{loc}}(\Omega)$
- $J_f(z) \in L^1_{\text{loc}}(\Omega)$
- $|Df(z)|^2 \leq J_f(z)K(z)$ almost everywhere in Ω,

for a measurable function $K(z) \geq 1$, which is finite almost everywhere. The smallest such function is denoted by $K_f(z)$ and called the *distortion* of the mapping f.

As mentioned before, this thesis is centered around rotational and stretching properties of mappings of finite distortion. To this end the full generality of these mappings is far too weak, as the definition does not even guarantee continuity. This in turn makes it impossible to even discuss what the pointwise stretching means. Thus we restrict to homeomorphic mappings and impose some standard additional assumptions, which control growth of the distortion function. In my thesis I consider two canonical subfamilies of mappings of finite distortion, mappings with q-exponentially integrable distortion and mappings with p-integrable distortion.

To define the first family we fix $q > 0$ and say that a mapping of finite distortion $f : \Omega \to \mathbb{C}$, where $\Omega \subset \mathbb{C}$ is an arbitrary domain, is a mapping with q-exponentially integrable distortion if it is a sense-preserving homeomorphism and the distortion satisfies

$$e^{K_f(z)} \in L^q_{\text{loc}}(\Omega). \tag{1.1}$$

To define the second family we fix $p \geq 1$ and say that a mapping of finite distortion $f : \Omega \to \mathbb{C}$ is a mapping with p-integrable distortion if it is a sense-preserving homeomorphism and the distortion satisfies

$$K_f(z) \in L^p_{\text{loc}}(\Omega). \tag{1.2}$$

Since David's article [12], mappings of finite distortion have been under active research by many prominent mathematicians both in Finland and abroad. My own research in this area has been influenced greatly by the works of Astala, Clop, Iwaniec, Herron, Koskela, Martio, Onninen and Saksman, to name just a few of the many authors whose inspirational results I have encountered.

The modern methods in the theory of quasiconformal mappings and mappings of finite distortion have been well captured, for example, in the monograph [5] by Astala, Iwaniec and Martin, in the monograph [27] by Iwaniec and Martin and in the monograph [25] by Hencl and Koskela. These books provide an excellent background for the topics of this thesis.

The contribution of this thesis falls into two parts. First, to finding the sharp pointwise bounds for rotation of mappings with integrable or exponentially integrable distortion using new methods based on the modulus of path families. These methods work also for pointwise stretching, enabling sharpening of some previously known results. This research has been conducted in the articles [B] and [C].

Establishing the sharp pointwise bounds for stretching and rotation enables us to fix some specific combination of them and ask what is the maximal size of sets in which the corresponding mappings of finite distortion can attain them. Solving this for all possible combinations of stretching and rotation amounts to finding the multifractal spectra of these mappings.

The second part of my thesis consists of describing multifractal spectra both in the case of quasiconformal mappings and in the case of mappings with integrable distortion. These questions have been considered in the articles [A] and [D]. Furthermore, in the article [D] we use the stretching multifractal spectra of mappings with integrable distortion to improve the compression of Hausdorff measure result by Clop and Herron, presented in [11].

1.2 Measure theory

While studying the multifractal spectra we encounter the notion of a size of a set, which naturally leads us to measure theory. We give here a short overview which is sufficient for the needs of this thesis, full details can be found, for example, in Mattila's book [37].

We call an increasing function $h : (0, \infty) \to (0, \infty)$ a Hausdorff dimension gauge function if $\lim_{r \to 0} h(r) = 0$. Given an arbitrary gauge function h and a set $A \subset \mathbb{C}$ we define the generalized Hausdorff measure $H^h(A)$ by

(1.3)
$$H^h(A) = \lim_{r \to 0} \left[\inf \left\{ \sum_j h\left(\mathrm{diam}\,(U_j)\right) : A \subset \bigcup_j U_j, \mathrm{diam}\,(U_j) \le r. \right\} \right].$$

If we choose $h(r) = r^s$ we obtain the usual s-dimensional Hausdorff measure $H^s(A)$, which is non-trivial when $s \in [0, 2]$. Using these measures we define the Hausdorff dimension of a set $A \subset \mathbb{C}$ by

$$\dim(A) = \inf\{s > 0 : H^s(A) = 0\}.$$

Usually we are interested only in whether the Hausdorff measure $H^h(A)$ is infinite, finite or zero. This thesis considers gauge functions that are either of form

$$h(r) = \left(\frac{1}{\log\left(\frac{1}{r}\right)} \right)^a,$$

where $a > 0$, or of form $h(r) = r^s$. Hence it is easy to see that we can restrict the sets U_j in (1.3) to balls without affecting the positivity or finiteness of the measure $H^h(A)$.

2. Pointwise bounds

The pointwise stretching properties of quasiconformal mappings are captured by the classical Hölder continuity results, which date back to the works of Ahlfors [2] and Mori [38]. They state that given an arbitrary K-quasiconformal mapping $f : \mathbb{C} \to \mathbb{C}$, normalized by $f(0) = 0$ and $f(1) = 1$, we have bounds

$$\frac{1}{c_K}|z|^K \leq |f(z)| \leq c_K |z|^{\frac{1}{K}}, \tag{2.1}$$

for all $|z| < 1$. The K-quasiconformal radial stretching mappings

$$f(z) = z|z|^{K-1}$$

and

$$f(z) = z|z|^{\frac{1}{K}-1}$$

show that the Hölder exponents in (2.1) are optimal.

For mappings with q-exponentially integrable distortion the analog to (2.1) follows from the modulus of continuity estimates by Herron and Koskela in [26] and by Onninen and Zhong in [40]. Their results show that given an arbitrary mapping $f : \mathbb{C} \to \mathbb{C}$ with q-exponentially integrable distortion, normalized by $f(0) = 0$, we have

$$e^{-\frac{c_{f,q}}{q}\log^2\left(\frac{1}{|z|}\right)} \lesssim |f(z)| \lesssim \frac{c_{f,q}}{\log^{\frac{q}{2}}\left(\frac{1}{|z|}\right)}, \tag{2.2}$$

when $|z| > 0$ is small enough. Moreover, these bounds are essentially sharp. That is, the exponent 2 on the left hand side can not be made any smaller, and the exponent $\frac{q}{2}$ on the right hand side can not be made any bigger.

Finally, the analog for (2.1) in the case of mappings with p-integrable distortion has been developed by Koskela and Takkinen in [31]. They proved that any mapping $f : \mathbb{C} \to \mathbb{C}$ with p-integrable distortion, normalized by $f(0) = 0$, satisfies

$$|f(z)| \geq e^{-c_{f,p}|z|^{-\frac{2}{p}}}, \tag{2.3}$$

when $|z| > 0$ is sufficiently small. Furthermore, the bound is again sharp in the sense that the exponent $\frac{2}{p}$ can not be made any smaller. There are no non-trivial upper bounds established for pointwise stretching of mappings with integrable distortion, and examples seem to indicate that such bound can not exist.

The sharp bounds (2.1), (2.2) and (2.3) show that the pointwise stretching is well-understood for these classes of mappings.

2.1 Quasiconformal pointwise rotation

The classical approach for studying rotation of mappings of finite distortion considers only mappings from annulus to annulus. More precisely, it relies on restricting to mappings that fix some given annulus, keep the outer circle in place and rotate the inner circle. For quasiconformal mappings the extremal rotation in the classical sense was established by Gutlyanskiĭ and Martio in [23], see also earlier work [29] by John for Bilipschitz case. Later on Balogh, Fässler and Platis in [7] extended this result for a more general class of mappings, while simultaneously considering mappings between annuli that are not conformally invariant.

Recently Astala, Iwaniec, Prause and Saksman proposed in [6] an alternative approach for studying pointwise rotation of quasiconformal mappings.

They dropped the restriction to annuli altogether and instead asked for the maximal pointwise rotation of a general quasiconformal mapping.

Theorem 2.1 (Theorem 3.1 in [6]). *Let $f : \mathbb{C} \to \mathbb{C}$ be a K-quasiconformal mapping, which is normalized by $f(0) = 0$ and $f(1) = 1$. Then for any $0 < r < 1$*

$$|\arg(f(r))| \leq \frac{1}{2}\left(K - \frac{1}{K}\right)\log\left(\frac{1}{r}\right) + c_K, \tag{2.4}$$

where the branch of the argument is determined by $\arg(1) = 0$*. Moreover, there exists a K-quasiconformal mapping that satisfies* (2.4) *as an equality with $c_K = 0$.*

In fact, the result in [6] considers more universal combination of pointwise rotation and stretching, but this formulation of Theorem 2.1 captures the parts which are relevant to us when studying pointwise rotation of more general classes of mappings.

The inequality (2.4) is the right analog for the Hölder exponents (2.1) when discussing pointwise rotation of quasiconformal mappings. This naturally leads to the question of generalizing the pointwise bounds for rotation, in the spirit of [6], for more general classes of mappings of finite distortion.

To this end, we have to develop new methods utilising the modulus of path families.

2.2 Modulus of path families

We call a continuous function $\gamma : I \to \mathbb{C}$, where $I \subset \mathbb{R}$ is an interval, a path. Denote both the function and its image by γ and let Γ be a family of paths. We say that a Borel measurable function $\rho : \mathbb{C} \to [0, \infty]$ is admissible with respect to Γ if

$$\int_\gamma \rho(z)\,|dz| \geq 1,$$

for every locally rectifiable $\gamma \in \Gamma$. Finally, denote the modulus of a path family Γ by $M(\Gamma)$ and define it by

$$M(\Gamma) = \inf_{\rho \text{ admissible}} \int_{\mathbb{C}} \rho^2(z)\, dz. \tag{2.5}$$

We will additionally need a weighted version of (2.5), where the weight $\omega : \mathbb{C} \to [0, \infty]$ is measurable and locally integrable. In this case we define

$$M_\omega(\Gamma) = \inf_{\rho \text{ admissible}} \int_{\mathbb{C}} \omega(z)\rho^2(z)\, dz.$$

In our applications the distortion function $K_f(z)$ will take the role of the weight function.

The properties of the modulus of path families are well presented, for example, in the books of Vuorinen [45] and Väisälä [44].

2.3 Exponentially integrable distortion

As a first step in generalizing Theorem 2.1 the article [B] introduced the optimal bound for pointwise rotation of mappings with q-exponentially integrable distortion.

Theorem 2.2 (Theorem 1.1 in [B])**.** *Fix an arbitrary $q > 0$ and let $f : \mathbb{C} \to \mathbb{C}$ be a mapping with q-exponentially integrable distortion, normalized by $f(0) = 0$ and $f(1) = 1$. Then for small enough $|z| > 0$*

$$|\arg(f(z))| \leq \frac{c}{q} \log^2\left(\frac{1}{|z|}\right), \tag{2.6}$$

where c is a fixed constant that does not depend on the parameter q or on the mapping f and the branch of the argument is fixed by $\arg(1) = 0$.

Moreover, the bound (2.6) *is optimal, up to the exact value of the constant c.*

The optimality of (2.6) is shown by finding for any given $\epsilon > 0$ a radial mapping h, which satisfies the assumptions of Theorem 2.2, such that

$$|\arg(h(r))| = \frac{1-\epsilon}{2q} \log^2\left(\frac{1}{r}\right),$$

for every $0 < r < \frac{1}{2}$.

The key idea for proving Theorem 2.2 was to develop new methods utilizing the modulus of path families instead of using the methods from [6], which rely on holomorphic motions of quasiconformal mappings. In the proof we fixed a growth rate for pointwise rotation and found suitable path families Γ such that the modulus $M(f(\Gamma))$ on the image side grows proportionally to this rotation, while the weighted modulus $M_{K_f}(\Gamma)$ on the domain side is not affected by it. One can then calculate the exact relation between the growth rate of pointwise rotation and the modulus $M(f(\Gamma))$, and simultaneously estimate the weighted modulus $M_{K_f}(\Gamma)$. Coupling this with a Väisälä type modulus inequality from [30] by Koskela and Onninen, which states that under the assumptions of Theorem 2.2

$$M(f(\Gamma)) \leq M_{K_f}(\Gamma)$$

for any path family Γ, leads to the result.

2.4 Integrable distortion

Finally, in article [C] we generalized Theorem 2.2 further for mappings with p-integrable distortion.

Theorem 2.3 (Theorems 1.3 and 1.4 in [C])**.** *Fix an arbitrary $p \geq 1$ and let $f : \mathbb{C} \to \mathbb{C}$ be a mapping with p-integrable distortion, normalized by $f(0) = 0$ and $f(1) = 1$. Then*

$$|\arg(f(z))| \leq \frac{c_{f,p}}{|z|^{\frac{2}{p}}} \tag{2.7}$$

for small enough $|z| > 0$*, where the constant* $c_{f,p}$ *does not depend on* z *and the branch of the argument is fixed by* $\arg(1) = 0$.

The main idea of the proof is similar to that of Theorem 2.2. So, we again estimate the moduli $M_{K_f}(\Gamma)$ and $M(f(\Gamma))$ for suitable path families Γ and note that assuming stronger pointwise rotation will increase the modulus on the image side but will not affect the modulus on the domain side. This allows us to proceed as with the exponentially integrable distortion, and after some delicate estimates we arrive at (2.7). The key difference to the proof of Theorem 2.2 lies in the fact that we need the Väisälä type modulus inequality in a more general setting than considered by Koskela and Onninen. Thus we rely on the following planar result.

Theorem 2.4 (Theorem 1.1 in [C]). *Let* $\Omega \subset \mathbb{C}$ *be a domain and* $f : \Omega \to \mathbb{C}$ *a mapping with 1-integrable distortion. Then, given any path family* Γ *of paths* $\gamma \subset \Omega$ *we have the inequality*

$$M(f(\Gamma)) \leq M_{K_f}(\Gamma). \tag{2.8}$$

The proof of this theorem is based on the work of Martio, Ryazanov, Srebro and Yakubov in [36], where they proved the modulus inequality (2.8) with the additional assumption that $f \in W^{1,2}_{\text{loc}}(\Omega)$. Their result in turn stems from the modulus theory of quasiconformal mappings, which is beautifully captured in, for example, Väisälä's book [44].

Remark. Theorem 2.4 follows also from the new result by Guo, see [22] chapter 3, that I became aware of only recently.

To relax the Sobolev assumption in Theorem 2.4 we used a deep result by Hencl and Koskela, see [24], which states that the inverse of a mapping with integrable distortion can have better Sobolev regularity than the mapping itself. Moreover, the proof used the result by Greco, Sbordone and Trombetti, see [20], which shows that the distortions of a planar homeomorphic mapping of finite distortion and its inverse are coupled with

$$K_{f^{-1}}(z) = K_f\left(f^{-1}(z)\right).$$

This result is non-trivial as the assumptions on the mapping f are very weak, for example, it does not have to satisfy the Lusin N condition.

Furthermore, we prove that Theorem 2.3 is sharp in the following, very strong, sense.

Theorem 2.5 (Theorem 1.5 in [C]). *Fix $p \geq 1$ and let $h : (0,1] \to (0,\infty)$ be an arbitrary function for which $h(r) \to 0$ when $r \to 0$. Then we can find a mapping $f : \mathbb{C} \to \mathbb{C}$ with p-integrable distortion, normalized by $f(0) = 0$ and $f(1) = 1$, and a sequence of positive radii r_n converging to zero such that*

$$|\arg(f(r_n))| \geq h(r_n) \left(\frac{1}{r_n}\right)^{\frac{2}{p}},$$

for every n. Here we again fix the branch of the argument by $\arg(1) = 0$.

The rotational bounds (2.6) and (2.7) are the correct counterparts for the modulus of continuity results (2.2) and (2.3). Thus the articles [B] and [C] together with the well known stretching bounds provide a comprehensive understanding of pointwise geometric properties of the corresponding mappings.

In conclusion, the transition from quasiconformal mappings to mappings with exponentially integrable distortion does not change the maximal rotation significantly as both (2.4) and (2.6) are logarithmic. On the other hand, the assumption that the distortion is merely integrable increases the growth of maximal rotation (2.7) significantly, from logarithmic to polynomial.

2.5 Applications to stretching

The modulus methods developed in the articles [B] and [C] can also be applied to pointwise stretching, and thus we can give new proofs for many previously known results. In this direction we also obtained the following sharpening for the bound (2.3) by Koskela and Takkinen.

Theorem 2.6 (Theorem 1.6 in [C]). *Let $f : \mathbb{C} \to \mathbb{C}$ be a mapping with 1-integrable distortion, normalized by $f(0) = 0$. Then for any $z \in B\left(0, \frac{1}{2}\right)$ holds*

$$|f(z)| \geq e^{-\frac{c_f(|z|)}{|z|^2}},$$

where $c_f(|z|) \to 0$ as $|z| \to 0$.

So, in this setting the pointwise bound (2.3) can not be achieved. Moreover, this result is sharp in a strong sense.

Theorem 2.7 (Theorem 1.6 in [C]). *Fix an arbitrary $p \geq 1$ and let $h : (0, 1] \to (0, \infty)$ be an arbitrary function for which $h(r) \to 0$ when $r \to 0$. Then we can find a mapping $f : \mathbb{C} \to \mathbb{C}$ with p-integrable distortion and a sequence of positive radii r_n converging to zero such that*

$$|f(r_n) - f(0)| \leq e^{-h(r_n) r_n^{-\frac{2}{p}}}, \tag{2.9}$$

for every n.

The mappings f in (2.9) are constructed iteratively using radial quasiconformal mappings and families of nested annuli whose distortion constant and diameter depend on the given function h. This ensures that the distortion of the limit map f, which is non-trivial only inside these annuli, stays p-integrable even when functions h decay slowly.

3. Multifractal spectra

The pointwise bounds for rotation and stretching, presented in the previous chapter, provide a starting point for the study of the multifractal spectra of mappings of finite distortion. That is, they enable us to study the maximal size of sets in which these mappings can admit a specific stretching, rotation or both of them simultaneously. Finding the sharp bounds for the multifractal spectra gives a deep understanding of the geometrical properties of these mappings. For instance, we will see that the maximal stretching or rotation can only occur in sets of zero Hausdorff dimension.

As an example of a concrete interesting application for the multifractal spectra, we improved the result on compression of sets under mappings with p-integrable distortion by Clop and Herron, see [11].

Research on the multifractal spectra of mappings of finite distortion was initiated by Astala, Iwaniec, Prause and Saksman in [6]. There the authors gave a complete description of the joint rotational and stretching multifractal spectra for quasiconformal mappings, in terms of the Hausdorff dimension of the corresponding sets.

To recall this, fix a stretching parameter $\alpha > 0$ and a rotation parameter $\delta \in \mathbb{R}$, and let $f : \mathbb{C} \to \mathbb{C}$ be an arbitrary K-quasiconformal mapping. Consider then the points $z \in \mathbb{C}$ for which there exists a decreasing sequence of radii $r_n \to 0$, such that

$$\begin{cases} \alpha = \lim_{n\to\infty} \frac{\log|f(z+r_n)-f(z)|}{\log r_n} \\ \delta = \lim_{n\to\infty} \frac{\arg(f(z+r_n)-f(z))}{\log|f(z+r_n)-f(z)|} \end{cases} \tag{3.1}$$

Note, that the rotational limit δ is independent of the choice of the branch of the argument.

The limits (3.1) are naturally constrained by the sharp pointwise bounds (2.1) and (2.4). Indeed, Theorem 3.3 in [6] shows that if there exist radii $r_n \to 0$ such that the limits (3.1) exist, then

$$\alpha(1+i\delta) \in \overline{B}_K,$$

where

$$B_K = \left\{\tau \in \mathbb{C} : \left|\tau - \frac{1}{2}\left(K + \frac{1}{K}\right)\right| < \frac{1}{2}\left(K - \frac{1}{K}\right)\right\}. \tag{3.2}$$

Moreover, Theorem 3.3 in [6] is sharp, and hence it describes accurately how much rotation and stretching can occur simultaneously at a pointwise level for a general K-quasiconformal mapping.

Given any K-quasiconformal mapping $f : \mathbb{C} \to \mathbb{C}$ denote by $E_f = E_{f,\alpha,\delta}$ the set of points that satisfy (3.1). Finding the joint rotational and stretching multifractal spectra for quasiconformal mappings amounts to identifying the maximal size of these sets E_f.

Theorem 3.1 (Theorem 5.1 in [6]). *Let $f : \mathbb{C} \to \mathbb{C}$ be a K-quasiconformal mapping and fix parameters $\alpha > 0$ and $\delta \in \mathbb{R}$ such that $\alpha(1+i\delta) \in \overline{B}_K$. Then*

$$\dim(E_f) \le 1 + \alpha - \frac{K+1}{K-1}\sqrt{(1-\alpha)^2 + \frac{4K}{(K+1)^2}\alpha^2\delta^2}, \tag{3.3}$$

and this bound is sharp. Moreover, if $\alpha(1+i\delta) \notin \overline{B}_K$ the sets E_f are empty, so in this case there are no points z satisfying (3.1) for any K-quasiconformal mapping f.

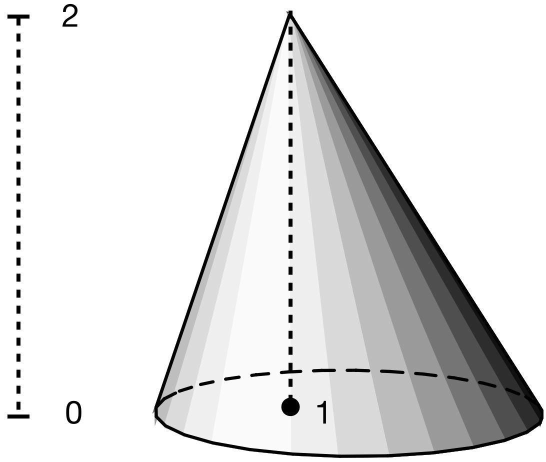

Figure 3.1: Function (3.3) as a cone, picture originally presented in [6].

Remark. As a function of the variable $\alpha(1 + i\delta)$ the function (3.3) is determined as the unique cone-like function which obtains the value 2 at the point 1, vanishes at the boundary of the ball B_K and is linear on every segment connecting the boundary to the point 1. For the illustration, see figure 3.1.

The proof of Theorem 3.1 relies on the complex integrability of the differential f_z, which was also established in [6].

Theorem 3.1 gives a complete description of the multifractal spectra in the sense of the Hausdorff dimension, but more delicate methods are needed to extend this result to the level of the Hausdorff measures. This was the aim of the article [A].

Theorem 3.2 (Theorem 1.2 in [A]). *Let $\alpha > 0$, $\delta \in \mathbb{R}$ and $K \geq 1$ be given such that $\alpha(1 + i\delta) \in \overline{B}_K$. Then there exists a K-quasiconformal mapping*

$f : \mathbb{C} \to \mathbb{C}$ such that

$$H^d(E_f) > 0,$$

where

$$d = 1 + \alpha - \frac{K+1}{K-1}\sqrt{(1-\alpha)^2 + \frac{4K}{(K+1)^2}\alpha^2\delta^2}$$

is the optimal Hausdorff dimension given by Theorem 3.1.

The proof of Theorem 3.2 was influenced by Uriarte-Tuero's work in [42]. In this article he studied area distortion of quasiconformal mappings by constructing examples which distort the size of a non-self-similar Cantor-like set. These sets were constructed by changing drastically the number and sizes of balls used in the construction at every step. We used a similar construction in the proof of Theorem 3.2, carefully adding the right amount of rotation at every step.

Theorems 3.1 and 3.2 raise a natural question of generalizing the results on multifractal spectra for more general classes of mappings, which was the aim of the final article [D].

3.1 Mappings with integrable distortion

The sharp stretching bound (2.3) by Koskela and Takkinen together with the sharp rotational bound (2.7), which was presented in [C], provide a starting point for studying the multifractal spectra of mappings with integrable distortion.

Unfortunately the methods used in [6] rely heavily on quasiconformality and thus do not readily extend to mappings with integrable distortion. Hence we had to develop new methods applying the modulus inequality presented in Theorem 2.4. Here the main obstacle, compared to the pointwise case, was finding the suitable path families to use in the modulus inequality. Indeed, the classical path families used in the pointwise case are not applicable for measuring stretching or rotation simultaneously at multiple points. Thus we have to modify the path families used in the pointwise case and separate them from each other.

The first result in this direction considers the stretching multifractal spectra.

Theorem 3.3 (Theorem 1.1 in [D]). *Fix $b > 0$, $p \geq 1$ and $s \in (0,2)$, and assume that $f : \mathbb{C} \to \mathbb{C}$ is a mapping with p-integrable distortion. Let $A \subset \mathbb{C}$ be the set of points $z \in \mathbb{C}$ for which there exists a sequence of complex numbers $\lambda_{z,n}$, where the moduli $|\lambda_{z,n}| \to 0$ form a decreasing sequence, satisfying*

$$|f(z+\lambda_{z,n}) - f(z)| \leq e^{-b\left(\frac{1}{|\lambda_{z,n}|}\right)^{\frac{2-s}{p}}}, \tag{3.4}$$

for every n. Then $\dim(A) \leq s$.

Thus we see that the pointwise bound (2.3) can indeed hold only in sets with Hausdorff dimension zero. Moreover, according to the inequality (3.4) the stretching exponent $\frac{2-s}{p}$ decays linearly as the dimension s grows.

Article [D] introduced also the rotational multifractal spectra, which can be established in a similar manner as the stretching spectra, but requires more attention to technical details.

Theorem 3.4 (Theorem 1.2 in [D]). *Let $p \geq 1$, $b > 0$ and $s \in (0,2)$ be given and assume that $f : \mathbb{C} \to \mathbb{C}$ is a mapping with p-integrable distortion. Let $A \subset \mathbb{C}$ be the set of points $z \in \mathbb{C}$ for which there exists a branch of the argument and a sequence of complex numbers $\lambda_{z,n}$, with moduli $|\lambda_{z,n}| \to 0$ forming a decreasing sequence and satisfying*

$$|\arg(f(z+\lambda_{z,n}) - f(z))| \geq b\left(\frac{1}{|\lambda_{z,n}|}\right)^{\frac{2-s}{p}}, \tag{3.5}$$

for every n. Then $\dim(A) \leq s$.

Note, that the choice of the branch of the argument in Theorem 3.4 plays very little role, since any change to the branch in (3.5) changes the left hand side by a fixed constant, which is insignificant as the radii $|\lambda_{z,n}|$ converge to zero.

Theorem 3.4 is a complete analog for Theorem 3.3 and shows that the pointwise rotational bound (2.7) can be achieved only in sets with Hausdorff dimension zero. Moreover, the rotational exponent in (3.5) again decays linearly as the dimension s grows.

Finally, we construct examples which show that Theorems 3.3 and 3.4 are optimal.

Theorem 3.5 (Theorem 1.3 in [D]). *Let $p \geq 1$ and $s \in (0,2)$ be given. Then we can find a mapping $f : \mathbb{C} \to \mathbb{C}$ with p-integrable distortion and a set $A \subset \mathbb{C}$, for which* $\dim(A) = s$, *such that for every point $z \in A$ there exists a branch of the argument and a sequence $\lambda_{z,n}$, where $|\lambda_{z,n}| \to 0$, satisfying*

$$|f(z+\lambda_{z,n}) - f(z)| \leq e^{-\left(\frac{1}{|\lambda_{z,n}|}\right)^{\frac{2-s}{p}}} \tag{3.6}$$

and

$$|\arg(f(z+\lambda_{z,n}) - f(z))| \geq \left(\frac{1}{|\lambda_{z,n}|}\right)^{\frac{2-s}{p}}, \tag{3.7}$$

for every n.

In addition to the sharpness of Theorems 3.3 and 3.4, Theorem 3.5 shows that the optimal stretching and rotation can happen simultaneously.

To prove Theorem 3.5 we construct classical self-similar Cantor sets using families of nested annuli. Then we build iteratively a mapping of finite distortion that is radial and quasiconformal inside these annuli and conformal elsewhere, taking special care in keeping the distortion just barely p-integrable. Similar construction have been widely used in literature to produce extremal examples, see, for example, [6], [11] and [42].

3.2 Area distortion

To finish, I want to present an application that highlights the strength of Theorem 3.3 when studying geometric properties of mappings of finite distortion. But first, let us present some background.

The classical question of distortion of the Hausdorff dimension under K-quasiconformal mappings was solved by Astala in [3].

Theorem 3.6 (Corollary 1.3 and Theorem 1.4 in [3]). *Let $f : \mathbb{C} \to \mathbb{C}$ be K-quasiconformal and let $E \subset \mathbb{C}$ be compact. Then*

$$\frac{1}{K}\left(\frac{1}{\dim(E)} - \frac{1}{2}\right) \leq \frac{1}{\dim(f(E))} - \frac{1}{2} \leq K\left(\frac{1}{\dim(E)} - \frac{1}{2}\right), \tag{3.8}$$

and these bounds are optimal in the sense that the equality can occur in either of them.

By the time of the article [3] quasiconformal mappings had already established themselves as an important tool in the theory of planar complex dynamics, but Astala was first to turn this relationship to the other direction and used ideas from dynamical systems and holomorphic motions to study planar quasiconformal mappings.

The inequalities (3.8) were first conjectured by Gehring and Väisälä in [19], where important partial answers to this problem were given. Further developments in the case of quasiconformal mappings, for example in the form of generalizing (3.8) to the level of the Hausdorff measures, were made in [4] and [42].

Theorem 3.6 invites a natural question of finding similar bounds for compression of sets under more general families of mappings, which has been studied, for example, by Zakeri in [46]. My contribution in this direction lies within the theory of mappings with integrable distortion. For these mappings the question of maximal compression was recently considered by Clop and Herron in [11], where they used the pointwise stretching bound (2.3) to estimate compression of small balls and obtained the following result.

Theorem 3.7 (Theorem B in [11]). *Fix $p > 1$ and $s \in (0, 2]$ and assume that $f : \mathbb{C} \to \mathbb{C}$ is a mapping with p-integrable distortion. Let $A \subset \mathbb{C}$ be such that $H^s(A) > 0$. Then*

$$H^h(f(A)) > 0,$$

where the gauge function h is defined by

$$h(t) = \left(\frac{1}{\log\left(\frac{1}{t}\right)} \right)^{\frac{ps}{2}}. \tag{3.9}$$

Moreover, given any $p \geq 1$, $s \in (0, 2)$ and $\epsilon > 0$ Clop and Herron constructed examples of mappings $f : \mathbb{C} \to \mathbb{C}$ with p-integrable distortion that can map a set $A \subset \mathbb{C}$, with $H^s(A) > 0$, to a set satisfying

$$H^{\bar{h}}(f(A)) = 0,$$

where

$$\bar{h}(t) = \left(\frac{1}{\log\left(\frac{1}{t}\right)} \right)^{\frac{ps}{2-(s+\epsilon)}}. \tag{3.10}$$

Since there was a gap left between the gauge functions h and $\bar{h}$, see the exponents in (3.9) and (3.10), it is an interesting question if Theorem 3.7 can be improved.

In article [D] we approached this problem by combining Theorem 3.3 with ideas from [11]. The key point was to note that the sharp bound for the stretching multifractal spectra, provided by Theorem 3.3, gives better control for compression of small balls than the pointwise bound (2.3) used in [11]. This, in turn, leads to a better estimate for the gauge function in (3.9).

Theorem 3.8 (Theorem 1.4 in [D]). *Fix $p \geq 1$ and let $f : \mathbb{C} \to \mathbb{C}$ be a mapping with p-integrable distortion. Choose $s \in (0, 2)$ and $\epsilon > 0$, and define the gauge function*

$$h(t) = \left(\frac{1}{\log\left(\frac{1}{t}\right)} \right)^{\frac{ps}{2-(s-\epsilon)}}. \tag{3.11}$$

Then every $A \subset \mathbb{C}$ with $H^h(f(A)) = 0$ satisfies $H^s(A) = 0$.

Theorem 3.8 together with the examples presented by Clop and Herron show that the gauge function

$$h(t) = \left(\frac{1}{\log\left(\frac{1}{t}\right)} \right)^{\frac{ps}{2-s}}$$

is indeed the critical one when measuring compression of Hausdorff measure under mappings with integrable distortion.

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