Non-Gaussian Cosmological Perturbations from Hybrid Inflation and Preheating

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ACADEMIC DISSERTATION

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All science is cosmology – Karl Popper
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Abstract

This thesis consists of four research papers and an introduction providing some background.

The structure in the universe is generally considered to originate from quantum fluctuations in the very early universe. The standard lore of cosmology states that the primordial perturbations are almost scale-invariant, adiabatic, and Gaussian. A snapshot of the structure from the time when the universe became transparent can be seen in the cosmic microwave background (CMB). For a long time mainly the power spectrum of the CMB temperature fluctuations has been used to obtain observational constraints, especially on deviations from scale-invariance and pure adiabacity. Non-Gaussian perturbations provide a novel and very promising way to test theoretical predictions. They probe beyond the power spectrum, or two point correlator, since non-Gaussianity involves higher order statistics.

The thesis concentrates on the non-Gaussian perturbations arising in several situations involving two scalar fields, namely, hybrid inflation and various forms of preheating. First we go through some basic concepts – such as the cosmological inflation, reheating and preheating, and the role of scalar fields during inflation – which are necessary for the understanding of the research papers. We also review the standard linear cosmological perturbation theory.

The second order perturbation theory formalism for two scalar fields is developed. We explain what is meant by non-Gaussian perturbations, and discuss some difficulties in parametrisation and observation. In particular, we concentrate on the nonlinearity parameter. The prospects of observing non-Gaussianity are briefly discussed. We apply the formalism and calculate the evolution of the second order curvature perturbation during hybrid inflation. We estimate the amount of non-Gaussianity in the model and find that there is a possibility for an observational effect. The non-Gaussianity arising in preheating is also studied. We find that the level produced by the simplest model of instant preheating is insignificant, whereas standard preheating with parametric resonance as well as tachyonic preheating are prone to easily saturate and even exceed the observational limits.

We also mention other approaches to the study of primordial non-Gaussianities, which differ from the perturbation theory method chosen in the thesis work.
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I am very grateful to my supervisor prof. Kari Enqvist for his invaluable guidance on my journey into science. He has given advise when needed but he has also encouraged independent thinking. His ability to quickly see the guiding physical principles, as well as his reassuring optimism, while I have felt bound by the fog of details have been vital on several occasions. I would also like to thank lecturer Hannu Kurki-Suonio for giving several courses on standard cosmology, including my first one, and doc. Esko Keski-Vakkuri for interesting and useful special courses and advise on more exotic physics.

I wish to thank Kari, Asko Jokinen, Anupam Mazumdar, and Tuomas Multamäki for fruitful and inspirational collaboration. In particular I acknowledge the advise and all-round support Anupam has provided, as well as his enthusiasm. The numerous discussions I have had with Asko, for example on non-localities, are greatly appreciated.

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Finally, I wish to express my deepest gratitude to my parents and my brother, and of course to Helena.

Antti Väihkönen
Helsinki, February 2006
List of Publications

[1] Kari Enqvist, Antti Väihkönen
   NON-GAUSSIAN PERTURBATIONS IN HYBRID INFLATION.

   NON-GAUSSIANITY FROM PREHEATING.

   NON-GAUSSIANITY FROM INSTANT AND TACHYONIC PREHEATING.

   COMMENT ON NON-GAUSSIANITY IN HYBRID INFLATION.
   astro-ph/0506304

The contribution of the present author to joint publications

My contribution to publication [1]: The idea to study the non-Gaussianity of two-field models of inflation was mainly due to Kari Enqvist, although it was based on our joint deliberations. I did the calculations and wrote most of the paper, except for the abstract and parts of introduction and discussion. Kari Enqvist provided for supervision and guidance.

My contribution to publication [2]: The idea for the study came originally from Anupam Mazumdar. The rough contents of the paper were formed and the idea was matured during joint discussions between Asko Jokinen, Anupam Mazumdar, Tuomas Multamäki and myself while I was visiting Nordita in Copenhagen. I provided
for my knowledge on second order cosmological perturbations and non-Gaussianity. Most of the calculations were done by Asko Jokinen and myself. The draft version of the paper was written by Anupam Mazumdar, and the polishing of the article was done jointly.

My contribution to publication [3]: The study originated from the same discussions and considerations as the previous preheating paper. The contributions of the authors were similar except that this time most of the calculations were done by Asko Jokinen.

In all papers authors are listed alphabetically according to the particle physics convention.
Chapter 1

Introduction

The standard lore of cosmology states that the seeds of the structure in the universe are quantum fluctuations of a scalar field, or fields, during inflationary phase in the early universe. Furthermore, these fluctuations are generically considered to be almost scale-invariant, adiabatic, and Gaussian [6]. The possible deviation from perfect scale-invariance is parameterised by the spectral index of the primordial fluctuations and is routinely used to test inflationary models against observational data, e.g., from WMAP satellite [7–9], (the web page of the satellite [10]). The physics of the early universe is also tested with the observational limits on the isocurvature fraction of the fluctuations. The data is consistent with purely adiabatic fluctuations [7], but see also [11].

In the wake of observational constraints (see e.g. [12, 13]) deviations from Gaussianity have become a novel way to distinguish between different theoretical models and scenarios. Observations are consistent with purely Gaussian fluctuations, but it is nevertheless reasonable to consider deviations from Gaussianity. Firstly, the definition of non-Gaussian perturbation is very general: everything that is not Gaussian, is non-Gaussian. This implies that the agreement with the data may strongly depend on what kind of parametrisation and tests have been applied. Furthermore, theoretically different kind of inflationary models and scenarios predict additional non-Gaussian component to the otherwise Gaussian perturbations.

The thesis concentrates on the non-Gaussian perturbations produced during several multi-field cases, namely, hybrid inflation and preheating, including instant and tachyonic preheating. Already, when the first enclosed paper [1] appeared it had become clear that single-field inflation does not produce significant non-Gaussianity, [14, 15]. The formalism for two-field situations was developed in [1] and immediately applied to hybrid inflation. Non-Gaussianity arising in preheating is studied in papers [2] and [3]. It is found that in these multi-field scenarios there is a possibility for generating significant amount of non-Gaussianity. These findings are in concordance with other studies, see Sections 3.4 and 3.6.

The thesis consists of an introductory part and four papers, [1–4]. The introduction is organised as follows. In the first chapter we set up notation and present some
basic concepts necessary in understanding the work done in the enclosed papers. The hot big bang is presented along with its historical problems. We define inflation and show how it is realized with scalar fields and discuss briefly inflationary models. Especially, we are interested in the hybrid model. The basic idea of preheating, including instant and tachyonic preheating, is presented along with some necessary formalism. We also show two example calculations of scalar field perturbations during inflation.

The second chapter is devoted to the cosmological perturbation theory. We review the well established linear perturbation theory. The necessary gauge transformations are given, and some useful quantities defined. We also give an example calculation of the evolution of the perturbations in single-field inflation. The second order theory is presented as it is developed in [15] and [1]. The second order treatment is naturally more involved but, in addition, we present a two-field case. The necessary second order gauge transformations are given. The aim of the chapter is to derive an evolution equation for the second order metric perturbation, which is necessary for the evaluation of the second order curvature perturbation. In the end we discuss the connection between different definitions of the curvature perturbation found in the literature.

The third chapter is about non-Gaussianity. We start by discussing the statistics in general and defining what is meant by non-Gaussianity. We also say a few words about observational aspects, especially regarding the nonlinearity parameter $f_{NL}$. We apply the formalism of the second chapter and show how non-Gaussianity arises in hybrid inflation and in preheating, including instant and tachyonic preheating, as studied in papers [1–4]. Also, some other methods in calculating non-Gaussianities are mentioned. Finally, chapter four presents the summary.

1.1 Notation

The natural units $c \equiv \hbar \equiv k_B \equiv 1$ are used throughout the thesis. We define Planck mass $M_P \equiv (8\pi G_N)^{-1/2}$ without unnecessarily calling it 'reduced'; $G_N$ is the Newton’s constant. Conformal time $\tau$ is defined by $d\tau = dt/a(t)$. Overdot denotes derivation with respect to the coordinate time $t$, i.e. $\dot{\equiv} \partial/\partial t$, whereas derivation with respect to conformal time $\tau$ is usually denoted by prime, i.e. $' \equiv \partial/\partial \tau$. The Hubble parameter in coordinate time is $H \equiv \frac{\dot{a}}{a}$ and in conformal time $\mathcal{H} \equiv \frac{a'}{a}$. Summation over repeated indices is understood; Greek letters (\(\alpha, \beta, \gamma, \ldots\)) run over the values 0, 1, 2, 3 whereas Latin indices stand for spatial coordinates and run over the values 1, 2, 3. The signature of the metric is (−, +, +, +).

Other notations, as well as deviations from the ones presented here, are mentioned in the text.
1.2 Hot Big Bang

The aim of the thesis is to put the studies [1–4] into context, and not to give a complete description of standard cosmology. Therefore, we only briefly go through the key aspects of the standard hot big bang theory as well as the inflationary scenario. For a more thorough treatment the reader is advised to see, e.g., the textbooks [6, 16–18].

One might easily think that a reliable description of our entire universe is almost impossible because we can not even deal with many physical phenomena appearing on Earth without great difficulties. The situation, however, is quite the opposite. The universe is astonishingly simple on the large scale average. This is somewhat similar to thermodynamics: the dynamics of a vast number, say $10^{23}$, of point particles is practically impossible to calculate over reasonably long periods of time, but their averaged behaviour, e.g. that of temperature and pressure, can still be rather simple.

This simplicity relies on the fact that, to a high degree of precision, we observe the universe to be highly isotropic and homogeneous on large scales. Actually the isotropy is much easier to observe, especially in the cosmic microwave background (CMB) than homogeneity, but the cosmological principle, which states that the universe looks the same regardless of the location of the observer, allows us to deduce large scale homogeneity from the observed isotropy. The isotropy and homogeneity enables us to apply the Robertson–Walker metric [16]

$$ds^2 = -dt^2 + a^2(t)\left[\frac{dr^2}{1-kr^2} + r^2d\theta^2 + r^2\sin^2\theta d\phi^2\right]$$ (1.1)

in the study of the universe. Here $t, r, \theta, \text{ and } \phi$ are the (polar) coordinates, $a(t)$ is the scale factor of the universe, and $k$ is the parameter used to distinguish an open, a flat, and a closed universe, with $k$ getting values $-1, 0,$ and $1$ respectively. In this thesis, however, we will mainly be working with a flat universe, $ds^2 = -dt^2 + a^2(t)dx^2 = a^2(\tau)(-d\tau^2 + dx^2)$, with perturbations added.

Eq. (1.1) can be written $ds^2 = g_{\mu\nu}dx^\mu dx^\nu$ and it gives the geometry of the universe through the metric $g_{\mu\nu}$. At the core of the study of the universe lies the theory of general relativity, which couples the evolution and dynamics of the universe to its energy density contents and distribution via the Einstein equation [16]

$$G_{\mu\nu} = \frac{1}{M_P^2}T_{\mu\nu} + g_{\mu\nu}\Lambda ,$$ (1.2)

where $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R$ is the Einstein tensor, $T_{\mu\nu}$ is the energy-momentum tensor, and $\Lambda$ is the cosmological constant\(^1\).

Taking into account the symmetries of the metric the energy-momentum tensor is diagonal with equal spatial components. Using the perfect fluid description as an

\(^1\)The cosmological constant $\Lambda$ is often not written explicitly but included implicitly in $T_{\mu\nu}$. 
example, $T_{\mu \nu}$ can be completely characterised by the time-dependent (total) energy density, $\rho(t)$, and pressure, $p(t)$, as follows

$$T_{\mu \nu} = \text{diag}(-\rho, p, p, p).$$

(1.3)

We are now in the position of easily obtaining dynamical equations describing the universe and its energy density as a whole. Firstly, from the $\mu = 0$ component of the conservation of the energy-momentum tensor, $\nabla_\nu T^{\mu \nu} = 0$, where $\nabla_\nu$ is the covariant derivative (see e.g. [19]), we get the continuity equation

$$d(\rho a^3) = -pd(a^3) \iff \dot{\rho} = -3H(\rho + p),$$

(1.4)

where $H \equiv \frac{\dot{a}}{a}$ is the Hubble parameter. Secondly, the $0 - 0$ component of the Einstein equation (1.2) with the metric (1.1) and energy-momentum tensor gives us the so-called Friedmann equation

$$H^2 = \frac{\rho}{3M_p^2} - \frac{k}{a^2},$$

(1.5)

where $k$ is the same as in Eq. (1.1). The cosmological constant, $\Lambda$, has been absorbed into the total energy density, $\rho$. The $i - i$ component of the Einstein equation (1.2) gives

$$2\frac{\ddot{a}}{a} + H^2 = -\frac{p}{M_p^2} - \frac{k}{a^2}.$$

(1.6)

The two equations, (1.5) and (1.6), can be subtracted to give the acceleration equation

$$\frac{\ddot{a}}{a} = -\frac{\rho + 3p}{6M_p^2},$$

(1.7)

which is often more useful than Eq. (1.6).

We also need the equation of state, $p = w\rho$. Generally $w$ depends on time but for now we assume it to be constant. The continuity equation immediately gives $\rho \propto a^{-3(1+w)}$, and the Friedmann equation (1.5), in the flat case $k = 0$, yields $a \propto t^{2/3(1+w)}$. For radiation, matter, and vacuum energy $w = \frac{1}{3}$, 0, and $-1$, respectively. The time evolutions of the energy densities of radiation, matter and vacuum energy are therefore $\rho_r \propto a^{-4}$, $\rho_m \propto a^{-3}$, and $\rho_\Lambda = \text{const}$, respectively.

The curvature of the universe can conveniently be described with the aid of the density parameter $\Omega \equiv \rho/\rho_c$, where $\rho_c \equiv 3M_p^2H^2$ is the critical density for which the universe is spatially flat, i.e. $k = 0$. This can easily be seen from the Friedmann equation (1.5), which can be written

$$\Omega_{\text{tot}}(t) - 1 = \frac{k}{a^2(t)H^2(t)}.$$

(1.8)

Here $\Omega_{\text{tot}}$ is the density contrast for the total energy density of the universe. It is, however, customary to isolate the contribution from the cosmological constant and to write

$$\Omega_{\text{tot}} = \Omega + \Omega_\Lambda.$$

(1.9)
This division is useful due to observational reasons. The CMB observations naturally constrain $\Omega_{\text{tot}}$ whereas some other observations, such as the supernova observations, constrain $\Omega_{\Lambda}$ [8].

### 1.2.1 Problems of hot big bang

The hot big bang theory is extremely successful, but there are also problems. We briefly present the three problems that were the original motivation for the inflationary scenario in the seminal paper by Guth [20].

First the horizon problem. Since photons travel along null paths, $ds^2 = -dt^2 + a^2dx^2 = 0$, the comoving distance travelled by a freely moving photon, emitted at time $t_1$ and observed at time $t_2$, is

$$x(t_1, t_2) \equiv \int_{t_1}^{t_2} \frac{dt}{a(t)} .$$  \hfill (1.10)

The universe is initially radiation dominated ($a \propto t^{1/2}$) and becomes matter dominated ($a \propto t^{2/3}$) some time before the photon decoupling. Therefore, up to a numerical factor we can estimate the size of the comoving causal patch at the last scattering surface (LSS) as

$$D_{\text{hor}} \equiv \int_{0}^{t_{\text{LSS}}} \frac{dt}{a(t)} \sim \frac{1}{a_{\text{LSS}}H_{\text{LSS}}} .$$  \hfill (1.11)

On the other hand, a photon emitted at the last scattering and observed today ($t_0$) has travelled a comoving distance

$$D_{\text{obs}} \equiv \int_{t_{\text{LSS}}}^{t_0} \frac{dt}{a(t)} = \frac{2}{a_0H_0} \left( 1 - \sqrt{\frac{a_{\text{LSS}}}{a_0}} \right) \sim \frac{2}{a_0H_0} ,$$  \hfill (1.12)

where we have made use of matter dominance ($a \propto t^{2/3}$ and $H_0 = 2/3t_0$) and the fact that the redshift of CMB is known to be ([6]) $1 + z = a_0/a_{\text{LSS}} \simeq 1100$.

Since during matter domination

$$H^2 \propto \rho_m \propto a^{-3} ,$$  \hfill (1.13)

we immediately obtain

$$\frac{D_{\text{obs}}}{D_{h}} \sim \frac{2a_{\text{LSS}}H_{\text{LSS}}}{a_0H_0} = 2\sqrt{\frac{a_0}{a_{\text{LSS}}}} = 2\sqrt{1 + z} \gg 1 .$$  \hfill (1.14)

Thus, according to the pure big bang, the universe we observe at the CMB consists of a large amount of causally disconnected regions. The startling homogeneity of the

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2In addition to these problems the inhomogeneities observed in the universe are left unexplained by the big bang theory alone and, explaining them is nowadays considered the most important aspect of inflation (see Sec. 1.3.6).
background can not be explained by some causal process but has to be put in as an initial condition.

The second classic shortcoming of big bang is the flatness problem. By definition $aH = \dot{a}$, and we see that during usual expansion the density contrast Eq. (1.8) evolves away from unity\(^3\), unless it is exactly one, in which case it stays that way forever. For example, during radiation domination (for slightly open or closed universe)

$$a \propto t^{1/2} \quad \text{and} \quad |\Omega_{\text{tot}} - 1| \propto t,$$

and during matter domination (again for slightly open or closed universe)

$$a \propto t^{2/3} \quad \text{and} \quad |\Omega_{\text{tot}} - 1| \propto t^{2/3}.$$  \hspace{1cm} (1.15)

(1.16)

The observed value of the density contrast differs from unity of the order few percent [8, 9]. Thus, one has to fine tune the initial conditions to a very high degree. For example, at the time of nucleosynthesis it is required that [6]

$$|\Omega_{\text{tot}} - 1| \lesssim 10^{-16}.$$  \hspace{1cm} (1.17)

Unless there is some process driving $\Omega_{\text{tot}}$ towards one, or some principle or symmetry reason for $\Omega_{\text{tot}} = 1$, one is tempted to consider the initial condition unlikely.

Lastly, there is the relic density problem. In the distant past the universe was very hot and many symmetries are assumed to have been unbroken. The breaking of these symmetries would have caused the formation of topological relics, such as magnetic monopoles, domain walls, textures, and cosmic strings [6, 17, 21]. These relics would affect the formation of structure in the universe or, at least, cause observable effects in the CMB, unless there is some way to get rid of them or dilute them.

Historically the main concern were the magnetic monopoles [20]. If a simple gauge group is broken down to a group that contains a $U(1)$ part, such as $SU(3) \times SU(2) \times U(1)$, monopoles will be produced [6, 17, 21]. These monopoles present a problem because of their huge mass and abundance [6, 20]. After being produced their energy density evolves like their number density $\propto a^{-3}$. In the early universe during radiation domination the total energy density is $\propto a^{-4}$. Thus, the magnetic monopoles may become the dominant constituents of the universe unless their production is somehow suppressed or their number density is reduced afterwards.

In modern view the dangerous primordial relics include supersymmetric particles such as the gravitino, and moduli fields [6]. These relics, however, are not topological and not caused by symmetry breaking.

### 1.3 Inflation

In this section we define inflation and concisely describe how it remedies the problems mentioned in the previous section. We also discuss the realization of inflation

\(^3\)This is true also in more general circumstances, such as during a thermal phase transition [6].
with scalar fields and consider different models of inflation. The important topic of producing cosmological inhomogeneities during inflation is covered in the end.

### 1.3 Inflation

Inflation is very easy to define: it is an epoch during which the scale factor of the universe is accelerating

\[
\text{Inflation} \iff \ddot{a} > 0 ,
\]

where the overdot means derivative with respect to cosmic time \( t \), i.e. the coordinate time in Eq. (1.1). This definition is rather general and the universe seems to be in a state of accelerated expansion also at the moment, since \( \Omega_\Lambda \sim 0.7 \) [8]. In this thesis, as usually is the case, the term ‘inflation’ refers to the era of accelerated expansion in the very early universe.

There are also other useful equivalent forms of the condition for inflation. Differentiating \( \frac{d}{dt}(aH)^{-1} \equiv \frac{d}{dt}a^{-1} = -a^{-2}\dot{a} \) we obtain

\[
\text{Inflation} \iff \frac{d}{dt} \left( \frac{1}{aH} \right) < 0 .
\]  

(1.19)

Physically this means that during inflation the comoving Hubble length, \( 1/aH \), decreases with time. From Eq. (1.7) we obtain the third equivalent condition for inflation

\[
\text{Inflation} \iff p < -\frac{1}{3} \rho ,
\]

(1.20)

i.e. the equation of state for inflationary universe must obey \( w < -1/3 \). This is a violation of the strong energy condition [19].

We can now briefly describe how a period of inflation in the early universe may alleviate the problems of big bang. The decreasing of the comoving Hubble scale, Eq. (1.19), means that according to Eq. (1.8) the density contrast \( \Omega_{\text{tot}} \) is driven towards unity during inflation. In fact, inflation usually makes the universe practically flat and, hence, the curvature term involving \( k \) does not appear in the Friedmann equation (1.5) when inflation is considered. The horizon problem is solved requiring that

\[
\int_{t_*}^{t_{\text{ls}}} dt a^{-1} \gg \int_{t_{\text{ls}}}^{t_0} dt a^{-1} ,
\]

(1.21)

where \( t_* \) is the earliest moment in the universe when our theories make sense; usually one just writes \( t_* = 0 \). With the fulfilment of the requirement in Eq. (1.21) causal processes are able to give rise to the observed homogeneity of CMB. For the relic density problem we will consider only massive particles, (for a more detailed discussion see e.g. [17]). Since \( \frac{\rho_{\text{em}}}{\rho_{\text{inf}}} \propto a^{3w} \), and during inflation \( w < -1/3 \), we see
that the energy density of massive particles gets redshifted away at least as fast as a with respect to the inflationary energy density.\textsuperscript{4}

The quantity describing the amount of inflation is the number of e-foldings, $N$, defined by [6]

$$N(t) \equiv \ln \frac{a(t_{\text{end}})}{a(t)},$$

(1.22)

where $a(t_{\text{end}})$ and $a(t)$ are, respectively, the scale factors at the end of inflation and at some time $t$ during inflation. In order to get rid of the problems at least $N \sim 50 \ldots 70$ e-foldings are required. In many single field inflationary models the expansion parameter can acquire even an enormous value such as $N \sim 10^{12}$ [17].

CMB observations only span seven to ten e-foldings, which take place some 60 e-foldings before the end of inflation.\textsuperscript{5}

1.3.2 Inflation and scalar fields

In general relativity the action can be written

$$S = \int d^4x \sqrt{-g} \mathcal{L},$$

(1.23)

where $g \equiv \det(g_{\mu\nu})$ is determinant of the metric. The Lagrangian consists of two parts $\mathcal{L} = \mathcal{L}_{EH} + \mathcal{L}_{\text{mat}}$: the Einstein–Hilbert part, $\mathcal{L}_{EH} = \frac{1}{2}M_p^2 R$, where $R$ is the Ricci scalar, and the matter part $\mathcal{L}_{\text{mat}}$. These Lagrangians yield the Einstein equation [6, 19, 23]

$$G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \frac{1}{M_p^2} T_{\mu\nu},$$

(1.24)

provided we interpret the energy-momentum tensor as

$$T_{\mu\nu} = \frac{1}{2} g_{\mu\nu} \mathcal{L}_{\text{mat}} \partial_{\mu} \phi \partial_{\nu} \phi - V(\phi).$$

(1.25)

Inflation, with the conditions described in Eqs. (1.18), (1.19), and (1.20), can be realized with a scalar field, or scalar fields.\textsuperscript{6} The Lagrangian density for a single (canonically normalised) scalar field is [6]

$$\mathcal{L}_{\text{mat}} = -\frac{1}{2} g^{\mu\nu} \partial_{\mu} \phi \partial_{\nu} \phi - V(\phi).$$

(1.26)

\textsuperscript{4}A further restriction imposed by the cosmological relics is on the temperature of the universe after inflation. This so called reheating temperature has to be low enough so that after inflation relics are not reproduced too abundantly.

\textsuperscript{5}We do not have any way to directly observe the total number of e-foldings, unless there would be observable effects from the transplanckian region, see e.g. [22] and references therein.

\textsuperscript{6}Instead of a fundamental scalar we could have a fermion condensate or a vector meson condensate, or the role of of the order parameter (i.e. scalar field) could be played by the scalar curvature $R$ or even the radius of the compactified space [17].
Eq. (1.25) thus gives us
\[ T_{\mu \nu} = \partial_\mu \varphi \partial_\nu \varphi - g_{\mu \nu} \left( \frac{1}{2} g^{\alpha \beta} \partial_\alpha \varphi \partial_\beta \varphi + V(\varphi) \right) . \] (1.27)

In the local rest frame, i.e. when \( T^{00} = 0 \), we get the components of the energy-momentum tensor of an isotropic system: \( T_0^0 = -\rho \) and \( T_i^j = \delta^i_j \rho \). The isotropy implies \( (\partial_i \varphi)^2 = \frac{1}{3} (\nabla \varphi)^2 \), for \( i = 1, 2, 3 \). In the comoving coordinates with metric \( g_{\mu \nu} = \text{diag}(-1, a^2, a^2, a^2) \) we obtain [18]

\[
\rho = \frac{1}{2} \dot{\varphi}^2 + \frac{1}{2} a^{-2} (\nabla \varphi)^2 + V(\varphi) , \tag{1.28}
\]

\[
p = \frac{1}{2} \dot{\varphi}^2 - \frac{1}{6} a^{-2} (\nabla \varphi)^2 - V(\varphi) . \tag{1.29}
\]

From the definition of the inflation, Eq. (1.18), the acceleration equation, Eq. (1.7), and the expressions for the energy density and pressure, Eqs. (1.28) and (1.29), we see that a scalar field can realize inflation provided that \( \dot{\varphi}^2 < V(\varphi) \).

From Eq. (1.5) we obtain
\[
\left( \frac{\dot{a}}{a} \right)^2 = \frac{1}{3 M_P^2} \left( \frac{1}{2} \dot{\varphi}^2 + \frac{1}{2} a^{-2} (\nabla \varphi)^2 + V(\varphi) \right) , \tag{1.30}
\]

where we have dropped the curvature term. If we assume the scalar field to be spatially almost constant\(^7\), at least at some scale, we may approximate \( \nabla \varphi \sim 0 \) and obtain

\[
\rho = \frac{1}{2} \dot{\varphi}^2 + V(\varphi) , \tag{1.31}
\]

\[
p = \frac{1}{2} \dot{\varphi}^2 - V(\varphi) , \tag{1.32}
\]

and
\[
H^2 = \frac{1}{3 M_P^2} \left( \frac{1}{2} \dot{\varphi}^2 + V(\varphi) \right) . \tag{1.33}
\]

The Friedmann equation (1.33) is not enough. We also need to know the time evolution of the scalar field. That can be obtained from the energy-momentum conservation equation \( \nabla_\mu T^{\mu \nu} = 0 \), and the result is the Klein-Gordon equation \( \Box \varphi + V'(\varphi) = 0 \), where the d’Alembertian is [6]
\[
\Box \varphi = g^{\mu \nu} \nabla_\mu \partial_\nu = -\frac{1}{\sqrt{-g}} \partial_\mu (\sqrt{-g} g^{\mu \nu} \partial_\nu \varphi) . \tag{1.34}
\]

\(^7\)Perturbations of the scalar field are discussed later, Sec. 1.3.6 and Chapter 2.
so that the field equation is
\[ \ddot{\varphi} + 3H \dot{\varphi} - \frac{1}{a^2} \nabla^2 \varphi + V'(\varphi) = 0. \] (1.35)

For a homogeneous field one obtains
\[ \ddot{\varphi} + 3H \dot{\varphi} + V'(\varphi) = 0. \] (1.36)

### 1.3.3 Slow-roll

Eqs. (1.33) and (1.36) are very important when considering a universe driven by a scalar field. In the study of the inflationary universe it is customary to resort to the slow-roll approximation, (see e.g. [6]). Applying the slow-roll conditions
\[ \frac{1}{2} \dot{\varphi}^2 \ll V(\varphi) \quad \text{and} \quad \ddot{\varphi} \ll 3H \dot{\varphi}, \] (1.37)
we can approximate Eqs. (1.33) and (1.36) and obtain the slow-roll equations:
\[ H^2 \simeq \frac{V(\varphi)}{3M_P^2}, \] (1.38)
\[ 3H \dot{\varphi} \simeq -V'(\varphi). \] (1.39)

The slow-roll parameters, \( \epsilon \) and \( \eta \), are defined by [6]
\[ \epsilon(\varphi) \equiv \frac{M_P^2}{2} \left( \frac{V'(\varphi)}{V(\varphi)} \right)^2, \] (1.40)
and
\[ \eta(\varphi) \equiv M_P^2 \frac{V''(\varphi)}{V(\varphi)}. \] (1.41)

Now, necessary conditions for the slow-roll approximation to hold are
\[ \epsilon \ll 1 \quad \text{and} \quad |\eta| \ll 1. \] (1.42)

These are not sufficient conditions, though. This can be seen by comparing the Friedmann and field equations before and after the slow-roll approximation. The order of the field equation has been reduced by one, thus leaving \( \dot{\varphi} \) as a seemingly free parameter. Because the slow-roll conditions (1.42) only restrict the form of the potential and not the properties of dynamic solutions, we could initially choose \( \dot{\varphi} \) so that it would violate Eq. (1.37) while still fulfilling Eq. (1.42).

Fortunately, however, the inflationary solutions pose an attractor behaviour, which means that all solutions approach each other asymptotically [24]. It can be shown that linear perturbations, or linear deviations from a solution \( H_0(\varphi) \), die out at least exponentially fast (as a function of \( \varphi \)). The slow-roll equation (1.39) can
be said to be the attractor behaviour of the solution.\(^8\) Thus, during inflation the outline of the events is roughly the following. At the beginning of inflation the attractor does not yet apply but rapidly all solutions approach each other. Once the deviations between two solutions reaches the linear regime, the different solutions converge at least exponentially.

The relation between slow-roll and inflation is simple: slow-roll implies inflation. This can be seen by writing the condition for inflation, Eq. (1.18), as

\[
\frac{\ddot{a}}{a} = \dot{H} + H^2 > 0 ,
\]

which is equivalent to

\[
-\frac{\dot{H}}{H^2} < 1 .
\]

Now, assuming the validity of slow-roll, and using the slow-roll equations, we may write

\[
-\frac{\dot{H}}{H^2} \simeq \frac{M_P^2}{2} \frac{V'}{V^2} \equiv \epsilon .
\]

Hence, the slow-roll (\(\epsilon \ll 1\)) implies inflation (\(\ddot{a} > 0\)). Inflation is usually taken to end when the slow-roll conditions are violated, i.e. when \(\epsilon \sim 1\) or \(\eta \sim 1\), even though it may in principle continue after that. On the other hand, inflation may end without violating the conditions, for example in the hybrid scenario, where the so called waterfall field ends inflation while \(\epsilon\) may still be well below one [6].

### 1.3.4 Models of inflation

Before the actual inflationary scenario was introduced Starobinsky took important steps towards it [25, 26]. His aim was to solve the problem of initial singularity in the Friedmann-Robertson-Walker universe by adding a preceding era of exponential expansion, i.e. de Sitter era.

However, the true beginning of the study of inflationary cosmology is generally considered to be Guth’s article in 1981, Ref. [20], where the so called old inflation was introduced. The model is first potential-driven inflation and based on false vacuum decay. Initially, a scalar field \(\varphi\) lies at a metastable local minimum and the universe is inflating. At some point the field tunnels to the true vacuum, with \(V(\varphi) = 0\), thus ending inflation. Old inflation suffered from severe inhomogeneities [20, 27] and was quickly replaced with new inflation [28, 29]. This time, instead of tunneling straight into the true vacuum, the field \(\varphi\) rolls there slowly. The actual inflation takes place during this slow-roll.

The chaotic inflation [30] meant a departure from both the old and new inflation. The scenario is not based on a cosmological high-temperature phase transition but

\(^8\) Actually, it is not precisely the attractor that every solution to the full equations, (1.33) and (1.36), approach but it is generally a good approximation whenever the slow-roll conditions are satisfied [6].
on “chaotic” initial conditions. The initial state before inflation is supposed to be some sort of quantum gravity or string state. It is assumed that within this “space-time foam” a large enough and smooth enough patch emerges and inflation takes off. The evolution of that domain can be described classically with the Friedmann equation (1.30) while the inflaton field complies with Eq. (1.35).

The chaotic scenario does not depend on the precise form of the potential, but the form is often considered to be monomial, \( V(\varphi) \propto \varphi^n \), the most common examples being \( V(\varphi) = \frac{1}{2} m^2 \varphi^2 \) and \( V(\varphi) = \frac{1}{4} \lambda \varphi^4 \). It is, however, difficult to combine single field chaotic inflation and conventional supergravity since gravity corrections to the potential tend to render it too steep for values of \(|\varphi|\) of the order Planck mass and larger [31], which are required with monomial single-field potentials.\(^9\)

The chaotic scenario is in fact so general that it is still the way the initial conditions are considered to emerge, and nowadays “model of inflation” refers to the form of the effective potential during inflation. It can, in addition, refer to the possible non-trivial kinetic terms and a specific way of ending inflation. For a thorough discussion on the model building of inflation, see [32].

1.3.5 Hybrid inflation

Hybrid inflation [33], studied in the enclosed papers [1] and [4], consists of at least two scalar fields. The inflaton, \( \varphi \), is still rolling slowly during inflation but now there is an additional field, \( \sigma \). The trajectory in the field space is such that \( \varphi = 0 \) does not correspond to \( V = 0 \). The field \( \sigma \) is kept in place (usually \( \sigma = 0 \)) while the inflaton rolls slowly until it reaches some critical value \( \varphi_c \) at which time \( \sigma \) destabilises and relatively suddenly rolls down to \( V = 0 \) thus causing inflation to end abruptly. A typical example of the hybrid potential is (see e.g. [32])

\[
V(\varphi, \sigma) = \frac{1}{4} \lambda (M^2 - \sigma^2)^2 + \frac{1}{2} m^2 \varphi^2 + \frac{1}{2} g^2 \varphi^2 \sigma^2 \\
= V_0 - \frac{1}{2} m^2 \varphi^2 + \frac{1}{4} \lambda \sigma^4 + \frac{1}{2} m^2 \varphi^2 + \frac{1}{2} g^2 \varphi^2 \sigma^2 ,
\]

(1.46)

where \( \varphi \) is the inflaton and \( \sigma \) is another field. With values \( \varphi^2 > \varphi_c^2 = \lambda M^2 / g^2 \) the inflationary trajectory in the field space is along the bottom of the valley at \( \sigma = 0 \). At that time the potential is dominated by \( V_0 = \frac{1}{4} \lambda M^4 \). At \( \varphi^2 = \varphi_c^2 \) the valley at \( \sigma = 0 \) becomes an unstable ridge and \( \sigma \) quickly rolls down towards \( V = 0 \) thus ending inflation. This is why \( \sigma \) is often called waterfall field.

The main advantage of hybrid inflation is that there is no need for large field values, since the inflation is supported by the constant term, \( V_0 \), in the potential. Therefore, one can have inflation with \(|\varphi| \ll M_P\).

\(^9\)In order to have \( \epsilon < 1 \) with potential \( V \propto \varphi^n \) one must have \( \varphi > M_P \).
1.3 Inflation

1.3.6 Inflation and scalar field perturbations

The historical motivation for inflation was to cure the problems of the standard big bang theory [20]. The most important feature of inflation, however, is that it provides a natural way to produce the seeds for the large scale structure through quantum fluctuations of scalar fields, (see e.g. [6]). During inflation every scalar field experiences quantum fluctuations. These fluctuations are stretched along with the enormous expansion of the universe. Soon after the scale of a particular quantum fluctuation ends up outside the Hubble horizon, i.e. \( k \approx aH \), the fluctuation becomes a classical perturbation. Once outside horizon, the perturbation ceases to evolve if its mass is not too large compared to the Hubble parameter. If the mass is too large (comparable or larger than the Hubble parameter) the amplitude of the perturbation is suppressed and decreases quickly [1, 34].

According to the standard inflationary scenario the quantum fluctuations of the inflaton field become classical and freeze once they end up outside the horizon. These inflaton perturbations cause perturbations to the metric. When inflation has ended and the matter content of the universe is produced in reheating the matter starts evolving according to these metric perturbations. The remnant of the matter perturbations, from the time they still were in the linear regime, can be observed in the CMB radiation. Later the matter perturbations evolve nonlinearly and produce galaxy clusters, galaxies, and other large scale structure, which also provide observable constraints to inflationary models, see e.g. [6].

Massless scalar field in a de Sitter space

Let us first, as an example, consider a generic massless scalar field, \( \chi \), which is not the inflaton, in a pure de Sitter space. We divide the field into a homogeneous part and a perturbation part

\[
\chi(\tau, \mathbf{x}) = \chi_0(\tau) + \delta \chi(\tau, \mathbf{x}) ,
\]

where \( \tau \) is the conformal time and is related to the cosmic time \( t \) via relation \( d\tau = dt/a(t) \). In a de Sitter space the scale factor \( a \propto e^{Ht} \) with a constant \( H \), (note the cosmic time \( t \)). With proper boundary conditions \( (a_{\tau \to 0} \to \infty) \) we obtain

\[
a(\tau) = \frac{1}{H\tau} , \quad (\tau < 0) .
\]

Now, we define a rescaled perturbation

\[
\delta \sigma = a \delta \chi ,
\]

which we could make an operator and expand [13]

\[
\tilde{\delta} \sigma(\tau, \mathbf{x}) = \int \frac{d^3 k}{(2\pi)^3} \left[ \sigma_k(\tau) \hat{a}_k e^{ik \cdot \mathbf{x}} + \sigma_k^*(\tau) \hat{a}_k^\dagger e^{-ik \cdot \mathbf{x}} \right] ,
\]
where $\hat{a}_k$ and $\hat{a}_k^\dagger$ are annihilation and creation operators, respectively; $\sigma_k(\tau)$ is the mode function obeying

$$\sigma''_k + \left( k^2 - \frac{a''}{a} \right) \sigma_k = 0 \ . \quad (1.51)$$

Here the prime denotes derivative with respect to the conformal time, $' \equiv \partial_\tau$. We are not, however, interested in the precise nature of the quantum-to-classical transition of the fluctuations and we are content with the Fourier notation (see Sec. 3.2)

$$\delta \sigma(\tau, x) = \int \frac{d^3k}{(2\pi)^3} e^{i k \cdot x} \sigma_k(\tau) \ . \quad (1.52)$$

According to Eq. (1.48) we can recast Eq. (1.51) into

$$\sigma''_k + \left( \frac{k^2}{\tau^2} - 2 \right) \sigma_k = 0 \ . \quad (1.53)$$

On a sub-Hubble scale, $k \gg aH$ ($-k\tau \gg 1$), Eq. (1.53) becomes

$$\sigma''_k + k^2 \sigma_k = 0 \ , \quad (1.54)$$

which is the same equation as in a flat Minkowski space. The solutions are plane waves [6, 34]

$$\sigma_k = \frac{e^{-ik\tau}}{\sqrt{2k}} , \quad (k \gg aH) \ . \quad (1.55)$$

On a super-Hubble scale, $k \ll aH$ ($-k\tau \ll 1$), Eq. (1.53) becomes

$$\sigma''_k - \frac{a''}{a} \sigma_k = 0 \ , \quad (1.56)$$

which has a solution [13, 34]

$$\sigma_k = B(k) a + C(k) a^{-2} \simeq B(k) a , \quad (k \ll aH) \ , \quad (1.57)$$

where the decaying part quickly becomes negligible. By matching the two solutions\(^\text{10}\), Eq. (1.55) and Eq. (1.57), at the (Hubble) horizon crossing, $k = aH$ ($-k\tau = 1$), we can solve $|B| = H/\sqrt{2k^3}$, (we are not interested in the phase of the solution). Thus, for the modes of the original scalar field, $\chi_k = \sigma_k/a$, outside horizon we obtain

$$|\chi_k| \simeq \frac{H}{\sqrt{2k^3}} , \quad (k \ll aH) \ . \quad (1.58)$$

\(^\text{10}\)Note that we could have made the matching by using an exact solution, $\sigma_k = e^{-ik\tau}/\sqrt{2k}(1 + \frac{i}{k\tau})$, of Eq. (1.53) around horizon crossing. The result, however, would in the end be the same, Eq. (1.58).
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With the spectrum defined as in Eq. (3.4), the above solution, Eq. (1.58), with a constant $H$ gives an exactly scale invariant spectrum

$$ P_{\delta\chi} \equiv \frac{k^3}{2\pi^2}|\chi_k|^2 = \left( \frac{H}{2\pi} \right)^2 $$

(1.59)

with a spectral index

$$ n_\chi - 1 \equiv \frac{d\ln P_{\delta\chi}}{d\ln k} = 0. $$

(1.60)

**Massive scalar field in a quasi de Sitter space**

Now we consider a more realistic example. Let the scalar field $\chi$ have a mass $m$, whence the equation of motion (1.51) becomes [13, 34]

$$ \sigma''_k + \left( k^2 - \frac{a''}{a} + m^2 a^2 \right) \sigma_k = 0. $$

(1.61)

In a pure de Sitter space there is an exact solution to Eq. (1.61). We consider instead a case where the Hubble parameter does not remain constant but changes according to $\dot{H} = -\epsilon H^2$, where $\epsilon$ is the slow roll parameter. Again, with the proper boundary conditions we obtain

$$ a(\tau) = -\frac{1}{H\tau(1-\epsilon)}, \quad (\tau < 0). $$

(1.62)

The mode equation (1.61) can now be recast into [34]

$$ \sigma''_k + \left[ k^2 - \frac{1}{\tau^2} \left( \nu^2 - \frac{1}{4} \right) \right] \sigma_k = 0, $$

(1.63)

where $\nu \simeq \frac{3}{2} + \epsilon - \eta_\chi$. We have denoted $\eta_\chi \equiv m^2/3H^2$. Note, that we are assuming the slow roll parameters to be small, $\epsilon, |\eta_\chi| \ll 1$, and the form of Eq. (1.63) is approximate.

For real $\nu$ solution to Eq. (1.63) reads [34,35]

$$ \sigma_k = \sqrt{-\tau} \left[ c_1(k)H^{(1)}_\nu(-k\tau) + c_2(k)H^{(2)}_\nu(-k\tau) \right], $$

(1.64)

where $H^{(1)}_\nu$ and $H^{(2)}_\nu$ are the Hankel functions of the first and second kind, respectively [36]. The asymptotic behaviour of the Hankel functions with $x \gg 1$ is [34]

$$ H^{(1)}_\nu(x) \sim \sqrt{\frac{2}{\pi x}} e^{i(x-\frac{\nu}{2}-\frac{\pi}{4})}, \quad H^{(2)}_\nu(x) \sim \sqrt{\frac{2}{\pi x}} e^{-i(x-\frac{\nu}{2}+\frac{\pi}{4})}. $$

(1.65)

We use the asymptotics, Eq. (1.65), and again demand that we obtain the plane wave solution, Eq. (1.55), deep inside the Hubble horizon, $k \gg aH (-k\tau \gg 1)$. 

Thus, we can set $c_2(k) = 0$ and $c_1(k) = \frac{\sqrt{\pi}}{2} \exp[i(\nu + \frac{1}{2})\frac{\pi}{2}]$ and obtain solution on sub-horizon scales,

$$
\sigma_k \simeq \frac{\sqrt{\pi}}{2} e^{i(\nu + \frac{1}{2})\frac{\pi}{2}} \sqrt{-\tau} H^{(1)}_\nu(-k\tau), \quad (-k\tau \gg 1).
$$

(1.66)

For $x \ll 1$ the asymptotic behaviour of the Hankel function is [34]

$$
H^{(1)}_\nu(x) \sim \sqrt{\frac{2}{\pi}} e^{-i\frac{\pi}{2}} 2^{\nu-\frac{3}{2}} \frac{\Gamma(\nu)}{\Gamma(3/2)} x^{-\nu},
$$

(1.67)

where $\Gamma$ is the Euler Gamma function [36]. Thus, on large scales, $k \ll aH$ ($-k\tau \ll 1$), the mode function becomes

$$
\sigma_k \simeq e^{i(\nu - \frac{1}{2})\frac{\pi}{2}} 2^{\nu-\frac{3}{2}} \frac{\Gamma(\nu)}{\Gamma(3/2)} \frac{1}{\sqrt{2k}} (-k\tau)^{\frac{1}{2} - \nu}.
$$

(1.68)

For the modes of the original scalar field, $\chi_k = \sigma_k/a$, outside horizon we can now write

$$
|\chi_k| \simeq \frac{H}{\sqrt{2k^3}} \left( \frac{k}{aH} \right)^{\frac{3}{2} - \nu}, \quad (k \ll aH),
$$

(1.69)

where $\nu \simeq \frac{3}{2} + \epsilon - \eta_\chi$; and $\epsilon, |\eta_\chi| \ll 1$. This leads to the spectrum

$$
P_{\delta\chi} \equiv \frac{k^3}{2\pi^2} |\chi_k|^2 = \left( \frac{H}{2\pi} \right)^2 \left( \frac{k}{aH} \right)^{3 - 2\nu}
$$

(1.70)

and the spectral index

$$
n_{\chi} - 1 \equiv \frac{d \ln P_{\delta\chi}}{d \ln k} = 3 - 2\nu \simeq 2\eta_\chi - 2\epsilon.
$$

(1.71)

1.4 Preheating

Preheating is a vast and complex topic, see e.g. [37–50] The basic idea, however, is quite simple. We do not intend to give an exhaustive discussion on the subject, but to present the features which are important for the non-Gaussianity treatment in Sec. 3.5. Preheating is studied in the enclosed Papers [2] and [3]. Here, we will mainly follow the treatment in [47], (see [51] for a more recent discussion).

1.4.1 Reheating

At the end of inflation the universe is cold and practically empty. The enormous expansion has diluted the number densities of particles as well as redshifted the
energy density of radiation. There is, however, still a substantial energy density stored in the scalar field potential, in the form of a homogeneous inflaton field.

After the inflationary epoch the standard hot big bang era has to begin. To achieve this, there has to be a way to transfer most of the energy stored in the homogeneous scalar field(s) into the energy and particle content of the universe. In this subsection we describe the traditional process of reheating [52, 53], (for a review see [17, 47, 51]), i.e. a theory describing the decay of scalar field(s) after inflation.

In the traditional reheating picture the inflaton field, $\phi$, starts to oscillate near the minimum of its effective potential immediately after the end of inflation [47]. Inflaton is coupled to other fields, scalars $\chi$ or fermions $\psi$, for example via terms $\nu \sigma \phi \chi^2$ and $h \phi \bar{\psi} \psi$ in the Lagrangian [51]. Here $\nu$ and $h$ are dimensionless coupling constants and $\sigma$ has the dimension of mass. In the case of adequately massive inflation, i.e. $m_\phi \gg m_\chi, m_\psi$, the decay rate, $\Gamma$, can be given by the expressions [51]

$$
\Gamma(\phi \to \chi \chi) = \frac{\nu^2 \sigma^2}{8\pi m_\phi} \\
\Gamma(\phi \to \psi \bar{\psi}) = \frac{h^2 m_\phi}{8\pi} .
$$

The decay of the inflaton may phenomenologically be described by adding a friction term $\Gamma_{\text{tot}} \dot{\phi}$ in the equation of motion, Eq. (1.36): [47]

$$
\ddot{\phi} + 3H \dot{\phi} + \Gamma_{\text{tot}} \dot{\phi} + m_\phi^2 \phi = 0 ,
$$

where $\Gamma_{\text{tot}} \equiv \Gamma(\phi \to \chi \chi) + \Gamma(\phi \to \psi \bar{\psi})$ [51].

While the inflaton field is oscillating the energy density of the universe evolves as if it was matter dominated. This gives $H \approx 2/3t$, a monotonically decreasing function of time. As long as $\Gamma_{\text{tot}} < H$ the expansion of the universe prevents its thermalization. We obtain an upper limit on the reheating temperature of the universe by assuming that when $\Gamma_{\text{tot}} = H$, the inflaton suddenly decays into ultrarelativistic particles, whose energy density $\rho \approx g_* \pi^2 T^4 / 30; g_*$ is the number of degrees of freedom at temperature $T$, ($g_* \approx 10^2$ $- 10^3$). $\Gamma_{\text{tot}} = H = (\rho / 3M_P^2)^{1/2}$ yields $\rho = 3\Gamma_{\text{tot}}^2 M_P^2$. Comparing these two expressions for the energy density we obtain an estimate for the reheating temperature

$$
T_r \approx 0.5 \left( \frac{100}{g_*} \right)^{1/4} \sqrt{\Gamma_{\text{tot}} M_P} .
$$

In case all decay products are not ultrarelativistic, the reheat temperature is different since the relation between $H$ and $T$ is different [51].

The reheating temperature should be below the GUT scale, $T_r \lesssim 10^{16}$ GeV, in order to prevent the production of monopoles, but there exist also other constraints depending on what kind of model is considered, see e.g. [51] and references therein.
1.4.2 Parametric resonance

There is a possibility that coherently oscillating classical inflaton field decays into other bosons due to parametric resonance, see e.g. [54], (for a parametric resonance in quantum field theory, see [55]). This stage is extremely rapid and is called preheating. Secondly, these bosons, which are massive and far away from thermal equilibrium immediately after preheating, decay further. This results in a multitude of particles creating the particle content of the universe. Finally, the universe thermalizes and the hot big bang era takes off.

Now we proceed to present parametric resonance, which is the key addition that makes the standard reheating and preheating different. Let us start with a simple two-field potential\footnote{For a thorough discussion on a conformally invariant theory, $V = \frac{1}{4} \lambda \varphi^4 + \frac{1}{2} g^2 \varphi^2 \sigma^2$, see [56].}

\begin{equation}
V = \frac{1}{2} m_\varphi^2 \varphi^2 + \frac{1}{2} g^2 \varphi^2 \sigma^2 ,
\end{equation}

where $\varphi$ is the inflaton (with mass $m_\varphi$) and $\sigma$ is another scalar field, and $g$ is dimensionless coupling constant. The mass term of the inflaton dominates the energy density of the universe over the coupling term. After the end of inflation the inflaton starts oscillating with a decaying amplitude $\Phi(t)$ according to [47, 51]

\begin{equation}
\varphi(t) \simeq \Phi(t) \sin(m_\varphi t), \quad \Phi(t) \simeq \sqrt{\frac{8}{3}} \frac{M_P}{m_\varphi} .
\end{equation}

Now, the second scalar field, $\sigma$, has an oscillating effective mass\footnote{If needed, for example to avoid having zero mass for $\sigma$ occasionally, one could easily just add a mass term for the field $\sigma$ without complications, i.e. $m_\sigma = m_{\sigma,0} + g^2 \varphi^2$.}

\begin{equation}
m_\sigma \equiv \frac{\partial^2 V}{\partial \sigma^2} = g^2 \varphi(t)^2 .
\end{equation}

The homogeneous inflaton field acts as an oscillating force on the perturbations of the field $\sigma$. The equation of motion for the Fourier modes of the field $\sigma$, see Eq. (2.30), can thus be written (with the metric perturbations neglected) [47, 51]

\begin{equation}
\ddot{\sigma}_k + 3H \dot{\sigma}_k + \left( \frac{k^2}{a^2} + g^2 \varphi^2 \right) \sigma_k = 0 .
\end{equation}

The oscillation of the inflaton field gives a natural time scale $m_\varphi^{-1}$ for the system. Since the oscillations begin after the Hubble scale has dropped below the inflaton mass, $m_\varphi \gg H$, (i.e. the friction term $3H \dot{\sigma}_k$ has become subdominant), we neglect the expansion of the universe here (and take $a = 1$). Physically this means that several oscillations take place during one Hubble time. In Minkowski space Eq. (1.79) can be recast as [47]

\begin{equation}
\ddot{\sigma}_k + [k^2 + g^2 \Phi^2 \sin^2(m_\varphi t)] \sigma_k = 0 ,
\end{equation}
with \( \Phi \) now considered a constant, or

\[
\frac{d^2 \sigma_k}{dz^2} + [A_k - 2q \sin(2z)] \sigma_k = 0 ,
\]

where \( z \equiv m_{\varphi} t \) and

\[
A_k \equiv 2q + \frac{k^2}{m_{\varphi}^2}, \quad q \equiv \frac{g^2 \Phi^2}{4m_{\varphi}^2} .
\]

Eq. (1.81) is the so called Mathieu equation [57], and is known to have instability bands for certain values of the parameters \( A_k \) and \( q \). Within these instability, or resonance, bands the solution for Eq. (1.81) grows exponentially,

\[
\sigma_k \propto \exp(\mu_k z),
\]

where \( \mu_k(n) \) is the so called Floquet index in a resonance band labeled by an integer \( n \) [47, 51, 57]. The first resonance band is between \( k = (m_{\varphi}/2)(1 \pm q/2) \) with the Floquet index [47]

\[
\mu_k = \sqrt{\left( \frac{q}{2} \right)^2 - \left( \frac{2k}{m_{\varphi}} - 1 \right)^2}
\]

taking its maximum value \( \mu_k = q/2 = g^2 \Phi^2/8m_{\varphi}^2 \) at \( k = m_{\varphi}/2 \).

The exponential growth of the amplitude of the modes \( \sigma_k \) can physically be interpreted as exponential production of particles with a momentum \( k \). The occupation number (density) of the \( \sigma \) particles in mode \( k \) is [47]

\[
n_k = \frac{w_k}{2} \left( \frac{\dot{\sigma}_k^2}{w_k^2} + |\sigma_k|^2 \right) - \frac{1}{2} ,
\]

which can be considered to be the sum of the kinetic energy, \( |\dot{\sigma}_k|^2 \), and the potential energy, \( w_k^2|\sigma_k|^2 \), divided by the energy (or frequency), of one particle

\[
w_k = \sqrt{k^2 + g^2 \varphi(t)^2}.
\]

In the middle of the first resonance band, at \( k = m_{\varphi}/2 \), the occupation number grows as \( n_k \propto \exp(qz) = \exp(g^2 \Phi^2 t/4m_{\varphi}) \).

The occupation number is an adiabatic invariant. The adiabacity condition can be quantified by a dimensionless ratio [51]

\[
R_a \equiv \dot{w_k}/w_k^2 .
\]

If \( |R_a| \ll 1 \) we are within adiabatic region and the occupation number does not change, i.e., there is no particle production. Another way to put this is, that according to WKB theory, if the frequency \( w_k \) is changing slowly, the solutions of Eq. (1.80) are close to those of the equation in which \( w_k \) is constant [51]. Thus, the condition for particle production is the absence of adiabacity, i.e. one must have \( |R_a| \gtrsim 1 \).

\footnote{For the connection to Bogoliubov transformations, see e.g. [58].}
Narrow resonance

There are two distinct types of parametric resonances depending on the value of the parameter \( q \). The case \( q \ll 1 \) is called narrow resonance, since the resonance bands are narrow (the width of the first band \( \Delta k = m_\varphi q / 2 \)). In the large scale limit, \( k \to 0 \), we have \( \varphi = \Phi \sin(m_\varphi t) \) with \( w_k = g\varphi \leq g\Phi \). Now, we have

\[
|R_a| = \left| \frac{\Phi m_\varphi \cos(m_\varphi t)}{g\varphi^2} \right| \geq \frac{m_\varphi}{g\Phi} |\cos(m_\varphi t)|, \tag{1.87}
\]

which is \( \geq 1 \) when \( \cos^2(m_\varphi t) \geq 4q \). For sufficiently small \( q \) this condition is satisfied essentially throughout each oscillation. Physically \( q \ll 1 \) means that the mass of the inflaton, \( m_\varphi \), is much larger than the maximum effective mass of \( \sigma \) field, \( g\Phi \). Therefore, the inflaton is capable of decaying into \( \sigma \)-particles all the time. Since in narrow resonance the adiabacity condition is never satisfied the occupation number increases exponentially without settling to a constant value at any stage of each oscillation [47].

Broad resonance

The case \( q \gg 1 \) is called broad resonance region. Let us again take the large scale limit, \( k \to 0 \). If we now consider periods of small field value \( \varphi \), i.e. \( |\varphi/\Phi| \ll 1 \), we have \( \dot{\varphi} = m_\varphi \Phi \cos(m_\varphi t) \simeq m_\varphi \Phi \). This way we can estimate

\[
|R_a| \simeq \left| \frac{g m_\varphi \Phi}{g^2 \varphi^2} \right| = \frac{m_\varphi}{g\Phi} \frac{1}{\sin^2(m_\varphi t)}, \tag{1.88}
\]

which is \( \geq 1 \) when \( \sin^2(m_\varphi t) \leq 1/(2\sqrt{q}) \). With \( q \gg 1 \) this condition is satisfied briefly but periodically every time \( \varphi \) oscillates through \( \varphi = 0 \). Physically \( q \gg 1 \) means that the mass of the inflaton, \( m_\varphi \), is too small to decay into (too massive) \( \sigma \) particles for the most of the time. However, decaying of \( \varphi \) becomes viable every time \( \varphi \) goes through zero. This indeed produces huge periodic bursts of \( \sigma \) particles and, actually, the particle production during broad resonance is much more efficient than during narrow resonance [47]. This is basically due to both stronger coupling between \( \varphi \) and \( \sigma \) (given by \( g^2 \)) and wider resonance bands.

Stochastic resonance

Taking properly into account the expansion of the universe the previous analysis changes considerably. The equation to be considered now is Eq. (1.79). By introducing a new quantity \( X_k \equiv a^{3/2} \sigma_k \) we may define a comoving occupation number [47, 51]

\[
n_k = \frac{w_k}{2} \left( \frac{|\dot{X}_k|^2}{w_k^2} + |X_k|^2 \right) - \frac{1}{2}. \tag{1.89}
\]
The Mathieu equation becomes

$$\frac{d^2 X_k}{dz^2} + [A_k - 2q \sin(2z)] X_k = 0 \quad ,$$

(1.90)

with

$$A_k \equiv 2q + \frac{k^2}{a^2 m^2_\sigma}, \quad q \equiv \frac{g^2 \Phi^2(t)}{4m^2_\phi} \quad ,$$

(1.91)

depending now explicitly on time, and a term $-(3/4)(2\dot{a}/a + \dot{a}^2/a^2)$ neglected. The time behaviour of $\Phi(t)$ is given by Eq. (1.77). The most obvious new features are now the redshifting of the parameters $A_k$ and $q$. This causes modes to drift in the stability-instability chart of the Mathieu equation. A particular mode is occasionally within a stable region and occasionally within an unstable region. Since the parameters $A_k$ and $q$ are not constants, using the analyses of the Mathieu equation directly is not entirely justified. However, as long as the parameters are not changing too rapidly the results should be applicable [51].

There is however a fundamental difference between parametric resonance in a Minkowski space and in an expanding space. In Eq. (1.81) the oscillating force term is correlated with the oscillation of $\sigma$. Every time a mode $\sigma_k$ gets a kick it is in an appropriate phase, and obtains a boost to its amplitude. This is not the case in an expanding universe, where the phase of $\sigma_k$ is uncorrelated with the oscillation of $\varphi$. Sometimes the kick given by the oscillating force boosts the amplitude of $\sigma_k$ but sometimes the amplitude gets damped. This can be considered as a successive scattering on parabolic potentials [47]. This process is called stochastic resonance.

The analysis of the evolution of a single mode, $\sigma_k$, during stochastic resonance does not make much sense. Tracking the situation is difficult analytically and numerical methods are usually needed [47]. The important point, however, is that when all the relevant modes are considered, the number of produced $\sigma$ particles still grows exponentially with some effective Floquet index, which depends on the parameters of the system, (see [47] for a table of Floquet indices with different values of $g$).

### 1.4.3 Instant preheating

The basic idea of instant preheating [59] is quite simple. Instead of having the usual preheating with parametric resonance instant preheating may also take place in models where parametric resonance cannot be realized. Immediately after the end of inflation the inflaton field, $\varphi$, starts to quickly roll down its potential, i.e., the inflaton starts its first oscillation. The inflaton is coupled to another scalar field, $\sigma$, whose effective mass is dictated by the inflaton. At the bottom of the oscillation the effective mass of $\sigma$ reaches its minimum and light $\sigma$ particles are copiously produced. As $\varphi$ starts to climb up its potential, however, the mass of $\sigma$ starts to grow and, finally, when the mass is large enough it decays into other lighter particles. This way the preheating may take place “instantaneously” during single inflaton oscillation.
Instant preheating is also typically accompanied by a production of heavy particles, whose masses may even be as large as $10^{17} - 10^{18}$ GeV [59].

Let us now go through the mechanism in a more detail using a toy model presented in [59]. The starting point is to have a simple effective potential for the inflaton $m^2 \phi^2$. There are also two other important terms, $g^2 \phi^2 \sigma^2$ and $h \bar{\psi} \psi \sigma$, coupling the inflaton to another scalar $\sigma$ and $\sigma$ to a fermion field $\psi$, respectively. Thus, the effective potential for the toy model is

$$V(\phi, \sigma, \psi) = m^2 \phi^2 + g^2 \phi^2 \sigma^2 + h \bar{\psi} \psi \sigma,$$  

(1.92)

where $g$ and $h$ are coupling strengths. The effective mass of $\sigma$ is now $m^2 \sigma = g^2 \phi^2$. We have assumed that $\sigma$ does not have a bare mass but that could easily be inserted.

The inflation has been driven by the inflaton mass term, $m^2 \phi^2$, and therefore, the CMB observations require $m \sim 10^{-6} M_P$. Since effective reheating demands [47, 59] $g \gtrsim 10^{-4}$, and at the end of inflation $\phi \sim M_P$, the effective mass $m^2_\sigma$ is initially much larger than $m^2$. Also, the adiabacity condition $|i m_\sigma| \ll m^2_\sigma$ is initially satisfied, and no $\sigma$ particles are produced. The situation, however, becomes nonadiabatic when $|i m_\sigma| = g |\dot{\phi}| \simeq g m \Phi$ becomes less than $g^2 \phi^2 \sigma^2$, i.e. when $g \phi \lesssim m \Psi$. Here $\Phi \sim 10^{-1} M_P$ is the amplitude of the first inflaton oscillation. The particle production, therefore, takes place when $|\varphi| \lesssim \varphi_* \equiv \sqrt{m \Phi / g}$, which is small with values of $g \gg 10^{-4}$ necessary for efficient preheating. This can be considered to be instantaneous process taking a time [59]

$$\Delta t_* \sim \frac{\varphi_*}{|\dot{\varphi}|} \sim (g m \Phi)^{-1/2},$$  

(1.93)

which is much smaller than the age of the universe. During this instant the occupation number of $\sigma$ particles with a momentum $k$ jumps from zero to [47]

$$n_k = \exp \left( - \frac{\pi k^2}{g m \Phi} \right).$$  

(1.94)

The possible bare mass of $\sigma$, $m_{\sigma,0}$, is easily added here and the occupation number becomes [47]

$$n_k = \exp \left( - \frac{\pi (k^2 + m^2_{\sigma,0})}{g |\dot{\varphi}_0|} \right).$$  

(1.95)

We have also written explicitly $|\dot{\varphi}_0|$ to emphasise the fact that in order to calculate the occupation number one only needs to know the velocity of the inflaton field at $\varphi = 0$; no knowledge of the actual form of the potential $m^2 \phi^2$, or of the parameters $m$ and $\Phi$, is needed.

The number density of produced particles can be obtained by integrating over all modes $k$

$$n_\sigma = \frac{1}{2 \pi^2} \int_0^\infty dk \, k^2 n_k = \frac{(g |\dot{\varphi}_0|)^{3/2}}{8 \pi^3} \exp \left( - \frac{\pi m^2_{\sigma,0}}{g |\dot{\varphi}_0|} \right),$$  

(1.96)

This is mainly for illustrational purposes, the exact form of the potential is not important.
reducing to

\[ n_\sigma \sim \frac{(g|\dot{\varphi}_0|)^{3/2}}{8\pi^3} \]  (1.97)

for an effectively massless \((m_\sigma^2 \ll g|\dot{\varphi}_0|)\) case. With each particle having a typical energy of \(\sim (g|\dot{\varphi}_0|/\pi)^{1/2}\) their total energy density is given by [59]

\[ \rho_\sigma \sim \frac{(g|\dot{\varphi}_0|)^2}{8\pi^{7/2}}. \]  (1.98)

This energy density can be compared to the energy density of the inflaton field, \(\rho_\varphi = \frac{1}{2}|\dot{\varphi}_0|^2\). The result is

\[ \frac{\rho_\sigma}{\rho_\varphi} \sim \frac{g^2}{4\pi^{7/2}} \sim 5 \times 10^{-3} g^2. \]  (1.99)

Note that this result does not depend on the mass of the inflaton; in fact similar result may be derived for the inflaton potential \(\lambda \varphi^4\) independent of the value of \(\lambda\) [59].

The fraction in Eq. (1.99) is the energy density at the moment when the \(\sigma\) particles are produced at \(\varphi = 0\). After their production the effective mass \(m_\sigma\) starts to increase as \(\varphi\) starts to climb up the potential. The most efficient instant preheating takes place if the particles \(\sigma\) decay further into fermions \(\psi\) via the fermion coupling in Eq. (1.92), when the inflaton reaches its maximum \(\Phi\). The decay rate is given by [47, 59]

\[ \Gamma(\sigma \rightarrow \bar{\psi}\psi) = \frac{h^2 m_\sigma}{8\pi} = \frac{h^2 g|\varphi|}{8\pi}, \]  (1.100)

which increases as \(|\varphi|\) increases.

With the model depicted here there is a possibility to produce particles with masses up to \(m_\psi \sim 4 \times 10^{16}\) GeV (with \(g \sim 10^{-1}\) and \(h \sim 7 \times 10^{-2}\), or \(m_\psi \sim 4 \times 10^{17}\) GeV \((g \sim 1, h \sim 2 \times 10^{-2})\) [59]. Since the mechanism is insensitive to the exact form of the inflaton potential \(V(\varphi)\) (the part of the potential (1.92) without the coupling terms) using “quintessential” type of potentials there is a possibility to produce even heavier particles (for details, see [59]). If one considers potential \(V(\varphi)\) which behaves, for example, as \(\varphi^2\) when \(\varphi < 0\) and gradually vanishes at larger \(\varphi\), one may produce particles with masses \(m_\psi \sim 10^{17} - 10^{18}\) GeV. These kind of heavy particles could constitute a part of dark matter and their late decay could even be the cause of ultra energy high cosmic rays (UHECR), see [60].

1.4.4 Tachyonic preheating

There is another important model of extremely fast preheating which does not require parametric resonance. This preheating is due to tachyonic instabilities in the effective potential and is therefore dubbed tachyonic preheating [61, 62], ([63]). Tachyonic preheating typically occurs during a single oscillation, as the field rolls down the tachyonic potential.
There are several types of realizations of tachyonic preheating [61, 62]: the term responsible for the instability may be quadratic, cubic, quartic, or $\propto \varphi^n$; the scalar field may be real or complex. Topological defects, such as domain walls or cosmic strings, are typically produced during tachyonic preheating. Part of the energy density of the inflaton is, at first, converted into the energy density of the topological defects, but one must assume that the defects decay at some point into particles in order not to cause problems for the evolution of the universe.

We go through the points relevant for us in tachyonic preheating by considering a simple toy model. We do not consider topological defects and the exact mechanism of thermalizing the universe. This is a complicated issue and numerical lattice simulations are needed in order to study it properly, see [61, 62].

It is important to notice that tachyonic preheating is easily fulfilled at the end of hybrid inflation. However, the toy model we go through here is the simplest one presented in [61], namely

$$V(\varphi) = \frac{\lambda}{4}(\varphi^2 - v^2)^2 \equiv -\frac{m^2}{2}\varphi^2 + \frac{\lambda}{4}\varphi^4 + \frac{m^4}{4\lambda},$$

(1.101)

where $m^2 \equiv \lambda v^2$, and the coupling $\lambda \ll 1$. The potential (1.101) presents almost all the features needed. The main addition coming from hybrid inflation is that there the initial conditions, namely, the starting point at a local maximum at $\varphi = 0$, can easily be motivated. Another additional feature in hybrid inflation is the presence of another scalar field, which will be important later when we consider the non-Gaussianity arising from tachyonic preheating. Another scalar field enables nonadiabatic perturbations and may lead to super-Hubble evolution of the curvature perturbation.

Consider Eq. (1.101) and its local maximum at $\varphi = 0$. \(^{15}\) We study the Fourier modes of its perturbations, $\varphi_k$, whose evolution equation is

$$\ddot{\varphi}_k + (k^2 + V''(\varphi))\varphi_k = 0,$$

(1.102)

where $V''(\varphi) = -m^2$. Initially, at a time $t = 0$, the tachyonic mass term is assumed to be “turned off”, and the mode functions are assumed to be the same as for the massless scalar field, $\varphi_k = \frac{1}{\sqrt{2k}}\exp(-ikt + ik \cdot x)$, where $k \equiv |k|$. Only the modes $k < m$ will be subject to the instability. The initial dispersion of these modes is given by [61, 62]

$$\langle \delta\varphi^2 \rangle = \int \frac{dk}{k} P_\varphi(k) = \int_0^m \frac{dk^2}{8\pi^2} = \frac{m^2}{8\pi^2}.$$  

(1.103)

The average amplitude of the perturbations is $|\delta\varphi(x)| \sim m/2\pi$.

When $t > 0$ the tachyonic term can be considered to be “turned on”. This causes an instability and an exponential expansion of the modes with $k < m$, namely

\(^{15}\)The order parameter, or the scalar field, $\varphi$ is assumed to lack the homogeneous part, i.e. $\langle \varphi \rangle = 0$. This will be a useful feature later when we consider the non-Gaussianities produced during tachyonic preheating.
$|\varphi_k| \propto \exp(t\sqrt{m^2 - k^2})$. The dispersion becomes \[61, 62]\]

$$
\langle \delta \varphi^2 \rangle = \frac{m^2}{8\pi^2} e^{2\sqrt{m^2-k^2}} = \frac{e^{2mt(2mt-1)} + 1}{16\pi^2 t^2}.
$$

(1.104)

The expansion continues until $\sqrt{\langle \delta \varphi^2 \rangle}$ becomes $\sim v/2$. From the relation $m^2 \equiv \lambda v^2$ and Eq. (1.104) we can estimate that this happens at $t_* \sim (1/2m) \ln(C\pi^2/\lambda)$, where $C \sim 2mt_*$. Here we assume that $C \sim 2mt_*$ > 1. Since the possible values of the coupling constant $\lambda$ may vary over several orders of magnitude and since the factor $C$ appears within a logarithm, for simplicity, we may conclude that the duration of the tachyonic instability is

$$
t_* \sim \frac{1}{2m} \ln \frac{\pi^2}{\lambda}.
$$

(1.105)

The tachyonic instability manifests itself in an exponential production of $\varphi$ quanta with $k < m$ \[61, 62\]. We use again the concept of occupation number, Eq. (1.84). This time, however, the definition of the frequency has to be changed from $w_k = \sqrt{k^2 + V''(\varphi)} = \sqrt{k^2 - m^2}$ to $w_k = \sqrt{k^2 + |m|^2}$ (see Eq. (1.85)). The notion of occupation number actually does not make sense during the tachyonic period, but Eq. (1.84) can be formally used with the frequency redefined whenever $m^2 < 0$. At $\varphi = v/\sqrt{3}$ the effective mass of the scalar field $V''(\varphi) = -m^2 + 3\lambda \varphi^2$ actually vanishes briefly. After this stage the occupation number again is well defined (and well interpreted), and the modes start their usual oscillating behaviour. The transition from the tachyonic stage into the oscillating stage does not pose any problems and the two differently defined occupation numbers match very well. After the tachyonic era the occupation number becomes an adiabatic invariant and, as discussed previously, stays constant unless something dramatic, e.g. parametric oscillation, takes place. During tachyonic preheating, i.e. during time $t_*$, the occupation number of the modes $k \ll m$ grows from being insignificant to $16$ \[61, 62\]

$$
n_k \sim \exp(2mt_*) \sim \exp \left( \ln \frac{\pi^2}{\lambda} \right) = \frac{\pi^2}{\lambda} \gg 1.
$$

(1.106)

### 1.4.5 A Comment on causality

Preheating is a vast and complex topic. It is therefore inevitable that many interesting and important issues are here left uncommented. These include the backreaction and scattering of the produced particles, geometric preheating, fermionic preheating, and thermalization and the actual transition from (p)reheating era into FRW era. For these and many other topics, see a recent review \[51\] and references therein.

However, let us briefly comment on causality during preheating, and on how preheating may affect the CMB spectrum.

\[16\]The presence of topological defects may change the number of produced particles, but the error is not very large \[62\].
There has been much discussion on the effect of preheating on CMB, see e.g. [64, 65] and also [66–80]. Questions about causality aside, the outcome seems to be that CMB may indeed be affected but only provided certain criteria are met. The important point is that preheating can only amplify pre-existing perturbations and, in order to enhance super-Hubble perturbations, there has to be entropy perturbations present [51, 81], which are not suppressed outside the Hubble scale. This, together with the requirement for a strong resonance \((q \gg 1)\), leads to the conclusion that in many models preheating does not have an effect on CMB, see e.g. [51].

One should note that when we later discuss generation of non-Gaussianity during preheating we are dealing with an inherently second order effect. Therefore, the discussion on preheating and CMB power spectrum does not concern us directly. And indeed, it is plausible that while not having an effect on the CMB spectrum preheating may give rise to a significant second order effect, and therefore to non-negligible non-Gaussianity. After all, preheating is supposed to end because second order effects become significant in backreaction and rescattering [47, 51].

By construction the end of inflation takes place some \(\sim 50\) e-foldings after the observable scales have exited the horizon. There can, therefore, arise some concern on causality, i.e. whether preheating can have an effect on large (super-Hubble) scales. Bassett et al. [67] discuss the causal issues of preheating quite extensively. Indeed, causality is known to constrain the shape of the spectrum outside the horizon, see e.g. [82] and in the context of preheating [51, 67]. However, one has to make a distinction between the true causal particle horizon, \(d_H\), and the Hubble distance, \(H^{-1}\), [82].

The exponential expansion of the universe during inflation expands the causal particle horizon to be orders of magnitude larger than the Hubble distance. This expansion also sets the initial conditions for the perturbations on a much larger region than one Hubble patch. So, inflation sets the seemingly “acausal” initial conditions. On the other hand the field equations are derived from inherently covariant Einstein equations. Therefore, a solution which satisfies the initial conditions set by inflation and fulfils the relativistic equations must be causal [67]. Also, preheating enhances pre-existing perturbation, it does not create them from scratch. This kind of enhancement is familiar in the presence of entropy perturbations, see e.g. [51]. As Bassett et al. point out [67], the study of causality should involve unequal-time correlation function in real space, (see e.g. [82]). Causality is not easily addressed using Fourier space considerations.

Surely, there is plenty of room for further studies in the case of second order perturbations and non-Gaussianity with respect to issues involving causality and preheating. However, a straightforward denial of preheating influence due to causal concerns is not well founded. After all, inflation has set up initial conditions such that there exists a spatially huge oscillating inflaton condensate, which is the source of the energy during preheating. This oscillating condensate pumps its energy into pre-

\(^{17}\)Of the size of the causal particle horizon as dictated by the required amount of inflationary expansion.
existing perturbations, which thus get enhanced in circumstances known to affect the super-Hubble perturbations, i.e. in the presence of entropy perturbations.
Chapter 2

Cosmological perturbations

2.1 General

In this chapter we will go through the cosmological perturbation theory using the Bardeen approach [83] and its extension to higher orders. We will only consider the perturbations during and immediately after inflation. In the first order the connection to CMB is well established, see e.g. [6, 35]. In the second order a proper theoretical connection does not exist. First steps into that direction have, however, been taken in [84–86], where the second order Sachs–Wolfe effect has been studied, and most recently in [87], where the second order transfer function on large scales has been computed.

We also only consider scalar perturbations. The first order scalar, vector, and tensor perturbations evolve independently and can be considered separately. In the second and higher orders the same does not apply. The mixing of scalar, vector and tensor perturbations in the second order, however, has not been studied in the literature.

In cosmology one is considering a homogeneous universe, the background, and small deviations on top of that, the perturbations. Thus, one needs to compare two different manifolds, the real physical spacetime with perturbations, and the abstract homogeneous background spacetime. In order to compare these two manifolds there must be some way to map the manifolds onto each other [88, 89]. In other words, there must be some diffeomorphism between the two different manifolds providing a prescription for identifying the points. This map between the background manifold and the physical spacetime is called gauge choice.1

The first order cosmological perturbation theory dates back to the 1960’s, when Sachs and Wolfe studied the CMB perturbations [94], and it has become a standard textbook material, see e.g. [6, 35]. The study of the second order gauge issues and

1The usual Bardeen way [83] of studying perturbations introduces coordinates. There is, however, also a covariant way of studying cosmological perturbations initiated by Bruni and Ellis in [90], and recently developed by Langlois and Vernizzi [91–93]. In the first order the two approaches are computationally equivalent but at the nonlinear level there are differences.
gauge invariant perturbations, however, started quite recently in the mid-1990’s, in the context of back reaction in [95], and more generally in [88, 89]. Recently the prospects of observing non-Gaussianity have stimulated research. The second order gauge invariant perturbations have been studied in [15, 96–100].

The proper treatment of the gauge issues is outside the scope of the thesis, and we will not dwell upon it. Instead, we only very briefly mention the main points. Let us consider two coordinate systems $x^\mu$ and $\tilde{x}^\mu$, with a relating coordinate transformation
\begin{equation}
\tilde{x}^\mu = e^{\lambda L_\xi} x^\mu , \tag{2.1}
\end{equation}
where $L_\xi$ is Lie derivative with respect to the vector field $\xi$, and $\lambda$ is a dimensionless expansion parameter [88, 89]. Usually in cosmology $\lambda$ is set to one, as in the metric perturbation expansion Eqs. (2.9), (2.10), and (2.11). The change of coordinates, Eq. (2.1), transforms a tensor $T$ as:
\begin{equation}
\tilde{T} = e^{\lambda L_\xi} T . \tag{2.2}
\end{equation}

Now, if we expand the vector field $\xi^\mu$ to second order and write $\xi^\mu = \xi^\mu_{(1)} + \frac{1}{2}\xi^\mu_{(2)}$ we obtain the expansion\(^2\) (up to second order) [88, 89]
\begin{equation}
\tilde{x}^\mu = x^\mu + \lambda \xi^\mu_{(1)} + \frac{\lambda^2}{2}(\xi^\mu_{(1)}\xi^\nu_{(1)} + \xi^\mu_{(2)}) , \tag{2.4}
\end{equation}
and
\begin{equation}
\tilde{T} = T + \lambda L_{\xi_{(1)}} T + \frac{\lambda^2}{2}(L_{\xi_{(1)}} + L_{\xi_{(2)}}) T . \tag{2.5}
\end{equation}
We can now set $\lambda = 1$ and expand $T = T_0 + \delta T_1 + \frac{1}{2}\delta T_2$ to obtain
\begin{equation}
\tilde{\delta T}_1 = \delta T_1 + L_{\xi_{(1)}} T_0 , \tag{2.6}
\end{equation}
and
\begin{equation}
\tilde{\delta T}_2 = \delta T_2 + L_{\xi_{(2)}} T_0 + L_{\xi_{(1)}}^2 T_0 + 2L_{\xi_{(1)}} \delta T_1 . \tag{2.7}
\end{equation}

The starting point of the formalism is the unperturbed spatially flat Robertson–Walker metric (for a non-flat treatment in the linear perturbation theory see, e.g., [35])
\begin{equation}
ds^2 = a^2(\tau)(-d\tau^2 + dx^2) . \tag{2.8}
\end{equation}
\(^2\)Ordinary passive coordinate transformation can be defined by introducing another coordinate system $y^\mu$ which is related to the coordinate system $x^\mu$ point by point by $y^\mu(q) \equiv x^\mu(p)$. The transformation becomes [88, 89]
\begin{equation}
y^\mu = x^\mu - \lambda \xi^\mu_{(1)} + \frac{\lambda^2}{2}(\xi^\mu_{(1)}\xi^\nu_{(1)} - \xi^\mu_{(2)}) , \tag{2.3}
\end{equation}
which is used in Sec. 2.3.6 when the curvature perturbation $R$ is expanded to second order.
2.2 First order

With perturbations added up to an arbitrary order the metric can be written \[ g_{00} = -a(\tau)^2 \left( 1 + 2 \sum_{r=1}^{\infty} \frac{1}{r!} \phi^{(r)} \right), \quad (2.9) \]
\[ g_{0i} = a(\tau)^2 \sum_{r=1}^{\infty} \frac{1}{r!} \hat{\omega}_i^{(r)}, \quad (2.10) \]
\[ g_{ij} = a(\tau)^2 \left[ \left( 1 - 2 \sum_{r=1}^{\infty} \frac{1}{r!} \psi^{(r)} \right) \delta_{ij} + \sum_{r=1}^{\infty} \frac{1}{r!} \hat{\chi}_{ij}^{(r)} \right], \quad (2.11) \]

where the functions \( \phi^{(r)}, \hat{\omega}_i^{(r)}, \psi^{(r)}, \) and \( \hat{\chi}_{ij}^{(r)} \) are the \( r \)th order perturbations of the metric. The perturbations can be split into scalar, vector, and tensor parts according to their behaviour under coordinate transformations

\[ \omega_i^{(r)} = \partial_i \omega^{(r)} + \omega_i^{(r)}, \quad \hat{\chi}_{ij}^{(r)} = D_{ij} \chi^{(r)} + \partial_i \chi_j^{(r)} + \partial_j \chi_i^{(r)} + \chi_{ij}^{(r)}, \quad (2.12) \]

where \( \partial_i \omega_i^{(r)} = \partial^i \chi_i^{(r)} = 0, \partial^i \chi_{ij}^{(r)} = 0, \chi_{ij}^{(r)} = 0, \) and \( D_{ij} = \partial_i \partial_j - \frac{1}{3} \delta_{ij} \partial^k \partial_k. \) In what follows we will study the first and second order terms only.

2.2 First order

In this section we go through the standard first order cosmological perturbation theory. The linearised theory of cosmological perturbations is well established, see e.g. \[6, 35\]; here we have also followed \[34\].

2.2.1 Metric

The first order perturbed metric, Eqs. (2.9), (2.10), and (2.11), reads

\[ g_{00} = -a(\tau)^2 \left( 1 + 2 \phi^{(1)} \right), \quad (2.14) \]
\[ g_{0i} = a(\tau)^2 \left( \partial_i \omega^{(1)} + \omega_i^{(1)} \right), \quad (2.15) \]
\[ g_{ij} = a(\tau)^2 \left[ \left( 1 - 2 \psi^{(1)} \right) \delta_{ij} + D_{ij} \chi^{(1)} + \left( \partial_i \chi_j^{(1)} + \partial_j \chi_i^{(1)} + \chi_{ij}^{(1)} \right) \right]. \quad (2.16) \]

In the first order the splitting into scalar, vector, and tensor parts is especially useful. All the equations are linear in the perturbations and all three different kind of perturbations evolve independently; e.g., it does not matter for the evolution of the scalar perturbations how the vector and tensor perturbations behave. Within inflationary context the scalar perturbations are the most important. They experience instabilities and may grow in time, and are finally responsible for the CMB temperature fluctuations and act as seeds for the formation of large scale structure
in the universe. The vector perturbations die out kinematically during inflation, and the tensor perturbations just represent gravitational waves [35].

In the following we consider the scalar perturbations only and, moreover, we choose the longitudinal gauge [35]

\[
g_{\mu\nu} = a^2 \begin{pmatrix} -\frac{1}{2} (1 + 2\phi) & 0 \\ 0 & (1 - 2\psi) \delta_{ij} \end{pmatrix},
\]

where we have for now dropped the superscript (1). The inverse metric, \(g^{\mu\nu}\), can be obtained by demanding \(g_{\mu\alpha}g^{\alpha\nu} = \delta_{\nu}^{\mu}\). From the metric (2.17) one gets the left hand side of the Einstein equation, i.e. the Einstein tensor, up to the first order in perturbations, \(G_{\mu\nu} = G^{(0)}_{\mu\nu} + \delta^{(1)}G_{\mu\nu}\). The background components are [1, 15, 35]

\[
G^{(0)}_{00} = \frac{3}{2} a^2 \left( \frac{a'}{a} \right)^2, \\
G^{(0)}_{ij} = \frac{1}{2a^2} \left[ 2 \frac{a''}{a} - \left( \frac{a'}{a} \right)^2 \right] \delta_{ij}, \\
G^{(0)}_{i} = G^{(0)}_{i0} = 0,
\]

while the first order components read as

\[
\delta^{(1)}G_{00} = a^{-2} \left( 6 \left( \frac{a'}{a} \right)^2 \phi^{(1)} + 6 \frac{a'}{a} \psi^{(1)} - 2 \partial_i \partial^i \psi^{(1)} \right), \\
\delta^{(1)}G_{0i} = a^{-2} \left( -2 \frac{a'}{a} \partial_i \phi^{(1)} - 2 \partial_i \psi^{(1)} \right), \\
\delta^{(1)}G_{ij} = a^{-2} \left[ \left( 2 \frac{a'}{a} \phi^{(1)} + 4 \frac{a''}{a} \phi^{(1)} - 2 \left( \frac{a'}{a} \right)^2 \phi^{(1)} + \partial_k \partial^k \phi^{(1)} + 4 \frac{a'}{a} \psi^{(1)} \right) \delta_{ij} - \partial^i \partial_j \phi^{(1)} + \partial^i \partial_j \psi^{(1)} \right].
\]

### 2.2.2 Energy–momentum tensor

Next we consider the energy–momentum tensor, \(T_{\mu\nu}\), for two scalar fields, which are minimally coupled to gravity. The tensor is given by [6]

\[
T_{\mu\nu} = \partial_{\mu} \varphi \partial_{\nu} \varphi + \partial_{\mu} \sigma \partial_{\nu} \sigma - g_{\mu\nu} \left( \frac{1}{2} g^{\alpha\beta} \partial_{\alpha} \varphi \partial_{\beta} \varphi + \frac{1}{2} g^{\alpha\beta} \partial_{\alpha} \sigma \partial_{\beta} \sigma + V(\varphi, \sigma) \right),
\]

where \(V(\varphi, \sigma)\) is the potential for the scalar fields; \(\varphi\) is the inflaton. The scalar fields are divided into a homogeneous part, denoted by subscript 0, and a perturbation, denoted by subscript 1, as follows:

\[
\varphi(\tau, x) = \varphi_0(\tau) + \varphi_1(\tau, x), \\
\sigma(\tau, x) = \sigma_0(\tau) + \sigma_1(\tau, x).
\]
As with the Einstein tensor previously the energy–momentum tensor can be divided into homogeneous part and perturbation, \( T_{\mu\nu} = T_{\mu\nu}^{(0)} + \delta^{(1)}T_{\mu\nu} \). In the longitudinal gauge, Eq. (2.17), the background components are \[ T_{00}^{(0)} = a^{-2} \left( -\frac{1}{2} \phi_0' \phi_0' - \frac{1}{2} \sigma_0' \sigma_0' - a^2 V_0 \right), \] \[ T_{ij}^{(0)} = a^{-2} \left[ \frac{1}{2} \left( \phi_0'^2 + \sigma_0'^2 \right) - a^2 V_0 \right] \delta_{ij}, \] and the perturbation components read as

\[ \delta^{(1)} T_{00} = a^{-2} \left[ -\phi_0' \phi_1' - \sigma_0' \sigma_1' + \left( \phi_0'^2 + \sigma_0'^2 \right) \phi^{(1)} \right] - a^2 \left( \frac{\partial V}{\partial \phi} \phi_1 + \frac{\partial V}{\partial \sigma} \sigma_1 \right), \]

\[ \delta^{(1)} T_{ij} = a^{-2} \left[ \phi_0' \phi_1' + \sigma_0' \sigma_1' - \left( \phi_0'^2 + \sigma_0'^2 \right) \phi^{(1)} \right] - a^2 \left( \frac{\partial V}{\partial \phi} \phi_1 + \frac{\partial V}{\partial \sigma} \sigma_1 \right) \delta_{ij}, \]

where we denote \( V_0 \equiv V(\phi_0, \sigma_0) \).

The first order perturbation of the energy–momentum conservation, \( \nabla_{\mu} T^{\mu\nu} = 0 \), yields\(^3\)

\[ \phi''_1 + 2H \phi'_1 - \partial_i \partial^i \phi_1' + a^2 \left( \frac{\partial^2 V}{\partial \phi^2} \phi_1' + \frac{\partial^2 V}{\partial \phi \partial \sigma} \sigma_1 \right) = 4\phi^{(1)'} \phi'_0 - 2a^2 \phi^{(1)} \frac{\partial V}{\partial \phi}, \]

\[ \sigma''_1 + 2H \sigma'_1 - \partial_i \partial^i \sigma_1' + a^2 \left( \frac{\partial^2 V}{\partial \sigma^2} \sigma_1' + \frac{\partial^2 V}{\partial \phi \partial \sigma} \phi_1 \right) = 4\phi^{(1)'} \sigma'_0 - 2a^2 \phi^{(1)} \frac{\partial V}{\partial \sigma}, \]

which are often applied without the terms with metric perturbations [35].

### 2.2.3 Gauge transformations in the first order

Since we are only interested in the scalar perturbations, we only consider transformation \( \xi^{(1)} = (\alpha_1, \partial^i \beta_1) \) [89], i.e., we have dropped the intrinsic vector part from \( \xi^{(1)} \).

\(^3\)Remember that in conformal time, \( \tau \), the Hubble parameter \( H \equiv a'/a \).
We write to the first order \( g_{\mu\nu} = g_{\mu\nu}^{(0)} + \delta^{(1)} g_{\mu\nu} \). According to Eq. (2.6) we have
\[
\delta^{(1)} g_{\mu\nu} = \delta^{(1)} g_{\mu\nu} + \mathcal{L}_{\xi^{(1)}} g_{\mu\nu}^{(0)} .
\] (2.31)
This gives the transformation properties of the scalar metric perturbations [89]
\[
\tilde{\phi}^{(1)} = \phi^{(1)} + \mathcal{H} \alpha_1 + \alpha' \quad \text{(2.32)}
\]
\[
\tilde{\psi}^{(1)} = \psi^{(1)} - \mathcal{H} \alpha_1 - \frac{1}{3} \nabla^2 \beta_1 .
\] (2.33)
For a scalar, such as the energy density \( \rho \) or the scalar field \( \varphi \), Eq. (2.23), we have [89]
\[
\tilde{\varphi}_1 = \varphi_1 + \varphi' \alpha_1 .
\] (2.34)
The first order metric perturbation \( \omega^{(1)} \), Eq. (2.12), which we need in Sec. 2.3.6, changes simply by [89]
\[
\tilde{\omega}^{(1)} = \omega^{(1)} - \alpha_1 ,
\] (2.35)
when \( \beta_1 = 0 \).

### 2.2.4 Some useful quantities

We are now ready to write the Einstein equations (1.24) up to first order in perturbations, but let us first define some useful quantities.

For a fixed time, \( \tau \), one can consider the spatial part of the metric, \( g_{ij} \), and calculate the corresponding spatial curvature scalar \([6, 34]\)
\[
(3) R = \frac{4}{a^2} \nabla^2 \psi .
\] (2.36)
This quantity, however, is not gauge independent since under a change of the time slicing \( \tau \rightarrow \tau + \delta \tau \) one has \( \psi \rightarrow \psi + \mathcal{H} \delta \tau \). Instead one can define the comoving curvature perturbation, or the curvature perturbation on hypersurfaces orthogonal to the worldlines of comoving observers, in the presence of a single scalar field \( \varphi \): \([6, 34]\)
\[
\mathcal{R} \equiv \psi + \mathcal{H} \frac{\varphi_1}{\varphi'} \left( = \psi + H \frac{\varphi_1}{\varphi} \right)
\] (2.37)
by considering time transformations between a generic slicing and a comoving slicing with \( \varphi_1 = 0 \). In the case of two scalar fields, \( \varphi \) and \( \sigma \), the quantity generalises to \([101]\)
\[
\mathcal{R} \equiv \psi + \mathcal{H} \left( \frac{\varphi' \varphi_1 + \sigma' \sigma_1}{\varphi'^2 + \sigma'^2} \right) .
\] (2.38)
Another useful, and related, quantity is the curvature perturbation on slices of uniform energy density \([34, 101]\)
\[
-\zeta \equiv \psi + \mathcal{H} \frac{\delta \rho}{\rho'} \left( = \psi + H \frac{\delta \rho}{\rho} \right) ,
\] (2.39)
2.2 First order

where the sign is just a matter of convention and varies between different authors. The definition (2.39) does not care for the actual contents of the universe, just the energy density. There is a relation between the two curvature perturbations\(^4\) [101]

\[-\zeta = \mathcal{R} + \frac{2\rho}{3(\rho + p)} \left( \frac{k}{aH} \right)^2 \Psi, \quad (2.40)\]

which can be obtained by a gauge transformation; \(\Psi\) is the usual gauge invariant Bardeen potential [34, 83]. The important point to notice here is that the two definitions (barring the different sign) coincide on large scales, \(k \to 0\).

In order for the curvature perturbation to evolve outside horizon, there has to be isocurvature perturbations present. In Papers [1–4] we are considering models with two scalar fields and, therefore, we have isocurvature (or entropy) degrees of freedom, and possibility for super-horizon evolution of \(\mathcal{R}\).

Sasaki–Mukhanov variable is defined by [34, 102, 103]

\[Q \equiv \varphi_1 + \frac{\varphi'}{H} \psi \quad \left( = \varphi_1 + \frac{\varphi'}{H} \psi \right) \quad (2.41)\]

and is obviously related to the curvature perturbation \(\mathcal{R}\) by \(Q = (\varphi'/H)\mathcal{R}\). Physically one can describe the different quantities as follows [34]: \(\mathcal{R}\) is the gravitational potential on comoving hypersurfaces, \(\mathcal{R} = \psi|_{\delta\rho = 0}\); \(\zeta\) is the gravitational potential on uniform density slices, \(-\zeta = \psi|_{\delta\rho = 0}\); and \(Q\) is the scalar field perturbation on a spatially flat slice, \(Q = \varphi_1|_{\psi = 0}\).

2.2.5 Evolution of the first order metric perturbation

The evolution of the metric perturbations is governed by the Einstein equations. In the longitudinal gauge (2.17) with two scalar fields the first order perturbed equations \(\delta^{(1)}G^0_0 = \frac{1}{M_P^2} \delta^{(1)}T^0_0\), \(\delta^{(1)}G^i_i = \frac{1}{M_P^2} \delta^{(1)}T^i_i\) and \(\delta^{(1)}G^i_j = \frac{1}{M_P^2} \delta^{(1)}T^i_j\) take respectively the forms

\[
6H^2\phi + 6H\psi' - 2\partial_i\psi^i = \frac{1}{M_P^2} \left[ -\varphi_0' \varphi_1' - \sigma_0' \sigma_1' + w^2\phi - a^2 \left( \frac{\partial V}{\partial \varphi} \varphi_1 + \frac{\partial V}{\partial \sigma} \sigma_1 \right) \right],
\]

\[
2\partial_i\phi + 2\partial_i\psi' = \frac{1}{M_P^2} \left( \varphi_0' \partial_i\varphi_1 + \sigma_0' \partial_i\sigma_1 \right),
\]

\[
\left( 2\partial^2 + \frac{a''}{a} \phi - 2H^2 \phi + \partial_k \partial^k \phi + 4H\psi' + 2\psi'' - \partial_k \partial^k \psi \right) \delta^i_j
\]

\[
+ \partial^i \partial_j (\psi - \phi) = \frac{1}{M_P^2} \left[ \varphi_0' \varphi_1' + \sigma_0' \sigma_1' - w^2\phi - a^2 \left( \frac{\partial V}{\partial \varphi} \varphi_1 + \frac{\partial V}{\partial \sigma} \sigma_1 \right) \right] \delta^i_j,
\]

\(^4\)Note that the relation is in cosmic time \(t\).
where we denote $w^2 \equiv \varphi_0^2 + \sigma_0^2$. Eqs. (2.42) can readily be read from the perturbed Einstein tensor and energy-momentum tensor, Eqs. (2.21) and (2.28), respectively.

Let us next, as an example, consider single field inflation and drop the terms involving $\sigma$. When the energy density of the universe consists of scalar field(s) the metric perturbations are equal, $\psi = \phi$ [35].

The 00 component and the diagonal part ($i = j$) of the $ij$ component of the Einstein equation can be subtracted to obtain

$$2\phi'' + 12H\phi' + \left(4H^2 + \frac{a''}{a}\right)\phi - 2\partial_i\partial^i\phi = -\frac{2}{M_P^2}a^2V'\varphi_1.$$  \hspace{1cm} (2.43)

From the 0$i$ component we obtain

$$2\phi' + 2H\phi = \frac{1}{M_P^2}\varphi_1.$$ \hspace{1cm} (2.44)

Eq. (2.44) and the background field equation $\varphi'' + 2H\varphi' = -a^2V'$ are used to get rid of the R.H.S. of Eq. (2.43). The result finally reads

$$\phi'' + 2\left(H - \frac{\varphi''}{\varphi}\right)\phi' + 2\left(\frac{H'}{H} - \frac{\varphi''}{\varphi}\right)\phi - \partial_i\partial^i\phi = 0.$$  \hspace{1cm} (2.45)

This is an important equation describing the evolution of the gravitational potential $\phi$.

We write Eq. (2.45) in cosmic time $t$, $(dt = ad\tau)$,

$$\ddot{\phi} + \left(H - \frac{2\dot{\varphi}}{\varphi}\right)\dot{\phi} + 2\left(\frac{H'}{H} - \frac{\dot{\varphi}}{\varphi}\right)\dot{\phi} - \partial_i\partial^i\phi = 0.$$ \hspace{1cm} (2.46)

This way we can readily apply the slow-roll relations [34]

$$\frac{\dot{H}}{H^2} = -\epsilon,$$ \hspace{1cm} (2.47)

$$\frac{\dot{\varphi}}{H\dot{\varphi}} = \epsilon - \eta.$$ \hspace{1cm} (2.48)

and write Eq. (2.46) as

$$\ddot{\phi} + (1 + 2\eta - 2\epsilon)\dot{\phi} + 2H^2(\eta - 2\epsilon)\phi \simeq 0,$$ \hspace{1cm} (2.49)

which is valid for large scales ($k \ll aH$). Here we have dropped the gradient term. Eq. (2.49) is fulfilled by $\dot{\phi} \sim 2H(2\epsilon - \eta)\phi$ up to first order in slow-roll parameters. Thus, we can conclude that on large scales $\phi \sim constant$.

Eq. (2.44) becomes $\dot{\phi} + H\phi = \epsilon H^2\varphi_1/\varphi$. Thus, on large scales we can write

$$\psi_k = \phi_k \simeq \epsilon H\frac{\varphi_k}{\varphi}.$$ \hspace{1cm} (2.50)
for the Fourier modes, ($\varphi_k$ is the Fourier transform of $\varphi_1$). This gives the Fourier modes of the comoving curvature perturbation $R$ on large scales \[34\]

$$ R_k = \psi_k + H \frac{\varphi_k}{\dot{\varphi}} = (1 + \epsilon)H \frac{\varphi_k}{\dot{\varphi}} \simeq H \frac{\varphi_k}{\dot{\varphi}} . \quad (2.51) $$

Indeed, we know that on large scales $\dot{R}_k \sim 0$, or more exactly $\dot{R} \simeq \frac{H}{\rho_p} \delta p_{nad}$, where $\delta p_{nad}$ is the non-adiabatic pressure perturbation \[34, 101\]. It is zero in the case of single scalar field.

Eq. (2.51) enables us to write the spectrum for the curvature scalar $R$:

$$ P_R = \left( \frac{H}{\dot{\varphi}} \right)^2 P_{\varphi_1} = \frac{k^3}{2\pi^2} \left( \frac{H}{\dot{\varphi}} \right)^2 |\varphi_k|^2 . \quad (2.52) $$

Thus, we simply need the spectrum for $\varphi_1$. Instead of the simplified examples before, in Sec. 1.3.6, we use the proper perturbed Klein–Gordon equations, where the metric perturbations are included. In the longitudinal gauge \[34\]

$$ \ddot{\varphi}_k + 3H \dot{\varphi}_k + \frac{k^2}{a^2} \varphi_k + V'' \varphi_k = -2\psi_k V' + 4 \dot{\psi}_k \dot{\varphi} . \quad (2.53) $$

Now, on large scales $|4 \dot{\psi}_k \dot{\varphi}| \ll |2\psi_k V'|$. By applying the slow roll equation $V' \simeq -3H \dot{\varphi}$ and Eq. (2.50) we end up with

$$ \ddot{\varphi}_k + 3H \dot{\varphi}_k + \frac{k^2}{a^2} \varphi_k + (V'' + 6\epsilon H^2) \varphi_k \simeq 0 . \quad (2.54) $$

Since the slow roll parameters are considered to be roughly constant, by switching to conformal time $\tau$ and making a change of variable $u_k \equiv \varphi_k/a$, we end up with

$$ u_k'' + \left[ k^2 - \frac{1}{\tau^2} \left( \nu^2 - \frac{1}{4} \right) \right] u_k' = 0 , \quad (2.55) $$

where $\nu \simeq \frac{3}{2} + 3\epsilon - \eta$. This the same equation as Eq. (1.63) but with a slightly different $\nu$. Thus, by considering Eqs. (1.63), (1.71), and (2.52) we can immediately conclude that the spectral index of the curvature perturbation $R$ is

$$ n_R - 1 = 3 - 2\nu = 2\eta - 6\epsilon . \quad (2.56) $$

### 2.3 Second order

In this section we go through the second order perturbation theory formalism as described in the case of single scalar field in \[15\] and two scalar fields in \[1\].
2.3.1 Metric

Again, the starting point is the metric perturbation up to arbitrary order, Eqs. (2.9), (2.10), and (2.11), with the standard splitting into scalar, vector, and tensor components, Eqs. (2.12), and (2.13). As in the first order case we consider only scalar field perturbations here. We, therefore, neglect $\omega_i^{(1)}$, $\chi_i^{(1)}$, and $\chi_{ij}^{(1)}$, thus obtaining the metric [1]

\[
g_{00} = -a(\tau)^2 \left( 1 + 2\phi^{(1)} + \phi^{(2)} \right), \tag{2.57}
\]

\[
g_{0i} = a(\tau)^2 \left( \partial_i \omega^{(1)} + \frac{1}{2} \partial_i \omega^{(2)} + \frac{1}{2} \omega_i^{(2)} \right), \tag{2.58}
\]

\[
g_{ij} = a(\tau)^2 \left[ (1 - 2\psi^{(1)} - \psi^{(2)}) \delta_{ij} + D_{ij} \left( \chi^{(1)} + \frac{1}{2} \chi^{(2)} \right) \right. \\
\left. + \frac{1}{2} \left( \partial_i \chi_j^{(2)} + \partial_j \chi_i^{(2)} + \chi_{ij}^{(2)} \right) \right]. \tag{2.59}
\]

In addition we will adopt the generalised longitudinal gauge [88, 89], and set $\omega^{(1)} = \omega^{(2)} = 0$ and $\chi^{(1)} = \chi^{(2)} = 0$. This renders the metric into the form

\[
g_{00} = -a(\tau)^2 \left( 1 + 2\phi^{(1)} + \phi^{(2)} \right), \tag{2.60}
\]

\[
g_{0i} = 0, \tag{2.61}
\]

\[
g_{ij} = a(\tau)^2 \left[ (1 - 2\psi^{(1)} - \psi^{(2)}) \delta_{ij} + \frac{1}{2} \left( \partial_i \chi_j^{(2)} + \partial_j \chi_i^{(2)} + \chi_{ij}^{(2)} \right) \right]. \tag{2.62}
\]

The components if the Einstein tensor can be computed using this metric, and by expanding $G^{\mu \nu} = G^{\mu \nu}^{(0)} + \delta^{(1)} G^{\mu \nu} + \frac{1}{2} \delta^{(2)} G^{\mu \nu}$. We do not, however, write them here explicitly. They can be found in Paper [1] or originally in [15].

2.3.2 Energy–momentum tensor

The energy–momentum tensor, $T_{\mu \nu}$, for two scalar fields is given by Eq. (2.22), as before. Only this time the expansion, Eqs. (2.23) and (2.24) is done to second order

\[
\varphi(\tau, \mathbf{x}) = \varphi_0(\tau) + \varphi_1(\tau, \mathbf{x}) + \frac{1}{2} \varphi_2(\tau, \mathbf{x}), \tag{2.63}
\]

\[
\sigma(\tau, \mathbf{x}) = \sigma_0(\tau) + \sigma_1(\tau, \mathbf{x}) + \frac{1}{2} \sigma_2(\tau, \mathbf{x}), \tag{2.64}
\]

and $T^{\mu \nu} = T^{\mu \nu}^{(0)} + \delta^{(1)} T^{\mu \nu} + \frac{1}{2} \delta^{(2)} T^{\mu \nu}$. We do not write the components of the energy–momentum tensor explicitly here, either. They can be read from the appendix of Paper [1].
2.3 Second order

2.3.3 Gauge transformations in the second order

We write to the second order $g_{\mu\nu} = g^{(0)}_{\mu\nu} + \delta^{(1)} g_{\mu\nu} + \frac{1}{2} \delta^{(2)} g_{\mu\nu}$ and, according to Eq. (2.7), we have

$$
\delta^{(2)} g_{\mu\nu} = \delta^{(2)} g_{\mu\nu} + \mathcal{L}_{\xi(2)} g^{(0)}_{\mu\nu} + \mathcal{L}^2_{\xi(1)} g^{(0)}_{\mu\nu} + 2 \mathcal{L}_{\xi(1)} \delta^{(1)} g_{\mu\nu} .
$$

(2.65)

For our purposes it is sufficient to consider only coordinate transformation $\xi^\mu \equiv (\alpha_r, 0)$, both in first and second order, $r = 1, 2$, respectively. This is the only thing needed in Sec. 2.2.3, where we consider the second order curvature perturbation.

The transformation of the scalar metric perturbation $\psi^{(2)}$ becomes [1, 89]

$$
\widetilde{\psi}^{(2)} = \psi^{(2)} + 2\alpha_1 \left( \psi^{(1)}\prime + 2\mathcal{H}^{(1)} \right) - \left( \mathcal{H}' + 2\mathcal{H}^2 \right) (\alpha_1)^2 - \mathcal{H} \alpha_1' \alpha_1 - \mathcal{H} \alpha_2 - \frac{1}{3} \left( 2\mathcal{D} \omega^{(1)} - \partial' \alpha_1 \right) \partial \alpha_1 .
$$

(2.66)

For a scalar, such as energy density $\rho$ or the field $\varphi$, Eq. (2.63), we have [89]

$$
\widetilde{\varphi} = \varphi + \left( \mathcal{L}_{\xi(2)} + \mathcal{L}^2_{\xi(1)} \right) \varphi + 2 \mathcal{L}_{\xi(1)} \varphi_1 = \varphi + \alpha_1 (b_{0} \alpha + b_{0}' \alpha + b_{1}' \alpha_1 + b_{2}' \alpha_2) .
$$

(2.67)

2.3.4 Second order Einstein equations for two scalar fields

Next step is to write out the second order Einstein equations. Again, setting $\psi^{(1)} = \phi^{(1)}$, the components, $\delta^{(2)} G^{0}_{0} = \frac{1}{M^2} \delta^{(2)} T^{0}_{0}$, $\delta^{(2)} G^{i}_{0} = \frac{1}{M^2} \delta^{(2)} T^{i}_{0}$, and $\delta^{(2)} G^{i}_{j} = \frac{1}{M^2} \delta^{(2)} T^{i}_{j}$, respectively read as [1]

$$
\mathcal{H}^2 \phi^{(2)} + \frac{a''}{a} \phi^{(2)} + 3 \mathcal{H} \psi^{(2)} + - \partial_i \partial^i \phi^{(2)} - 12 \mathcal{H}^2 (\phi^{(1)})^2 - 3 \partial_i \phi^{(1)} \partial^i \phi^{(1)}
$$

- $8 \phi^{(1)} \partial_i \partial^i \phi^{(1)} - 3 (\phi^{(1)})^2 = \frac{1}{M^2} \left\{ - \frac{1}{2} (\phi_{0} \varphi_2 + \sigma_{0} \phi_2) - \frac{1}{2} (\phi_{1})^2 + (\phi_{1})^2 \right\}
$$

- $2 (\phi_{0} \phi_{1} + \sigma_{0} \phi_{1}) \phi^{(1)} - 2 w^2 (\phi^{(1)})^2 - \frac{1}{2} (\partial_i \phi_1 \partial^i \varphi_1 + \partial_i \sigma_1 \partial^i \sigma_1)
$$

- $\frac{a^2}{2} \left[ \frac{\partial V}{\partial \varphi} \varphi + \frac{\partial V}{\partial \sigma} \sigma + \frac{\partial^2 V}{\partial \varphi^2} (\varphi)^2 + \frac{\partial^2 V}{\partial \sigma^2} (\sigma)^2 + 2 \frac{\partial^2 V}{\partial \varphi \partial \sigma} \varphi \sigma \right] \right\} ,
$$

$$
\mathcal{H} \partial^i \phi^{(2)} + \partial^i \psi^{(2)} + \frac{1}{4} \partial_k \partial^i \chi^{(2)} + 2 \phi^{(1)} \partial^i \phi^{(1)} + 8 \phi^{(1)} \partial^i \phi^{(1)}
$$

$$
= \frac{1}{M^2} \left\{ \frac{1}{2} (\varphi_{0} \partial^i \varphi_2 + \sigma_{0} \partial^i \sigma_2) + \varphi_{1} \partial^i \varphi_1 + \sigma_{1} \partial^i \sigma_1
$$

+ $2 (\varphi_{0} \partial^i \varphi_1 + \sigma_{0} \partial^i \sigma_1) \phi^{(1)} \right\} ,
$$

(2.68)
\[
\begin{align*}
&\left[ \frac{1}{2} \partial_i \partial^k \phi^{(2)} + \mathcal{H} \phi^{(2)'} + \frac{a''}{a} \phi^{(2)} + \mathcal{H}^2 \phi^{(2)} - \frac{1}{2} \partial_i \partial^k \psi^{(2)} + \psi^{(2)''} \right. \\
&+ 2 \mathcal{H} \phi^{(2)'} + 4 \mathcal{H}^2 \left( \phi^{(1)} \right)^2 - 8 \frac{a''}{a} \left( \phi^{(1)} \right)^2 - 8 \mathcal{H} \phi^{(1)} \phi^{(1)'} - 3 \partial_i \phi^{(1)} \partial^i \phi^{(1)} \\
&- \left. 4 \phi^{(1)} \partial_i \partial^k \phi^{(1)} - \left( \phi^{(1)'} \right)^2 \right] \delta^i_j - \frac{1}{2} \partial_i \partial_j \phi^{(2)} + \frac{1}{2} \partial^i \partial_j \psi^{(2)} \\
&+ \frac{1}{2} \mathcal{H} \left( \partial^i \chi_j^{(2)'} + \partial_j \chi_i^{(2)'} + \chi_i^{(2)''} \right) + \frac{1}{4} \left( \partial^i \chi_j^{(2)''} + \partial_j \chi_i^{(2)''} + \chi_j^{(2)''} \right) \\
&- \frac{1}{4} \partial_i \partial^k \chi_j^{(2)} + 2 \partial^i \phi^{(1)} \partial_j \phi^{(1)} + 4 \phi^{(1)} \partial^i \partial_j \phi^{(1)} \\
&= \frac{1}{M_P^2} \left\{ \left[ \frac{1}{2} \left( \varphi_0' \varphi_2' + \sigma_0' \sigma_2' \right) + \frac{1}{2} \left( \left( \varphi_1' \right)^2 + \left( \sigma_1' \right)^2 \right) \right. \\
&- 2 \left( \varphi_0' \varphi_1' + \sigma_0' \sigma_1' \right) \phi^{(1)} + 2 w^2 \left( \phi^{(1)} \right)^2 \\
&- \frac{1}{2} \left( \partial_k \varphi_1 \partial^k \varphi_1 + \partial_k \sigma_1 \partial^k \sigma_1 \right) \\
&- \frac{a^2}{2} \left( \frac{\partial V}{\partial \varphi_2} + \frac{\partial V}{\partial \sigma_2} \right) - \frac{\partial^2 V}{\partial \varphi^2} \left( \varphi_1 \right)^2 + \frac{\partial^2 V}{\partial \sigma^2} \left( \sigma_1 \right)^2 + 2 \frac{\partial^2 V}{\partial \varphi \partial \sigma} \varphi_1 \sigma_1) \right] \delta^i_j \\
&+ \partial_i \varphi_1 \partial_j \varphi_1 + \partial^i \sigma_1 \partial_j \sigma_1 \right\},
\end{align*}
\]

where we have defined \( w^2 \equiv \varphi_0' \varphi_2' + \sigma_0' \sigma_2' \). In writing the 00 and \( ij \) components we have made use of the background equality \( \frac{1}{M_P^2} w^2/2 = \mathcal{H}^2 - \mathcal{H}' \) together with the relation \( a''/a = \mathcal{H}^2 + \mathcal{H}' \) which follows directly from the definition of \( \mathcal{H} \).

### 2.3.5 Master equation for \( \phi^{(2)} \)

Next we take a divergence (\( \partial_i = \delta_{ij} \partial^j \)) of the i0 component of Eq. (2.68) and act on the result with the inverse of the spatial Laplace operator, \( \Delta^{-1} \), which is an operator defined to cancel the effect of Laplacian, i.e. \( \Delta^{-1} \partial_i \partial^i A = A \). The result may be written as [1]

\[
\frac{1}{2} \left( \varphi_0 \varphi_2 + \sigma_0 \sigma_2 \right) = M_P^2 \left( \psi^{(2)'} + \mathcal{H} \phi^{(2)} + \Delta^{-1} \alpha \right) - \Delta^{-1} \beta, \tag{2.69}
\]

where

\[
\begin{align*}
\alpha &= 2 \phi^{(1)'} \partial_i \partial^i \phi^{(1)} + 10 \partial_i \phi^{(1)'} \partial^i \phi^{(1)} + 8 \phi^{(1)} \partial_i \partial^i \phi^{(1)}' \\
\beta &= \partial_i \varphi_1 \partial^i \varphi_1 + \partial_i \sigma_1 \partial^i \sigma_1 + \varphi_1' \partial_i \partial^i \varphi_1 + \sigma_1' \partial_i \partial^i \sigma_1 \\
&+ 2 \phi^{(1)} \partial_i \partial^i \left( \varphi_0 \varphi_1 + \sigma_0 \sigma_1 \right) + 2 \partial_i \phi^{(1)'} \partial^i \left( \varphi_0 \varphi_1 + \sigma_0 \sigma_1 \right).
\end{align*}
\tag{2.70}
\]

We also take a trace of the \( ij \) component of Eq. (2.68) and act on it with \( \Delta^{-1} \). The result reads [1]

\[
\psi^{(2)} = \phi^{(2)} - \Delta^{-1} \gamma, \tag{2.71}
\]
where

\[
\frac{1}{3} \gamma = \frac{8}{a} \frac{a''}{a'} (\phi^{(1)})^2 - 4 \mathcal{H}^2 (\phi^{(1)})^2 + 8 \mathcal{H} \phi^{(1)} \phi^{(1)'} + \frac{7}{3} \partial_i \phi^{(1)} \partial_i \phi^{(1)} \\
+ \frac{8}{3} \phi^{(1)} \partial_i \phi^{(1)} + (\phi^{(1)'})^2 + \Delta^{-1} a' + 2 \mathcal{H} \Delta^{-1} a - \frac{1}{M_P^2} \Delta^{-1} b' \\
- 2 \mathcal{H} \frac{1}{M_P^2} \Delta^{-1} b + \frac{1}{M_P^2} \left\{ \frac{1}{2} \left( (\sigma_1')^2 + (\sigma_1')^2 \right) \right\} \\
- 2 (\sigma_0 \sigma_1 + \sigma_0 \sigma_1') \phi^{(1)} + 2 w^2 (\phi^{(1)})^2 \\
- \frac{1}{6} \left( \partial_i \phi \partial_i \phi + \partial_i \sigma \partial_i \sigma \right) \\
- \frac{a^2}{2} \left[ \frac{\partial^2 V}{\partial \phi^2} (\phi_1)^2 + \frac{\partial^2 V}{\partial \sigma^2} (\sigma_1)^2 + 2 \frac{\partial^2 V}{\partial \phi \partial \sigma} \phi_1 \sigma_1 \right] \right\} .
\]

(2.72)

Now, we start with the 00 component of Eq. (2.68) and apply Eqs. (2.69) and (2.71). After some manipulation the result finally reads [1]

\[
\phi^{(2)''} - \partial_i \partial_i \phi^{(2)} + 2 \mathcal{H} \phi^{(2)'} + 2 \mathcal{H}^2 \phi^{(2)} = 12 \mathcal{H}^2 \phi^{(1)} \phi^{(1)'} + 3 \phi^{(1)} + 8 \phi^{(1)} \partial_i \partial_i \phi^{(1)} \\
+ 3 \partial_i \phi^{(1)} \partial_i \phi^{(1)} + 2 \mathcal{H} \Delta^{-1} a - 2 \mathcal{H} \frac{1}{M_P^2} \Delta^{-1} b - 2 \mathcal{H} \Delta^{-1} \gamma' - \Delta^{-1} a' + \frac{1}{M_P^2} \Delta^{-1} b' \\
+ \Delta^{-1} \gamma'' + 3 \mathcal{H} \Delta^{-1} \gamma' - \gamma + \frac{1}{M_P^2} \left\{ \phi_0 \phi_2 + \phi_0 \phi_2 - \frac{1}{2} \left( (\sigma')^2 + (\sigma')^2 \right) \right\} \\
- \frac{1}{2} \left( \partial_i \phi_1 \partial_i \phi_1 + \partial_i \sigma_1 \partial_i \sigma_1 \right) - 2 w^2 (\phi^{(1)})^2 + 2 (\sigma_0 \sigma_1') \phi^{(1)} \\
- \frac{a^2}{2} \left[ \frac{\partial^2 V}{\partial \phi^2} (\phi_1)^2 + \frac{\partial^2 V}{\partial \sigma^2} (\sigma_1)^2 + 2 \frac{\partial^2 V}{\partial \phi \partial \sigma} \phi_1 \sigma_1 \right] \right\} .
\]

(2.73)

This is our master equation for the second order metric perturbation \( \phi^{(2)} \). The important point is that this equation is exact, we have not made any approximations and it applies to all models. However, Eq. (2.73) is quite complicated and it is difficult to proceed without making some approximations or assumptions. This is not a unique problem for this particular equation, though, see e.g. [100] for a discussion on the general problems for obtaining a closed set of equation in the second order.

One possible way to proceed, and the one which we choose here, is to restrict the treatment to models, where \( \sigma_0 = 0 \) and \( \partial^2 V/\partial \phi \partial \sigma = 0 \). Even after this restriction our treatment still covers several interesting models, most notably hybrid inflation, see Section 1.3.5, and the Linde–Mukhanov model [104]. The main consequence of the restrictions is the decoupling of the second scalar field. In the first order Einstein equations \( \sigma \) does not appear at all, it is only described by its Klein–Gordon equation in the first order. In the second order equations the contribution of \( \sigma \) is completely additive, there are no terms which would mix \( \sigma \) and \( \phi \).

Most importantly, we can now solve \( \phi_2 \) from Eq. (2.69) and get rid of the second order scalar field fluctuations \( \phi_2 \) and \( \sigma_2 \). After some algebra the master equation
(2.73) for the second order metric perturbation $\phi^{(2)}$ reads [1]

$$
\phi^{(2)''} - \partial_i \partial^i \phi^{(2)} + 2 \left( \mathcal{H} - \frac{\phi^{(1)''}}{\phi^{(1)}} \right) \phi^{(2)'} + 2 \left( \mathcal{H}' - \frac{\phi^{(1)''}}{\phi^{(1)} H} \right) \phi^{(2)}
= 12 H^2 \left( \phi^{(1)} \right)^2 + 3 \left( \phi^{(1)'} \right)^2 + 8 \phi^{(1)} \partial_i \partial^i \phi^{(1)} + 3 \partial_i \phi^{(1)} \partial^i \phi^{(1)} + 2 \left( \mathcal{H} + \frac{\phi^{(1)''}}{\phi^{(1)}} \right) \Delta^{-1} \alpha - 2 \frac{1}{M^2_P} \left( \Delta^{-1} \beta' + \gamma' - \mathcal{H} - 2 \frac{\phi^{(1)''}}{\phi^{(1)}} \right) \Delta^{-1} \gamma' + \Delta^{-1} \gamma'' - \frac{1}{M^2_P} \left\{ - \frac{1}{2} \left( \phi^{(1)} \right)^2 \left( \sigma^{(1)} \right)^2 - \frac{1}{2} \left. \left( \partial_i \phi^{(1)} \partial^i \phi^{(1)} + \partial_i \sigma^{(1)} \partial^i \sigma^{(1)} \right) \right. - 2 \phi^{(1)''} \right\}
+ 2 \phi^{(1)'} \phi^{(1)} - \frac{a^2}{2} \left[ \frac{\partial^2 V}{\partial \varphi^2} (\varphi^{(1)})^2 + \frac{\partial^2 V}{\partial \sigma^2} (\sigma^{(1)})^2 \right] .
$$

(2.74)

Still, apart from the slight restrictions on the model considered, we have not made any approximations in Eq. (2.74). Also, one should note that we have been able to cast the equation in the form where the left hand side containing the second order metric perturbation $\phi^{(2)}$ is exactly of the same form as in the linear Eq. (2.45). The right hand side of Eq. (2.74), i.e. the source terms, consists completely of terms quadratic in the first order perturbations. Moreover, the contribution to the quadratic source from the field $\sigma$ is completely additive. This fact will turn out to be extremely useful later in Chapter 3 when we consider the non-Gaussianities of different models.

### 2.3.6 Curvature perturbation

Next we derive the second order gauge-invariant comoving curvature perturbation for two scalar fields, as done in [1], (see also [15, 105]). We denote the gauge-invariant curvature perturbation by $\mathcal{R}$ and expand it up to the second order in the already familiar way

$$
\mathcal{R} = \mathcal{R}^{(1)} + \frac{1}{2} \mathcal{R}^{(2)} .
$$

(2.75)

We are interested in the second order part. Our starting point is the first order quantity [101]

$$
\mathcal{R} = \psi + \mathcal{H} \left( \frac{\phi^{(1)'} \delta \varphi + \sigma^{(1)'} \delta \sigma}{\phi^{(1)} + \sigma^{(1)}} \right) .
$$

(2.76)

Instead of the first order quantities we write the expansion up to second order for the metric perturbation $\psi = \psi^{(1)} + \frac{1}{2} \psi^{(2)}$ and the scalar fields $\delta \varphi = \varphi_1 + \frac{1}{2} \varphi_2$ and $\delta \sigma = \sigma_1 + \frac{1}{2} \sigma_2$. We obtain

$$
\psi + \mathcal{H} \left( \frac{\phi^{(1)'} \delta \varphi + \sigma^{(1)'} \delta \sigma}{\phi^{(1)} + \sigma^{(1)}} \right) = \mathcal{R}^{(1)} + \frac{1}{2} \left[ \psi^{(2)} + \mathcal{H} \left( \frac{\phi^{(1)'} \varphi_2 + \sigma^{(1)'} \sigma_2}{\phi^{(1)} + \sigma^{(1)}} \right) \right] .
$$

(2.77)

5Recall that the curvature perturbation does not have a homogeneous part.
Consider then the following second-order shift of the time coordinate [15, 88, 89]

\[ \tau \to \tau - \xi^0_{(1)} + \frac{1}{2} \left( \xi^0_{(1)} \xi^0_{(2)} - \xi^0_{(2)} \right), \]  

(2.78)

which transforms \( \psi^{(2)} \), \( \varphi_2 \) and \( \sigma_2 \) into\(^6\)

\[ \tilde{\psi}^{(2)} = \psi^{(2)} + 2 \xi^0_{(1)} \left( \psi^{(1)'} + 2 \mathcal{H} \psi^{(1)} \right) - \left( \mathcal{H}' + 2 \mathcal{H}^2 \right) \left( \xi^0_{(1)} \right)^2 \]

\[ - \mathcal{H} \xi^0_{(1)} \xi^0_{(2)} - \frac{1}{3} \left( 2 \partial^i \omega^{(1)} - \partial^i \xi^0_{(1)} \right) \partial_i \xi^0_{(1)} , \]  

(2.79)

Thus, the expansion (2.77) transforms as\(^7\)

\[ \tilde{\psi} + \mathcal{H} \left( \varphi'_0 \delta \varphi + \sigma'_0 \delta \sigma \right) \]

\[ = \mathcal{R}^{(1)} + \frac{1}{2} \left[ \mathcal{H} \left( \varphi'_0 \varphi_2 + \sigma'_0 \sigma_2 \right) + \tilde{\psi}^{(2)} \right] \]

\[ = \psi + \mathcal{H} \left( \varphi'_0 \delta \varphi + \sigma'_0 \delta \sigma \right) + \xi^0_{(1)} T \]

\[ - \frac{1}{2} \left( \xi^0_{(1)} \right)^2 \left[ \mathcal{H}' + 2 \mathcal{H}^2 - \mathcal{H} \left( \frac{\varphi'_0 \varphi''_0 + \sigma'_0 \sigma''_0}{\varphi^2_0 + \sigma^2_0} \right) \right] - \frac{1}{6} \left( 2 \partial^i \omega^{(1)} - \partial^i \xi^0_{(1)} \right) \partial_i \xi^0_{(1)} , \]

where, following [15], we have denoted

\[ T = \psi^{(1)'} + 2 \mathcal{H} \psi^{(1)} + \mathcal{H} \left( \frac{\varphi'_0 \varphi'_0 + \sigma'_0 \sigma'_0}{\varphi^2_0 + \sigma^2_0} \right) . \]  

(2.81)

By virtue of the first order transformations \( \tilde{\psi}^{(1)} = \psi^{(1)} - \mathcal{H} \xi^0_{(1)} \), \( \tilde{\varphi}_1 = \varphi_1 + \varphi'_0 \xi^0_{(1)} \) and \( \tilde{\sigma}_1 = \sigma_1 + \sigma'_0 \xi^0_{(1)} \) we find

\[ T - \tilde{T} = \xi^0_{(1)} \left[ \mathcal{H}' + 2 \mathcal{H}^2 - \mathcal{H} \left( \frac{\varphi'_0 \varphi''_0 + \sigma'_0 \sigma''_0}{\varphi^2_0 + \sigma^2_0} \right) \right] . \]  

(2.82)

The expansion (2.80) can now be written as

\[ \tilde{\psi} + \mathcal{H} \left( \frac{\varphi'_0 \delta \varphi + \sigma'_0 \delta \sigma}{\varphi^2_0 + \sigma^2_0} \right) = \psi + \mathcal{H} \left( \frac{\varphi'_0 \delta \varphi + \sigma'_0 \delta \sigma}{\varphi^2_0 + \sigma^2_0} \right) \]

\[ + \frac{1}{2} \left( T + \tilde{T} \right) \xi^0_{(1)} - \frac{1}{6} \left( 2 \partial^i \omega^{(1)} - \partial^i \xi^0_{(1)} \right) \partial_i \xi^0_{(1)} . \]  

(2.83)

\(^6\)Note, that we are using more general metric, at the moment, and we have not set \( \omega^{(1)} = 0 \).

\(^7\)The first order part \( \mathcal{R}^{(1)} \) remains unchanged under the transformation Eq. (2.78).
We also solve $\xi^{(0)}_0$ from Eq. (2.82) and insert it into the $T + \tilde{T}$ term above. Note that by virtue of the first order transformation $\tilde{\omega}^{(1)} = \omega^{(1)} - \xi^{(1)}_0$, the last term can be written as

$$ -\frac{1}{6} \left( 2 \partial^i \omega^{(1)} - \partial^i \xi^{(0)}_0 \right) \partial_i \xi^{(0)}_0 = -\frac{1}{6} \left( \partial^i \omega^{(1)} \partial_i \xi^{(0)}_0 - \partial^i \tilde{\omega}^{(1)} \partial_i \tilde{\omega}^{(1)} \right). \quad (2.84) $$

Therefore, after some algebra we see that

$$ \tilde{\psi} + H \left( \frac{\varphi'_0 \delta \varphi + \sigma'_0 \delta \sigma}{\varphi'^2_0 + \sigma'^2_0} \right) + \frac{1}{2} \frac{\tilde{T}^2}{\mathcal{H}' + 2\mathcal{H}^2 - \mathcal{H} \left( \frac{\varphi'_0 \delta \varphi + \sigma'_0 \delta \sigma}{\varphi'^2_0 + \sigma'^2_0} \right)} - \frac{1}{6} \partial^i \omega^{(1)} \partial_i \tilde{\omega}^{(1)} \quad (2.85) $$

The treatment above shows that the comoving curvature perturbation $R = R^{(1)} + \frac{1}{2} R^{(2)}$, which is invariant under the time shift $\tau \rightarrow \tau - \xi^{(1)}_0 + \frac{1}{2} \left( \xi^{(1)}_1 - \xi^{(1)}_2 \right)$, reads in the case of two scalar fields as

$$ R^{(2)} = R^{(1)} + \frac{1}{2} \left( \mathcal{H} \frac{\varphi'_0 \varphi_2 + \sigma'_0 \sigma_2}{\varphi'^2_0 + \sigma'^2_0} + \psi^{(2)} \right) + \frac{1}{2} \left( \psi^{(1)} \mathcal{H}^{(1)} + \mathcal{H} \frac{\varphi'_0 \varphi'_1 + \sigma'_0 \sigma'_1}{\varphi'^2_0 + \sigma'^2_0} \right)^2 \right) \qquad \text{2.86} $$

$$ -\frac{1}{6} \partial^i \omega^{(1)} \partial_i \omega^{(1)}, \quad \text{2.86} $$

where

$$ R^{(1)} = \psi^{(1)} + \mathcal{H} \frac{\varphi'_0 \varphi_1 + \sigma'_0 \sigma_1}{\varphi'^2_0 + \sigma'^2_0}. \quad \text{2.87} $$

This result coincides with the one obtained in [96] once one takes into account the field redefinitions there.

### 2.4 Relation between $R$ and $\zeta$

The comoving curvature perturbation, Eq. (2.86), which is an extension of the single-field case presented in [15], is not the only quantity used to study the curvature perturbations in the second order.\(^8\) Using the spatial metric with uniform density slicing we may define curvature perturbation $\zeta$ non-perturbatively as\(^9\) [106, 107]

$$ g_{ij} = a^2(\eta)e^{2\zeta} \gamma_{ij}, \quad \text{2.88} $$

\(^{8}\)In [106] there is a nice summary on three different curvature perturbations.

\(^{9}\)The discussion here applies in the super-Hubble regime.
where $\gamma_{ij}$ has a unit determinant and can thus be written $\gamma_{ij} = I e^h$, where $I$ is unit matrix and $h$ is traceless. In the inflationary context $h$ corresponds to gravitational waves. Following Lyth and Rodríguez [106] we drop $h$ and consider the spatial metric

$$g_{ij} = a^2(\eta) e^{2\zeta} \delta_{ij} .$$  

(2.89)

The non-perturbative expression, Eq. (2.89), can readily be expanded to give

$$g_{ij} = a^2(\eta) \delta_{ij} (1 + 2\zeta) .$$  

(2.90)

Up to sign conventions this is the definition of the well-known first order quantity. As discussed in [108], the uniform density perturbation is given by

$$\zeta_1 = -\psi^{(1)} - \mathcal{H} \frac{\rho_1}{\rho_0}$$  

(2.91)

and the comoving curvature perturbation by

$$\mathcal{R}_1 = \psi^{(1)} + \mathcal{H} \frac{\phi_1}{\phi_0} ,$$  

(2.92)

and they coincide on large scales

$$\zeta_1 + \mathcal{R}_1 \simeq 0 .$$  

(2.93)

Both are constants on large scales in case only adiabatic perturbations exist.

The second order expansion of Eq. (2.89) reads

$$g_{ij} = a^2(\eta) \delta_{ij} (1 + 2\zeta + 2\zeta^2) ,$$  

(2.94)

which is the definition for the second order quantity $\zeta_2 \equiv \zeta_{2LR}$ used in [106], whereas Malik and Wands [105] use Eq. (2.90) with $\zeta_2 \equiv \zeta_{2MW}$. Thus, there is a relation between the quantities

$$\zeta_{2LR} = \zeta_{2MW} + 2(\zeta_1)^2 .$$  

(2.95)

Both $\zeta_{2LR}$, $\zeta_{2MW}$, and $\zeta_1$, are constants on large scales in the absence of entropy perturbations [105, 106, 108].

On large scales the Malik–Wands uniform density curvature perturbation is [105] (see also [108])

$$\zeta_{2MW} = -\psi^{(2)} - \mathcal{H} \frac{\rho_2}{\rho'} + 2\mathcal{H} \frac{\rho_1}{\rho''} + 2 \frac{\rho_1}{\rho'} (\psi^{(1)} + 2 \mathcal{H} \psi^{(1)})$$

$$+ \frac{\rho_2}{\rho'} (\mathcal{H}' + 2 \mathcal{H} - \mathcal{H} \frac{\rho''}{\rho'}) .$$  

(2.96)

Following Eq. (2.96) Vernizzi [108] defines a comoving curvature perturbation $\mathcal{R}_2^V$ by changing the sign of the right hand side and replacing $\rho_2 \rightarrow \phi_2$. In a single field
inflation these quantities are shown to coincide on large scales [108] $\zeta^\text{MW}_2 + \mathcal{R}_2^V \simeq 0$ and they are both conserved.

Acquaviva et al. [15] define $\mathcal{R}^{(2)}$ which is related to $\mathcal{R}_2^V$ by [108]

$$
\mathcal{R}^{(2)} = \mathcal{R}_2^V + \frac{(2\mathcal{R}^{(1)} + 2H\mathcal{R}^{(1)})^2}{H^2 - \dot{\phi}/\dot{\phi}}.
$$

(2.97)

Such a definition gives rise to an artificial evolution on large scales

$$
\dot{\mathcal{R}}^{(2)} \sim (2\dot{\epsilon} - \dot{\eta})(\mathcal{R}^{(1)})^2.
$$

(2.98)

Since the curvature expansion used in this thesis is an extension of that of Acquaviva et al. [15] the same artificial evolution is also present. However, since we are considering multi-field scenarios with entropy degrees of freedom, there exists a possibility for a real super-Hubble evolution, see also [106].
Chapter 3

Non-Gaussianity

3.1 General

After some early attempts [104, 109–114] the study of primordial non-Gaussianity has become an extremely active field of cosmology, see e.g. [1, 12, 14, 15, 115–125], the review [13] and references therein, and also more recent work [2, 3, 5, 51, 84–87, 91–93, 99, 100, 106, 108, 126–163]. The main conclusion is that single-field inflation does not produce significant non-Gaussianity, whereas in multi-field models the possibility exists.

In this chapter we will define non-Gaussianity and briefly explain the parameterisation relevant for the studies in Papers [1–4], as well as the observational situation of the given parametrisation. Finally, we will apply the cosmological perturbation theory to calculate the second order perturbations, and therefrom, non-Gaussianity arising in hybrid inflation and preheating, including instant and tachyonic preheating.

3.2 Statistics of a cosmological random field

The random fields in cosmology have diverse properties. They may be continuous or discrete (at least in simulations), the number of dimensions may be one (Ly-α), two (CMB), three (galaxy distribution data), or even higher [164]. Also, the random fields may have varying degrees of non-Gaussianity.

The main assumptions when the statistics of cosmological perturbations are studied are statistical homogeneity, statistical isotropy, and ergodicity [164, 165]. Statistical homogeneity and isotropy mean that the statistical properties of the perturbations do not change in spatial translations and rotations, respectively. This can be seen as a consequence of the ‘normal’ homogeneity and isotropy assumptions. Ergodicity means that the ensemble average of a random field (such as perturbation field) can be replaced with a spatial average. Since we only can observe one universe the ensemble average is out of reach and we hope that a sufficiently large
volume (ultimately the observable universe) can be used for measurements of statistical properties. The ubiquitous angle brackets in cosmology, and in this thesis in particular, represent ensemble averages in this sense.

We follow the traditional approach here, and consider the statistics of cosmological perturbation fields using \(N\)-point functions (\(N\)-point correlators). Since the first order moment of cosmological perturbation field is usually zero, i.e. the average of a cosmological perturbation is zero by definition on large enough volume, the first informative, or non-trivial, statistical quantity is the (two-point) correlation function, defined in real space as [165]

\[
\xi(r) \equiv \langle \delta(x)\delta(x + r) \rangle ,
\]

where \(\delta\) is a general cosmological random field (with \(\langle \delta(x) \rangle = 0\)). The Fourier transformation

\[
\delta(x) = \frac{1}{(2\pi)^3} \int d^3 k \delta_k e^{ikx}
\]

allows us to write [165]

\[
\langle \delta_k \delta_{k'} \rangle = \int d^3 x \, d^3 r \, \xi(r) e^{-i(k+k')x-ik'r} = (2\pi)^3 \delta^{(3)}(k + k') \int d^3 r \xi(r) e^{-ik'r} \equiv (2\pi)^3 \delta^{(3)}(k + k') P(k) ,
\]

where in the last step we have defined power spectrum \(P(k)\), (as the Fourier transform of the two point correlator). Another widely used quantity, also called the spectrum is

\[
P_g(k) \equiv \frac{k^3}{2\pi^2} |g_k|^2 ,
\]

with spectral index defined as

\[
n_g - 1 \equiv \frac{d \ln P_g}{d \ln k} .
\]

1Presented here for a general perturbation \(g\).

2Here, conventions regarding the \(-1\) vary, depending on the quantity in question.
Higher order correlators are defined using the connected joint moments of the random field in different points (in real space), or in Fourier space for $N$ modes \cite{164, 165}

$$\langle \delta_{k_1} \ldots \delta_{k_N} \rangle = (2\pi)^3 \delta^{(3)}(k_1 + \ldots + k_N) P_N(k_1, \ldots, k_N). \quad (3.6)$$

Here $P_N(k_1, \ldots, k_N)$ is called $N-1$-spectrum. The most important special case is $N = 3$, the three point correlator and the bispectrum, which is conventionally denoted with $B(k_1, k_2, k_3)$. Observationally higher order statistics are more difficult than the spectrum, and they are more sensitive to, e.g., systematics. Also, in principle the three point function depends on nine coordinates, but symmetries in cosmology reduce it to three parameters in real space \cite{164}.

Finally, we are ready to define Gaussian and non-Gaussian statistics. A random field, such as cosmological perturbation field, is called \textit{Gaussian}, when its statistical properties are fully described by its two point correlator, or spectrum\footnote{We are assuming that the spatial average of the field is zero. Otherwise one would also need its value.} \cite{164, 165}. (This is sometimes called the Wick theorem). On the other hand, a random field is called \textit{non-Gaussian} if it has at least one non-vanishing higher order connected moment. This is clearly very general definition, since everything that is not Gaussian is non-Gaussian. It obviously poses problems for parameterising and, therefore, for theoretically predicting and observationally constraining non-Gaussianities.

### 3.3 Connection to observations

As was mentioned in the previous section the description of non-Gaussian random field in principle requires infinite amount of parameters. This poses an immediate difficulty in parameterising and observing the deviations from Gaussianity. Here we follow the standard method of using the three-point correlation function, or equivalently bispectrum, and particularly the nonlinearity parameter $f_{NL}$, defined later. This is the quantity best constrained by observations, and it is also possible to compute theoretical predictions for the parameter, which is not so easy for many other methods. Also, the three-point function, or bispectrum, vanishes for Gaussian field so any signal indicates deviation from Gaussianity.

We are not aiming to discuss the observational side thoroughly, only to present the quantity we aim to use in our study of non-Gaussianities. For more on the observational side of the subject, see the review \cite{13} and references therein.

#### 3.3.1 Nonlinearity parameter $f_{NL}$

The bispectrum contains the lowest order statistics capable of distinguishing deviations from Gaussianity. The nonlinearity parameter was originally introduced to purely phenomenologically parameterise the non-Gaussianity level in the bispectrum
There were no theoretical motivations for its functional form. Later it has been shown that it has important, and non-trivial, scale dependence which calls for reanalysis of the previous limits \cite{166}. Clear majority of the present observational limits, however, are for constant $f_{NL}$.

The nonlinearity parameter, $f_{NL}$, is defined as \cite{13}

$$\Phi = \Phi_L + f_{NL} \ast (\Phi_L)^2 ,$$  \hspace{1cm} (3.7)

where the star, $\ast$, denotes convolution and represents the fact that in general the nonlinearity parameter has non-trivial scale dependence. Here $\Phi$ is the Bardeen potential and it is connected to the temperature anisotropies of the CMB as $^{4}$

$$\frac{\Delta T}{T} = -\frac{1}{3}\Phi .$$  \hspace{1cm} (3.8)

The connection to curvature perturbation $\zeta$, discussed in Sec. 2.4 comes from the relation \cite{167} $\Phi = -\frac{5}{3}\zeta$. The connection to CMB is achieved using first order perturbation theory; for first steps towards second order treatment, see \cite{84–87}.

If we make a splitting $\Phi(k) = \Phi_L(k) + \Phi_{NL}(k)$ the non-Gaussian part can be expressed as double convolution \cite{166},

$$\Phi_{NL}(k_3) = \frac{1}{(2\pi)^3} \int d^3k_1d^3k_2 \delta^3(k_1 + k_2 - k_3)\Phi_L(k_1)\Phi_L(k_2)f_{NL}(k_1,k_2,k_3) ,$$  \hspace{1cm} (3.9)

where $\Phi_L(k)$ is the linear, Gaussian, part, and the nonlinearity parameter $f_{NL}$ appears as a kernel. With the non-Gaussian term present, the bispectrum now acquires contribution \cite{166}

$$\langle \Phi(k_1)\Phi(k_2)\Phi(k_3) \rangle = \langle \Phi_L(k_1)\Phi_L(k_2)\Phi_{NL}(k_3) \rangle = 2(2\pi)^3\delta^3(k_1 + k_2 + k_3)f_{NL}(k_1,k_2,k_3)P(k_1)P(k_2) ,$$  \hspace{1cm} (3.10)

where we have not written the permutations explicitly.

As already mentioned, the observational constraints are imposed upon a constant $f_{NL}$. Thus, the definition of the parameter, Eq. (3.7), can be written (in real space, again up to a constant offset) \cite{166}

$$\Phi(x) = \Phi_L(x) + f_{NL}(\Phi_L(x))^2 .$$  \hspace{1cm} (3.11)

The strictest limits come from WMAP satellite (\cite{10}); WMAP team reports \cite{12}

$$-58 < f_{NL} < 134 , \text{ at 95\% confidence level.}$$  \hspace{1cm} (3.12)

Recently Creminelli \textit{et al.} \cite{168} reported slightly improved result

$$-27 < f_{NL} < 121 , \text{ at 95\% confidence level}$$  \hspace{1cm} (3.13)

\textit{This connection is made at the matter dominated era before horizon entry, see e.g. [6].}
after reanalysis of WMAP data.

There are projections on the sensitivity of different observational experiments (e.g., WMAP, Planck) with and without polarisation, see e.g. [169] (see also [157] for a recent discussion). The formalism, however, is still quite undeveloped and, also, in the papers included in the thesis, [1–4], we have just obtained preliminary results on the prospects of the overall level of non-Gaussianity in different theoretical scenarios and models. In that sense we are not in immediate need of numbers exact to several decimal places.

Finally, we may conclude what different theoretically predicted values of the nonlinearity parameter, \( f_{NL} \), mean. In case a model predicts \( f_{NL} \gg 1 \) there are good prospects of observing the non-Gaussianity, if not in the immediate future with WMAP, then later with Planck. In case a model predicts \( f_{NL} \sim 1 \) more work needs to be done (for example on finding out the scale dependence\(^5\)) before anything conclusive can be said. The non-Gaussianity might be observable. In case a model predicts \( f_{NL} \ll 1 \) there is very little chance of observing the non-Gaussianity, at least through \( f_{NL} \), unless something radically different comes up.

### 3.3.2 Other tests of (non-)Gaussianity

The obvious next step in statistics, while using \( N \)-point correlators, is the trispectrum. There already are approaches in this direction where the expansion in Eq. (3.7) is extended to contain a term \( \propto \Phi^3 \), and becoming observable in the four-point correlator: in [170] the parameter is called \( \tau_{NL} \) and in [86] \( g_{NL} \) (see also [171]). This parameter, however, is not yet well constrained by observations and not computed for many models.

There are also other kind of tests for non-Gaussianity, such as Minkowski functionals [12, 172–174], properties of hot and cold spots [175], geometrical estimators [176, 177], extrema correlation function [174, 178–180], goodness of fit tests [129, 181], multifractals [182], phase analysis [183–185], and wavelet techniques [186], (for more references, see e.g. [13, 186]). These methods, however, are usually not directly linked to the dynamics of the system at hand and making theoretical predictions is difficult, often imposing a need for comparing simulated CMB maps with the observed ones.

### 3.4 Non-Gaussianity from hybrid inflation

The previous studies have shown that the non-Gaussianity arising during single field inflation is small [13–15, 136, 147]. Here we, thus, do not consider the non-Gaussianity from the inflaton, but focus on the additional non-Gaussianity arising\(^5\) taking properly into account the scale dependence, both in the theoretical predictions of models and in observations, things seem to be quite promising in distinguishing between different models and scenarios, see [166].

\(^5\)Taking properly into account the scale dependence, both in the theoretical predictions of models and in observations, things seem to be quite promising in distinguishing between different models and scenarios, see [166].
from the $\sigma$ field in hybrid inflation. This is studied in the enclosed Papers [1, 4].

We apply now our master equation (2.74) to hybrid inflation. By switching to the cosmic time, $dt = a d\tau$, and dropping terms next to leading order in slow roll parameters, the master equation (2.74) can be written as [1]

$$\ddot{\phi}^{(2)} + H \dot{\phi}^{(2)} + 2H^2 \left( \frac{\dot{H}}{H^2} - \frac{\dot{\phi}_0}{H \dot{\phi}_0} \right) \phi^{(2)} - \frac{1}{a^2} \partial_i \partial^i \phi^{(2)} = \text{[ inflaton source ]} + \text{[ } \sigma \text{ source ]},$$

(3.14)

where

$$\text{[ } \sigma \text{ source ]} = + 6 \frac{1}{M_P^2} H \Delta^{-1} \partial_i (\dot{\sigma}_1 \partial^i \sigma_1) + 4 \frac{1}{M_P^2} \Delta^{-1} \partial_i (\dot{\sigma}_1 \partial^i \sigma_1) - 2 \frac{\dot{\phi}_0}{\dot{\phi}_0} \Delta^{-1} \gamma_{\sigma} + \Delta^{-1} \gamma_{\sigma},$$

(3.15)

and

$$\gamma_{\sigma} = - \frac{3}{M_P^2} \Delta^{-1} \partial_i (\partial_k \partial^k \sigma_1 \partial^i \sigma_1) - \frac{1}{2M_P^2} \partial_i \sigma_1 \partial^i \sigma_1.$$

(3.16)

The inflaton source contains the terms involving first order metric and inflaton perturbations. Due to the complete decoupling of $\sigma$ from the first order Einstein equations, as discussed previously in Sec. 2.3.5, the terms with first order metric perturbations do not contain $\sigma$ even implicitly. The form of the inflaton source can be computed from Eq. (2.74) or looked up from [15]. Since the contribution is known and small we do not concern ourselves with its actual form. Thus, from now on we focus on the contribution from $\sigma$.

First, let us consider the curvature perturbation, Eq. (2.86). With the general longitudinal gauge, here essentially $\omega^{(2)} = 0$, and the hybrid inflation condition $\sigma_0 = 0$ the curvature perturbation acquires the same functional form as in the single-field case [1, 15]

$$\mathcal{R} = \mathcal{R}^{(1)} + \frac{1}{2} \left( \frac{\mathcal{H} \dot{\varphi}_0^2}{\varphi_0'} + \psi^{(2)} \right) + \frac{1}{2} \left( \frac{\psi^{(1)'} + 2 \mathcal{H} \psi^{(1)} + \mathcal{H} \varphi_1'/\varphi_0'}{\mathcal{H}' + 2 \mathcal{H}^2 - \mathcal{H} \varphi_0'/\varphi_0} \right)^2,$$

(3.17)

where

$$\mathcal{R}^{(1)} = \psi^{(1)} + \mathcal{H} \frac{\varphi_1}{\varphi_0}.$$

(3.18)

Rewriting $\mathcal{R}^{(2)}$ in terms of the cosmic time $dt = a d\tau$ and applying the condition $\psi^{(1)} = \phi^{(1)}$, we obtain

$$\mathcal{R}^{(2)} = \mathcal{H} \frac{\varphi_2}{\varphi_0} + \psi^{(2)} + \left( \frac{\dot{\phi}^{(1)} + 2H \phi^{(1)} + H \varphi_1'/\varphi_0'}{H^2 \left( 2 + \dot{H}/H - \dot{\varphi}_0/H \dot{\varphi}_0 \right)} \right)^2.$$

(3.19)
Making use of the relations (2.69) and (2.71) we find

\[
R^{(2)} = \frac{2HM_p^2}{\dot{\varphi}_0^2} \left[ \dot{\phi}^{(2)} + H\phi^{(2)} - \Delta^{-1} \left( \frac{1}{M_P^2} a - \frac{\alpha}{a} \right) \right] + \phi^{(2)} - \frac{2HM_p^2}{\dot{\varphi}_0^2} \Delta^{-1} \dot{\gamma}
\]

\[
- \Delta^{-1} \gamma + \frac{\left( \dot{\phi}^{(1)} + 2H\phi^{(1)} + H\dot{\varphi}_1/\dot{\varphi}_0 \right)^2}{H^2 \left( 2 + \dot{H}/H - \dot{\varphi}_0/H\dot{\varphi}_0 \right)}.
\]  

(3.20)

Acquaviva et al. [15] point out that the last term gives a subdominant contribution in the single field case. Moreover, it does not contain any dependence on \( \sigma \), not even implicitly through \( \dot{\phi}^{(1)} + 2H\phi^{(1)} \). Therefore, in what follows we shall neglect this term.

Since \( 2HM_p^2/\dot{\varphi}_0^2 = 1/\epsilon \), the term \( \phi^{(2)} \) outside the square brackets is subdominant to the one inside; hence we discard it. Thus, up to the leading order in the slow-roll parameters we may write the curvature perturbation as

\[
R^{(2)} \approx \frac{2H}{\kappa^2 \dot{\varphi}_0^2} \left[ \dot{\phi}^{(2)} + H\phi^{(2)} - \Delta^{-1} \left( \kappa^2 \frac{\beta}{a} - \frac{\alpha}{a} \right) \right] - \frac{2H}{\kappa^2 \dot{\varphi}_0^2} \Delta^{-1} \dot{\gamma} - \Delta^{-1} \gamma.
\]  

(3.21)

As with the master equation, Eq. (3.14), we may isolate the contributions coming from the inflaton and \( \sigma \) in \( \mathcal{R} \). We thus write the comoving curvature perturbation as

\[
\mathcal{R} = \mathcal{R}^{(1)} + \frac{1}{2} \mathcal{R}^{(2)} = \mathcal{R}_\varphi + \frac{1}{2} \mathcal{R}_\varphi^{(2)} + \frac{1}{2} \mathcal{R}_\sigma^{(2)} = \mathcal{R}_\varphi + \frac{1}{2} \mathcal{R}_\sigma^{(2)}.
\]  

(3.22)

\( \mathcal{R}_\varphi \) contribution has already been calculated by Acquaviva et al. [15] taking into account that at large scales, \( k \ll aH \), \( \psi^{(1)} \) can be taken constant and

\[
\psi^{(1)} = \frac{\kappa^2}{2H} \dot{\varphi}_0 = \epsilon H \frac{\varphi_1}{\varphi_0}.
\]  

(3.23)

This makes it possible to set \( \psi^{(1)} = \epsilon \mathcal{R}^{(1)} \) so that the result can be written in a deceptively simple looking way as [13, 15]

\[
\mathcal{R}_\varphi^{(2)} = (\eta - 3\epsilon) \left( \mathcal{R}^{(1)} \right)^2 + \mathcal{I}_\varphi,
\]  

(3.24)

where

\[
\mathcal{I}_\varphi = -\frac{2}{\epsilon} \int \frac{1}{a^2} \psi^{(1)} \partial_i \psi^{(1)} dt - \frac{4}{\epsilon} \int \frac{1}{a^2} \partial_i \psi^{(1)} \partial^i \psi^{(1)} dt - \frac{4}{\epsilon} \int \left( \psi^{(1)} \right)^2 dt + (\epsilon - \eta) \Delta^{-1} \partial_i \mathcal{R}^{(1)} \partial^i \mathcal{R}^{(1)}.
\]  

(3.25)

The important point to stress here is that the single field contribution to the curvature perturbation, including the integral part \( \mathcal{I}_\varphi \), is proportional to the slow-roll parameters and hence naturally small in hybrid inflation [13, 15].
However, in hybrid inflation the waterfall field $\sigma$ yields an additional contribution to $\mathcal{R}$. We may calculate it from Eq. (3.21) by plugging in the $\sigma$ dependent parts of $\alpha, \beta, \gamma$, and integrating the $\sigma$ dependent part of our master equation (3.14) to obtain $(\dot{\phi}^{(2)} + H\dot{\phi}^{(2)})_\sigma$ (see [15] for the inflaton part). The result finally reads [1]

$$
\mathcal{R}^{(2)}_{\sigma} = \frac{1}{\epsilon H M^2_p} \left\{ \int [6H \Delta^{-1}\partial_i(\dot{\sigma}_i \dot{\sigma}_1) + 4\Delta^{-1}\partial_i(\dot{\sigma}_1 \dot{\sigma}_1)^* - 2(\dot{\sigma}_1)^2 + m^2_\sigma(\sigma_1)^2 + (\epsilon - \eta)6H \Delta^{-2}\partial_i(\dot{\partial}_k \dot{\sigma}_1 \dot{\sigma}_1)^* \\
+ (\epsilon - \eta)H \Delta^{-1}(\partial_k \sigma_1 \partial^k \sigma_1)^* - 3\Delta^{-2}\partial_i(\partial_k \partial^k \sigma_1 \partial^i \sigma_1)^* \\
- \frac{1}{2}\Delta^{-1}(\partial_k \sigma_1 \partial^k \sigma_1)^* \Delta^{-1}\partial_i(\dot{\sigma}_1 \dot{\sigma}_1) \\
+ 3\Delta^{-2}\partial_i(\partial_k \partial^k \sigma_1 \partial^i \sigma_1)^* \Delta^{-1}(\partial_i \partial^i \sigma_1) + \frac{\epsilon H}{2} \Delta^{-1}(\partial_i \partial^i \sigma_1)] \right\},
$$

(3.26)

where we have used the shorthand notation $m^2_\sigma \equiv \partial^2 V/\partial \sigma^2$ and the slow-roll relations Eqs. (2.47) and (2.48). By noting that [4]

$$
6H \Delta^{-1}\partial_i(\dot{\sigma}_1 \dot{\sigma}_1) + 2\Delta^{-1}\partial_i(\dot{\sigma}_1 \dot{\sigma}_1)^* - (\dot{\sigma}_1)^2 + m^2_\sigma(\sigma_1)^2 = 2\Delta^{-1}\partial_i[(3H \dot{\sigma}_1 + \ddot{\sigma}_1 + m^2_\sigma \sigma_1) \dot{\sigma}_1 \dot{\sigma}_1] = 0,
$$

(3.27)

since $3H \dot{\sigma}_1 + \ddot{\sigma}_1 + m^2_\sigma \sigma_1 = 0$ outside horizon, Eq. (3.26) can be written [4]

$$
\mathcal{R}^{(2)}_{\sigma} = \frac{1}{\epsilon H M^2_p} \left\{ \int [2\Delta^{-1}\partial_i(\dot{\sigma}_1 \dot{\sigma}_1)^* - (\dot{\sigma}_1)^2 + (\epsilon - \eta)6H \Delta^{-2}\partial_i(\dot{\partial}_k \dot{\sigma}_1 \dot{\sigma}_1)^* \\
+ (\epsilon - \eta)H \Delta^{-1}(\partial_k \sigma_1 \partial^k \sigma_1)^* - 3\Delta^{-2}\partial_i(\partial_k \partial^k \sigma_1 \partial^i \sigma_1)^* \\
- \frac{1}{2}\Delta^{-1}(\partial_k \sigma_1 \partial^k \sigma_1)^* \Delta^{-1}\partial_i(\dot{\sigma}_1 \dot{\sigma}_1) \\
+ 3\Delta^{-2}\partial_i(\partial_k \partial^k \sigma_1 \partial^i \sigma_1)^* \Delta^{-1}(\partial_i \partial^i \sigma_1) + \frac{\epsilon H}{2} \Delta^{-1}(\partial_i \partial^i \sigma_1)] \right\}.
$$

(3.28)

An exact evaluation of the second order curvature $\mathcal{R}^{(2)}_{\sigma}$ would be extremely difficult. Instead, we make an order of magnitude estimate of the expression (3.28). To that end, we write the hybrid potential, Eq. (1.46), in a more suggestive form, keeping only the relevant terms, as [4,167]

$$
V = V_0(1 + \frac{1}{2} \eta \frac{\varphi^2}{M^2_p} + \frac{1}{2} \eta_\sigma \frac{\sigma^2}{M^2_p}).
$$

(3.29)

For the estimation purposes the slow roll parameters $\eta$ and $\eta_\sigma$, and also the Hubble
parameter $H$, are set to be constants. For hybrid inflation this is a good approximation. Eq. (3.28) now simplifies to

$$
R^{(2)}_\sigma = \frac{1}{\epsilon H M_P^2} \left\{ \int \left[ - (\dot{\sigma}_1)^2 + 2 H \tilde{\epsilon} \right] dt + \Delta^{-1} \partial_i(\dot{\sigma}_1 \partial^i \sigma_1) + H(\epsilon - 2\eta)\tilde{\gamma}_\sigma \right\},
$$

(3.30)

where we have denoted

$$
\tilde{\gamma}_\sigma \equiv -M_P^2 \gamma_\sigma = 3 \Delta^{-2} \partial_i(\partial_k \partial^k \sigma_1 \partial^i \sigma_1) + \frac{1}{2} \Delta^{-1}(\partial_k \sigma_1 \partial^k \sigma_1).
$$

(3.31)

Even though $\eta$, $\eta_\sigma$, and $H$ are constants, the time evolution of some other quantities is important. Using the slow roll equations (outside horizon) we obtain

$$
\sigma_1(t) = \sigma_1(t_i)e^{-\eta_\sigma \Delta N},
\varphi_1(t) = \varphi_1(t_i)e^{-\eta_\varphi \Delta N},
\varphi_0(t) = \varphi_0(t_i)e^{-\eta_\varphi \Delta N},
$$

(3.32)

where $\Delta N = H\Delta t$ is the number of e-folds since $t_i$; thus, we obtain $\dot{\sigma}_1 = -\eta_\sigma H \sigma_1$, $\dot{\varphi}_1 = -\eta_\varphi H \varphi_1$, and $\dot{\varphi}_0 = -\eta_\varphi H \varphi_0$. Since $\epsilon \sim \varphi_0^2/H^2 M_P^2$ we can also readily write

$$
\epsilon(t) = \epsilon_i e^{-2\eta_\varphi \Delta N},
$$

(3.33)

where we have denoted $\epsilon_i \equiv \epsilon(t_i)$. Now, it is immediately clear that $|R^{(1)}| = |H \varphi_1/\varphi_0|$ stays constant, but one also sees that $\Delta N$ e-folds after horizon exit

$$
|H \sigma_1/\varphi_0| \sim |R^{(1)}| |\sigma_1/\varphi_1| \sim \epsilon^{\Delta N(\eta_\varphi - \eta_\sigma)}|R^{(1)}|,
$$

(3.34)

where in the last step we have used the knowledge that the amplitude of the perturbation of any effectively massless field $f$ is $|\delta f| \sim H$ immediately after exiting horizon, i.e., at $\Delta N = 0$ in this case.

For obtaining an order of magnitude estimate for the second order curvature perturbation $R^{(2)}$, Eq. (3.30), we are not, at this point, interested in the precise scale dependence of the second order curvature. Thus, we neglect the cancelling orders of spatial derivative operators and estimate, for example, $|\Delta^{-1} \partial_i R^{(1)} \partial^i R^{(1)}| \sim |R^{(1)}|^2$. Because of this, both terms in $\tilde{\gamma}_\sigma$, Eq. (3.31), are essentially the same. Furthermore, since the order of magnitude estimate anyway gives an upper limit, and since $\epsilon \leq \epsilon_i$ for any time $t \geq t_i$, we replace $\epsilon$ with $\epsilon_i$ except in the prefactor $1/\epsilon$. We explicitly replace $\dot{\sigma}_1 = -\eta_\sigma H \sigma_1$. The estimate for the second order curvature perturbation, Eq. (3.30), thus becomes

$$
R^{(2)}_\sigma \sim \frac{1}{\epsilon H M_P^2} \left\{ \int \left[ \eta_\sigma^2 H^2 |\sigma_1|^2 + \epsilon_i \eta_\sigma H^2 |\sigma_1|^2 \right] dt + \eta_\sigma H |\sigma_1|^2 + \epsilon_i H |\sigma_1|^2 + \eta H |\sigma_1|^2 \right\}
$$

$$
\sim \frac{1}{\epsilon H M_P^2} \left\{ \mathcal{O}(\epsilon_i, \eta_\sigma) \int \eta_\sigma H^2 |\sigma_1|^2 dt + \mathcal{O}(\epsilon_i, \eta, \eta_\sigma) H |\sigma_1|^2 \right\}.
$$

(3.35)
We have now two terms to evaluate, namely

\[
\frac{1}{\epsilon H M_P^2} H |\sigma_1|^2 = \frac{\sigma_1(t_i)^2}{\epsilon_i M_P^2} e^{2\Delta N(\eta - \eta_\sigma)} = \left| H \frac{\sigma_1(t_i)}{\varphi_0(t_i)} \right|^2 e^{2\Delta N(\eta - \eta_\sigma)}
\]

and

\[
\frac{1}{\epsilon H M_P^2} \int_{t_0}^{t} H^2 \eta_\sigma |\sigma_1|^2 \, dt = \frac{|\sigma_1(t_i)|^2}{\epsilon M_P^2} \eta_\sigma \int_{0}^{\Delta N} e^{-2\eta_\sigma N} dN
\]

\[
\sim \frac{|\sigma_1(t_i)|^2}{\epsilon M_P^2} e^{-2\eta_\sigma \Delta N} = e^{2\Delta N(\eta - \eta_\sigma)} |\mathcal{R}^{(1)}|^2.
\]

Therefore, our final estimate for the second order curvature perturbation from the \(\sigma\) field reads

\[
\mathcal{R}^{(2)}_\sigma \sim \mathcal{O}(\epsilon_i, \eta, \eta_\sigma) e^{2\Delta N(\eta - \eta_\sigma)} |\mathcal{R}^{(1)}|^2.
\]

The non-Gaussianity in hybrid inflation has also been calculated by Lyth and Rodríguez [167] and by Malik [100]. In [167] the Sasaki-Stewart \(\delta N\) formalism [187] was applied, after being extended to second order in scalar field perturbations. A different definition, and notation, for the curvature perturbation was used, Sec. 2.4, the result being [167]

\[
\zeta_2,\sigma \approx \eta_\sigma e^{2\Delta N(\eta - \eta_\sigma)} |\zeta_1|^2.
\]

The two results, Eqs. (3.38) and (3.39), obtained using the perturbation approach and \(\delta N\) approach, respectively, differ by the existence of the non-local terms. If one would neglect the non-local terms in Eq. (3.30), there would be a coefficient \(\mathcal{O}(\eta_\sigma)\) instead of \(\mathcal{O}(\epsilon_i, \eta, \eta_\sigma)\) in Eq. (3.38) and the two results would agree completely. However, the perturbation theory approach seems to produce non-local terms by construction, see e.g. [13, 166] and also [188] Sec. III, and there should be some justification for dropping them.6

Malik [100] obtains a result which agrees with the one by Lyth and Rodríguez [167]. He uses an approach which looks like a cosmological perturbation theory but resembles the separate universe approach, or the so called gradient expansion method [109], in the sense that it considers the super-Hubble scales only and the gradient terms are dropped already at the beginning. Also in [100] the nonlocal terms are encountered, but there the slow-roll approximation is used to get rid of them.

As concluded in [167], by suitably choosing the parameters of the model there is a possibility for non-Gaussianity even at the level of \(|f_{NL}| > 1\), which might be observable in the near future [169].

Even though the different approaches agree on the magnitude of the produced non-Gaussianity, the question on the non-local terms still remains. One problem is that the usual scalar-vector-tensor decomposition of the perturbations is inherently

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6For a recent development, see Barnaby and Cline [160].
non-local [188]. These non-localities do not appear in the first order, but in the second order equations there are terms like $\Delta^{-1}(\partial_i g \partial^i g)$ and $\Delta^{-1}(g \Delta g)$, where $g$ represents a generic perturbation. The physical interpretation of these terms is not clear, and by using the separate universe approach they do not even appear.

3.5 Non-Gaussianity from preheating

It seems quite natural that non-Gaussianity in preheating could be significant. After all, the way preheating is assumed to end is by backreaction effects, and backreaction, by definition, means that the second order effects become at least as significant as the first order effects.

The possibility of using the non-Gaussianity as an observational signal of preheating was already mentioned in [69]. The first actual computation of the amount of non-Gaussianity arising from preheating was presented in [2], soon followed by estimates of non-Gaussianity in tachyonic and instant preheating [3], and its implications on string scale and coupling [5]. See also recent work on massless preheating [159].

First, we consider the case of standard preheating, i.e., preheating realized by parametric resonance, as described in Sec. 1.4.2. For simplicity we will neglect the expansion of the universe. This may seem rather restrictive, but since the homogeneous condensate oscillates coherently with a frequency larger than the Hubble expansion rate during the initial stages of preheating, the only relevant time scale is the mass of the coherently oscillating field. Of course, for detailed numerical values, the expansion as well as backreaction effects will be important.

Let us consider a two-field model with the potential

$$V = \frac{1}{2} m_{\phi}^2 \phi^2 + \frac{1}{2} g^2 \phi^2 \sigma^2 , \quad (3.40)$$

where $\phi$ is the homogeneous scalar condensate coherently oscillating with a mass $m_{\phi}$. We assume $\phi$ to be our inflaton. The $\sigma$ field will be created resonantly by virtue of the coupling term. We will adopt the metric

$$g_{\tau\tau} = -a(\tau)^2 \left( 1 + 2\phi^{(1)} + \phi^{(2)} \right) , \quad \quad (3.41)$$

$$g_{\tau i} = 0 , \quad \quad (3.42)$$

$$g_{ij} = a(\tau)^2 \left( 1 - 2\psi^{(1)} - \psi^{(2)} \right) \delta_{ij} , \quad \quad (3.43)$$

and divide the scalar fields, $\phi$ and $\sigma$, into background and perturbations

$$\phi(\tau, \mathbf{x}) = \phi_0(\tau) + \phi_1(\tau, \mathbf{x}) + \frac{1}{2} \phi_2(\tau, \mathbf{x}) , \quad \quad (3.44)$$

$$\sigma(\tau, \mathbf{x}) = \sigma_1(\tau, \mathbf{x}) + \frac{1}{2} \sigma_2(\tau, \mathbf{x}) . \quad \quad (3.45)$$

For simplicity and for the sake of clarity we have assumed above that the vacuum expectation value (VEV) of $\sigma$ vanishes, $\langle \sigma \rangle = 0$ or $\sigma_0 = 0$, which makes it possible
to apply the second order perturbation formalism, as described in Sec. 2.3. Such a situation occurs if the $\sigma$ field is driven to the minimum of its potential during inflation.

Now we are ready to tackle the situation with the cosmological perturbation formalism, Chapter 2. As mentioned before, Sec. 2.3.5, $\sigma$ decouples from the first order equations, thus making the situation appear to be similar to a single field inflation from the point of view of the inflaton. The first order field equation for $\sigma$, however, leads to the Mathieu equation (1.81), and to parametric resonance, as described in Sec. 1.4.2.

The first order metric perturbation at the end of inflation for a single field is given by [35]

$$\phi^{(1)} = \frac{H}{\varphi_0} \delta^{(1)} \varphi \simeq -\frac{\varphi_0}{2M_p^2} \delta^{(1)} \varphi,$$

(3.46)

where the quantities on the right-hand side are calculated at the horizon crossing; the last step is obtained using slow-roll equations of motion. Since $g \ll 1$ we can effectively take $\sigma_1 \sim \varphi_1$ during inflation. This determines the initial conditions before preheating starts.

In the second order we are only interested in the gravitational perturbation, whose equation can be written in an expanding background as (see, Eq. (2.74))

$$\phi^{(2)''} + 2 \left( \mathcal{H} - \frac{\varphi_0''}{\varphi_0'} \right) \phi^{(2)'} + 2 \left( \mathcal{H}' - \frac{\varphi_0''}{\varphi_0'} \mathcal{H} \right) \phi^{(2)} - \partial_i \partial^i \phi^{(2)} = \mathcal{J}_{\sigma,\text{local}} + \mathcal{J}_{\sigma,\text{non-local}} + \mathcal{J}_{\text{rest}},$$

(3.47)

where the source terms $\mathcal{J}$ are quadratic combinations of the first order perturbations; in particular,

$$\mathcal{J}_{\sigma,\text{local}} = -\frac{2}{M_p^2} (\sigma_1')^2 + \frac{a^2}{M_p^2} \partial^2 V \partial^2 \sigma_1.$$

(3.48)

$\mathcal{J}_{\sigma,\text{non-local}}$ involves an inverse spatial Laplacian, thus rendering it non-local, while $\mathcal{J}_{\text{rest}}$ consists of metric and $\varphi$ perturbations.

Fourier transforming $\mathcal{J}_{\sigma,\text{local}} \rightarrow \mathcal{J}_k$ we end up with the convolutions

$$\mathcal{J}_k = -\frac{2}{M_p^2 (2\pi)^3} \int d^3k' \sigma_{1k'} \sigma_{1k-k'} + \frac{1}{M_p^2} \frac{\partial^2 V}{\partial \sigma^2} \frac{1}{(2\pi)^3} \int d^3k' \sigma_{1k'} \sigma_{1k-k'}.$$

(3.49)

The object is then to compute the convolutions. We are interested in their contributions at large scales and, to that end, take the limit $k \to 0$; we also assume $a = 1$. For our purposes it is sufficient to estimate the solution to be independent of $k$ in the resonance band. Therefore, we estimate $\sigma_1 \simeq \sigma_{1\text{eff}} \equiv A \exp(\mu_{\text{eff}} m_{\varphi} t)$, where

---

We use the convention $f(x) = (1/2\pi)^3 \int d^3k e^{ikx} f(k)$. 

---
\[ \mu_{\text{eff}} = \mu_{\text{max}}/2 = q/4, \] in the resonance band and \( \sigma_1 = 0 \) otherwise; here \( A \) is an amplitude after the end of inflation.

Since the mode function \( \sigma_{\text{eff}} \) only depends on the magnitude of the vector \( \mathbf{k} \), the angular integration can be carried out trivially. The time derivative only produces a constant factor. Thus, we obtain

\[
\mathcal{J}_k = -2 \frac{\mu_{\text{max}}}{2M_c^2} \frac{4\pi}{(2\pi)^3} \int dk' k'^2 (\sigma_1')^2 \\
+ \frac{a^2}{M_c^2} \frac{\partial^2 V}{\partial \sigma^2} \frac{4\pi}{(2\pi)^3} \int dk' k'^2 (\sigma_1')^2 \\
= \left[ -\frac{2\mu_{\text{eff}} m_{\phi}^2}{M_c^2} + \frac{1}{M_c^2} \frac{\partial^2 V}{\partial \sigma^2} \right] \sigma_{\text{eff}}^2 \frac{4\pi}{(2\pi)^3} \int_{k_-}^{k_+} dk' k'^2,
\]

where in the last step we have assumed that the \( k \)-dependence of the amplitude \( A \) can be ignored. If we are working in a narrow resonance regime with \( q < 1 \), the integral can be written as

\[
\int_{k_-}^{k_+} dk' k'^2 = \frac{1}{3} \left( \frac{m_{\phi}}{2} \right)^3 \left[ 3q + 2 \left( \frac{q}{2} \right)^3 \right] \simeq q \left( \frac{m_{\phi}}{2} \right)^3.
\]

We can now write the source term as

\[
\mathcal{J}_k = \frac{4\pi}{(2\pi)^3} \left( \frac{m_{\phi}}{2} \right)^3 \left[ -\frac{q^2 m_{\phi}^2}{8M_c^2} + \frac{1}{M_c^2} \frac{\partial^2 V}{\partial \sigma^2} \right] A^2 e^{q m_{\phi} t/2}
\]

\[
= \frac{2m_{\phi}^2 q}{M_c^2} \left[ 1 - \frac{q}{16} \right] \times B e^{q m_{\phi} t/2}.
\]

where \( B = (q/8\pi^2) (m_{\phi}/2)^3 A^2 \). It is worth noting that the source \( \mathcal{J}_k \), which we study at the large scale limit \( k \sim 0 \), is actually generated by first order, local perturbations on much smaller scales \( k_- < k < k_+ \).

Consider Eq. (3.47) in \( k \)-space. The homogeneous part is the same as in the first order. Therefore we know that the homogeneous solutions are well-behaved. Barring accidental cancellations, we may assume that the local terms we have considered are representative of the exponential behaviour of the source; see Eq. (3.48). There is also a nonexponential part which naturally becomes quickly insignificant. In order to estimate the behaviour of \( \phi^{(2)} \) at large scales \( k \sim 0 \) we neglect the expansion of the Universe and drop the terms with \( \mathcal{H} \). The approximated metric perturbation then reads

\[
\phi_k^{(2)''} - 2 \frac{\dot{\phi}_k^{(2)}}{\phi_0} \phi_k^{(2)'} = \frac{2m_{\phi}^2 q}{M_c^2} \left[ 1 - \frac{q}{16} \right] \times B e^{q m_{\phi} t/2}
\]

\[
+ \text{[non-exponential source]},
\]

\[
(3.53)
\]
Assuming that on average after some oscillations, the fraction $\varphi_0'/\varphi_0$ can be approximated by the frequency of the coherent oscillations $\sim m_\varphi$, we readily obtain an exponential behaviour for the solution of Eq. (3.53). We may thus write

$$\phi_k^{(2)} \approx -\frac{2m_\varphi^2(1 - q/16) \cdot B}{M_p^2 m_\varphi^2(1 - q/4)} e^{q m_\varphi t/2}.$$  \hspace{1cm} (3.54)

Let us use the following definition [13] for the constant non-linearity parameter $f_{NL}^{\varphi}$:

$$f_{NL}^{\varphi} \equiv \phi = \phi^{(1)} + f_{NL}^{\varphi}(\phi^{(1)})^2.$$  However, the definition is given in $x$-space and we have performed our calculations in $k$-space. In principle we could transform $\phi_k^{(2)}$ back to $x$-space, but then we would need to know it for all $k$ and we have evaluated only the super-horizon mode $k = 0$. Instead, we can carry the definition of $f_{NL}^{\varphi}$ over to $k$-space by

$$\phi_k^{(2)} = f_{NL}^{\varphi} \frac{1}{(2\pi)^3} \int d^3 k' \phi_k^{(1)} \phi_{k-k'}^{(1)}$$

where we have treated $f_{NL}^{\varphi}$ as a constant.

In the present scenario, where the first order perturbations are equivalent to that of a single-field case, we have $\sigma_1 \sim \varphi_1$ right after inflation. Since, during inflation and preheating $\phi^{(1)}$ stays roughly constant, we immediately obtain an order of magnitude estimate from Eq. (3.46)

$$f_{NL}^{\varphi} \sim \frac{\phi_k^{(2)}}{(\phi^{(1)} * \phi^{(1)})_k} \approx -8 \frac{1 - q/16}{1 - q/4} \left( \frac{M_p}{\varphi_0} \right)^2 e^{N q/2}, \hspace{1cm} (3.55)$$

where we have written $N = t \omega$, where $N$ is the number of oscillations during preheating and $\omega$ is the frequency of the oscillations. On the average the frequency of the oscillations is given by $\omega \sim m_\varphi$. Inflation ends when $\varphi_0 \sim M_p$, therefore the coefficient in front of the exponential is of order one. The factor $B$ in the coefficient of Eq. (3.54), whose origin lies in the source terms, Eqs. (3.50), (3.51), and (3.52), cancels out completely from the contribution coming from $(\phi^{(1)} * \phi^{(1)})_k$, because of the initial evolution for $\varphi_1$ and $\sigma_1$ is the same.

In our case the amplitude of the oscillations remains constant. Therefore $N$ is an indicative number, valid until the backreaction kicks in and shuts off the parametric resonance. Obviously $N$ depends on the potential and on the expansion of the Universe. The parametric resonance would shift with the expansion as $q \sim \Phi^2 t^{-2}$ and thus becomes narrower. However, for a simple single field chaotic type inflation model with $V \sim (m^2_\varphi/2)\varphi^2$ one would typically have many oscillations within one Hubble time: $\omega H^{-1} \sim M_p/\Phi \gg 1$. It seems therefore that backreaction would be more decisive as far as the magnitude of the non-Gaussian amplitude $f_{NL}^{\varphi}$ is concerned. This requires more study, but let us point out that in chaotic inflation backreaction becomes important after 10-30 oscillations [47]. Hence $N = \mathcal{O}(10)$ might be a reasonable number, and should we for illustrative purposes choose $q = 0.8$, we would obtain $f_{NL} \approx e^4 \approx 55$. This should certainly be at an observable level for the Planck Surveyor Mission [169]; see also [189].

---

*The actual observationally constrained parameter is $f_{NL} = -f_{NL}^{\varphi} + (11/6)$, see [13]. This subtlety, however, is not relevant with the level of uncertainties in the present considerations.
The expansion of the Universe changes the situation in two ways. First, the parametric resonance can be broad with $q > 1$ a time dependent quantity $[47]$. Second, because of the expansion the momenta and the oscillation amplitude redshift, see Eqs. (1.77), (1.82), and (1.83). However, the amplitude of the first order metric perturbation still undergoes resonant amplification as the momentum modes drift through the broad resonance regime $[66, 67, 71, 72, 78]$. This ensures that the second order metric perturbations also grow exponentially, but one has to ensure that the amplitude of the initial perturbations for $\sigma$ does not damp away during inflation. A detailed study would require numerical simulation, but nevertheless we may conclude that our result hints at the possibility of exciting the second order metric perturbations during the first few oscillations of the inflaton, hence linking preheating with possibly observable non-Gaussianities.$^9$

### 3.5.1 Non-Gaussianity from instant preheating

The instant preheating scenario $[59]$ is described in Sec. 1.4.3. The non-Gaussianity produced during instant preheating was first studied in $[3]$. This section follows the treatment in the paper.

Here we have the same notation as in the previous section, Sec. 3.5. We have a potential

$$V = \frac{1}{2} m^2 \varphi^2 + g^2 \varphi^2 \sigma^2, \quad (3.56)$$

where $\varphi$ is the inflaton condensate with a mass $m_{\varphi}$, and $\sigma$ is another scalar field. We split the scalar fields into background and perturbations

$$\varphi = \varphi_0(\eta) + \varphi_1(\eta, x) + \frac{1}{2} \varphi_2(\eta, x), \quad (3.57)$$

$$\sigma = \sigma_1(\eta, x) + \frac{1}{2} \sigma_2(\eta, x), \quad (3.58)$$

with $\sigma_0$ again vanishing.

The relevant equations are (we denote $\kappa^2 \equiv M_P^{-2}$): for the background quantities

$$3\mathcal{H}^2 = \frac{\kappa^2}{2} \varphi_0'' + \frac{1}{2} \kappa^2 a^2 V(\varphi_0), \quad (3.59)$$

$$0 = \varphi_0'' + 2\mathcal{H}\varphi_0' + a^2 V'(\varphi_0), \quad (3.60)$$

for the first order quantities

$$\phi^{(1)''} - \partial_i \partial^i \phi^{(1)} + 2 \left( \mathcal{H} - \frac{\varphi_0''}{\varphi_0} \right) \phi^{(1)'} + 2 \left( \mathcal{H}' - \frac{\varphi_0''}{\varphi_0} \mathcal{H} \right) \phi^{(1)} = 0, \quad (3.61)$$

$$\sigma_1'' + 2\mathcal{H}\sigma_1' - \partial_i \sigma_1 \partial^i \sigma_1 + g^2 \varphi_0^2 \sigma_1 = 0, \quad (3.62)$$

$^9$Recently, Jokinen and Mazumdar $[159]$ studied massless preheating, $V = \frac{1}{2} \lambda \varphi^4 + \frac{1}{2} g^2 \varphi^2 \sigma^2$. They found that even the present observational limits of non-Gaussianity were exceeded with a certain range of parameter values.
and in the second order we have

$$\phi^{(3)\prime\prime} + 2 \left( \mathcal{H} - \varphi_0'' \right) \phi^{(2)\prime} + 2 \left( \mathcal{H}' - \mathcal{H} \varphi_0'' \right) \phi^{(2)} - \partial_i \partial^i \phi^{(2)} = J_{\varphi,\text{local}} + J_{\sigma,\text{local}} + J_{\text{non-local}},$$

(3.63)

where the source terms $J$ are quadratic combinations of first order perturbations

$$J_{\varphi,\text{local}} = \kappa^2 \left[ -2 (\varphi')^2 - 8 (\varphi_0')^2 (\phi^{(1)})^2 + 8 \varphi_0 (\phi^{(1)}) \varphi'_1 + a^2 \frac{\partial^2 V}{\partial \varphi^2} (\varphi_1)^2 \right] - 24 \mathcal{H}' (\phi^{(1)})^2 - 24 \mathcal{H} \phi^{(1)} \phi^{(1)\prime},$$

(3.64)

$$J_{\sigma,\text{local}} = -2 \kappa^2 (\sigma')^2 + \kappa^2 a^2 \frac{\partial^2 V}{\partial \sigma^2} (\sigma_1)^2,$$

(3.65)

$$J_{\text{non-local}} = \Delta^{-1} f(\varphi_1, \sigma_1, \phi^{(1)})$$

(3.66)

where $f$ is a quadratic function of the first order fluctuations and the coefficients depend on background quantities. Because of the inverse Laplacian the last source term is non-local. Typically such term contains: $\Delta^{-1} (\phi^{(1)} \Delta \phi^{(1)})$, $\Delta^{-1} (\partial_i \varphi_1 \partial^i \varphi_1)$, \ldots

In instant preheating the particle production occurs during one oscillation of the inflaton when it passes through the minimum of the potential $\varphi = 0$. In this case the process can be approximated by writing

$$\varphi = \dot{\varphi}_m (t - t_m),$$

(3.67)

where $\dot{\varphi}_m$ is the velocity of the field when it passes through the minimum of the potential at time $t_m$. The time interval within which the production of $\sigma$ quanta occurs is [59]

$$\Delta t^* = (g |\dot{\varphi}_m|)^{-1/2},$$

(3.68)

which is much smaller than the Hubble expansion rate; thus expansion can be neglected. Note that by virtue of the coupling $g^2 \varphi^2 \sigma^2$ the $\sigma$ field acquires a mass and provided that $g \leq H_{inj}/\varphi \sim 10^{-5}$ for $H_{inj} \sim 10^{13}$ GeV and $\varphi \sim \kappa^{-1} \sim 10^{18}$ GeV, the fluctuations in $\sigma$ field were already present on large scales during inflation with $\sigma_1 \sim H_{inj}/2\pi$.

The occupation number of produced particles jumps from its initial value zero to a non-zero value during $-\varphi_* \leq \varphi \leq \varphi_*$. In the momentum space the occupation number is given by [59],

$$n_k = \exp \left( - \frac{\pi k^2}{g |\dot{\varphi}_m|} \right),$$

(3.69)

and the largest number density of produced particles in $x$-space reads

$$n_\sigma \approx \frac{(g |\dot{\varphi}_m|)^{3/2}}{8\pi^3},$$

(3.70)
with the particles having a typical energy of \((g|\dot{\phi}_m|/\pi)^{1/2}\), so that their total energy density is given by

\[
\rho_\sigma \sim \frac{1}{2} (\dot{\sigma}_1)^2 \sim \frac{(g|\dot{\phi}_m|)^2}{8\pi^{7/2}}.
\] (3.71)

These expressions are valid if \(m^2_\sigma < g|\dot{\phi}_m|\), a condition that we assume for the rest of our calculation.

Ignoring the expansion of the Universe \((a = 1, \eta = t)\), and using Eq. (3.67), the second order gravitational perturbation, Eq. (3.63), reads at large scales

\[
\ddot{\phi}^{(2)} \sim -2\kappa^2 (\dot{\sigma}_1)^2.
\] (3.72)

The non-Gaussianity in the gravitational potential, parameterised as (see e.g. [13]) \(\phi^{(2)} = f_{\phi}^{NL}(\phi^{(1)})^2\), can be estimated by solving the second order gravitational potential from Eq. (3.72) using Eq. (3.71). We obtain

\[
f_{\phi}^{NL} = \left| \frac{g^2|\dot{\phi}_m|^2 \Delta \nu_s^2}{8\pi^{7/2} M_P^2 (\phi^{(1)})^2} \right|.
\] (3.73)

It is a simple exercise to estimate the right hand side for a chaotic inflaton potential with \(m = 10^{13}\) GeV and \(\phi^{(1)} \sim 10^{-5}\). The velocity of the scalar field at the potential minimum turns out to be \(|\dot{\phi}_m| \approx 10^{-7} M_P^2\); using these values and Eq. (3.68) we obtain an estimate for the upper limit of the non-Gaussianity parameter in the case of instant preheating:

\[
f_{\phi}^{NL} \sim 2g.
\] (3.74)

Thus, Eq. (3.74) implies that \(f_{\phi}^{NL} \ll 1\) and instant preheating therefore is unlikely to yield any detectable non-Gaussian signal in the forthcoming CMB experiments.\(^{10}\)

### 3.5.2 Non-Gaussianity from tachyonic preheating

The tachyonic preheating scenario is presented in Sec. 1.4.4. The non-Gaussianity produced during tachyonic preheating was first studied in [3]. This section follows the treatment in the paper.

In order to understand the non-Gaussianity triggered by the tachyonic instability, let us assume a simple toy model where there is an inflationary sector \((\varphi)\) and a symmetry breaking phase transition with a negative mass squared term:

\[
V = V(\varphi) + V_0 - \frac{1}{2} m^2 \chi^2 + \frac{\lambda}{4} \chi^4.
\] (3.75)

We assume that the inflaton potential is some polynomial potential with a vanishing VEV, \(V(\varphi) \sim f(\varphi^a)\). Inflation is supported by \(V(\varphi) + V_0\). During inflation we assume

\(^{10}\)In [158] Byrnes and Wands also consider non-Gaussianity produced by instant preheating. They conclude that the level is likely to be unmeasurable, but that there is a possibility for large \(f_{\phi}^{NL}\). However, their model is more complicated and involves a complex scalar field with different coupling constants for real and imaginary components. The level of produced non-Gaussianity is sensitive to the difference between the coupling constants and the phase of the complex field.
that the tachyon field is sitting at the maximum $\chi = 0$ by virtue of large friction. The mass of $\chi$ is such that the tachyonic instability is triggered when $m \geq H \sim V_0/3M_P^2$. During this period we assume that the inflaton settles down to $\langle \varphi \rangle = 0$. This will allow us to separate the tachyon fluctuations from that of the inflaton. This also allows us to use the same equations (3.57) - (3.66) but now the tachyon field $\chi$ obeys Eq. (3.61) and the inflaton field $\varphi$ obeys Eq. (3.62). So we replace $\varphi \rightarrow \chi$ in Eq. (3.61) and in $\sigma \rightarrow \varphi$ in Eq. (3.62), and we make similar replacements in Eq. (3.63) and the expressions for the source terms $J_{\chi,\text{local}}$ and $J_{\varphi,\text{local}}$ respectively.

The rolling of the tachyon results in an exponential instability in the perturbations of $\chi$ with physical momenta smaller than the mass. The tachyonic growth takes place within a short time interval, $t_* \sim (1/2m) \ln(\pi^2/\lambda)$ (see [61] and Sec. 1.4.4). During this short period the occupation number of $\chi$ quanta grows exponentially for modes $k < m$ up to $n_k \sim \exp(2mt_*) \sim \exp(\ln(\pi^2/\lambda)) \sim \pi^2/\lambda$. For very small self-coupling, which is required for a successful inflation, the occupation number, which depends inversely on the coupling constant, can become much larger than one.

The scalar field fluctuations, which are responsible for exponentially enhancing the occupation number for $\chi$ quanta, also couple to the metric fluctuations. If we assume that the modes grow within a time interval much smaller than the Hubble rate, we can set $H = 0$ in Eq. (3.61). Then, in the long wavelength limit, we get from Eq. (3.61),

$$\ddot{\phi}^{(1)} - 2A \dot{\phi}^{(1)} = 0,$$

(3.76)

where $A = \dot{\chi}/\dot{\chi}_0$. With the assumption of a brief tachyonic stage we can take $A$ to be effectively constant. Note that although during rolling tachyon the long wavelength modes are excited it is important that the tachyon perturbations must exist during inflation. In this respect the tachyon fluctuations have isocurvature nature. In order to further simplify our calculation we neglect the inflaton perturbations in our subsequent analysis.

There are two solutions to Eq. (3.76); a constant $\phi^{(1)} \sim 10^{-5}$, and an exponentially growing solution $\phi^{(1)} \propto \exp(2At)$. If the isocurvature component at the end of inflation is small, then the first derivative of $\phi^{(1)}$ is also small but non-vanishing. Hence we may neglect the exponential solution of the first order metric perturbation. With these simplified approximations we can then estimate the amount of generated non-Gaussianity by following a logic similar to the case of instant preheating.

First, the number density of the produced particles in $x$-space is given by $n_\chi \sim m^3/(8\pi\lambda)$. Hence the total energy density stored in produced $\chi$ quanta is given by

$$\rho_\chi \sim \frac{1}{2}(\dot{\chi}_1)^2 \sim mn_\chi \sim \frac{1}{8\pi} \frac{m^4}{\lambda}.$$  (3.77)

The main contribution to the second order metric perturbation comes from the excitations of the tachyonic instability. The inflaton fluctuations are subdominant compared to the exponential growth of $\chi_1$, when the inflaton is settled around its
VEV $\langle \varphi \rangle = 0$. In the long wavelength regime the perturbation equation (3.63) reads as

$$\ddot{\phi}^{(2)} \sim -\frac{1}{\pi^2} \frac{m^4}{\lambda}. \quad (3.78)$$

Integrating the above equation over the time interval $t_\ast \sim (1/2m) \ln(\pi^2/\lambda)$, we find $\phi^{(2)} \sim (m/M_P)^2 \ln^2(\pi^2/\lambda)/(4\pi \lambda)$. In case the first order metric perturbation stays constant the non-Gaussianity parameter for tachyonic preheating is roughly given by [3]

$$f_{\phi}^{NL} \sim \frac{1}{4\pi} \left( \frac{m}{M_P} \right)^2 \frac{2}{\lambda} \ln \left( \frac{\pi^2}{\lambda} \right), \quad (3.79)$$

where we substitute $\kappa \sim M_P^{-1}$. Writing this in terms of $V_0 = m^4/(4\lambda)$, as in hybrid inflation Sec. 1.3.5, and taking $\phi^{(1)} \sim 10^{-5}$, we obtain

$$f_{\phi}^{NL} \sim 1.6 \times 10^9 \lambda^{-1/2} \left( \frac{V_0^{1/4}}{M_P} \right)^2 \ln \left( \frac{\pi^2}{\lambda} \right). \quad (3.80)$$

This expression should again be compared with the observationally constrained one: $f_{NL} = -f_{NL}^{\phi} + 11/6$. WMAP observations set the limit $-132 < f_{NL}^{\phi} < 60$, at 95% confidence level [12]. Adopting the upper limit $|f_{NL}^{\phi}| < 132$ and rearranging, we arrive at the bound $V_0^{1/4}/M_P \leq 3 \times 10^{-4} \lambda^{1/4} \ln^{-1} (\pi^2/\lambda)$.

For an effective field theory to remain perturbative we should require that $\lambda \ll 1$, which yields the interesting constraint $V_0^{1/4}/M_P \ll 10^{-4}$. Note that compared to the usual bound $V_0^{1/4} \leq 10^{16} \text{GeV}$ from COBE normalisation, the absence of observable non-Gaussianity implies a bound on the scale of tachyonic instability $V_0^{1/4}$ which is more stringent by two orders of magnitude. The parameter space allowed by WMAP data is given by the region below the sloped lower curve in Fig. 3.1.

When obtaining the result above we have assumed that the first order metric perturbations are roughly given by the constant value determined by the inflationary epoch. Let us investigate the other limit when the first order metric fluctuations also obtain an exponentially growing solution by virtue of the tachyon excitations. To check this possibility, let us assume that the exponential solution actually dominates over the constant one. Following the second order analysis and assuming that the main contribution to the second order perturbation arises from the tachyonic instability, we obtain from Eq. (3.63)

$$\ddot{\phi}^{(2)} - 2A \dot{\phi}^{(2)} = -\kappa^2 \left[ 2\chi_1^2 + 8\dot{\chi}_0^2(\phi^{(1)})^2 - V_{,\chi\chi}(\chi_1)^2 - 8\dot{\chi}_0 \phi^{(1)} \dot{\chi}_1 \right], \quad (3.81)$$

where $V_{,\chi\chi} = -m^2 + 3\lambda \chi^2$ and where $\chi_1$ can be solved through the Einstein constraint, Eq. (2.44), $\chi_1 = (2/\kappa^2 \dot{\chi}_0) \left( \dot{\phi}^{(1)} + H\phi^{(1)} \right)$.

For the tachyonic region $3\lambda \chi^2 < m^2$ and we can take $V_{,\chi\chi} \sim -m^2$. Now $\phi^{(2)}$ contains the homogeneous solution $\sim \exp(2At)$ together with a source part $\sim$
Figure 3.1: The parameter space in tachyonic preheating allowed by the WMAP data for single field inflation (below the sloped lower curve). The horizontal upper line is the usual limit $V_0^{1/4} = 10^{16}$ GeV coming from the COBE normalisation.

The tachyonic growth persists until $\chi \sim m/(2\sqrt{\lambda})$, which happens at a time $t_* \sim (1/2m) \ln(\pi^2/\lambda)$. With these approximations, and writing again $V_0 = m^4/(4\lambda)$, we obtain

$$f_{NL}^\phi \approx 8 - 8\sqrt{\frac{\lambda M_P^4}{V_0}} - \frac{1}{2} \sqrt{\frac{V_0}{\lambda M_P^4}} - \sqrt{\frac{\lambda M_P^4}{V_0}} \ln^2 \left(\frac{\pi^2}{\lambda}\right). \quad (3.83)$$

If we assume COBE normalisation $V_0^{1/4} \leq 10^{16}$ GeV, the minimum of $V_0$ is given by the conditions $V_0^{1/4}/M_P = \lambda^{1/4} \sqrt{8 - f_{NL}^\phi}$ and $\lambda = \pi^2 \exp(-\sqrt{(8 - f_{NL}^\phi)^2/2 - 8})$. These imply the limit $f_{NL}^\phi < -37$ regardless of the value of $\lambda$, well within the

\[^{11}\]The second order metric perturbations always have a growing source term by virtue of the non-vanishing background motion of the scalar field, i.e. $\chi_1, \dot{\chi}_1$.

\[^{12}\]Assuming $A$ is constant; in reality there will be a small time variation but we may assume that most of the interesting modes are growing within a time interval which is short compared to the variation in $\dot{\chi}_0/\chi_0$ and the Hubble rate.
observational capabilities of WMAP.

The non-Gaussianity from tachyonic stage provides a handle on many string motivated inflationary models, e.g. on inflation first driven by brane-anti-brane interaction and then coming to an end when the tachyonic instability is triggered; see [5] for a recent work on this.

3.6 Other approaches to non-Gaussianities

In the literature there exists other approaches to studying non-Gaussianity in addition to the (pure) cosmological perturbation theory applied in the papers included in this thesis ([1–4]); see [13] and references therein for a review on the perturbation theory approach. Some of the other approaches are closely related to perturbative approaches and some are not. Here we briefly describe the main ideas and results of the approaches without going into details.

3.6.1 Separate universe approach / $\delta N$ formalism

The $\delta N$ approach by Sasaki and Stewart [187] (see also [190] for a recent extension to multi-field inflation) has also been used to study non-Gaussianities. The general idea of the separate universe approach (see e.g. [81] for a concise description) is to consider each point in space as being surrounded by a homogeneous Friedmann–Robertson–Walker universe. Each point then has its own expansion parameter $N$, i.e. local number of e-folds, independent of the value of the expansion parameter (or any quantity) in other points. The complete, inhomogeneous, behaviour of the universe is obtained when all the separately treated points are patched together.

Let us now consider the non-perturbative form of the spatial metric [167]

$$
g_{ij} = a^2(t)e^{2\zeta(t,x)}\gamma_{ij}(t, x),
$$

(3.84)

where, within inflationary context, $\gamma_{ij}$ contains the tensor perturbation which we do not consider here, (see e.g. [106] for more details). Eq. (3.84) now defines the curvature perturbation $\zeta$, which has been shown to be conserved using $\delta N$ approach, (see also [191]). The (local) amount of expansion $N$ can now be connected to the curvature perturbation $\zeta$ [167]

$$
\zeta(t, x) = \delta N \equiv N(t, x) - N_0(t),
$$

(3.85)

where $N(t, x)$ is the local amount of expansion and $N_0(t)$ is the unperturbed value, i.e. the value of the 'surrounding' universe. In [167] it is expanded up to second order in scalar field perturbations ($\delta\phi_i \equiv \delta\phi_i(t, x)$) as

$$
\zeta(t, x) = \sum_i N_i(t)\delta\phi_i + \frac{1}{2} \sum_{ij} N_{ij}(t)\delta\phi_i\delta\phi_j,
$$

(3.86)
where $N_i \equiv \frac{\partial N}{\partial \phi_i}$ and $N_{ij} \equiv \frac{\partial^2 N}{\partial \phi_i \partial \phi_j}$.

The nonlinearity parameter $f_{NL}$ defined (up to a constant offset) by

$$\zeta(x) = \zeta_g(x) - \frac{3}{5} f_{NL} \zeta_g^2(x),$$  

(3.87)

($\zeta_g(x)$ is the Gaussian curvature) is obtained from the formula [167] (see also [170])

$$-\frac{3}{5} f_{NL} = \sum_{ij} N_{i} N_{j} N_{ij} \frac{3}{2} \sum_{i} \left[ N_{i}^2 \right]^2 + \ln(kL) \frac{P_{\zeta} \sum_{ijk} N_{ij} N_{jk} N_{ki}}{2 \left[ \sum_{i} N_{i}^2 \right]^3},$$  

(3.88)

where $P_{\zeta}$ is the spectrum of $\zeta$. According to [167], $k^{-1}$ is a typical scale under consideration and $L$ is the size of the region within which the stochastic properties are specified and, therefore, the logarithm can be taken to be of order 1. The important point is that (except for the logarithm) the $f_{NL}$ is momentum independent.

This formalism has been used, for example, to work out the non-Gaussianity in the curvaton scenario and in two field model ([167]), whose non-Gaussianity was first considered in [1], see also [4].

### 3.6.2 Stochastic approach

There is an approach by Rigopoulos et al. [126, 128, 139, 143, 155] where the stochastic formalism, previously studied within linear order (see e.g. [110, 192-199]), is extended to nonlinear order and used to study cosmological non-Gaussianities.

The starting point is to make the long wavelength approximation, which corresponds to dropping all non-leading terms in the gradient expansion [128, 191], (see also [91-93, 107, 109]), and construct fully nonlinear equations. These equations are valid outside (Hubble) horizon. The short wavelength quantum fluctuations are taken into account by consistently introducing stochastic sources. The stochastic sources are solutions to the linear perturbation equations on small scales, which means that the second order effects on small scales are neglected. The authors of [126, 128, 139, 143, 155] claim that the error made this way is not significant.

The formalism is well suited for numerical implementation and numerical solutions can be obtained directly without additional approximations. For analytical solutions one needs to make perturbative expansion and additional approximations, e.g., the slow roll approximation.

This formalism is applied to both single-field and multi-field inflation [128, 143, 155]. It is found that the nonlinearity parameter $f_{NL}$ in general has nontrivial momentum dependence. This result has also been established and studied in [14, 136, 141, 147, 168, 200-202]. Also, the previous result [14] that no significant non-Gaussianity is produced during single field inflation is confirmed. There is, however, a small discrepancy in the result with [14] when the momenta are comparable, $k_1 \approx k_2 \approx k_3$ [128]. This is said to be due to the fact that the stochastic sources are solutions to the linear equations, i.e., the second order effects inside the (Hubble) horizon are neglected [143].
The multi-field inflation, on the other hand, is found to be capable of producing significant non-Gaussianity. In [143] an explicit two-field model, \( V = \frac{m_1^2}{2} \phi_1^2 + \frac{m_2^2}{2} \phi_2^2 \), is studied and a realistic possibility for \( f_{NL} \gtrsim 1 \) is found. This is in agreement with other results. The same model is studied further in [155], where the isocurvature contribution to \( f_{NL} \) is worked out. A possibility for a large \( f_{NL} \) in models where there is a turn in field trajectory in the last 60 e-foldings is found. This holds even without interactions in the potential.

### 3.6.3 Field theoretical approach: interactions in the Lagrangian

Quantum field theoretical tools have been utilised in [14, 120, 136, 141], where the approach has been to start from the action (or Lagrangian) and work out the tree level interactions. Later Weinberg [147] reviewed the formalism, which he calls “in-in”-formalism, and worked out the loop contribution.

According to [147] there are certain main characteristics of the “in-in”-formalism when applied to cosmology. Firstly, it is not \( S \) matrix elements which are calculated but expectation values of products of fields at a fixed time (but generally at different points in space)\(^{13}\). Secondly, the boundary conditions on the fields are imposed at early times, when the scales in question are well inside horizon, not on both early and late times. And, finally, the time dependence of the field fluctuations is governed by fluctuation Hamiltonian with explicit time dependence, although the actual Hamiltonian generating the time dependence of the fields is itself time independent.

The approach was initiated in [14], where three-point function for primordial scalar and tensor perturbations in single-field inflation was computed. The non-Gaussianities were seen to come from cubic terms in the action, and they arise both from the nonlinearities in the Einstein action and from the nonlinearities in the scalar field potential. The momentum dependence of the nonlinearity parameter \( f_{NL} \) was also established in [14].

Later the effect of higher derivative interactions of the form \( (\nabla \phi)^4 \) on non-Gaussianities were studied in [120]. In [136] the three-point function for primordial scalar fields was calculated in the case when the scalar field Lagrangian is completely general function of the field and its first derivative. The conclusion of these studies is that the level of non-Gaussianity is usually small in single-field inflation, even though the actual momentum dependence may vary according to how the canonical Lagrangian is extended. In [141] the treatment was generalised for multiple scalar field (with canonical Lagrangian), and the intrinsic 3-point function of the coupled scalar-gravity system was computed at horizon crossing. This result could be used as an initial condition for the super-horizon evolution in the \( \Delta N \) approach.

\(^{13}\)As Maldacena said in [14] he is not calculating transition amplitude but expectation value.
Chapter 4

Summary

Non-Gaussianity is a novel tool in studying the physics of the very early universe. It goes beyond the CMB power spectrum, which previously has been the main source of observational data. Present observations are consistent with purely Gaussian primordial perturbations. Different theoretical models and scenarios, however, predict different amount (and type) of non-Gaussianity in addition to the Gaussian perturbations. This provides a new method of distinguishing different scenarios of the early universe. Observationally the situation seems quite promising. Already with the present data and analyses several models are within reach. For example, the non-Gaussianity arising during preheating or in the curvaton scenario seems to be capable of saturating, and even exceeding, the observational limits.

The thesis work consists of studying the amount of non-Gaussian perturbation arising in certain two-field scenarios. In [1] we have developed and extended the cosmological perturbation theory formalism to second order for two scalar fields. Then, we have applied the formalism and studied hybrid inflation [1, 4], preheating with parametric resonance [2], as well as instant and tachyonic preheating [3].

For single-field models the produced non-Gaussianity is known to be small. We have found real possibility for significant non-Gaussian perturbation in two-field scenarios. This is in accordance with other studies as well. In hybrid inflation large non-Gaussianity may possibly arise but by no means inevitably. Standard preheating with parametric resonance, however, seems prone to produce significant level of non-Gaussian perturbation, enough to easily saturate the present observational limits. Although, further understanding of the backreaction and rescattering, and nonlinear effects in general, during preheating is clearly needed. We have also found that the simplest model of instant preheating is not capable of producing observable levels of non-Gaussianity, but then again, tachyonic preheating easily reaches observational sensitivity. The present observational limits can already be used to constrain models. The constraints on tachyonic preheating have been applied to string scale and coupling.

There is a clear need for further development of the formalism, or formalisms, to study non-Gaussianities. And indeed, there is an active industry and lot of studies
aiming to get a handle on nonlinear effects and non-Gaussianities. Different approaches and formalisms also need to be compared and tested against each others, e.g., by applying them to the same models and scenarios. For example, the issue of non-local terms is far from settled. The perturbation theory formalism inevitably seems to produce them, while the separate universe approach does not. The non-local terms are quite difficult to interpret, but on the other hand, in second (and higher) order of perturbation expansion the mode couplings in general are difficult to decipher. Especially in preheating, where one is studying the evolution of the Fourier modes, the mode-mode couplings become extremely important. Our understanding of the processes during preheating is clearly lacking and further studies are needed. The effect and importance of backreaction and re-scattering on the production of non-Gaussianities needs clarification, and numerics would likely be of help.

Also observationally there is room for improvement. The present constraints on the nonlinearity parameter are mainly on constant $f_{NL}$, and are likely to be improved by taking the theoretical predictions for the scale dependence properly into account.
References


[157] P. Cabella et. al., The integrated bispectrum as a test of CMB non-Gaussianity: detection power and limits on $f_{NL}$ with WMAP data, astro-ph/0512112.


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