LIGHTCURVE INVERSION FOR ASTEROID SPINS AND SHAPES

Johanna Torppa

Department of Astronomy
Faculty of Science
University of Helsinki, Finland

Academic dissertation

To be presented, with the permission of the Faculty of Science of the University of Helsinki, for public criticism in the large auditorium at Bulevardi 18 on December 14, 2007, at 12 noon.

ISBN 978-952-10-4410-6 (paperback)
ISBN 978-952-10-4416-8 (PDF)
Helsinki, 2007
Yliopistopaino

Abstract

Knowledge of the physical properties of asteroids is crucial in many branches of solar-system research. Knowledge of the spin states and shapes is needed, e.g., for accurate orbit determination and to study the history and evolution of the asteroids. In my thesis, I present new methods for using photometric lightcurves of asteroids in the determination of their spin states and shapes. The convex inversion method makes use of a general polyhedron shape model and provides us at best with an unambiguous spin solution and a convex shape solution that reproduces the main features of the original shape when an abundant data set is available. Deriving information about the non-convex shape features is, in principle, also possible, but usually requires a priori information about the object. Alternatively, a distribution of non-convex solutions, describing the scale of the non-convexities, is also possible to be obtained. Due to insufficient number of absolute observations and inaccurately defined asteroid phase curves the flatness of an individual shape is somewhat ill-defined for spheroidal objects. In the case of elongated objects, on the other hand, all the axis ratios are often reasonably well constrained, even in the case when only relative lightcurves are available. The results prove that it is, contrary to the earlier misbelief, possible to derive shape information from the lightcurve data if a sufficiently wide range of observing geometries is covered by the observations. Along with the more accurate shape models, also the rotational states, i.e., spin vectors and rotation periods, are defined with improved accuracy. When only a highly limited data set is available, fast semi-analytical methods can be used to obtain a distribution of possible spin and shape solutions. The distributions can be used, for example, in planning for additional observations required for unambiguous spin and shape determination. The shape solutions obtained so far reveal a population of irregular objects whose most descriptive shape characteristics, however, can be expressed with only a few parameters. Preliminary statistical analyses for the shapes suggests that there are correlations between shape and other physical properties, such as the size, rotation period and taxonomic type of the asteroids. More shape data of, especially, the smallest and largest asteroids, as well as the fast and slow rotators is called for in order to be able to study the statistics more thoroughly.

Acknowledgements

I want to thank Karri Muinonen, who acted as the supervisor for the latter half of my thesis, and who so often came to me and asked how things are. He always had time, when I needed advise. Neither did he give up hope, though I now and then told that I will be off work for a while, again. I also owe thanks to Mikko Kaasalainen, the supervisor of the first half of my thesis work, and the pre-examiners Markku Lehtinen and Alan Harris.

I am grateful to my colleagues in the Planetary research group, especially Jyri Näränen, Mikael Granvik and Antti Penttilä, who all have contributed my work in one way or another. Special thanks to Jenni Virtanen, who originally dragged me in the group, and with whom I also have shared so many nice moments outside work.

Finnish amateur astronomers are to be thanked for their enthusiasm in carrying out scientific observations. Special thanks to Petri Kehusmaa, Pertti Pääkkönen, Tommi Itkonen, Jani Silvennoinen, Teemu Vartiainen, Veli-Pekka Hentunen, Markku Nissinen and Arto Oksanen for carrying out asteroid lightcurve observations for me.

For financial support I am grateful to Emil Aaltonen foundation, Jenny and Antti Wihuri foundation, Vilho, Yrjö, and Kalle Väisälä foundation, Finnish graduate school in astronomy and space physics, and The University of Helsinki. All the work for this thesis has been carried out at the Observatory of the University of Helsinki.

And, of course, no words can describe my gratitude to my family. Especially, I am thankful to three little people, Niilo, Onni and Vilpas, without whom this thesis would have been published many years ago.

List of the publications included in the thesis

Paper I: Kaasalainen M. and **Torppa J.**, 2001. Optimization methods for asteroid lightcurve inversion. I. Shape determination, Icarus, 153, 24.

Paper II: Kaasalainen M., **Torppa J.**, and Muinonen K., 2001. Optimization methods for asteroid lightcurve inversion. II. The complete inverse problem, Icarus, 153, 37.

Paper III: Torppa J., Kaasalainen M., Michalowski T., Kwiatkowski T., Kryszczyńska A., Denchev P. and Kowalski R., 2003. Shapes and rotational properties of thirty asteroids from photometric data, Icarus 164, 346.

Paper IV: Torppa J. and Muinonen K., 2005. Statistical Inversion of Gaia Photometry for Asteroid Spins and Shapes, in *Proceedings of the Gaia Symposium The Three-Dimensional Universe with Gaia (ESA SP-576).* Held at the Observatoire de Paris-Meudon, 4-7 October 2004. Editors: C. Turon, K.S. O'Flaherty, M.A.C. Perryman, p. 321.

Paper V: Muinonen K., **Torppa J.**, Virtanen J., and 15 co-authors, 2007. Spins, shapes, and orbits for near-Earth objects by Nordic NEON, In *Proceedings of IAU Symposium No 236, Near-Earth Objects, our Celestial Neighbors: Opportunity and Risk* (A. Milani, G. Valsecchi, and D. Vokrouhlicky, eds.), p 309.

Paper VI: Torppa J., Hentunen V-P., Pääkkönen P., Kehusmaa P., and Muinonen K., 2007. Asteroid shape and spin statistics from convex models, submitted to Icarus.

Abbreviations used in the text

MBA	main-belt asteroid
NEA	near-Earth asteroid
NEO	near-Earth object
NEON	Near-Earth-Object Network
ISO	The Infrared Space Observatory
NASA	National Aeronautics and Space Administration
JAXA	Japan Aerospace Exploration Agency
PanSTARRS	Panoramic Survey Telescope and Rapid Response System

Contents

1 Introduction

The physical properties of asteroids, such as spin states and shapes, reflect the history and evolution of these small solar-system objects. The most abundant data source for determining the physical properties are the photometric observations. For spin and shape determination, it is sufficient to study the integrated brightness of the target, which has been known for more than a hundred years to vary with time. A natural explanation for the variation is rotation combined with irregular shape and varying scattering properties across the surfaces. Russell (1906) was the first to analyze the inverse problem of determining the pole orientation, rotation period, three-dimensional shape, and scattering properties of the surface of a body from disk-integrated photometry. Russell mainly studied the stars which always radiate from every point of their surfaces. Thus, when considering asteroids, a natural choice for him was to treat them at opposition (i.e., when the Sun and the Earth are in the same direction, as seen from the asteroid), where they, as well, scatter light from every visible point of their surface. The opposition geometry was a relevant assumption also since, in the beginning of the twentieth century, asteroids were most often observed near opposition. Russell assumed, to a good approximation that, at opposition, the body scatters light geometrically, i.e., the total brightness depends only on the projected surface area. The quite pessimistic conclusion he made was that one cannot distinguish the shape and scattering effects to any acceptable accuracy from photometric opposition lightcurve data alone. Only information about possible existence of albedo variegation or non-spherical shape can be obtained.

Later, the theoretical and observational techniques have become more and more powerful, and observations at larger phase angles (i.e., angles between the direction to the Sun and to the Earth, as seen from the asteroid) have been carried out; even phase angles over 100° have been reached for NEAs (near-Earth asteroids). Thus, the information content in the lightcurves is currently much larger than at Russell's time. Methods for interpreting asteroid lightcurves have been under constant development, but the problem of the traditional methods has been the simplicity of, especially, the shape models, in part due to a common misbelief that more detailed information cannot be obtained. Also, the efficiency of present-day computers has inspired the development of alternatives to analytical methods which necessarily limit the variety of possible models.

In the thesis, I present a novel lightcurve inversion method, called *convex inversion*, that provides us a convex hull -like shape model of the target and a spin solution which is more accurate than those obtained with other methods. Also, the possibility to derive information about the non-convex features on the asteroid's surface is discussed. The study was inspired by the work of Kaasalainen et al. (1992) who showed that it is possible to obtain shape models for asteroids based on the lightcurve data if a wide enough range of observing geometries is covered; the present thesis is the implementation of the practical numerical methods. A crucial part of convex inversion is also the numerical method for constructing the shape from the Gaussian surface density (Lamberg, 1993). The groundwork for deriving convex polyhedron shape models from lightcurve data was carried out by Torppa (1999). In the following, I give a brief review of the included papers:

Paper I: Kaasalainen and Torppa, 2001. Optimization methods for asteroid lightcurve inversion. I. Shape determination, Icarus, 153, 24.

Paper II: Kaasalainen M., Torppa J., and Muinonen K., 2001. Optimization methods for asteroid lightcurve inversion. II. The complete inverse problem, Icarus, 153, 37.

Paper III: Torppa J., Kaasalainen M., Michalowski T., Kwiatkowski T., Kryszczyńska A., Denchev P. and Kowalski R., 2003. Shapes and rotational properties of thirty asteroids from photometric data, Icarus 164, 346.

Paper IV: Torppa et al., 2005. Statistical inversion of GAIA photometry for asteroid spins and shapes, in *Proceedings of the Gaia Symposium The Three-Dimensional Universe with Gaia (ESA SP-576).* Held at the Observatoire de Paris-Meudon, 4-7 October 2004. Editors: C. Turon, K.S. O'Flaherty, M.A.C. Perryman, p. 321.

Paper V: Muinonen K., Torppa J., Virtanen J., and 15 co-authors, 2007. Spins, shapes, and orbits for near-Earth objects by Nordic NEON, In *Proceedings of IAU Symposium No 236, Near-Earth Objects, our Celestial Neighbors: Opportunity and Risk* (A. Milani, G. Valsecchi, and D. Vokrouhlicky, eds.), p 309.

Paper VI: Torppa J., Hentunen V-P., Pääkkönen P., Kehusmaa P., and Muinonen K., 2007. Asteroid shape and spin statistics from convex models, submitted to Icarus.

In Paper I, the theory of determining the shapes using the convex inversion method is presented and applied to simulated lightcurve data. Methods for obtaining non-convex shapes are also presented. In Paper II, the spin state is added to the inverse problem, and the complete convex inversion procedure is applied to asteroids with shape models available from spacecraft flybys or radar observations. The choice of a suitable scattering law is discussed, as well as the possibility to generate reference phase curves if absolute photometry is available. In Paper III, the convex inversion method is applied to 30 asteroids, revealing a population of irregular and elongated shapes. In Paper IV, a data set typical for the future all-sky surveys is analysed with the convex inversion and spherical-harmonics methods, the latter of which is largely similar to the non-convex inversion method presented in Paper I. In Paper V, the convex inversion method is applied to the first NEA observations of the Nordic NEON (Near-Earth-Object Network) observing program. A novel method is also presented to derive a distribution of possible spin states and shapes from very limited data. In Paper VI, the shape models derived so far are collected together, and a statistical analysis is carried out to find correlations between the shape, size, rotation period, and taxonomic type.

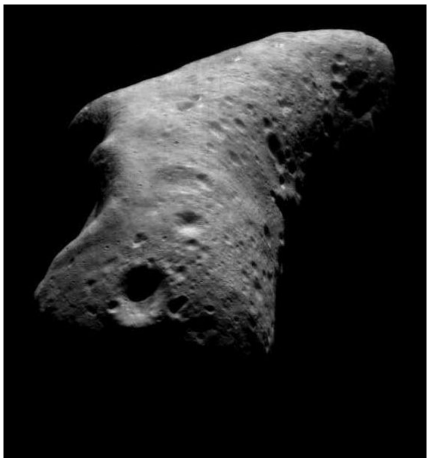
2 Asteroids

2.1 Overview

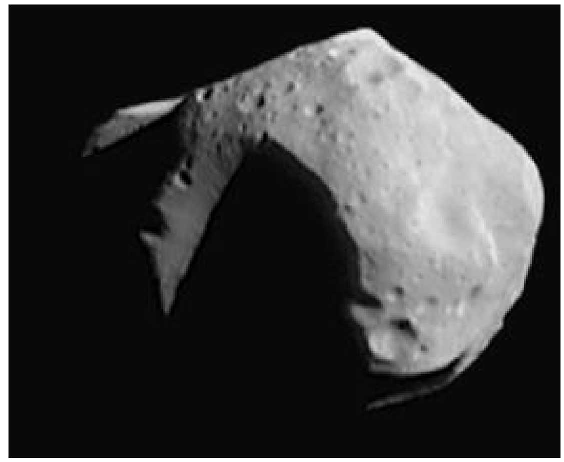
In addition to the eight planets, there are populations of smaller objects orbiting the Sun: comets, asteroids, and trans neptunian objects. These populations differ significantly from each other by the orbits and physical properties of their members. Comets are icy bodies in the outer Solar system with stellar and galactic perturbations delivering some of them periodically close to the Sun. Trans neptunian objects, as well, are icy, but have orbits that constantly keep them in the outer borders of the Solar system. Pluto, which was previously known as the ninth planet, is now considered to be one of the large trans neptunian objects called *dwarf planets*. Asteroids, which are the subject of this thesis, are rocky bodies, that lie mostly in the inner Solar system. The terminology is ambiguous since, e.g., the compositional differences between asteroids and dormant comets are thought to be no larger than the differences between different asteroid classes. Thus, in the present thesis, we do not make a distinction between these populations, but call all of the rocky small bodies asteroids. Asteroids were discovered in 1801, when (1) Ceres was first observed by Piazzi. The name *asteroid* ('star-like') was given by William Herschel in 1802, because these objects are so small compared to their distance from the Earth that they are observed as point sources, just like stars. The next three asteroids (2) Pallas, (3) Juno, and (4) Vesta were discovered within the next seven years, after which it took almost 40 years to discover the next one, (5) Astraea. Ever since, increasing numbers of new asteroids have been discovered per each year. At the moment, there are about 160000 numbered asteroids (as of September, 2007), that is to say, asteroids with well-defined orbits. The estimated total number of 1-km-sized objects or larger is 1.1-1.9 million, according to the results from the ISO (The Infrared Space Observatory) Deep Asteroid Search.

Most of the asteroids orbit the Sun in the main belt between the orbits of Mars and Jupiter but, due to planetary resonances, some of the main-belt asteroids (MBAs) are ejected to unstable orbits outside Jupiter's or inside Mars' orbits. A few populations of asteroids also have stable orbits at or outside that of Jupiter's. Trojans, for example, leading and following Jupiter on its orbit, are only slightly less numerous than MBAs. The largest known asteroid is Ceres, with a diameter about 1000 km. A lower limit is not strictly defined for asteroid size and, in practice, the size distribution is continuous from one thousand kilometers to dust grains. The smallest particles, however, have traditionally been called *meteoroids* or *dust*.

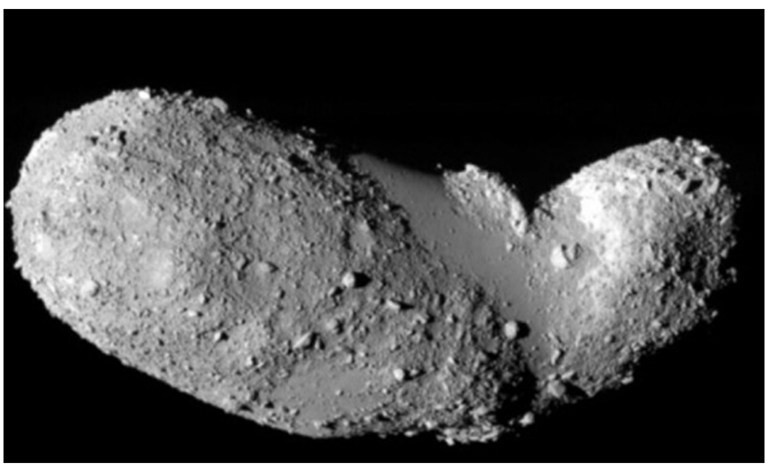
Asteroids are thought to be remnants from the early stages of the Solar system evolution, when most of the material around the young Sun accreted to form the eight planets. Part of the remaining dust particles ended up as building blocks of asteroids. One scenario of the formation of asteroids is the collisional disruption of one large or a number of smaller planetesimals. Another widely accepted explanation is that the formation of the largest planets (Jupiter and Saturn) prevented further accretion of planetesimals and left a number of small fragments to orbit the Sun. Both theories probably are correct, since part of the asteroids show spectral features of primitive material (targets that have never grown very large), while others seem to represent differentiated, metamorphosed material (suggesting that they have been part of a larger asteroid at some epoch). Independent of the process that initially formed asteroids, collisions have been fracturing them ever since. Depending on their size, asteroids are thus either irregularly shaped fractures, or just covered with craters of various sizes (Fig. 1). The most detailed shape models for asteroids have been obtained from spacecraft encounters or flybys. All of the targets have turned out to be irregular in shape, or covered with global-scale craters (see, e.g., Thomas et al. (1999) for (253) Mathilde, Thomas et al. (2002) for (433) Eros, and Demura et al. (2006) for (25143) Itokawa).



(a)



(b)



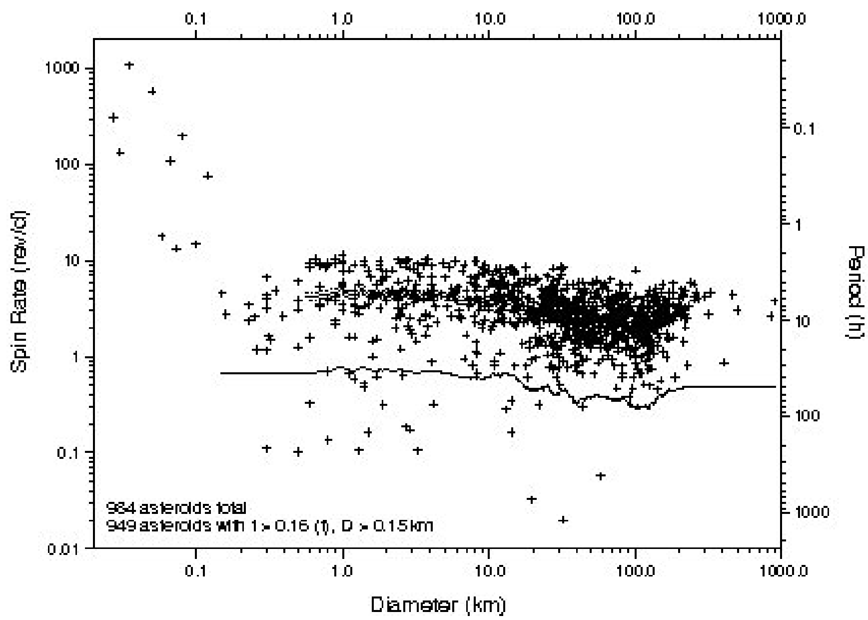
(c)

Figure 1: Images of asteroids a) 433 Eros (33 km $\times$ 13 km $\times$ 13 km) and b) 253 Mathilde ($D = 52$ km), taken by the NEAR spacecraft (NASA). c) Image of asteroid (25143) Itokawa (540 m $\times$ 270 m $\times$ 210 m), taken by the Hayabusa-spacecraft (JAXA).

The spin states of asteroids are also affected by mutual collisions, causing, for example, the observed nearly random distribution of pole directions. However, there is also another process gradually changing the rotation periods and pole directions of asteroids, namely, the thermal radiation force, existence of which was realized already in the 19^{th} century by the Russian civil engineer Ivan Osipovich Yarkovsky (1844-1902). The effect that changes gradually the semimajor axes of asteroid orbits is called the Yarkovsky effect, whereas the effect spinning up or down the rotation and affecting the spin axis orientation of asteroids is known as the YORP effect. Rubincam (2000) gave a thorough theoretical presentation of the YORP effect and Vokrouhlický et al. (2003) suggested that it is the cause of the spin-vector alignment of the Koronis family asteroids. The YORP effect was directly measured, e.g., for asteroid (54509) 2000 PH5 (now named YORP) by Taylor et al. (2007). Since the rate of change in the rotation period is extremely slow, compared to the time span of any set of asteroid lightcurve observations, the change can be modelled as a linear function of time. The YORP effect is most likely to be detected for small asteroids with data sets spanning over a long period of time. Thus, the linear change in period should be included in the spin-state analyses of small asteroids, if it otherwise is impossible to simultaneously fit the rotational phase to all the data lightcurves.

By investigating the dynamical and physical properties of asteroids, we can obtain a more accurate picture of the early evolution of the Solar system. The rotational states of individual objects, together with their shapes and orbits, reflect the collisional history of the objects. Studying the statistical distributions of and correlations between different properties provides information about asteroids as a population of primordial Solar-system bodies. Just to give a few examples of the statistical studies of asteroid properties, Tedesco and Zappalà (1980), for example, studied the dependence between various properties. They detect correlations between the rotation period and size, period and amplitude, and taxonomic type and semimajor axis, but indicate the need for further observations to minimize the possible biasing effects. Their dataset consists of physical properties for 134 MBAs. Pravec et al. (2002) review the results of detailed analyses of asteroid rotations that have revealed certain structures in the spin-size distribution. As seen in Fig. 2, for example, only asteroids below 0.15 km in diameter show rotation periods less than 2 h. Such objects must be monolithic, since fractured rubble-pile bodies cannot be held together by self-gravitation at such high rotation rates. Alvarez-Candal et al. (2004) studied the rotational periods, lightcurve amplitudes, and sizes of Themis-, Eos-, and Maria-family asteroids. They found a weak negative correlation between the size and the spin period. Hartmann et al. (1988) observed that Trojans, in the 1:1 mean-motion resonance with Jupiter, and Hildas show a greater incidence of high lightcurve amplitudes than MBAs of comparable size, suggesting more elongated shapes for Trojans and Hildas than for MBAs. In Paper VI, we add the shape of the asteroid to the statistical study of the properties and, as well, find correlations between various properties. However, the sample for which the shape has been determined does not represent evenly the entire asteroid population, and more observations are required to perform more thorough analyses. Future all-sky surveys, such as the Gaia-satellite and PanSTARRS, will provide new results for thousands of asteroids, which will be enough to carry out a thorough statistical study of various asteroid populations.

Considering NEAs, which are the population of asteroids with Earth-approaching orbits, there is also a practical aspect in studying spins and shapes of these objects: they represent a hazard to the Earth's biosphere. According to Stuart and Binzel (2004) and Harris (personal



(a)

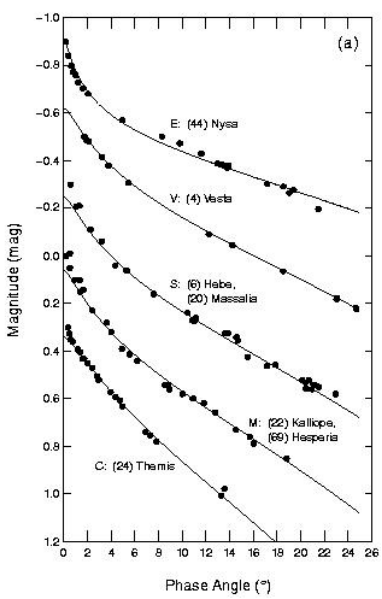
Figure 2: Rotation periods of asteroids compared to their diameters (Pravec et al. 2002).

communication), an object greater than 200 m in diameter impacts the Earth on average every 35000-56000 years. The explosive yield of such an impact is about 30000 times the Hiroshima atomic bomb, and the consequences can be devastating depending on the location of the impact. The more accurate the orbit determination of the asteroid is, the longer in advance we can predict the impact. On the other hand, computation of an accurate orbit requires the barycenter correction to be made for the astrometric observations, which, again requires knowledge about the shape of the asteroid. Physical properties of a hazardous asteroid should also be known for any actions that aim at preventing a possible collision.

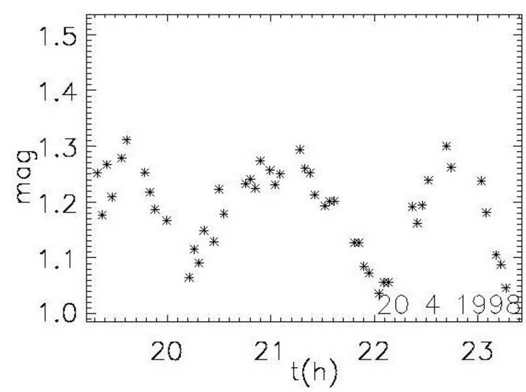
2.2 Photometric observations

The most important sources of information on the shapes and spin states of asteroids are ground-based optical, interferometric, and radar observations, stellar occultations, and space missions. Ground-based radar observations reach asteroids that orbit close to the Earth, and have provided a large number of non-convex models for NEAs (see, e.g., Magri et al., 2007). Spacecraft flybys and encounters have, during the recent years, provided invaluable information about the physical properties of asteroids (e.g., Demura et al., 2006). These missions are, however, rare, and cannot be used alone as the source of information of the asteroid belt as a whole.

Of the aforementioned types of observations, the ones that encounter the least hinders (financial or technical) are ground-based optical observations. Recently, in a few cases when an asteroid has come very close to the Earth, the largest optical telescopes provided with adaptive optics have been able to carry out disk-resolved imaging, which provides information about the overall dimensions and, especially, about the possible multiplicity of the object from the CCD images alone. However, due to their small size, and usually (fortunately!) long distance from the Earth, the asteroids mostly appear star-like to optical ground-based telescopes. Thus, in the visible wavelengths, we observe the total amount of sunlight reflected from the surface of the asteroid at each epoch of observation. The intensity of the reflected radiation depends on the phase angle, i.e., the angle between the Earth and the Sun, as seen from the asteroid; due to the scattering properties of the surface material and shadowing effects on the surface, more light is reflected at small phase angles than at larger ones. Especially, near opposition (zero phase angle), the intensity increases rapidly; the phenomenon is called the *opposition effect*. The rotational phase of the asteroid, as well, affects the amount of observed radiation at a certain epoch; since asteroids are irregular, the part of its surface area which is both visible and illuminated, changes as it rotates. Thus, the total amount of reflected sunlight, seen by the observer, varies, unless we have a pole-on view to the asteroid. A sequence of brightness measurements taken during a revolution is called a *lightcurve*. Examples of a phase curve and a lightcurve of an asteroid are shown in Fig. 3.



(a)



(b)

Figure 3: a) Phase curves of seven asteroids representing different taxonomic classes (Muinonen et al. 2002). b) Lightcurve of asteroid 1862 (Apollo) (Paper V).

The more densely sampled the lightcurve points are, the more detailed information is obtained about the shape and albedo of the target. The restrictions of ground-based optical observations come mostly from the long exposure times needed to achieve an adequate signal-to-noise ratio. With slowly moving MBAs, this is usually no problem, and they represent the majority of the observed asteroid population. NEAs, on the other hand, often move fast in the field-of-view of the telescope, making long exposure times useless. This is why the faint-end of NEAs is rarely observed with optical telescopes, and should be observed with radar instead. Photometric observations are also sensitive to the weather, unlike radar observations. Especially, when carrying out absolute photometric observations, the sky should be completely clear to allow for a good enough signal-to-noise ratio. Absolute photometry also requires the use of standard filters, which decreases the amount of radiation incident on the CCD, and thus decreases the number of observable asteroids. In addition, standard star observations have to be carried out throughout the night, making absolute brightness measurements laborious. Absolute photometry is required for obtaining information on the scattering properties of the surface, as well as of the c/b ratio of the shape. The number of absolute brightness observations, however, is usually too small to be of much use.

One challenge is the planning of the observations. As mentioned above, MBAs move slowly, and remain visible for long periods of time, giving flexibility to the timing of the observations. They are also relatively bright, and carrying out MBA observations does not often require the use of large professional telescopes, from which it is thus very hard to obtain observing time for MBAs. Amateur observers provide a significant contribution to MBA photometry, since MBAs are one of the few astronomical targets of which scientific observations can be carried out with small telescopes. Compared to MBAs, NEA observations are much harder to carry out. They are observable only at most a few months during each apparition and apparitions can be years apart. Thus, to obtain photometric data from a wide range of observing geometries required, e.g., for spin-axis and shape estimation, long-term or especially intensive observing programs are usually needed. One of these programs is the Nordic near-Earth object network (NEON), and its first results are presented in Paper V. These kinds of observing programs are not easy to be organized at large professional telescopes and, due to the faintness and fast motion of NEAs, they can be rarely observed by amateur telescopes. Hence, photometric data sets of only a marginal sample of NEAs are sufficient for generating a useful physical model. NEAs, however, are valuable photometric targets, since they are observable also at large phase angles, while observations of MBAs are restricted to phase angles below approximately thirty degrees. Observations carried out across a wide range of phase angles provide us with valuable information about the shapes of the targets.

A problem with using photometric data of asteroids is that there is no place where all the observations would be collected at the moment. There is a huge amount of data that is sedimentated in the drawers of the observers and maybe single scientists, and finding even most of the observations of a certain object takes time, and so does interpreting all the differently formatted data files. The Uppsala Asteroid Photometric Catalogue (UAPC) served as the main database of photometric data until its last update (Lagerkvist et al., 2001). Ever since, the amount of observed lightcurve data has grown enormously, and a paperback version of the catalogue is now too difficult to keep up to date; an electronic version of the catalogue would be more practical for a number of reasons: 1) it would be updated continuously as observations are carried out or published, 2) the data would be in a standard electronic format,

3) the data could be browsed with various criteria. Such catalogues are under developement; for example, the Standard Asteroid Photometric Catalogue, SAPC, developed and currently maintained at the University of Helsinki Observatory (http://www.astro.helsinki.fi/SAPC) and NASA's Planetery Data System, PDS (http://pds.jpl.nasa.gov/data_services/). SAPC is suitable for both the observers and the scientists to submit data to the data base and to search data with specific criteria, while PDS is adequate to serve as a long-term archive.

2.3 Scattering and brightness

All the visible light detected from the asteroids is scattered sunlight. To be able to find out, what kind of an object produces a certain lightcurve, we need to compute the model brightness. The amount of radiation reflected from a surface is the differential brightness integrated over the visible and illuminated part of the surface

$$L_m = F_{\odot} \int\int_{A_+} S da, \tag{1}$$

where $F_{\odot}$ is the flux density of the incident light, da is the area of a surface element and S is the scattering function at the surface element. A_+ refers to the part of the surface that is both visible and illuminated. The corresponding equation for a discretized polyhedron surface is

$$L_m = F_{\odot} \sum_i S_i a_i, \tag{2}$$

where the index i refers to the sum over all the visible and illuminated facets.

For most of the asteroids, with the exception of the largest ones, the albedo is usually assumed to be uniform across the surface in global scale. The choice of the scattering law, however, is not straightforward. Currently, there is no universally accepted scattering law that would explain all the features in the variation of the observed brightness of asteroids. Determining the physical parameters of the available scattering laws requires accurate absolute photometry, which is not always available. Examples of such scattering laws are the well-known Lumme-Bowell and Hapke laws (Lumme and Bowell, 1981a,b and Hapke 1986). Some of the parameters of the laws are related to the properties of the regolith, some only to the shape of the phase curve. Neither the Lumme-Bowell nor the Hapke law can fully explain the scattering of asteroid surfaces, which motivates continuous development of scattering models. Traditional explanations for the opposition effect, for example, are the shadowing effects in the structure of the surface, while within the last two decades the role of coherent backscattering has been intensively studied. Muinonen et al. (2002) give a review of the advances in investigating scattering from asteroid surfaces.

In addition to the physical scattering laws, there are also scattering laws with no direct physical meaning of the parameters. Such empirical functions are generally of the form

$$S = S(\mu, \mu_0, \tilde{\omega}, \alpha), \tag{3}$$

where $\mu = \mathbf{E} \cdot \mathbf{n}$ and $\mu_0 = \mathbf{E}_0 \cdot \mathbf{n}$, $\mathbf{E}$ and $\mathbf{E}_0$ being the unit vectors pointing to the observer and to the Sun, as seen from the asteroid, and $\mathbf{n}$ the unit surface normal. $\tilde{\omega}$ is the albedo and α the phase angle. The most simple expression for the scattering, with zero parameters, is geometric scattering, according to which the reflected brightness depends only on the area projected towards the observer, i.e., $S = \mu$. Geometric scattering is applicable at the zero-phase-angle observing geometry, and it is often applied in methods that assume small-phase-angle observations. One example of a one-parameter scattering law is the combination of the Lommel-Seeliger and Lambert laws

$$S(\mu, \mu_0) = \frac{\mu\mu_0}{\mu + \mu_0} + c\mu\mu_0, \tag{4}$$

where the first term is the Lommel-Seeliger part and the second term the Lambert part. Lambertian law is often applied in the case of bright surfaces, whereas the Lommel-Seeliger law is more suitable for low-albedo objects. Considering asteroids, a combination of the two laws, with the Lambertian weight depending on the assumed albedo of the object, has proven to be suitable for reproducing the desired lightcurve features. Especially, at small phase angles the maxima and minima of the lightcurves generated with different scattering laws deviate from each other only by a few percent. Thus, the shape models derived from small-phase-angle observations (e.g., for MBAs) are not very sensitive to the choice of the scattering law; models obtained using different scattering laws most likely differ from each other less than from the real shape of the target asteroid. Thus, it is justified to use a simple model, such as that of Eq. (4), in lightcurve inversion, at least for MBAs. If absolute photometric data is available, Eq. (4) should be multiplied with a phase function $f(\alpha)$ that accounts for the brightening towards small phase angles. In inverse problems, any scattering parameters should usually be constrained to realistic limits, since they do not affect strongly the *rms* of the model fit, and often achieve extreme values if incorporated as free parameters to be fitted.

For computing the model brightness corresponding to a certain data point, we need to know the observing geometry $(\mathbf{E}, \mathbf{E}_0)$. If we assume, that the asteroid rotates around a fixed rotation axis with rotation period P, we obtain a vector in asteroid's own rotation frame from that expressed in ecliptical coordinates, by the rotation sequence

$$\mathbf{r}_{ast} = R_z(\phi_0 + \omega(t - t_0))R_y(\tilde{\beta})R_z(\lambda)\mathbf{r}_{ecl}, \tag{5}$$

where λ and $\beta = 90^\circ - \tilde{\beta}$ are the ecliptic longitude and latitude of the spin axis, t is the time of the observation, ϕ_0 the rotation phase of the asteroid at time t_0, and ω the angular velocity of rotation. $R_k(\alpha)$ is the rotation matrix for rotation through angle α in a positive direction about axis k. Using Eq. (5), we obtain the directions of the Sun and the Earth in the asteroid's own rotation frame at the epoch of each data point and, thus, can compute the corresponding model brightness.

3 Asteroid spins and shapes uncovered

3.1 Convex inversion

Quantum leap in the field of lightcurve inversion during this decade has been the convex inversion method presented in this thesis. The method has been used in various publications to derive the spin state and the convex shape model describing the global shape features of asteroids. In some cases, information about the non-convex features is possible to be obtained. Examples of the results are presented in, e.g., Paper III, Kaasalainen et al. (2002a), and Kaasalainen et al. (2004).

Basically, we compare the brightnesses calculated for the model shape to data brightnesses, and find the shape and spin state that produces the best fit to the data. In the following, I explain the procedure, step by step:

Determining the approximate spin state

In the initial spin-state search, we use a low-order functional series representation for the logarithm of the Gaussian curvature of the model shape's surface

$$G(\vartheta, \psi) = \exp\big(\sum_{lm} a_{lm} Y_l^m(\vartheta, \psi)\big), \tag{6}$$

where ϑ and ψ are the spherical coordinates of the surface normal direction and a_{lm} are the coefficients of the spherical harmonics series Y_l^m. We prefer the exponential form since it is a convenient way to force the curvature to be positive when searching for the best-fit coefficients a_{lm}. The total brightness for a discretized surface at the observing geometry $(\mathbf{E}, \mathbf{E}_0)$ is then

$$L_m(\mathbf{E}, \mathbf{E}_0) = \sum_i S(\mu_i, \mu_{0,i}) G(\vartheta_i, \psi_i)\sigma_i, \tag{7}$$

where the sum is over the visible and illuminated surface elements, and σ_i is the area of the i^{th} surface element on the unit sphere, corresponding to that on the model shape, i.e., the one with the same surface normal.

Since we want to find a shape and spin state that minimizes the difference between the data and model brightnesses, we minimize the function

$$\chi^2 = \sum_j (L_{m,j} - L_{dat,j})^2, \tag{8}$$

where $L_{dat,j}$ is the value of the j^{th} data point. In practice, to keep the shape solution convex, we have to add a reqularization term to Eq. (8). When using relative photometry, we also have to divide each lightcurve sequence by the mean of the data lightcurve to make curves comparable to one another. The form of the χ^2 function in the case of relative photometry is, thus,

$$\chi^2 = \sum_{cp} (\frac{L_{m,cp}}{\bar{L}_c} - \frac{L_{dat,cp}}{\bar{L}_c})^2 + w \sum_{ik} n_{ik} G(\vartheta_i, \psi_i)\sigma_i, \tag{9}$$

where the index c corresponds to the lightcurves and p to the data points in each lightcurve and $\bar{L}_c$ is the mean brightness of the c^{th} lightcurve. The second term is the convexity regularization, which increases χ^2 for non-convex shapes with a weighting factor w. $\mathbf{n}_i$ is the unit surface normal of facet i and index k refers to the x, y, and z coordinates. A suitable method to find the minimum of χ^2, when G is represented as in Eq. (27), is the Levenberg-Marquardt optimization method (Press et al. 1994).

The free parameters in the initial spin-state search are the coefficients of the spherical-harmonics series representation of the Gaussian curvature in Eq. (27). Since we do not need to know the detailed shape of the object for obtaining an approximate spin state, the degree of the series can be as low as $l_{max} = 2$ ($l = 0, \ldots, l_{max}$ and $m = 0, \ldots, l$), which produces an almost ellipsoidal shape. The possible spin states are found by scanning through the period and spin-axis space, while fitting the shape for each sampled spin state. From the set of solutions, the ones that produce satisfactory fits to the data are accepted. The previous estimates for the rotation period are useful to constrain the range of periods that has to be tested. The time step in period sampling should be a few times ΔP, which is the difference between the local minima in P, χ^2 -space,

$$\Delta P = \frac{1}{2}\frac{P^2}{T}, \tag{10}$$

where P is the rotation period and T the total time range of observations. These minima are due to that, if P is changed by ΔP, the lightcurve is phase-shifted by π during T. Thus, if we have a lightcurve with two roughly equal maxima and minima that coincide in the first and the last of the observed lightcurves for period P, then they coincide also for period $P + \Delta P$. Although lightcurves often are non-sinusoidal enough to provide an irregular shape model, this equation gives a rough estimate about how densely the χ^2 minima are located in the χ^2, P-space, and how densely P should be sampled. The spin axis should be sampled in at least eight uniformly distributed directions.

Improving the spin solution

To improve the spin solution, we use the spin states, accepted in the initial spin state search, as a starting point. The procedure is otherwise the same as above, but the number of shape parameters is increased to, e.g., $l_{max} = 6$, and the spin-axis direction and rotation period are set as free parameters. From the set of solutions we, again, accept the ones satisfying certain criteria. If the data set is abundant, a single best-fit solution is found. The number of such targets is restricted, however, and usually we get two or three, or even a distribution of solutions.

Improving the shape solution

When the number of acceptable spin solutions is small, we can still improve the corresponding shape solutions by expressing the model shapes as polyhedra, whose facet areas are solved for. Such a model allows expressing in practice any convex shape. The total model brightness is then

$$L_m(\mathbf{E}, \mathbf{E}_0) = \sum_i S(\mu_i, \mu_{0,i}) \exp(a_i), \tag{11}$$

where $\exp(a_i)$ is the facet area of the i^{th} surface element. The surface normals are fixed according to a suitable discretization method. One way to obtain a uniform distribution of facets is to apply the octant triangulation routine, described, e.g., in Paper I. The number of facets has to be large to ensure that the shape solution does not depend on the discretization method, i.e., on the exact choice of the surface normals of the facet elements. The number of shape parameters is now much larger than when expressing the shape as spherical-harmonics series and, thus, we fix the spin parameters.

A suitable method to find the minimum χ^2, when L_m is expressed as in Eq. (11), is the conjugate gradient method (Press et al. 1994). From the best-fit solution, we obtain a set of facet areas and normal directions. To derive the actual shape information, i.e., vertices of the shape, from the facet information, we can apply, e.g., the Minkowksy minimization method (Lamberg 1993 and Kaasalainen et al. 1992), described below.

Reconstruction of the shape from facet information

The Gaussian surface density of a convex surface corresponds to one and only one surface. To define the surface from the Gaussian surface density, the Minkowski minimization method employs two quantities, one of which is the support function

$$\rho(\vartheta, \psi) = \mathbf{n}(\vartheta, \psi) \cdot \mathbf{r}(\vartheta, \psi), \tag{12}$$

where $\mathbf{n}(\vartheta, \psi)$ is the surface normal directed to (ϑ, ψ), and $\mathbf{r}(\vartheta, \psi)$ is the corresponding radius vector. In other words, $\rho(\vartheta, \psi)$ is the distance of the tangent plane, with normal directed to (ϑ, ψ), from the origin. The other quantity is the mixed volume that can be defined for two strictly convex bodies R and S as

$$V(R, S) = \frac{1}{3} \int_0^{2\pi} \int_0^{\pi} \rho_R(\vartheta, \psi) G_S(\vartheta, \psi) \sin\vartheta d\vartheta d\psi. \tag{13}$$

Thus, e.g., $V(S, S)$ is the volume of S and $3V(U, S)$ is the surface area of S, when U is the unit sphere. Minkowski (1903) showed that the mixed volume reaches its minimum when ρ_R and G_S correspond to the same surface, i.e., $R = S$. Thus, by minimizing V, with known G, we can solve for ρ. The only constraint required is that the volume of R be constant. If we know the discretized Gaussian surface density, i.e., facet normals and areas, we can express the mixed volume as

$$V(R, S) = \frac{1}{3} \sum_{j=1}^{n} l_j(R) a_j(S), \tag{14}$$

where a_j is the area of the facet j, and l_j is its distance from the origin. An analogous, but computationally more efficient way to minimizing $V(R, S)$, is to maximize $V(R)$ while keeping the inner product $\mathbf{a} \cdot \mathbf{l}$ constant. In that case, we minimize

$$V(R) = \frac{1}{3} \sum_{j=1}^{n} l_j A_j, \tag{15}$$

where A_j are the facet areas as computed from $\mathbf{l}$. The constraint of keeping $\mathbf{a} \cdot \mathbf{l}$ constant is fulfilled by calculating the gradient projected onto the constraint plane

$$\mathbf{f} = \mathbf{A} - \frac{\langle \mathbf{A}, \mathbf{a} \rangle}{\langle \mathbf{a}, \mathbf{a} \rangle}, \tag{16}$$

when applying numerical minimization methods that apply gradient information.

Convex inversion can be used, when the number of lightcurves and observing geometries is large enough to constrain the spin axis and shape properly. If, for example, only one lightcurve was available, there would be a shape solution, acceptably fitting the data, for any pole direction, since the dimensions of the model shape are not constrained in any way. If a large number of possible spin states is found, it would be too time consuming to check if each corresponding shape model is realistic. Thus, within a wide distribution of solutions, there are also spurious solutions included. The convex inversion method is, hence, most applicable to objects, for which a clearly constrained distribution of possible spin states can be found, and the dimensions for each shape model can be checked.

The computing time with a typical present-day computer, including all the steps of convex inversion, is approximately $20-45$ minutes per one solution, depending on the density of the period minima. In the case when the data are not sufficient for an unambiguous solution, it may takes about one day to get the distribution of spin solutions.

3.2 Non-convex inversion

In Papers I and IV, we discuss deriving information about the non-convex features of asteroids. In Paper I, we express the radius of the surface as an exponental spherical harmonics series in the same manner as in Eq. (31) (Muinonen, 1998), but without the statistical parameters:

$$r(\theta, \phi) = \exp\big(\sum_{lm} c_{lm} Y_l^m(\theta, \phi)\big), \tag{17}$$

where θ and ϕ are the spherical coordinates. To avoid porcupine-shaped solutions, we prefer the functional series representation, instead of directly fitting the radii. We use the shape model from the convex inversion as the first guess, and minimize the difference between the model and data brightnesses. Regularization is necessary to prevent the existence of unrealistically deep vallies. A convenient way to smoothen the model shape is to force it to be as convex as possible.

In Paper IV, we use a statistical approach to derive information about the nonconvexities from sparse data. We express the shape as in Eq. (31), and use the convex inversion solution as a starting point. In the vicinity of the initial solution, we incorporate higher spherical harmonics for the logarithmic radius and map the parameter distributions using a trial and error method based on existing knowledge of the statistical properties of asteroid shapes and spins. We repeat the analysis for varying initial conditions in order to assess the potential clumpiness in the a posteriori probability density of the parameters.

3.3 Limited data sets

In Paper V, we present a method to derive spin and shape information from very limited data sets (in the sense of observing geometries); even based on a single lightcurve, the distribution of possible solutions can be somewhat constrained. The shape model we use for the method is a combination of a sphere, two cylinders, and four plane elements, for each of which the brightness can be calculated analytically, using a suitable scattering law. We sampled the spin state with a Monte Carlo technique and fitted the shape parameters, constrained to reasonable limits. Thousands of possible solutions can be sampled in a moderate time, showing the distribution of the possible spin and shape solutions. The method should be used in planning future observations of a certain object, since the correct choice of observing geometries reduces the number of required lightcurve sequences.

3.4 Background and other advances

In various publications considering the interpretation of photometric lightcurve data, the term *shape* has a variety of meanings. The first, most simple shape model that has been used for decades, is the triaxial ellipsoid. The radius of an ellipsoid can be expressed as

$$r(\theta, \varphi) = \frac{abc}{\sqrt{b^2c^2\sin^2\theta\cos^2\varphi + a^2c^2\sin^2\theta\sin^2\varphi + a^2b^2\cos^2\theta}}, \tag{18}$$

where θ and φ are the spherical coordinates of the radius vector and a, b, and c are the ellipsoid axes in decreasing order.

The first available publications applying an ellipsoid model are from the mid 20^{th} century. For example, Cuffey (1953) analysed a single lightcurve of (4) Vesta and, based on the amplitude of the lightcurve, derived relative ellipsoid axis ratios $b/a = 0.88$ and $c/b = 1$. From the deviation of the lightcurve from sinusoidal shape, he concluded that, on the ellipsoidal shape, there are either a number of albedo spots or large mountain ranges. They, however, called for further observations to distinguish between the two theories. They, of course, had no intention to solve for the spin state, since only one lightcurve was available.

The earliest methods utilizing a set of lightcurves, observed at various observing geometries, actually aimed at defining the spin-axis direction and/or the rotation period of the asteroid. The ellipsoid axis ratios, if needed, acted only as fitting parameters, but were often published as the axis ratios of the model shape. In the photometric astrometry method, or the epoch method, no shape model is used. It is assumed that a specific feature on the surface of the asteroid (RFO, recognizable feature origin) can be connected to a feature (LRF, a lightcurve recognizable feature) in each of the data lightcurves (notations RFO and LRF are from Detal et al., 1994). Since the synodic period of an asteroid depends on its pole direction, the spin axis can be solved for by investigating the epochs of LRFs. The time difference between the epochs of two LRFs, corresponding to the same RFO is

$$J_j - J_i = \big(n_{ij} + \frac{\phi_{ij}}{2\pi}\big) P_{sid}, \tag{19}$$

where n_{ij} is the number of full revolutions between the two epochs, P_{sid} is the sideral period, and ϕ_{ij} the azimuthal angle, in the equatorial coordinates, between the direction of RFO at the two epochs. ϕ_{ij} depends on the spin axis direction, which is determined by finding the pole coordinates that produce the best fit to the observed LRF epochs.

The amplitude and magnitude (AM) methods, which are often used complementary to the epoch method, derive from the fact that the amplitude and magnitude of a lightcurve at a certain epoch depends on the aspect angle of the observation (i.e., the angle between the spin vector and the vector pointing to the observer). There are various versions of the AM methods that differ, e.g., in the choice of the model shape and in the way to take into account the brightening along with the decreasing phase angle. Zappalà (1981), for example, used a triaxial ellipsoid model with axes $a > b > c$, and thus the method they presented is restricted to asteroids with regular lightcurves with well-defined maxima and minima. Small-phase-angle observations, where the brightness is approximately proportional to the projected area of the asteroid only, are required as well. With no loss of generality, they set $c = 1$, and calculated the difference between the maximum magnitudes at aspect angle $\xi = 90°$ and the observed aspect angle ξ from

$$\Delta V(\xi) = 2.5 \log \frac{P_{eq}}{P(\xi)}, \tag{20}$$

where $P_{eq} = \pi a$ is the projected area at $\xi = 90°$ and $P(\xi) = \pi a d = \pi a \sqrt{b^2 \cos^2 \xi + \sin^2 \xi}$ is that corresponding to the observation. Assuming, that the phase behaviour of the brightness obeys a linear change in the phase-angle range $8.5° < \alpha < 25°$ and a parabolic increase for $\alpha < 8.5°$, they transformed all the lightcurves to a comparable magnitude level. The amplitude of the model was presented as

$$A(\xi) = 2.5 \log(ad/bd'), \tag{21}$$

where $d' = \sqrt{a^2 \cos^2 \xi + \sin^2 \xi}$. Assuming, that the maximum observed amplitude corresponds to the equatorial viewing geometry, i.e., $A_{max} = 2.5 \log(a/b)$, they obtained the b/a-ratio of the ellipsoid axis. Then they plotted the magnitude versus the amplitude, calibrating the axis ranges so that $\Delta V = 0$ for $A = A_{max}$. The aspect angle of the observations can be obtained from either the magnitudes (Eq. (20)) or amplitudes (Eq. (21)) as

$$\cos \xi = \pm \sqrt{(V^\star - 1)/(b^2 - 1)}, \tag{22}$$

$$\cos \xi = \pm \sqrt{(1 - A^\star)/(A^\star a^2 - A^\star - b^2 + 1)} \tag{23}$$

where $V^\star = 10^{2\Delta V/2.5}$ and $A^\star = 10^{2(A-A_{max})/2.5}$. From the relation between the coordinates of the pole and those of the asteroid, they calculated the longitude of the pole while sampling the latitude with a step of 10°. Usually two possible pole solutons are obtained using the amplitude-magnitude (-aspect) method.

However, it was evident from the beginning that a plain triaxial ellipsoid is too simple to model most asteroids. A more detailed shape model, to be used for the AM and epoch methods, was proposed by Cellino and Zappalà (1989), who combined eight octants of ellipsoids having different semiaxes so that the adjacent ellipsoids have two equal semiaxes in common. Lightcurves that they computed for their model were similar to those observed for asteroids.

An ellipsoid, whose ends were sharper than those of an ellipsoid was used by Magnusson et al. (1996), who applied the AM and epoch methods to define the spin vector of (1620) Geographos. As a near-Earth asteroid, Geographos was observed at large phase angles, and its lightcurves showed irregular features, most dominating being the unequal maxima and minima. For the AM methods, they combined different ways to define the amplitudes and magnitudes. They also took into account nongeometric scattering, and the fact that only the part of the surface that is both illuminated and visible contributes to the observed brightness.

Lumme et al. (1989) presented a pole determination method based on the series expansion of lightcurves. They extrapolated all the lightcurves to zero phase angle to reduce two (phase angle and obliquity) of the four angles defining the observing geometry (obliquity is the angle between the vector normal to the plane defined by the asteroid, the Earth and the Sun and the plane defined by the spin vector and the direction of the Earth). Thus, the brightness of the asteroid depends only on the aspect angle and rotational phase of the asteroid. The series expansion of the lightcurves, extrapolated to the zero phase angle, is represented as

$$L(\xi, \phi) = \sum_{lm} P_l^m(\cos\xi)[x_{lm}\cos(m\phi + \delta_l) + y_{lm}\sin(m\phi + \delta_l)] \tag{24}$$

where ξ is the aspect angle, ϕ the rotational phase, P_l^m are the associated Legendre functions, and δ_l the absolute rotational phases. x_{lm} and y_{lm} are determined by the shape and albedo distribution of the asteroid. Lumme et al. restrict themselved to the case of sinusoidal lightcurves with two maxima and two minima, since the power spectrum of such curves is dominated by the second-order amplitude term H_2

$$H_2 = \sum_{n=2}^{N} h_{2n} P_n^2(\mu), \tag{25}$$

where $\mu = \cos\xi$. Since H_2 is the amplitude of the function it represents and, as Lumme et al. showed, the series can be truncated already at $N = 2$, the amplitude of each curve is

$$A_c = H_{2,c} = h_{22} 3(1 - \mu_c), \tag{26}$$

where the index c corresponds to the c^{th} lightcurve. The unknowns are h_{22} and $\mu = -\sin\beta_0 \sin\beta - \cos\beta_0 \cos\beta \cos(\lambda - \lambda_0)$, whence the pole direction (λ_0, β_0) can be solved from the set of observed amplitudes. λ and β are the ecliptic coordinates of the asteroid.

Kaasalainen et al. (1992) introduce a novel approach to shape modelling by expressing the combined Gaussian surface density and the dependence of the scattering law on the location on the surface of an asteroid as a functional series,

$$G(\vartheta, \psi) = \sum_{lm} a_{lm} Y_l^m(\vartheta, \psi), \tag{27}$$

where ϑ and ψ are the spherical coordinates of the surface normal direction and a_{lm} are the coefficients of spherical harmonics series Y_l^m. The Gaussian surface density is related to the surface radii through the equation

$$G(\vartheta, \psi) = \frac{\mid \frac{\partial \mathbf{r}}{\partial \vartheta} \times \frac{\partial \mathbf{r}}{\partial \psi} \mid}{\sin \vartheta}, \tag{28}$$

where $\mathbf{r}(\vartheta, \psi)$ is the radius vector of the surface. Using the Gaussian surface density, the model brightness is

$$L_m = \int \int_{\mu, \mu 0 \geq 0} S(\mu, \mu_0, \alpha, \varpi) G(\vartheta, \varphi) \sin \vartheta d\vartheta d\varphi \tag{29}$$

according to Eq. (1). By expressing both L_m and the observed lightcurves as Fourier series, the corresponding coefficients can be compared and shape parameters (i.e., coefficients a_{lm} in Eq. (27)) solved. Kaasalainen et al. assumed ovaloid shapes, i.e., convex shapes with no planar sections, where every direction of the surface normal corresponds to one and only one point on the surface. When deriving the shape parameters from the lightcurve data, no a priori shape was assumed but, due to the instability of the inversion problem, regularization had to be used. They also considered expressing the surface density as a set of discretized values, but found it too large a computational effort, since the number of discrete density values should be at least hundreds.

Barucci et al. (1992) performed shape determination on asteroid Gaspra using both the method by Kaasalainen et al. (1992) and the octant ellipsoid method by Cellino and Zappalà (1989), plus a third method by Fulchignoni and Barucci (1988), where some reshaping was imposed on the best-fit ellipsoid by cutting off a piece of it, making craters, etc. They compared each shape result with the Galileo spacecraft image of this asteroid and it was evident that all models describe the overall shape well. They conclude that each method has specific advantages and drawbacks, none of them being absolutely preferable to the others.

Detal et al. (1994) presented three shape and/or spin determination methods: the RAMA, FAM, and FS methods. With the RAMA (revised AM) method, they assumed an ellipsoidal shape, geometric scattering, and small-phase-angle observations. They took into account the opposition spike, and expressed the model brightness as

$$L_m(\phi, \alpha) = \frac{F_s \varpi A(\phi)}{g(Q, \alpha)}, \tag{30}$$

where F_s is the incident solar flux, ϖ the geometric albedo, A the projected area, and $g(Q, \alpha)$ the distribution of single and multi-scattered light (Lumme and Bowell, 1981a,b). ϕ is the rotational phase, Q the multiple-scattering parameter, and α the phase angle. By fitting the model brightness to the observed lightcurves, they obtained values for the pole direction, ellipsoid axis (both included in A), and Q. In the FAM (free albedo map) method, the model is a discretized sphere, whose facet albedos and spin axes are adjusted to reproduce the observed lightcurves. Analogously with the RAMA method, Detal et al. assumed only small-phase-angle observations. Brightness of the model was the sum of the brightnesses of the visible surface elements, including the multiple-scattering function. For the last, FS (free shape) method, they applied a convex polyhedron shape model. They ensured convexity either minimizing the surface to volume ratio, or by maximizing the radii entropy, and calculated the total brightness analogously with Eq. (30). The parameters that they obtained were the vertices of the polyhedron, Q, and the spin-axis direction. The number of facets, however, was enough to reproduce only a very rough estimate of the shape. In all three methods they sampled the spin axis direction over the entire space with $1^\circ - 10^\circ$ interval while fitting other parameters.

Non-convex features on asteroid surfaces were studied by Karttunen (1989) and Karttunen and Bowell (1989), who investigated the effect of small-scale concavities by implanting spherical craters on ellipsoidal surfaces. The message was that the effect of these possibly quite wide, but relatively shallow craters on lightcurves could not be distinguished from noise.

Muinonen (1998) expressed the radius of a shape as an exponential series

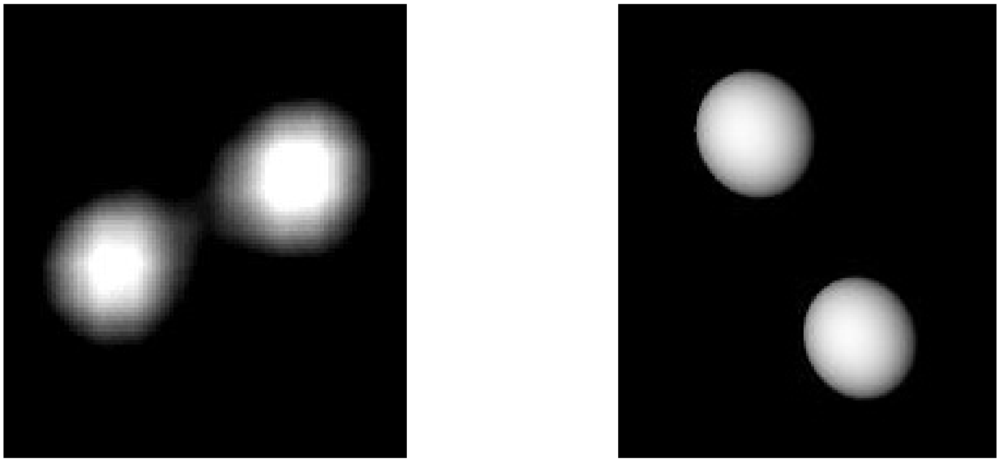
$$r(\theta, \varphi) = a \exp[s(\theta, \varphi) - \frac{\beta^2}{2}], \tag{31}$$

where θ and φ are the spherical coordinates of the radius vector, a is the mean radius and β the standard deviation of the logarithmic radius $s(\theta, \varphi)$, which is represented as a spherical harmonics series

$$s(\theta, \varphi) = \sum_{lm} s_{lm} Y_{lm}(\theta, \varphi), \tag{32}$$

where Y_{lm} are the spherical harmonics functions, and coefficients s_{lm} are independent Gaussian random variables. The exponential form is a convenient way to restrict the radii to positive values. Muinonen generated sample shapes by selecting the coefficients s_{lm} randomly from a Gaussian distribution. Muinonen and Lagerros (1998) presented a method to derive the covariance function for the logarithmic radii for a sample of small solar-system bodies.

The special case of interpreting lightcurve data of binary asteroids has become an important field of study, since the number of binaries in, e.g., the NEA population, is assumed to be about 15%. Binary asteroids are generally observed as point sources from the Earth distance. Only adaptive optics may, in some cases, provide a disk resolved image, where the two components can be detected separately (Fig. 4). Kaasalainen et al. (2002a) applied the convex inversion method to detect binary asteroids from lightcurve observations. They interpret cone-like models, with one end much smaller than the other, to be of binary origin. They, however, predict that, for MBAs, which are observed only at small phase angles, it is mostly impossible to deduce any information of the non-convex features of the binaries based on lightcurves only. Descamps et al. (2007) used a different, three-step approach for binary analysis of the main-belt asteroid (3169) Ostro. In the first step, they assumed that the lightcurve with the largest amplitude corresponds to the equatorial observing geometry, and obtained the possible pole latitudes as a function of pole longitude. Using the same



(a) (b)

Figure 4: a) Adaptive optics image of the binary asteroid (90) Antiope. b) A model of asteroid (90) Antiope, based on the adaptive optics images and lightcurve observations. (Descamps et al., 2007.)

assumption, they obtained an initial guess for the Roche parameters of the binary system. In the second step, they searched for the correct pole using the initial shape model and fitting the model to the observed lightcurve minima. In the third step, they held the spin solution fixed, and improved the shape parameters. They conclude that it should be possible to detect the binary shape of even MBAs, if the aspect coverage of the observations is large enough. However, when performing binary analysis, one must be careful not to over-interpret the lightcurve data. As Descamps et al. state, additional observations are needed also to verify the binary character of asteroid Ostro.

Analysing data sparse in time has become important along with the future all-sky surveys, such as the ESA astrometric cornerstone mission Gaia (launch 2011) and PanSTARRS (Panoramic Survey Telescope and Rapid Response System). Both will produce highly accurate absolute photometric data as solitary points, distributed unevenly in time. For example, Gaia will carry out 50-100 brightness measurements of each of the about 300,000 asteroids to be observed during its five-year mission. Cellino et al. (2006) propose a method applying a triaxial ellipsoid model to analyze the spin state and shape of asteroids based on such data. They first sample randomly the parameters, i.e., the ellipsoid axis ratios, spin axis, and rotation period. Of all the solutions, they accept the ones that fit the data to a reasonable accuracy. Then they improve the accepted solutions, again, with random sampling. The fits to the data are good, and they conclude that the ellipsoid model is accurate enough to model the asteroids with such sparse data.

4 Summary of papers

4.1 Paper I

Optimization methods for asteroid lightcurve inversion. I. Shape determination. In Paper I, we present a new method to construct convex shape models of asteroids from their photometric lightcurves, and investigate how well this model represents the shape of the original nonconvex object. We discuss the state of art in asteroid shape modelling, and compare the new method to the earlier ones.

We use two approaches to represent the model shape: a convex polyhedron and a series representation. Both methods are thoroughly described and the computation of the brightness of convex as well as nonconvex objects is presented. We studied how the scale of the nonconvexities in the original shape affects the quality of the convex shape solution by using four different nonconvex shapes: from nearly convex to a binary shape. We compared the lightcurves and shapes of the original, nonconvex objects, to those of the convex model shapes and the convex hulls of the original shapes. For situations, where a convex solution cannot be found, we present a method to interpret the residual nonconvexity as albedo variegation. Then again, for cases where we have highly informative data, a method to derive knowledge about the nonconvex features of the original shape is presented. We also discuss the effect of observing geometry and amount, structure and quality of data to the shape solution as well as the difference of considering the data relative or absolute. If we use absolute data, the function to be minimized is of square form, and thus is formally unique (has only one solution). A beneficial result was, however, that even when considering data as relative, thus removing the formal uniqueness, the shape solution is stable. This stability is one of the strengths of the convex inversion method.

The main result of the paper is that the convex shape solutions obtained with the convex inversion method really describe the overall features of the original, non-convex shapes with the valleys and depressions covered with large planar areas.

The author wrote the Fortran programs for the convex-hull computation and convex inversion with the polyhedron model as well as the routines for extracting the data from the data files. All the simulations were carried out by the author.

4.2 Paper II

Optimization methods for asteroid lightcurve inversion. II. The complete inverse problem. As a continuation to Paper I, we present the method to find the spin state of asteroids in addition to their shape, and also discuss the scattering models suitable for lightcurve inversion as well as phase behaviour of the brightness.

To solve for the rotation state of an asteroid, we set the pole direction and rotation period as free parameters in the inverse problem, otherwise keeping the mathematical form of the problem unaltered. This way we make the minimized function formally non-unique (like when considering data as relative), but this, again, does not remove the uniqueness of the solution. We show how we can approach the final spin and shape solution gradually, by first finding the approximate spin solution using an approximate shape solution, and then by improving in two steps the spin and shape solutions. We also discuss the stability and accuracy of the solution. Finally, we test the applicability of the convex inversion method by comparing its results to those obtained from radar observations or space missions. The conclusion is clearly that the convex inversion method gives reliable solutions for both the shape and the spin state of the asteroid.

The choice of the scattering model in the inverse problem is discussed, and it is pointed out that a simple but versatile scattering law, like the combination of the Lommel-Seeliger and Lambert laws, is a suitable choice. This is an empirical scattering law with no physical meaning of the parameters. The possibility of obtaining the phase curve of the target from the scale factors of the absolute observations is discussed. We also test how the phase curve of an asteroid depends on its shape, and draw the conclusion that phase behaviour is highly independent on the global shape of the target, leaving the explanation of the differing phase curves solely to the small-scale roughness and material of the surface.

The author carried out most of the simulations as well as contributed to the inversion for the sample targets.

4.3 Paper III

Shapes and rotational properties of thirty asteroids from photometric data. In paper III, we present results of spin and shape analysis with convex inversion for thirty main-belt asteroids. This is one of the series of papers scanning through the lightcurve data in the Uppsala Asteroid Photometric Catalogue (UAPC). Also some new data was used. Results show, that even large asteroids ($>$ 100 km in diameter) show remarkable irregular features that cannot be modelled with a triaxial ellipsoid.

We introduce the growing amateur-professional co-operation and point out the enormous observing potential that we have in the amateur observers. We discuss the quality of the data, and information content in MBA data. A general conclusion is that MBA data cannot contain much information of the non-convexities, since they can be observed only at small phase angles. The data were not as good as in the previous papers and, thus, a unique model was not found for all the targets, but many obtained a double spin solution.

The author carried out inversion for most of the targets and undertook the major part of writing the paper.

4.4 Paper IV

Statistical inversion of Gaia photometry for asteroid spins and shapes. Traditionally, photometric asteroid data has consisted of more or less complete lightcurves. Gaia, however, as well as ground-based all-sky survey programs, such as PanSTARRS, will start to produce accurate absolute photometric data sparse in time in the near future. For example, a typical data set observed by Gaia of an asteroid is about 50-100 brightness values distributed over the time span of five years. In Paper IV, we study the applicability of two lightcurve inversion methods, the convex inversion method and the spherical-harmonics method, to photometric data sparse in time. We use the convex inversion method to determine the spin state and convex shape from the data reproduced by a nonconvex, asteroid-like object. Two possible spin states are obtained, one of which is the correct one, and the other one is the mirror solution. The shape solution describes well the overall shape of the original body. We describe briefly the spherical-harmonics method and, in the vicinity of the convex-inversion spin and shape solution, find nonconvex shape solutions that produce an acceptable fit to the data. Although the non-convex solution is not as stable as the convex one, the concave features in the spherical-harmonics model were reasonably close to their real locations.

The author was responsible for the convex inversion part and for writing sections other than Sect. 4.

4.5 Paper V

Spins, shapes, and orbits for near-Earth objects by Nordic NEON. In Paper V, we apply the convex inversion method to the lightcurve data of NEAs. The observations were carried out as part of the Nordic NEON (Near-Earth-Object Network) observing program at the NOT (Nordic Optical Telescope). We obtained an unambiguous solution for one asteroid and a set of possible spin states for two. We also present a new method for deriving spin and shape information from limited data. The shape model is chosen to allow analytic computation of the disk-integrated brightness. Thus, it is possible to obtain thousands of possible solutions, with a random sampling of spin states while fitting the shape, in a reasonable time. The obtained distribution of the possible solutions shows the part of the spin and shape space, where the correct solution lies. Astrometric observations were carried out as well, and during the program four NEAs were recovered and orbits were improved for 76 objects.

The author had a significant contribution to applying for the observing time, planning, carrying out, and reducing the photometric observations. The convex inversion was performed by the author.

4.6 Paper VI

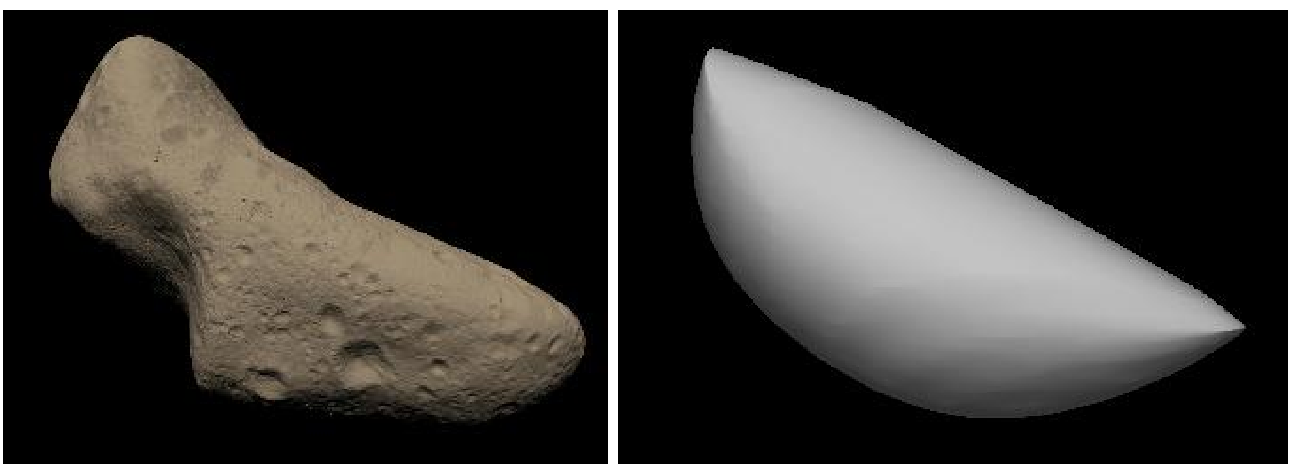
Asteroid shape and spin statistics from convex models. Paper VI deals with the problem of characterizing the shape solutions from convex inversion with a small number of parameters. We present three methods to find characteristic parameters describing the overall dimensions of the shape and calculate the inertia tensor of the model shape to evaluate its reliability. The irregularity of the shape is expressed as the deviation from the best-fit ellipsoid, as well as by the distribution of the facet areas on the surface of the model. With the aforementioned methods, we calculate characteristic parameters for the model shapes of 87 asteroids, and study the distributions of various properties. We find correlations between shape and other physical properties, such as size, taxonomic type, and rotation period. The amount of data, however, is still small, and more observations are called for, especially of extreme populations, such as small and large asteroids as well as slow and fast rotators, to perform a more thorough statistical analysis of the asteroid population.

We also present new spin and shape solutions for eight asteroids. Since the data sets of the asteroids generally are not necessarily sufficient for deriving an unambiguous spin and shape solution, we present a semi-statistical approach to find all the solutions that produce an acceptable fit to the data. For one asteroid, we obtained a wide distribution of possible spin solutions, for one target an unambiguous solution, and for the remaining six objects two or three solutions, some of which were double solutions (with the same pole latitude and longitudes 180° apart).

The author has been responsible for everything else but carrying out and reducing the observations.

5 Conclusions and future prospects

The aim of the present study was to develop methods to derive more accurate spin and shape models for asteroids from their photometric observations. With the novel convex inversion method, it is possible to obtain from an extensive data set the spin state, to an accuracy of 5°, and a convex shape model that resembles the convex hull of the target (Paper I). Some of the lightcurve data sets allow the determination of only the best-fit ellipsoid axis ratios, but most contain more shape information. An example representing the typical quality of convex models is the model of asteroid (433) Eros, shown in Fig. 5 together with the model reconstructed from spacecraft observations. Results from the convex inversion of the nearly hundred asteroids analyzed so far suggests that most of the asteroids are irregular, and it is possible to obtain a significant amount of information of the shape features if we are not constrained to ellipsoidal models.



(a) (b)

Figure 5: a) Model of asteroid (433) Eros, reconstructed from the NEAR-spacecraft observations (NASA). b) A convex inversion model of asteroid (433) Eros from the same viewing direction as in fig. a.

In Paper IV, we apply the convex inversion method for sparse, Gaia-like data, and obtain a shape that closely resembles the shape of the original simulated object. The results are not in full agreement with those of Cellino et al. (2006) who, as mentioned above, concluded that the triaxial ellipsoid model is applicable when interpreting sparse data. Although the convex shape solution is not as accurate as that derived, e.g., from radar observations, the number of solutions is larger, about 100 at the moment, and as all-sky surveys, such as PanSTARRS and Gaia, start to produce photometric data in the future, the number of solutions will increase dramatically. Thus, for statistical studies of asteroid spins and shapes, the photometric observations remain the main source of information in the future.

In some cases, information about the non-convex features can be derived, but the inverse problem is not as stable as the convex inversion, and usually a priori information is required to obtain a reliable solution (Paper II and Paper IV). Important for non-convex inversion is the wide range of observing geometries, especially large-phase-angle observations (typical for NEAs). It has to be kept in mind that, although methods for obtain more and more detailed shape solutions will be developed, the smallest scale features of the shape are surely drowned in the noise of the data. Even though the inversion methods have developed in twenty years, one must still be careful when interpreting the detailed shape solutions and avoid fitting the data noise.

A qualitative determination of the uncertainty of each convex inversion model has not yet been carried out. However, empirical studies show that the spin solution can be considered to be accurate within $5°$, since it is not sensitive to moderate changes in the model shape or scattering law (Paper II). The uncertainty of the rotation period for each pole solution is ΔP of Eq. (10). The accuracy of the shape model is the hardest to define, but the uncertainty of the b/a-ratio for each solution can be assumed to be the same as that for the support function (Paper II), i.e., a few percent. Solving for the c/b-ratio would require rotation about an axis perpendicular to the z-axis, which corresponds to absolute observations carried out at a large range of aspect angles. Thus, especially when only relative photometry is available, c/b is poorly constrained. However, even absolute photometry does not guarantee the correct c/b-ratio, since the phase behaviour of the brightness is not exactly known for asteroids. A good example of an asteroid with poorly-defined c/b-ratio is (93) Minerva, for which a value of $c/b = 0.97$ is obtained when using relative photometry only, while including absolute photometry suggests a more flattened shape with $c/b = 0.6$ (Harris et al., 1999). The b/a-ratio is unity according to both solutions. In this case, the Harris et al.'s solution is most probably correct, since the absolute magnitude at various oppositions varied markedly. According to simulations, however, most of the shape models with $b/a << 1$ obtained using relative photometry are in agreement with the shape of the target asteroid, since the amplitude of the lightcurves of elongated shapes clearly changes along with the changing c/b-ratio at a certain observing geometry. The $b \simeq a$ case, like that of (93) Minerva, is more difficult, since all the lightcurves are more or less flat and it is hard to detect differences is amplitude for changing c/b-ratio. Absolute photometry should be included if possible; even if not providing an accurate c/b-ratio (due to the inaccurately known phase curve), it most probably improves the solution. Development of more accurate scattering laws is called for, as well as calibration of all the absolute lightcurve observations to the same system in order to be able to deduce as much information from the photometric data as possible.

Observations, especially of small and large asteroids as well as slow and fast rotators, should be carried out for performing a thorough statistical analysis of asteroid shapes, spin states, and compositions. Correlations between shape and other properties have been detected, but the sample of asteroids with well-determined shape is still small.

The main outcome of the present thesis is that lightcurves of asteroids contain a significant amount of information of their spin states and shapes. Along with additional well-planned absolute photometric observations, current lightcurve inversion methods can provide a valuable contribution to the knowledge of the properties of the asteroid population, of the evolution of the asteroid belt, and of the formation of the Solar system.

References

Barucci A., Cellino A., Fulchignoni M., Lumme K., Zappala V., and Magnusson P., 1992. Ground-based Gaspra modelling: Comparison with the first Galileo image, A&A, 266, 385.

Cellino A., Zappalà V., and Farinella P., 1989. Asteroid shapes and lightcurve morphology, Icarus, 78,298.

Cellino A., Delbò M., Zappalà V., Dell'Oro A., and Tanga P., 2006. Rotational properties of asteroids from Gaia disk-integrated photometry: A ¨geneticälgorithm, Adv. in Space Res. 38 (9), 2000.

Cuffey J., 1953. Pallas, Vesta, Ceres and Victoria; photoelectric photometry, AJ, 58, 212.

Demura H., Kobayashi S., Nemoto E., and 17 co-authors, 2006. Pole and global shape of 25143 Itokawa, Science 312, 1330.

Descamps P., Marchis F., Michalowski T., and 12 co-authors, 2007. Nature of the small main belt Asteroid 3169 Ostro, Icarus, 189, 362.

Detal A., Hainaut O., Pospieszalska-Surdej A., Schils P., Schober H.J., and Surdej J., 1994. Pole, albedo and shape of the minor planets 624 Hektor and 43 Ariadne: two tests for comparing four different pole determination methods, A&A, 281, 269.

Fulchignoni M. and Barucci A., 1988. Numerical Algorithms Modeling Asteroids, BAAS, 20, 866.

Hapke B., 1986. Bidirectional reflectance spectroscopy 4. The extinction coefficient and the opposition effect, Icarus, 67, 264.

Harris A., Young J., Bowell E., and Tholen D., 1999. Asteroid lightcurve observations from 1981 to 1983, Icarus 142, 173.

Hartmann W. K., Tholen D. J., Goguen J., Binzel R. P. and Cruikshank D. P., 1988. Trojan and Hilda asteroid lightcurves. I - Anomalously elongated shapes among Trojans (and Hildas?), Icarus 73, 487.

Kaasalainen M., Lamberg L., Lumme K., and Bowell E., 1992. Interpretation of lightcurves of atmosphereless bodies. I. General theory and new inversion schemes, A&A, 259, 318.

Kaasalainen M. and Torppa J., 2001a. Optimization methods for asteroid lightcurve inversion. I. Shape determination, Icarus, 153, 24. (Paper I)

Kaasalainen M., Torppa J., and Muinonen K., 2001b. Optimization methods for asteroid lightcurve inversion. II. The complete inverse problem, Icarus, 153, 37. (Paper II)

Kaasalainen M., Torppa J., and Piironen J., 2002a. Binary structures among large asteroids, A&A, 383, L19.

Kaasalainen M., Torppa J., and Piironen J., 2002b. Models of twenty asteroids from photometric data, Icarus, 159, 369.

Kaasalainen M., Pravec P., Krugly Y., and 19 co-authors, 2004. Photometry and models of eight near-Earth asteroids, Icarus, 167, 178.

Karttunen H., 1989. Modelling asteroid brightness variations. I. Numerical methods, A&A, 208, 314.

Karttunen H. and Bowell E., 1989. Modelling asteroid brightness variations. II. The uninterpretability of light curves and phase curves, A&A, 208, 320.

Lagerkvist C-I., Piironen J., and Erikson A., 2001. *Asteroid Photometric Catalogue*, 5^{th} update, Uppsala University Press, Uppsala.

Lamberg L., 1993. *On the Minkowski problem and the lightcurve operator.* Academica Scientiarum Fennica, Series A, I. Mathematica dissertationes 87, University of Helsinki.

Lumme K. and Bowell E., 1981a. Radiative transfer in the surfaces of atmosphereless bodies. I - Theory. II - Interpretation of phase curvesAJ, 86, 1694

Lumme K. and Bowell E., 1981b. Radiative transfer in the surfaces of atmosphereless bodies. II. Interpretation, AJ, 86, 1705

Lumme K., Karttunen H., and Bowell E., 1990. A spherical harmonics method for asteroid pole determination, A&A, 229, 228.

Magnusson P., Barucci A., Drummond J., Lumme K., Ostro S., Surdej J., Taylor R., and Zappala V,1989. Determination of pole orientations and shapes of asteroids. In *Asteroids II* (R.P. Binzel, T. Gehrels, and M.S. Matthews, Eds.) pp 67.97, Univ. of Arizona press, Tuscon.

Magnusson P., and 46 colleagues, 1996. Photometric observations and modeling of asteroid 1620 Geographos, Icarus, 123, 227.

Magri C., Ostro S., Scheeres D., Nolan M., Giorgini J., Benner L., Margot J-L., 2007. Radar observations and a physical model of Asteroid 1580 Betulia, Icarus, 186, 152.

Michalowski T., 1996. A new model of the asteroid 532 Herculina, A&A, 309,970.

Minkowski H., 1903. Volumen und Oberfläche, Math. Ann., 57, 447.

Muinonen K., 1998. Introducing the Gaussian shape hypothesis for asteroids and comets, A&A, 332, 1087.

Muinonen K. and Lagerros J., 1998. Inversion of shape statistics for small solar system bodies, A&A, 333, 753.

Muinonen K., Piironen J., Shkuratov Y., Ovcharenko A., and Clark B., 2002. Asteroid photometric and polarimetric phase effects, in *Asteroids III* (Bottke W., Cellino A., Paolicci P., and Binzel R., eds.), p. 123.

Muinonen K., Torppa J., Virtanen J., Näränen J., Niemelä J., Granvik M., Laakso T., Parviainen H., Aksnes K., Dai Z., Lagerkvist C.-I., Rickman H., Karlsson O., Hahn G., Michelsen R., Grav, T., Pravec P., and Jørgensen U. G., 2007. Spins, shapes, and orbits for potentially hazardous near-Earth objects by NEON. In *Proceedings of IAU Symposium No 236, Near-Earth Objects, our Celestial Neighbors: Opportunity and Risk* (A. Milani, G. Valsecchi, and D. Vokrouhlicky, eds.), p 309. (Paper V)

Press W.H., Flannery B.P., Teukolsky S.A., and Vetterling W.T., 1994. *Numerical recipes in fortran.* Cambridge University Press, Cambridge, UK.

Rubincam DP., 2000. Radiative spin-up and spin-down of small asteroids, Icarus, 148, 2.

Russell H. N., 1906. On the light-variations of asteroids and satellites, ApJ, 24, 1.

Stuart J. and Binzel R., 2004. Bias-corrected population, size distribution, and impact hazard for the near-Earth objects, Icarus, 170, 295.

Taylor P., Margot J-L., Vokrouhlick D., Scheeres D., Pravec P., Lowry S., Fitzsimmons A., Nolan M., Ostro S., Benner L., Giorgini J., and Magri C., 2007. Spin Rate of Asteroid (54509) 2000 PH5 Increasing Due to the YORP Effect, Science, 316, 274.

Thomas P., Veverka J., Bell J., Clark B., Carcich B., Joseph J., Robinson M., McFadden L., Malin M., Chapman C., Merline W., Murchie S., 1999. Mathilde: Size, Shape, and Geology, Icarus, 140, 17.

Thomas P., Joseph J., Carcich B., and 16 co-authors, 2002. Eros: Shape, Topography, and Slope Processes, Icarus, 155, 18.

Torppa J., 1999, Asteroid lightcurve inversion: methods for obtaining a unique and stable shape solution. Master's thesis, University of Helsinki.

Torppa J., Kaasalainen M., Michalowski T., Kwiatkowski T., Kryszczyńska A., Denchev P. and Kowalski R., 2003. Shapes and rotational properties of thirty asteroids from photometric data, Icarus 164, 346-383. (Paper III)

Torppa J. and Muinonen K., 2005. Statistical Inversion of Gaia Photometry for Asteroid Spins and Shapes, in *Proceedings of the Gaia Symposium The Three-Dimensional Universe with Gaia (ESA SP-576).* Held at the Observatoire de Paris-Meudon, 4-7 October 2004. Editors: C. Turon, K.S. O'Flaherty, M.A.C. Perryman, p. 321. (Paper IV)

Torppa J., Hentunen V-P., Pääkkönen P., Kehusmaa P., and Muinonen K., 2007. Asteroid shape and spin statistics from convex models, submitted to Icarus. (Paper VI)

Vokrouhlický D., Nesvorný D., and Bottke W., 2003. The vector alignments of asteroid spins by thermal torques, Nature, 425, 147.