

# LEARNING DENSITY OF NUMBERS IN ELEMENTARY TEACHER EDUCATION

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*Infinity is an important concept in mathematics, which students find difficult to learn. This paper will report Finnish elementary teacher students' understanding of density at the beginning of their studies and the development of that understanding during a mathematics methods course. The results show that even quite limited teaching can initiate significant improvement. Moreover, students can make progress even if their initial level of performance is low. Yet, only 60 % of elementary education students reach satisfactory content knowledge and pedagogical content knowledge by the end of the course.*

## INTRODUCTION

Infinity is an important concept in mathematics. We encounter infinity already in counting as it has no endpoint. Such ongoing processes without an end are usually the first examples of infinity for children; such processes are called *potential infinity*. However, the interesting cases in mathematics are, when infinity is conceptualised as a realised 'thing' – the so-called *actual infinity* (Fishbein, 1987; Tsamir and Dreyfus, 2002). The set of all natural numbers is an example of actual infinity. Rational numbers have infinitely many elements between any two rational numbers and therefore no number has a unique successor. This property of rational numbers is called *density*.

In the primary curriculum, infinity is implicitly present in many of the topics, e.g. in arithmetic, when dealing with fractions, or when introducing straight line in geometry. Infinity awakes curiosity in children already before they enter school (Wheeler, 1987). However, this early interest is not often met by school mathematics curriculum, and infinity remains mysterious for most students throughout school years. This is reflected also among students who enter teacher education: only 40 % of them answer correctly to questions about density of numbers (Pehkonen & Hannula, 2006). Could a reason for the low levels of understanding of infinity be that teachers in elementary schools lack sufficient level of mathematical understanding and self confidence required for teaching about infinity?

The purpose of this paper is to find out how well elementary education students learn ideas about density during their mathematics methods course.

## **STUDENTS' CONCEPTIONS OF DENSITY**

Infinity has been a difficult concept for mathematicians. It is no wonder, that also students have had difficulties with it. Previous research has identified typical problems and constructive teaching approaches to cardinality of infinite sets. Students use intuitively the same methods for the comparison of infinite sets as they use for the comparison of finite sets. (Tsamir & Tirosh, 2006; Tsamir & Dreyfus 2002)

Vamvakoussi's (2010) studies indicate that students treat decimal fractions and common fractions as if they were different numbers and they are reluctant to accept that there can be decimals between fractions and vice versa. She also gave an example of a 9<sup>th</sup> grader "who answered that there are 9 numbers between .001 and .01, but stated without hesitation that there are infinitely many numbers between  $\frac{3}{8}$  and  $\frac{5}{8}$ ."

Among 411 German 7th graders, 10 % of the students could correctly name one decimal fraction between  $\frac{1}{2}$  and  $\frac{2}{3}$  and 18 % of them were able to name a common fraction between numbers 0.3 and 0.6. In a task based interview of 28 students only four students realized that there are infinitely many responses to the first task and in the second task there were only two such students. (Neumann, 1998)

Vamvakoussi (2010) suggests that density has two aspects that may not be equivalent from the learner's point of view of, namely the "infinity of intermediates" and the "no successor" aspect. This increasing difficulty of ideas from potential infinity of counting to no successor aspect of density is reflected in empirical results.

Among Finnish 5<sup>th</sup> graders 15 % of students used infinity when responding to a question about the largest number that exists and 6 % knew that there is no largest number. Yet, only 2 % said that there are infinitely many numbers between 0.8 and 1.1, and less than 1 % saw that there is no unique predecessor for number one. Among 7<sup>th</sup> graders the respective figures were 24 %, 13 %, 12 % and 2 % (Hannula, Pehkonen, Maijala & Soro, 2006). In another similar test, 66 % of grade 11 students could handle the infinitely large, and 55 % knew that there are infinitely many intermediates between given two decimal numbers (note that only 60 % of population continue to grade 11) (Pehkonen & Hannula, 2006).

The idea of successor is deeply grounded in students intuitions about numbers (e.g. Merenluoto, 2005). Vamvakoussi (2010, p. 211) concludes that "It is amply documented that the idea of discreteness constraints students' understanding of the density property of rational and real numbers."

## **ELEMENTARY TEACHER STUDENTS CONCEPTIONS OF DENSITY**

It is well recognized that prospective elementary teachers lack conceptual understanding and have misconceptions in several branches of mathematics (e.g. Llinares & Krainer, 2006; Oliveira & Hannula, 2008).

In a Cypriot study, 43 pre-service and in-service elementary education teachers were asked to compare the cardinality of some infinite sets. Approximately half of the respondents were able to give correct responses to such tasks as to compare the cardinality of natural and even numbers. Less than half were able to justify their responses in a valid way. It should be noted that the respondents were participating a mathematics education seminar at the local university, and the results may not be representative to the teacher population in general. (Kattou, Michael, Kontoyianni, Christou & Philippou, 2010)

In UK, 69 teacher trainees were asked to “define a number line”. Most of the definitions implied a finite partitioning, i.e. a discrete perspective to numbers. Only five definitions implied an understanding of infinity and density of number line. (Doritou & Grey, 2010)

At the University of Turku, 70 elementary teacher students’ participated in a study that included a task with unlimited amount of solutions. About one fourth (24 %) found correctly two alternative solutions and recognized that there are even more, but noted cautiously to it with words like “many” or “several”. Almost one third of the students (27 %) had an idea of unbounded, uncountable or infinite number of solutions, but they were not able to reason their answer. Only two students presented a high-level solution where the infinite number of the solutions was based on the relation between the unknown variables. (Merenluoto & Pehkonen, 2002)

## **FOCUS OF THE PAPER**

We want to find out what is the level of elementary education students’ understanding on density of numbers in the beginning and at the end of their mathematics methods course. We will distinguish three levels of students understanding of density. The lowest level is when their response does not deal infinity at all. In the intermediate level, the students understand that there are infinitely many numbers within a given interval. The highest level of understanding is to know that there is no unique successor for any given number.

## **METHODS**

The research is based on data collected for the project “LOMA”, supported by the Academy of Finland (project 8201695). The project was a three-year longitudinal investigation during the academic years 2003-2006.

In the sample of elementary teacher students, we had all participants of their mathematics methods course from three Finnish universities (Helsinki, Turku, and Lapland). In Helsinki there were two groups: normal quota and a supplementary quota. In the universities of Lapland and Helsinki, the methods course was the only compulsory course in mathematics. In the University of Turku, the students had also a problem solving course that covered content areas of mathematics and sciences. The focus of the methods course was on methods for teaching mathematics on the elementary school level. In our study, we measured students’ mathematical beliefs and understanding of some basic mathematical concepts both at the beginning and in

the end of their mathematics methods course. In this study we shall look at elementary education students' learning of density of rational numbers. We got responses from 255 students both at the beginning and end of their mathematics methods course.

The pre-test questionnaire consisted of a belief survey instrument and a skills test containing 12 mathematical tasks. Four tasks measured mathematical understanding and eight tasks measured calculation skills. One of the items measuring student' mathematical understanding was the following density task:

How many numbers are there between numbers 0.4 and 1.1? Explain?

In our analysis of the student responses we identified the following response categories

- Incorrect or no response: Reasoning on integral level (i.e. 1), no answer or other incorrect answers (e.g. 0.3). Alternatively, no answer.
- No infinity: This covers a variety of finite level answers, from 6 to millions.
- Density: Infinitely many, or an unending number, e.g. 9999....

After the methods course, the students responded to a post-test, which consisted of four tasks that measured students' understanding of infinity, division, scale and percentage. Students' understanding of infinity was measured using the following task:

Marika claims that the largest fraction that is still smaller than 3, is  $2\frac{99}{100}$ . Give a reasoned response to her statement.

Both tasks above were formulated in a way that they do not indicate infinity explicitly in the task context. This was a deliberate choice, because we knew the tendency of students to perceive rational numbers as discrete. It should be noted that there are some important differences between the pre-test and post-test tasks. In the pre-test, the task addresses the *infinity of intermediates* aspect of density, while the post-test task addresses the *no successor* aspect. In the pre-test, the rational numbers are presented in decimal form, in the post-test as fraction. Students often perceive decimals and fractions as different types of numbers. Lastly, while the pre-test measured mathematical content knowledge, the post-test is framed as a pedagogical content question. The elementary education students are familiar with this task type. The name contextualizes the task into a teaching context, where Marika is a pupil and the respondent should give a response as her teacher. Therefore, we assumed that the students would give more thorough explanations in the post-test. Specifically, a good response is expected to deal also the *no successor* aspect of rational numbers.

The responses were categorized according to how they indicate an understanding of density:

*Incorrect or no response:* The respondent has either accepted the statement as correct, or the reasoning is incorrect or incomprehensible. Alternatively, student gave no response.

*No infinity*: The respondent has given one example of a number between  $2\frac{99}{100}$  and 3, but there is no further argumentation related to the infinite number of intermediates.

*Density*: The respondent states that there are infinitely many intermediate numbers and gives a reasonable argument. Sample response: “No, because one can find infinitely many numbers larger than  $2\frac{99}{100}$  before 3. One can write infinitely many times the number 9 after the decimal point.”

*No successor*: The respondent states that there is no number, or it can not be determined. In addition a reasonable argument is given. Sample response: “The number cannot be determined, because between 3 and  $2\frac{99}{100}$  there fits infinite number of numbers.”

First analysis was made on the level of understanding indicated by the frequencies of different responses, i.e. if any learning took place during the course.

Secondly, a GLM univariate test was used to determine which factors and covariates could explain the student’s level of performance in the post-test, i.e. possible explanatory factors for the learning that took place. A full factorial model included two infinity tasks of the first test, the arguments provided for the infinity tasks, gender and university of the respondent. The lack of fit statistics were calculated for custom models (each including a different subset of the variables in the full factorial model) in order to determine which variables to include in the model. We will present below results only from the model with best fit.

Thirdly, it was analysed how the students’ knowledge in the beginning influenced their learning. Would there be clear disadvantage among those who had weaker results in the beginning, or could there be radical improvement?

Finally, it was analysed how the different universities differed in the learning that took place.

## **RESULTS**

The scoring scales of the density tasks are not completely compatible. Most notably, the first task did not measure the *no successor* aspect of density. However, results indicate a clear decrease of students whose reasoning remains on a finite level (Figure 1). Despite this clear development, it is disheartening that immediately after mathematics methods courses, 40 % of the future teachers did not use infinity in their argumentation in a situation that clearly calls for it.

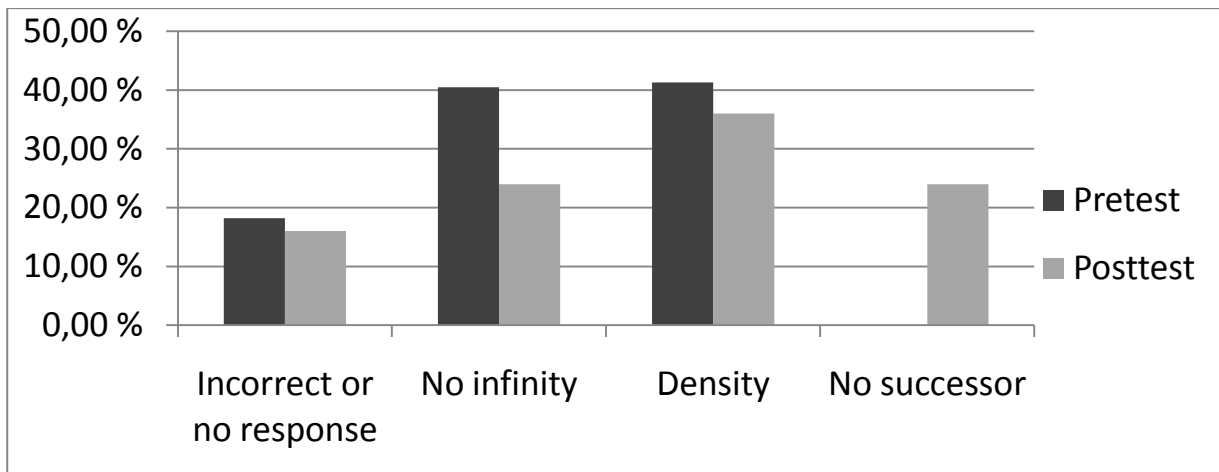


Figure 1. Elementary education students' results in a task measuring understanding of density. Note! In the pre-test "no successor" was not measured.

In a GLM univariate analysis the best fit was received when the model included the pre-test density score, density explanation score and the university. The lack of fit statistics for the model were ( $F= 1.791$ ,  $df =28$ ,  $p=.012$ ,  $\text{partial } \eta^2= .208$ ). In this model, the largest effect size was for university, which was more powerful predictor for post-test performance than pre-test performance (Table 1).

Source	df	F	Sig.	Partial Eta <sup>2</sup>
Density	1	17.443	.000	.074
Density explanation	1	10.408	.001	.045
University	3	20.618	.000	.220

Table 1. The results of the GLM Univariate test with best fit, indicating variables that predict post-test results for density and their effect sizes.

When the relationships between responses to the density tasks in the pre-test and the post-test were analysed, it was observed that those who had indicated understanding of density in the pre-test, tended to give more density and no successor answers in the post-test than others (Figure 2). However, among those who indicated no sense of density in the pre-test, improvement in the post-test was almost as likely as lack of it.

A crosstabulation analysis was used to reveal how the density tasks in the beginning and end were related in each university. The test revealed that the level of density understanding in the beginning of studies was a statistically significant predictor in all universities except in University of Lapland.

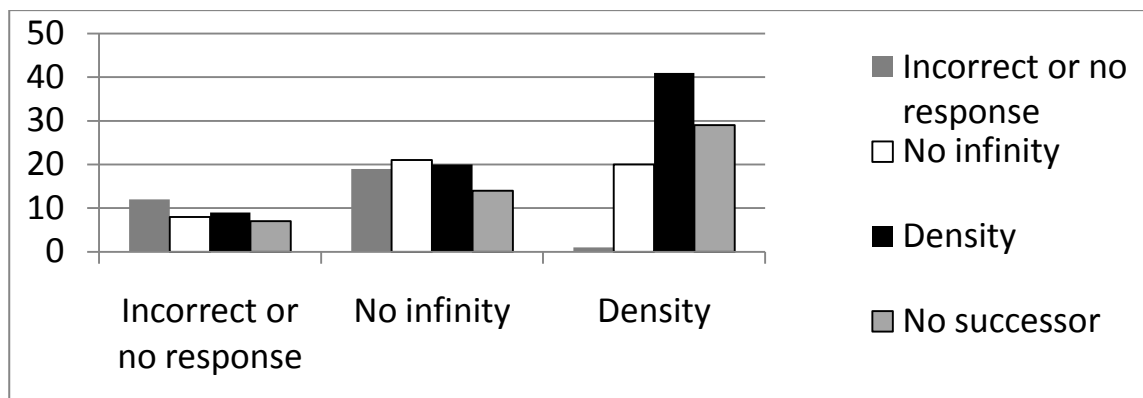


Figure 2. Post-test result frequencies among students of different pre-test results.

The pre- and post-test results from all universities indicate clear differences between their results (Figure 3). The space does not allow a detailed analysis of these differences, but we point out the high performance of University of Turku students in the pre-test and the progress made in university of Lapland from pre- to post-test. Moreover, in Helsinki, the normal quota students made practically no progress.

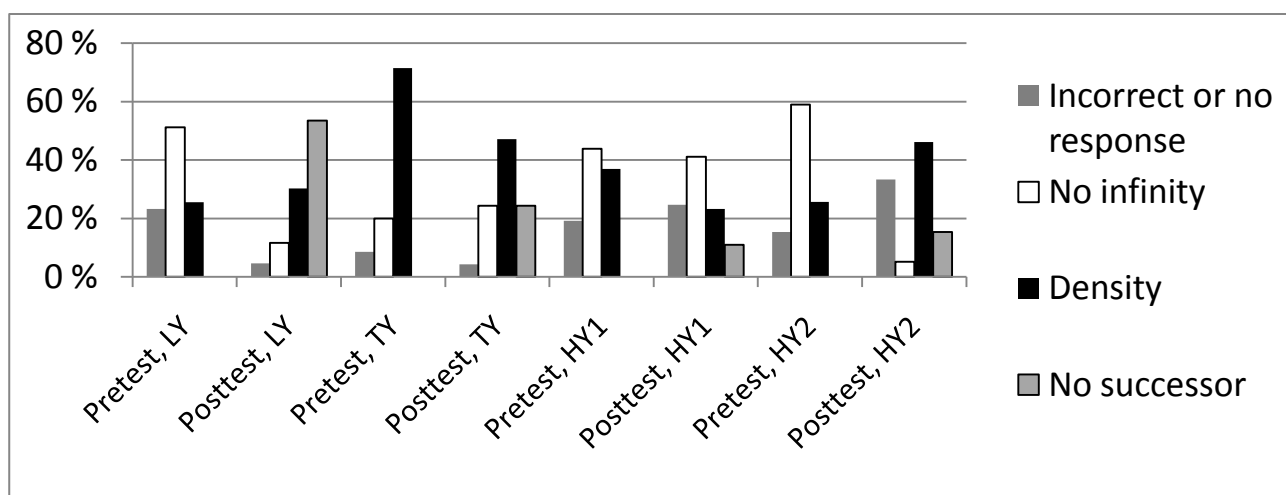


Figure 3. Pre- and post-test results of all four groups of the study.

## CONCLUSIONS AND DISCUSSION

Infinity is one of the mathematical ‘Fundamental Ideas’ that need to be introduced to children early on (Schweiger, 1992). It is essential that elementary education teachers are familiar with ideas of infinity and that they are ready to discuss these ideas in the class. Teachers are likely to encounter these issues as questions from their students. They should understand at least basic ideas of potential infinity, including the *infinitely many intermediate numbers* aspect of the density of rational numbers. Unfortunately, this seems not to be the case. In the post-test of our study, 40 % of elementary education students gave an unsatisfactory response to a fictional teaching situation considering density of rational numbers. That is better than before the mathematics methods courses, but still not satisfactory.

We found a surprisingly strong influence of the place of study, which calls for closer analysis. First of all, there was no major difference in the scope of the methods course. All covered the same basic content with some variation and infinity and density were marginal topics. It should be noted that the post-test conditions were not the same for all groups. In Lapland and in the University of Helsinki supplementary quota, the test was part of mathematics methods exam. In Helsinki normal quota and in Turku, the test was an exercise. This might have had an effect on students in exam situation putting more effort and giving more thorough responses. However, the development of students understanding of division did not follow the same pattern although it was measured in the same test (Laine, Huhtala, Hannula, Kaasila & Pehkonen, forthcoming).

Teacher education programs in Finland are selective (approximately 10 % of applicants are accepted). Mathematics has not been a selection criterion, except in University of Turku. In Turku, the teacher education program used a special test on mathematical and scientific thinking, which selected students with better mathematics skills than in other programs in our study (Kaasila *et al.* 2008a). In Turku, the pre-test results were better than in the other universities. However, in Turku there was hardly any advancement in the students' understanding of density during the studies. In the post-test the results in Turku were only marginally better than in Helsinki.

In the University of Lapland the progress was strikingly better than in other places. In order to explore the effect of teaching, we contacted all the instructors and they explored their teaching materials to see how they had handled infinity in programs. It became evident, that in Lapland the instructor had emphasised the density aspect slightly more than the instructors in Helsinki. Ironically, his teaching material on infinity was based mainly on the research of the instructor in Turku. On the other hand, University of Helsinki normal quota did not explicitly address density at all and there was practically no development in students' understanding of density either.

Oliveira and Hannula (2008) found in a literature review, that elementary education students' learning of content knowledge is often intertwined with the changing beliefs about the nature of mathematics. Instructors in Lapland and Helsinki supplementary quota acknowledge mathematics anxiety of many of elementary education students and address it explicitly from the beginning of the course. Could this "therapeutic" approach explain better progress in these groups? Sadly, we cannot separate the effects of instruction and testing condition mentioned earlier.

The results of Lapland are very encouraging. They show that it is possible to teach these difficult issues about density even to students who lack foundation at the beginning of their studies. The next stage would be to address the problems of teaching density as part of the mathematics methods course. Vamvakoussi, Katsigiannis and Vosniadou (2009) have received encouraging results using infinitely extendable rubber line as an analogy to rational number line.



## References

- Doritou, M., & Gray, E. (2010). Teachers' subject knowledge: The number line representation. In V. Durand-Guerrier, S. Soury-Lavergne & F. Arzarello (Eds.) *Proceedings of CERME 6, January 28th-February 1st 2009, Lyon France*, pp. 1734-1743. INRP 2010 <downloaded 15.6.2011: [www.inrp.fr/editions/cerme6](http://www.inrp.fr/editions/cerme6)>
- Fischbein, E. (1987). *Intuition in Science and Mathematics*. Dordrecht, The Netherlands: Reidel.
- Hannula, M. S., Pehkonen, E., Maijala, H., & Soro, R. (2006). Levels of students' understanding of infinity. *Teaching Mathematics and Computer Science* 4/2, 317 - 337.
- Kaasila, R., Hannula, M. S., Laine, A. & Pehkonen, E. (2008b). Socio-emotional orientations and teacher change. *Educational Studies in Mathematics* 67, 111-123.
- Kaasila, R., Hannula, M.S., Laine, A. & Pehkonen, E. (2008a). Evaluating admission procedures for teacher education in Finland. *Teaching Mathematics and Computer Science* 6/1, 231-243.
- Kattou M., Michael T., Kontoyianni K., Christou C., & Philippou G. (2010). Teachers' perceptions about infinity: a process or an object? In V. Durand-Guerrier, S. Soury-Lavergne & F. Arzarello (Eds.) *Proceedings of CERME 6, January 28th-February 1st 2009, Lyon France*, pp. 1771-1780. INRP 2010 <downloaded 15.6.2011: [www.inrp.fr/editions/cerme6](http://www.inrp.fr/editions/cerme6)>
- Laine, A., Huhtala, S., Hannula, M. S., Kaasila, R. & Pehkonen, E. (Forthcoming). Luokanopettajaopiskelijoiden jakolaskun ymmärtämisestä. Accepted to be published in the proceedings of Ainedidaktiikan symposium, University of Helsinki, February 2<sup>nd</sup>, 2010.
- Llinares, S., & Krainer, K. (2006). Mathematics (student) teachers and teacher educators as learners. In A. Gutierrez & P. Boero (Eds.), *Handbook of research on the psychology of mathematics education: Past, present and future* (pp. 429-459). Rotterdam, The Netherlands: Sense.
- Merenluoto, K. (2005). Discussion about conceptual change in mathematics. *Nordic Studies in Mathematics Education* 10 (2), 17–33.
- Merenluoto, K. & Pehkonen, E. 2002. Elementary teacher students' mathematical understanding explained via conceptual change. In: *Proceedings of the PME-NA XXIV* (ed. D. Mewborne, P. Sztajn, D.Y. White, H.G. Wiegel, R.L. Bryant & K. Nooney), 1936–1939. Columbus (OH): ERIC.
- Neumann, R. (1998). Students' ideas on the density of fractions. In: *The Annual Conference on Didactics of Mathematics*, Munich, p. 97–104. <http://webdoc.sub.gwdg.de/ebook/e/gdm/1998/neumann3.pdf>
- Oliveira, H. & Hannula, M. S. (2008). Individual prospective mathematics teachers; Studies on their professional growth. In K. Krainer & T. Wood (Eds.) *Participants in Mathematics Teacher Education*, pp. 13-34. Rotterdam, The Netherlands: Sense.
- Pehkonen, E. & Hannula, M. S. (2006). Infinity of numbers: a complex concept to be learnt? In S. Alatorre, J. L. Cortina, M. Sáiz & A. Méndez (Eds.), *Proceedings of PME-*

*NA in Merida, Mexico Vol. 2*, 152–154. Merida, Mexico: Universidad Pedagógica Nacional.

Schweiger, F. (1992). Fundamentale Ideen. Eine geistesgeschichtliche Studie zur Mathematikdidaktik. *Journal für Mathematik-Didaktik* 13, 199 – 214.

Tsamir, P. & Tirosh, D. (2006). PME 1 to 30 – Summing up and looking ahead: A personal perspective on infinite sets. In J. Novotna, H. Moraová, M. Krátká & N. Stehliková (Eds.), *Proceedings of PME-30 in Prague Vol. 1*, pp. 49–63. Prague: Charles University.

Tsamir, P. & Dreyfus, T. (2002). Comparing infinite sets – a process of abstraction. The case of Ben. *The Journal of Mathematical Behavior* 21 (1), 1–23.

Vamvakoussi, X. (2010). The “numbers are points on the line” analogy: Does it have an instructional value? In L. Verschaffel, E. De Corte, T. de Jong, & J. Elen (Eds.), *Use of external representations in reasoning and problem solving: Analysis and improvement*, pp. 209-224. New York: Routledge.

Vamvakoussi, X., Katsigiannis, K., & Vosniadou, S. (2009). Bridging the gap between discreteness and density. In M. Tzekaki, M. Kaldrimidou, & C. Sakonidis (Eds.), *Proceedings of the 33<sup>rd</sup> Conference of the International Group for the Psychology of Mathematics Education* (Vol. 5, pp. 273-280). Thessaloniki, Greece: PME

Wheeler, M.M. (1987). Children’s understanding of zero and infinity. *Arithmetic teacher*, Nov. 1987, 42–44.