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Bonnesen's inequality for s-John domains in $\mathbb{R}^n$

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Tiivistelmä — Referat — Abstract

Tämän tutkielman tavoitteena on esitellä isoperimetrinen epäyhtälö ja joitakin kvantitatiivisia isoperimetrisiä epäyhtälöitä. Tutkielma koostuu kahdesta osasta. Ensimmäinen osa on yleiskatsaus isoperimetriseen epäyhtälöön avaruudessa $\mathbb{R}^n$ ja joihinkin tunnettuihin kvantitatiivisiin isoperimetrisiin epäyhtälöihin. Ensimmäisessä luvussa esitellään isoperimetrinen epäyhtälö ja joitakin todistusmenetelmiä sekä tasossa että korkeammissa ulottuvuuksissa. Toisessa luvussa tutustutaan joihinkin tunnettuihin kvantitatiivisiin isoperimetrisiin epäyhtälöihin sekä niiden todistuksiin.

Tutkielman toinen osa on tätä tutkielmaa varten kirjoitettu artikkeli, jossa todistetaan Bonnesenin epäyhtälö niin kutsutuille s-John alueille, $s > 1$, avaruudessa $\mathbb{R}^n$. Artikkelissa osoitetaan, että menetelmät, joita on kirjallisuudessa sovellettu John alueille voidaan käyttää myös s-John alueiden tapauksessa. Tulos on uusi ja se antaa perheen Bonnesenin epäyhtälöitä, jotka riippuvat parametrista $s > 1$.

The goal of this thesis is to introduce the isoperimetric inequality and various quantitative isoperimetric inequalities. The thesis has two parts. The first one is an overview of the isoperimetric inequality in $\mathbb{R}^n$ and some of the known quantitative isoperimetric inequalities. In the first chapter we introduce the isoperimetric inequality and show some possible methods of proving the isoperimetric inequality in $\mathbb{R}^n$ for both $n = 2$ and $n \geq 3$. In the second chapter we discuss some known quantitative isoperimetric inequalities as well as their proofs.

The second part of the thesis is a paper. In this paper we prove a Bonnnesen type inequality for so called s-John domains, $s > 1$, in $\mathbb{R}^n$. We show that the methods that have been applied to John domains in the literature, suitably modified, can be applied to s-John domains. Our result is new and gives a family of Bonnesen type inequalities depending on the parameter $s > 1$.

Avainsanat — Nyckelord — Keywords

Isoperimetrinen epäyhtälö, Bonnesenin epäyhtälö, Geometrinen mittateoria

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Contents

Chapter 1

Isoperimetric inequality

The classical isoperimetric inequality in $\mathbb{R}^n$ says roughly that, among the sets with given volume, the ball has the smallest perimeter. In this chapter we give a precise mathematical statement of the isoperimetric inequality and discuss its proofs. The planar case of the isoperimetric inequality was already known to Archimedes, see [20]. However, the rigorous proofs of the isoperimetric inequality in the plane started to appear in the beginning of the 20th century. The first mathematicians working on this problem include for example Sturm, Schwarz, Steiner, Weierstrass, Hurwitz, Minkowski and many others. We present the proof of the planar isoperimetric inequality in Section 1.1. Rigorous proofs for the isoperimetric inequality in higher dimensions are based on geometric measure theory and calculus of variations. The general form of the isoperimetric inequality in arbitrary dimension was proven by De Giorgi in [4]. We give the precise statement of De Giorgi's result and present the proof in Section 1.2.

For any Lebesgue measurable set $E \subset \mathbb{R}^n$, the n-dimensional Lebesgue measure of E is denoted $|E|$. If $E \subset \mathbb{R}^n$ is polyhedral or a set with smooth boundary, then the natural notion of perimeter is the $(n-1)$-dimensional Hausdorff measure of the topological boundary of E, denoted $\mathcal{H}^{n-1}(\partial E)$. We consider some notion of perimeter, denoted $p(E)$, which extends the natural perimeter, that is, $p(E) = \mathcal{H}^{n-1}(\partial E)$ if E has smooth boundary or if E is polygonal. In general, isoperimetric inequality refers to an inequality of the form

$$n\omega_n^{1/n}|E|^{(n-1)/n} \leq p(E). \tag{1.1}$$

Different choices of p can be considered. See Sections 1.2 and 1.3 for two different extensions of the perimeter.

1.1 Isoperimetric inequality in the plane

Let us first deal with the planar case of the isoperimetric inequality in more detail. We need to choose some natural concept of perimeter and the family of sets we consider to state the problem precisely. In the planar case it is natural to consider domains bounded by rectifiable Jordan curves. Recall that a curve $\sigma : [0,1] \to \mathbb{R}^2$ is a Jordan curve if it has no self intersections and it is closed (i.e. $\sigma(0) = \sigma(1)$). Curve σ is rectifiable if

$$\sup \sum_{k=1}^{m} |\sigma(a_k) - \sigma(a_{k-1})| < \infty, \tag{1.2}$$

where the supremum is taken over all partitions $0 = a_0 < a_1 < \cdots < a_m = 1$. If σ is a rectifiable curve then the length of σ, denoted $\ell(\sigma)$, is defined to be the quantity in (1.2). The precise statement of the classical isoperimetric inequality in the plane is the following.

Theorem 1.1. *Let $\Omega \subset \mathbb{R}^2$ be a domain bounded by a rectifiable Jordan curve. Then*

$$\ell(\partial\Omega)^2 \geq 4\pi|\Omega| \tag{1.3}$$

and the equality holds if and only if Ω is a disc.

There are many ways to prove the isoperimetric inequality in the plane. These include symmetrization, calculus of variations, Minkowski sums of sets, trigonometric series and integral geometry (for more exhaustive list see [21]). Symmetrization methods are demonstrated in the proof of the isoperimetric inequality in higher dimensions (see Section 1.2). The approach using Minkowski sums leads to the Brunn–Minkowski inequality that implies the isoperimetric inequality also in the higher dimensions (see Section 1.3 for a brief discussion). Calculus of variations was used by Schwarz to obtain the first rigorous proof of the isoperimetric inequality in the plane.

Next we present a proof of Theorem 1.1 by Hurwitz in [1] and [2], see also [20]. The key result which we will need is the following Wirtinger inequality. This inequality can be proven for example by Parseval's theorem for Fourier series. This proof is given below.

Lemma 1.2. *Let $u : [0, 2\pi] \to \mathbb{R}$ be absolutely continuous. If u' is square integrable, $u(0) = u(2\pi)$ and*

$$\int_0^{2\pi} u(t)\ \mathrm{d}t = 0, \tag{1.4}$$

then

$$\int_0^{2\pi} u(t)^2\ \mathrm{d}t \leq \int_0^{2\pi} u'(t)^2\ \mathrm{d}t. \tag{1.5}$$

Moreover, the inequality above is an equality, if and only if

$$u(t) = a\cos(t) + b\sin(t) \tag{1.6}$$

for some $a, b \in \mathbb{R}$ and all $t \in [0, 2\pi]$.

Proof. Let u be as in the statement of the lemma. We use the Fourier expansion of u. By Parseval's identity we have

$$\int_0^{2\pi} u(t)^2 \,\mathrm{d}t = 2\pi \sum_{k=-\infty}^{\infty} |a_k|^2, \tag{1.7}$$

where a_k is the kth Fourier coefficient of u, that is,

$$a_k = \frac{1}{2\pi} \int_0^{2\pi} u(t) e^{-ikt} \,\mathrm{d}t.$$

Similarly, we use Parseval's identity for u' to obtain that

$$\int_0^{2\pi} u'(t)^2 \,\mathrm{d}t = 2\pi \sum_{k=-\infty}^{\infty} |b_k|^2, \tag{1.8}$$

where b_k is given by

$$b_k = \frac{1}{2\pi} \int_0^{2\pi} u'(t) e^{-ikt} \,\mathrm{d}t.$$

Since u is absolutely continuous and periodic, integration by parts yields

$$b_k = -\frac{1}{2\pi} \int_0^{2\pi} u(t)(-ik) e^{-ikt} \,\mathrm{d}t = ika_k \tag{1.9}$$

for each $k \in \mathbb{Z}$. By the assumption (1.4) we have

$$a_0 = \frac{1}{2\pi} \int_0^{2\pi} u(t) \,\mathrm{d}t = 0. \tag{1.10}$$

By (1.7), (1.8), (1.9) and (1.10), we arrive at

$$\begin{aligned}
\int_0^{2\pi} u(t)^2 \,\mathrm{d}t &= 2\pi \sum_{k=1}^{\infty} \left(|a_{-k}|^2 + |a_k|^2\right) \\
&\leq 2\pi \sum_{k=1}^{\infty} \left((-k)^2 |a_{-k}|^2 + k^2 |a_k|^2\right) \\
&= \int_0^{2\pi} u'(t)^2 \,\mathrm{d}t.
\end{aligned}$$

Furthermore, the equality occurs if and only if $a_k = 0$ for all $|k| > 1$. Using the Fourier series representation for u, we see that equality occurs if and only if

$$u(t) = a\cos(t) + b\sin(t),$$

for some $a, b \in \mathbb{R}$ and all $t \in [0, 2\pi]$. □

The isoperimetric theorem in the plane, Theorem 1.1, follows from Lemma 1.2.

Proof of Theorem 1.1. Let $\delta\Omega$ be $\sigma : [0, L] \to \mathbb{R}^2$, parametrized by arclength. Thus

$$|\sigma'(t)| = 1 \tag{1.11}$$

for almost every $t \in [0, L]$. Let $(x_1, x_2) \in \mathbb{R}^2$ be the center of mass of $\partial\Omega$, that is,

$$(x_1, x_2) = \int_0^L \sigma(t)\ \mathrm{d}t. \tag{1.12}$$

By Green's formula, we have

$$|\Omega| = \int_\Omega \mathrm{d}x = \frac{1}{2}\Big|\int_0^L (x_2 - \sigma_2(t), \sigma_1(t) - x_1) \cdot \sigma'(t)\ \mathrm{d}t\Big|.$$

By the Cauchy–Schwarz inequality and (1.11), we have that

$$\begin{aligned} |\Omega| &\leq \frac{1}{2}\Big(\int_0^L (x_2 - \sigma_2(t))^2 + (\sigma_1(t) - x_1)^2\ \mathrm{d}t\Big)^{1/2}\Big(\int_0^L |\sigma'(t)|^2\ \mathrm{d}t\Big)^{1/2} \\ &= \frac{\sqrt{L}}{2}\Big(\int_0^L (x_2 - \sigma_2(t))^2 + (\sigma_1(t) - x_1)^2\ \mathrm{d}t\Big)^{1/2}. \end{aligned} \tag{1.13}$$

Because σ is closed, we have $\sigma_1(0) = \sigma_1(L)$ and $\sigma_2(0) = \sigma_2(L)$. Furthermore, by (1.12) we have that

$$\int_0^L \sigma_1(t) - x_1\ \mathrm{d}t = 0 \quad \text{and} \quad \int_0^L \sigma_2(t) - x_2\ \mathrm{d}t = 0.$$

Hence the assumptions of the Wirtinger inequality are satisfied. Wirtinger inequality, Lemma 1.2, gives

$$\int_0^L (\sigma_i(t) - x_i)^2\ \mathrm{d}t \leq \Big(\frac{L}{2\pi}\Big)^2 \int_0^L |\sigma_i'(t)|^2 dt$$

for $i = 1, 2$. Inserting this estimate into the inequality (1.13) and recalling (1.11), we obtain

$$|\Omega| \leq \frac{\sqrt{L}}{2}\frac{L}{2\pi}\Big(\int_0^L |\sigma'(t)|^2\ \mathrm{d}t\Big)^{1/2} = \frac{L^2}{4\pi}. \tag{1.14}$$

Thus (1.3) is proved. Furthermore, we have equality in (1.14) if and only if

$$\sigma_i(t) - x_i = a_i \cos((2\pi/L)t) + b_i \sin((2\pi/L)t) \tag{1.15}$$

for some constants $a_i, b_i \in \mathbb{R}$ and $i = 1, 2$. Because $|\sigma'(t)| \equiv 1$ and σ is closed, equation (1.15) gives

$$\sigma(t) = (x_1, x_2) + \frac{L}{2\pi}(\cos(2\pi t/L + \theta), \sin(2\pi t/L + \theta))$$

for some $\theta \in [0, 2\pi)$ and all $t \in [0, L]$. This means that σ is a circle. Hence (1.14) is an equality if and only if Ω is a disc. $\square$

1.2 Isoperimetric inequality in higher dimensions

Next we discuss the isoperimetric inequality in higher dimensions. De Giorgi [4] (see also [3]) proved the isoperimetric inequality for the sets of finite perimeter. We start by giving a brief introduction to sets of finite perimeter and then we give the main steps of the proof by De Giorgi. For an alternative extension of the perimeter and a different approach to the problem see Section 1.3.

Let E be a Borel set. For any domain $\Omega \subset \mathbb{R}^n$ the perimeter of E inside Ω is defined as

$$P(E,\Omega) = \sup\Big\{ \int_E \operatorname{div}\varphi \, \mathrm{d}\mathcal{H}^n : \varphi \in C_0^1(\Omega;\mathbb{R}^n), \ \|\varphi\|_\infty \le 1 \Big\}.$$

The perimeter in the whole space is denoted by $P(E)$. The perimeter defined this way has many desirable properties. It agrees with the $(n-1)$-dimensional Hausdorff measure for polygonal sets and sets with smooth boundary, so it is an extension of the natural concept of perimeter.

We define a (pseudo)metric d on Borel sets by setting $d(E,F) = |E \triangle F|$, where $\triangle$ denotes the symmetric difference, for any Borel sets $E, F \subset \mathbb{R}^n$. (Actually, d is just a pseudometric as $d(E,F) = 0$ for any Borel sets $E, F \subset \mathbb{R}^n$ that are the same up to a set of measure zero.) One of the most important properties of the perimeter, as defined by De Giorgi, is the lower semicontinuity with respect to this metric.

Theorem 1.3. *Let $(E_k)_{k=1}^\infty$ be a sequence of Borel sets converging to a Borel set $E \subset \mathbb{R}^n$ with respect to the metric d. Then*

$$\liminf_{k\to\infty} P(E_k) \ge P(E).$$

The family of sets of finite perimeter has the following useful compactness property.

Theorem 1.4. *Let $\Omega \subset \mathbb{R}^n$ be a bounded Borel set and let $(E_k)_{k=1}^\infty$ be a sequence of Borel subsets of Ω. If there exists $M > 0$, such that $P(E_k) \ge M$ for all $k \in \mathbb{N}$, then the sequence $(E_k)_{k=1}^\infty$ has a subsequence that converges with respect to the metric d.*

De Giorgi proved the following isoperimetric inequality. We denote by ω_n the n-dimensional Lebesgue measure of the unit ball in $\mathbb{R}^n$. The perimeter of a unit ball is $n\omega_n$.

Theorem 1.5. *Let $E \subset \mathbb{R}^n$ be a Borel set and $|E| < \infty$. Then*

$$n\omega_n^{1/n}|E|^{(n-1)/n} \le P(E), \tag{1.16}$$

and the equality occurs if and only if E is, up to a set of measure zero, a ball.

Remark 1.6. Clearly (1.16) is equivalent to

$$P(B) \le P(E),$$

where B is a ball such that $|B| = |E|$.

To prove the isoperimetric theorem, we show that among the class of all sets of finite perimeter having the given volume there exists a set minimizing the perimeter. Moreover, the minimizer is a ball. Theorems 1.3 and 1.4 prove the existence of a minimizer. We also need the following theorem.

Theorem 1.7. *Let E be a set of finite perimeter. Then there exists a sequence $(E_k)_{k=1}^{\infty}$ of polygonal domains converging to E, such that*

$$P(E) = \lim_{k\to\infty} P(E_k).$$

The above theorem shows that it is enough to deal with polygonal domains. The benefit of dealing with poygonal domains is that they are necessarily bounded which allows us to apply Theorem 1.4. These results allow us to deal with the existence part of the isoperimetric theorem.

The proof that the minimizer is a ball is based on symmetrization methods. We introduce the concept of Steiner symmetrization which is the basis of De Giorgi's proof.

Definition 1.8. Let $E \subset \mathbb{R}^n$ be a Borel set. Define $l(\tilde{x}) = \mathcal{H}^1(E \cap (\{\tilde{x}\} \times \mathbb{R}))$, for any $\tilde{x} \in \mathbb{R}^{n-1}$. The Steiner symmetral E^* of E with respect to the hyperplane $\{(\tilde{x}, x_n) \in \mathbb{R}^n : x_n = 0\}$ is defined as

$$E^* = \{(\tilde{x}, x_n) \in \mathbb{R}^{n-1} \times \mathbb{R} : |x_n| < l(\tilde{x})/2\}.$$

Steiner symmetrization has a couple of obvious but important properties. First, by Fubini's theorem it is easy to see that E^* is Borel and has the same measure as E. Second, the set E^* is symmetric with respect to the hyperplane. Moreover, one of the most important properties of Steiner symmetrization is that it does not increase the perimeter of the set. De Giorgi's proof is based on the following Steiner's inequality.

Theorem 1.9. *Let $E \subset \mathbb{R}^n$ be a Borel set and E^* its Steiner symmetrization with respect to the hyperplane $\{(\tilde{x}, x_n) \in \mathbb{R}^n : x_n = 0\}$. Then*

$$P(E) \geq P(E^*).$$

If $P(E) = P(E^*)$ then $E \cap (\{\tilde{x}\} \times \mathbb{R})$ is, up to a set of $\mathcal{H}^1$-measure zero, a line segment for $\mathcal{H}^{n-1}$-a.e. $\tilde{x} \in \mathbb{R}^{n-1}$. In addition, if E is convex and $P(E) = P(E^*)$, then there exists $\nu \in \mathbb{R}^n$, orthogonal to the hyperplane, such that $E = E^* + \nu$.

Theorem 1.9 is technically the most difficult part of the proof of the isoperimetric inequality. We refer to [11] for the proof of Theorem 1.9. Now we are ready to give the proof of the isoperimetric theorem. As mentioned before, we use essentially the same method as De Giorgi in [4].

Proof of Theorem 1.5. Let $E \subset \mathbb{R}^n$ be a polygonal domain with finite measure. Then E is bounded, that is, there exists $R > 0$ such that $E \subset B(0, R)$. Consider the minimization problem

$$m = \inf \Big\{ P(F) : F \subset B(0,R),\ P(F) \le P(E),\ |F| = |E| \Big\} \tag{1.17}$$

among Borel sets $F \subset \mathbb{R}^n$. We choose a minimizing sequence. That is, we choose a sequence $(F_k)_{k=1}^{\infty}$ of admissible sets, such that the perimeters converge to the infimum (1.17). By Theorem 1.4 such a minimizing sequence has a converging subsequence and the limit set $F \subset \mathbb{R}^n$ is a set of finite perimeter. By the lower semicontinuity of the perimeter, Theorem 1.3, we conclude that

$$P(F) \le \liminf_{k\to\infty} P(F_k) = m \le P(E).$$

Next we show that F is a ball. If F^* is the Steiner symmetral of F with respect to any hyperplane then $F^* \subset B(0, R)$ and $|F^*| = |E|$. Recalling Theorem 1.9 and the fact that F is a minimizer, we see that $P(F^*) = P(F)$. We can use the second result of Theorem 1.9 to conclude that F can be modified in a set of measure zero to obtain a convex set. Indeed, for any two points $x, y \in F$ choose the hyperplane orthogonal to $x - y$. Then we know that the line segment connecting x and y lies in F. Thus F is convex. Now the last conclusion of Theorem 1.9 shows that F is a translation of its Steiner symmetrization with respect to any hyperplane. In particular, the set F is symmetric with respect to any hyperplane going through the center of mass of F. We conclude that F is ball. Thus we have

$$P(E) \ge P(F) = n\omega_n^{1/n}|E|^{(n-1)/n}.$$

Next we consider the general case. Let $E \subset \mathbb{R}^n$ be a set of finite perimeter. By Theorem 1.7 we can find a sequence $(E_k)_{k=1}^{\infty}$ of polyhedral approximations to E. Applying the result obtained above to these polygonal domains we have

$$P(E) = \lim_{k\to\infty} P(E_k) \ge \lim_{k\to\infty} n\omega_n^{1/n}|E_k|^{(n-1)/n} = n\omega_n^{1/n}|E|^{(n-1)/n}.$$

If E itself is a minimizer we may apply the symmetrization argument to E to conclude that E is a ball. Hence the equality holds if and only if E is a ball. □

1.3 The Brunn–Minkowski inequality

We discuss briefly the Brunn–Minkowski inequality, which is closely related to the isoperimetric inequality. Let $A, B \subset \mathbb{R}^n$ be non-empty sets. Their Minkowski sum is the set

$$A + B = \{a + b : a \in A, b \in B\}.$$

The Brunn–Minkowski inequality is stated in the following theorem.

Theorem 1.10. *Let $A, B \subset \mathbb{R}^n$ be non-empty Borel sets. Then*

$$|A + B|^{1/n} \geq |A|^{1/n} + |B|^{1/n}.$$

For the proof of Theorem 1.10 see [19] and the references therein. We will only discuss the connection of the Brunn–Minkowski inequality to the isoperimetric inequality. In this setting we are going to use a different concept of perimeter called the Minkowski content of the boundary. Let $E \subset \mathbb{R}^n$ be a Borel set. For any $h > 0$ consider the h-neighbourhood of E, that is,

$$B(E, h) = E + B(0, h).$$

The Minkowski content of the boundary of E is defined as

$$P_M(E) = \liminf_{h \to 0^+} \frac{|B(E, h)| - |E|}{h}.$$

Although, in general, Minkowski content differs from the distributional perimeter, it agrees with the usual definition of the perimeter for smooth and polyhedral sets. Hence Minkowski content is also a valid generalisation of the usual perimeter. We assume that the boundary E has finite Minkowski content. The Brunn–Minkowski inequality shows that

$$|B(E, h)|^{1/n} \geq |E|^{1/n} + h\omega_n^{1/n}.$$

Rearranging the terms and taking the limit we find that

$$\begin{aligned}
\omega_n^{1/n} &\leq \liminf_{h \to 0^+} \frac{|B(E, h)|^{1/n} - |E|^{1/n}}{h} \\
&= \frac{1}{n} |E|^{(1-n)/n} \liminf_{h \to 0^+} \frac{|B(E, h)| - |E|}{h} \\
&= \frac{1}{n} |E|^{(1-n)/n} P_M(E).
\end{aligned}$$

This proves the isoperimetric inequality. However, more careful anaysis is needed to prove that the ball is the only minimizer.

Chapter 2

Quantitative isoperimetric inequalities

The isoperimetric inequality, Theorem 1.5, states that the ball is the unique set that minimizes the perimeter among sets with given volume. In this chapter we consider different quantitative isoperimetric inequalities. Quantitative isoperimetric inequalities were studied by Bonnesen (see [7] and [6]). He derived several quantitative versions of the isoperimetric inequality for convex planar domains. One of the inequalities Bonnesen derived for convex domains, that was later extended to more general setting, is stated in the following theorem.

Theorem 2.1. *Let $\Omega \subset \mathbb{R}^2$ be a domain bounded by a rectifiable Jordan curve and assume $|\Omega| < \infty$. Then*

$$\ell(\Omega)^2 - 4\pi|\Omega| \geq 4\pi(R-\rho)^2, \tag{2.1}$$

where $R > 0$ is the circumradius of Ω and $\rho > 0$ the inradius of Ω.

The circumradius of a bounded domain $\Omega \subset \mathbb{R}^n$ is the radius of the smallest ball containing Ω. The inradius of Ω is the radius of the largest ball contained in Ω. The quantity $R - \rho$ in the theorem above essentially measures how far Ω is from a ball. The circumradius and the inradius are equal if and only if Ω is a ball. Hence this Bonnesen's inequality is a quantitative isoperimetric inequality. Many inequalities, similar to (2.1), were proven by Bonnesen and others. Osserman gave a comprehensive survey of different planar Bonnesen inequalities and their relations in [18]. In Section 2.1 we prove Theorem 2.1 for convex domains.

We introduce a general form of quantitative isoperimetric inequalities. The isoperimetric deficit $\delta(\Omega)$ of a Borel set $E \subset \mathbb{R}^n$ is defined as

$$\delta(E) = \frac{P(E)}{n\omega_n^{1/n}|E|^{(n-1)/n}} - 1.$$

We note that the isoperimetric deficit is scaling invariant. The isoperimetric theorem implies that isoperimetric deficit is non-negative and vanishes when E is a ball. We study quantitative isoperimetric inequalities of the form

$$D(E) \leq f(\delta(E)), \tag{2.2}$$

where D is some geometric quantity that measures the distance of E from a ball and $f : [0, \infty) \to [0, \infty)$ is a given function. We assume that f is strictly increasing and satisfies $f(0) = 0$. We also assume that D has the following properties:

(i) $D(E) \geq 0$,

(ii) $D(E) = 0$ if and only if E is a ball and

(iii) $D(E)$ is scaling invariant and has some geometric meaning.

Fusco, Maggi and Pratelli [10] proved a sharp quantitative isoperimetric inequality of the form (2.2), where the geometric quantity D is the Fraenkel asymmetry. See Section 2.2 for a sketch of the proof. In Sections 2.3 and 2.4 we consider quantitative isoperimetric inequalities of the form (2.2), where the geometric quantity D is the metric distortion. In Section 2.3 we consider convex domains and in Section 2.4 we study domains with controlled cusps, that is, John domains and so called s-John domains.

2.1 Bonnesen's inequality in the plane

According to [18], the first quantitative isoperimetric inequality in the plane is actually from Bernstein [5] who considered subsets of spheres of increasing radius. As a limiting case he was able to derive a quantitative isoperimetric inequality for convex sets in the plane. Bernstein proved Theorem 2.1 but with much smaller (non-sharp) constant on the right hand side of the inequality. In [6] Bonnesen improved the constant to π^2. This is still not the optimal constant. The best possible constant is 4π (as in the statement of Theorem 2.1) as was proven by Bonnesen in [7].

Next we give a proof of Theorem 2.1 (with non-sharp constant π^2) for convex domains. We follow Osserman's [18] presentation, which is based on the idea of Bonnesen's original proof.

Proof of Theorem 2.1. The idea of the proof is to first deal with polygonal convex domains and then use approximation argument for general convex domains. The reason for this is that it is possible to use elementary geometric arguments when dealing with polygonal domains. Let $F \subset \mathbb{R}^2$ be a convex polygonal domain. We will consider the t-neighbourhood $F_t = B(F, t)$ of F for every $\rho < t < R$, where $\rho > 0$ is are the inradius, and $R > 0$ is the circumradius of F. As F is convex polygon it is easy to see that ∂F_t consists of lines parralel to the sides of F together with circular arcs centered at the corners of F. These circular arcs together form a full circle and the flat parts of the boundary have the same lengths as the corresponding sides of F. With this observation we may easily conclude that

$$|F_t| = |F| + \ell(F)t + \pi t^2 \tag{2.3}$$

and

$$\ell(F_t) = \ell(F) + 2\pi t.$$

Now we divide F_t into subsets which are easier to handle. For any $k \in \mathbb{N}$ define the set $F_t^k \subset F$ by setting $x \in F_t^k$ if and only if $x \in F_t$ and the circle of radius t centered at x intersects ∂F at exactly k points. The fact that $\rho < t < R$ has two important consequences. First, it is important to note that any circle of radius t centered at any point of F_t intersects ∂F at least once. Second, we note that $F_t^1 = \emptyset$. Indeed, any circle of radius t whose center lies in F_t and that intersects ∂F at exactly one point must lie either inside the closure of F or outside F. Hence, we would have either $t \leq \rho$ or $t \geq R$, neither of which is possible. We conclude that

$$F_t = \bigcup_{k=2}^{\infty} F_t^k. \tag{2.4}$$

Next we show that

$$\sum_{k=2}^{\infty} k|F_t^k| = 4t\ell(F). \tag{2.5}$$

We start by first considering a line segment $[0, L] \times \{0\} \subset \mathbb{R}^2$, where $L > 0$. Any circle of radius $t > 0$ can intersect the segment up to two times. It is easy to see that the circle intersects the segment twice if its center lies a distance t away from both of the endpoints. Also the circle cannot intersect the line segment if the center is less than distance t away from both endpoints. Hence the circle intersect the line segment twice if the center of the circle lies in the set

$$E^2 = ([0, L] \times (-t, t)) \setminus (\overline{B}((0, 0), t) \cup \overline{B}((L, 0), t))$$

and once if the center lies in the set

$$E^1 = \overline{B}((0, 0), t) \triangle \overline{B}((L, 0), t).$$

We observe that

$$\begin{aligned} |E^1| + 2|E^2| =& \Big|\Big([0, L] \times [-t, t] \cup B\big((0, 0), t\big) \cup B\big((L, 0), t\big) \setminus B\big((0, 0), t\big)\Big)\Big| \\ &+ \Big|\Big([0, L] \times [-t, t] \cup B\big((0, 0), t\big) \cup B\big((L, 0), t\big) \setminus B\big((L, 0), t\big)\Big)\Big| \\ =& 4Lt. \end{aligned} \tag{2.6}$$

Now we consider all the sides of the polygon F and apply the observation above. Summing the left hand side of (2.6) over all sides of the polygon we obtain the quantity

$$\sum_{k=2}^{\infty} k|F_t^k|.$$

Moreover, the right hand side of (2.6), summed over all the sides of the polygon, obviously gives the quantity

$$4t\ell(F).$$

Hence we have proven (2.5).

Using (2.4) and (2.5) we get the inequality

$$2|F_t| = 2\sum_{k=2}^{\infty} |F_t^k| \leq \sum_{k=2}^{\infty} k|F_t^k| = 4t\ell(F).$$

By (2.3) we may write this as

$$2t\ell(F) \geq |F_t| = |F| + \ell(F)t + \pi t^2 \tag{2.7}$$

for any $\rho < t < R$. Taking the limits as $t \to \rho^+$ and $t \to R^-$ we conclude that

$$\rho\ell(F) \geq |F| + \pi\rho^2$$

and

$$R\ell(F) \geq |F| + \pi R^2.$$

If $\Omega \subset \mathbb{R}^2$ is any convex Jordan domain we may approximate it by convex polygons so that the areas, lengths of the boundaries and both the circumradiuses and inradiuses converge to those of Ω. Hence the results obtained above easily extend to the case of convex domains. Extension to general Jordan domains is more involved and we shall not deal with it here. With the above inequalities we may conclude that

$$R \leq \frac{\ell(F) + \sqrt{\ell(F)^2 - 4\pi|F|}}{2\pi}$$

and

$$\rho \geq \frac{\ell(F) - \sqrt{\ell(F)^2 - 4\pi|F|}}{2\pi}.$$

Simply subtracting these inequalities gives

$$(R-\rho)^2 \leq \frac{4\ell(F)^2 - 16\pi|F|}{4\pi^2},$$

which can be written as

$$\ell(F)^2 - 4\pi|F| \geq \pi^2(R-\rho)^2.$$

□

We mention here a quantitative isoperimetric inequality of the form (2.2) in $\mathbb{R}^2$. The metric distortion of a bounded domain $\Omega \subset \mathbb{R}^n$ is defined as

$$\beta(\Omega) = \inf \Big\{ \frac{R-\rho}{\rho} \ : \text{ there exists } x \in \mathbb{R}^n \text{ such that } B(x,\rho) \subset \Omega \subset B(x,R) \Big\}. \tag{2.8}$$

We have the following theorem.

Theorem 2.2. *Let $\Omega \subset \mathbb{R}^2$ be a domain bounded by a rectifiable Jordan curve and $\delta_0 > 0$. If $\delta(\Omega) < \delta_0$, then*

$$\beta(\Omega) \le C\delta(\Omega)^{1/2}, \tag{2.9}$$

where $C > 0$ depends only on δ_0.

Proof of Theorem 2.2: Let $\Omega \subset \mathbb{R}^2$ be a bounded domain whose boundary is a rectifiable Jordan curve. In [13] Fuglede proved that (2.1) holds with right hand side replaced by a strictly bigger quantity. Namely, we have

$$\ell(\partial\Omega)^2 - 4\pi|\Omega| \ge 4\pi d^2, \tag{2.10}$$

where d is the width of the unique annulus bi-enclosing the boundary of Ω. We refer to [13] for more detailed discussion of (2.10) and the definition of bi-enclosing annulus. In particular, the bi-enclosing annulus contains the boundary of Ω. By the definition of $\beta(\Omega)$ we have

$$\beta(\Omega) \le \frac{R^* - \rho^*}{\rho^*}. \tag{2.11}$$

where $R^* > 0$ and $\rho^* > 0$ are the outer and inner radius, respectively, of the unique annulus bi-enclosing Ω. We write inequality (2.10) in the form

$$\frac{R^* - \rho^*}{r} \le \pi^{1/2}\delta(\Omega)^{1/2}(\delta(\Omega) + 2)^{1/2}, \tag{2.12}$$

where $r > 0$ is the volume radius of Ω. As $R^* \ge r$, the inequality (2.12) gives the estimate

$$\rho^* \ge (1 - \pi^{1/2}\delta(\Omega)^{1/2}(\delta(\Omega) + 2)^{1/2})r. \tag{2.13}$$

Assume that $\delta(\Omega)$ is so small that the left hand side of the inequality (2.13) is positive. Inserting the estimate (2.13) back into inequality (2.12) and recalling (2.11) gives the estimate

$$\beta(\Omega) \le \frac{R^* - \rho^*}{\rho^*} \le \frac{\pi^{1/2}(\delta(\Omega) + 2)^{1/2}}{1 - \pi^{1/2}\delta(\Omega)^{1/2}(\delta(\Omega) + 2)^{1/2}}\delta(\Omega)^{1/2},$$

when $\delta(\Omega)$ is not too large. □

2.2 Fraenkel asymmetry

In this section we consider a quantitative isoperimetric inequality for arbitrary Borel sets in all dimensions. To this end, we need to consider a geometric quantity, called the Fraenkel asymmetry.

Definition 2.3. The Fraenkel asymmetry $\lambda(E)$ of a Borel set $E \subset \mathbb{R}^n$ is defined as

$$\lambda(E) = \min_{x \in \mathbb{R}^n} \frac{|E \setminus B(x,r)|}{|E|}, \tag{2.14}$$

where r is the volume radius of E.

We have the following sharp quantitative isoperimetric inequality involving Fraenkel asymmetry.

Theorem 2.4. *Let E be Borel measurable and assume $0 < |E| < \infty$. Then*

$$\lambda(E) \leq C\delta(E)^{1/2} \tag{2.15}$$

where C depends only on n.

Hall, Hayman and Weitsman studied the Fraenkel asymmetry in [8]. In [9], Hall proved Theorem 2.4 for axially symmetric sets and conjectured that (2.15) holds for all Borel sets. Fusco, Maggi and Pratelli [10] proved Theorem 2.4 for general Borel sets and thus confirmed Hall's conjecture. Considering a family of ellipses which have one axis of length $1 + \epsilon$ and others of unit length, one easily sees that the exponent in (2.15) is optimal and cannot be replaced by any larger exponent.

In the following we briefly explain Hall's method and go through the main steps of the proof of Theorem 2.4 in [10]. Maggi showed in [11] that mass transportation methods can also be used to prove Theorem 2.4. Similarly to De Giorgi's proof of the isoperimetric inequality, the proof of Theorem 2.4 relies on symmetrization. However, instead of Steiner symmetrization we need the concept of Schwarz symmetrization with respect to a line. By a rotation it is enough to consider Schwarz symmetrization with respect to the nth coordinate axis.

Definition 2.5. Let $E \subset \mathbb{R}^n$ be a measurable set. Let $r(t)$ be such that $\omega_{n-1} r(t)^{n-1} = \mathcal{H}^{n-1}(E \cap (\mathbb{R}^{n-1} \times \{t\}))$ for all $t \in \mathbb{R}$. The Schwarz symmetral E^* of E with respect to the nth coordinate axis is defined as

$$E^* = \{(x,t) \in \mathbb{R}^{n-1} \times \mathbb{R} : t \in \mathbb{R}, |x| < r(t)\},$$

Fubini's theorem shows that Schwarz symmetrization does not change the measure of the set. Moreover, Schwarz symmetrization does not increase the perimeter.

As stated above, Hall was able to prove Theorem 2.4 for axially symmetric sets (see [9] for the proof), that is, sets that are rotationally symmetric with respect to some line. The natural idea to deal with general Borel set $E \subset \mathbb{R}^n$ is to perform Schwarz symmetrization on E to obtain an axially symmetric set $E^* \subset \mathbb{R}^n$. Because Schwarz symmetrization does not increase the isoperimetric deficit, Theorem 2.4 can be applied to the axially symmetric set E^* to obtain

$$\lambda(E^*) \leq C\delta(E^*)^{1/2} \leq C\delta(E)^{1/2}. \tag{2.16}$$

The problem is that $\lambda(E^*)$ does not, in general, control $\lambda(E)$. That is, the Fraenkel asymmetry of the Schwarz symmetral might be zero even for a set of non-zero asymmetry. However, Hall, Hayman and Weitsman proved in [8] that it is always possible to choose a line such that the Schwarz symmetral E^* of E with respect to the line satisfies

$$\lambda(E) \leq C\lambda(E^*)^{1/2}. \tag{2.17}$$

Combining (2.16) and (2.17) proves the non-sharp inequality

$$\lambda(E) \leq C\lambda(E^*)^{1/2} \leq C\delta(E)^{1/4}.$$

The proof of Theorem 2.4 in [10] is also based on symmetrization and reduces to the analysis of axially symmetric sets as in [9]. We outline the main steps of the proof following [10].

Sketch of the proof of Theorem 2.4. Let $E \subset \mathbb{R}^n$ be a Borel set. There exists a ball $B(x, r) \subset \mathbb{R}^n$ that realizes the Fraenkel asymmetry, that is

$$\frac{|E \setminus B|}{|E|} = \lambda(E) \tag{2.18}$$

and r is the volume radius of E. One of the problems with estimating the Fraenkel asymmetry is that there is little a priori knowledge on the ball that realizes the Fraenkel asymmetry. As a result the first step is to show that it is enough to prove the theorem for n-symmetric sets, that is, sets that have n orthogonal hyperplanes of symmetry. The benefit of dealing with n symmetric sets is that the ball centered at the intersection of the n hyperplanes of symmetry realizes the Fraenkel asymmetry of an n-symmetric set up to a constant depending only on n.

The reduction of a general Borel set to an n-symmetric one proceeds as follows. Choose any hyperplane that divides E to two parts of equal measure. Then we may reflect either part of E with respect to the hyperplane to obtain a reflectionally symmetric set with the same measure as E. In fact one of the reflected sets has isoperimetric deficit comparable to the isoperimetric deficit of E. However, the Fraenkel asymmetry of the reflected set does not need to be comparable to the Fraenkel asymmetry of E. The key result in this step is to prove that if we choose two orthogonal hyperplanes then one of the four possible reflections will give a set E' satisfying

$$\lambda(E) \leq C\lambda(E') \quad \text{and} \quad \delta(E') \leq C\delta(E). \tag{2.19}$$

Of course we may iterate this result to obtain a set E' with $n-1$ hyperplanes of symmetry and satisfying (2.19). Now there exists just one hyperplane orthogonal to the $n-1$ hyperplanes of symmetry of E' so the same argument cannot be applied again. However, it can be proven that one of the two possible reflections with respect to the hyperplane gives an n-symmetric set (still denoted E') that satisfies (2.19).

With the discussion above, we may assume that E is n-symmetric. Because Fraenkel asymmetry and isoperimetric deficit are scaling invariant we may assume $|E| = 1$. In this case we have

$$\lambda(E) \leq d(E, B) \leq C\lambda(E), \tag{2.20}$$

where $d(E, B) = |E \triangle B|$ and B is the ball centered at the intersection of the n hyperplanes of symmetry, such that $|B| = |E| = 1$. Thus it is enough to study the much simpler quantity $d(E, B)$ instead of $\lambda(E)$.

The next step is to apply Schwarz symmetrization on E to obtain an axially symmetric set E^*. By triangle inequality we have

$$d(E, B) \leq d(E, E^*) + d(E^*, B). \tag{2.21}$$

As E^* is axially symmetric (and n-symmetric), we may prove that

$$d(E^*, B) \leq C\delta(E^*)^{1/2} \leq C\delta(E)^{1/2} \tag{2.22}$$

using Hall's result in [9]. However, a simpler proof, which utilizes the fact that E^* is n-symmetric, is presented in [10]. We will briefly discuss the elementary method used in [10]. Provided that the axis of symmetrization is chosen appropriately, we may also prove that

$$d(E, E^*) \leq C\delta(E)^{1/2}. \tag{2.23}$$

The proof is based on induction over the dimension of the space and is discussed below. Estimates (2.23), (2.22) and (2.21) finish the proof of Theorem 2.4.

Let us first discuss the proof of (2.23). Assume that the axis of symmetrization is orthogonal to one of the hyperplanes of symmetry. By rotation we may assume that the axis of symmetrization is the nth coordinate axis. By Fubini's theorem we have

$$d(E, E^*) = \int_{\mathbb{R}} \mathcal{H}^{n-1}(E_t \triangle E_t^*) \, \mathrm{d}t, \tag{2.24}$$

where $E_t = E \cap (\mathbb{R}^{n-1} \times \{t\})$ and $E_t^* = E^* \cap (\mathbb{R}^{n-1} \times \{t\})$ are the $(n-1)$-dimensional slices of E and E^*, respectively. Because E is n-symmetric and the axis of symmetrization is orthogonal one of the hyperplanes of symmetry, it follows immediately that the slices E_t are $(n-1)$-symmetric subsets of $\mathbb{R}^{n-1}$ (we identify $\mathbb{R}^{n-1} \times \{t\}$ with $\mathbb{R}^{n-1}$). Moreover, the intersection of the $n-1$ hyperplanes of symmetry lies on the axis of symmetrization. By the definition of Schwarz symmetrization (see Definition 2.5) the slices of E^* are $n-1$ dimensional balls centered on the axis of symmetrization and $\mathcal{H}^{n-1}(E_t^*) = \mathcal{H}^{n-1}(E_t)$. Thus we can apply Theorem 2.4 in dimension $n-1$ to obtain

$$\mathcal{H}^{n-1}(E_t \triangle E_t^*) \le C\delta(E_t)^{1/2}. \tag{2.25}$$

In dimension $n = 2$ the above inequality is easy to prove directly. Inserting (2.25) into (2.24) gives

$$d(E, E^*) \le C \int_{\mathbb{R}} \delta_{n-1}(E_t)^{1/2}\, \mathrm{d}t. \tag{2.26}$$

It can be proven that

$$\int_{\mathbb{R}} \delta_{n-1}(E_t)^{1/2}\, \mathrm{d}t \le C\delta(E)^{1/2}, \tag{2.27}$$

if the axis of symmetrization is chosen carefully. Combining the two estimates above we obtain (2.23).

The idea of the proof of (2.22) is to reduce the general case to a simple situation of two overlapping balls. This can be done as follows. First we show that it is possible to choose $(n-1)$-dimensional sections E_t^* of E^* and B_s of B such that $d(E^*, B)$ is equivalent to $d_{n-1}(E_t^*, B_s)$. Then we show that it is enough to consider central sections $t = s = 0$. Recall that in particular E^* is symmetric with respect to the hyperplane $\{x_n = 0\}$. Now we may replace both halves of E^* with parts of a ball without changing the section. Hence we only need to analyse $d(E', B)$, where $E' \subset \mathbb{R}^n$ is a domain consisting of two overlapping balls. This concrete case can be estimated explicitly. □

2.3 Bonnesen's inequality for convex domains in $\mathbb{R}^n$

Theorem 2.2 shows that metric distortion can be controlled by the isoperimetric deficit in dimension $n = 2$. However, this result does not extend to arbitrary sets in higher dimensions. For example, consider the following family of domains. For each $\eta > 0$ define the function $f_\eta : [0, 1] \to [0, \infty)$ by

$$f_\eta(t) = \begin{cases} \eta^\eta - t^\eta + \sqrt{1-\eta^2}, & \text{if } t < \eta; \\ \sqrt{1-t^2}, & \text{if } t \ge \eta. \end{cases} \tag{2.28}$$

We consider the family of domains

$$\Omega_\eta = \{(\tilde{x}, x_n) \in \mathbb{R}^n : |x_n| < f_\eta(|\tilde{x}|)\}.$$

We show that if $n \ge 3$ then

$$\delta(\Omega_\eta) \to 0, \tag{2.29}$$

as $\eta \to 0$. However, note that the inradius of Ω_η approaches to one and the circumradius of Ω_η approaches to two, as η goes to zero. Thus it is easy to see that

$$\beta(\Omega_\eta) \to 1 \neq 0,$$

as $\eta \to 0$. This example shows that the isoperimetric deficit does not control the difference of the circumradius and inradius of an arbitrary set in higher dimensions.

If $n = 2$ we see that $P(\Omega_\eta) \to 2\pi + 4$. Hence the isoperimetric deficit does not approach to zero in the plane. However, if $n \geq 3$ we may compute the measure and perimeter of Ω_η in cylindrical coordinates as follows

$$|\Omega_\eta| - \omega_n = 2\omega_{n-2} \int_0^\eta (\eta^\eta - t^\eta) t^{n-2}\, \mathrm{d}t + o(\eta^{n+1}) = 2\omega_{n-2} \frac{\eta^{n+\eta}}{(n-1)(n-1+\eta)} + o(\eta^{n+1})$$

and

$$\begin{aligned} P(\Omega_\eta) - n\omega_n &= 2\omega_{n-2} \int_0^\eta (1 + (\eta t^{\eta-1})^2)^{1/2} t^{n-2}\, \mathrm{d}t - 2\omega_{n-2}/(n-1)\eta^{n-1} + o(\eta^{n-1}) \\ &\leq \frac{C}{\eta + n - 2} \eta^{n-1} + o(\eta^{n-1}). \end{aligned}$$

Hence the isoperimetric deficit is

$$\delta(\Omega_\eta) = \frac{P(\Omega_\eta)}{n\omega_n^{1/n} |\Omega_\eta|^{(n-1)/n}} - 1 = \frac{C}{\eta + n - 2} \eta^{n-1+\eta} + o(\eta^{n-1+\eta}).$$

This proves (2.29) when $n \geq 3$.

With this example in mind, there is no hope to generalize Theorem 2.2 to arbitrary sets in dimension $n \geq 3$. However, we can ask if such an inequality holds for some smaller class of domains. First development in this direction is due to Fuglede [12]. Fuglede derived an analogous inequality for convex domains in any dimension. The decay rates in the following theorem 2.6 are sharp, see [12] for examples.

Theorem 2.6. *Let $\Omega \subset \mathbb{R}^n$, $n \geq 3$, be a bounded convex domain. Then*

$$\beta(\Omega) \leq \begin{cases} C\Big(\delta(\Omega) \log \frac{1}{\delta(\Omega)}\Big)^{1/2}, & n = 3; \\ C\delta\,(\Omega)^{2/(n+1)}, & n \geq 4, \end{cases}$$

where the constants $C > 0$ depend only on n and are explicitly calculable.

Actually, Fuglede's result for convex domains is a consequence of a more general theorem. Fuglede introduced the concept of nearly spherical domains. They are domains that are starshaped with respect to its center of mass and satisfy some quantitative bounds. A domain is starshaped with respect to a point (called the center) if any point of the domain can be connected to the center with a line segment contained in the domain. For any starshaped domain Ω we may move the center to origin and normalize $|\Omega| = \omega_n$. Then the boundary of Ω can be represented in polar coordinates $(r, \theta) \in [0, \infty) \times S^{n-1}$ by the equation $r = 1 + u(\theta)$ for some Sobolev function $u : S^{n-1} \to [0, \infty)$. We say that Ω is nearly spherical if

$$\|u\|_\infty \le \frac{3}{20n}, \quad \|\nabla u\|_\infty \le \frac{1}{2}. \tag{2.30}$$

Fuglede proved two core estimates for nearly spherical domains. The first shows that shows that the isoperimetric deficit of Ω controls the Sobolev norm of u. We denote by $\mathrm{d}\mu$ the surface measure on S^{n-1}, normalized so that $\int_{S^{n-1}} \mathrm{d}\mu = 1$.

Theorem 2.7. *Let $\Omega \subset \mathbb{R}^n$ be a nearly spherical domain and u a parametrization of $\partial\Omega$ as defined above. Then*

$$\frac{1}{10}\Big(\int_{S^{n-1}} |u(x)|^2 \, \mathrm{d}\mu + \int_{S^{n-1}} |\nabla u(x)|^2 \, \mathrm{d}\mu \Big) \le \delta(\Omega) \le \frac{3}{5} \int_{S^{n-1}} |\nabla u(x)|^2 \, \mathrm{d}\mu.$$

The second theorem shows that there is stability with respect to the L^∞-norm of u.

Theorem 2.8. *Let $\Omega \subset \mathbb{R}^n$ be a nearly spherical domain and u a parametrization of $\partial\Omega$ as defined above. Then*

$$\|u\|_\infty^{n-1} \le \begin{cases} C\delta(\Omega) \log \frac{\|\nabla u\|_\infty^2}{\delta(\Omega)}, & n = 3, \\ C\delta\,(\Omega)\, \|\nabla u\|_\infty^{n-3}, & n \ge 4. \end{cases}$$

Where $C > 0$ depends only on n.

Fuglede shows convex domains with small isoperimetric deficit are nearly spherical domains and that Theorem 2.8 together with some estimates for convex domains implies Theorem 2.6.

We give the main steps of the proof of Theorem 2.8. See [12] for details. We will show why we need to impose the quantitative bounds (2.30) in the definition of nearly spherical domains.

Sketch of the proof of Theorem 2.8. Let $\Omega \subset \mathbb{R}^n$ be a normalized nearly spherical domain and define $u : S^{n-1} \to [0, \infty)$ as before. Suppose that

$$\|u\|_\infty + \|u\|_\infty \le \epsilon, \tag{2.31}$$

where $\epsilon > 0$ will be chosen later depending only on n. Using spherical coordinates we may write the volume, perimeter and the center of mass of Ω as

$$|\Omega| = \omega_n \int_{S^{n-1}} (1 + u)^n \, \mathrm{d}\mu \tag{2.32}$$

$$P(\Omega) = n\omega_n \int_{S^{n-1}} (1 + u)^{n-1} (1 + (1 + u)^{-2} |\nabla u|^2)^{1/2} \, \mathrm{d}\mu \tag{2.33}$$

$$b = \int_{S^{n-1}} (1 + u(x))^{n+1} x \, \mathrm{d}\mu \tag{2.34}$$

Recall that $|\Omega| = \omega_n$ and $b = 0$. We estimate the integral on the right hand side of (2.32) by the Taylor expansion. We see that u satisfies the equation

$$\omega_n = \omega_n \int_{S^{n-1}} 1 + nu + \big(n(n-1)/2 + O(\epsilon)\big)u^2 \,\mathrm{d}\mu,$$

which implies that

$$\int_{S^{n-1}} u \,\mathrm{d}\mu = \big(-(n-1)/2 + O(\epsilon)\big) \int_{S^{n-1}} u^2 \,\mathrm{d}\mu. \tag{2.35}$$

By equation (2.35), we see that u satisfies the equation

$$\begin{aligned} \int_{S^{n-1}} (1+u)^{n-1} \,\mathrm{d}\mu &= \int_{S^{n-1}} 1 + (n-1)u + \big((n-1)(n-2)/2 + O(\epsilon)\big)u^2 \,\mathrm{d}\mu \\ &= \int_{S^{n-1}} 1 + \big(-(n-1)/2 + O(\epsilon)\big)u^2 \,\mathrm{d}\mu. \end{aligned} \tag{2.36}$$

We have the trivial inequality $(1+t)^{1/2} \geq 1 + t/2 - t^2/8$ for $t \geq 0$. This inequality, (2.36) and (2.33) give the following lower bound for the perimeter

$$\begin{aligned} P(\Omega)/(n\omega_n) &\geq \int_{S^{n-1}} (1+u)^{n-1} + (1+u)^{n-3}|\nabla u|^2/2 + (1+u)^{n-5}|\nabla u|^4/8 \,\mathrm{d}\mu \\ &= 1 + \int_{S^{n-1}} \big(-(n-1)/2 + O(\epsilon)\big)u^2 \,\mathrm{d}\mu \\ &\quad + \int_{S^{n-1}} \big(1/2 + O(\epsilon)\big)|\nabla u|^2 \,\mathrm{d}\mu. \end{aligned}$$

The particular choice of $\epsilon > 0$ will be made in the following.

The lower bound for the perimeter gives us an estimate of the isoperimetric deficit from below

$$\delta(\Omega) = \frac{P(\Omega)}{n\omega_n} - 1 \geq \big(-(n-1)/2 - O(\epsilon)\big) \int_{S^{n-1}} u^2 \,\mathrm{d}\mu + \big(1/2 - O(\epsilon)\big) \int_{S^{n-1}} |\nabla u|^2 \,\mathrm{d}\mu \tag{2.37}$$

The tricky part of the proof is to obtain conclusion of Theorem 2.7 from estimate (2.37). If we compare this to the proof of the planar isoperimetric inequality by Hurwitz (see the proof of Theorem 1.1), we see that we need something similar to the Wirtinger inequality (see Lemma 1.2). But in order to obtain quantitative estimates we would also need to control the error in the Wirtinger inequality. Fuglede's proof is essentially based on this idea. While the Wirtinger inequality is proven by using Fourier series, Fuglede uses spherical harmonics. Spherical harmonics could be called higher dimensional analogues of trigonometric functions.

Spherical harmonics form an orthonormal basis of the space of L^2 functions on the sphere. Hence we may represent u as the series

$$u = \sum_{k=0}^{\infty} \langle a_k, Y_k \rangle,$$

where Y_k is a $\wedge^k$-valued function whose component functions are orthogonal eigenfunctions of the Laplace operator $-\Delta$ on the sphere corresponding to the eigenvalue $k(k+n-2)$, and

$$a_k = \int_{S^{n-1}} u(x)Y_k(x)\,\mathrm{d}\mu.$$

The Parseval identity shows that

$$\int_{S^{n-1}} u(x)^2\,\mathrm{d}\mu = \sum_{k=0}^{\infty} \|a_k\|^2.$$

Using integration by parts and the fact that the component functions of Y_k are eigenfunctions of the Laplace operator we may compute

$$\begin{aligned}\int_{S^{n-1}} |\nabla u|^2\,\mathrm{d}\mu &= \int_{S^{n-1}} u\Delta u\,\mathrm{d}\mu \\ &= \sum_{k=0}^{\infty} k(k+n-2)\langle a_k, \int_{S^{n-1}} uY_k\,\mathrm{d}\mu\rangle \\ &= \sum_{k=0}^{\infty} k(k+n-2)\|a_k\|^2.\end{aligned}$$

We note that $Y_0 \equiv 1$. The equation (2.35) gives

$$\|a_0\|^2 = O(\epsilon)\int_{S^{n-1}} u^2\,\mathrm{d}\mu.$$

The important step is to note that a_1 is also small. We have $Y_1 = x$, and consequently (2.34) gives

$$0 = \int_{S^{n-1}} (1+u)^{n+1}Y_1\,\mathrm{d}\mu = \int_{S^{n-1}} Y_1 + (n+1)uY_1 + \big(n(n+1)/2 + O(\epsilon)\big)u^2Y_1\,\mathrm{d}\mu \tag{2.38}$$

The symmetry of the function Y_1 implies that

$$\int_{S^{n-1}} Y_1\,\mathrm{d}\mu = 0.$$

Consequently, equation (2.38), the facts that $\|Y_1\|_\infty \le 1$ and Y_1 is a restriction of a linear form give

$$\|a_1\|^2 = \left\|\int_{S^{n-1}} uY_1\,\mathrm{d}\mu\right\|^2 \le O(\epsilon)\int_{S^{n-1}} u^2\,\mathrm{d}\mu.$$

Using the series representation for u and the estimates for a_0 and a_1 we see that

$$\int_{S^{n-1}} u^2\,\mathrm{d}\mu \le (1+O(\epsilon))\sum_{k=2}^{\infty}\|a_k\|^2. \tag{2.39}$$

Now we would like to find a small number $c_n > 0$ such that

$$\Big(-\frac{1}{2}(n-1) - O(\epsilon)\Big)\int_{S^{n-1}} u^2 \,\mathrm{d}\mu + \Big(\frac{1}{2} - O(\epsilon)\Big)\int_{S^{n-1}} |\nabla u|^2 \,\mathrm{d}\mu \\ \geq c_n\Big(\int_{S^{n-1}} u^2 \,\mathrm{d}\mu + \int_{S^{n-1}} |\nabla u|^2 \,\mathrm{d}\mu\Big).$$

By (2.39) it is enough to find $c_n > 0$ such that

$$\sum_{k=2}^{\infty} C_k \|a_k\|^2 \geq 0,$$

where

$$C_k = -\frac{1}{2}(1+O(\epsilon))(n-1+CO(\epsilon)) + \frac{1}{2}k(k+n-2) - c_n(1+k(k+n-2)),$$

and the constant $C > 0$ depends only on n. We note that the sequence C_k is increasing in k. So it is enough to choose c_n so that the first coefficient (corresponding to $k = 2$) is non-negative, that is

$$-\frac{1}{2}(1+O(\epsilon))(n-1+CO(\epsilon)) + n - c_n(1+2n) \geq 0.$$

Such a choice of c_n is possible as long as ϵ is chosen small enough. The a priori bounds on $\|u\|_\infty$ and $\|\nabla u\|_\infty$ in the definition of nearly spherical domains are essentially chosen so that this condition is satisfied. Now equation (2.37) implies that

$$\delta(\Omega) \geq c_n\Big(\int_{S^{n-1}} u^2 \,\mathrm{d}\mu + \int_{S^{n-1}} |\nabla u|^2 \,\mathrm{d}\mu\Big).$$

The upper bound in the conclusion of Theorem 2.7 follows using similar estimates as those used to prove (2.37). These considerations, when made precise, finish the proof of Theorem 2.7.

Theorem 2.8 is a straightforward corollary of Theorem 2.7 once we are able to estimate the supremum norm of u in terms of the L^2 norm of ∇u. For that purpose we have the following lemma.

Lemma 2.9. *Let $v : S^{n-1} \to \mathbb{R}$ be a Lipschitz function. If*

$$\int_{S^{n-1}} v \,\mathrm{d}\mu = 0 \tag{2.40}$$

then we have the estimate

$$\|v\|_\infty^{n-1} \leq \begin{cases} \pi\|\nabla v\|_2^2, & \text{if } n = 2, \\ 4\|\nabla v\|_2^2 \log(8e\|\nabla v\|_\infty^2/\|\nabla v\|_2^2), & \text{if } n = 3, \\ C\|\nabla v\|_2^2\|\nabla v\|_\infty^{n-3}, & \text{if } n \geq 4, \end{cases}$$

where $C > 0$ depends only on n.

Note that if $u : S^{n-1} \to [0, \infty)$ corresponds to a nearly spherical domain as before, it need not satisfy (2.40). However, by (2.32) the integral of the function $v = ((1+u)^n - 1)/n$ over the sphere vanishes. It is also easy to show that we have the pointwise estimates

$$c|u| \le |v| \le C|u| \quad \text{and} \quad c|\nabla u| \le |\nabla v| \le C|\nabla u|, \tag{2.41}$$

where $c, C > 0$ are constants. It follows that the conclusion of Lemma 2.9 applies to u. The upper and lower bounds for the isoperimetric deficit given by Theorem 2.7, together with Lemma 2.9 give the inequality

$$\begin{aligned} \|u\|_\infty^{n-1} &\le C\|\nabla u\|_2^2 \log(8e\|\nabla u\|_\infty / \|\nabla u\|_2^2) \\ &\le C\delta(\Omega) \log(\|\nabla u\|_\infty / \delta(\Omega)) \end{aligned} \tag{2.42}$$

if $n = 3$ and

$$\|u\|_\infty^{n-1} \le C\|\nabla u\|_2^2 \|\nabla u\|_\infty^{n-3} \le C\delta(\Omega)\|\nabla u\|_\infty^{n-3} \tag{2.43}$$

if $n \ge 4$. This proves Theorem 2.8. □

We are now ready to consider convex domains.

Sketch of the proof of Theorem 2.6. Let $\Omega \subset \mathbb{R}^n$ be a convex domain. We assume that Ω is normalized as before. We note that the boundary of Ω can be represented as before with a Lipschitz function $u : S^{n-1} \to \mathbb{R}$. We want to show that if the isoperimetric deficit of Ω is small enough then Ω is actually a nearly spherical domain. First we show that in the case of convex domain, the supremum norm of u controls the supremum norm of the gradient of u.

Lemma 2.10. *Let $\Omega \subset \mathbb{R}^n$ be a convex domain and $u : S^{n-1} \to \mathbb{R}$ a Lipschitz function as defined above. Then*

$$\|\nabla u\|_\infty \le \|u\|_\infty^{1/2} \frac{1 + \|u\|_\infty}{1 - \|u\|_\infty}.$$

This lemma has two important consequences. First, it shows that if $\|u\|_\infty$ is small then $\|\nabla u\|_\infty$ is also small. Hence, to show that Ω with small isoperimetric deficit is nearly spherical, we only need to show that $\|u\|_\infty$ can be controlled by $\delta(\Omega)$. Second, if Ω is indeed nearly spherical, then the estimate given by Lemma 2.10 shows that $\|\nabla u\|_\infty$ has an upper bound independent of u. Together with Theorem 2.8 this proves the inequality in Theorem 2.6 (note that $\beta(\Omega) \le 2\|u\|_\infty$ if $\delta(\Omega)$ is small enough).

All that is left to prove is that a convex domain with small isoperimetric deficit has small spherical deviation $\|u\|_\infty$. We prove that if $\delta(\Omega)$ is small enough (depending only on n) then $\|u\|_\infty \le 3/(20n)$. We only give the sketch of the proof to make the idea as clear as possible (see [12] for details). Let Ω be a normalized convex set as before. The key idea is to study the parallel bodies $B(\Omega, \lambda)$, for each $\lambda \ge 0$. These parallel bodies are also convex. Fuglede proves that the spherical deviation $d(\lambda) = \|u_\lambda\|_\infty$ of $B(\Omega, \lambda)$ is a continuous function of λ and $d(\lambda) \to 0$ as $\lambda \to \infty$. Moreover, he proves that $\delta(B(\Omega, \lambda))$ is decreasing in λ. Define the function

$$f(t) = \begin{cases} C\big(t \log(1/t)\big)^{1/2}, & n = 3 \\ Ct^{2/(n+1)}, & n \geq 4 \end{cases} \tag{2.44}$$

Choose $\eta > 0$ such that $f(\eta) < 3/(20n)$. Assume Ω is convex and $\delta(\Omega) < \eta$. As $\delta(\lambda)$ is decreasing we have $\delta(\lambda) < \eta$ for all $\lambda \geq 0$. Because $d(\lambda)$ approaches to 0 there exists $\lambda_0 \geq 0$ such that $B(\Omega, \lambda_0)$ is nearly spherical. As discussed the conclusion of Theorem 2.6 applies to $B(\Omega, \lambda_0)$, that is

$$d(\lambda_0) \leq f(\delta(\lambda_0)) < 3/(20n). \tag{2.45}$$

We are ready to apply a standard continuity argument. By the continuity of $d(\lambda)$ we know that $B(\Omega, \lambda)$ is nearly spherical if λ is close to λ_0. The continuity of $d(\lambda)$ also shows that the set of $\lambda \geq 0$ such that $B(\Omega, \lambda)$ is nearly spherical is closed. Hence $B(\Omega, \lambda)$ is nearly spherical for every $\lambda \geq 0$. In particular Ω is nearly spherical. This finishes the proof of Theorem 2.6. □

2.4 Bonnesen's inequality for s-John domains in $\mathbb{R}^n$

Theorem 2.6 shows that the isoperimetric deficit controls the metric distortion in the class of convex domains. Convexity is rather limiting assumption so it is natural to ask whether similar inequality can be proved in a larger class. This is indeed possible, as was shown by Rajala and Zhong in [15]. They extended the result to John domains which exclude cusps (see Theorem 2.12). In [16] the author of this thesis further extended this type of results to more general case of s-John domains, $s > 1$, which allow controlled cusp behaviour (see Theorem 2.13).

Definition 2.11. Let $\Omega \subset \mathbb{R}^n$ be a bounded domain, $x_0 \in \Omega$, $s \geq 1$ and $b > 0$. We say that Ω is an s-John domain with the center x_0 and constant b if for each $x \in \Omega$ there exists a rectifiable curve $\gamma : [0, l] \to \Omega$ from x to x_0 that is parametrized by arclength and satisfies

$$\operatorname{dist}(\gamma(t), \mathbb{R}^n \setminus \Omega) \geq b \operatorname{diam}(\Omega)^{1-s} t^s$$

for all $t \in [0, l]$.

Note that for $s = 1$ the definition agrees with the classical definition of a John domain. We say that a path γ satisfying the condition in the definition is an s-John path. The constant b in the definition is called the s-John domain constant of the domain. It is also important to note that we have introduced a normalization in the condition so that the s-John domain constant is scaling invariant and hence geometrically meaningful. In the literature s-John domains have been studied in connection to Sobolev–Poincare type inequalities (see [23, 24, 25] and the references therein).

The following sharp result was proven in [15]. The outer metric distortion $\alpha(\Omega)$ of a bounded domain $\Omega \subset \mathbb{R}^n$ is defined as

$$\alpha(\Omega) = \frac{R-r}{r},$$

where $R > 0$ is the circumradius of Ω and $r > 0$ is the volume radius of Ω.

Theorem 2.12. *Let $\Omega \subset \mathbb{R}^n$, $n \geq 3$, be a John domain with center x_0 and constant $b > 0$. Then there exists a constant $C > 0$, depending only on n and b, such that*

$$\alpha(\Omega) \leq \begin{cases} C\Big(\delta(\Omega)\log(\frac{1}{\delta(\Omega)})\Big)^{1/2}, & n = 3, \\ C\delta(\Omega)^{1/(n-1)}, & n \geq 4. \end{cases}$$

In addition, if $\Omega_0^c := B(x_0, 2\operatorname{diam}(\Omega))$ is also a John domain, then $\alpha(\Omega)$ can be replaced by $\beta(\Omega)$ with C depending also on the John domain constant of Ω_0^c.

The proof of Theorem 2.12 is based on symmetrization and a sharp quantitative isoperimetric inequality for axially symmetric sets that are symmetric with respect to a hyperplane orthogonal to the axis of symmetry. In [16] the methods introduced in [15] were applied to s-John domains to obtain the following sharp theorem. See [16] for more detailed discussion of Theorems 2.12 and 2.13.

Theorem 2.13. *Let $\Omega \subset \mathbb{R}^n$, $n \geq 3$, be an s-John domain, $s > 1$. Then there exists a constant $C > 0$, depending only on n, s and the s-John domain constant, such that*

$$\alpha(\Omega) \leq C\delta(\Omega)^{1/(1+s(n-2))}.$$

In addition, if $\Omega_0^c := B(x_0, 2\operatorname{diam}(\Omega))$ is also an s-John domain, then $\alpha(\Omega)$ can be replaced by $\beta(\Omega)$ with C depending also on the s-John domain constant of Ω_0^c.

The exponents in Theorems 2.12 and 2.13 are optimal. In dimension three the sharpness of theorems by Rajala and Zhong is shown by the example of convexs domains given in [12]. For any $s \geq 1$ the sharpness of Theorem 2.13 can be verified by considering a unit ball with caps at both poles replaced by cusps as was discussed in [15] and [16]. Let $n \geq 3$ and $s \geq 1$ be such that $s + n > 4$. For any $0 < \eta < 1/2$, define the function $f_\eta : [0, 1] \to [0, \infty)$ by setting

$$f_\eta(t) = \begin{cases} \eta^{1/s} - t^{1/s} + \sqrt{1-\eta^2}, \ t < \eta, \\ \sqrt{1-t^2}, \ t \geq \eta. \end{cases}$$

Study the family of domains

$$\Omega_\eta = \{(\tilde{x}, x_n) \in \mathbb{R}^n : |x_n| < f_\eta(|\tilde{x}|)\}.$$

It is straightforward to see that $\delta(\Omega_\eta) \to 0$ as $\eta \to 0$, each Ω_η is an s-John domain and the s-John domain constant of Ω_η has a uniform lower bound for all $0 < \eta < 1/2$. Let us first calculate the volume of Ω_η. The combined volume of the two cusps is

$$V_c = 2(n-1)\omega_{n-1} \int_0^\eta (\eta^{1/s} - t^{1/s}) t^{n-2} \, \mathrm{d}t = \frac{2\omega_{n-1}}{1 + s(n-1)} \eta^{1/s+n-1},$$

while the combined volume of the two removed spherical caps is

$$\begin{aligned} V_s &= 2(n-1)\omega_{n-1} \int_0^\eta \big(\sqrt{1-t^2} - \sqrt{1-\eta^2}\big) t^{n-2} \, \mathrm{d}t \\ &= \frac{\omega_{n-1}}{n+1} \eta^{n+1} + o(\eta^{n+1}). \end{aligned}$$

Hence

$$|\Omega_\eta| = \omega_n + V_c - V_s = \omega_n + \frac{2\omega_{n-1}}{1+s(n-1)} \eta^{1/s+n-1} + o(\eta^{1/s+n-1}) \tag{2.46}$$

Next we calculate the perimeter of Ω_η. The combined perimeter of the two cusps is

$$\begin{aligned} P_c &= 2(n-1)\omega_{n-1} \int_0^\eta \Big(1 + \big(\frac{1}{s} t^{1/s-1}\big)^2\Big)^{1/2} t \, \mathrm{d}t \\ &= \frac{2(n-1)\omega_{n-1}}{s} \int_0^\eta \Big(1 + o(1)\Big) t^{1/s+n-3} \, \mathrm{d}t \\ &= \frac{2(n-1)\omega_{n-1}}{1+s(n-2)} \eta^{1/s+n-2} + o\big(\eta^{1/s+n-2}\big) \end{aligned}$$

and the combined perimeter of the two removed spherical caps is

$$\begin{aligned} P_s &= 2(n-1)\omega_{n-1} \int_0^\eta \big(1 + (-\frac{t}{\sqrt{1-t^2}})^2\big)^{1/2} t^{n-2} \, \mathrm{d}t \\ &= 2\omega_{n-1} \eta^{n-1} + o\big(\eta^{n-1}\big). \end{aligned}$$

Hence the perimeter of Ω_η is

$$\begin{aligned} P(\Omega_\eta) &= n\omega_n + P_c - P_s \\ &= n\omega_n + \frac{2(n-1)\omega_{n-1}}{1+s(n-2)} \eta^{1/s+n-2} + o(\eta^{1/s+n-2}). \end{aligned} \tag{2.47}$$

By (2.47) and (2.46) we have

$$\delta(\Omega_\eta) = \frac{2(n-1)\omega_{n-1}}{n\omega_n(1+s(n-2))} \eta^{1/s+n-2} + o\big(\eta^{n-1}\big). \tag{2.48}$$

Clearly

$$\alpha(\Omega_\eta) = \eta^{1/s} + o\big(\eta^{1/s}\big) \tag{2.49}$$

and when η is small we have

$$\beta(\Omega_\eta) = \eta^{1/s}. \tag{2.50}$$

By (2.49) and (2.48) we have

$$\frac{\alpha(\Omega)}{\delta(\Omega)^{\frac{1}{1+s(n-2)}}} = \Big(\frac{n\omega_n(1+s(n-2))}{2(n-1)\omega_{n-1}}\Big)^{\frac{1}{1+s(n-2)}} + o(1). \tag{2.51}$$

Similarly (2.50) and (2.48) give

$$\frac{\beta(\Omega)}{\delta(\Omega)^{\frac{1}{1+s(n-2)}}} = \Big(\frac{n\omega_n(1+s(n-2))}{2(n-1)\omega_{n-1}}\Big)^{\frac{1}{1+s(n-2)}} + o(1). \tag{2.52}$$

Inequalities (2.51) and (2.52) prove the sharpness of Theorem 2.12 and Theorem 2.13.

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Bonnesen's inequality for s-John domains in $\mathbb{R}^n$

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Abstract

We prove Bonnesen type inequalities for s-John domains, $s > 1$, which allow certain kind of cusps. We use methods introduced by Rajala and Zhong [5] and obtain similar quantitative isoperimetric inequalities as those obtained for John domains, with sharp decay rates depending continuously on parameter s.

1 Introduction

One of the fundamental geometric inequalities is the sharp isoperimetric inequality. It says that

$$n\omega_n^{1/n}|E|^{(n-1)/n} \leq P(E) \tag{1.1}$$

for any Borel set $E \subset \mathbb{R}^n$ with finite n-dimensional Lebesgue measure $|E|$, where $\omega_n = |B(0,1)|$ and $P(E)$ is the distributional perimeter of E (see Section 2). The equality holds if and only if E is, up to set of measure zero, a ball. For a Borel set $E \subset \mathbb{R}^n$, such that $|E| < \infty$, we introduce the isoperimetric deficit

$$\delta(E) = \frac{P(E)}{n\omega_n^{1/n}|E|^{(n-1)/n}} - 1.$$

The isoperimetric inequality (1.1) implies that the isoperimetric deficit of E is non-negative and vanishes if and only if E is a ball (up to set of measure zero). Quantitative isoperimetric inequalities are inequalities showing that a set with small isoperimetric deficit is close to ball with respect to some geometric quantity.

First results in the study of quantitative isoperimetric inequalities were proven by Bernstein [1] and Bonnesen [2] who considered convex Jordan domains in the plane. They proved that for any convex domain $\Omega \subset \mathbb{R}^2$, whose boundary is a rectifiable Jordan curve, we have the following inequality

$$\ell(\partial\Omega)^2 - 4\pi|\Omega| \geq 4\pi(R-\rho)^2, \tag{1.2}$$

where $\ell(\partial\Omega)$ is the length of the boundary, R is the circumradius and ρ the inradius of Ω. Many similar inequalities were derived by Bonnesen and the results were later extended for non-convex domains by various authors (see [13] for details). For any domain $\Omega \subset \mathbb{R}^2$, whose boundary is a rectifiable Jordan curve, we also have the inequality

$$\ell(\partial\Omega)^2 - 4\pi|\Omega| \geq 4\pi d^2, \tag{1.3}$$

where d is the width of the narrowest annulus containing the boundary of Ω, that is

$$d = \inf\big\{R - \rho : \text{there exists } x \in \mathbb{R}^2 \text{ such that } B(x,\rho) \subset \Omega \subset B(x,R)\big\}.$$

We refer to [12] for more detailed discussion of (1.3). We introduce two geometric quantities. The outer metric distortion of a bounded domain $\Omega \subset \mathbb{R}^n$ is defined as

$$\alpha(\Omega) = \frac{R - r}{r},$$

where R is the circumradius and r the volume radius of Ω. Furthermore, the metric distortion of a bounded domain $\Omega \subset \mathbb{R}^n$ is defined as

$$\beta(\Omega) = \inf\Big\{\frac{R-\rho}{\rho} : \text{ there exists } x \in \mathbb{R}^n \text{ such that } B(x,\rho) \subset \Omega \subset B(x,R)\Big\}.$$

Note that $\beta(\Omega) \geq \alpha(\Omega)$ for any bounded domain $\Omega \subset \mathbb{R}^n$. The following theorem, which is also stated in [11], is an easy consequence of (1.3).

Theorem 1.1. *Let $\Omega \subset \mathbb{R}^2$ be a domain bounded by a rectifiable Jordan curve and let $\delta_0 > 0$ be a constant. If $\delta(\Omega) < \delta_0$, then*

$$\alpha(\Omega) \leq \beta(\Omega) \leq C\delta(\Omega)^{1/2}, \tag{1.4}$$

where $C > 0$ depends only on δ_0.

In higher dimensions $n \geq 3$, Theorem 1.1 does not hold, in general, for bounded domains $\Omega \subset \mathbb{R}^n$. That is, in higher dimensions the isoperimetric deficit cannot, in general, control the metric distortion. Counterexamples can be given by gluing thin spikes to the unit ball. For this reason recent results have instead focused on deriving Bonnesen style inequalities (1.4) for some classes of sets in the higher dimensions.

First inequality of the type (1.4) for $n \geq 3$ was proven by Fuglede in [11]. In this paper he proved the following result for convex domains. Fuglede actually considered more general family of domains, called nearly spherical domains, see [11].

Theorem 1.2. *Let $\Omega \subset \mathbb{R}^n$, $n \geq 3$, be a bounded convex domain. Then*

$$\beta(\Omega) \leq \begin{cases} C(\delta(\Omega)\log\frac{1}{\delta(\Omega)})^{1/2}, & n = 3, \\ C\delta(\Omega)^{2/(n+1)}, & n \geq 4, \end{cases}$$

where the constants $C > 0$ depend only on n and are explicitly calculable.

In [5] Rajala and Zhong derived similar inequalities for domains that exclude outward and inward cusps. Namely, they considered John domains (see definition 1.4) whose complement with respect to a ball is also a John domain and proved the following theorem.

Theorem 1.3. *Let $\Omega \subset \mathbb{R}^n$, $n \geq 3$, be a John domain with center $x_0 \in \Omega$ and constant $b > 0$. Then there exists a constant $C > 0$, depending only on n and b, such that*

$$\alpha(\Omega) \leq \begin{cases} C\big(\delta(\Omega)\log\frac{1}{\delta(\Omega)}\big)^{1/2}, & n = 3, \\ C\delta(\Omega)^{1/(n-1)}, & n \geq 4. \end{cases} \tag{1.5}$$

In addition, if $\Omega_0^c := B(x_0, 4\,\mathrm{diam}(\Omega)) \setminus \Omega$ is also a John domain, then (1.5) holds with $\alpha(\Omega)$ replaced by $\beta(\Omega)$ and with C depending also on the John domain constant of Ω_0^c.

The proof of Theorem 1.3 in [5] is based on symmetrization and a sharp quantitative isoperimetric inequality for domains symmetric with respect to a line. Our main goal is to show that the methods introduced in [5] are not limited to John domains. In fact, we will show that similar results can be obtained for much larger class of domains using the same approach. Indeed, we will derive Bonnesen type inequalities for s-John domains, $s > 1$, which allow certain kind of cusps.

Definition 1.4. Let $\Omega \subset \mathbb{R}^n$ be a bounded domain, $x_0 \in \Omega$, $s \geq 1$ and $b > 0$. We say that Ω is an s-John domain with the center x_0 and constant b if for each $x \in \Omega$ there exists a rectifiable curve $\gamma : [0, l] \to \Omega$ from x to x_0 that is parametrized by arclength and satisfies

$$\mathrm{dist}(\gamma(t), \mathbb{R}^n \setminus \Omega) \geq b\,\mathrm{diam}(\Omega)^{1-s} t^s \tag{1.6}$$

for all $t \in [0, l]$.

Note that for $s = 1$ the definition agrees with the classical definition of a John domain. We say that a rectifiable curve γ that is parametrized by arclength and satisfies (1.6) is an s-John path. The constant b in the definition is called the s-John domain constant. The concept of s-John domain was introduced in [6]. In the literature s-John domains have been studied in connection to Sobolev–Poincare type inequalities, see [14, 15, 16] and the references therein. We remark here that, comparing to the definitions in the literature, we have introduced a normalization by the diameter of Ω in the condition (1.6). The normalization is necessary for the s-John domain constant to be scaling invariant and hence geometrically meaningful. Our first theorem gives bounds for the outer metric distortion of s-John domains.

Theorem 1.5. *Let $\Omega \subset \mathbb{R}^n$, $n \geq 3$, be an s-John domain, $s > 1$. Then there exists a constant $C > 0$, depending only on n, s and the s-John domain constant of Ω, such that*

$$\alpha(\Omega) \leq C\delta(\Omega)^{1/(1+s(n-2))}. \tag{1.7}$$

As in the case of John domains, to control the metric distortion we also need to have some control for the inward cusps. Our second theorem is the following.

Theorem 1.6. *Let $\Omega \subset \mathbb{R}^n$, $n \geq 3$, be an s-John domain, $s > 1$, with center $x_0 \in \Omega$. If $\Omega_0^c = B(x_0, 4\,\mathrm{diam}(\Omega))$ is also an s-John domain then there exists a constant $C > 0$, depending only on n, s and the s-John domain constants of Ω and Ω_0^c, such that*

$$\beta(\Omega) \leq C\delta(\Omega)^{1/(1+s(n-2))}. \tag{1.8}$$

Our results extend the results of the type considered in [11] and [5] to an even larger class of domains in dimension $n \geq 3$. The sharp decay rates in (1.7) and (1.8) depend continuously on the parameter $s > 1$.

In Section 3, similar to the proof in [5], we will first study model domains that are symmetric with respect to a line. However, our assumptions allow controlled cusp behaviour near the line of symmetry. Our main tool is the quantitative isoperimetric inequality for such symmetric domains (see Lemma 3.2), proven in [5]. We show that even under our assumptions it is possible to control the explicit error terms given by Lemma 3.2. In Section 4 we use the symmetrization scheme used in [5] to reduce the proof of Theorem 1.5 to domains of the form that was dealt with in Section 3. In Section 5 we prove Theorem 1.6 by using spherical symmetrization and a reflection argument to reduce the problem of controlling the metric distortion to a problem of controlling the outer metric distortion, as was done in [5]. We also use Theorems 1.2 and 2.1 in the symmetrization part of the proof.

As in (1.5), the decay rates in (1.7) and (1.8) are optimal and the constants in these theorems are explicitly calculable. In dimension three the sharpness of (1.5) is shown by the examples of convex domains given in [11]. In dimensions $n \geq 4$, the sharpness of (1.5) was proven in [5] by considering the unit ball with a cone attached at the pole. To show the sharpness of (1.7) and (1.8) we consider the unit ball with caps at both poles replaced by cusps. More precisely, let $n \geq 3$ and $s > 1$. For any $0 < \eta < 1/2$, define the function $f_\eta : [0, 1] \to [0, \infty)$ by

$$f_\eta(t) = \begin{cases} \eta^{1/s} - t^{1/s} + \sqrt{1-\eta^2}, & t < \eta, \\ \sqrt{1-t^2}, & t \geq \eta. \end{cases}$$

We consider the family of domains

$$\Omega_\eta = \{(\tilde{x}, x_n) \in \mathbb{R}^n : |x_n| < f_\eta(|\tilde{x}|)\}.$$

Each Ω_η is an s-John domain and the s-John domain constant of Ω_η has a uniform lower bound for all $0 < \eta < 1/2$. A simple calculation shows that

$$\delta(\Omega_\eta) = \frac{2(n-1)\omega_{n-1}}{n\omega_n(1+s(n-2))}\eta^{1/s+n-2} + O\big(\eta^{n-1}\big). \tag{1.9}$$

and

$$\alpha(\Omega_\eta) = \eta^{1/s} + o\big(\eta^{1/s}\big). \tag{1.10}$$

When η is small we have

$$\beta(\Omega_\eta) = \eta^{1/s}. \tag{1.11}$$

By (1.10) and (1.9) we have

$$\frac{\alpha(\Omega_\eta)}{\delta(\Omega_\eta)^{\frac{1}{1+s(n-2)}}} = \Big(\frac{n\omega_n(1+s(n-2))}{2(n-1)\omega_{n-1}}\Big)^{\frac{1}{1+s(n-2)}} + o(1). \tag{1.12}$$

Similarly (1.11) and (1.9) give

$$\frac{\beta(\Omega_\eta)}{\delta(\Omega_\eta)^{\frac{1}{1+s(n-2)}}} = \Big(\frac{n\omega_n(1+s(n-2))}{2(n-1)\omega_{n-1}}\Big)^{\frac{1}{1+s(n-2)}} + o(1). \tag{1.13}$$

Equations (1.12) and (1.13), respectively, prove that the decay rates in (1.7) and (1.8) are sharp.

2 Preliminaries

Let $E \subset \mathbb{R}^n$ be a Borel set and $\Omega \subset \mathbb{R}^n$ a domain. The perimeter of E in Ω is defined as

$$P(E, \Omega) = \sup\Big\{ \int_E \operatorname{div}\varphi \,\mathrm{d}x : \varphi \in C_c^1(\Omega; \mathbb{R}^n),\ \|\varphi\|_\infty \le 1 \Big\}.$$

As usual, we denote $P(E, \mathbb{R}^n)$ by $P(E)$. The perimeter is natural in the sense that it agrees with $\mathcal{H}^{n-1}$ measure of the boundary for smooth sets. For further properties of the perimeter see [3].

We need the following sharp quantitative isoperimetric inequality that was proven by Fusco, Maggi and Pratelli in [7]. They considered the Fraenkel asymmetry of a Borel set $E \subset \mathbb{R}^n$ that is defined as

$$\lambda(E) = \min_{x\in\mathbb{R}^n} \frac{|E \setminus B(x,r)|}{|E|},$$

where r is the volume radius of E.

Theorem 2.1. *Let $E \subset \mathbb{R}^n$ be Borel measurable. Then*

$$\lambda(E) \leq C\delta(E)^{\frac{1}{2}}$$

where C depends only on n.

We prove some basic properties of s-John domains. Let $\Omega \subset \mathbb{R}^n$ be an s-John domain, $s \geq 1$, with center $x_0 \in \Omega$ and constant $b > 0$ (see Definition 1.4). There exists a point $x \in \Omega$ satisfying

$$\operatorname{dist}(x, x_0) > \operatorname{diam}(\Omega)/4. \tag{2.1}$$

Because Ω is an s-John domain, there exists an s-John path $\gamma : [0, l] \to \Omega$ connecting x to x_0. The path γ satisfies

$$\operatorname{dist}(\gamma(t), \mathbb{R}^n \setminus \Omega) \geq b \operatorname{diam}(\Omega)^{1-s} t^s \tag{2.2}$$

for every $t \in [0, l]$. As γ is parametrized by arclength (2.1) shows that $l \geq \operatorname{diam}(\Omega)/4$. Hence (2.2) with $t = l$ gives

$$\operatorname{dist}(x_0, \mathbb{R}^n \setminus \Omega) \geq 4^{-s} b \operatorname{diam}(\Omega).$$

Thus the volume radius r of Ω has a lower bound

$$r \geq 4^{-s} b \operatorname{diam}(\Omega). \tag{2.3}$$

It is easy to prove that we may choose any point in an s-John domain as the center. Next lemma shows that the s-John domain constant with respect to the new center can be controlled by the distance of the new center to the boundary.

Lemma 2.2. *Let $\Omega \subset \mathbb{R}^n$ be an s-John domain, $s > 1$, with respect to $x_0 \in \Omega$ and with s-John domain constant $b > 0$. Assume $x \in \Omega$ and $\operatorname{dist}(x, \mathbb{R}^n \setminus \Omega) \geq \eta \operatorname{diam}(\Omega)$ for a constant $\eta > 0$. Then Ω is an s-John domain with respect to x with s-John domain constant depending only on s, b and η.*

Proof. The s-John domain constant of Ω is scaling invariant, so we may assume that $\operatorname{diam}(\Omega) = 1$. By the definition of an s-John domain (Definition 1.4) there exists a path $\gamma_0 : [0, l_0] \to \Omega$ from x to x_0 that is parametrized by arclength and satisfies

$$\operatorname{dist}\left(\gamma_0(t), \mathbb{R}^n \setminus \Omega\right) \geq bt^s \tag{2.4}$$

for all $t \in [0, l_0]$. As $\eta \leq 1$ and $\operatorname{dist}(x, \mathbb{R}^n \setminus \Omega) \geq \eta$, we have

$$\operatorname{dist}\left(\gamma_0(t), \mathbb{R}^n \setminus \Omega\right) \geq \eta/2 \geq (\eta/2)^s, \tag{2.5}$$

for all $t < \eta/2$. Furthermore, (2.4) shows that

$$\operatorname{dist}\left(\gamma_0(t), \mathbb{R}^n \setminus \Omega\right) \geq b(\eta/2)^s \tag{2.6}$$

for all $t \geq \eta/2$. Estimates (2.5) and (2.6) imply that

$$\operatorname{dist}\big(\gamma_0(t), \mathbb{R}^n \setminus \Omega\big) \geq \delta \tag{2.7}$$

for all $t \in [0, l_0]$, where $\delta = \min\{b(\eta/2)^s, (\eta/2)^s\}$.

Let $y \in \Omega$. Then there exists a path $\gamma_1 : [0, l_1] \to \Omega$ that connects y to x_0, is parametrized by arclength and satisfies

$$\operatorname{dist}\big(\gamma_1(t), \mathbb{R}^n \setminus \Omega\big) \geq bt^s \tag{2.8}$$

for all $t \in [0, l_1]$. Joining γ_1 and γ_0 we have a path $\gamma : [0, l_1 + l_0] \to \Omega$ that is parametrized by arclength and connects y to x. Note that (2.4) with $t = l_0$ gives

$$bl_0^s \leq \operatorname{dist}(x_0, \mathbb{R}^n \setminus \Omega) \leq \operatorname{diam}(\Omega) = 1,$$

that is $l_0 \leq b^{-\frac{1}{s}}$. Similarly, (2.8) with $t = l_1$ shows that $l_1 \leq b^{-1/s}$. Hence the length of γ is bounded from above by

$$l_0 + l_1 \leq 2b^{-1/s}. \tag{2.9}$$

We have

$$\operatorname{dist}\big(\gamma(t), \mathbb{R}^n \setminus \Omega\big) \geq ct^s \tag{2.10}$$

for all $t \in [l_1 + l_0]$, where $c = \min\{2^{-s}\delta b, b\}$. Indeed, (2.8) shows that (2.10) holds for all $t \in [0, l_1]$. Furthermore, the estimate (2.7), together with (2.9), proves that

$$\operatorname{dist}\big(\gamma(t), \mathbb{R}^n \setminus \Omega\big) \geq \delta \geq c(l_0 + l_1)^s \geq ct^s$$

for all $t \in [l_1, l_1 + l_0]$. Thus γ satisfies the condition (1.6) with constant c. Consequently Ω is an s-John domain with respect to x with s-John domain constant c, that depends only on s, b and η. □

3 Isoperimetric deficit for domains symmetric with respect to a line

Let $f : [0, 1] \to [0, \infty)$ be a non-increasing and non-negative function with $f(0) \geq 1/2$ and $f(1) = 0$. Assume there exist $M > 0$ and $s > 1$ such that

$$f(0) - f(t) \leq Mt^{\frac{1}{s}} \tag{3.1}$$

for all $t \in [0, 1]$. Define

$$\Omega_f := \{(\tilde{x}, x_n) \in \mathbb{R}^{n-1} \times \mathbb{R} : |x_n| < f(|\tilde{x}|)\}.$$

Theorem 3.1. *Let f and Ω_f be as above. Then there exists a constant $C > 0$, depending only on n, s and M, such that*

$$\frac{f(0) - r}{r} \leq C\delta(\Omega_f)^{\frac{1}{1+s(n-2)}}, \tag{3.2}$$

where r is the volume radius of Ω_f. Consequently, if $f(0) \geq 1$ and f also satisfies

$$f(t) \leq \sqrt{f(0)^2 - t^2} \tag{3.3}$$

for all $t \in [0,1]$, then

$$\alpha(\Omega_f) \leq C\delta(\Omega_f)^{\frac{1}{1+s(n-2)}}. \tag{3.4}$$

We begin with notation and a few remarks. Clearly, if f satisfies (3.3) we know that the circumradius of Ω_f is $f(0)$ and

$$\alpha(\Omega_f) = \frac{f(0) - r}{r}.$$

So the second claim (3.4) in Theorem 3.1 follows immediately from the first claim (3.2).

We note that (3.1) together with the fact that $f(0)$ is bounded from below guarantees that there exists $\epsilon > 0$, depending only on M and s, such that $B(0, \epsilon f(0)) \subset \Omega_f$. Hence the volume radius of Ω_f is bounded from below by $\epsilon f(0)$. It follows that

$$\frac{f(0) - r}{r} \leq \frac{f(0) - \epsilon f(0)}{\epsilon f(0)} = \frac{1 - \epsilon}{\epsilon}.$$

Note that ϵ depends only on M and s. So to prove (3.2) we only need to consider the case $\delta(\Omega_f) < 1$.

By approximation, we may assume that f is smooth. Indeed, by applying [3, (3.47)] in cylindrical coordinates to the surface of revolution $\partial\Omega_f$, it is enough to prove that f can be approximated in BV by smooth functions satisfying the assumptions. This can be done as follows: Extend f to $f(0)$ in $(-\infty, 0)$ and to 0 in $(1, \infty)$ and consider the standard convolutions g_ϵ for all $\epsilon > 0$. By construction $g_\epsilon(-\epsilon) = f(0)$ and $g_\epsilon(1 + \epsilon) = 0$. The smooth approximations $f_\epsilon(t) = g_\epsilon(-\epsilon + (1 + 2\epsilon)t)$ converge to f in BV and in addition $f_\epsilon(0) = f(0)$, $f_\epsilon(1) = 0$ and each f_ϵ is non-increasing. For $\epsilon > 0$ small enough we also see that f_ϵ satisfies condition (3.1) with constant $2M$.

Define auxiliary functions $\phi, \psi : [0,1] \to \mathbb{R}$ by setting

$$\phi(\tau) = \int_\tau^1 \left(1 + f'(t)^2\right)^{\frac{1}{2}} t^{n-2} \, \mathrm{d}t, \qquad \psi(\tau) = \int_\tau^1 f(t) t^{n-2} \, \mathrm{d}t,$$

for all $\tau \in [0,1]$. We note that $\phi(t)$ represents (up to a constant) the surface area and $\psi(t)$ the volume of a Ω_f outside the the cylinder $B^{n-1}(t) \times \mathbb{R}$. In particular we have

$$P(\Omega_f) = 2(n-1)\omega_{n-1}\phi(0), \qquad |\Omega_f| = 2(n-1)\omega_{n-1}\psi(0).$$

Define similar functions corresponding to the unit ball by setting

$$\phi_0(\tau) = \int_\tau^1 \frac{t^{n-2}}{\sqrt{1-t^2}}\,\mathrm{d}t, \qquad \psi_0(\tau) = \int_\tau^1 \sqrt{1-t^2}t^{n-2}\,\mathrm{d}t,$$

for all $\tau \in [0,1]$. The definitions of ϕ_0 and ψ_0 show that

$$\omega_n = |B(0,1)| = 2(n-1)\omega_{n-1}\psi_0(0)$$

and

$$n\omega_n = P(B(0,1)) = 2(n-1)\omega_{n-1}\phi_0(0).$$

In particular, we have the identity

$$\phi_0(0) = n\psi_0(0).$$

We will also denote

$$a = \big(\phi(0)/\phi_0(0)\big)^{\frac{1}{n-1}}.$$

As ϕ_0 is strictly increasing, it has a strictly increasing inverse function ϕ_0^{-1}. We introduce a reparametrization $\omega : [0,1] \to [0,1]$ by setting

$$\omega(t) = \phi_0^{-1}\big(\frac{\phi_0(0)}{\phi(0)}\phi(t)\big). \tag{3.5}$$

We observe that, indeed, ω is a strictly increasing differentiable function and, furthermore, $\omega(0) = 0$ and $\omega(1) = 1$. We may express the isoperimetric deficit of Ω_f using the auxiliary functions defined above. We find that

$$\delta(\Omega_f) = \frac{P(\Omega_f)}{n\omega_n^{1/n}|\Omega_f|^{(n-1)/n}} - 1 = \frac{\psi_0(0)^{(n-1)/n}}{\phi_0(0)}\frac{\phi(0)}{\psi(0)^{(n-1)/n}} - 1. \tag{3.6}$$

Expressing the volume radius of Ω_f in terms of the auxiliary functions, we conclude that

$$\frac{f(0)-r}{r} = \left(\frac{\psi_0(0)}{\psi(0)}\right)^{\frac{1}{n}} f(0) - 1. \tag{3.7}$$

The most important part of the proof of Theorem 3.1 is the following sharp quantitative isoperimetric inequality which is proved in [5]. Although [5] covers only the case $s = 1$, we note that exactly the same proof works also in our case as property (3.1) is not needed in the proof of the inequality.

Lemma 3.2. *Let f be as above. Then*

$$\psi(0) \le a^n \psi_0(0) - c(n)(F+G),$$

where $c(n) > 0$,

$$F = \int_0^1 \Big(- f'(t) - \frac{1}{a}\big(1+f'(t)^2\big)^{\frac{1}{2}} t\Big)^2 \frac{t^{n-2}}{\big(1+f'(t)^2\big)^{\frac{1}{2}}}\,\mathrm{d}t,$$

$G = 0$ when $n = 3$; and

$$G = \int_0^1 \Big(a^{\frac{1-n}{2}} \omega(t)^{\frac{3-n}{2}} t^{\frac{n-1}{2}} - \omega(t)\Big)^2 \big(1+f'(t)^2\big)^{\frac{1}{2}} t^{n-2}\,\mathrm{d}t$$

when $n \ge 4$, where ω is defined as in (3.5).

As in [5], we can rewrite Lemma 3.2 in terms of the isoperimetric deficit $\delta(\Omega_f)$ as the following corollary.

Corollary 3.3. *Let f be as in Theorem 3.1 and suppose that $\delta(\Omega) < 1$. Then there exists a constant $C(n) > 0$, depending only on n, such that*

$$F + G \le C(n) a^{n-1} \delta(\Omega_f),$$

where F and G are defined as in the statement of the Lemma 3.2.

Theorem 3.1 can now be proven by estimating F and G from below as in [5]. We will show that condition (3.1) gives the desired decay rate (3.2) for the quantity $(f(0) - r)/r$.

Proof of Theorem 3.1. Let $\theta \in (0,1)$ be arbitrary and define

$$\varrho(\theta) = \int_\theta^1 \big(1+f'(t)^2\big)^{\frac{1}{2}} t\,\mathrm{d}t.$$

The exact value of θ will be chosen later. First, we consider the integral F. By the Cauchy–Schwarz inequality we have

$$\begin{aligned} \big|f(\theta) - \frac{\varrho(\theta)}{a}\big| &= \Big|\int_\theta^1 \Big(-f'(t) - \frac{1}{a}\big(1+f'(t)^2\big)^{\frac{1}{2}} t\Big)\,\mathrm{d}t\Big| \\ &\le F^{\frac{1}{2}} \Big(\int_\theta^1 \frac{\big(1+f'(t)^2\big)^{\frac{1}{2}}}{t^{n-2}}\,\mathrm{d}t\Big)^{\frac{1}{2}}. \end{aligned} \tag{3.8}$$

We estimate the integral on the right hand side of (3.8) as follows. Define $\varphi : [0,1] \to \mathbb{R}$ as

$$\varphi(\tau) = \int_0^\tau \big(1+f'(t)^2\big)^{\frac{1}{2}}\,\mathrm{d}t, \quad t \in [0,1].$$

The estimate $\varphi(\tau) \geq \tau$ holds trivially for all $\tau \in [0,1]$. Furthermore, because f is non-increasing and satisfies (3.1), we obtain the estimate

$$\varphi(\tau) \leq \int_0^\tau \big(1 - f'(t)\big)\,\mathrm{d}t \leq \tau + M\tau^{\frac{1}{s}} \leq (M+1)\tau^{\frac{1}{s}} \tag{3.9}$$

for all $\tau \in [0,1]$. By integration by parts we have

$$\begin{aligned}\int_\theta^1 \frac{\big(1+f'(t)^2\big)^{\frac{1}{2}}}{t^{n-2}}\,\mathrm{d}t &= \int_\theta^1 \frac{\varphi'(t)}{t^{n-2}}\,\mathrm{d}t \\ &= \varphi(1) - \varphi(\theta)\theta^{2-n} + (n-2)\int_\theta^1 \varphi(t)t^{1-n}\,\mathrm{d}t.\end{aligned} \tag{3.10}$$

It follows from estimate (3.9) and equation (3.10) that

$$\begin{aligned}\int_\theta^1 \frac{\big(1+f'(t)^2\big)^{\frac{1}{2}}}{t^{n-2}}\,\mathrm{d}t &\leq M+1+(M+1)(n-2)\int_\theta^1 t^{\frac{1}{s}+1-n}\,\mathrm{d}t \\ &= M+1+\frac{(M+1)(n-2)}{n-2-\frac{1}{s}}\big(\theta^{\frac{1}{s}+2-n}-1\big).\end{aligned} \tag{3.11}$$

As $\theta \leq 1$ and $1/s + 2 - n < 0$, the inequality (3.11) gives the estimate

$$\int_\theta^1 \frac{\big(1+f'(t)^2\big)^{\frac{1}{2}}}{t^{n-2}}\,\mathrm{d}t \leq C(n,s,M)\theta^{\frac{1}{s}-(n-2)}. \tag{3.12}$$

Combining estimates (3.8) and (3.12), we obtain that

$$\big|f(\theta) - \frac{\varrho(\theta)}{a}\big| \leq CF^{\frac{1}{2}}\theta^{\frac{1}{2s}-\frac{n-2}{2}}. \tag{3.13}$$

Next, we consider the integral G. We will show that

$$\big|\varrho(\theta) - a^2\big| \leq C\Big(G^{\frac{1}{2}}\delta(\Omega)^{\frac{1}{2s}-\frac{n-2}{2}} + \theta^{1+\frac{1}{s}}\Big). \tag{3.14}$$

In the case $n = 3$, the estimate (3.14) follows from a simple calculation. Indeed, if $n = 3$ then $G = 0$, $\varrho(\theta) = \phi(\theta)$ and $a^2 = \phi(0)/\phi_0(0) = \phi(0)$. Thus

$$\big|\varrho(\theta) - a^2\big| = \big|\phi(\theta) - \phi(0)\big| \leq \int_0^\theta \big(1+f'(t)^2\big)^{\frac{1}{2}} t\,\mathrm{d}t \leq \theta\varphi(\theta) \leq C\theta^{1+\frac{1}{s}},$$

as claimed.

Next we prove the inequality (3.14) in the case $n \geq 4$. We argue exactly as in the proof of the estimate (3.9) to estimate the quantity

$$\phi(0) - \phi(\tau) = \int_0^\theta \big(1+f'(t)^2\big)^{\frac{1}{2}} t^{n-2}\,\mathrm{d}t$$

as follows

$$\frac{1}{n-1}\tau^{n-1} \le \phi(0) - \phi(\tau) \le (M+1)\tau^{n-2+\frac{1}{s}}. \tag{3.15}$$

Applying estimate (3.15) for $\tau = 1$, we have

$$\big((n-1)\phi_0(0)\big)^{-\frac{1}{n-1}} \le a = \big(\phi(0)/\phi_0(0)\big)^{\frac{1}{n-1}} \le \big((M+1)/\phi_0(0)\big)^{\frac{1}{n-1}}. \tag{3.16}$$

Thus a is bounded above and below by constants depending only on n and M. We note that the inverse function of ϕ_0 satisfies

$$ct^{1/(n-1)} \le \phi_0^{-1}\big(\phi_0(0) - t\big) \le Ct^{1/(n-1)}$$

for all $t \in [0,1]$, which together with estimates (3.15) and (3.16) shows that

$$\omega(t) = \phi_0^{-1}\big(\frac{\phi_0(0)}{\phi(0)}\phi(t)\big) \ge c\Big(\frac{\phi_0(0)}{\phi(0)}(\phi(0) - \phi(t))\Big)^{1/(n-1)} \ge ct. \tag{3.17}$$

Define $I : [0,1] \to \mathbb{R}$ by

$$I(t) = a^{3-n}\omega(t)^{3-n} - t^{3-n}.$$

We will show that

$$\big|I(t)\big| \le Ct^{2-n}\big|a^{\frac{1-n}{2}}\omega(t)^{\frac{3-n}{2}}t^{\frac{n-1}{2}} - \omega(t)\big|. \tag{3.18}$$

For that purpose we need an elementary inequality. For any $\gamma > 0$ there exists a constant $C(\gamma) > 0$ such that the inequality

$$\big|x^\gamma - y^\gamma\big| \le C(\gamma)|x-y|(x+y)^{\gamma-1}$$

holds for all $x, y \ge 0$. We apply this inequality with $x = a^{\frac{1-n}{2}}\omega(t)^{\frac{1-n}{2}}$, $y = t^{\frac{1-n}{2}}$ and $\gamma = 2(n-3)/(n-1) > 0$ to obtain the estimate

$$\big|I(t)\big| \le C\big|a^{\frac{1-n}{2}}\omega(t)^{\frac{1-n}{2}} - t^{\frac{1-n}{2}}\big|\big(a^{\frac{1-n}{2}}\omega(t)^{\frac{1-n}{2}} + t^{\frac{1-n}{2}}\big)^{\frac{n-5}{n-1}}, \tag{3.19}$$

which can be written as

$$\big|I(t)\big| \le Ct^{\frac{1-n}{2}}\omega(t)^{-1}\big|a^{\frac{1-n}{2}}\omega(t)^{\frac{3-n}{2}} - \omega(t)\big|\big(a^{\frac{1-n}{2}}\omega(t)^{\frac{1-n}{2}} + t^{\frac{1-n}{2}}\big)^{\frac{n-5}{n-1}} \tag{3.20}$$

If $n = 4$, estimate (3.17), together with (3.16), shows that

$$\big(a^{-\frac{3}{2}}\omega(t)^{-\frac{3}{2}} + t^{-\frac{3}{2}}\big)^{-\frac{1}{3}} \le C\omega(t)^{\frac{1}{2}}. \tag{3.21}$$

We combine the estimates (3.20) and (3.21) to conclude that

$$\big|I(t)\big| \le Ct^{-\frac{3}{2}}\omega(t)^{-\frac{1}{2}}\big|a^{-\frac{3}{2}}\omega(t)^{-\frac{1}{2}}t^{\frac{3}{2}} - \omega(t)\big|. \tag{3.22}$$

We apply (3.17) to estimate $\omega(t)$ from below in (3.22). This gives the estimate

$$\big|I(t)\big| \le Ct^{-2}\big|a^{-\frac{3}{2}}\omega(t)^{-\frac{1}{2}}t^{\frac{3}{2}} - \omega(t)\big|,$$

which is exactly the claimed inequality (3.18) in dimension $n = 4$.

In case $n \ge 5$ the estimate (3.17), together with (3.16), shows that

$$\big(a^{\frac{1-n}{2}}\omega(t)^{\frac{1-n}{2}} + t^{\frac{1-n}{2}}\big)^{\frac{n-5}{n-1}} \le Ct^{\frac{5-n}{2}}. \tag{3.23}$$

Inserting (3.23) into (3.20), we have the estimate

$$\big|I(t)\big| \le Ct^{3-n}\omega(t)^{-1}\big|a^{\frac{1-n}{2}}\omega(t)^{\frac{3-n}{2}}t^{\frac{n-1}{2}} - \omega(t)\big|. \tag{3.24}$$

We apply (3.17) to estimate $\omega(t)$ from below in (3.24). This gives the estimate

$$\big|I(t)\big| \le t^{2-n}\big|a^{\frac{1-n}{2}}\omega(t)^{\frac{3-n}{2}}t^{\frac{n-1}{2}} - \omega(t)\big|,$$

in dimension $n \ge 5$, as claimed.

Recall that

$$G = \int_0^1 \big(a^{\frac{1-n}{2}}\omega(t)^{\frac{3-n}{2}}t^{\frac{n-1}{2}} - \omega(t)\big)^2\big(-\phi'(t)\big)\,\mathrm{d}t.$$

Applying estimate (3.18) and the Cauchy–Schwarz inequality, together with (3.12), we have

$$\begin{aligned}\Big|\int_\theta^1 I(t)\big(-\phi'(t)\big)\,\mathrm{d}t\Big| &\le CG^{\frac{1}{2}}\Big(\int_\theta^1 \frac{\big(1+f'(t)^2\big)^{\frac{1}{2}}}{t^{n-2}}\,\mathrm{d}t\Big)^{\frac{1}{2}} \\ &\le CG^{\frac{1}{2}}\theta^{\frac{1}{2s}-\frac{n-2}{2}}.\end{aligned} \tag{3.25}$$

To estimate the integral on the left hand side of the inequality (3.25), we note that

$$\int_\theta^1 I(t)\big(-\phi'(t)\big)\,\mathrm{d}t = a^{3-n}\int_\theta^1 \omega(t)^{3-n}\big(-\phi'(t)\big)\,\mathrm{d}t - \varrho(\theta). \tag{3.26}$$

Differentiating ω and ϕ_0 we see that

$$\omega'(t) = \frac{a^{1-n}}{\phi_0'\big(\omega(t)\big)}\phi'(t) = \frac{a^{1-n}\sqrt{1-t^2}}{t^{n-2}}\phi'(t). \tag{3.27}$$

By change of variables and equation (3.27) we have

$$\int_0^1 \omega(t)^{3-n}\big(-\phi'(t)\big)\,\mathrm{d}t = a^{n-1}\int_0^1 \frac{t}{\sqrt{1-t^2}}\,\mathrm{d}t = a^{n-1}. \tag{3.28}$$

The estimate (3.17) and the proof of (3.9) show that

$$\int_0^\theta \omega(t)^{3-n}\big(-\phi'(t)\big)\,\mathrm{d}t \le C\int_0^\theta \big(1+f'(t)^2\big)^{\frac{1}{2}}t\,\mathrm{d}t \le C\theta^{1+\frac{1}{s}}. \tag{3.29}$$

Equations (3.26) and (3.28) imply that

$$\int_\theta^1 I(t)\big(-\phi'(t)\big)\,\mathrm{d}t = a^2 - a^{3-n}\int_0^\theta \omega(t)^{3-n}\big(-\phi'(t)\big)\,\mathrm{d}t - \varrho(\theta). \tag{3.30}$$

A simple application of the triangle inequality, together with (3.30), (3.29) and (3.16), proves that

$$\Big|\int_\theta^1 I(t)\phi'(t)dt\Big| \geq |\varrho(\theta) - a^2| - C\theta^{1+\frac{1}{s}}. \tag{3.31}$$

The claimed inequality (3.14) follows immediately from the estimates (3.25) and (3.31).

Estimates (3.13), (3.14) and (3.16), imply that

$$\begin{aligned} \big|f(\theta) - a\big| &\leq \big|f(\theta) - \frac{\varrho(\theta)}{a}\big| + \frac{1}{a}\big|\varrho(\theta) - a^2\big| \\ &\leq C\big(\delta(\Omega_f)^{\frac{1}{2}}\theta^{\frac{1}{2s}-\frac{n-2}{2}} + \theta^{\frac{1}{s}+1}\big), \end{aligned} \tag{3.32}$$

where we have used Corollary 3.3 to estimate F and G from above by the isoperimetric deficit. Assumption (3.1) allows us to control the behaviour of f near 0. Hence (3.1) and (3.32) give

$$\big|f(0) - a\big| \leq \big|f(0) - f(\theta)\big| + \big|f(\theta) - a\big| \leq C\big(\delta(\Omega_f)^{\frac{1}{2}}\theta^{\frac{1}{2s}-\frac{n-2}{2}} + \theta^{\frac{1}{s}}\big). \tag{3.33}$$

Recalling formulae (3.6) and (3.7), and utilizing estimate (3.16), we see that

$$\frac{f(0) - r}{r} = (1 + \delta(\Omega_f))^{\frac{1}{n-1}}\frac{f(0)}{a} - 1 \leq C(|f(0) - a| + \delta(\Omega_f)).$$

We use (3.33) to estimate $|f(0) - a|$ from above in the previous inequality and choose $\theta = \delta(\Omega_f)^{\frac{s}{1+s(n-2)}} \in (0, 1)$ to obtain the estimate

$$\frac{f(0) - r}{r} \leq C\delta(\Omega_f)^{\frac{1}{1+s(n-2)}},$$

which is the claimed inequality (3.2). □

4 Proof of theorem 1.5

In this section, we prove Theorem 1.5. Let $\Omega \subset \mathbb{R}^n$ be a bounded s-John domain with center $x_0 \in \Omega$ and constant $b > 0$. Choose a ball $B(z, r)$ that realizes the Frankel asymmetry, that is, the equation $|\Omega \backslash B(z, r)|/|\Omega| = \lambda(\Omega)$ holds. By translation we may assume that $z = 0$. Thus we have

$$|\Omega \setminus B(r)|/|\Omega| = \lambda(\Omega). \tag{4.1}$$

As the s-John domain constant is scaling invariant, we may assume $\operatorname{diam}(\Omega) = 1$. We denote by u the radius of the smallest ball, centered at 0, that contains Ω. Obviously the circumradius R of Ω satisfies $R \leq u$. Thus, to prove Theorem 1.5, it is enough to prove that

$$\frac{u-r}{r} \leq C\delta(\Omega)^{\frac{1}{1+s(n-2)}}. \tag{4.2}$$

This stronger result is needed in the proof of Theorem 1.6.

Let $\delta_0 > 0$ be a small number to be chosen later. Assume first that $\delta(\Omega) \geq \delta_0$. As in the proof of (2.3), there exists a ball

$$B(x_0, \epsilon) \subset \Omega, \tag{4.3}$$

where $0 < \epsilon < 1$ depends only on s and the s-John domain constant of Ω. Clearly u is bounded above by the diameter so (4.3) gives

$$\frac{u-r}{r} \leq \frac{1-\epsilon}{\epsilon} \leq C(\epsilon, \delta_0)\delta(\Omega)^{\frac{1}{1+s(n-2)}}. \tag{4.4}$$

Next we will make a suitable choice of δ_0, depending only on n, s and b, and prove the claim in the case $\delta(\Omega) < \delta_0$.

Assume $\delta(\Omega) \leq \delta_0$. By the choice of u there exists $a \in \partial\Omega \cap S^{n-1}(u)$ and, rotating if necessary, we may assume that $a = ue_n$, where e_n is the nth standard basis vector and $S^{n-1}(u)$ is the sphere of radius u centered at origin.

We may assume, without loss of generality, that the s-John domain center of Ω satisfies

$$x_0 \in B(\epsilon r/4) \subset B(r/4), \tag{4.5}$$

where $0 < \epsilon < 1$ is as in (4.3). Indeed, if the original s-John domain center x_0 of Ω does not lie in $B(\epsilon r/4)$ we argue as follows. Theorem 2.1 shows that $\lambda(\Omega) \leq \delta(\Omega)^{1/2}$. We choose δ_0 so small that $\lambda(\Omega) < (8\epsilon)^{-n}$, which by (4.1) implies that $\Omega \cap B(\epsilon r/8) \neq \emptyset$. Next we choose a point $x \in \Omega \cap B(\epsilon r/8)$ and connect x to x_0 with an s-John path γ. As $x_0 \notin B(\epsilon r/4)$ and $x \in B(\epsilon r/8)$ there exists $\tilde{x}_0 \in S^{n-1}(x, 3\epsilon r/16)$ on the path γ. As $|\tilde{x}_0 - x| \geq \epsilon r/16$ and $\tilde{x}_0$ lies on γ, which is parametrized by arclength, we have

$$\tilde{x}_0 = \gamma(t_0) \tag{4.6}$$

for some $t_0 \geq \epsilon r/16$. Because we have (4.6) and γ satisfies (1.6), we obtain

$$\operatorname{dist}(\tilde{x}_0, \mathbb{R}^n \setminus \Omega) = \operatorname{dist}(\gamma(t_0), \mathbb{R}^n \setminus \Omega) \geq b(\epsilon r/16)^s \geq C, \tag{4.7}$$

where $C > 0$ depends only on s and b, and where in the second inequality we have estimated the volume radius from below by (2.3). The estimate above, together with Lemma 2.2, shows that Ω is an s-John domain with center $\tilde{x}_0 \in B(x, \epsilon r/4)$ and a constant that depends only on n, s and the original constant.

Denote by μ the maximum of those $t \in \mathbb{R}$ for which

$$\Omega \subset \{x_n > t\}. \tag{4.8}$$

If $0 \geq \mu \geq -r$ an elementary lower bound for the measure of the spherical cap, together with (4.8) and (4.1), shows that

$$(\mu - (-r))^n \leq C\,|B(r) \cap \{x_n \leq \mu\}| \leq C\,|B(r) \setminus \Omega| \leq C\lambda(\Omega)r^n.$$

By Theorem 2.1 and the estimate above, we may choose δ_0 so small that $\mu < -3r/4$. In particular, there exists a point $y \in \Omega \cap \{x_n < -3r/4\}$. We connect y to the s-John domain center x_0 of Ω with an s-John path γ. By (4.5) and the fact that $y_n < -3r/4$, there exists a point v on γ, such that $v_n = -r/2$. As γ is an s-John path and $|v-y| \geq r/4$, we may argue similarly as in the proof of (4.7) to obtain

$$\operatorname{dist}(v, \mathbb{R}^n \setminus \Omega) \geq b(r/4)^s \geq C,$$

where we also used (2.3). By lemma 2.2 the domain Ω is an s-John domain with center v and a constant that depends only on n, s and the the original s-John domain constant. In the following we denote by b the minimum of the original s-John domain constant and the s-John domain constant of Ω with respect to v.

Next we prove that for each $-r/2 \leq t < u$ there exists a point $y(t) \in \Omega$, such that $(y(t))_n = t$ and

$$\operatorname{dist}(y(t), \mathbb{R}^n \setminus \Omega) \geq C(u-t)^s. \tag{4.9}$$

Indeed, let $-r/2 \leq t < u$. Choose a point $y \in \Omega$, such that $u - y_n \leq (u-t)/2$. As Ω is an s-John domain with center v, there exists an s-John path γ connecting y to v. As $v_n = -r/2$ and $y_n > t$ we may choose a point $y(t)$ on γ, such that $(y(t))_n = t$. Because γ is an s-John path and $|y(t) - y| \geq (u-t)/2$, we argue similarly as in the proof of (4.7) to obtain

$$\operatorname{dist}(y(t), \mathbb{R}^n \setminus \Omega) \geq 2^{-s} b(u-t)^s,$$

which proves (4.9).

We perform Schwarz symmetrization on Ω with respect to x_n-axis. This gives a domain Ω', such that $\Omega' \cap \{x_n = t\}$ is an $(n-1)$-ball with center $(0, .., 0, t)$ and $\mathcal{H}^{n-1}$-measure equal to that of $\Omega \cap \{x_n = t\}$. By Fubini's theorem we have $|\Omega'| = |\Omega|$ and obviously $a \in \partial\Omega'$. By [4] we know that the perimeter decreases under Schwarz symmetrization, hence

$$\delta(\Omega') \leq \delta(\Omega). \tag{4.10}$$

By Fubini's theorem we also have

$$|B(r) \setminus \Omega'| \leq |B(r) \setminus \Omega| = \omega_n r^n \lambda(\Omega), \tag{4.11}$$

where we also used (4.1). By the construction of Ω', (4.3) and (4.5) we have

$$B(\epsilon/2) \subset B(\epsilon - \epsilon r/4) \subset \Omega', \tag{4.12}$$

where we also used the fact that $r < \operatorname{diam}(\Omega) = 1$.

Let $\nu \in \mathbb{R}$ be such that

$$|\Omega_\nu^+| = |\Omega' \setminus \Omega_\nu^+| = |B(r)|/2, \tag{4.13}$$

where $\Omega_\nu^+ = \Omega' \cap \{x_n \geq \nu\}$. We claim that

$$\min\{|\nu|, r\} \leq Cr\lambda(\Omega) \leq Cr\delta(\Omega)^{1/2} \leq Cr\delta(\Omega)^{1/(1+s(n-2))}. \tag{4.14}$$

Indeed, the second inequality is a consequence of Theorem 2.1 and the first one is proven as follows. We observe that

$$\begin{aligned} |B(r) \setminus \Omega'| &\geq |B(r) \cap \{x_n < \nu\} \setminus \Omega'| \\ &\geq |B(r) \cap \{x_n < \nu\}| - |\Omega' \cap \{x_n < \nu\}|. \end{aligned} \tag{4.15}$$

If $\nu \geq 0$, we have the elementary estimate

$$|B(r) \cap \{x_n < \nu\}| \geq |B(r)|/2 + C\min\{\nu, r\} r^{n-1},$$

which, together with (4.11), (4.13) and (4.15) gives the first inequality in (4.14) for $\nu \geq 0$. If $\nu < 0$, a similar argument, with $\{x_n < \nu\}$ replaced by $\{x_n > \nu\}$, proves (4.14).

If $(u - r)/2 \leq |\nu|$, then we may apply estimate (4.14) to conclude that

$$\frac{u - r}{r} \leq C\delta(\Omega)^{1/2},$$

which proves the claimed inequality (4.2). In the following we assume

$$|\nu| \leq (u - r)/2. \tag{4.16}$$

By (4.14) we may also choose δ_0 so small that

$$|\nu| \leq r/4. \tag{4.17}$$

Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be the reflection with respect to the hyperplane $\{x_n = \nu\}$. Define

$$\Omega^* = \Omega_\nu^+ \cup T(\Omega_\nu^+).$$

By (4.13) it is clear that $|\Omega^*| = |\Omega'|$. We also notice that $a = ue_n \in \partial\Omega^*$ and $T(a) = (2\nu - u)e_n \in \partial\Omega^*$. By the relative isoperimetric inequality in the open half space we have

$$P(\Omega', \{x_n < \nu\}) \geq n\omega_n |\Omega'|^{(n-1)/n}/2.$$

Furthermore, by the construction of Ω^* we have

$$P(\Omega^*) = 2P(\Omega_\nu^+, \{x_n > \nu\}).$$

The two inequalities above give the estimate

$$\begin{aligned} P(\Omega') &\geq P(\Omega', \{x_n > \nu\}) + P(\Omega_\nu^+, \{x_n > \nu\}) \\ &\geq n\omega_n^{1/n} |\Omega|^{(n-1)/n}/2 + P(\Omega^*)/2. \end{aligned}$$

As $|\Omega^*| = |\Omega'|$, it follows from the previous inequality that

$$P(\Omega^*) - n\omega_n^{\frac{1}{n}} |\Omega^*|^{(n-1)/n} \leq 2(P(\Omega') - n\omega_n^{1/n}|\Omega'|^{(n-1)/n}).$$

Thus

$$\delta(\Omega^*) \leq 2\delta(\Omega') \leq 2\delta(\Omega), \tag{4.18}$$

where the second inequality follows from (4.10). Because $a, T(a) \in \partial\Omega^*$, we may apply (4.16) to estimate the circumradius of Ω^* from below by

$$|a - T(a)|/2 = (u - (2\nu - u))/2 \geq (u + r)/2.$$

It follows that

$$\alpha(\Omega^*) \geq \frac{(u + r)/2 - r}{r} = \frac{u - r}{2r}. \tag{4.19}$$

By (4.14) and the fact that $r \leq \text{diam}(\Omega) = 1$ we may choose δ_0 so small that

$$|\nu| \leq \epsilon r/4 \leq \epsilon/4,$$

which, together with (4.12) and the construction of Ω^*, shows that

$$B(\nu e_n, \epsilon/4) \subset \Omega^*. \tag{4.20}$$

The condition (4.9), together with (4.17), imply that in particular

$$\mathcal{H}^{n-1}(\Omega^* \cap \{x_n = t\}) \geq C(u - t)^{s(n-1)} \tag{4.21}$$

for $\nu \leq t < u$. We perform Steiner symmetrization on Ω^*, with respect to the hyperplane $\{x_n = \nu\}$, to obtain a domain Ω^{**}, such that $\Omega^{**} \cap (\{\tilde{x}\} \times \mathbb{R})$ is an open line segment with center $(\tilde{x}, \nu)$ and 1-dimensional Hausdorff measure equal to $H^1(\Omega^* \cap (\{\tilde{x}\} \times \mathbb{R}))$, for each $\tilde{x} \in \mathbb{R}^{n-1}$. By Fubini's theorem we have $|\Omega^{**}| = |\Omega^*|$ and by [4], Steiner symmetrization does not increase the perimeter. Hence

$$\delta(\Omega^{**}) \leq \delta(\Omega^*) \leq \delta(\Omega). \tag{4.22}$$

Translating so that $\nu = 0$, we see that Ω^{**} is actually of the form

$$\Omega_f = \{(\tilde{x}, x_n) \in \mathbb{R}^{n-1} \times \mathbb{R} : |x_n| < f(|\tilde{x}|)\},$$

where $f : [0, \infty) \to [0, \infty)$ is a decreasing function. We have $f(0) = u - \nu$. Furthermore, because Ω^* is symmetric with respect to $\{x_n = \nu\}$ and satisfies (4.21), we have

$$f(t) \geq \mathcal{H}^1\big(\{0 \leq \tau \leq u - \nu : C(u - (\tau + \nu))^s \geq t\}\big) = u - \nu - C^{-1/s} t^{1/s},$$

that is

$$f(t) \geq f(0) - M t^{1/s}, \tag{4.23}$$

for all $t \geq 0$, where $M > 0$ is a constant depending only on n, s and the s-John domain constant of Ω. Denote $\tau := \inf\{t > 0 : f(t) = 0\}$ and consider the function $g : [0, 1] \to [0, \infty)$, defined by $g(t) = \tau^{-1} f(\tau t)$ for all $t \in [0, 1]$. The domain

$$\Omega_g = \{(\tilde{x}, x_n) \in \mathbb{R}^{n-1} \times \mathbb{R} : |x_n| < g(|\tilde{x}|)\},$$

is actually $\tau^{-1}\Omega_f$. By the choice of τ we have

$$g(1) = 0. \tag{4.24}$$

Note that $f(0) = u - \nu \geq r - |\nu|$. As the circumradius of Ω_f is bounded from above by $u + |\nu| \leq r + |\nu|$, we have $\tau \leq r + |\nu|$. The previous two estimates for $f(0)$ and τ, together with (4.17), show that

$$g(0) \geq 1/2. \tag{4.25}$$

By (4.20) and the construction of Ω_f we have $B(\epsilon/4) \subset \Omega_f$, where $\epsilon > 0$ depends only on s and the s-John domain constant of Ω. Hence τ is bounded from below by

$$\tau \geq \epsilon/4. \tag{4.26}$$

By (4.26) and (4.23) we have

$$g(t) = \tau^{-1} f(\tau t) \geq g(0) - M \tau^{1/s - 1} t^{1/s} \geq g(0) - \tilde{M} t^{1/s}, \tag{4.27}$$

where $\tilde{M} > 0$ depends only on n, s and the s-John domain constant of Ω. Because f is decreasing g is also decreasing, which together with (4.27), (4.24) and (4.25), shows that the function g satisfies the assumptions of Theorem 3.1. Thus we conclude that

$$\frac{f(0) - r}{r} = \frac{g(0) - b^{-1} r}{b^{-1} r} \leq C \delta(\Omega_g)^{1/(1+s(n-2))} = C \delta(\Omega_f)^{1/(1+s(n-2))}, \tag{4.28}$$

where C depends only on n, s and the s-John domain constant of Ω, and where we also used the facts that the volume radius of Ω_g is $b^{-1}r$ and the isoperimetric deficit is scaling invariant. The circumradius of Ω^* is bounded from above by $u + |\nu|$. Hence (4.28), (4.14) and (4.22) give the estimate

$$\begin{aligned}\alpha(\Omega^*) &\leq \frac{u + |\nu| - r}{r} = \frac{f(0) - r}{r} + \frac{\nu + |\nu|}{r} \\ &\leq C\delta(\Omega)^{1/(1+s(n-2))}.\end{aligned} \tag{4.29}$$

Combining this inequality with (4.19) proves the claimed inequality (4.2) and finishes the proof of Theorem 1.5.

5 Proof of theorem 1.6

Now we use Theorem 1.5 to prove Theorem 1.6. Let $\Omega \subset \mathbb{R}^n$ be an s-John domain with center x_0, such that $\Omega_0^c = B(x_0, 4\,\mathrm{diam}(\Omega)) \setminus \Omega$ is also an s-John domain. As in the proof of Theorem 1.5 we may assume that $\delta(\Omega) < \delta_0$, where $\delta_0 > 0$ is a small number to be chosen later and depending only on n, s and the s-John domain constant of Ω. Let r be the volume radius of Ω and let $B(z, r)$ be a ball realizing Frankel asymmetry. By translation, we may assume that $z = 0$. As the s-John domain constant is scaling invariant we may assume that $\mathrm{diam}(\Omega) = 1$. As in (4.5) we can also assume that the s-John domain center of Ω is inside $B(r/2)$, if δ_0 is chosen small enough. Let u be the smallest radius and ρ the largest radius, such that

$$B(\rho) \subset \Omega \subset B(u).$$

Arguing similarly as in the proof of (4.5) we can assume, without loss of generality, that the s-John domain center of Ω_0^c is outside $B(2u)$. By (4.2), we have

$$\frac{u - r}{r} \leq C\delta\,(\Omega)^{1/(1+s(n-2))}\,. \tag{5.1}$$

We will need a preliminary lemma that allows us to control ρ. Note that the conclusion of the lemma is not sharp.

Lemma 5.1. *Let Ω, r and ρ be as above. We have*

$$\frac{r - \rho}{r} \leq C\lambda(\Omega)^{1/(ns)} \leq C\delta(\Omega)^{1/(2ns)},$$

where $C > 0$ depends only on n, s and the s-John domain constant of Ω and Ω_0^c.

Proof. We will use the s-John domain property of Ω_0^c. As $S^{n-1}(\rho) \setminus \Omega \neq \emptyset$ there exists $x \in S^{n-1}(\rho + (r - \rho)/4) \cap \Omega_0^c$. We connect x to the s-John domain center of Ω_0^c, which lies outside $B(2u)$, with an s-John path γ and argue similarly as in the proof of (4.9) to find a point $y \in S(\rho+(r-\rho)/2) \cap \Omega_0^c$ on γ, such that

$$B(y, C(r-\rho)^s) \subset B(r) \setminus \Omega,$$

where $C > 0$ depends only on s and the s-John domain constant of Ω_0^c. Consequently, we can argue similarly as in the proof of (4.7) to conclude that

$$(r-\rho)^{ns} \leq C\lambda(\Omega) r^n.$$

The previous inequality and the lower bound (2.3) for r give

$$r - \rho \leq C\lambda(\Omega)^{1/(ns)} r^{1/s} \leq C\lambda(\Omega)^{1/(ns)} r.$$

By the previous inequality and Theorem (2.1), we have

$$\frac{r-\rho}{r} \leq C\lambda(\Omega)^{1/(ns)} \leq C\delta(\Omega)^{1/(2ns)}.$$

□

By the previous lemma we can choose δ_0 so small that

(5.2) $$\rho \geq r/2.$$

Inequality (5.1), together with (5.2), gives

(5.3) $$\beta(\Omega) \leq \frac{u-\rho}{\rho} \leq 2\left(\frac{u-r}{r} + \frac{r-\rho}{r}\right) \leq C\delta(\Omega)^{1/(1+s(n-2))} + 2\frac{r-\rho}{r},$$

so we only need to estimate $(r-\rho)/r$ to prove Theorem 1.6.

Next we define functions $g, h : [0,\infty) \to [0,\infty)$ by

$$g(t) = \mathcal{H}^{n-1}(S^{n-1}(t) \setminus \Omega) \quad \text{and}$$
$$h(t) = \mathcal{H}^{n-1}(S^{n-1}(t) \cap \Omega) = n\omega_n u^{n-1} - g(t).$$

For any $t < \rho$ we have $g(t) = 0$ and for any $t > u$ we have $h(t) = 0$. If $\rho < t < u$ we may estimate $h(t)$ and $g(t)$ as follows. By (5.2) and the s-John domain properties of Ω_0^c and Ω, respectively, we may argue similarly as in the proof of (4.9) to obtain the estimates

(5.4) $$g(t) \geq C(t-\rho)^{(n-1)s}$$

and

(5.5) $$h(t) \geq C(u-t)^{(n-1)s}.$$

We perform spherical symmetrization on Ω to obtain a domain Ω' such that $S^{n-1}(t) \cap \Omega'$ is a relatively open spherical cap with center $-te_n$ and

$$\mathcal{H}^{n-1}(S^{n-1}(t) \cap \Omega') = h(t)$$

for every $t \geq 0$. Obviously Fubini's theorem gives $|\Omega'| = |\Omega|$. By [9, Lemma 3] the spherical symmetrization decreases the perimeter for polyhedral sets. By [8, Theorem 2.4.2] we may approximate Ω by polyhedral sets and conclude that

$$P\left(\Omega'\right) \leq P\left(\Omega\right).$$

As $|\Omega'| = |\Omega|$ the inequality above gives

$$\delta(\Omega') \leq \delta(\Omega). \tag{5.6}$$

We choose the smallest $\rho \leq \mu \leq u$ that satisfies

$$\Omega' \subset \{x_n < \mu\}.$$

Then there exists a Borel set $E \subset \mathbb{R}^n$, such that E minimizes the perimeter among Borel sets in

$$\left\{F \subset B^{n-1}(u) \times \{-u < x_n < \mu\} : |F| = |\Omega'|\right\}.$$

In particular $P(E) \leq P(\Omega')$ and $|E| = |\Omega'|$, which implies

$$\delta(E) \leq \delta(\Omega') \leq \delta(\Omega), \tag{5.7}$$

where the second inequality follows from (5.6). Moreover, as $B^{n-1}(u) \times \{-u < x_n < \mu\}$ is convex we may apply [10] to conclude that E is convex. Hence we can apply Theorem 1.2 and (5.7) to obtain

$$\beta(E) \leq \delta(E)^{1/(1+s(n-2))} \leq C\delta(\Omega)^{1/(1+s(n-2))}. \tag{5.8}$$

The volume radius of E is r and the inradius of E is bounded above by $(u+\mu)/2$. Of course, the inradius is bounded above by the volume radius and the circumradius is bounded below by the volume radius. It follows that

$$\beta(E) \geq \frac{r - (u+\mu)/2}{r} = \frac{r-\mu}{2r} - \frac{u-r}{2r},$$

which, together with (5.8) and (5.1), implies that

$$\frac{r-\mu}{r} \leq C\delta(\Omega)^{1/(1+s(n-2))}. \tag{5.9}$$

By the choice of μ there exists at least one $\tilde{v} > 0$ such that

$$S^{n-1}(\tilde{v}) \cap \{x_n = \mu\} \subset \partial\Omega' \cap \{x_n = \mu\}.$$

We choose the smallest such $\tilde{v}$ and denote it by v. Because $\mu \geq \rho$, the set $B(v) \cap \{x_n < \mu\} \setminus \Omega'$ has a unique component, denoted by U, containing $\{(0, \ldots, x_n) : \rho < x_n < \mu\}$. Let U^* be the reflection of U with respect to hyperplane $\{x_n = \mu\}$ and define Ω^* as the interior of

$$\Omega' \cup \bar{U} \cup U^*.$$

The boundary of U inside $B(v)$ is in the interior of Ω^* and the only new boundary created in the construction of Ω^* comes from the boundary of U^*. Hence we have $P(\Omega^*) \leq P(\Omega')$, which, together with (5.6) and the fact that $|\Omega^*| \geq |\Omega'|$, gives

$$\delta(\Omega^*) \leq \delta(\Omega') \leq \delta(\Omega). \tag{5.10}$$

By the construction of Ω^* we have $-ue_n \in \partial\Omega^*$ and $(\mu + (\mu - \rho))e_n \in \partial\Omega^*$. Hence the circumradius R^* of Ω^* can be estimated from below as follows

$$R^* \geq (u + \mu + (\mu - \rho))/2 \geq (r + 2\mu - \rho)/2. \tag{5.11}$$

Moreover, the volume of Ω^* can be estimated from above by

$$\begin{aligned} |\Omega^*| = |\Omega| + 2|U| &\leq |\Omega| + 2|U \cap B(r)| + 2|B(u) \setminus B(r)| \\ &\leq \omega_n r^n + 2\omega_n r^n \lambda(\Omega) + 2\omega_n(u^n - r^n). \end{aligned}$$

By estimate (5.1) and Theorem 2.1 the above inequality gives

$$|\Omega^*| = |\Omega| + 2|U| \leq \omega_n r^n \big(1 + C\delta(\Omega)^{1/(1+s(n-2))}\big).$$

Hence the volume radius r^* of Ω^* has the upper bound

$$r^* \leq r\big(1 + C\delta(\Omega)^{1/(1+s(n-2))}\big). \tag{5.12}$$

By (5.11) and (5.12) we have

$$\begin{aligned} \alpha(\Omega^*) = \frac{R^* - r^*}{r^*} &\geq \frac{(r + 2\mu - \rho)/2 - r(1 + C\delta(\Omega)^{1/(1+s(n-2))})}{r} \\ &= \frac{r - \rho}{2r} - \frac{r - \mu}{r} - C\delta\,(\Omega)^{1/(1+s(n-2))}. \end{aligned} \tag{5.13}$$

We apply (5.9) in the inequality above to obtain

$$\frac{r - \rho}{2r} \leq \alpha(\Omega^*) + C\delta(\Omega)^{1/(1+s(n-2))}. \tag{5.14}$$

Estimates (5.4) and (5.5) show that we may use the proof of Theorem 1.5 on Ω^* to control its outer metric distortion. We conclude that

$$\alpha(\Omega^*) \leq C\delta(\Omega^*)^{1/(1+s(n-2))} \leq C\delta(\Omega)^{1/(1+s(n-2))},$$

where we also used (5.10). By (5.14) and the previous inequality we have

$$\frac{r - \rho}{r} \leq C\delta(\Omega)^{1/(1+s(n-2))}.$$

Together with (5.3) this finishes the proof of theorem 1.6.

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