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<p>We introduce a new model for contingent convertibles. The write-down, or equity conversion, and default of the contingent convertible are modeled as states of conditional Markov process. Valuation formulae for different financial contracts, like CDS and different types of contingent convertibles, are derived. The Model can be thought of as an extension to reduced form models with an additional state. For practical applications, this model could be used for new type of contingent convertible derivatives in a similar fashion than reduced form models are used for credit derivatives.</p>			
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A CONDITIONAL MARKOV MODEL FOR PRICING CONTINGENT CONVERTIBLES

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1 Introduction

After the financial crisis banking regulation has been developing to improve the safety of financial institutions and to prevent disruptive bankruptcies of banks. Among these new regulatory innovations a new type of financial instrument has been introduced, called contingent convertible.

Contingent convertibles are bonds, which convert into shares, or are written down, if a trigger event takes place. This event can be triggered either when the capital ratio of the issuer bank deteriorates below some predetermined threshold (accounting trigger), or it can be forced by regulators (regulatory trigger). This mechanism is meant to work as a buffer to improve solvency of banks and prevent bankruptcy, but at the same time it imposes losses on bond holders.

In this paper we develop a new type of model for contingent convertibles. The write-down feature of contingent convertibles makes it natural to model them using a conditional Markov-process, which drives the migration between different states of the bond, one of which is the write-down state. We do this in a similar fashion as in the credit migration framework initially introduced by works like [18] and [19] and further formalized by [1], [15] and [13] among others. Similar approach has also been used in modeling restructuring previously by [14].

In the most common approach to modeling contingent convertibles, the trigger event happens, when share price hits some boundary, like in [10], [12], [11], [22] and [23]. However, since the trigger event can be a regulatory trigger, it can be enforced by banking regulators at any point of time. Therefore we take a different approach and try to take this into account modeling it as an exogenous event avoiding assumptions about trigger event being related to asset prices.

One of our motivations is to build a bridge with models that are commonly used in practice to evaluate credit derivatives, so called reduced form credit models. In these models the credit event, or time of default, is modelled as a stopping time. Reduced form models have been suggested to contingent convertible modeling before by [12], [8] and [9]. These approaches model the write-down event as a stopping time. In our approach this stopping time, and the time of default, are constructed using the conditional Markov-process. In our approach we also don't need to distinguish regulatory trigger and accounting trigger as in [9], who model these two triggers as two separate stopping times.

Another class of credit models are so called structural models. We refer to [5] for good discussion about the benefits of structural models, as opposed to reduced form models, and [20] for a literature review of contingent convertible models with structural approach.

Although contingent convertibles with equity conversion have been the focus point of academic research, bonds with write-down mechanism are another important class of contingent convertibles. This paper tries to extend pricing to these types of bonds. For instance we want to incorporate the possibility of a temporary write-down, because the economic value of this feature might be of interest in practice. If the write-down is temporary, there is a possibility, that

bonds notional value will be written back up if the capital ratio of the issuer bank improves.

This paper is divided into four sections. In section 2 we introduce contingent convertibles and formalize their features. After this, we construct our framework with a conditional Markov model in section 3. In this section we will derive results, that are needed for asset pricing in our setting. These results are applied in section 4, which focuses on asset pricing and deriving pricing formulas for different financial instruments and, in particular, contingent convertibles. Our model has also applications to new types of derivatives, related to contingent convertibles, and we briefly touch this topic. We will demonstrate the applicability of our model in the final section 5 where illustrate the use of the model with similar assumptions as in the ISDA standard model for CDS pricing.

2 Features of Contingent Convertibles

In this paper we define contingent convertibles as capital instruments, which fulfill the regulatory requirements of Additional Tier 1 (AT1) capital as specified in [21]. This by no means restricts the model, represented here, to be used for other types of financial instruments, like Tier 2 contingent convertibles. In order to be eligible for AT1 designation, contingent convertibles have to fulfill certain features and we will give a formal representation of these features in this section.

Let $(\Omega, \mathcal{F}, \mathbf{F} = (\mathcal{F}_t)_{t \geq 0}, \mathbf{P})$ be a filtered probability space. We assume that filtration $(\mathcal{F}_t)_{t \geq 0}$ satisfies the usual conditions, i.e. it is complete and right-continuous.

We define different states of the bond as elements of a set $\mathcal{K} = \{1, 2, K\}$. State 1 is the normal state of the bond. State 2 is the write-down state when bond is fully or partially written down or converted to equity. State $K = 3$ is the default state. We model migration between states as a continuous-time Markov chain C with state space \mathcal{K} and the default state $K = 3$ is absorbing state of C . On the day of issue the contingent convertible will be in the normal state, $C_0 = 1$ a.s.

We define two stopping times in the filtration generated by the migration process C_t

$$\tau^{\text{CoCo}} = \inf\{t > 0 : C_t = 2\}, \quad \tau^{\text{D}} = \inf\{t > 0 : C_t = K\},$$

where τ^{CoCo} is the time when trigger event happens the first time and bond is written down, or converted to equity. Stopping time τ^{D} is the time of default.

Let $(\Lambda_t)_{t \geq 0}$ be a matrix-valued stochastic process

$$\Lambda_t = \begin{bmatrix} \lambda_t^{1,1} & \lambda_t^{1,2} & 0 \\ \lambda_t^{2,1} & \lambda_t^{2,2} & \lambda_t^{2,3} \\ 0 & 0 & 0 \end{bmatrix}, \quad (2.1)$$

where $\lambda_t^{i,j} : \Omega \times [0, \infty) \rightarrow \mathbb{R}_+$ are bounded, \mathbf{F} -progressively measurable stochastic processes. For every $i, j \in \mathcal{K}$, $i \neq j$, the processes $\lambda_t^{i,j}$ are non-negative and $\lambda_t^{i,i} = -\sum_{j \in \mathcal{K} \setminus \{i\}} \lambda_t^{i,j}$, for $t \in [0, \infty)$. We also define the initial probability distribution of C on \mathcal{K} , $\mu = (\mu_1, \mu_2, \mu_3)$, which is a one-point mass on the state 1.

Row 3 is zeros, because default is an absorbing state. Since contingent convertibles will always be written down before the issuer defaults, which is a key feature of these bonds, we have set $\lambda_t^{1,3} = 0$. Bonds with permanent write-down or equity conversion will never return to their normal state and in their case $\lambda_t^{2,1} = 0$.

In Basel III framework Additional Tier 1 instruments are required to be subordinated to depositors, general creditors and subordinated debt of the bank. This means, that in case of liquidation, even subordinated bonds are compensated before AT1 bonds. Furthermore, AT1 instruments can't be secured. These features make it very unlikely that AT1 bonds have any recovery at default and we will assume zero recovery rate for contingent convertibles.

Payoff of a coupon payment of a contingent convertible depends on the state of the bond and we define write-down fraction $(1 - q^{\text{CoCo}}) \in (0, 1]$ which tells how much bond's face value will be written down when trigger event happens. In this paper we focus on instruments, for which write-down fraction is a predetermined constant specified in the bond contract. We leave the possibility of stochastic q^{CoCo} for further research. For bonds with full write-down or equity conversion $(1 - q^{\text{CoCo}}) = 1$.

As specified in [21], the issuer can cancel coupon payments without triggering default. Coupon cancellation could be modeled as a state of it's own, as suggested by [9]. However, we don't want to introduce any more states in our model, so we implicitly assume, that coupon cancellation happens only when bond is in the write-down state. We also refer to [11] for a model, where coupon cancellation is triggered by share price.

Coupon payments c_k are paid at times $T_k, T_{k-1} < T_k, k \in \mathbb{N}$. Payoff of a coupon c_k maturing at time T_k is given by

$$c_k \mathbf{1}_{\{C_{T_k}=1\}} N + c_k \mathbf{1}_{\{C_{T_k}=2\}} q^{\text{CoCo}} N,$$

where N is the notional of the bond.

In practice contingent convertibles usually consist of both fixed and floating rate coupons. For floating rate coupons we set $c_k = \Delta_{T_k} (L_{T_{k-1}, T_k} + Z_0)$, where $\Delta_{T_k} := T_k - T_{k-1}$. Here Z_0 is a predefined constant specified in the bond contract, $Z_0 \in \mathbb{R}_+$. We define forward Libor rate at time $t < T_k$ for the accrual period $[T_k, T_{k+1}]$ as

$$L_{t, T_k} := \frac{1}{\Delta_{T_{k+1}}} \left(\frac{D(t, T_k)}{D(t, T_{k+1})} \right),$$

where discount factors $D(t, u) := \mathbf{E}^{\mathbb{Q}} \left[\exp \left(- \int_t^u r_s ds \right) \mid \mathcal{F}_t \right]$ are risk-free and $D(t, t) = 1$.

Banking regulation [21] requires contingent convertibles to be perpetual. The issuer will typically have a possibility to redeem bonds (that is, to pay them off) in predetermined dates, T_m, T_{m+1}, \dots, T_n , which have to take place after a minimum of five years from issue date. However, the redemption is conditional to capital requirements and regulatory approval making contingent convertibles different from traditional callable bonds. Therefore, we require that bond be in state 1 for redemption to occur. In case of redemption, the issuer will pay both the final coupon and the notional amount. If the bond is not redeemed, the issuer will pay only a coupon (taking into account the partial write-down). The payoff from the first possible redemption is $\mathbf{1}_{\{C_{T_m}=1\}} N$. The rest of the redemption payoffs and their corresponding coupons form a sequence dependent on whether the bond has been redeemed or not:

$$\left(\mathbf{1}_{\{C_{T_k}=1\}} (1 + c_k) N + \mathbf{1}_{\{C_{T_k}=2\}} c_k q^{\text{CoCo}} N \right) \prod_{j=m}^{k-1} \mathbf{1}_{\{C_{T_j}=2\}},$$

for $k \in \{m + 1, \dots, n\}$. Typically, if the bond is not redeemed at the first redemption date T_m , any subsequent coupons are floating.

Notice, that coupons after the first possible redemption are also conditional on whether the bond has been redeemed or not. The same goes for any coupons after the last possible redemption date for which the payoff is of the form

$$\left(\mathbf{1}_{\{C_{T_k}=1\}} c_k N + \mathbf{1}_{\{C_{T_k}=2\}} c_k q^{\text{CoCo}} N \right) \prod_{j=m}^n \mathbf{1}_{\{C_{T_j}=2\}},$$

where $k > n$. In case the bond hasn't been redeemed at any redemption date, coupon payments continue to perpetuity.

Here we implicitly assume, that the issuer will redeem the bond if it is in state 1. In literature there are alternative approaches to modeling this decision. One could make the redemption decision dependent on the bond price, as in typical callable bond pricing models. However, in case of contingent convertibles (or other credit risky bonds), there are other factors, which might affect issuer's decision, like market frictions or issuer's capital position. Therefore it is not uncommon to see callable bonds being redeemed even when their price is lower, than the redemption price. In [17] it is suggested to use a random time τ_C as a time of issuer's redemption and [9] apply this approach to contingent convertibles. To avoid further states in our model, we assume that bond will be redeemed if in state 1. This is a common practice in many credit applications, but it might not be realistic enough, because redemption will require approval from banking regulators. We leave this under further discussion.

Contingent convertibles, with possibility for equity conversion, form a different class of these financial instruments. For these instruments we set only one possibility of redemption at time T_m , since the transition from state 2 back to state 1 is not possible. Once the bond has been

converted to shares, any possible coupon or notional payments are canceled. We define share price $(S_t)_{t \geq 0}$ as a stochastic process adapted to \mathbf{F} . Payoff from equity conversion is

$$\frac{N}{\phi(S_{\tau^{\text{CoCo}}})} S_{\tau^{\text{CoCo}}} \mathbf{1}_{\{\tau^{\text{CoCo}} \leq T_m\}},$$

where function $\phi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is the conversion price of the bond and it is a $\mathcal{F}_{\tau^{\text{CoCo}}}$ -measurable random variable.

Different types of conversion prices can be defined. The conversion price is said to be floating, if it is the share price at the time of conversion, $\phi(S_{\tau^{\text{CoCo}}}) = S_{\tau^{\text{CoCo}}}$ a.s. If the conversion price is fixed, it is equal to the share price on the issue date of the contingent convertible, $\phi(S_{\tau^{\text{CoCo}}}) = S_0$ a.s. A third type of conversion price used in practice is the floored conversion, $\phi(S_{\tau^{\text{CoCo}}}) = \max(S_{\tau^{\text{CoCo}}}, \bar{S})$ for some constant \bar{S} , which is defined in the bond contract. Ratio $N/\phi(\cdot)$ is called the conversion ratio and it is the amount of shares investor receives in conversion.

Having introduced different components of contingent convertibles, we now give separate definitions for contingent convertibles with write-down feature and contingent convertibles with equity conversion.

2.2 Definition. A contingent convertible with temporary write-down and write-down fraction $(1 - q^{\text{CoCo}})$ is a contract, that pays coupons c_k at times T_k , $T_{k-1} < T_k$, $k \in \mathbb{N}$ and is redeemable by issuer at times T_m, T_{m+1}, \dots, T_n .

The payoff of this contract is given by

$$\begin{aligned} & \sum_{k=1}^m \left(c_k \mathbf{1}_{\{C_{T_k}=1\}} N + c_k \mathbf{1}_{\{C_{T_k}=2\}} q^{\text{CoCo}} N \right) + \mathbf{1}_{\{C_{T_m}=1\}} N \\ & + \sum_{k=m+1}^n \left(\mathbf{1}_{\{C_{T_k}=1\}} (1 + c_k) N + \mathbf{1}_{\{C_{T_k}=2\}} c_k q^{\text{CoCo}} N \right) \prod_{j=m}^{k-1} \mathbf{1}_{\{C_{T_j}=2\}} \\ & + \sum_{k=n+1}^{\infty} \left(\mathbf{1}_{\{C_{T_k}=1\}} c_k N + \mathbf{1}_{\{C_{T_k}=2\}} c_k q^{\text{CoCo}} N \right) \prod_{j=m}^n \mathbf{1}_{\{C_{T_j}=2\}}. \end{aligned}$$

In practice the last redemption time, T_n , is usually set decades after the issue date and for practical implementations one might approximate the last term with

$$\sum_{k=n+1}^{\hat{m}} \left(\mathbf{1}_{\{C_{T_k}=1\}} c_k N + \mathbf{1}_{\{C_{T_k}=2\}} c_k q^{\text{CoCo}} N \right) \prod_{j=m}^n \mathbf{1}_{\{C_{T_j}=2\}}$$

with some sufficiently large $\hat{m} \in \mathbb{N}$.

One could view contingent convertibles with permanent write-down as a special case of the ones with temporary write-down by setting $n = m$ and $\lambda_t^{2,1} = 0$. We will nevertheless give them a definition of their own.

2.3 Definition. A contingent convertible with permanent write-down and write-down fraction $(1 - q^{\text{CoCo}})$ is a contract, that pays coupons c_k at times $T_k, T_{k-1} < T_k, k \in \mathbb{N}$ and is redeemable by issuer at time T_m .

The payoff of this contract is given by

$$\sum_{k=1}^m \left(c_k \mathbf{1}_{\{C_{T_k}=1\}} N + c_k \mathbf{1}_{\{C_{T_k}=2\}} q^{\text{CoCo}} N \right) + \mathbf{1}_{\{C_{T_m}=1\}} N \\ + \sum_{k=m+1}^{\infty} \mathbf{1}_{\{C_{T_k}=2\}} c_k q^{\text{CoCo}} N.$$

2.4 Definition. A contingent convertible with equity conversion is a contract, that pays coupons c_k at times $T_k, T_{k-1} < T_k, k \in \mathbb{N}$ and is redeemable by issuer at time T_m . It converts to equity at the trigger event at conversion ratio $N/\phi(\cdot)$.

The payoff of this contract is given by

$$\sum_{k=1}^m c_k \mathbf{1}_{\{C_{T_k}=1\}} N + \mathbf{1}_{\{C_{T_m}=1\}} N \\ + \frac{N}{\phi(S_{\tau^{\text{CoCo}}})} S_{\tau^{\text{CoCo}}} \mathbf{1}_{\{\tau^{\text{CoCo}} \leq T_m\}}$$

Notice, that in reality instruments with permanent write-down, or equity conversion, might have multiple possible redemption dates. However, these are not meaningful in our framework, since we have assumed, that bank is not allowed to redeem a bond which has been written-down. On the other hand we have assumed, that redemption will happen in any case, if bond is in state 1 at time T_m . This means, that if bond hasn't been redeemed at time T_m , it has been written-down and can't be redeemed anymore, since write-down is permanent.

3 Conditional Markov Model for Contingent Convertible modeling

In this section we follow [1] and [13] and construct useful results for migration between different states of the contingent convertible. Previous works have been focusing on migration between different credit ratings, but we'll be borrowing their ideas and applying them in a new way.

We denote by $\mathbf{F}^C := (\mathcal{F}_t^C)_{t \geq 0}$ the natural filtration of process C , where

$$\mathcal{F}_t^C := \bigvee_{0 \leq s \leq t} \sigma(C_s),$$

where $\sigma(C_s)$ is the smallest σ -algebra containing elements $\{C_s = i\}$, $i \in \mathcal{K}$. We define $\mathbf{G} = (\mathcal{G}_t)_{t \geq 0}$ where $\mathcal{G}_t := \mathcal{F}_t \vee \mathcal{F}_t^C$. From here on, we will be working with an enlarged probability space $(\tilde{\Omega}, \mathcal{G}, \mathbb{Q})$. This is called the canonical construction of C and we refer to [14] and [15] for more detailed construction of C .

One can interpret σ -algebra \mathcal{F}_t as all the information of market observables, like the share price, interest rates or credit spreads, up until time t . On the other hand \mathcal{F}_t^C contains the history of which states process C has been at any given time until t . Here $\sigma(C_t)$ contains the information about at which state C is at time t .

The process C is an \mathbf{F} -conditional Markov process and has the conditional Markov property

$$\mathbf{E}^{\mathbb{Q}} [h(C_s) \mid \mathcal{F}_t \vee \mathcal{F}_t^C] = \mathbf{E}^{\mathbb{Q}} [h(C_s) \mid \mathcal{F}_t \vee \sigma(C_t)] \quad (3.1)$$

for every $0 \leq t \leq s$ and any function $h : \mathcal{K} \rightarrow \mathbf{R}$. The process C also has a stronger property

$$\mathbf{E}^{\mathbb{Q}} [h(C_s) \mid \mathcal{F}_u \vee \mathcal{F}_t^C] = \mathbf{E}^{\mathbb{Q}} [h(C_s) \mid \mathcal{F}_u \vee \sigma(C_t)] \quad (3.2)$$

for every $0 \leq t \leq s \leq u$ and any function $h : \mathcal{K} \rightarrow \mathbf{R}$. This property follows from the canonical construction of C .

The interpretation of Markov property is, that when evaluating the expected future state of the bond, only present state matters and history is irrelevant. One analogy would be, that it doesn't matter if the bank had a strong balance sheet last year, if it's balance sheet is weak now.

This following Lemma will be our first step into calculating conditional expected values of \mathcal{G}_t -measurable random variables.

3.3 Lemma. *If Y is a \mathcal{G}_t -measurable random variable, then*

$$\mathbf{E}^{\mathbb{Q}} [Y \mid \mathcal{F}_s \vee \sigma(C_t)] = \sum_{i=1}^K \mathbf{1}_{\{C_t=i\}} \frac{\mathbf{E}^{\mathbb{Q}} [Y \mathbf{1}_{\{C_t=i\}} \mid \mathcal{F}_s]}{\mathbf{E}^{\mathbb{Q}} [\mathbf{1}_{\{C_t=i\}} \mid \mathcal{F}_s]},$$

for $0 \leq t \leq s$, $i \in \mathcal{K}$.

Proof. We show that

$$\mathbf{1}_{\{C_t=i\}} \mathbf{E}^{\mathbb{Q}} [Y \mid \mathcal{F}_s \vee \sigma(C_t)] = \mathbf{1}_{\{C_t=i\}} \frac{\mathbf{E}^{\mathbb{Q}} [Y \mathbf{1}_{\{C_t=i\}} \mid \mathcal{F}_s]}{\mathbf{E}^{\mathbb{Q}} [\mathbf{1}_{\{C_t=i\}} \mid \mathcal{F}_s]}. \quad (3.4)$$

Equality applies if and only if

$$\mathbf{E}^{\mathbb{Q}} \left[Y \mathbf{1}_G \mathbf{1}_{\{C_t=i\}} \right] = \mathbf{E}^{\mathbb{Q}} \left[\mathbf{1}_{\{C_t=i\}} \mathbf{1}_G \frac{\mathbf{E}^{\mathbb{Q}} \left[Y \mathbf{1}_{\{C_t=i\}} \mid \mathcal{F}_s \right]}{\mathbf{E}^{\mathbb{Q}} \left[\mathbf{1}_{\{C_t=i\}} \mid \mathcal{F}_s \right]} \right]$$

for every $G \in \mathcal{F}_s \vee \sigma(C_t)$. This σ -algebra is generated by sets $\{C_t = j\} \cap F$, where $F \in \mathcal{F}_s$, so we need to show that

$$\mathbf{E}^{\mathbb{Q}} \left[Y \mathbf{1}_F \mathbf{1}_{\{C_t=i\}} \mathbf{1}_{\{C_t=j\}} \right] = \mathbf{E}^{\mathbb{Q}} \left[\mathbf{1}_{\{C_t=i\}} \mathbf{1}_{\{C_t=j\}} \mathbf{1}_F \frac{\mathbf{E}^{\mathbb{Q}} \left[Y \mathbf{1}_{\{C_t=i\}} \mid \mathcal{F}_s \right]}{\mathbf{E}^{\mathbb{Q}} \left[\mathbf{1}_{\{C_t=i\}} \mid \mathcal{F}_s \right]} \right],$$

for any $\{C_t = j\}$ and $F \in \mathcal{F}_s$. When $i \neq j$, both sides are equal to 0. When $i = j$, we have

$$\begin{aligned} \mathbf{E}^{\mathbb{Q}} \left[\mathbf{1}_{\{C_t=i\}} \mathbf{1}_F \frac{\mathbf{E}^{\mathbb{Q}} \left[Y \mathbf{1}_{\{C_t=i\}} \mid \mathcal{F}_s \right]}{\mathbf{E}^{\mathbb{Q}} \left[\mathbf{1}_{\{C_t=i\}} \mid \mathcal{F}_s \right]} \right] &= \mathbf{E}^{\mathbb{Q}} \left[\mathbf{E}^{\mathbb{Q}} \left[\mathbf{1}_{\{C_t=i\}} \mathbf{1}_F \frac{\mathbf{E}^{\mathbb{Q}} \left[Y \mathbf{1}_{\{C_t=i\}} \mid \mathcal{F}_s \right]}{\mathbf{E}^{\mathbb{Q}} \left[\mathbf{1}_{\{C_t=i\}} \mid \mathcal{F}_s \right]} \mid \mathcal{F}_s \right] \right] \\ &= \mathbf{E}^{\mathbb{Q}} \left[\frac{\mathbf{E}^{\mathbb{Q}} \left[Y \mathbf{1}_F \mathbf{1}_{\{C_t=i\}} \mid \mathcal{F}_s \right]}{\mathbf{E}^{\mathbb{Q}} \left[\mathbf{1}_{\{C_t=i\}} \mid \mathcal{F}_s \right]} \mathbf{E}^{\mathbb{Q}} \left[\mathbf{1}_{\{C_t=i\}} \mid \mathcal{F}_s \right] \right] = \mathbf{E}^{\mathbb{Q}} \left[\mathbf{E}^{\mathbb{Q}} \left[Y \mathbf{1}_F \mathbf{1}_{\{C_t=i\}} \mid \mathcal{F}_s \right] \right] \\ &= \mathbf{E}^{\mathbb{Q}} \left[Y \mathbf{1}_F \mathbf{1}_{\{C_t=i\}} \right]. \end{aligned}$$

Lemma 3.3 is a corollary of equality 3.4. □

As a corollary of Lemma 3.3 the conditional Markov property can be written in the form

$$\mathbf{E}^{\mathbb{Q}} \left[h(C_s) \mid \mathcal{F}_t \vee \mathcal{F}_t^C \right] = \sum_{i=1}^K \mathbf{1}_{\{C_t=i\}} \frac{\mathbf{E}^{\mathbb{Q}} \left[h(C_s) \mathbf{1}_{\{C_t=i\}} \mid \mathcal{F}_t \right]}{\mathbf{E}^{\mathbb{Q}} \left[\mathbf{1}_{\{C_t=i\}} \mid \mathcal{F}_t \right]}, \quad (3.5)$$

for $t \leq s$, and in particular

$$\mathbf{E}^{\mathbb{Q}} \left[\mathbf{1}_{\{C_s=j\}} \mid \mathcal{G}_t \right] = \sum_{i=1}^K \mathbf{1}_{\{C_t=i\}} \frac{\mathbf{E}^{\mathbb{Q}} \left[\mathbf{1}_{\{C_s=j\}} \mathbf{1}_{\{C_t=i\}} \mid \mathcal{F}_t \right]}{\mathbf{E}^{\mathbb{Q}} \left[\mathbf{1}_{\{C_t=i\}} \mid \mathcal{F}_t \right]}. \quad (3.6)$$

Because of the canonical construction, random state C_s is influenced by information from the filtration \mathbf{F} up to time s only. We express this feature formally in the following proposition.

3.7 Proposition. Let C be a conditional Markov chain obtained by the canonical construction. Then for any $0 \leq t \leq s_1 \leq s_2 \leq u \leq T$ and $i_1, i_2 \in \mathcal{K}$:

(a)

$$\mathbf{E}^{\mathbb{Q}} \left[\mathbf{1}_{\{C_{s_1}=i_1\}} \mathbf{1}_{\{C_{s_2}=i_2\}} \mid \mathcal{F}_u \vee \mathcal{F}_t^C \right] = \mathbf{E}^{\mathbb{Q}} \left[\mathbf{1}_{\{C_{s_1}=i_1\}} \mathbf{1}_{\{C_{s_2}=i_2\}} \mid \mathcal{F}_{s_2} \vee \mathcal{F}_t^C \right]$$

(b)

$$\mathbf{E}^{\mathbb{Q}} \left[\mathbf{1}_{\{C_{s_1}=i_1\}} \mathbf{1}_{\{C_{s_2}=i_2\}} \mid \mathcal{F}_u \right] = \mathbf{E}^{\mathbb{Q}} \left[\mathbf{1}_{\{C_{s_1}=i_1\}} \mathbf{1}_{\{C_{s_2}=i_2\}} \mid \mathcal{F}_{s_2} \right]$$

Proof. (a) The proof of part (a) relies on the canonical construction of C and we refer to proof of Proposition 2.18 in [15].

(b) Setting $t = 0$ into (a), we have

$$\mathbf{E}^{\mathbb{Q}} \left[\mathbf{1}_{\{C_{s_1}=i_1\}} \mathbf{1}_{\{C_{s_2}=i_2\}} \mid \mathcal{F}_u \vee \mathcal{F}_0^C \right] = \mathbf{E}^{\mathbb{Q}} \left[\mathbf{1}_{\{C_{s_1}=i_1\}} \mathbf{1}_{\{C_{s_2}=i_2\}} \mid \mathcal{F}_{s_2} \vee \mathcal{F}_0^C \right].$$

Applying Lemma 3.3 to the left side, we get

$$\begin{aligned} \mathbf{E}^{\mathbb{Q}} \left[\mathbf{1}_{\{C_{s_1}=i_1\}} \mathbf{1}_{\{C_{s_2}=i_2\}} \mid \mathcal{F}_u \vee \mathcal{F}_0^C \right] &= \sum_{i=1}^K \mathbf{1}_{\{C_0=i\}} \frac{\mathbf{E}^{\mathbb{Q}} \left[\mathbf{1}_{\{C_{s_1}=i_1\}} \mathbf{1}_{\{C_{s_2}=i_2\}} \mathbf{1}_{\{C_0=i\}} \mid \mathcal{F}_u \right]}{\mathbf{E}^{\mathbb{Q}} \left[\mathbf{1}_{\{C_0=i\}} \mid \mathcal{F}_u \right]} \\ &= \mathbf{E}^{\mathbb{Q}} \left[\mathbf{1}_{\{C_{s_1}=i_1\}} \mathbf{1}_{\{C_{s_2}=i_2\}} \mid \mathcal{F}_u \right]. \end{aligned}$$

The same argument can be used for the right side of the equation as well. Hence,

$$\begin{aligned} \mathbf{E}^{\mathbb{Q}} \left[\mathbf{1}_{\{C_{s_1}=i_1\}} \mathbf{1}_{\{C_{s_2}=i_2\}} \mid \mathcal{F}_u \right] &= \mathbf{E}^{\mathbb{Q}} \left[\mathbf{1}_{\{C_{s_1}=i_1\}} \mathbf{1}_{\{C_{s_2}=i_2\}} \mid \mathcal{F}_u \vee \mathcal{F}_0^C \right] \\ &= \mathbf{E}^{\mathbb{Q}} \left[\mathbf{1}_{\{C_{s_1}=i_1\}} \mathbf{1}_{\{C_{s_2}=i_2\}} \mid \mathcal{F}_{s_2} \vee \mathcal{F}_0^C \right] = \mathbf{E}^{\mathbb{Q}} \left[\mathbf{1}_{\{C_{s_1}=i_1\}} \mathbf{1}_{\{C_{s_2}=i_2\}} \mid \mathcal{F}_{s_2} \right]. \end{aligned}$$

□

We follow notation of [13] and denote

$$\mathbf{E}^{\mathbb{Q}} [Y \mid \mathcal{F}_s; C_t = i] := \frac{\mathbf{E}^{\mathbb{Q}} [Y \mathbf{1}_{\{C_t=i\}} \mid \mathcal{F}_s]}{\mathbf{E}^{\mathbb{Q}} [\mathbf{1}_{\{C_t=i\}} \mid \mathcal{F}_s]}, \quad (3.8)$$

where Y is a \mathcal{G}_s -measurable random variable and $0 \leq t \leq s$. When $Y = \mathbf{1}_{\{C_s=j\}}$, we have conditional expectation

$$\mathbf{E}^{\mathbb{Q}} \left[\mathbf{1}_{\{C_s=j\}} \mid \mathcal{F}_s; C_t = i \right] = \mathbb{Q} (C_s = j \mid \mathcal{F}_s; C_t = i). \quad (3.9)$$

This is the conditional probability with respect to \mathbf{F} of the process C being in state j at time s if it was in state i at time t .

3.10 Definition. The \mathbf{F} -conditional transition matrix of C is defined as

$$\mathbb{Q}(t, s) = \left[\mathbb{Q}^{i,j}(t, s) \right]_{i,j \in \mathcal{K}},$$

where

$$\mathbb{Q}^{i,j}(t, s) := \mathbb{Q} (C_s = j \mid \mathcal{F}_s; C_t = i).$$

3.11 Lemma. For each $i, j, k \in \mathcal{K}$ and any $0 \leq t \leq u \leq s$ we have

$$\mathbb{Q} (C_s = j \mid \mathcal{F}_s; C_u = k, C_t = i) = \mathbb{Q} (C_s = j \mid \mathcal{F}_s; C_u = k). \quad (3.12)$$

Proof. From conditional Markov properties 3.1 and 3.2 we get

$$\mathbf{E}^{\mathbb{Q}} \left[\mathbf{1}_{\{C_s=j\}} \mid \mathcal{F}_s \vee \sigma(C_u) \vee \sigma(C_t) \right] = \mathbf{E}^{\mathbb{Q}} \left[\mathbf{1}_{\{C_s=j\}} \mid \mathcal{F}_s \vee \sigma(C_u) \right].$$

For any arbitrary $F \in \mathcal{F}_s$, this is equivalent to

$$\mathbf{E}^{\mathbb{Q}} \left[\mathbf{1}_{\{C_s=j\}} \mathbf{1}_{\{C_u=k\}} \mathbf{1}_{\{C_t=i\}} \mathbf{1}_F \right] = \mathbf{E}^{\mathbb{Q}} \left[\mathbf{E}^{\mathbb{Q}} \left[\mathbf{1}_{\{C_s=j\}} \mid \mathcal{F}_s \vee \sigma(C_u) \right] \mathbf{1}_{\{C_u=k\}} \mathbf{1}_{\{C_t=i\}} \mathbf{1}_F \right].$$

The right hand side is equal to

$$\begin{aligned} & \mathbf{E}^{\mathbb{Q}} \left[\frac{\mathbf{E}^{\mathbb{Q}} \left[\mathbf{1}_{\{C_s=j\}} \mathbf{1}_{\{C_u=k\}} \mid \mathcal{F}_s \right]}{\mathbf{E}^{\mathbb{Q}} \left[\mathbf{1}_{\{C_u=k\}} \mid \mathcal{F}_s \right]} \mathbf{1}_{\{C_u=k\}} \mathbf{1}_{\{C_t=i\}} \mathbf{1}_F \right] \\ &= \mathbf{E}^{\mathbb{Q}} \left[\mathbf{E}^{\mathbb{Q}} \left[\frac{\mathbf{E}^{\mathbb{Q}} \left[\mathbf{1}_{\{C_s=j\}} \mathbf{1}_{\{C_u=k\}} \mid \mathcal{F}_s \right]}{\mathbf{E}^{\mathbb{Q}} \left[\mathbf{1}_{\{C_u=k\}} \mid \mathcal{F}_s \right]} \mathbf{1}_{\{C_u=k\}} \mathbf{1}_{\{C_t=i\}} \mathbf{1}_F \right] \right] \\ &= \mathbf{E}^{\mathbb{Q}} \left[\frac{\mathbf{E}^{\mathbb{Q}} \left[\mathbf{1}_{\{C_s=j\}} \mathbf{1}_{\{C_u=k\}} \mid \mathcal{F}_s \right]}{\mathbf{E}^{\mathbb{Q}} \left[\mathbf{1}_{\{C_u=k\}} \mid \mathcal{F}_s \right]} \mathbf{E}^{\mathbb{Q}} \left[\mathbf{1}_{\{C_u=k\}} \mathbf{1}_{\{C_t=i\}} \mid \mathcal{F}_s \right] \mathbf{1}_F \right]. \end{aligned}$$

Therefore, we have

$$\mathbf{E}^{\mathbb{Q}} \left[\mathbf{1}_{\{C_s=j\}} \mathbf{1}_{\{C_u=k\}} \mathbf{1}_{\{C_t=i\}} \mathbf{1}_F \right] = \mathbf{E}^{\mathbb{Q}} \left[\frac{\mathbf{E}^{\mathbb{Q}} \left[\mathbf{1}_{\{C_s=j\}} \mathbf{1}_{\{C_u=k\}} \mid \mathcal{F}_s \right]}{\mathbf{E}^{\mathbb{Q}} \left[\mathbf{1}_{\{C_u=k\}} \mid \mathcal{F}_s \right]} \mathbf{E}^{\mathbb{Q}} \left[\mathbf{1}_{\{C_u=k\}} \mathbf{1}_{\{C_t=i\}} \mid \mathcal{F}_s \right] \mathbf{1}_F \right],$$

which is equivalent to

$$\mathbf{E}^{\mathbb{Q}} \left[\mathbf{1}_{\{C_s=j\}} \mathbf{1}_{\{C_u=k\}} \mathbf{1}_{\{C_t=i\}} \mid \mathcal{F}_s \right] = \frac{\mathbf{E}^{\mathbb{Q}} \left[\mathbf{1}_{\{C_s=j\}} \mathbf{1}_{\{C_u=k\}} \mid \mathcal{F}_s \right]}{\mathbf{E}^{\mathbb{Q}} \left[\mathbf{1}_{\{C_u=k\}} \mid \mathcal{F}_s \right]} \mathbf{E}^{\mathbb{Q}} \left[\mathbf{1}_{\{C_u=k\}} \mathbf{1}_{\{C_t=i\}} \mid \mathcal{F}_s \right].$$

By re-arranging we get

$$\frac{\mathbf{E}^{\mathbb{Q}} \left[\mathbf{1}_{\{C_s=j\}} \mathbf{1}_{\{C_u=k\}} \mathbf{1}_{\{C_t=i\}} \mid \mathcal{F}_s \right]}{\mathbf{E}^{\mathbb{Q}} \left[\mathbf{1}_{\{C_u=k\}} \mathbf{1}_{\{C_t=i\}} \mid \mathcal{F}_s \right]} = \frac{\mathbf{E}^{\mathbb{Q}} \left[\mathbf{1}_{\{C_s=j\}} \mathbf{1}_{\{C_u=k\}} \mid \mathcal{F}_s \right]}{\mathbf{E}^{\mathbb{Q}} \left[\mathbf{1}_{\{C_u=k\}} \mid \mathcal{F}_s \right]}.$$

Notice, that

$$\mathbb{Q} (C_s = j \mid \mathcal{F}_s; C_u = k, C_t = i) = \frac{\mathbf{E}^{\mathbb{Q}} \left[\mathbf{1}_{\{C_s=j\}} \mathbf{1}_{\{C_u=k\}} \mathbf{1}_{\{C_t=i\}} \mid \mathcal{F}_s \right]}{\mathbf{E}^{\mathbb{Q}} \left[\mathbf{1}_{\{C_u=k\}} \mathbf{1}_{\{C_t=i\}} \mid \mathcal{F}_s \right]},$$

which leads to our claim. \square

The following proposition will be the key in solving a representation to transition probabilities, which will be needed in applications.

3.13 Proposition. (a) $\mathbb{Q}(\cdot, \cdot)$ satisfies the **F**-conditional Chapman-Kolmogorov equation

$$\mathbb{Q}(t, s) = \mathbb{Q}(t, u) \mathbb{Q}(u, s), \quad t \leq u \leq s. \quad (3.14)$$

(b) $\mathbb{Q}(\cdot, \cdot)$ satisfies the **F**-conditional forward Kolmogorov equation

$$\frac{d\mathbb{Q}(t, s)}{ds} = \mathbb{Q}(t, s) \Lambda_s, \quad \mathbb{Q}(t, t) = I. \quad (3.15)$$

Proof. (a) For part (a), we have to show, that for each $i, j \in \mathcal{K}$ and any $0 \leq t \leq u \leq s$ we have

$$\mathbb{Q}^{i,j}(t, s) = \sum_{k=1}^K \mathbb{Q}^{i,k}(t, u) \mathbb{Q}^{k,j}(u, s)$$

on the set $\{C_t = i\}$.

We can use Lemma 3.11:

$$\begin{aligned}
\mathbb{Q}^{i,j}(t, s) &= \mathbb{Q}(C_s = j \mid \mathcal{F}_s; C_t = i) = \frac{\mathbb{Q}(C_s = j, C_t = i \mid \mathcal{F}_s)}{\mathbb{Q}(C_t = i \mid \mathcal{F}_s)} \\
&= \frac{1}{\mathbb{Q}(C_t = i \mid \mathcal{F}_s)} \sum_{k=1}^K \mathbb{Q}(C_s = j, C_u = k, C_t = i \mid \mathcal{F}_s) \\
&= \frac{1}{\mathbb{Q}(C_t = i \mid \mathcal{F}_s)} \sum_{k=1}^K \mathbb{Q}(C_u = k, C_t = i \mid \mathcal{F}_s) \frac{\mathbb{Q}(C_s = j, C_u = k, C_t = i \mid \mathcal{F}_s)}{\mathbb{Q}(C_u = k, C_t = i \mid \mathcal{F}_s)} \\
&= \sum_{k=1}^K \mathbb{Q}(C_u = k \mid \mathcal{F}_s; C_t = i) \mathbb{Q}(C_s = j \mid \mathcal{F}_s; C_u = k, C_t = i) \\
&= \sum_{k=1}^K \mathbb{Q}(C_u = k \mid \mathcal{F}_s; C_t = i) \mathbb{Q}(C_s = j \mid \mathcal{F}_s; C_u = k) \\
&= \sum_{k=1}^K \mathbb{Q}^{i,k}(t, u) \mathbb{Q}^{k,j}(u, s),
\end{aligned}$$

where the last equality follows from using Proposition 3.7 and the definition of conditional probabilities in 3.9:

$$\mathbb{Q}(C_u = k \mid \mathcal{F}_s; C_t = i) = \mathbb{Q}(C_u = k \mid \mathcal{F}_u; C_t = i).$$

(b) Part (b) follows from definition 3.7 in [16]. □

In our application we will find useful the following proposition, which gives us a possibility to separate \mathcal{F}_T -measurable variables, like discount factors or share prices, from \mathcal{F}_s^C -measurable variables, like $\mathbf{1}_{\{C_s=i\}}$, when calculating conditional expectations with respect to \mathcal{F}_s .

3.16 Proposition. *Let C be a canonically constructed conditional Markov chain. Let X be a bounded \mathcal{F}_T -measurable random variable and Y a bounded \mathcal{F}_s^C -measurable random variable, $s \in [0, T]$. Then the following three equivalent statements hold:*

- (a) $\mathbf{E}^{\mathbb{Q}}[XY \mid \mathcal{F}_s] = \mathbf{E}^{\mathbb{Q}}[X \mid \mathcal{F}_s] \mathbf{E}^{\mathbb{Q}}[Y \mid \mathcal{F}_s]$, i.e., the σ -algebras \mathcal{F}_T and \mathcal{F}_s^C are conditionally independent given \mathcal{F}_s .
- (b) $\mathbf{E}^{\mathbb{Q}}[Y \mid \mathcal{F}_T] = \mathbf{E}^{\mathbb{Q}}[Y \mid \mathcal{F}_s]$

$$(c) \mathbf{E}^{\mathbb{Q}} [X \mid \mathcal{F}_s \vee \mathcal{F}_s^C] = \mathbf{E}^{\mathbb{Q}} [X \mid \mathcal{F}_s]$$

Proof. The equivalence between (a) and (b) and the equivalence between (a) and (c) follow from Theorem 2.3 in [15].

To show that (b) is true for C , one can use Proposition 3.7(b), by passing from $Y = \mathbf{1}_{\{C_{s_1}=i\}} \mathbf{1}_{\{C_{s_2}=j\}}$, where $0 \leq s_1 \leq s_2 \leq s$ and $i, j \in \mathcal{K}$, to any bounded \mathcal{F}_s^C -measurable Y using standard probabilistic arguments. \square

3.17 Lemma. *Let X a bounded \mathcal{F}_T -measurable random variable and $0 \leq t \leq s \leq T$. Then following items hold:*

(a)

$$\frac{\mathbf{E}^{\mathbb{Q}} [X \mathbf{1}_{\{C_s=j\}} \mathbf{1}_{\{C_t=i\}} \mid \mathcal{F}_t]}{\mathbf{E}^{\mathbb{Q}} [\mathbf{1}_{\{C_t=i\}} \mid \mathcal{F}_t]} = \mathbf{E}^{\mathbb{Q}} [X \mathbb{Q}^{i,j}(t, s) \mid \mathcal{F}_t] \quad (3.18)$$

and

(b)

$$\mathbf{E}^{\mathbb{Q}} [X \mathbf{1}_{\{C_s \neq K\}} \mid \mathcal{G}_t] = \sum_{i=1}^{K-1} \mathbf{E}^{\mathbb{Q}} [X (1 - \mathbb{Q}^{i,K}(t, s)) \mid \mathcal{F}_t] \mathbf{1}_{\{C_t=i\}}. \quad (3.19)$$

Proof. (a) We use the properties of conditional expectations, 3.16 and the definition of $\mathbb{Q}^{i,j}(t, s)$:

$$\begin{aligned} \frac{\mathbf{E}^{\mathbb{Q}} [X \mathbf{1}_{\{C_s=j\}} \mathbf{1}_{\{C_t=i\}} \mid \mathcal{F}_t]}{\mathbf{E}^{\mathbb{Q}} [\mathbf{1}_{\{C_t=i\}} \mid \mathcal{F}_t]} &= \mathbf{E}^{\mathbb{Q}} \left[\frac{\mathbf{E}^{\mathbb{Q}} [X \mathbf{1}_{\{C_s=j\}} \mathbf{1}_{\{C_t=i\}} \mid \mathcal{F}_s]}{\mathbf{E}^{\mathbb{Q}} [\mathbf{1}_{\{C_t=i\}} \mid \mathcal{F}_s]} \mid \mathcal{F}_t \right] \\ &= \mathbf{E}^{\mathbb{Q}} \left[X \frac{\mathbf{E}^{\mathbb{Q}} [\mathbf{1}_{\{C_s=j\}} \mathbf{1}_{\{C_t=i\}} \mid \mathcal{F}_s]}{\mathbf{E}^{\mathbb{Q}} [\mathbf{1}_{\{C_t=i\}} \mid \mathcal{F}_s]} \mid \mathcal{F}_t \right] = \mathbf{E}^{\mathbb{Q}} [X \mathbb{Q}^{i,j}(t, s) \mid \mathcal{F}_t]. \end{aligned}$$

(b) Using (a), equation 3.6, and the fact, that $\mathbf{1}_{\{C_s \neq K\}} = \sum_{j=1}^{K-1} \mathbf{1}_{\{C_s=j\}}$ we get:

$$\begin{aligned}
\mathbf{E}^{\mathbb{Q}} \left[X \mathbf{1}_{\{C_s \neq K\}} \mid \mathcal{G}_t \right] &= \sum_{j=1}^{K-1} \mathbf{E}^{\mathbb{Q}} \left[X \mathbf{1}_{\{C_s=j\}} \mid \mathcal{G}_t \right] \\
&= \sum_{j=1}^{K-1} \sum_{i=1}^K \mathbf{1}_{\{C_t=i\}} \frac{\mathbf{E}^{\mathbb{Q}} \left[X \mathbf{1}_{\{C_s=j\}} \mathbf{1}_{\{C_t=i\}} \mid \mathcal{F}_t \right]}{\mathbf{E}^{\mathbb{Q}} \left[\mathbf{1}_{\{C_t=i\}} \mid \mathcal{F}_t \right]} \\
&= \sum_{i=1}^K \sum_{j=1}^{K-1} \mathbf{E}^{\mathbb{Q}} \left[X \mathbb{Q}^{i,j}(t,s) \mid \mathcal{F}_t \right] \mathbf{1}_{\{C_t=i\}} = \sum_{i=1}^K \mathbf{E}^{\mathbb{Q}} \left[X \sum_{j=1}^{K-1} \mathbb{Q}^{i,j}(t,s) \mid \mathcal{F}_t \right] \mathbf{1}_{\{C_t=i\}} \\
&= \sum_{i=1}^K \mathbf{E}^{\mathbb{Q}} \left[X (1 - \mathbb{Q}^{i,K}(t,s)) \mid \mathcal{F}_t \right] \mathbf{1}_{\{C_t=i\}} = \sum_{i=1}^{K-1} \mathbf{E}^{\mathbb{Q}} \left[X (1 - \mathbb{Q}^{i,K}(t,s)) \mid \mathcal{F}_t \right] \mathbf{1}_{\{C_t=i\}},
\end{aligned}$$

where the last equality follows from $\mathbb{Q}^{K,K}(t,s) = 1$, since K is an absorbing state. \square

4 Asset prices in Conditional Markov model

In this section we will derive prices to financial instruments relevant in our setup, that is bonds, credit default swaps and contingent convertibles. All the results will be derived conditional on set $\{C_t = i\}$ for some $i \neq K$.

For practical reasons, it is important to notice, that cash flows from traditional senior bonds are not affected by write-down or equity conversion. This information can be used in model calibration of Λ_t using bond prices. We define senior bond with maturity T_m as a security paying coupons c_k at times T_1, \dots, T_m . The payoff of a senior bond coupon is

$$c_k \mathbf{1}_{\{C_{T_k} \neq K\}} N$$

and the payoff of the redemption of a senior bond maturing at time T_m is

$$\mathbf{1}_{\{C_{T_m} \neq K\}} N.$$

The assumption of zero recovery is not very realistic with senior bonds, in fact their recovery rate might be quite high. We define recovery rate as a constant $q^D \in [0, 1]$. The discounted payoff from recovery is

$$D(t, \tau) \mathbf{1}_{\{\tau^D \leq T_m\}} q^D N.$$

Here we assume, that recovery payment is paid at time of default, τ^D .

Our assumptions of constant recovery rate paid at time of default are motivated by reduced form models used in practice for pricing credit derivatives. A standard assumption of recovery rate in the derivatives markets is, that it is a constant (usually 0.4). In reality, the bankruptcy and liquidation process involves a great deal of uncertainty of how large payments bond investors will receive and when they will be paid. In a case of a large, international bank this process might take several years.

One way to model recovery rate would be to model it as a stochastic process. After the default event investors might be able to sell their bonds to a speculative investor (say, a hedge fund specialized in distressed debt) for some price. It is our view, that this would be the most realistic approach, but it might also induce too much complexity for the model to be of any practical use. For a comprehensive comparison of different types of recovery rate assumptions we refer to [2] and [3].

A common practice in literature is to approximate the discount factor $D(t, \tau^D)$ with the discounting factor of the following cash flow $D(t, T_l)$, $T_l = \inf\{T_k, T_k > \tau^D, k = 1, \dots, m\}$. This simplifies the derivation of some formulas and avoids numerical integration, but we will refrain from this assumption.

4.1 Proposition. *The price at time $t < T_m$ of a senior bond with maturity T_m and recovery rate q^D , which pays coupons c_k at times T_k , $k = 1, \dots, m$, is given by*

$$\begin{aligned} \sum_{k=1}^m D(t, T_k) c_k N \mathbf{E}^{\mathbb{Q}} [1 - \mathbb{Q}^{i,K}(t, T_k) \mid \mathcal{F}_t] + D(t, T_m) N \mathbf{E}^{\mathbb{Q}} [1 - \mathbb{Q}^{i,K}(t, T_m) \mid \mathcal{F}_t] \\ + q^D N \mathbf{E}^{\mathbb{Q}} \left[\int_t^{T_m} D(t, u) d\mathbb{Q}^{i,K}(t, u) \mid \mathcal{F}_t \right]. \end{aligned}$$

Proof. According to the risk-neutral valuation formula, the price of coupon payments is given by

$$\begin{aligned} \mathbf{E}^{\mathbb{Q}} \left[\sum_{k=1}^m D(t, T_k) c_k N \mathbf{1}_{\{C_{T_k} \neq K\}} \mid \mathcal{G}_t \right] &= \sum_{k=1}^m D(t, T_k) c_k N \mathbf{E}^{\mathbb{Q}} \left[\mathbf{1}_{\{C_{T_k} \neq K\}} \mid \mathcal{G}_t \right] \\ &= \sum_{k=1}^m D(t, T_k) c_k N \left(\mathbf{E}^{\mathbb{Q}} [1 - \mathbb{Q}^{1,K}(t, T_k) \mid \mathcal{F}_t] \mathbf{1}_{\{C_t=1\}} + \mathbf{E}^{\mathbb{Q}} [1 - \mathbb{Q}^{2,K}(t, T_k) \mid \mathcal{F}_t] \mathbf{1}_{\{C_t=2\}} \right), \end{aligned}$$

where we use 3.17 (b). At time t , given that $C_t = i$, we have

$$\sum_{k=1}^m D(t, T_k) c_k N \mathbf{E}^{\mathbb{Q}} [1 - \mathbb{Q}^{i,K}(t, T_k) \mid \mathcal{F}_t].$$

The price of the redemption payment is derived in the same way:

$$\begin{aligned}
& D(t, T_m) N \left(\mathbf{E}^{\mathbb{Q}} \left[\mathbf{1}_{\{C_{T_m}=1\}} \mid \mathcal{G}_t \right] + \mathbf{E}^{\mathbb{Q}} \left[\mathbf{1}_{\{C_{T_m}=2\}} \mid \mathcal{G}_t \right] \right) \\
&= D(t, T_m) N \left(\mathbf{E}^{\mathbb{Q}} \left[1 - \mathbb{Q}^{1,K}(t, T_m) \mid \mathcal{F}_t \right] \mathbf{1}_{\{C_t=1\}} + \mathbf{E}^{\mathbb{Q}} \left[1 - \mathbb{Q}^{2,K}(t, T_m) \mid \mathcal{F}_t \right] \mathbf{1}_{\{C_t=2\}} \right) \\
&= D(t, T_m) N \mathbf{E}^{\mathbb{Q}} \left[1 - \mathbb{Q}^{i,K}(t, T_m) \mid \mathcal{F}_t \right].
\end{aligned}$$

That leaves us with final term, which is the recovery:

$$\begin{aligned}
\mathbf{E}^{\mathbb{Q}} \left[D(t, \tau^D) \mathbf{1}_{\{\tau^D \leq T_m\}} q^D N \mid \mathcal{G}_t \right] &= q^D N \mathbf{E}^{\mathbb{Q}} \left[D(t, \tau^D) \mathbf{1}_{\{\tau^D \leq T_m\}} \mid \mathcal{F}_t \right] \mathbf{1}_{\{C_t=1\}} \\
&\quad + q^D N \mathbf{E}^{\mathbb{Q}} \left[D(t, \tau^D) \mathbf{1}_{\{\tau^D \leq T_m\}} \mid \mathcal{F}_t \right] \mathbf{1}_{\{C_t=2\}}.
\end{aligned}$$

Assuming that discount factors are independent of intensity processes $\lambda_t^{i,j}$, the recovery term becomes

$$\begin{aligned}
& \sum_{i=1}^{K-1} q^D N \mathbf{E}^{\mathbb{Q}} \left[D(t, \tau^D) \mathbf{1}_{\{\tau^D \leq T_m\}} \mid \mathcal{F}_t \right] \mathbf{1}_{\{C_t=i\}} \\
&= \sum_{i=1}^{K-1} q^D N \mathbf{E}^{\mathbb{Q}} \left[\int_t^{T_m} D(t, u) d\mathbb{Q}^{i,K}(t, u) \mid \mathcal{F}_t \right] \mathbf{1}_{\{C_t=i\}}
\end{aligned}$$

and since $\mathbf{1}_{\{C_t=i\}}$, we have

$$q^D N \mathbf{E}^{\mathbb{Q}} \left[\int_t^{T_m} D(t, u) d\mathbb{Q}^{i,K}(t, u) \mid \mathcal{F}_t \right].$$

□

Credit default swap is a financial instrument, where protection buyer pays periodical payments, with rate $Z \in \mathbf{R}_+$, to protection seller for protection against an issuer or a bond going into default. Payments are paid periodically at times T_1, \dots, T_m , until the event of default, τ^D , or until maturity T_m , whichever becomes first. In case of default, protection buyer will pay accrued interest from period between the previous payment and the default time. This chain of cash flows is called the premium leg:

$$\sum_{k=1}^m \left(N D(t, T_k) Z \Delta_{T_k} \mathbf{1}_{\{C_{T_k} \neq K\}} + N D(t, \tau^D) Z (\tau^D - T_{k-1}) \mathbf{1}_{\{T_{k-1} < \tau^D \leq T_k\}} \right),$$

where $\Delta_{T_k} := T_k - T_{k-1}$. Constant Z is often called contract spread, or coupon and it is specified in the CDS contract.

On the other hand, the protection seller agrees to cover losses with fraction $(1 - q^D)$ of nominal amount N provided that the default occurs before maturity. This is called the protection leg:

$$ND(t, \tau^D) (1 - q^D) \mathbf{1}_{\{\tau^D \leq T_m\}}.$$

4.2 Proposition. *The prices of CDS-legs at time $t < T_m$ with maturity T_m , recovery rate q^D and paying coupons at rate Z at times T_k , $k = 1, \dots, m$, are given by*

(a) *Premium leg:*

$$\begin{aligned} & \sum_{k=1}^m \left(ND(t, T_k) Z \Delta_{T_k} \mathbf{E}^{\mathbb{Q}} [1 - \mathbb{Q}^{i,K}(t, T_k) \mid \mathcal{F}_t] \right. \\ & \left. + NZ \mathbf{E}^{\mathbb{Q}} \left[\int_{T_{k-1}}^{T_k} D(t, u) (u - T_{k-1}) d\mathbb{Q}^{i,K}(t, u) \mid \mathcal{F}_t \right] \right) \end{aligned} \quad (4.3)$$

(b) *Protection leg:*

$$(1 - q^D) N \mathbf{E}^{\mathbb{Q}} \left[\int_t^{T_m} D(t, u) d\mathbb{Q}^{i,K}(t, u) \mid \mathcal{F}_t \right] \quad (4.4)$$

Proof. (a) For the premium leg we have:

$$\begin{aligned} & \mathbf{E}^{\mathbb{Q}} \left[\sum_{k=1}^m \left(ND(t, T_k) Z \Delta_{T_k} \mathbf{1}_{\{C_{T_k} \neq K\}} + ND(t, \tau^D) Z (\tau^D - T_{k-1}) \mathbf{1}_{\{T_{k-1} < \tau^D \leq T_k\}} \right) \mid \mathcal{G}_t \right] \\ & = \sum_{k=1}^m NZ \left(\mathbf{E}^{\mathbb{Q}} \left[D(t, T_k) \Delta_{T_k} \mathbf{1}_{\{C_{T_k} \neq K\}} \mid \mathcal{G}_t \right] + \mathbf{E}^{\mathbb{Q}} \left[D(t, \tau^D) (\tau^D - T_{k-1}) \mathbf{1}_{\{T_{k-1} < \tau^D \leq T_k\}} \mid \mathcal{G}_t \right] \right). \end{aligned}$$

Using the same deduction as in the proof of Proposition 4.1, the first term becomes

$$\mathbf{E}^{\mathbb{Q}} \left[D(t, T_k) \Delta_{T_k} \mathbf{1}_{\{C_{T_k} \neq K\}} \mid \mathcal{G}_t \right] = D(t, T_k) \Delta_{T_k} \mathbf{E}^{\mathbb{Q}} [1 - \mathbb{Q}^{i,K}(t, T_k) \mid \mathcal{F}_t].$$

The second term becomes

$$\begin{aligned} & \mathbf{E}^{\mathbb{Q}} \left[D(t, \tau^D) (\tau^D - T_{k-1}) \mathbf{1}_{\{T_{k-1} < \tau^D \leq T_k\}} \mid \mathcal{G}_t \right] \\ & = \mathbf{E}^{\mathbb{Q}} \left[\int_{T_{k-1}}^{T_k} D(t, u) (u - T_{k-1}) d\mathbb{Q}^{i,K}(t, u) \mid \mathcal{F}_t \right]. \end{aligned}$$

(b) Price of the protection leg can be derived in a similar fashion as the recovery term in the proof of Proposition 4.1:

$$\begin{aligned}
& (1 - q^D) N \mathbf{E}^{\mathbb{Q}} \left[D(t, \tau^D) \mathbf{1}_{\{\tau^D \leq T_m\}} \mid \mathcal{F}_t \right] \\
&= \sum_{i=1}^{K-1} (1 - q^D) N \mathbf{E}^{\mathbb{Q}} \left[D(t, \tau^D) \mathbf{1}_{\{\tau^D \leq T_m\}} \mid \mathcal{F}_t \right] \mathbf{1}_{\{C_i=1\}} \\
&= \sum_{i=1}^{K-1} (1 - q^D) N \mathbf{E}^{\mathbb{Q}} \left[\int_t^{T_m} D(t, u) d\mathbb{Q}^{i,K}(t, u) \mid \mathcal{F}_t \right] \mathbf{1}_{\{C_i=i\}} \\
&= (1 - q^D) N \mathbf{E}^{\mathbb{Q}} \left[\int_t^{T_m} D(t, u) d\mathbb{Q}^{i,K}(t, u) \mid \mathcal{F}_t \right].
\end{aligned}$$

□

The pricing formulas for senior bonds and credit default swaps are model independent in the sense, that we don't make any specific assumptions about the evolution of processes $\lambda_t^{i,j}$. For model independent valuation of CDS in reduced form models, see [4] chapter 21.

We will now represent the main results for this paper, prices of contingent convertibles, starting with a contingent convertible with temporary write-down.

4.5 Proposition. *The price at time t of a contingent convertible with temporary write-down and write-down fraction $(1 - q^{\text{CoCo}})$, paying coupons c_k at times $T_k, T_{k-1} < T_k, k \in \mathbb{N}$, that is redeemable at times T_m, T_{m+1}, \dots, T_n is given by:*

$$\begin{aligned}
& \sum_{k=1}^m D(t, T_k) c_k N \left(\mathbf{E}^{\mathbb{Q}} \left[1 - \mathbb{Q}^{i,2}(t, T_k) - \mathbb{Q}^{i,K}(t, T_k) \mid \mathcal{F}_t \right] + q^{\text{CoCo}} \mathbf{E}^{\mathbb{Q}} \left[\mathbb{Q}^{i,2}(t, T_k) \mid \mathcal{F}_t \right] \right) \\
&+ D(t, T_m) N \mathbf{E}^{\mathbb{Q}} \left[1 - \mathbb{Q}^{i,2}(t, T_m) - \mathbb{Q}^{i,K}(t, T_m) \mid \mathcal{F}_t \right] \\
&+ \sum_{k=m+1}^n D(t, T_k) N \left((1 + c_k) \mathbf{E}^{\mathbb{Q}} \left[\mathbb{Q}^{i,2}(t, T_m) \mathbb{Q}^{2,1}(T_{k-1}, T_k) \prod_{j=m+1}^{k-1} \mathbb{Q}^{2,2}(T_{j-1}, T_j) \mid \mathcal{F}_t \right] \right. \\
&\quad \left. + c_k q^{\text{CoCo}} \mathbf{E}^{\mathbb{Q}} \left[\mathbb{Q}^{i,2}(t, T_m) \mathbb{Q}^{2,2}(T_{k-1}, T_k) \prod_{j=m+1}^{k-1} \mathbb{Q}^{2,2}(T_{j-1}, T_j) \mid \mathcal{F}_t \right] \right) \\
&+ \sum_{k=n+1}^{\infty} D(t, T_k) N c_k \left(\mathbf{E}^{\mathbb{Q}} \left[\mathbb{Q}^{i,2}(t, T_m) \mathbb{Q}^{2,1}(T_n, T_k) \prod_{j=m+1}^n \mathbb{Q}^{2,2}(T_{j-1}, T_j) \mid \mathcal{F}_t \right] \right. \\
&\quad \left. + q^{\text{CoCo}} \mathbf{E}^{\mathbb{Q}} \left[\mathbb{Q}^{i,2}(t, T_m) \mathbb{Q}^{2,2}(T_n, T_k) \prod_{j=m+1}^n \mathbb{Q}^{2,2}(T_{j-1}, T_j) \mid \mathcal{F}_t \right] \right).
\end{aligned}$$

Proof. For the coupon payments up to first redemption time, T_m , we have:

$$\begin{aligned}
& \mathbf{E}^{\mathbb{Q}} \left[\sum_{k=1}^m D(t, T_k) c_k \mathbf{1}_{\{C_{T_k}=1\}} N + D(t, T_k) c_k \mathbf{1}_{\{C_{T_k}=2\}} q^{\text{CoCo}} N \mid \mathcal{G}_t \right] \\
&= \sum_{k=1}^m D(t, T_k) c_k N \left(\mathbf{E}^{\mathbb{Q}} \left[\mathbf{1}_{\{C_{T_k}=1\}} \mid \mathcal{G}_t \right] + q^{\text{CoCo}} \mathbf{E}^{\mathbb{Q}} \left[\mathbf{1}_{\{C_{T_k}=2\}} \mid \mathcal{G}_t \right] \right) \\
&= \sum_{k=1}^m D(t, T_k) c_k N \left(\sum_{i=1}^K \mathbf{1}_{\{C_t=i\}} \frac{\mathbf{E}^{\mathbb{Q}} \left[\mathbf{1}_{\{C_{T_k}=1\}} \mathbf{1}_{\{C_t=i\}} \mid \mathcal{F}_t \right]}{\mathbf{E}^{\mathbb{Q}} \left[\mathbf{1}_{\{C_t=i\}} \mid \mathcal{F}_t \right]} \right. \\
&\quad \left. + q^{\text{CoCo}} \sum_{i=1}^K \mathbf{1}_{\{C_t=i\}} \frac{\mathbf{E}^{\mathbb{Q}} \left[\mathbf{1}_{\{C_{T_k}=2\}} \mathbf{1}_{\{C_t=i\}} \mid \mathcal{F}_t \right]}{\mathbf{E}^{\mathbb{Q}} \left[\mathbf{1}_{\{C_t=i\}} \mid \mathcal{F}_t \right]} \right),
\end{aligned}$$

where we used equation 3.6. On the set $\{C_t = i\}$ we have

$$\frac{\mathbf{E}^{\mathbb{Q}} \left[\mathbf{1}_{\{C_{T_k}=j\}} \mathbf{1}_{\{C_t=i\}} \mid \mathcal{F}_t \right]}{\mathbf{E}^{\mathbb{Q}} \left[\mathbf{1}_{\{C_t=i\}} \mid \mathcal{F}_t \right]} = \mathbf{E}^{\mathbb{Q}} \left[\frac{\mathbf{E}^{\mathbb{Q}} \left[\mathbf{1}_{\{C_{T_k}=j\}} \mathbf{1}_{\{C_t=i\}} \mid \mathcal{F}_{T_k} \right]}{\mathbf{E}^{\mathbb{Q}} \left[\mathbf{1}_{\{C_t=i\}} \mid \mathcal{F}_{T_k} \right]} \mid \mathcal{F}_t \right] = \mathbf{E}^{\mathbb{Q}} \left[\mathbb{Q}^{i,j}(t, T_k) \mid \mathcal{F}_t \right],$$

where we have used Proposition 3.16 (b) and Definition 3.10. This leaves us with

$$\begin{aligned}
& \sum_{k=1}^m D(t, T_k) c_k N \left(\mathbf{E}^{\mathbb{Q}} \left[\mathbb{Q}^{i,1}(t, T_k) \mid \mathcal{F}_t \right] + q^{\text{CoCo}} \mathbf{E}^{\mathbb{Q}} \left[\mathbb{Q}^{i,2}(t, T_k) \mid \mathcal{F}_t \right] \right) \\
&= \sum_{k=1}^m D(t, T_k) c_k N \left(\mathbf{E}^{\mathbb{Q}} \left[1 - \mathbb{Q}^{i,2}(t, T_k) - \mathbb{Q}^{i,K}(t, T_k) \mid \mathcal{F}_t \right] + q^{\text{CoCo}} \mathbf{E}^{\mathbb{Q}} \left[\mathbb{Q}^{i,2}(t, T_k) \mid \mathcal{F}_t \right] \right).
\end{aligned}$$

The value of the first redemption payment is:

$$\mathbf{E}^{\mathbb{Q}} \left[D(t, T_m) \mathbf{1}_{\{C_{T_m}=1\}} N \mid \mathcal{G}_t \right] = D(t, T_m) N \mathbf{E}^{\mathbb{Q}} \left[1 - \mathbb{Q}^{i,2}(t, T_m) - \mathbb{Q}^{i,K}(t, T_m) \mid \mathcal{F}_t \right].$$

For the rest of the redemption payments and their corresponding coupon payments, we have:

$$\begin{aligned}
& \mathbf{E}^{\mathbb{Q}} \left[\sum_{k=m+1}^n \left(D(t, T_k) \mathbf{1}_{\{C_{T_k}=1\}} (1 + c_k) N + D(t, T_k) \mathbf{1}_{\{C_{T_k}=2\}} c_k q^{\text{CoCo}} N \right) \prod_{j=m}^{k-1} \mathbf{1}_{\{C_{T_j}=2\}} \middle| \mathcal{G}_t \right] \\
&= \sum_{k=m+1}^n D(t, T_k) N \mathbf{E}^{\mathbb{Q}} \left[\mathbf{1}_{\{C_{T_k}=1\}} (1 + c_k) \prod_{j=m}^{k-1} \mathbf{1}_{\{C_{T_j}=2\}} + \mathbf{1}_{\{C_{T_k}=2\}} c_k q^{\text{CoCo}} \prod_{j=m}^{k-1} \mathbf{1}_{\{C_{T_j}=2\}} \middle| \mathcal{G}_t \right] \\
&= \sum_{k=m+1}^n D(t, T_k) N \left((1 + c_k) \mathbf{E}^{\mathbb{Q}} \left[\prod_{j=m}^{k-1} \mathbf{1}_{\{C_{T_k}=1\}} \mathbf{1}_{\{C_{T_j}=2\}} \middle| \mathcal{G}_t \right] \right. \\
&\quad \left. + c_k q^{\text{CoCo}} \mathbf{E}^{\mathbb{Q}} \left[\prod_{j=m}^{k-1} \mathbf{1}_{\{C_{T_k}=2\}} \mathbf{1}_{\{C_{T_j}=2\}} \middle| \mathcal{G}_t \right] \right) \\
&= \sum_{k=m+1}^n D(t, T_k) N \left((1 + c_k) \sum_{i=1}^K \mathbf{1}_{\{C_t=i\}} \frac{\mathbf{E}^{\mathbb{Q}} \left[\prod_{j=m}^{k-1} \mathbf{1}_{\{C_{T_k}=1\}} \mathbf{1}_{\{C_{T_j}=2\}} \mathbf{1}_{\{C_t=i\}} \middle| \mathcal{F}_t \right]}{\mathbf{E}^{\mathbb{Q}} \left[\mathbf{1}_{\{C_t=i\}} \middle| \mathcal{F}_t \right]} \right. \\
&\quad \left. + c_k q^{\text{CoCo}} \sum_{i=1}^K \mathbf{1}_{\{C_t=i\}} \frac{\mathbf{E}^{\mathbb{Q}} \left[\prod_{j=m}^{k-1} \mathbf{1}_{\{C_{T_k}=2\}} \mathbf{1}_{\{C_{T_j}=2\}} \mathbf{1}_{\{C_t=i\}} \middle| \mathcal{F}_t \right]}{\mathbf{E}^{\mathbb{Q}} \left[\mathbf{1}_{\{C_t=i\}} \middle| \mathcal{F}_t \right]} \right). \tag{4.6}
\end{aligned}$$

We can now use Proposition 3.16 (b):

$$\begin{aligned}
& \frac{\mathbf{E}^{\mathbb{Q}} \left[\prod_{j=m}^{k-1} \mathbf{1}_{\{C_{T_k}=1\}} \mathbf{1}_{\{C_{T_j}=2\}} \mathbf{1}_{\{C_t=i\}} \middle| \mathcal{F}_t \right]}{\mathbf{E}^{\mathbb{Q}} \left[\mathbf{1}_{\{C_t=i\}} \middle| \mathcal{F}_t \right]} = \mathbf{E}^{\mathbb{Q}} \left[\frac{\mathbf{E}^{\mathbb{Q}} \left[\prod_{j=m}^{k-1} \mathbf{1}_{\{C_{T_k}=1\}} \mathbf{1}_{\{C_{T_j}=2\}} \mathbf{1}_{\{C_t=i\}} \middle| \mathcal{F}_{T_k} \right]}{\mathbf{E}^{\mathbb{Q}} \left[\mathbf{1}_{\{C_t=i\}} \middle| \mathcal{F}_{T_k} \right]} \middle| \mathcal{F}_t \right] \\
&= \mathbf{E}^{\mathbb{Q}} \left[\mathbb{Q} (C_{T_k} = 1, C_{T_m} = 2, \dots, C_{T_{k-1}} = 2 \middle| \mathcal{F}_{T_k}; C_t = i) \middle| \mathcal{F}_t \right]
\end{aligned}$$

and for the conditional probability we use the fact, that

$$\mathbb{Q} (C_{T_k} = 1, C_{T_m} = 2, \dots, C_{T_{k-1}} = 2 \middle| \mathcal{F}_{T_k}; C_t = i) = \mathbb{Q}^{i,2} (t, T_m) \mathbb{Q}^{2,2} (T_m, T_{m+1}) \dots \mathbb{Q}^{2,1} (T_{k-1}, T_k).$$

The same deduction can be used for the case $C_{T_k} = 2$ in 4.6. It follows that

$$\begin{aligned}
& \sum_{k=m+1}^n D(t, T_k) N \left((1 + c_k) \sum_{i=1}^K \mathbf{1}_{\{C_i=i\}} \frac{\mathbf{E}^{\mathbb{Q}} \left[\prod_{j=m}^{k-1} \mathbf{1}_{\{C_{T_k}=1\}} \mathbf{1}_{\{C_{T_j}=2\}} \mathbf{1}_{\{C_i=i\}} \mid \mathcal{F}_t \right]}{\mathbf{E}^{\mathbb{Q}} \left[\mathbf{1}_{\{C_i=i\}} \mid \mathcal{F}_t \right]} \right. \\
& \quad \left. + c_k q^{\text{CoCo}} \sum_{i=1}^K \mathbf{1}_{\{C_i=i\}} \frac{\mathbf{E}^{\mathbb{Q}} \left[\prod_{j=m}^{k-1} \mathbf{1}_{\{C_{T_k}=2\}} \mathbf{1}_{\{C_{T_j}=2\}} \mathbf{1}_{\{C_i=i\}} \mid \mathcal{F}_t \right]}{\mathbf{E}^{\mathbb{Q}} \left[\mathbf{1}_{\{C_i=i\}} \mid \mathcal{F}_t \right]} \right) \\
&= \sum_{k=m+1}^n D(t, T_k) N \left((1 + c_k) \sum_{i=1}^K \mathbf{1}_{\{C_i=i\}} \mathbf{E}^{\mathbb{Q}} \left[\mathbb{Q}^{i,2}(t, T_m) \mathbb{Q}^{2,1}(T_{k-1}, T_k) \prod_{j=m+1}^{k-1} \mathbb{Q}^{2,2}(T_{j-1}, T_j) \mid \mathcal{F}_t \right] \right. \\
& \quad \left. + c_k q^{\text{CoCo}} \sum_{i=1}^K \mathbf{1}_{\{C_i=i\}} \mathbf{E}^{\mathbb{Q}} \left[\mathbb{Q}^{i,2}(t, T_m) \mathbb{Q}^{2,2}(T_{k-1}, T_k) \prod_{j=m+1}^{k-1} \mathbb{Q}^{2,2}(T_{j-1}, T_j) \mid \mathcal{F}_t \right] \right) \\
&= \sum_{k=m+1}^n D(t, T_k) N \left((1 + c_k) \mathbf{E}^{\mathbb{Q}} \left[\mathbb{Q}^{i,2}(t, T_m) \mathbb{Q}^{2,1}(T_{k-1}, T_k) \prod_{j=m+1}^{k-1} \mathbb{Q}^{2,2}(T_{j-1}, T_j) \mid \mathcal{F}_t \right] \right. \\
& \quad \left. + c_k q^{\text{CoCo}} \mathbf{E}^{\mathbb{Q}} \left[\mathbb{Q}^{i,2}(t, T_m) \mathbb{Q}^{2,2}(T_{k-1}, T_k) \prod_{j=m+1}^{k-1} \mathbb{Q}^{2,2}(T_{j-1}, T_j) \mid \mathcal{F}_t \right] \right).
\end{aligned}$$

The value of the rest of the coupon payments can be calculated in a similar fashion

$$\begin{aligned}
& \mathbf{E}^{\mathbb{Q}} \left[\sum_{k=n+1}^{\infty} D(t, T_k) \left(\mathbf{1}_{\{C_{T_k}=1\}} c_k N + \mathbf{1}_{\{C_{T_k}=2\}} c_k q^{\text{CoCo}} N \right) \prod_{j=m}^n \mathbf{1}_{\{C_{T_j}=2\}} \middle| \mathcal{G}_t \right] \\
&= \sum_{k=n+1}^{\infty} D(t, T_k) N c_k \left(\mathbf{E}^{\mathbb{Q}} \left[\mathbf{1}_{\{C_{T_k}=1\}} \prod_{i=m}^n \mathbf{1}_{\{C_{T_i}=2\}} \middle| \mathcal{G}_t \right] \right. \\
&\quad \left. + q^{\text{CoCo}} \mathbf{E}^{\mathbb{Q}} \left[\mathbf{1}_{\{C_{T_k}=2\}} \prod_{j=m}^n \mathbf{1}_{\{C_{T_j}=2\}} \middle| \mathcal{G}_t \right] \right) \\
&= \sum_{k=n+1}^{\infty} D(t, T_k) N c_k \left(\sum_{i=1}^K \mathbf{1}_{\{C_i=i\}} \frac{\mathbf{E}^{\mathbb{Q}} \left[\mathbf{1}_{\{C_{T_k}=1\}} \prod_{i=m}^n \mathbf{1}_{\{C_{T_i}=2\}} \mathbf{1}_{\{C_i=i\}} \middle| \mathcal{F}_t \right]}{\mathbf{E}^{\mathbb{Q}} \left[\mathbf{1}_{\{C_i=i\}} \middle| \mathcal{F}_t \right]} \right. \\
&\quad \left. + q^{\text{CoCo}} \sum_{i=1}^K \mathbf{1}_{\{C_i=i\}} \frac{\mathbf{E}^{\mathbb{Q}} \left[\mathbf{1}_{\{C_{T_k}=2\}} \prod_{j=m}^n \mathbf{1}_{\{C_{T_j}=2\}} \mathbf{1}_{\{C_i=i\}} \middle| \mathcal{F}_t \right]}{\mathbf{E}^{\mathbb{Q}} \left[\mathbf{1}_{\{C_i=i\}} \middle| \mathcal{F}_t \right]} \right) \\
&= \sum_{k=n+1}^{\infty} D(t, T_k) N c_k \left(\mathbf{E}^{\mathbb{Q}} \left[\mathbb{Q}^{i,2}(t, T_m) \mathbb{Q}^{2,1}(T_n, T_k) \prod_{j=m+1}^n \mathbb{Q}^{2,2}(T_{j-1}, T_j) \middle| \mathcal{F}_t \right] \right. \\
&\quad \left. + q^{\text{CoCo}} \mathbf{E}^{\mathbb{Q}} \left[\mathbb{Q}^{i,2}(t, T_m) \mathbb{Q}^{2,2}(T_n, T_k) \prod_{j=m+1}^n \mathbb{Q}^{2,2}(T_{j-1}, T_j) \middle| \mathcal{F}_t \right] \right).
\end{aligned}$$

□

4.7 Proposition. *The price at time t of a contingent convertible with permanent write-down and write-down fraction $(1 - q^{\text{CoCo}})$, paying coupons c_k at times T_k , $T_{k-1} < T_k$, $k \in \mathbb{N}$, that is redeemable at time T_m is given by:*

$$\begin{aligned}
& \sum_{k=1}^m D(t, T_k) c_k N \left(\mathbf{E}^{\mathbb{Q}} \left[1 - \mathbb{Q}^{i,2}(t, T_k) - \mathbb{Q}^{i,K}(t, T_k) \middle| \mathcal{F}_t \right] + q^{\text{CoCo}} \mathbf{E}^{\mathbb{Q}} \left[\mathbb{Q}^{i,2}(t, T_k) \middle| \mathcal{F}_t \right] \right) \\
&+ D(t, T_m) N \mathbf{E}^{\mathbb{Q}} \left[1 - \mathbb{Q}^{i,2}(t, T_m) - \mathbb{Q}^{i,K}(t, T_m) \middle| \mathcal{F}_t \right] \\
&+ \sum_{k=m+1}^{\infty} D(t, T_k) c_k N q^{\text{CoCo}} \mathbf{E}^{\mathbb{Q}} \left[\mathbb{Q}^{i,2}(t, T_k) \middle| \mathcal{F}_t \right]
\end{aligned}$$

Proof. One can use similar deduction than in the proof of Proposition 4.5. \square

4.8 Proposition. *The price at time t of a contingent convertible with equity conversion and conversion ratio $N/\phi(\cdot)$, paying coupons c_k at times T_k , $T_{k-1} < T_k$, $k \in \mathbb{N}$, that is redeemable at time T_m is given by:*

$$\begin{aligned} & \sum_{k=1}^m D(t, T_k) c_k NE^{\mathbb{Q}} [1 - Q^{i,2}(t, T_k) - Q^{i,K}(t, T_k) \mid \mathcal{F}_t] \\ & + D(t, T_m) NE^{\mathbb{Q}} [1 - Q^{i,2}(t, T_m) - Q^{i,K}(t, T_m) \mid \mathcal{F}_t] \\ & + NE^{\mathbb{Q}} \left[\int_t^{T_m} D(t, u) \frac{S_u}{\phi(S_u)} dQ^{i,2}(t, u) \mid \mathcal{F}_t \right] \end{aligned}$$

Proof. The first two components follow the proof of Proposition 4.5 by setting $q^{\text{CoCo}} = 0$. It is sufficient to derive the price at time t of equity conversion:

$$\begin{aligned} & E^{\mathbb{Q}} \left[D(t, \tau^{\text{CoCo}}) \frac{N}{\phi(S_{\tau^{\text{CoCo}}})} S_{\tau^{\text{CoCo}}} \mathbf{1}_{\{\tau^{\text{CoCo}} \leq T_m\}} \mid \mathcal{G}_t \right] \\ & = NE^{\mathbb{Q}} \left[D(t, \tau^{\text{CoCo}}) \frac{S_{\tau^{\text{CoCo}}}}{\phi(S_{\tau^{\text{CoCo}}})} \mathbf{1}_{\{\tau^{\text{CoCo}} \leq T_m\}} \mid \mathcal{G}_t \right] \\ & = NE^{\mathbb{Q}} \left[\int_t^{T_m} D(t, u) \frac{S_u}{\phi(S_u)} dQ^{i,2}(t, u) \mid \mathcal{F}_t \right]. \end{aligned}$$

\square

It is worth noting, that in case of a floating conversion price, $\phi(S_{\tau^{\text{CoCo}}}) = S_{\tau^{\text{CoCo}}}$, the stock price is canceled out and the conversion term reduces to $NE^{\mathbb{Q}} \left[\int_t^{T_m} D(t, u) dQ^{i,2}(t, u) \mid \mathcal{F}_t \right]$. This means, that the price of a contingent convertible with floating conversion price is not dependent on stock price directly.

Other types of dependencies could be embedded into the model, however. For instance, one could have intensities and stock price being driven by some common factor X_t , $\lambda_t^{i,j}(X_t)$, $S_t(X_t)$, where X_t is a stochastic process. This has been discussed in some of the original works in the credit migration framework like [19]. One could also introduce correlation parameters between $\lambda_t^{i,j}$ and S_t .

There exist a vast literature on joint stock and credit models, where the stock price follows a jump-diffusion process prior to default and jumps to 0 at time τ^D (see for example [6] and [7]). This could give a good starting point for further developments of our model. In this paper we don't go any deeper into modeling the stock price or equity conversion and leave this to later research.

As a final illustration of our framework, we propose a new type of credit default swap, which gives protection to buyer in case of a write-down of contingent convertible. Although these types of instruments don't exist yet, they have been discussed in the financial media. To best of our knowledge this is the first formal treatment of how these securities could be constructed.

4.9 Definition. Contingent credit default swap is a financial instrument, where protection buyer pays periodical payments, with rate $Z \in \mathbf{R}_+$, to protection seller for protection against write-down of a contingent convertible. Payments are paid periodically at times T_1, \dots, T_m , until the event of write-down, τ^{CoCo} , or until maturity T_m , which ever becomes first. In case of write-down, protection buyer will pay accrued interest from period between the previous payment and the time of write-down. This chain of cash flows is called the premium leg:

$$\sum_{k=1}^m \left(ND(t, T_k) Z \Delta_{T_k} \mathbf{1}_{\{T_k < \tau^{\text{CoCo}}\}} + ND(t, \tau^{\text{CoCo}}) Z (\tau^{\text{CoCo}} - T_{k-1}) \mathbf{1}_{\{T_{k-1} < \tau^{\text{CoCo}} \leq T_k\}} \right),$$

where $\Delta_{T_k} := T_k - T_{k-1}$.

In case of write-down, the protection seller agrees to cover losses with write-down fraction $(1 - q^{\text{CoCo}})$ of nominal amount N provided that the write-down occurs before maturity. Then the protection leg is of the form:

$$ND(t, \tau^{\text{CoCo}}) (1 - q^{\text{CoCo}}) \mathbf{1}_{\{\tau^{\text{CoCo}} \leq T_m\}}.$$

4.10 Proposition. The prices of contingent credit default swap legs at time $t < T_m$ with maturity T_m , write-down fraction $(1 - q^{\text{CoCo}})$ and paying coupons at rate Z at times T_k , $k = 1, \dots, m$, conditional on set $\{C_t = 1\}$, are given by

(a) Premium leg:

$$\sum_{k=1}^m \left(ND(t, T_k) Z \Delta_{T_k} \mathbf{E}^{\mathbb{Q}} [\mathbb{Q}^{1,1}(t, T_k) \mid \mathcal{F}_t] + NZ \mathbf{E}^{\mathbb{Q}} \left[\int_{T_{k-1}}^{T_k} D(t, u) (u - T_{k-1}) d\mathbb{Q}^{1,2}(t, u) \mid \mathcal{F}_t \right] \right)$$

(b) Protection leg:

$$(1 - q^{\text{CoCo}}) N \mathbf{E}^{\mathbb{Q}} \left[\int_t^{T_m} D(t, u) d\mathbb{Q}^{1,2}(t, u) \mid \mathcal{F}_t \right]$$

Proof. Proof can be derived in a similar fashion as with regular CDS-contracts in Proposition 4.2 by noticing, that none of the payoff components are affected by the default state. \square

Here we haven't differentiated between permanent and temporary write-down. To keep it simple, we have defined the contract so, that it is triggered at the first time of write-down. As with contingent convertibles with permanent write-down or equity conversion, for these instruments $\lambda_t^{2,1} = 0$.

Our definition of contingent CDS is perfectly analogous with regular CDS, when the triggering event τ^{CoCo} is used instead of the default τ^{D} and recovery rate is replaced by fraction q^{CoCo} . Since contingent CDS cash flows are not affected by default, one can price them without knowing probabilities $\mathbb{Q}^{i,K}(t, \cdot)$. In fact, one can ignore the default state K entirely. Due to this, existing reduced form models could be used to price contingent CDS instruments by replacing intensity process with $-\lambda_t^{1,1}$.

To bring more originality into these securities, one can also define a version with equity conversion. In this case the protection leg could be defined as

$$D(t, \tau^{\text{CoCo}}) \left(N - \frac{N}{\phi(S_{\tau^{\text{CoCo}}})} S_{\tau^{\text{CoCo}}} \right) \mathbf{1}_{\{\tau^{\text{CoCo}} \leq T_m\}}$$

with present value of

$$NE^{\mathbb{Q}} \left[\int_t^{T_m} D(t, u) \left(1 - \frac{S_u}{\phi(S_u)} \right) d\mathbb{Q}^{i,2}(t, u) \middle| \mathcal{F}_t \right].$$

5 The Case of Piecewise Constant Intensities

In this section we will demonstrate our model using similar assumptions, than in the so called ISDA standard model, which is widely used in practice for CDS and credit derivatives valuations. We refer to [24] for a detailed review and discussion about the ISDA standard model.

We start by defining a sequence of times $\mathcal{T} = \{t_1, \dots, t_n\}$, $n \in \mathbb{N}$, $t_l \leq t_{l+1}$. Let $\mathcal{L} = \{\Lambda_1, \dots, \Lambda_n\}$ be a sequence of matrices,

$$\Lambda_l = \begin{bmatrix} \lambda_l^{1,1} & \lambda_l^{1,2} & 0 \\ \lambda_l^{2,1} & \lambda_l^{2,2} & \lambda_l^{2,3} \\ 0 & 0 & 0 \end{bmatrix}.$$

For every $i, j \in \mathcal{K}$, $i \neq j$, the elements $\lambda_l^{i,j}$ are non-negative and $\lambda_l^{i,i} = -\sum_{j \in \mathcal{K} \setminus \{i\}} \lambda_l^{i,j}$.

We set

$$\Lambda_t = \Lambda_{t_l} = \Lambda_l,$$

for all $t_{l-1} < t \leq t_l$. That is, Λ_t is piecewise constant in time.

In general, there is no closed form solution to forward Kolmogorov equation 3.15 and the transition probabilities might not have any convenient analytical form. However, assuming

piecewise constant intensities, the solution takes the form of matrix exponential

$$\mathbb{Q}(t, s) = e^{\Lambda_l(s-t)} := \sum_{k=1}^{\infty} \frac{(\Lambda_l(s-t))^k}{k!},$$

for all $t_{l-1} < t \leq s \leq t_l$ and $t_{l-1}, t_l \in \mathcal{T}$.

Furthermore, due to the properties given to Λ_l , it can be shown that any $\Lambda_l \in \mathcal{L}$ is diagonalizable and can be written as

$$\Lambda_l = \Xi_l H_l \Xi_l^{-1},$$

where H_l is a diagonal matrix with elements $h_l^{i,j}$ ($h_l^{i,j} = 0$, for all $i \neq j$, $i, j \in \mathcal{K}$) and Ξ_l is a matrix whose columns are the corresponding eigenvectors of Λ_l . We denote $\xi_l^{i,k}$ as the elements of $\Xi_l = [\xi_l^{i,j}]_{i,j \in \mathcal{K}}$ and $\bar{\xi}_l^{i,j}$ as the elements of $\Xi_l^{-1} = [\bar{\xi}_l^{i,j}]_{i,j \in \mathcal{K}}$. We also define $h_l^i := -h_l^{i,i}$.

A property of matrix exponential for diagonalizable matrices gives us

$$e^{\Lambda_l(s-t)} = e^{\Xi_l H_l (s-t) \Xi_l^{-1}} = \Xi_l e^{H_l (s-t)} \Xi_l^{-1},$$

so that we can now express transition probabilities as

$$\mathbb{Q}^{i,j}(t, s) = \sum_{k=1}^K \xi_l^{i,k} \bar{\xi}_l^{k,j} e^{-h_l^k (s-t)}$$

for all $t_{l-1} < t \leq s \leq t_l$. There is an analogy between h_l^m and the so called (forward) hazard rate in reduced form models. The most important feature of ISDA standard model is, that these forward hazard rates are assumed to be piecewise constant.

In general form, the integrals in pricing formulas in section 4 have to be solved by numerical integration. Next we will illustrate how they can be solved when intensities are piecewise constant. We will use CDS instruments as an example, but similar results can be derived for other instruments as well. To simplify our notations, we will assume, that assets are priced at time $t_0 := t$, and that we start at state $i \neq K$, $C_t = i$. Another assumption we make (in accordance with the ISDA standard model), is that forward rates are piecewise constant. This means, that discount factors have a representation

$$D(s, u) = e^{-f_l(u-s)},$$

for all $t_{l-1} < s \leq u \leq t_l$, where f_l are constants.

Let the maturity of a CDS contract be $T_m = t_n$. The pricing equation 4.4 for CDS protection leg becomes

$$\begin{aligned} (1 - q^D) N \int_t^{T_m} D(t, u) d\mathbb{Q}^{i,K}(t, u) &= (1 - q^D) N \int_t^{T_m} D(t, u) \lambda_u^{2,3} \mathbb{Q}^{i,2}(t, u) du \\ &= (1 - q^D) N \sum_{l=1}^n \int_{t_{l-1}}^{t_l} D(t, u) \lambda_l^{2,3} \mathbb{Q}^{i,2}(t, u) du, \end{aligned}$$

where the first equality follows from the forward Kolmogorov equation 3.15. We can also apply the Chapman-Kolmogorov equation 3.14 to obtain

$$\begin{aligned}
& (1 - q^D) N \sum_{l=1}^n \sum_{j=1}^K \lambda_l^{2,3} D(t, t_{l-1}) \mathbb{Q}^{i,j}(t, t_{l-1}) \int_{t_{l-1}}^{t_l} D(t_{l-1}, u) \mathbb{Q}^{j,2}(t_{l-1}, u) \, du \\
&= (1 - q^D) N \sum_{l=1}^n \sum_{j=1}^K \lambda_l^{2,3} D(t, t_{l-1}) \mathbb{Q}^{i,j}(t, t_{l-1}) \int_{t_{l-1}}^{t_l} e^{-f_l(u-t_{l-1})} \sum_{m=1}^K \xi_l^{j,m} \bar{\xi}_l^{m,2} e^{-h_l^m(u-t_{l-1})} \, du \\
&= (1 - q^D) N \sum_{l=1}^n \sum_{j=1}^K \lambda_l^{2,3} D(t, t_{l-1}) \mathbb{Q}^{i,j}(t, t_{l-1}) \sum_{m=1}^K \xi_l^{j,m} \bar{\xi}_l^{m,2} \int_{t_{l-1}}^{t_l} e^{-(f_l+h_l^m)(u-t_{l-1})} \, du \\
&= (1 - q^D) N \sum_{l=1}^n \sum_{j=1}^K \lambda_l^{2,3} D(t, t_{l-1}) \mathbb{Q}^{i,j}(t, t_{l-1}) \sum_{m=1}^K \xi_l^{j,m} \bar{\xi}_l^{m,2} \frac{1}{f_l + h_l^m} \left(1 - e^{-(f_l+h_l^m)(t_l-t_{l-1})}\right).
\end{aligned}$$

For the premium leg in formula 4.3 we only focus on the integrals in the recovery term. To help with notations, we define a new set $\tilde{\mathcal{T}} := \{t_l \in \mathcal{T}\} \cup \{T_{k-1}, T_k\}$. Again, using the forward Kolmogorov equation 3.15 and the Chapman-Kolmogorov equation 3.14, these integrals can be written as

$$\begin{aligned}
& \int_{T_{k-1}}^{T_k} D(t, u) (u - T_{k-1}) \, d\mathbb{Q}^{i,K}(t, u) = \int_{T_{k-1}}^{T_k} D(t, u) (u - T_{k-1}) \lambda_u^{2,3} \mathbb{Q}^{i,2}(t, u) \, du \\
&= \sum_{\substack{T_{k-1} < t_l \leq T_k \\ t_l \in \tilde{\mathcal{T}}}} \int_{t_{l-1}}^{t_l} D(t, u) (u - T_{k-1}) \lambda_l^{2,3} \mathbb{Q}^{i,2}(t, u) \, du \\
&= \sum_{\substack{T_{k-1} < t_l \leq T_k \\ t_l \in \tilde{\mathcal{T}}}} \sum_{j=1}^K \lambda_l^{2,3} D(t, t_{l-1}) \mathbb{Q}^{i,j}(t, t_{l-1}) \int_{t_{l-1}}^{t_l} D(t_{l-1}, u) (u - T_{k-1}) \mathbb{Q}^{j,2}(t_{l-1}, u) \, du \\
&= \sum_{\substack{T_{k-1} < t_l \leq T_k \\ t_l \in \tilde{\mathcal{T}}}} \sum_{j=1}^K \lambda_l^{2,3} D(t, t_{l-1}) \mathbb{Q}^{i,j}(t, t_{l-1}) \int_{t_{l-1}}^{t_l} e^{-f_l(u-t_{l-1})} (u - T_{k-1}) \sum_{m=1}^K \xi_l^{j,m} \bar{\xi}_l^{m,2} e^{-h_l^m(u-t_{l-1})} \, du \\
&= \sum_{\substack{T_{k-1} < t_l \leq T_k \\ t_l \in \tilde{\mathcal{T}}}} \sum_{j=1}^K \lambda_l^{2,3} D(t, t_{l-1}) \mathbb{Q}^{i,j}(t, t_{l-1}) \sum_{m=1}^K \xi_l^{j,m} \bar{\xi}_l^{m,2} \int_{t_{l-1}}^{t_l} (u - T_{k-1}) e^{-(f_l+h_l^m)(u-t_{l-1})} \, du.
\end{aligned}$$

Using integration by parts, the integral term becomes

$$\begin{aligned} & \int_{t_{l-1}}^{t_l} (u - T_{k-1}) e^{-(f_l + h_l^m)(u - t_{l-1})} du \\ &= \frac{1 + T_{k-1}}{f_l + h_l^m} \left(e^{-(f_l + h_l^m)(t_l - t_{l-1})} - 1 \right) - \frac{1}{f_l + h_l^m} \left(t_l e^{-(f_l + h_l^m)(t_l - t_{l-1})} - t_{l-1} \right). \end{aligned}$$

Typically nodes t_l are maturities of instruments used in calibration and they lie further apart, than single coupon payments, which will simplify pricing formula even further.

While model calibration is beyond the scope of this paper, a few words are in place. While the reduced form models can be calibrated using CDS data only, this more general conditional Markov model requires contingent convertible prices as well. If a market for contingent credit default swaps (as defined in Definition 4.9) existed, their market quotes could be used for calibration purposes. Future research should be done on model calibration and how to determine parameters Λ_l and whether any issues might arise from having more parameters, than in reduced form models. A good starting point would be contingent convertibles with permanent (and preferably full) write-down to reduce complexity of Λ_l .

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