Vector-valued BMOA and Composition Operators

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Licentiate Thesis
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February 2004

Acknowledgements

I wish to thank my supervisor Docent Hans-Olav Tylli for continuous support and encouragement. I am also grateful to Docent Mikael Lindström for reading the manuscript and for helpful comments. I thank my colleagues, especially my officemates Ph.Lic. Juha Kontinen and Ph.Lic. Pekka Pankka for the inspiring working environment. Finally, I wish to thank Minna for her support, patience and everything.

For financial support I am indepted to the Vilho, Yrjö and Kalle Väisälä Foundation and the Academy of Finland, project 53893.

Contents

Chapter 1

Introduction

Let φ be an analytic function mapping the unit disk $\mathbb{D} = \{z \in \mathbb{C} \colon |z| < 1\}$ into itself. The composition operator C_φ induced by φ is the linear map defined by

$$C_\varphi f = f \circ \varphi,$$

for analytic functions f on $\mathbb{D}$. A fundamental problem concerning composition operators is to relate function theoretic properties of φ to operator theoretic properties of C_φ on various Banach spaces of analytic functions. Thus the study of composition operators lies on the interface of analytic function theory and operator theory. We refer to the monographs [Sha93] and [CM95] for an overview of the various aspects of the theory. Although there exists a vast literature on analytic composition operators, the study of composition operators on spaces of analytic functions taking values in an arbitrary complex Banach space is of fairly recent origin. In this setting weak compactness of composition operators has been studied previously in [LST98] and [BDL01].

In this work we study the boundedness, compactness and weak compactness of the composition operator when $f \colon \mathbb{D} \to X$ belongs to $BMOA(\mathbb{D}, X)$, the Banach space of analytic functions on $\mathbb{D}$ taking values in the complex Banach space X that are of bounded mean oscillation on the unit circle, or to its closed subspace $VMOA(\mathbb{D}, X)$. Using the theory of vector-valued Hardy spaces we show that the composition operator C_φ is always bounded on $BMOA(\mathbb{D}, X)$. This gives rise to the natural follow-up question of when this operator is compact or weakly compact. It turns out that if the Banach space X is infinite dimensional, then the composition operator is never compact. However, weak compactness of the operator turns out to be an equally interesting smallness property. One goal of this work is to find necessary and sufficient conditions for weak compactness of the composition operator on $BMOA(\mathbb{D}, X)$. In particular, we will characterize the weakly compact composition operators C_φ on $BMOA(\mathbb{D}, X)$ whose symbol φ is univalent. We also consider the conditions for a composition operator to be a Rosenthal (weakly conditionally compact) operator on $BMOA(\mathbb{D}, X)$ and discuss the analogous questions concerning composition operators on $VMOA(\mathbb{D}, X)$.

The theory of composition operators on vector-valued spaces is closely related to the results on composition operators on classical spaces, that is, on spaces of analytic complex-valued functions. Compact composition operators on the complex-valued space $VMOA(\mathbb{D}, \mathbb{C})$ have previously been studied by Tjani [Tja96], and on the spaces $BMOA(\mathbb{D}, \mathbb{C})$ and $VMOA(\mathbb{D}, \mathbb{C})$ by Smith [Smi99], and Bourdon, Cima and Matheson [BCM99].

This work is organized as follows. In Chapter 2 we consider Banach spaces of vector-valued analytic functions, emphasizing the theory from the point of view of composition operators. As in the complex-valued setting, the theory of vector-valued $BMOA(\mathbb{D}, X)$ is closely related to vector-valued Hardy spaces. There is more than one way to generalize the scalar-valued Hardy spaces to the vector-valued case and it is known that in the vector-valued case the generalizations do not always lead to same spaces (see e.g. [Bla88b]). In Section 2.1 we concentrate on two of the well-known generalizations, namely the space $\mathcal{H}^p(\mathbb{D}, X)$ of the analytic functions f on $\mathbb{D}$ that satisfy $\sup_{0<r<1} \int_0^{2\pi} \|f(re^{i\theta})\|_X^p < \infty$ and the space $H^p(\mathbb{D}, X)$ of the analytic functions g on $\mathbb{D}$ such that the almost everywhere defined radial limit function $\lim_{r\to 1} g(re^{i\theta})$ belongs to the space $L^p(\mathbb{T}, X)$ (here $1 \leq p < \infty$). In Sections 2.2 and 2.3 we introduce the corresponding vector-valued analogues of $BMOA$ and $VMOA$ and use the vector-valued Hardy space theory to derive the basic properties of these spaces. It is worth noting that a systematic theory of vector-valued $BMOA$ has not appeared before (although different versions of the vector-valued space have appeared in the works of Blasco [Bla97] and Chen and Ouyang [CO01]). Moreover, it seems that the vector-valued $VMOA$ functions have not been considered explicitly before.

In Chapter 3 we establish the boundedness of the composition operator on vector-valued Hardy spaces and $BMOA(\mathbb{D}, X)$. The boundedness of composition operators on Hardy spaces has been established in [LST98], but we give also a (different) proof of this fact to motivate our characterization of the bounded composition operators on $BMOA(\mathbb{D}, X)$ and $VMOA(\mathbb{D}, X)$.

In Chapter 4 we consider the more difficult problem of characterizing the weakly compact composition operators on $BMOA(\mathbb{D}, X)$ and $VMOA(\mathbb{D}, X)$. We notice first that the composition operator C_φ is compact on the vector-valued space if and only if X is finite dimensional and the same composition operator is compact on the complex-valued space. Moreover, if C_φ is weakly compact on the vector-valued space, then X must be reflexive and the composition operator is weakly compact on the complex-valued space. In Section 4.2 we prove a sufficient condition for the weak compactness of C_φ on $BMOA(\mathbb{D}, X)$: If X is reflexive and the composition operator is compact on $BMOA(\mathbb{D}, \mathbb{C})$, then C_φ is weakly compact on $BMOA(\mathbb{D}, X)$ (Theorem 4.6). Here we apply the recent characterization of compact composition operators on $BMOA(\mathbb{D}, \mathbb{C})$ due to Smith [Smi99]. We are not able to characterize all weakly compact composition operators on $BMOA(\mathbb{D}, X)$. However, in Section 4.3 we characterize the weakly compact composition operators C_φ on $BMOA(\mathbb{D}, X)$ whose symbol φ is univalent

(Theorem A), and the weakly compact composition operators on $BMOA(\mathbb{D}, X)$ whose symbol φ maps $\mathbb{D}$ inside a polygon inscribed in the unit circle (Theorem B). In Section 4.4 we consider the question when the composition operator is a Rosenthal operator on $BMOA(\mathbb{D}, X)$ and $VMOA(\mathbb{D}, X)$. Recall that a Rosenthal (or weakly conditionally compact) operator is a bounded linear operator mapping bounded sequences to sequences that have weakly Cauchy subsequences.

In Appendix A we establish the theory of vector-valued BMO and VMO spaces that will be needed in Chapter 2. The theory follows closely the well-known theory of scalar-valued BMO and VMO spaces. However, since a systematic treatment of vector-valued BMO has not appeared in the literature before, we want to develop the theory in detail. Moreover, it seems that the vector-valued VMO has not been discussed explicitly before.

Chapter 2

Spaces of vector-valued analytic functions

2.1 Basic results on vector-valued functions

We begin by recalling some basic facts about vector-valued analytic functions. We restrict ourselves only to the facts needed in the sequel. For further results we refer to Chapter III of [HP57]. The word "vector" refers always to a point in a Banach space. If X denotes a Banach space, we may also talk about X-valued functions. The theory of complex-valued functions is sometimes referred to as the scalar theory or the classical theory.

Let $\mathbb{D}$ denote the unit disk $\{z \in \mathbb{C}\colon |z| < 1\}$ and $\mathbb{T}$ the unit circle $\{z \in \mathbb{C}\colon |z| = 1\}$ of the complex plane. We denote by A the Lebesgue area measure on $\mathbb{D}$. By $X = (X, \|\cdot\|_X)$ we mean always a complex Banach space. Let B_X denote the closed unit ball $\{x \in X\colon \|x\|_X \leq 1\}$, let X^* denote the dual space of X, and put $\langle x, x^* \rangle = x^*(x)$, for $x \in X$ and $x^* \in X^*$. See the Appendix for basic facts about vector-valued integration and the spaces $L^p(X) = L^p(\mathbb{T}, X)$ of X-valued Bochner integrable functions on $\mathbb{T}$.

Recall that a function $f\colon \mathbb{D} \to X$ is *analytic* if the complex-valued function $z \mapsto \langle f(z), x^* \rangle$ is analytic (i.e. complex differentiable) in $\mathbb{D}$ for every $x^* \in X^*$. We denote the set of all analytic functions $f\colon \mathbb{D} \to X$ by $H(\mathbb{D}, X)$. It is known that the vector-valued analytic functions are continuous and strongly differentiable in $\mathbb{D}$, i.e. the limit

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} = f'(z_0)$$

exists in X for every $z_0 \in \mathbb{D}$ [HP57, Theorem 3.10.1]. Analogously to the scalar case, it follows that the vector-valued analytic functions can be represented by the *Taylor expansion*: If $f\colon \mathbb{D} \to X$ is analytic on $\mathbb{D}$, then

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} z^n,$$

where $f^{(n)}$ denotes the nth derivative of f and the series converges uniformly on compact subsets of $\mathbb{D}$. The existence of the Taylor expansion is based on the vector-valued Cauchy integral formula

$$f^{(n)}(z) = \frac{n!}{2\pi i}\int_\gamma \frac{f(\zeta)}{(\zeta - z)^{n+1}}d\zeta,$$

where $n \geq 0$ and γ is a closed path in $\mathbb{D}$ such that z is in the interior of γ, and the vector-valued Cauchy-Hadamard theorem: If $(x_n) \subset X$ is a sequence such that $\limsup_{n\to\infty} \|x_n\|_X^{1/n} \leq 1$, then the power series $\sum_{n=0}^\infty x_n z^n$ converges uniformly to an analytic function on compact subsets of $\mathbb{D}$ [HP57, p. 96-97].

Let now $g \in L^1(\mathbb{T}, X)$. The *Poisson integral* of g is the function $P[g]\colon \mathbb{D} \to X$ defined by

$$P[g](z) = \frac{1}{2\pi}\int_0^{2\pi} g(e^{i\theta})P_z(\theta)d\theta,$$

for every $z \in \mathbb{D}$, where P_z is the *Poisson kernel*

$$P_{re^{it}}(\theta) = \frac{1-r^2}{|1-re^{i(t-\theta)}|^2} = \sum_{n=-\infty}^{\infty} r^{|n|}e^{in(t-\theta)},$$

for $re^{i\theta} \in \mathbb{D}$.

We denote by

$$\widehat{g}(n) = \frac{1}{2\pi}\int_0^{2\pi} g(e^{i\theta})e^{-in\theta}d\theta$$

the nth *Fourier coefficient* of g. As in the scalar case one verifies that $P[g](re^{i\theta}) = \sum_{n=-\infty}^{\infty} \widehat{g}(n)r^{|n|}e^{in\theta}$, where the sum converges uniformly on compact subsets of $\mathbb{D}$ (see e.g. [Hen86, p. 12]). Writing $z = re^{i\theta}$ we obtain the identity

$$P[g](z) = \sum_{n=0}^{\infty} \widehat{g}(n)z^n + \sum_{n=1}^{\infty} \widehat{g}(-n)\overline{z}^n,$$

where $\overline{z}$ denotes the complex conjugate of $z \in \mathbb{D}$. Using this representation we notice that for $g \in L^1(\mathbb{T}, X)$ the following conditions are equivalent:

(i) $P[g]\colon \mathbb{D} \to X$ is analytic.

(ii) $\widehat{g}(n) = 0$ for every $n < 0$.

(iii) $P[g](z) = \sum_{n=0}^{\infty} \widehat{g}(n)z^n$ for $z \in \mathbb{D}$.

For any analytic function $f\colon \mathbb{D} \to X$ and $0 < r < 1$, let $f_r\colon \mathbb{D} \to X$ denote the bounded analytic function defined by $f_r(z) = f(rz)$. If the *radial limit* $f^*(e^{i\theta}) = \lim_{r\to 1} f_r(e^{i\theta})$ exists for every $e^{i\theta} \in E \subset \mathbb{T}$ where $m(\mathbb{T} \setminus E) = 0$, then we call the almost everywhere defined function f^* the *radial boundary function of* f.

If $g \in L^1(\mathbb{T}, X)$, then by Fatou's theorem, which holds also in the vector-valued case [Hen86, Corollar 1.9], the radial limit $P[g]^*(e^{i\theta}) = \lim_{r \to 1} P[g](re^{i\theta})$ exists and satisfies

$$P[g]^*(e^{i\theta}) = g(e^{i\theta}) \tag{2.1}$$

almost everywhere on $\mathbb{T}$. The proof of the vector-valued Fatou theorem is quite long but it follows closely the classical proof [Koo80, p. 14], [Rud87, Theorem 11.32].

2.2 Vector-valued Hardy spaces

The classical Hardy spaces H^p $(1 \leq p \leq \infty)$ are usually defined either as the spaces of

(i) the analytic functions $f \colon \mathbb{D} \to \mathbb{C}$ satisfying $\sup_{0<r<1} \|f_r\|_{L^p(\mathbb{T},\mathbb{C})} < \infty$, or

(ii) the functions $f \in L^p(\mathbb{T}, \mathbb{C})$ satisfying $\widehat{f}(n) = 0$ for every $n < 0$.

These spaces are isometrically isomorphic, where the identification is by means of the Poisson integral (see [Dur70, Chapter 3.2]). In the following section we consider Hardy spaces of X-valued functions that are defined analogously, by replacing the space $L^p(\mathbb{T}, \mathbb{C})$ in (i) and (ii) by $L^p(\mathbb{T}, X)$. However, in the vector-valued case (i) may give a strictly larger space than (ii). In fact, the spaces are the same only provided X has the so called analytic Radon-Nikodým property. Defined in either way, we obtain Banach spaces having their own interest. It will also be useful from the view of composition operators to introduce the two different definitions. Due to limitations of space we cannot prove all the results in detail. Our main reference for the vector-valued Hardy space theory is [Hen86] (see also [Hen91]) since it contains a systematic treatment of these spaces. Many results can also be found from the papers of Bukhvalov and Danilevich [BD82] and Blasco (e.g.) [Bla88a]. In many cases the proofs are straightforward generalizations of the classical theory (see [Dur70], [Gar81], or [Koo80]).

Definition 2.1. We define $\mathcal{H}^p(\mathbb{D}, X)$ $(1 \leq p < \infty)$ as the space of analytic functions $f \colon \mathbb{D} \to X$ with

$$\|f\|_{H^p(X)} = \sup_{0<r<1} \|f_r\|_{L^p(X)} < \infty.$$

Here $\|\cdot\|_{L^p(X)}$ denotes the usual norm of the space $L^p(\mathbb{T}, X)$ (see Appendix). By $\mathcal{H}^\infty(\mathbb{D}, X)$ we mean the space of bounded analytic functions $f \colon \mathbb{D} \to X$. The quantity $\|\cdot\|_{H^p(X)}$ defines a norm that makes $\mathcal{H}^p(\mathbb{D}, X)$ into a Banach space for every $1 \leq p \leq \infty$ [Hen86, p. 14-15] (c.f. [Rud87, Remark 17.8]). Let $H^p(\mathbb{D}, X)$ denote the subspace of $\mathcal{H}^p(\mathbb{D}, X)$ that consists of analytic functions $f \colon \mathbb{D} \to X$ such that the radial limit $f^*(e^{i\theta}) = \lim_{r \to 1} f(re^{i\theta})$ exists almost everywhere on

$\mathbb{T}$. It is easy to see that for $1 \le p \le \infty$, the space $H^p(\mathbb{D}, X)$ is a closed subspace of $\mathcal{H}^p(\mathbb{D}, X)$ and hence a Banach space. We will see that the space $H^p(\mathbb{D}, X)$ corresponds isometrically to the vector-valued version of definition (ii) above.

Note that in the literature one usually denotes the space of all bounded analytic functions by $H^\infty(\mathbb{D}, X)$ instead of $\mathcal{H}^\infty(\mathbb{D}, X)$. However, with the introduced notations $H^\infty(\mathbb{D}, X)$ is in general strictly smaller than $\mathcal{H}^\infty(\mathbb{D}, X)$.

First properties of the vector-valued Hardy spaces follow from the theory of subharmonic functions. Recall that a continuous real-valued function f on $\mathbb{D}$ is *subharmonic* if it satisfies the submean property: For every $z \in \mathbb{D}$ there exists $r_0 > 0$ such that $\{w \colon |w - z| < r_0\} \subset \mathbb{D}$, and

$$f(z) \le \frac{1}{2\pi} \int_0^{2\pi} f(z + re^{i\theta}) d\theta,$$

for every $0 < r < r_0$.

Lemma 2.2. *Let $1 \le p < \infty$ and let $f \colon \mathbb{D} \to X$ be an analytic function. Then*
(i) the function $z \mapsto \|f(z)\|_X^p$ is subharmonic on $\mathbb{D}$,
(ii) the function $r \mapsto \|f_r\|_{L^p(X)}$ is monotone increasing on $[0, 1)$, and
(iii) $\|f\|_{H^p(X)} = \lim_{r\to1} \|f_r\|_{L^p(X)} = \lim_{r\to1} \|f_r\|_{H^p(X)}$, for $1 \le p < \infty$.

Proof. We obtain (i) immediately from the vector-valued Cauchy formula and Hölder's inequality;

$$\begin{aligned} \|f(z)\|_X^p &= \left\| \frac{1}{2\pi} \int_0^{2\pi} f(z + re^{i\theta}) d\theta \right\|_X^p \\ &\le \left(\frac{1}{2\pi} \int_0^{2\pi} \|f(z + re^{i\theta})\|_X d\theta \right)^p \le \frac{1}{2\pi} \int_0^{2\pi} \|f(z + re^{i\theta})\|_X^p d\theta. \end{aligned}$$

Alternatively one may use the fact that the function $z \mapsto |\langle f(z), x^*\rangle|^p$ is subharmonic for every $x^* \in X^*$, and the function $x^* \mapsto |\langle f(z), x^*\rangle|^p$ is continuous on the compact space $(\overline{B_{X^*}}, w^*)$ for every $z \in \mathbb{D}$. Hence by [Ran95, Theorem 2.4.7], the function $\|f\|_X^p = \sup_{x^* \in \overline{B_{X^*}}} |\langle f, x^*\rangle|^p$ is subharmonic. Statements (ii) and (iii) follow from (i) and the properties of subharmonic functions in a well-known manner (see [Dur70, Theorem 1.6]). □

We show next that for $f \in H^p(\mathbb{D}, X)$, the almost everywhere existing radial limit f^* belongs to the space $L^p(\mathbb{T}, X)$ (c.f. [Hen86, p. 28]).

Theorem 2.3. *Let $f \in H^p(\mathbb{D}, X)$ $(1 \le p < \infty)$. Then $f^* \in L^p(\mathbb{T}, X)$ with $\|f^*\|_{L^p(X)} \le \|f\|_{H^p(X)}$.*

Proof. The radial limit function f^* is measurable as an almost everywhere pointwise limit of measurable functions. Moreover, by Lemma 2.2, Fatou's lemma and the continuity of $\|\cdot\|_X$, we have

$$\|f^*\|_{L^p(X)} = \frac{1}{2\pi} \int_0^{2\pi} \lim_{r\to1} \|f_r(e^{i\theta})\|_X d\theta \le \lim_{r\to1} \|f_r\|_{L^p(X)} = \|f\|_{H^p(X)},$$

so that $f^* \in L^p(\mathbb{T}, X)$. $\square$

Our definition of the Hardy space $H^p(\mathbb{D}, X)$ $(1 \leq p \leq \infty)$ as the subspace of the $\mathcal{H}^p(\mathbb{D}, X)$ functions having radial limits almost everywhere on $\mathbb{T}$ emphasizes the analytic nature of $H^p(\mathbb{D}, X)$. We may consider these spaces also from another point of view. Let

$$H^p(\mathbb{T}, X) = \{g \in L^p(\mathbb{T}, X) \colon \widehat{g}(n) = 0, \text{ for every } n < 0\},$$

where $1 \leq p \leq \infty$. Since the mapping $a_n \colon L^p(\mathbb{T}, X) \to X$, $a_n(g) = \widehat{g}(n)$ is clearly a bounded linear operator for any fixed n, we have that $H^p(\mathbb{T}, X) = \bigcap_{n<0} \ker a_n$ is a closed subspace of $L^p(\mathbb{T}, X)$, and hence a Banach space under the norm $\|\cdot\|_{L^p(X)}$.

In Section 2.1 we recalled that the Poisson integral $f := P[g]$ of a function $g \in H^p(\mathbb{T}, X)$ $(1 \leq p < \infty)$ is analytic. It is easy to see that actually f belongs to the space $H^p(\mathbb{D}, X)$ with $\|f\|_{H^p(X)} \leq \|g\|_{L^p(X)}$ (c.f. [Hen86, p. 18]). For a while, let $\widetilde{H}^p(\mathbb{D}, X)$ denote the image of $H^p(\mathbb{T}, X)$ under the Poisson integral $g \mapsto P[g]$. It is known that $g \mapsto P[g]$ is one-to-one. Since we also have $g = f^*$ and $\|g\|_{L^p(X)} \leq \|f\|_{H^p(X)}$, by Fatou's theorem (2.1) and Theorem 2.3, it follows that the Poisson integral defines an isometric isomorphism from $H^p(\mathbb{T}, X)$ to $\widetilde{H}^p(\mathbb{D}, X)$ where the inverse mapping of $g \mapsto P[g]$ is $f \mapsto f^* \colon \widetilde{H}^p(\mathbb{D}, X) \to H^p(\mathbb{T}, X)$. Consequently, $\widetilde{H}^p(\mathbb{D}, X)$ is a closed subspace of $H^p(\mathbb{D}, X)$, for $1 \leq p \leq \infty$.

We will use the following theorem (see e.g. [Hen86, Satz 2.7]). We only sketch the proof here.

Theorem 2.4. *Let $1 \leq p < \infty$, $f \in \mathcal{H}^p(\mathbb{D}, X)$ and $g \in L^p(\mathbb{T}, X)$. Then $f = P[g]$ if and only if $\lim_{r \to 1} f_r = g$ almost everywhere on $\mathbb{T}$.*

The implication "$\Rightarrow$" follows immediately from Fatou's theorem (2.1). For the implication "$\Leftarrow$" we use the identity $f(rz_0) = \frac{1}{2\pi}\int_0^{2\pi} f(re^{i\theta})P_{z_0}(\theta)d\theta$ for $0 < r < 1$ and a fixed $z_0 \in \mathbb{D}$. By the dominated convergence theorem, we obtain $\frac{1}{2\pi}\int_0^{2\pi} f(re^{i\theta})P_{z_0}(\theta)d\theta \to \frac{1}{2\pi}\int_0^{2\pi} g(e^{i\theta})P_{z_0}(\theta)d\theta = P[g](z_0)$, as $r \to 1$, since the majorant $\sup_{0<r<1} \|f(re^{i\theta})\|_X$ is integrable, by the vector-valued Hardy-Littlewood theorem [Hen86, Satz 1.10] (c.f. [Dur70, Theorem 1.8]). Hence $f(z_0) = \lim_{r \to 1} f(rz_0) = P[g](z_0)$.

From Theorems 2.3 and 2.4 we obtain immediately the isometric identity $\widetilde{H}^p(\mathbb{D}, X) = H^p(\mathbb{D}, X)$, for $1 \leq p < \infty$. In other words, we get the following corollary.

Corollary 2.5. *Let $1 \leq p < \infty$. Then*

$$H^p(\mathbb{D}, X) = \{f \in \mathcal{H}^p(\mathbb{D}, X) \colon f = P[g] \textit{ for some (unique) } g \in H^p(\mathbb{T}, X)\},$$

Consequently, the map $g \mapsto P[g]$ defines an isometric isomorphism between $H^p(\mathbb{T}, X)$ and $H^p(\mathbb{D}, X)$.

Note that although we have not stated it explicitely, the identity holds also for $p = \infty$, that is, $H^\infty(\mathbb{D}, X) = \{f \in \mathcal{H}^\infty(\mathbb{D}, X) \colon f = P[g], \text{ where } g \in H^\infty(\mathbb{T}, X)\}$ [Hen86]. However, we do not need this fact later.

We consider next more carefully the relationship of the spaces $H^p(\mathbb{D}, X)$ and $\mathcal{H}^p(\mathbb{D}, X)$. The following known example shows that the inclusion $H^p(\mathbb{D}, X) \subset \mathcal{H}^p(\mathbb{D}, X)$ may be strict.

Example 2.6. Define the analytic function $f \colon \mathbb{D} \to c_0$ by $f(z) = (z^n)_{n\in\mathbb{N}}$. Then for every $0 \leq r < 1$,

$$\|f_r\|^p_{L^p(c_0)} = \frac{1}{2\pi}\int_0^{2\pi} \|(r^n e^{in\theta})_{n\in\mathbb{N}}\|^p_\infty d\theta = \frac{1}{2\pi}\int_0^{2\pi} \sup_{n\in\mathbb{N}} |r^n|^p d\theta \leq 1,$$

so that $f \in \mathcal{H}^p(\mathbb{D}, c_0)$. However, for every $e^{i\theta} \in \mathbb{T}$, the sequence $(r^n e^{in\theta})_{n\in\mathbb{N}}$ converges coordinatewise to $(e^{in\theta})_{n\in\mathbb{N}} \notin c_0$, as $r \to 1$. Since norm convergence implies coordinatewise convergence in c_0, the radial limit $\lim_{r\to 1} f(re^{i\theta})$ does not exist in any $e^{i\theta} \in \mathbb{T}$. In particular, $f \notin H^p(\mathbb{D}, c_0)$.

With a similar argument one sees that $H^p(\mathbb{D}, X) \subsetneq \mathcal{H}^p(\mathbb{D}, X)$ if X is any Banach space that contains an isomorphic copy of c_0 (consider the analytic function $z \mapsto \sum_{n\in\mathbb{N}} z^n x_n$, where $(x_n) \subset X$ is such that $\|\sum_{n\in\mathbb{N}} a_n x_n\|_X \approx \|(a_n)\|_\infty$, for every $(a_n) \in c_0$).

The Banach spaces X satisfying $H^p(\mathbb{D}, X) = \mathcal{H}^p(\mathbb{D}, X)$ were characterized by Bukhvalov and Danilevitch ([Buk81], [BD82]):

Definition 2.7. A complex Banach space X is said to have the *analytic Radon-Nikodým property* (ARNP), if for every $f \in \mathcal{H}^p(\mathbb{D}, X)$ the radial limit f^* exists almost everywhere on $\mathbb{T}$, where $1 \leq p \leq \infty$.

Every $f \in \mathcal{H}^p(\mathbb{D}, X)$ $(1 \leq p \leq \infty)$ can be written as $f = g/h$, where $g \in \mathcal{H}^\infty(\mathbb{D}, X)$ and $h \in H^\infty(\mathbb{D}, \mathbb{C})$, h non-vanishing on $\mathbb{D}$. This is the vector-valued version of the well-known fact from the classical Hardy space theory [Hen86, Satz 2.8]. Since h has boundary values almost everywhere on $\mathbb{T}$ (by the classical Fatou theorem), the analytic Radon-Nikodým property is independent of p. The above definition is the original definition of the ARNP by Bukhvalov and Danilevitch, but there are are also many other useful characterizations of the ARNP (see [Hen86, p. 45]). Note that by definition, X has the ARNP if and only if $H^p(\mathbb{D}, X) = \mathcal{H}^p(\mathbb{D}, X)$ for some (every) $1 \leq p \leq \infty$.

It is useful to know which spaces possess the ARNP. As Example 2.6 shows, c_0 does not have the ARNP, nor does any Banach space that contains c_0 isomorphically. However, since the reflexive Banach spaces have the Radon-Nikodým property (RNP), we have the following theorem.

Theorem 2.8. *If X is a reflexive complex Banach space, then X has the ARNP.*

We do not prove Theorem 2.8 here since the proof involves concepts such as vector measures and vector-valued harmonic functions that would lead us far from our present subject. However, we give some comments on the proof at the end of the chapter.

We prove next some simple growth estimates that are familiar from the classical Hardy space theory.

Proposition 2.9. *Let $f \in \mathcal{H}^p(\mathbb{D}, X)$ $(1 \le p < \infty)$ and $z \in \mathbb{D}$. Then*

$$\|f(z) - f(0)\|_X \le \frac{|z|}{1-|z|}\|f\|_{H^p(X)}.$$

Proof. Let $f \in \mathcal{H}^p(\mathbb{D}, X)$ $(1 \le p < \infty)$. Then f is analytic in $\mathbb{D}$. By the vector-valued Cauchy integral formula,

$$f(z) - f(0) = \frac{1}{2\pi i}\int_\gamma \left(\frac{f(\zeta)}{\zeta - z} - \frac{f(\zeta)}{\zeta}\right) d\zeta = \frac{1}{2\pi}\int_0^{2\pi} \frac{e^{-i\theta} z f_r(e^{i\theta})}{r - e^{-i\theta} z} d\theta,$$

where $|z| < r < 1$, and $\gamma\colon [0, 2\pi) \to \mathbb{C}$ is the path $\gamma(\theta) = re^{i\theta}$. Thus,

$$\|f(z) - f(0)\|_X \le \sup_{\theta \in [0,2\pi)} \frac{|z|}{|r - e^{-i\theta} z|}\|f_r\|_{L^1(X)} = \frac{|z|}{r - |z|}\|f_r\|_{L^1(X)}.$$

The proof follows since $\|f_r\|_{L^1(X)} \le \|f_r\|_{L^p(X)} \le \|f\|_{H^p(X)}$ and r is arbitrary. □

Corollary 2.10. *Let $f \in \mathcal{H}^p(\mathbb{D}, X)$ $(1 \le p < \infty)$ and $z \in \mathbb{D}$. Then*

$$\|f(z)\|_X \le \frac{1}{1-|z|}\|f\|_{H^p(X)}.$$

Proof. For $0 < r < 1$,

$$\|f(0)\|_X = \left\|\frac{1}{2\pi}\int_0^{2\pi} f(re^{i\theta}) d\theta\right\|_X \le \|f_r\|_{H^1(X)} \le \|f\|_{H^p(X)}.$$

Hence

$$\|f(z)\|_X \le \|f(z) - f(0)\|_X + \|f(0)\|_X \le \left(\frac{|z|}{1-|z|} + 1\right)\|f\|_{H^p(X)}.$$

□

If $f \in \mathcal{H}^\infty(\mathbb{D}, X)$, then clearly also $f \circ \varphi \in \mathcal{H}^\infty(\mathbb{D}, X)$ for every analytic $\varphi\colon \mathbb{D} \to \mathbb{D}$. We conclude the section by showing that the same fact holds for any $1 \le p < \infty$, that is, if $f \in \mathcal{H}^p(\mathbb{D}, X)$, then $f \circ \varphi \in \mathcal{H}^p(\mathbb{D}, X)$ for every analytic $\varphi\colon \mathbb{D} \to \mathbb{D}$. This was shown by Liu, Saksman and Tylli in [LST98, Proposition 1]. Another proof of the fact can be found in [HJ99]. We give yet another proof that relies on the classical proof of [Dur70, p. 29]. In particular, we show that the norm of $f \circ \varphi$ is bounded by a constant times the norm of f.

Let $f : \mathbb{D} \to \mathbb{R}$ be subharmonic. A function $u : \mathbb{D} \to \mathbb{R}$ is a *harmonic majorant* of f, if u is harmonic, and $f(z) \le u(z)$ on $\mathbb{D}$. The harmonic majorant u_f is the *least harmonic majorant* of f, if $u_f(z) \le u(z)$ for every other harmonic majorant u of f. If f has a harmonic majorant, then it has the least one. Since $\|f(\cdot)\|_X$ is subharmonic for an analytic function $f : \mathbb{D} \to X$, the following result can be proved exactly as in the scalar case (see [Gar81, Theorem I.6.7], [Dur70, Theorem 2.12]). Hence we omit the proof here.

Proposition 2.11. *Let $f : \mathbb{D} \to X$ be analytic and let $1 \le p < \infty$. Then $f \in \mathcal{H}^p(\mathbb{D}, X)$ if and only if the function $z \mapsto \|f(z)\|_X^p$ has a harmonic majorant.*

In that case the least harmonic majorant u of $z \mapsto \|f(z)\|_X^p$ is defined by

$$u(z) = \lim_{r \to 1^-} \frac{1}{2\pi} \int_0^{2\pi} \|f(re^{i\theta})\|_X^p P_z(\theta) d\theta,$$

for $z \in \mathbb{D}$.

Theorem 2.12. *Let $f \in \mathcal{H}^p(\mathbb{D}, X)$ $(1 \le p < \infty)$ and let $\varphi : \mathbb{D} \to \mathbb{D}$ be analytic. Then $f \circ \varphi \in \mathcal{H}^p(\mathbb{D}, X)$, with*

$$\|f \circ \varphi\|_{H^p(X)} \le \left(\frac{1 + |\varphi(0)|}{1 - |\varphi(0)|} \right)^{1/p} \|f\|_{H^p(X)}.$$

Proof. Let f and φ be as above. Then by Proposition 2.11, the function

$$u(z) = \lim_{r \to 1^-} \frac{1}{2\pi} \int_0^{2\pi} \|f(re^{i\theta})\|_X^p P_z(\theta) d\theta \tag{2.2}$$

is a harmonic majorant of $\|f\|_X^p$. Thus $u \circ \varphi$ is a harmonic majorant of $\|f \circ \varphi\|_X^p$. Hence

$$\begin{aligned} \frac{1}{2\pi} \int_0^{2\pi} \|(f \circ \varphi)(re^{i\theta})\|_X^p d\theta &\le \frac{1}{2\pi} \int_0^{2\pi} (u \circ \varphi)(re^{i\theta}) d\theta \\ &= (u \circ \varphi)(0) \\ &= \lim_{r \to 1^-} \frac{1}{2\pi} \int_0^{2\pi} \|f(re^{i\theta})\|_X^p P_{\varphi(0)}(\theta) d\theta, \end{aligned}$$

for every $0 < r < 1$, by the mean value property of the harmonic function $u \circ \varphi$ and (2.2). Since $P_z(\theta) = (1 - |z|^2)/|z - e^{i\theta}|^2 \le (1 + |z|)/(1 - |z|)$ for $z \in \mathbb{D}$, we have

$$\begin{aligned} \frac{1}{2\pi} \int_0^{2\pi} \|(f \circ \varphi)(re^{i\theta})\|_X^p d\theta &= \lim_{r \to 1^-} \frac{1}{2\pi} \int_0^{2\pi} \|f(re^{i\theta})\|_X^p P_{\varphi(0)}(\theta) d\theta \\ &\le \frac{1 + |\varphi(0)|}{1 - |\varphi(0)|} \left(\lim_{r \to 1^-} \frac{1}{2\pi} \int_0^{2\pi} \|f(re^{i\theta})\|_X^p d\theta \right) \\ &= \frac{1 + |\varphi(0)|}{1 - |\varphi(0)|} \|f\|_{H^p(X)}^p, \end{aligned}$$

for every $0 < r < 1$. Taking the supremum over r completes the proof. □

2.3 Analytic functions of bounded and vanishing mean oscillation

Recall that the space $BMOA$ of complex-valued functions is often defined as the set of the Poisson extensions to the unit disk of the analytic BMO functions defined on $\mathbb{T}$. In the scalar theory there exist several useful characterizations for $BMOA$. Vector-valued Hardy space theory and its connections to the analytic Radon-Nikodým property suggest that in the vector-valued case we introduce two versions of the space, $BMOA(\mathbb{D}, X)$ and $\mathcal{BMOA}(\mathbb{D}, X)$. As one might expect, it turns out that $BMOA(\mathbb{D}, X) = \mathcal{BMOA}(\mathbb{D}, X)$ if and only if X has the ARNP. Moreover, both spaces will be useful from the view of the theory of composition operators. We proceed classically, that is, starting from the vector-valued BMO functions. We also introduce the vector-valued version of the space $VMOA$.

We define the vector-valued BMO space as follows.

$$BMO(\mathbb{T}, X) = \left\{ f \in L^1(\mathbb{T}, X) \colon \|f\|_{*,X} = \sup \frac{1}{m(I)} \int_I \|f(e^{it}) - f_I\|_X dt < \infty \right\},$$

where $f_I = \frac{1}{m(I)} \int_I f(e^{it}) dt$, and the supremum is taken over all subintervals I of $\mathbb{T}$. Endowed with the norm $\|f\|_{BMO(\mathbb{T},X)} = \|f_{\mathbb{T}}\|_X + \|f\|_{*,X}$ the space $BMO(\mathbb{T}, X)$ becomes a Banach space. We refer to the Appendix for further properties of the space $BMO(\mathbb{T}, X)$.

The following definition of the vector-valued $BMOA$ corresponds to the classical definition. For vector-valued functions this definition was introduced by Blasco ([Bla95], [Bla97]). Let

$$BMOA(\mathbb{D}, X) = \{f \in H(\mathbb{D}, X) \colon f = P[g] \text{ for some (unique) } g \in BMO(\mathbb{T}, X)\}$$

and endow the space $BMOA(\mathbb{D}, X)$ with the norm $\|f\|_{BMOA(\mathbb{D},X)} = \|g\|_{BMO(\mathbb{T},X)}$, where f and g are as in the definition. In particular, for given $1 \le p < \infty$ we get that

$$\|f\|_{BMOA(\mathbb{D},X)} \approx \|f(0)\|_X + \sup_{a \in \mathbb{D}} \|g \circ \sigma_a - f(a)\|_{L^p(X)}, \tag{2.3}$$

(see Corollary A.15 of the Appendix). We set

$$BMOA(\mathbb{T}, X) = \{f \in BMO(\mathbb{T}, X) \colon \widehat{f}(n) = 0, \text{ for every } n < 0\}.$$

Since $\|\widehat{f}(n)\|_X \le \|f\|_{L^1(X)} \le \|f\|_{BMO(\mathbb{T},X)}$ (see (A.2) of the Appendix), the mapping $a_n \colon BMO(\mathbb{T}, X) \to X$, $a_n(f) = \widehat{f}(n)$ is a bounded linear operator for any fixed n. Hence $BMOA(\mathbb{T}, X) = \bigcap_{n<0} \ker a_n$ is a closed subspace of $BMO(\mathbb{T}, X)$ and a Banach space under the norm $\|\cdot\|_{BMO(\mathbb{T},X)}$. Since $f \mapsto P[f]$ is a one-to-one map of $L^1(\mathbb{T},X)$ onto $H^1(\mathbb{T},X)$, it defines an isomorphism from $BMOA(\mathbb{T},X)$ to $BMOA(\mathbb{D},X)$. In particular, $BMOA(\mathbb{D},X)$ is a Banach space.

The norm $\|\cdot\|_{BMOA(\mathbb{D},X)}$ is in many situations not very practical. Using the properties of Garsia type seminorms from the $BMO(\mathbb{T},X)$ theory, we define a more practical norm on $BMOA(\mathbb{D},X)$ as follows. For $a \in \mathbb{D}$, let σ_a denote the *Möbius transformation* $\sigma_a(z) = (a-z)/(1-\overline{a}z)$. Set

$$\|f\|_{**,p,X} = \sup_{a\in\mathbb{D}} \|f\circ\sigma_a - f(a)\|_{H^p(X)},$$

and

$$\|f\|_{\mathcal{BMOA}(\mathbb{D},X),p} = \|f(0)\|_X + \|f\|_{**,p,X},$$

for $1 \le p < \infty$ where $f\colon \mathbb{D} \to X$ is analytic. It is easy to see that the quantity $\|\cdot\|_{\mathcal{BMOA}(\mathbb{D},X),p}$ defines a norm. Note also that we have the inequalities

$$\begin{aligned}\|f\|_{H^1(X)} &\le \|f - f(0)\|_{H^1(X)} + \|f(0)\|_X \\ &\le \sup_{a\in\mathbb{D}} \|f\circ\sigma_a - (f\circ\sigma_a)(0)\|_{H^1(X)} + \|f(0)\|_X = \|f\|_{\mathcal{BMOA}(\mathbb{D},X),1},\end{aligned} \tag{2.4}$$

and

$$\begin{aligned}\|f\|_{\mathcal{BMOA}(\mathbb{D},X),1} &= \sup_{a\in\mathbb{D}} \|f\circ\sigma_a - f(a)\|_{H^1(X)} + \|f(0)\|_X \\ &\le \sup_{a\in\mathbb{D}} \big(\|f\circ\sigma_a\|_{H^\infty(X)} + \|f(a)\|_X\big) + \|f(0)\|_X \le 3\|f\|_{H^\infty(X)},\end{aligned} \tag{2.5}$$

for an analytic function $f\colon \mathbb{D} \to X$. We will use the abbreviations $\|f\|_{**,X} = \|f\|_{**,1,X}$ and $\|f\|_{\mathcal{BMOA}(\mathbb{D},X)} = \|f\|_{\mathcal{BMOA}(\mathbb{D},X),1}$ for the case $p = 1$.

The following theorem is our main application of the $BMO(\mathbb{T},X)$ theory to the $BMOA(\mathbb{D},X)$ theory:

Theorem 2.13. *Let $1 \le p < \infty$. Then $f \in BMOA(\mathbb{D},X)$ if and only if f^* exists almost everywhere on $\mathbb{T}$ and*

$$\|f\|_{\mathcal{BMOA}(\mathbb{D},X),p} < \infty.$$

Moreover,

$$\|f\|_{BMOA(\mathbb{D},X)} \approx \|f\|_{\mathcal{BMOA}(\mathbb{D},X),p},$$

for analytic $f\colon \mathbb{D} \to X$ such that the radial limit f^* exists almost everywhere on $\mathbb{T}$.

Proof. Assume first that $f \in BMOA(\mathbb{D},X)$. Then there exists a unigue $g \in BMO(\mathbb{T},X)$ such that $f = P[g]$. By Fatou's theorem (2.1), the radial limit f^* exists almost everywhere on $\mathbb{T}$ with $f^* = g$. Thus $f^* \in BMO(\mathbb{T},X)$ and $f = P[f^*]$. By (2.3), we have

$$\|f\|_{BMOA(\mathbb{D},X)} \approx \|f(0)\|_X + \sup_{a\in\mathbb{D}} \|f^*\circ\sigma_a - P[f^*](a)\|_{L^p(X)} < \infty.$$

Since the Poisson integral defines an isometry from $L^p(\mathbb{T}, X)$ to $H^p(\mathbb{T}, X)$, we obtain

$$\|f(0)\|_X + \sup_{a\in\mathbb{D}} \|P[f^* \circ \sigma_a - P[f^*](a)]\|_{H^p(X)} \le C\|f\|_{BMOA(\mathbb{D},X)},$$

where C depends only on p. Now by linearity of the Poisson integral and Lemma A.14,

$$\begin{aligned}
\|f\|_{\mathcal{BMOA}(\mathbb{D},X),p} &= \|f(0)\|_X + \|f \circ \sigma_a - f(a)\|_{H^p(X)} \\
&= \|f(0)\|_X + \|P[f^*] \circ \sigma_a - P[f^*](a)\|_{H^p(X)} \\
&= \|f(0)\|_X + \|P[f^* \circ \sigma_a - P[f^*](a)]\|_{H^p(X)} \le C\|f\|_{BMOA(\mathbb{D},X)}.
\end{aligned}$$

Assume next that $f: \mathbb{D} \to X$ is analytic such that f^* exists almost everywhere on $\mathbb{T}$ and $\|f\|_{\mathcal{BMOA}(\mathbb{D},X),p} < \infty$. Then $f \in \mathcal{H}^1(\mathbb{D}, X)$, by (2.4) and Hölder's inequality. We have actually that $f = P[f^*] \in H^1(\mathbb{D}, X)$ and $f^* \in H^1(\mathbb{T}, X)$, by the fact that f^* exists almost everywhere on $\mathbb{T}$, the definition of the space $H^1(\mathbb{D}, X)$ and Corollary 2.5. By Theorem 2.12, we have $f^* \circ \sigma_a - P[f^*](a) \in H^1(\mathbb{T}, X)$ for every $a \in \mathbb{D}$. By linearity of the Poisson integral and Lemma A.14 one gets

$$\begin{aligned}
\|f^* \circ \sigma_a - P[f^*](a)\|_{L^p(X)} &= \|P[f^* \circ \sigma_a - f(a)]\|_{H^p(X)} \\
&= \|P[f^*] \circ \sigma_a - f(a)\|_{H^p(X)} = \|f \circ \sigma_a - f(a)\|_{H^p(X)}.
\end{aligned}$$

Since $f = P[f^*]$, we obtain

$$\begin{aligned}
\|f\|_{BMOA(\mathbb{D},X)} &\le C(\|f(0)\|_X + \sup_{a\in\mathbb{D}} \|f^* \circ \sigma_a - P[f^*](a)\|_{L^p(X)}) \\
&\le C(\|f(0)\|_X + \sup_{a\in\mathbb{D}} \|f \circ \sigma_a - f(a)\|_{H^p(X)}) \\
&= C\|f\|_{\mathcal{BMOA}(\mathbb{D},X)},
\end{aligned}$$

by (2.3), where C depends only on p. □

We introduce an analogous definition for the space of analytic X-valued functions of vanishing mean oscillation:

$$VMOA(\mathbb{D}, X) = \{f \in H(\mathbb{D}, X) : f = P[g], g \in VMO(\mathbb{T}, X)\},$$

where the space

$$VMO(\mathbb{T}, X) = \left\{ f \in BMO(\mathbb{T}, X) : \lim_{m(I)\to 0} \frac{1}{m(I)} \int_I \|f(e^{it}) - f_I\|_X dt = 0 \right\}$$

is a closed subspace of $BMO(\mathbb{T}, X)$ (Proposition A.18). Moreover, let

$$VMOA(\mathbb{T}, X) = \{f \in VMO(\mathbb{T}, X) : \widehat{f}(n) = 0, \text{ for every } n < 0\}.$$

Since $VMOA(\mathbb{T},X) = VMO(\mathbb{T},X) \cap BMOA(\mathbb{T},X)$ is a closed subspace of $BMO(\mathbb{T},X)$, it is a Banach space. Moreover, the map $f \mapsto P[f]$ defines an isometric isomorphism from $VMOA(\mathbb{T},X)$ to $VMOA(\mathbb{D},X)$. In particular, the space $VMOA(\mathbb{D},X)$ is a Banach space.

It is important to notice that $VMO(\mathbb{T},X)$ is the closure of the trigonometric polynomials in the space $BMO(\mathbb{T},X)$ (Theorem A.20). Consequently, the space $VMOA(\mathbb{T},X)$ is the closure in $BMOA(\mathbb{T},X)$ of the "analytic" trigonometric polynomials $s_N(e^{i\theta}) = \sum_{n=0}^{N} x_n e^{in\theta}$ ($x_n \in X$). Hence $VMOA(\mathbb{D},X)$ is the closure of the polynomials $\sum_{n=0}^{N} x_n z^n = P[s_N](z)$ in $BMOA(\mathbb{D},X)$.

We have also the following characterization of $VMOA(\mathbb{D},X)$ that follows easily from the computations of the proof of Theorem 2.13 and the equivalence $(i) \Leftrightarrow (vi)$ of Theorem A.20.

Theorem 2.14. *Let $1 \leq p < \infty$. Then $f \in VMOA(\mathbb{D},X)$ if and only if $f \in BMOA(\mathbb{D},X)$ and*

$$\lim_{|a|\to 1} \|f \circ \sigma_a - f(a)\|_{H^p(X)} = 0.$$

We conclude the section by studying briefly the invariance of the seminorm $\|\cdot\|_{**,X}$ under composition with Möbius transformations. The following lemma is familiar from the scalar theory.

Lemma 2.15. *Let $f\colon \mathbb{D} \to X$ be analytic and let $1 \leq p < \infty$. Then*

$$\|f\|_{**,p,X} = \sup_{\tau \in \mathcal{M}} \|f \circ \tau - f(\tau(0))\|_{H^p(X)},$$

where $\mathcal{M} = \{\xi\sigma_a \colon a \in \mathbb{D}, |\xi| = 1\}$.

Proof. Let $f\colon \mathbb{D} \to X$ be analytic and let $N(f) = \sup_{\tau\in\mathcal{M}} \|f \circ \tau - f(\tau(0))\|_{H^p(X)}$. Clearly $\|f\|_{**,p,X} \leq N(f)$. Let $\psi \in \mathcal{M}$. Then it takes an easy calculation to see that there exist $a \in \mathbb{D}$ and $\psi \in [0, 2\pi)$ such that $\tau(z) = \sigma_a(e^{i\psi}z)$. Thus

$$\begin{aligned}\int_0^{2\pi} \|f(\tau(re^{i\theta})) - f(\tau(0))\|_X^p d\theta &= \int_0^{2\pi} \|f(\sigma_a(re^{i(\theta+\psi)})) - f(\sigma_a(0))\|_X^p d\theta \\ &= \int_0^{2\pi} \|f(\sigma_a(re^{i\theta})) - f(\sigma_a(0))\|_X^p d\theta,\end{aligned}$$

by the translation invariance of the $L^p(\mathbb{T},X)$ norm. Taking the pth root and the supremum over $r \in (0,1)$ and $a \in \mathbb{D}$ in the equality gives $N(f) \leq \|f\|_{**,p,X}$. ☐

The following lemma is the $BMOA$ version of Corollary 2.10.

Lemma 2.16. *Let $f\colon \mathbb{D} \to X$ be analytic and $z \in \mathbb{D}$. Then*

$$\|f(z)\|_X \leq \|f(0)\|_X + \left(\frac{1}{2}\log\frac{1+|z|}{1-|z|}\right) \|f\|_{**,X}.$$

Proof. Let $f\colon \mathbb{D} \to X$ be analytic. Using the vector-valued Cauchy integral formula for the derivative, we obtain

$$\|f'(0)\|_X = \left\| \frac{1}{2\pi} \int_{\mathbb{D}} \frac{f(re^{i\theta})}{re^{i\theta}} d\theta \right\|_X \leq \frac{1}{r} \|f\|_{H^1(X)},$$

for any $0 < r < 1$, so that $\|f'(0)\|_X \leq \|f\|_{H^1(X)}$. Let now $g_z = f \circ \sigma_z - f(z)$ where $z \in \mathbb{D}$. Then we calculate that

$$\|(g_z)'(0)\|_X = \|f'(\sigma_z(0))\|_X |\sigma_z'(0)| = \|f'(z)\|_X (1 - |z|^2).$$

Thus we obtain

$$\|f'(z)\|_X (1 - |z|^2) \leq \|g_z\|_{H^1(X)} \leq \|f\|_{**,X}, \tag{2.6}$$

for any analytic function $f\colon \mathbb{D} \to X$ and $z \in \mathbb{D}$. If $z = re^{i\theta}$, then

$$f(z) - f(0) = \int_0^z f'(w)dw = e^{i\theta} \int_0^r f'(te^{i\theta})dt,$$

so that

$$\begin{aligned} \|f(z)\|_X &\leq \|f(0)\|_X + \int_0^r \|f'(te^{i\theta})\|_X dt \leq \|f(0)\|_X + \|f\|_{**,X} \int_0^r \frac{1}{1-t^2} dt \\ &= \|f(0)\|_X + \frac{1}{2} \|f\|_{**,X} \log \frac{1+r}{1-r}, \end{aligned}$$

by (2.6). □

Corollary 2.17. *Let $1 \leq p < \infty$ and let $a \in \mathbb{D}$. Then*

$$\|f \circ \sigma_a\|_{**,p,X} = \|f\|_{**,p,X}.$$

Moreover,

$$\|f \circ \sigma_a\|_{\mathcal{BMOA}(\mathbb{D},X),p} \leq \left(1 + \frac{1}{2} \log \frac{1+|a|}{1-|a|}\right) \|f\|_{\mathcal{BMOA}(\mathbb{D},X),p}.$$

Proof. Let $\tau \in \mathcal{M} = \{\xi \sigma_a \colon a \in \mathbb{D}, |\xi| = 1\}$ and let $a \in \mathbb{D}$. Then $\sigma_a \circ \tau = \widetilde{\tau} \in \mathcal{M}$, and

$$\begin{aligned} \|f \circ \sigma_a \circ \tau - f(\sigma_a(\tau(0)))\|_{H^p(X)} &= \|f \circ \widetilde{\tau} - f(\widetilde{\tau}(0))\|_{H^p(X)} \\ &\leq \sup_{\tau \in \mathcal{M}} \|f \circ \tau - f(\tau(0))\|_{H^p(X)}. \end{aligned}$$

Taking the supremum over $\tau \in \mathcal{M}$ gives the proof, by Lemma 2.15. Now,

$$\begin{aligned} \|f \circ \sigma_a\|_{\mathcal{BMOA}(\mathbb{D},X),p} &= \|f \circ \sigma_a\|_{**,p,X} + \|f(a)\|_X \\ &= \|f\|_{**,p,X} + \|f(a)\|_X \\ &\leq \|f\|_{**,p,X} + \|f(0)\|_X + \left(\frac{1}{2} \log \frac{1+|a|}{1-|a|}\right) \|f\|_{**,p,X} \\ &\leq \left(1 + \frac{1}{2} \log \frac{1+|a|}{1-|a|}\right) \|f\|_{\mathcal{BMOA}(\mathbb{D},X),p}, \end{aligned}$$

by Lemma 2.16. □

Finally, we need the following simple observation about the Taylor coefficients of $BMOA(\mathbb{D}, X)$ functions.

Lemma 2.18. *Let* $f(z) = \sum_0^\infty x_n z^n$ *be such that* $\|f\|_{\mathcal{BMOA}(\mathbb{D},X)} < \infty$. *Then* $\|x_n\|_X \le \|f\|_{\mathcal{BMOA}(\mathbb{D},X)}$, *for all* $n \ge 0$.

Proof. The claim follows from the stronger estimate for $\mathcal{H}^1(\mathbb{D}, X)$ functions. Let $n \ge 0$. Then

$$\|x_n\|_X = \left\| \frac{1}{2\pi} \int_0^{2\pi} \frac{f(re^{i\theta})}{r^n e^{in\theta}} d\theta \right\|_X \le \frac{1}{2\pi} \int_0^{2\pi} \frac{\|f(re^{i\theta})\|_X}{r^n} d\theta \le \frac{1}{r^n} \|f\|_{H^1(X)},$$

for $0 < r < 1$. Using the fact that $\|f\|_{H^1(X)} \le \|f\|_{\mathcal{BMOA}(\mathbb{D},X)}$ and letting $r \to 1$ we obtain the desired estimate. ☐

2.4 An alternative approach to BMOA

We consider next a vector-valued generalization of $BMOA$, called $\mathcal{BMOA}(\mathbb{D}, X)$, that is a larger space than $BMOA(\mathbb{D}, X)$ in the sense that $BMOA(\mathbb{D}, X) \subset \mathcal{BMOA}(\mathbb{D}, X)$ but $\mathcal{BMOA}(\mathbb{D}, X)$ may contain functions that do not have radial limits anywhere on $\mathbb{T}$.

Motivated by Theorems 2.13 and 2.14, and the definitions familiar from the scalar theory, we make the following definitions:

$$\mathcal{BMOA}(\mathbb{D}, X) = \{ f \in H(\mathbb{D}, X) \colon \|f\|_{\mathcal{BMOA}(\mathbb{D},X)} < \infty \},$$

and

$$\mathcal{VMOA}(\mathbb{D}, X) = \{ f \in \mathcal{BMOA}(\mathbb{D}, X) \colon \lim_{|a| \to 1} \|f \circ \sigma_a - f(a)\|_{H^1(X)} = 0 \},$$

where the norm in both spaces is given by $\|f\|_{\mathcal{BMOA}(\mathbb{D},X)} = \|f(0)\|_X + \|f\|_{**,X}$, where $\|f\|_{**,X} = \sup_{a \in \mathbb{D}} \|f \circ \sigma_a - f(a)\|_{H^1(X)}$. The above definition of $BMOA$ has appeared in [CO01] along with many basic results in the setting of vector-valued analytic functions on the unit ball of $\mathbb{C}^n$. As in the case of Hardy spaces, the motivation from the complex-valued theory comes from the facts $BMOA(\mathbb{D}, \mathbb{C}) = \mathcal{BMOA}(\mathbb{D}, \mathbb{C})$ and $VMOA(\mathbb{D}, \mathbb{C}) = \mathcal{VMOA}(\mathbb{D}, \mathbb{C})$, that follow from Fatou's theorem: for every $f \in H^1(\mathbb{D}, \mathbb{C})$ the radial limit exists almost everywhere on $\mathbb{T}$. Moreover, $\mathcal{BMOA}(\mathbb{D}, X)$ extends naturally the space $BMOA(\mathbb{D}, X)$ which becomes a closed subspace of $\mathcal{BMOA}(\mathbb{D}, X)$. We will see that the boundedness of the composition operator is easier to establish first on $\mathcal{BMOA}(\mathbb{D}, X)$ and the result may be used to establish the boundedness on $BMOA(\mathbb{D}, X)$.

We need first to assure that $\mathcal{BMOA}(\mathbb{D}, X)$ and $\mathcal{VMOA}(\mathbb{D}, X)$ are Banach spaces:

Theorem 2.19. $(\mathcal{BMOA}(\mathbb{D},X),\|\cdot\|_{\mathcal{BMOA}(\mathbb{D},X)})$ *is a Banach space whenever* X *is a complex Banach space.*

Proof. Let (f_n) be a Cauchy sequence in $\mathcal{BMOA}(\mathbb{D},X)$ and let $\varepsilon > 0$. Then there exists a positive integer n_ε such that

$$\|f_n - f_m\|_{H^1(X)} \le \|f_n - f_m\|_{\mathcal{BMOA}(\mathbb{D},X)} < \varepsilon, \tag{2.7}$$

for every $n, m \ge n_\varepsilon$, by (2.4). Hence (f_n) is a Cauchy sequence in $\mathcal{H}^1(\mathbb{D},X)$ and, since $\mathcal{H}^1(\mathbb{D},X)$ is complete, converges to an analytic function $f \in \mathcal{H}^1(\mathbb{D},X)$. Since, by Corollary 2.10, the $\mathcal{H}^1(\mathbb{D},X)$ norm convergence implies pointwise convergence on $\mathbb{D}$, we have for every $a \in \mathbb{D}$ that $\|f(a) - f_n(a)\|_X \to 0$, as $n \to \infty$. Fix one $a \in \mathbb{D}$. Then by Theorem 2.12, there exists an integer $n_a \ge n_\varepsilon$ such that

$$\begin{aligned}&\|f \circ \sigma_a - f(a) - (f_m \circ \sigma_a - f_m(a))\|_{H^1(X)}\\ &\le C_a \left(\|f - f_m\|_{H^1(X)} + \|f(a) - f_m(a)\|_X\right) < \varepsilon,\end{aligned}$$

for every $m \ge n_a$, where we can take $C_a = (1+|a|)/(1-|a|)$.

Let now $n \ge n_\varepsilon$. Then for every $a \in \mathbb{D}$ we have

$$\begin{aligned}&\|f \circ \sigma_a - f(a) - (f_n \circ \sigma_a - f_n(a))\|_{H^1(X)}\\ &\le \sup_{m \ge n_a} \|f \circ \sigma_a - f(a) - (f_m \circ \sigma_a - f_m(a))\|_{H^1(X)}\\ &\quad + \sup_{m \ge n_a} \|f_n \circ \sigma_a - f_n(a) - (f_m \circ \sigma_a - f_m(a))\|_{H^1(X)}\\ &\le \varepsilon + \sup_{m \ge n_\varepsilon} \|f_n - f_m\|_{\mathcal{BMOA}(\mathbb{D},X)} \le 2\varepsilon,\end{aligned}$$

by (2.7). Taking the supremum over $a \in \mathbb{D}$, gives $\|f - f_n\|_{**,X} \le 2\varepsilon$. Since $\|f(0) - f_n(0)\|_X \to 0$, as $n \to \infty$, we have $\|f - f_n\|_{\mathcal{BMOA}(\mathbb{D},X)} \to 0$, and $f \in \mathcal{BMOA}(\mathbb{D},X)$. ☐

Theorem 2.20. $\mathcal{VMOA}(\mathbb{D},X)$ *is a closed subspace of* $\mathcal{BMOA}(\mathbb{D},X)$.

Proof. Let $f \in \mathcal{BMOA}(\mathbb{D},X)$ belong to the norm closure of $\mathcal{VMOA}(\mathbb{D},X)$ and let $\varepsilon > 0$. Then there exists a function $g \in \mathcal{VMOA}(\mathbb{D},X)$ and a number $0 < r < 1$, such that $\|f - g\|_{**,X} < \varepsilon/2$, and $\|g \circ \sigma_a - g(a)\|_{H^1(X)} < \varepsilon/2$, for $a \in \mathbb{D}$ with $|a| \ge r$. Let a be such a point. Then

$$\|f \circ \sigma_a - f(a)\|_{H^1(X)} \le \|g \circ \sigma_a - g(a)\|_{H^1(X)} + \|f - g\|_{**,X} \le \varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, $f \in \mathcal{VMOA}(\mathbb{D},X)$. ☐

We have clearly always that $BMOA(\mathbb{D},X) \subset \mathcal{BMOA}(\mathbb{D},X)$. Moreover, as a Banach space $BMOA(\mathbb{D},X)$ is a closed subspace of $\mathcal{BMOA}(\mathbb{D},X)$. The fact $H^p(\mathbb{D},X) \subsetneq \mathcal{H}^p(\mathbb{D},X)$ whenever X does not have the ARNP, suggests that in general $\mathcal{BMOA}(\mathbb{D},X)$ is strictly larger than $BMOA(\mathbb{D},X)$. The following example shows that this is indeed the case (cf. Example 2.6):

Example 2.21. Let $f\colon \mathbb{D} \to c_0$ be the analytic function $f(z) = (z^n)_{n\in\mathbb{N}}$. Then for every $0 \le r < 1$,

$$\begin{aligned}\|(f\circ\sigma_a)_r - f(a)\|_{L^1(c_0)} &\le \|(f\circ\sigma_a)_r\|_{L^1(c_0)} + \|(f(a))_{n\in\mathbb{N}}\|_\infty \\ &= \frac{1}{2\pi}\int_0^{2\pi} \|(\sigma_a(re^{i\theta})^n)_{n\in\mathbb{N}}\|_\infty d\theta + \|(a^n)_{n\in\mathbb{N}}\|_\infty \le 2,\end{aligned}$$

since $\max\{|\sigma_a(re^{i\theta})|, |a|\} \le 1$, for any $0 \le r < 1$. Hence $f \in \mathcal{BMOA}(\mathbb{D}, c_0)$. However, by Example 2.6, the radial limit $\lim_{r\to 1} f(re^{i\theta})$ does not exist in c_0 for any $e^{i\theta} \in \mathbb{T}$. In particular, it follows that $f^* \notin BMOA(\mathbb{D}, c_0)$.

In fact, we have the following theorem, that follows easily from the continuous inclusions $\mathcal{H}^\infty(\mathbb{D}, X) \subset \mathcal{BMOA}(\mathbb{D}, X) \subset \mathcal{H}^1(\mathbb{D}, X)$ and $H^\infty(\mathbb{D}, X) \subset BMOA(\mathbb{D}, X) \subset H^1(\mathbb{D}, X)$.

Theorem 2.22. *Let X be a complex Banach space. Then X has the ARNP if and only if $\mathcal{BMOA}(\mathbb{D}, X) = BMOA(\mathbb{D}, X)$.*

Proof. Assume first that X has the ARNP and $f \in \mathcal{BMOA}(\mathbb{D}, X)$. Then $f \in \mathcal{H}^1(\mathbb{D}, X) = H^1(\mathbb{D}, X)$ (by (2.4)), so that $f \in \mathcal{BMOA}(\mathbb{D}, X) \cap H^1(\mathbb{D}, X) = BMOA(\mathbb{D}, X)$. Hence $\mathcal{BMOA}(\mathbb{D}, X) \subset BMOA(\mathbb{D}, X)$. This proves one direction, since $BMOA(\mathbb{D}, X) \subset \mathcal{BMOA}(\mathbb{D}, X)$, by definition.

Assume next that $\mathcal{BMOA}(\mathbb{D}, X) \subset BMOA(\mathbb{D}, X)$ and let $f \in \mathcal{H}^\infty(\mathbb{D}, X)$. By (2.5), we have $f \in \mathcal{BMOA}(\mathbb{D}, X)$. By assumption the radial boundary function f^* exists almost everywhere on $\mathbb{T}$. Hence $f \in H^\infty(\mathbb{D}, X)$. This means, by definition, that X has the ARNP. ☐

Theorem 2.22 implies trivially that if X has the ARNP, then $VMOA(\mathbb{D}, X) = \mathcal{VMOA}(\mathbb{D}, X)$. Unfortunately we do not know if there exist Banach spaces X such that the inclusion $VMOA(\mathbb{D}, X) \subset \mathcal{VMOA}(\mathbb{D}, X)$ is strict. We formulate this as an open problem.

Open problem 2.23. *Is $VMOA(\mathbb{D}, X) = \mathcal{VMOA}(\mathbb{D}, X)$ if and only if X has the ARNP?*

Notes

We give a brief outline of the proof of Theorem 2.8 as it appears in the work of Bukhvalov and Danilevich [BD82]. One can also prove the theorem by combining the results in [DU77, p. 76] (the fact that reflexive spaces have the RNP) and [Hen86, p. 44] (the fact that the RNP implies ARNP). Recall that a (complex) Banach space is said to have the Radon-Nikodým property (RNP) if for each finite measure space (Ω, Σ, μ) and for each μ-continuous vector measure $G\colon \Sigma \to X$ of bounded variation there exists $g \in L^1(\mu, X)$ such that $G(E) = \int_E g\,d\mu$ for all $E \in \Sigma$ [DU77, p. 61]. It is shown in [BD82] that the RNP is equivalent to the requirement that for every harmonic vector-valued function f belonging to the space $h^p(\mathbb{D}, X)$ $(1 < p < \infty)$ the boundary function f^* exists almost everywhere on $\mathbb{T}$. Here $h^p(\mathbb{D}, X)$ denotes the Hardy space of harmonic vector-valued functions, that is, the space of harmonic functions $f\colon \mathbb{D} \to X$ with $\sup_{0<r<1} \int_0^{2\pi} \|f(re^{i\theta})\|_X^p d\theta < \infty$. Recall that a function $f\colon \mathbb{D} \to X$ is harmonic if it is weakly harmonic (see e.g. [Hen91]). Since every analytic X-valued function is weakly analytic and hence harmonic, it follows that the RNP implies the ARNP. The space $L^1(\mathbb{T}, \mathbb{C})$ gives an example of a space that has the ARNP but does not have the RNP (see [BD82], [Hen86, Satz 3.1]).

Inequality (2.6) of Lemma 2.16 implies that $\mathcal{BMOA}(\mathbb{D}, X)$ is continuously embedded in the vector-valued Bloch space $\mathcal{B}(X)$ of analytic functions $f\colon \mathbb{D} \to X$ satisfying $\sup_{z\in\mathbb{D}} \|f'(z)\|_X(1 - |z|^2) < \infty$. Composition operators on the vector-valued Bloch space have earlier been considered in [LST98] and [BDL01].

Let $BMOA_{\mathcal{C}}(\mathbb{D}, X)$ be the space of analytic functions $f\colon \mathbb{D} \to X$ with

$$\|f\|_{\mathcal{C},X} = \sup_{a\in\mathbb{D}} \left(\int_{\mathbb{D}} \|f'(z)\|_X(1 - |\sigma_a(z)|^2)dA(z) \right) < \infty$$

endowed with the norm $\|f\|_{BMOA_{\mathcal{C}}(\mathbb{D},X)} = \|f(0)\|_X + \|f\|_{\mathcal{C},X}$. Recall that for complex-valued functions we have the identity $BMOA(\mathbb{D}, \mathbb{C}) = BMOA_{\mathcal{C}}(\mathbb{D}, \mathbb{C})$ with equivalent norms (see e.g. [Gir01]). The vector-valued space $BMOA_{\mathcal{C}}(\mathbb{D}, X)$ was introduced by O. Blasco in [Bla00] in connection with Fourier multipliers on $H^1(\mathbb{D}, X)$. One of his interesting results states that $BMOA(\mathbb{D}, X) = BMOA_{\mathcal{C}}(\mathbb{D}, X)$ if and only if X is a Hilbert space [Bla00, Corollary 1.1] (see also [Bla97]). Hence $BMOA_{\mathcal{C}}(\mathbb{D}, X)$ is yet another natural generalization of $BMOA$ to the vector-valued case. We will later discuss briefly the boundedness of the composition operator on $BMOA_{\mathcal{C}}(\mathbb{D}, X)$.

Chapter 3

Composition operators: Boundedness on BMOA

3.1 Boundedness on Hardy spaces

Let φ be an analytic function mapping the unit disk $\mathbb{D} = \{z\colon |z| < 1\}$ into itself (we call such a function also analytic *self-map* of the unit disk). Clearly, if $f\colon \mathbb{D} \to X$ is an analytic function then also $f \circ \varphi$ is analytic in $\mathbb{D}$. When restricted to a Banach space of analytic functions, we call the linear operator $C_\varphi\colon f \mapsto f \circ \varphi$ the analytic *composition operator* induced by φ. The function φ is called the *symbol* of C_φ. In this chapter we study the boundedness of the composition operators on vector-valued function spaces considered in the previous chapter.

Every analytic composition operator is bounded on $\mathcal{H}^p(\mathbb{D}, X)$ $(1 \le p < \infty)$ with

$$\|C_\varphi f\|_{H^p(X)} \le \left(\frac{1 + |\varphi(0)|}{1 - |\varphi(0)|}\right)^{1/p} \|f\|_{H^p(X)}, \tag{3.1}$$

for $f \in \mathcal{H}^p(\mathbb{D}, X)$, by Theorem 2.12. Moreover, every composition operator C_φ is clearly bounded on $\mathcal{H}^\infty(\mathbb{D}, X)$ with $\|C_\varphi\| \le 1$. We show next that all composition operators are bounded also on the space $H^p(\mathbb{D}, X)$ for $1 \le p \le \infty$. Since $H^p(\mathbb{D}, X)$ is a closed subspace of $\mathcal{H}^p(\mathbb{D}, X)$, it suffices to show that the composition operator maps the space $H^p(\mathbb{D}, X)$ into itself.

Theorem 3.1. *Let $1 \le p \le \infty$ and let φ be an analytic self-map of the unit disk. Then $C_\varphi(H^p(\mathbb{D}, X)) \subset H^p(\mathbb{D}, X)$.*

The proof of Theorem 3.1 follows immediately from Theorem 2.12 and the following proposition.

Proposition 3.2. *Let $f \in H^1(\mathbb{D}, X)$ and let $\varphi\colon \mathbb{D} \to \mathbb{D}$ be analytic. Then the radial limit function $(f \circ \varphi)^*$ exists almost everywhere on $\mathbb{T}$.*

Proof. Let $f \in H^1(\mathbb{D}, X)$. Then there exists a sequence of polynomials $p_n(z) = \sum_{i=0}^{n} a_n z^n$ such that $\|p_n - f\|_{H^1(X)} \to 0$, as $n \to \infty$. Since the radial limit $\varphi^*(e^{i\theta})$ exists almost everywhere on $\mathbb{T}$, also $(p_n \circ \varphi)^*(e^{i\theta}) = \sum_{j=0}^{n} a_j (\varphi^*(e^{i\theta}))^j$ exists almost everywhere on $\mathbb{T}$. Hence $p_n \circ \varphi \in H^1(\mathbb{D}, X)$. By Theorem 2.12, we have $\|p_n \circ \varphi - f \circ \varphi\|_{H^1(X)} \leq C\|p_n - f\|_{H^1(X)} \to 0$ and $f \circ \varphi$ belongs to $H^1(\mathbb{D}, X)$, since $H^1(\mathbb{D}, X)$ is a Banach space. In particular, $(f \circ \varphi)^*$ exists almost everywhere on $\mathbb{T}$. □

Proof of Theorem 3.1. Let $1 \leq p \leq \infty$ and $f \in H^p(\mathbb{D}, X)$. Then by Proposition 3.2, the radial limit $(f \circ \varphi)^*$ exists almost everywhere on $\mathbb{T}$. Moreover, we have $(f \circ \varphi)^* \in H^p(\mathbb{T}, X)$, by Theorem 2.3. Hence $f \circ \varphi \in H^p(\mathbb{D}, X)$, by definition. □

3.2 Boundedness on BMOA and VMOA

By Corollary 2.17, the composition operator induced by a Möbius transformation σ is bounded on $\mathcal{BMOA}(\mathbb{D}, X)$. Even more is true, as the following theorem shows.

Theorem 3.3. *Let φ be an analytic self-map of the unit disk. Then*

(i) $\|C_\varphi f\|_{**,X} \leq \|f\|_{**,X}$*,*

(ii) C_φ *is bounded on* $\mathcal{BMOA}(\mathbb{D}, X)$ *with*

$$\|C_\varphi f\|_{\mathcal{BMOA}(\mathbb{D},X)} \leq \left(1 + \frac{1}{2}\log\frac{1+|\varphi(0)|}{1-|\varphi(0)|}\right)\|f\|_{\mathcal{BMOA}(\mathbb{D},X)}$$

and

(iii) $C_\varphi(BMOA(\mathbb{D}, X)) \subset BMOA(\mathbb{D}, X)$*.*

The boundedness of the composition operator has been established in the scalar case by Stephenson [Ste80]. We need only the vector-valued upper estimate (3.1) to transfer Stephenson's proof to the vector-valued case. However, we give a slightly different proof, in the scalar case due to Smith [Smi99, p. 2716], to motivate our study of weak compactness in the next chapter. For the proof we need a change of variables formula due to Stanton. The formula involves the Nevanlinna counting function, which will be defined next. Let $\varphi\colon \mathbb{D} \to \mathbb{D}$ be analytic and $0 < r < 1$. The *partial Nevanlinna counting function* $N_r(\varphi, \cdot)\colon \mathbb{D} \to \mathbb{R}$ is defined by

$$N_r(\varphi, z) = \begin{cases} \sum_{w \in \varphi^{-1}(z)} \log^+ \frac{r}{|w|}, & z \in \mathbb{D} \setminus \{\varphi(0)\} \\ 0, & z = \varphi(0), \end{cases}$$

where $\varphi^{-1}(z)$ is the sequence of the preimages of $z \in \mathbb{D}$, each point being repeated according to its multiplicity. Clearly $N_r(\varphi, z)$ is an increasing function of r. The limit

$$N(\varphi, z) = \lim_{r \to 1^-} N_r(\varphi, z) = \sum_{w \in \varphi^{-1}(z)} \log \frac{1}{|w|}$$

is called *Nevanlinna counting function.* If σ_a is a Möbius transformation $\sigma_a(z) = (a-z)/(1-\overline{a}z)$, then we have $N_r(\sigma_a, z) = \log^+(r/|\sigma_a(z)|)$ for $0 < r < 1$. Moreover, it follows easily from the definition that

$$N(\varphi, \sigma_a(z)) = N(\sigma_a \circ \varphi, z)$$

for analytic $\varphi\colon \mathbb{D} \to \mathbb{D}$ and $a \in \mathbb{D}$ (c.f. [Smi99, Lemma 2.2]). Littlewood's inequality states that if $\varphi\colon \mathbb{D} \to \mathbb{D}$ is analytic, then

$$N(\varphi, w) \leq \log \left| \frac{1 - \overline{\varphi(0)}w}{\varphi(0) - w} \right| = N(\sigma_{\varphi(0)}, w), \tag{3.2}$$

for $w \in \mathbb{D} \setminus \{\varphi(0)\}$. This inequality is classical and we omit the proof here (see [CM95, Theorem 2.29]).

The following formula for a continuous subharmonic function $u\colon \mathbb{D} \to [-\infty, \infty)$ is due to Stanton ([Sta86, Theorem 2], [ESS85, Theorem 2]):

$$\frac{1}{2\pi} \int_{\mathbb{T}} u(\varphi(re^{i\theta}))d\theta = u(\varphi(0)) + \frac{1}{2\pi} \int_{\mathbb{D}} N_r(\varphi, w)d[\triangle u](w),$$

where $r \in (0, 1)$ and $\varphi\colon \mathbb{D} \to \mathbb{D}$ is analytic. Here $d[\triangle u](w)$ denotes integration with respect to the distributional Laplacian of u. This means that for every test function $\psi \in C_0^\infty(\mathbb{D})$ we have

$$\int_{\mathbb{D}} \psi(w)d[\triangle u](w) = \int_{\mathbb{D}} u(w)\triangle\psi(w)dA(w),$$

where $A(w)$ denotes the Lebesgue area measure $dA(x + iy) = dxdy$. It is known that the distributional Laplacian is a positive Radon measure [Ran95, 3.7].

If $f\colon \mathbb{D} \to X$ is analytic, then we may apply Stanton's formula to the function $\|f(\cdot)\|_X\colon \mathbb{D} \to \mathbb{R}_+$, that is subharmonic by Lemma 2.2. The following special cases of Stanton's formula were derived in [LST98, p. 300-301] with the additional assumption that $\varphi(0) = 0$. The general case follows immediately from Stanton's formula. The technique of applying Stanton's formula for the Hardy spaces was introduced by Shapiro and Sundberg [SS90, p. 446], where Lemma 3.4 appears in the case $X = \mathbb{C}$.

Lemma 3.4. *If* $f\colon \mathbb{D} \to X$ *and* $\varphi\colon \mathbb{D} \to \mathbb{D}$ *are analytic and* $0 < r < 1$ *then*

$$\frac{1}{2\pi} \int_{\mathbb{T}} \|(f \circ \varphi)(re^{i\theta})\|_X d\theta = \|f(\varphi(0))\|_X + \frac{1}{2\pi} \int_{\mathbb{D}} N_r(\varphi, w)d[\triangle\|f\|_X](w),$$

$$\|f \circ \varphi\|_{H^1(X)} = \|f(\varphi(0))\|_X + \frac{1}{2\pi} \int_{\mathbb{D}} N(\varphi, w)d[\triangle\|f\|_X](w). \tag{3.3}$$

The special case $\varphi(z) \equiv z$ yields the Littlewood-Paley type identities

$$\frac{1}{2\pi}\int_{\mathbb{T}} \|f(re^{i\theta})\|_X d\theta = \|f(0)\|_X + \frac{1}{2\pi}\int_{\mathbb{D}} \log^+\left(\frac{r}{|w|}\right) d[\triangle\|f\|_X](w),$$
$$\|f\|_{H^1(X)} = \|f(0)\|_X + \frac{1}{2\pi}\int_{\mathbb{D}} \log\left(\frac{1}{|w|}\right) d[\triangle\|f\|_X](w). \tag{3.4}$$

Moreover,

$$\begin{aligned}\|f\|_{**,X} &= \sup_{a\in\mathbb{D}} \|f\circ\varphi\circ\sigma_a - f(\varphi(a))\|_{H^1(X)} \\ &= \sup_{a\in\mathbb{D}} \frac{1}{2\pi}\int_{\mathbb{D}} N(\varphi\circ\sigma_a, w) d[\triangle\|f - f(\varphi(a))\|_X](w).\end{aligned}$$

We are now ready to prove Theorem 3.3.

Proof of Theorem 3.3. (i). Let $f\colon \mathbb{D}\to X$ and $\varphi\colon \mathbb{D}\to\mathbb{D}$ be analytic. It follows from identity (3.3) of Lemma 3.4 that

$$\|f\circ\varphi\circ\sigma_a - f(\varphi(a))\|_{H^1(X)} = \frac{1}{2\pi}\int_{\mathbb{D}} N(\varphi\circ\sigma_a, w) d[\triangle\|f - f(\varphi(a))\|_X](w),$$

for $a\in\mathbb{D}$. By Littlewood's inequality (3.2), we have $N(\varphi\circ\sigma_a, w) \le N(\sigma_{\varphi(a)}, w)$ for $w\in\mathbb{D}\setminus\{\varphi(a)\}$. Using (3.3) of Lemma 3.4 again, we get that

$$\begin{aligned}\|f\circ\varphi\circ\sigma_a - f(\varphi(a))\|_{H^1(X)} &\le \frac{1}{2\pi}\int_{\mathbb{D}} N(\sigma_{\varphi(a)}, w) d[\triangle\|f - f(\varphi(a))\|_X](w) \\ &= \|f\circ\sigma_{\varphi(a)} - f(\varphi(a))\|_{H^1(X)} \\ &\le \sup_{b\in\varphi(\mathbb{D})} \|f\circ\sigma_b - f(b)\|_{H^1(X)} \\ &\le \sup_{b\in\mathbb{D}} \|f\circ\sigma_b - f(b)\|_{H^1(X)} = \|f\|_{**,X}.\end{aligned}$$

Since the inequality holds for every $a\in\mathbb{D}$, we obtain the desired inequality

$$\|f\circ\varphi\|_{**,X} \le \|f\|_{**,X},$$

for $f\in\mathcal{BMOA}(\mathbb{D},X)$.

(ii). We apply the argument from the proof of Corollary 2.17. Namely, by part (i) and Lemma 2.16, we have

$$\begin{aligned}\|C_\varphi f\|_{\mathcal{BMOA}(\mathbb{D},X)} &= \|C_\varphi f\|_{**,X} + \|f(\varphi(0))\|_X \\ &\le \|f\|_{**,X} + \|f(\varphi(0))\|_X \\ &\le \|f\|_{**,X} + \|f(0)\|_X + \left(\frac{1}{2}\log\frac{1+|\varphi(0)|}{1-|\varphi(0)|}\right)\|f\|_{**,X} \\ &= \left(1 + \frac{1}{2}\log\frac{1+|\varphi(0)|}{1-|\varphi(0)|}\right)\|f\|_{\mathcal{BMOA}(\mathbb{D},X)}.\end{aligned}$$

(iii). By part (ii), C_φ maps $BMOA(\mathbb{D},X)$ boundedly to $\mathcal{BMOA}(\mathbb{D},X)$. If $f \in BMOA(\mathbb{D},X)$, then also $f \in H^1(\mathbb{D},X)$. Hence $(f\circ\varphi)^* \in H^1(\mathbb{T},X)$, by Proposition 3.2 and Theorem 2.3. Thus $f\circ\varphi \in \mathcal{BMOA}(\mathbb{D},X)\cap H^1(\mathbb{D},X) = BMOA(\mathbb{D},X)$. □

Since $VMOA(\mathbb{D},X)$ is a closed subspace of $BMOA(\mathbb{D},X)$, a composition operator is bounded on $VMOA(\mathbb{D},X)$ if it maps $VMOA(\mathbb{D},X)$ into $VMOA(\mathbb{D},X)$.

Theorem 3.5. *Let $\varphi\colon \mathbb{D}\to\mathbb{D}$ be an analytic self-map of the unit disk. Then $C_\varphi(VMOA(\mathbb{D},X))\subset VMOA(\mathbb{D},X)$ if and only if $\varphi\in VMOA(\mathbb{D},\mathbb{C})$.*

See [Smi99, p. 2718] or [BCM99, p. 2186] for the usual scalar case arguments. We prove first an auxiliary lemma:

Lemma 3.6. *Let X be a Banach space. If $\varphi\colon\mathbb{D}\to\mathbb{D}$ is analytic such that $\varphi\in VMOA(\mathbb{D},\mathbb{C})$ and p is an X-valued analytic polynomial, then $p\circ\varphi\in VMOA(\mathbb{D},X)$*

Proof. Since φ is a bounded analytic function on $\mathbb{D}$, the radial limit φ^* is defined almost everywhere on $\mathbb{T}$. It is clear that if $p(z)=\sum_{n=0}^N x_n z^n$ is a polynomial, then the function $p\circ\varphi=\sum_{n=0}^N x_n\varphi^n$ is analytic on $\mathbb{D}$ and the radial limit $(p\circ\varphi)^*$ exists almost everywhere on $\mathbb{T}$. By the formula $(z^n-w^n)=(z-w)\left(\sum_{j=0}^{n-1} z^j w^{n-1-j}\right)$ that is valid for $z,w\in\mathbb{C}$ and $n\in\mathbb{N}$, we get

$$\begin{aligned}\|\varphi^n\circ\sigma_a-\varphi^n(a)\|_{H^1(\mathbb{C})} &= \Big\|(\varphi\circ\sigma_a-\varphi(a))\big(\sum_{j=0}^{n-1}(\varphi\circ\sigma_a)^j(\varphi(a))^{n-1-j}\big)\Big\|_{H^1(\mathbb{C})}\\ &\le n\sup_{a\in\mathbb{D}}\|\varphi\circ\sigma_a-\varphi(a)\|_{H^1(\mathbb{C})}\to 0,\end{aligned}$$

as $|a|\to 1$, since $\varphi\in VMOA(\mathbb{D},\mathbb{C})$. Thus $\varphi^n\in VMOA(\mathbb{D},\mathbb{C})$ for every $n\in\mathbb{N}$. Hence $p\circ\varphi=\sum_{n=0}^N x_n\varphi^n\in VMOA(\mathbb{D},X)$. □

Proof of Theorem 3.5. Assume first that $\varphi\in VMOA(\mathbb{D},\mathbb{C})$. Then by Lemma 3.6, the operator C_φ maps analytic polynomials into $VMOA(\mathbb{D},X)$. Since the polynomials are dense in $VMOA(\mathbb{D},X)$, it follows that C_φ maps $VMOA(\mathbb{D},X)$ into itself.

Conversely, suppose that C_φ maps $VMOA(\mathbb{D},X)$ into itself. Let $0\neq x_0\in X$. Since the function $f(z)\equiv x_0 z$ belongs to $VMOA(\mathbb{D},X)$, also $C_\varphi f=x_0\varphi$ belongs to $VMOA(\mathbb{D},X)$. Using the $VMOA(\mathbb{D},X)$ condition it follows now easily that $\varphi\in VMOA(\mathbb{D},\mathbb{C})$. □

3.3 Ryff's theorem

In characterizing the bounded composition operators C_φ on Banach spaces of analytic functions $f\colon\mathbb{D}\to X$ having boundary values almost everywhere on $\mathbb{T}$, the existence of the radial limit $(f \circ \varphi)^*(e^{i\theta}) = \lim_{r\to 1}(f \circ \varphi)(re^{i\theta})$ plays a central role (c.f. Proposition 3.2). We prove next a theorem, in the scalar case due to Ryff [Ryf66, Theorem 2], that characterizes the radial limits $(f \circ \varphi)^*(e^{i\theta})$ almost everywhere on $\mathbb{T}$. We will not use this theorem later, although we will need the scalar version of the result. However, the result in itself is an interesting part of vector-valued complex analysis. It shows in particular that in connection with the study of composition operators on $H^1(\mathbb{D}, X)$ one may ignore the distinction between functions and their boundary values also in the vector-valued case (c.f. [CM95, p. 31]).

Theorem 3.7 (Vector-valued Ryff's theorem). *Let $f \in H^1(\mathbb{D}, X)$ and let φ be an analytic self-map of the unit disk. Then $(f \circ \varphi)^* = f^* \circ \varphi^*$ almost everywhere on $\mathbb{T}$, where f^* denotes the function defined almost everywhere on $\mathbb{T}$, and everywhere on $\mathbb{D}$, by $f^*(z) = \lim_{r\to 1} f(rz)$.*

We prove the theorem using a similar argument as Ryff. The key ingredient of Ryff's proof is Lindelöf's theorem which must first be extended to the vector-valued case.

Theorem 3.8 (Vector-valued Lindelöf's theorem). *Let f be a bounded analytic function from the unit disk $\mathbb{D}$ to a Banach space $(X, \|\cdot\|_X)$. If there exists a path $\Gamma\colon [0,\infty) \to \mathbb{D}$ such that $\lim_{t\to\infty} \Gamma(t) = \zeta \in \mathbb{T}$ and $\lim_{t\to\infty} f(\Gamma(t)) = x \in X$, then f has radial limit $\lim_{r\to 1} f(r\zeta) = x$ at ζ.*

The proof will be based on the following fact about subharmonic functions:

Lemma 3.9. *Let f be a subharmonic function on $\mathbb{D}$ such that $f < 0$ on $\mathbb{D}$. If there exists a path $\Gamma\colon [0,\infty) \to \mathbb{D}$ such that*

$$\lim_{t\to\infty} \Gamma(t) = \zeta \in \mathbb{T} \quad \textit{and} \quad \lim_{t\to\infty} f(\Gamma(t)) = \alpha \in [-\infty, 0),$$

then $\limsup_{r\to 1} f(r\zeta) \le \alpha$.

Lemma 3.9 is well-known but will be proved for completeness using the following half-plane result [Ran95, Theorem 4.3.11]: Let f be a subharmonic function on $\mathbb{H} = \{z\colon \operatorname{Im} z > 0\}$ such that $f < 0$ on $\mathbb{H}$. If there exists a path $\Gamma\colon [0,\infty) \to \mathbb{H}$ such that

$$\lim_{t\to\infty} \Gamma(t) = \infty \quad \text{and} \quad \lim_{t\to\infty} f(\Gamma(t)) = \alpha \in [-\infty, 0), \tag{3.5}$$

then $\limsup_{t\to\infty} f(it) \le \alpha$.

Proof of Lemma 3.9. Let f and Γ be as above, where we may assume that $\zeta = 1$. We transform (3.5) to the unit disk using conformal mappings. Let $g\colon \mathbb{H} \to \mathbb{D}$ be the conformal transformation defined by $g(z) = (z - i)/(z + i)$, for $z \in \mathbb{H}$. Then the inverse mapping $g^{-1} = h\colon \mathbb{D} \to \mathbb{H}$ is given by $h(z) = i(1+z)/(1-z)$, for $z \in \mathbb{D}$. Since f is subharmonic on $\mathbb{D}$, the function $\tilde{f} = f \circ g$ is subharmonic on $\mathbb{H}$

[Ran95, Theorem 2.7.4] and $\tilde{f} < 0$ on $\mathbb{H}$. Moreover, the path $\tilde{\Gamma} = h \circ \Gamma\colon [0,\infty) \to \mathbb{H}$ satisfies $\lim_{t\to\infty} \tilde{\Gamma}(t) = \infty$ (since for every $M > 0$ there exists t_0 such that $|1+\Gamma(t)|/|1-\Gamma(t)| > M$ for every $t > t_0$) and $\lim_{t\to\infty} \tilde{f}(\tilde{\Gamma}(t)) = \lim(f \circ g \circ h \circ \Gamma)(t) = \lim \Gamma(f(t)) = \alpha$. Hence by (3.5), $\limsup_{t\to\infty}(f \circ g)(it) \le \alpha$. Since the map $s \mapsto g(is)$ maps the interval $[t,\infty)$ onto $[r,1)$, where $r = (t-1)/(t+1)$, we get

$$\limsup_{r\to 1} f(r) = \lim_{r\to 1} \sup_{s\in[r,1)} f(s) = \lim_{t\to\infty} \sup_{s\in[t,\infty)} f(g(is)) = \limsup_{t\to\infty} f(g(it)) \le \alpha.$$

□

Proof of Theorem 3.8. Let $g\colon \mathbb{D} \to X$ be a bounded and non-constant analytic function. Assume that there exists a path $\Gamma\colon [0,\infty) \to \mathbb{D}$ such that $\lim_{t\to\infty} \Gamma(t) = \zeta$ and $\lim_{t\to\infty} g(\Gamma(t)) = x$, for some $x \in X$. Define the subharmonic function $f\colon \mathbb{D} \to \mathbb{R}$ by $f(z) = \log(\|g(z) - x\|_X/M)$, where $M = \sup_{z\in\mathbb{D}} \|g(z) - x\|_X < \infty$. Then $f < 0$ in $\mathbb{D}$, since by the maximum principle the subharmonic function $\|g - x\|_X$ does not attain its maximum in $\mathbb{D}$. Moreover,

$$\lim_{t\to\infty} f(\Gamma(t)) = \lim_{t\to\infty} \log(\|g(\Gamma(t)) - x\|_X/M) = -\infty.$$

Hence we get from Lemma 3.9 that $\lim_{r\to 1} \log(\|g(r) - x\|_X/M) = -\infty$, or equivalently, $\lim_{r\to 1} \|g(r) - x\|_X = 0$. □

Remark. Using [Ran95, Theorem 4.3.11] instead of the consequence (3.5), it is possible to show that the limit x in Theorem 3.8 is actually non-tangential (see also the proof of [Ran95, Corollary 4.3.12]).

Proof of Theorem 3.7. We may assume that $f \in H^\infty(\mathbb{D}, X)$. Namely, if $g \in H^p(\mathbb{D}, X)$ where $1 \le p < \infty$, then $g = h_1/h_2$, where $h_1 \in H^\infty(\mathbb{D}, X)$ and $h_2 \in H^\infty(\mathbb{D}, \mathbb{C})$, h_2 without zeros on $\mathbb{D}$ [Hen86, Satz 2.8]. Since by the scalar-valued Lindelöf's theorem [Rud87, Theorem 12.10] $(h_2 \circ \varphi)^* = h_2^* \circ \varphi^*$ almost everywhere on $\mathbb{T}$, we have $(g \circ \varphi)^* = g^* \circ \varphi^*$ almost everywhere on $\mathbb{T}$ if and only if $(h_1 \circ \varphi)^* = h_1^* \circ \varphi^*$ almost everywhere on $\mathbb{T}$.

By Proposition 3.2, there exists a set $E \subset \mathbb{T}$ of full measure such that φ and $f \circ \varphi$ have radial limits everywhere on E. Write $E = E_1 \cup E_2$, where $E_1 = \{e^{i\theta}\colon |\varphi^*(e^{i\theta})| = 1\}$ and $E_2 = \{e^{i\theta}\colon |\varphi^*(e^{i\theta})| < 1\}$. By the continuity of f on $\mathbb{D}$, we have $(f \circ \varphi)^* = f \circ \varphi^* = f^* \circ \varphi^*$ on E_2. On E_1, f has a limit along the path $\Gamma(t) = \varphi((t/(t+1))\zeta)$ $(0 \le t < \infty)$ that satisfies the assumptions of Lindelöf's theorem with $\lim_{t\to\infty} \Gamma(t) = \varphi^*(\zeta)$. Thus we have

$$f^*(\varphi^*(\zeta)) = \lim_{r\to 1} f(r\varphi^*(\zeta)) = \lim_{t\to\infty} f(\Gamma(t)) = \lim_{r\to 1} (f \circ \varphi)(r\zeta) = (f \circ \varphi)^*(\zeta),$$

by Lindelöf's theorem. Thus $f^* \circ \varphi^* = (f \circ \varphi)^*$ everywhere on E_1 and hence almost everywhere on $\mathbb{T}$. □

Notes

We sketch an easier alternative proof for Theorem 3.3 (i) using the argument of Stephenson [Ste80] (see also [Tja96, p. 22]). Suppose that φ is an analytic self-map of $\mathbb{D}$ and $f \in BMOA(\mathbb{D}, X)$. Then $\varphi = \sigma_{\varphi(0)} \circ \psi$ where $\psi = \sigma_{\varphi(0)} \circ \varphi$ satisfies $\psi(0) = 0$. Then by (3.1), we have

$$\begin{aligned}\|f \circ \varphi - f(\varphi(0))\|_{H^1(X)} &= \|f \circ \sigma_{\varphi(0)} \circ \psi - f(\varphi(0))\|_{H^1(X)} \\ &\leq \|f \circ \sigma_{\varphi(0)} - f(\varphi(0))\|_{H^1(X)} \leq \|f\|_{**,X}.\end{aligned}$$

Replacing φ by $\varphi \circ \sigma_a$ in the inequality yields $\|f \circ \varphi \circ \sigma_a - f(\varphi(a))\|_{H^1(X)} \leq \|f\|_{**,X}$, for all $a \in \mathbb{D}$. Thus,

$$\|f(\varphi(0))\|_X + \|f \circ \varphi\|_{**,X} \leq C(\|f(0)\|_X + \|f\|_{**,X}),$$

for all $f \in BMOA(\mathbb{D}, X)$, where C depends on $|\varphi(0)|$ as in Theorem 3.3.

However, using the arguments of our original proof of Theorem 3.3, we can show that all composition operators are bounded on the space $BMOA_{\mathcal{C}}(\mathbb{D}, X)$ (see the definition on p. 22). Indeed, we can repeat the calculation familiar from the scalar setting to obtain that in fact

$$\|f\|_{BMOA_{\mathcal{C}}(\mathbb{D},X)} \approx \|f(0)\|_X + \sup_{a \in \mathbb{D}} \int_{\mathbb{D}} \|f'(z)\|_X \log \frac{1}{|\sigma_a(z)|} dA(z)$$

(see [Gar81, Lemma IV.3.2]). In addition, by a non-univalent change of variables as done in [Sha93, Proposition, p. 186] (where the result is general enough to cover also the vector-valued case), we have

$$\int_{\mathbb{D}} \|(f \circ \varphi)'(z)\|_X \log \frac{1}{|\sigma_a(z)|} dA(z) = \int_{\mathbb{D}} \|f'(z)\|_X N(\varphi \circ \sigma_a, z) dA(z).$$

Hence, in view of the arguments used in the proof of Theorem 3.3, the boundedness of the composition operator on $BMOA_{\mathcal{C}}(\mathbb{D}, X)$ accounts to showing that

$$\sup_{a \in \mathbb{D}} \int_{\mathbb{D}} \|f'(z)\|_X N(\varphi \circ \sigma_a, z) dA(z) \leq \sup_{a \in \mathbb{D}} \int_{\mathbb{D}} \|f'(z)\|_X \log \frac{1}{|\sigma_a(z)|} dA(z),$$

for all analytic maps $\varphi \colon \mathbb{D} \to \mathbb{D}$. But this follows from Littlewood's inequality (3.2) and by taking the supremum over $a \in \mathbb{D}$, as was seen in the proof of Theorem 3.3. Hence all composition operators are bounded on the space $BMOA_{\mathcal{C}}(\mathbb{D}, X)$.

Our proof of the vector-valued Lindelöf's theorem is not new. Notice that Danilevich states the theorem in [Dan76, Lemma 1.6], but does not provide any details. In the scalar case Ryff's theorem (Theorem 3.7) can be proved also without Lindelöf's theorem (see [Mac85, Lemma 1.6]). Also this argument can be easily extended to the vector-valued case.

Chapter 4

Compactness and weak compactness on BMOA

4.1 A general observation

In the following sections we study necessary and sufficient conditions for a composition operator to be compact or weakly compact on the spaces $BMOA(\mathbb{D}, X)$ and $VMOA(\mathbb{D}, X)$ when X is a complex Banach space. In Section 4.2 we show that if X is reflexive and φ induces a compact composition operator on $BMOA(\mathbb{D}, \mathbb{C})$, then φ induces a weakly compact composition operator on $BMOA(\mathbb{D}, X)$. In Section 4.3 we characterize completely the weakly compact composition operators on $VMOA(\mathbb{D}, X)$ and $BMOA(\mathbb{D}, X)$ whose symbol is univalent. Finally, in Section 4.4 we discuss Rosenthal composition operators.

We start by observing that, unlike boundedness, the compactness properties of the composition operators depend on the Banach space X. It was noted in [LST98, p. 296] that if X is infinite dimensional, then the composition operator C_φ is never compact on $\mathcal{H}^p(\mathbb{D}, X)$ or $H^p(\mathbb{D}, X)$, for $1 \leq p \leq \infty$. Moreover, if X is not reflexive, then the composition operator is not weakly compact.

If E and F are Banach spaces, let $\mathcal{L}(E, F)$ denote the set of all bounded linear operators from E to F. Recall that a linear operator in $\mathcal{L}(E, F)$ is said to be *compact*, if it maps bounded sets of E into relatively norm compact sets in F. A linear operator in $\mathcal{L}(E, F)$ is said to be *weakly compact*, if it maps bounded sets of E into relatively weakly compact sets of F. Recall also that bounded, compact, and weakly compact operators form closed operator ideals [Pie80, p. 46-47, 64]. In particular, this implies that if $\mathcal{I}$ denotes any of the ideals of bounded, compact, or weakly compact operators, if E' and F' are Banach spaces, $T \in \mathcal{L}(E', E)$, $R \in \mathcal{L}(F, F')$, and $S \in \mathcal{I}(E, F)$, then $RST \in \mathcal{I}(E', F')$.

The following lemma is based on a general observation which applies to various spaces of vector-valued analytic functions (see [BDL01, Proposition 1]). Let $E(X) = \mathcal{BMOA}(\mathbb{D}, X)$, $E(X) = BMOA(\mathbb{D}, X)$, or $E(X) = VMOA(\mathbb{D}, X)$.

Lemma 4.1. *If $C_\varphi\colon E(X)\to E(X)$ belongs to an operator ideal $\mathcal{I}$, then the identity operator $id_X\colon X\to X$ and the scalar composition operator $\widetilde{C}_\varphi\colon E(\mathbb{C})\to E(\mathbb{C})$, $\widetilde{C}_\varphi f=f\circ\varphi$ belong to $\mathcal{I}$.*

Proof. Let $0\neq x_0\in X$, $x_0^*\in X^*$ such that $\langle x_0^*,x_0\rangle=1$. Define the linear maps

$$\begin{array}{ll} A\colon E(X)\to X, & A(f)=f(0);\\ B\colon X\to E(X), & B(x)=f_x;\\ T\colon E(\mathbb{C})\to E(X), & T(f)=x_0f;\ \text{and}\\ S\colon E(X)\to E(\mathbb{C}), & S(f)=x_0^*\circ f, \end{array}$$

where f_x is the constant function $f\equiv x$. Then A,B,T and S are bounded. Since C_φ fixes the constant functions, we have $(AC_\varphi B)x=x$, for every $x\in X$. Hence $C_\varphi\in\mathcal{I}$ implies that $id_X\in\mathcal{I}$. For every $f\in BMOA(\mathbb{D},\mathbb{C})$, we have $\widetilde{C}_\varphi=(TC_\varphi S)f=f\circ\varphi$. Hence $C_\varphi\in\mathcal{I}$ implies $\widetilde{C}_\varphi\in\mathcal{I}$. □

Proposition 4.2. *The composition operator C_φ is compact on $E(X)$ if and only if $\dim(X)<\infty$ and the composition operator $\widetilde{C}_\varphi$ is compact on $E(\mathbb{C})$.*

Proof. Assume first that C_φ is compact on $E(X)$. Then by Lemma 4.1, $id_X\colon X\to X$ is compact. Hence, by Riesz' lemma, X is finite dimensional. Moreover, $\widetilde{C}_\varphi$ is compact on $E(\mathbb{C})$, by Lemma 4.1.

Assume next that $\dim(X)=n<\infty$ and $\widetilde{C}_\varphi$ is compact on $E(\mathbb{C})$. Let $(x_k)\subset X$ be the linearly independent set in X such that $y=\sum_{k=1}^n a_kx_k$, for every $y\in X$, where $(a_k)_{k=1}^n\subset\mathbb{C}$ is unique. Then one can find functionals $(x_j)^*\in X^*$ biorthogonal to (x_k), such that $\max_{k=1,\dots,n}\max\{\|x_k\|_X,\|x_k^*\|_{X^*}\}\leq K<\infty$, and for every $y\in X$ we have $y=\sum_{k=1}^n\langle y,x_k^*\rangle x_k$. Let (f_j) be a sequence of functions $f_j\in E(X)$ such that $\|f_j\|_{\mathcal{BMOA}(\mathbb{D},X)}\leq 1$ for every $j\in\mathbb{N}$. Then the function $z\mapsto\langle f_j(z),x_k^*\rangle$ belongs to the space $E(\mathbb{C})$ with $\|\langle f_j,x_k^*\rangle\|_{BMOA(\mathbb{D},\mathbb{C})}\leq K$, where $1\leq k\leq n$. Moreover, since $\widetilde{C}_\varphi$ is compact on $E(\mathbb{C})$, the sequence $(\widetilde{C}_\varphi\langle f_j,x_k^*\rangle)=(\langle C_\varphi f_j,x_k^*\rangle)$ has a converging subsequence in $E(\mathbb{C})$, that we still denote by $(\langle C_\varphi f_j,x_k^*\rangle)$. That is, there exists a function $g\in E(\mathbb{C})$ such that $\|\langle C_\varphi f_j,x_k^*\rangle-g\|_{BMOA(\mathbb{D},\mathbb{C})}\to 0$, as $j\to\infty$. Now $\|\langle C_\varphi f_j,x_k^*\rangle x_k-gx_k\|_{\mathcal{BMOA}(\mathbb{D},X)}\leq K\|\langle C_\varphi f_j,x_k^*\rangle-g\|_{BMOA(\mathbb{D},\mathbb{C})}\to 0$, as $j\to\infty$, so that the linear operator $f\mapsto\langle C_\varphi f,x_k^*\rangle x_k$ is compact on $E(X)$. Consequently, the operator $C_\varphi=\sum_{k=1}^n\langle C_\varphi,x_k^*\rangle x_k$ is compact on $E(X)$. □

Proposition 4.2 implies that if X is infinite dimensional, then C_φ cannot be compact on $E(X)$. It turns out that when X is infinite dimensional, the weak compactness of C_φ on $E(X)$ remains as an interesting smallness property. Unfortunately the proof of Proposition 4.2 does not apply in the infinite dimensional case. However, since $id_X\colon X\to X$ is weakly compact if and only if X is reflexive, Lemma 4.1 yields the following corollary.

Corollary 4.3. *If C_φ is weakly compact on $E(X)$, then X is reflexive and φ induces a weakly compact composition operator on $E(\mathbb{C})$.*

Note in particular that since the reflexive Banach spaces have the analytic Radon-Nikodým property (Theorem 2.8), we have the identity $BMOA(\mathbb{D},X) = \mathcal{BMOA}(\mathbb{D},X)$ for reflexive X.

4.2 Weak compactness: sufficient conditions

In this section we study the sufficient conditions for a composition operator to be weakly compact on $BMOA(\mathbb{D},X)$ and $VMOA(\mathbb{D},X)$. We start the section by giving an example of weakly compact composition operators on $BMOA(\mathbb{D},X)$.

Theorem 4.4. *Let X be a reflexive complex Banach space and let φ be an analytic self-map of the unit disk such that $\|\varphi\|_\infty < 1$. Then C_φ is weakly compact on $BMOA(\mathbb{D},X)$.*

For the proof we need the following result which is one of the key results in this section. Similar results have been used in estimating the weak essential norms of composition operators on various spaces of analytic functions (c.f. [LST98] and [BDL01]). We give the argument after the proof of Theorem 4.4.

Proposition 4.5. *There exists a sequence $(V_n)_{n=1}^\infty$ of linear operators V_n mapping $\mathcal{BMOA}(\mathbb{D},X) \to VMOA(\mathbb{D},X)$ with the following properties:*
(i) $\|V_n f\|_{\mathcal{BMOA}(\mathbb{D},X)} \le 3\|f\|_{\mathcal{BMOA}(\mathbb{D},X)}$, for $n \in \mathbb{N}$.
(ii) For every $0 < r < 1$,

$$\sup_{f \in B_{BMOA(\mathbb{D},X)}} \sup_{|z| \le r} \|((I - V_n)f)(z)\|_X \to 0,$$

as $n \to \infty$.
(iii) If X is reflexive, then V_n is weakly compact, for $n \in \mathbb{N}$.

Proof of Theorem 4.4. Assume that X is reflexive and φ is an analytic self-map of the unit disk such that $\|\varphi\|_\infty < 1$. Then C_φ is bounded on $BMOA(\mathbb{D},X)$, by Theorem 3.3. Since X is reflexive, the operators V_n from Proposition 4.5 are weakly compact. Since the weakly compact operators form a closed operator ideal, the operator $C_\varphi V_n$ is weakly compact for every $n \ge 1$. Thus it suffices to show that $\|C_\varphi - C_\varphi V_n\|_{\mathcal{L}(BMOA(\mathbb{D},X))} \to 0$ as $n \to \infty$. Since $\|\varphi\|_\infty < 1$, there exists $0 < r < 1$ such that $\varphi(\mathbb{D}) \subset r\mathbb{D}$. Now,

$$\begin{aligned}
\|C_\varphi - C_\varphi V_n\|_{\mathcal{L}(BMOA(\mathbb{D},X))} &= \sup_{f \in B_{BMOA(\mathbb{D},X)}} \|(C_\varphi - C_\varphi V_n)f\|_{\mathcal{BMOA}(\mathbb{D},X)} \\
&= \sup_{f \in B_{BMOA(\mathbb{D},X)}} \|((I - V_n)f) \circ \varphi\|_{\mathcal{BMOA}(\mathbb{D},X)} \\
&\le 3 \sup_{f \in B_{BMOA(\mathbb{D},X)}} \|((I - V_n)f) \circ \varphi\|_{H^\infty(X)} \\
&\le 3 \sup_{f \in B_{BMOA(\mathbb{D},X)}} \sup_{|z| \le r} \|((I - V_n)f)(z)\|_X \to 0,
\end{aligned}$$

as $n \to \infty$, by (2.5) and Proposition 4.5, which completes the proof. ☐

We model the proof of Proposition 4.5 on the proof of [LST98, Proposition 2]. For $n \geq 0$, define the linear map V_n by setting

$$V_n f(z) = \sum_{k=0}^{n} a_k z^k + \sum_{k=n+1}^{2n-1} \frac{2n-k}{n} a_k z^k,$$

where $f\colon \mathbb{D} \to X$ is analytic with the Taylor expansion $f = \sum_{k=0}^{\infty} a_n z^n$. Then $V_n f = 2k_{2n-1}(f) - k_{n-1}(f)$, where

$$k_n(f)(z) = \sum_{k=0}^{n} \left(1 - \frac{k}{n+1}\right) a_k z^k = \frac{1}{2\pi} \int_0^{2\pi} K_n(\theta) f(ze^{-i\theta}) d\theta,$$

and K_n is the Fejér kernel

$$K_n(\theta) = \sum_{k=-n}^{n} \left(1 - \frac{|k|}{n+1}\right) e^{ik\theta}.$$

Recall that $K_n \geq 0$ and $\int_0^{2\pi} K_n(\theta) d\theta = 2\pi$ for every $n \in \mathbb{N}$ (see [Kat76, p. 12] or [Tor86]). Note that $V_n f$ equals the convolution of f with the de la Vallée-Poussin kernel $2K_{2n-1} - K_{n-1}$. Since $V_n f$ is a polynomial, it belongs to $VMOA(\mathbb{D}, X)$ for $n \in \mathbb{N}$. We show that the operator V_n satisfies the properties (i)$-$(iii) of Proposition 4.5.

Proof of Proposition 4.5. (i). We use a similar argument as in the proof of Theorem 4 in [HW86] for scalar-valued functions. Let $n \in \mathbb{N}$. Then

$$(k_n(f) \circ \sigma_a)(z) - k_n(f)(a) = \frac{1}{2\pi} \int_0^{2\pi} K_n(\theta)[f(\sigma_a(z)e^{-i\theta}) - f(ae^{-i\theta})] d\theta.$$

Hence,

$$\begin{aligned}
&\frac{1}{2\pi} \int_0^{2\pi} \|(k_n(f) \circ \sigma_a)(re^{it}) - k_n(f)(a)\|_X dt \\
&\quad \leq \frac{1}{(2\pi)^2} \int_0^{2\pi} \int_0^{2\pi} K_n(\theta) \|f(\sigma_a(re^{it})e^{-i\theta}) - f(ae^{-i\theta})\|_X d\theta dt \\
&\quad = \frac{1}{(2\pi)^2} \int_0^{2\pi} K_n(\theta) \int_0^{2\pi} \|f(\sigma_a(re^{it})e^{-i\theta}) - f(ae^{-i\theta})\|_X dt d\theta \\
&\quad \leq \|f\|_{**,X},
\end{aligned}$$

by Lemma 2.15, since $e^{-i\theta}\sigma_a \in \mathcal{M}$ and $\int_0^{2\pi} K_n(\theta) d\theta = 2\pi$. Hence we obtain $\|k_n(f)\|_{**,X} \leq \|f\|_{**,X}$ and $\|V_n f\|_{**,X} = \|2k_{2n-1}(f) - k_{n-1}(f)\|_{**,X} \leq 3\|f\|_{**,X}$. Since $(V_n f)(0) = f(0)$, we get that $\|V_n f\|_{\mathcal{BMOA}(\mathbb{D},X)} \leq 3\|f\|_{\mathcal{BMOA}(\mathbb{D},X)}$.

(ii). Let $0 < r < 1$ and $\varepsilon > 0$. Then $\|g(z)\|_X \leq 2(1-r)^{-1}\|g\|_{H^1(X)}$, for $|z| < r$ and any $g \in \mathcal{H}^1(\mathbb{D}, X)$, by Corollary 2.10. Choose n_0 large enough so that $8r^n(1-r)^{-1} < \varepsilon$ for $n \geq n_0$. Given $f(z) = \sum_{j=0}^{\infty} x_j z^j \in \mathcal{BMOA}(\mathbb{D}, X)$, we get

$$((I - V_n)f)(z) = \sum_{k=n+1}^{2n-1} \left(\frac{k-n}{n}\right) x_k z^k + \sum_{k=2n}^{\infty} x_k z^k = z^n g(z),$$

where g is analytic, $\|((I-V_n)f)(z)\|_X = |z|^n \|g(z)\|_X$, and $\|(I-V_n)f\|_{H^1(X)} = \lim_{r\to 1}(r^n \|g_r\|_{L^1(X)}) = \|g\|_{H^1(X)}$. Therefore,

$$\begin{aligned}
\|((I-V_n)f)(z)\|_X = |z|^n \|g(z)\|_X &\leq 2r^n(1-r)^{-1}\|g\|_{H^1(X)} \\
&\leq (\varepsilon/4)\|g\|_{H^1(X)} = (\varepsilon/4)\|(I-V_n)f\|_{H^1(X)} \\
&\leq (\varepsilon/4)\|(I-V_n)f\|_{\mathcal{BMOA}(\mathbb{D},X)} \leq \varepsilon \|f\|_{\mathcal{BMOA}(\mathbb{D},X)},
\end{aligned}$$

by (i) and (2.4), for any $|z| < r$ and $n \geq n_0$.

(iii). Assume that X is reflexive and denote by $l^2_{2n}(X)$ the space $\{x = (x_1, \dots, x_{2n}) \in \bigoplus_{i=1}^{2n} X \colon \|x\|^2_{l^2_{2n}(X)} = \sum_{j=1}^{2n} \|x_j\|_X^2 < \infty\}$. Then $l^2_{2n}(X)$ is reflexive since $(l^2_{2n}(X))^{**} = (l^2_{2n}(X^{**})) = l^2_{2n}(X)$, by the reflexivity of X. Define the linear maps $R_n \colon \mathcal{BMOA}(\mathbb{D}, X) \to l^2_{2n}(X)$ and $T_n \colon l^2_{2n}(X) \to \mathcal{BMOA}(\mathbb{D}, X)$ by

$$\begin{aligned}
&R_n\Big(\sum_{n=0}^{\infty} x_n z^n\Big) = (x_0, x_1, \dots, x_{2n-1}); \text{ and} \\
&(T_n(x_0, \dots, x_{2n-1}))(z) = \sum_{k=0}^{n} x_k z^k + \sum_{k=n+1}^{2n-1} \frac{2n-k}{n} x_k z^k,
\end{aligned}$$

where $z \in \mathbb{D}$. Since for every $f(z) = \sum_{n=0}^{\infty} x_n z^n$ we have

$$\begin{aligned}
\|R_n f\|_{l^2_{2n}(X)} &= \|(x_0, \dots, x_{2n-1})\|_{l^2_{2n}(X)} \\
&\leq (2n)^{1/2} \max_{0 \leq j \leq n} \|x_j\|_X \leq (2n)^{1/2}\|f\|_{\mathcal{BMOA}(\mathbb{D},X)},
\end{aligned}$$

by Lemma 2.18, the mapping R_n is bounded. Since $\|z^n\|_{\mathcal{BMOA}(\mathbb{D},\mathbb{C})} = 1$ for $n \in \mathbb{N}$, we have

$$\begin{aligned}
\|T(x_0, \dots, x_{2n-1})\|_{\mathcal{BMOA}(\mathbb{D},X)} &\leq \sum_{j=0}^{2n-1} \|x_j z^j\|_{\mathcal{BMOA}(\mathbb{D},X)} = \sum_{j=0}^{2n-1} \|x_j\|_X \\
&\leq (2n)^{1/2}\|(x_0, \dots, x_{2n-1})\|_{l^2_{2n}(X)},
\end{aligned}$$

by Hölder's inequality, so that the mapping $T_n \colon l^2_{2n}(X) \to \mathcal{BMOA}(\mathbb{D}, X)$ is bounded. Now the operator $V_n = T_n R_n$ is weakly compact since it factorizes through the reflexive space $l^2_{2n}(X)$. □

If the Banach space X is finite dimensional, then the operators V_n of Proposition 4.5 are actually compact (since in that case also the space $l^2_{2n}(X)$ in the proof of part (iii) is finite dimensional). Hence the proof of Theorem 4.4 shows that the analytic mappings φ with $\|\varphi\|_\infty < 1$ induce compact composition operators on $BMOA(\mathbb{D}, X)$ whenever X is finite dimensional. In the case $X = \mathbb{C}$ this fact was shown in [Tja96, Corollary 2.12]. This raises the question whether all compact composition operators on $BMOA(\mathbb{D}, \mathbb{C})$ are weakly compact on $BMOA(\mathbb{D}, X)$ if X is reflexive. Our next theorem answers this question in the affirmative.

Theorem 4.6. *Let X be a reflexive complex Banach space and let φ be an analytic map that induces a compact composition operator on $BMOA(\mathbb{D}, \mathbb{C})$. Then C_φ is weakly compact on $BMOA(\mathbb{D}, X)$.*

For the proof of Theorem 4.6 we need Smith's characterization of compact composition operators on $BMOA(\mathbb{D}, \mathbb{C})$ that we will consider next. If $\varphi\colon \mathbb{D} \to \mathbb{D}$ is analytic, then by Fatou's theorem, the limit $\lim_{r\to 1} \varphi_r = \varphi^*$ exists uniquely on a set $E \subset \mathbb{T}$ of full measure. We write $E_a(\varphi, t)$ for the set $E_a(\varphi, t) = \{\sigma_a(e^{i\theta})\colon |\varphi^*(e^{i\theta})| > t\}$, where σ_a denotes the Möbius transformation $\sigma_a(w) = (a-w)/(1-\overline{a}w)$ for $w \in \overline{\mathbb{D}}$. Note that actually $E_a(\varphi, t) = \{e^{i\theta}\colon |(\varphi\circ\sigma_a)^*(e^{i\theta})| > t\}$, since we have $\varphi^*\circ\sigma_a = \varphi^*\circ\sigma_a^* = (\varphi\circ\sigma_a)^*$, by Theorem 3.7.

Definition 4.7. We say that an analytic function $\varphi\colon \mathbb{D} \to \mathbb{D}$ satisfies *property* (S) if it satisfies the following conditions:

$$\lim_{r\to 1} \sup_{\{a\colon |\varphi(a)|>r\}} \sup_{w\in\mathbb{D}\setminus\{0\}} |w|^2 N(\sigma_{\varphi(a)}\circ\varphi\circ\sigma_a, w) = 0,$$

and for every $R < 1$,

$$\lim_{t\to 1} \sup_{\{a:|\varphi(a)|\le R\}} m(E_a(\varphi, t)) = 0.$$

The following result due to Smith characterizes the compact composition operators on $BMOA(\mathbb{D}, \mathbb{C})$ and $VMOA(\mathbb{D}, \mathbb{C})$ [Smi99, Theorem 1.1 and Corollary 1.3].

Theorem 4.8 ([Smi99]). *Let φ be an analytic self-map of $\mathbb{D}$. Then*

(i) C_φ is compact on $BMOA(\mathbb{D}, \mathbb{C})$ if and only if φ satisfies property (S).

(ii) C_φ is compact on $VMOA(\mathbb{D}, \mathbb{C})$ if and only if φ satisfies property (S) and $\varphi \in VMOA(\mathbb{D}, \mathbb{C})$.

We use Theorem 4.8 (i) without proof (see [Lai00] for a detailed proof). We need later only the implication "$\Rightarrow$" of (i). Theorem 4.6 generalizes the implication "$\Leftarrow$" to X-valued functions in the following sense: The proof of Theorem 4.6 shows that if X is a finite dimensional Banach space and φ satisfies property (S), then

$C_\varphi\colon BMOA(\mathbb{D},X) \to BMOA(\mathbb{D},X)$ is compact. In the proof of Theorem 4.6 we do not need Theorem 4.8, but we use similar arguments as Smith. The proof of implication "$\Rightarrow$" of (ii) follows from the fact that $BMOA(\mathbb{D},\mathbb{C})$ is isomorphic to the second dual of $VMOA(\mathbb{D},\mathbb{C})$ (see e.g. [Zhu90, §8]) and hence one can calculate that the second adjoint of the composition operator acting on $VMOA(\mathbb{D},\mathbb{C})$ is the composition operator acting on $BMOA(\mathbb{D},\mathbb{C})$ (see the argument on p. 939 of [CM02]). Now the implication follows from Gantmacher's theorem and Theorem 3.5. The implication "$\Leftarrow$" of (ii) follows from the density of the polynomials in $VMOA(\mathbb{D},\mathbb{C})$ (c.f. the proof of Corollary 4.13).

The rest of the section is devoted to proving Theorem 4.6. The idea of the proof is to combine Smith's method of the proof of Theorem 4.8 and the vector-valued arguments used in [LST98], such as the vector-valued Stanton's formula and Proposition 4.5. Proving the following proposition requires some work but it contains almost all the information that we need to prove Theorem 4.6.

Proposition 4.9. *Let φ satisfy property (S). Then*

$$\|C_\varphi(I - V_n)\|_{\mathcal{L}(BMOA(\mathbb{D},X))} \to 0$$

as $n \to \infty$, where V_n is the operator from Proposition 4.5.

To prove Proposition 4.9, we need some auxiliary results. The first one is a reformulation of [Smi99, Lemma 2.1].

Lemma 4.10. *Let φ be an analytic self-map of $\mathbb{D}$ with $\varphi(0) = 0$. If*

$$\sup_{0<|z|<1} |z|^2 N(\varphi, z) \le \varepsilon,$$

where $\varepsilon < e^{-2}$, then

$$N(\varphi, w) \le \begin{cases} \log(1/|w|), & 0 < |w| \le \varepsilon^{1/4} \\ 2e\varepsilon^{1/2}\log(1/|w|), & \varepsilon^{1/4} < |w| < 1. \end{cases}$$

Proof. Since $\varphi(0) = 0$, the estimate is Littlewood's inequality (3.2), for $0 < |w| \le \varepsilon^{1/4}$. For $\varepsilon^{1/4} < |w| \le e^{-1/2}$ the result follows from the assumption: We calculate that

$$N(\varphi, w) \le \varepsilon|w|^{-2} \le \varepsilon^{1/2} = 2\varepsilon^{1/2}\log(e^{1/2}) \le 2\varepsilon^{1/2}\log(1/|w|) \le 2e\varepsilon^{1/2}\log(1/|w|),$$

for $\varepsilon^{1/4} < |w| \le e^{-1/2}$. Note that the interval $(\varepsilon^{1/4}, e^{-1/2})$ is nonempty by the assumption $\varepsilon < e^{-2}$. In the remaining case, where $e^{-1/2} < |w| < 1$, we have $N_r(\varphi, w) \le \varepsilon|w|^{-2}$, for each $0 < r < 1$, since $N_r(\varphi, w)$ is an increasing function of r with $\lim_{r\to 1} N_r(\varphi, w) = N(\varphi, w)$. Thus $N_r(\varphi, w)$ is bounded above by εe when $|w| = e^{-1/2}$ and tends uniformly to zero as $|w| \to 1$, by Littlewood's inequality. Since it is subharmonic on $\mathbb{D}\setminus\{0\}$, it is bounded above by the harmonic function

h defined on $\{z : e^{-1/2} < |z| < 1\}$ which has the boundary value εe on $\partial(e^{-1/2}\mathbb{D})$ and zero on $\mathbb{T}$, namely $h(w) = 2e\varepsilon \log(1/|w|)$. Thus,

$$N(\varphi, w) = \lim_{r\to 1} N_r(\varphi, w) \le 2e\varepsilon^{1/2} \log(1/|w|),$$

for $e^{-1/2} < |w| < 1$. □

Lemma 4.11. *Let $\varepsilon > 0$, $\varphi\colon \mathbb{D} \to \mathbb{D}$ analytic, and $a \in \mathbb{D}$. Then there exists a number $\delta > 0$ such that if*

$$\sup_{0<|w|<1} |w|^2 N(\varphi_a, w) < \delta, \tag{4.1}$$

where $\varphi_a\colon \mathbb{D} \to \mathbb{D}$ is the analytic function defined by $\varphi_a = \sigma_{\varphi(a)} \circ \varphi \circ \sigma_a$, then

$$\|C_{\varphi\circ\sigma_a}(S_n f) - (S_n f)(\varphi(a))\|_{H^1(X)} \le \varepsilon \|f\|_{**,X},$$

for every $n \in \mathbb{N}$ and $f \in BMOA(\mathbb{D}, X)$, where $S_n = I - V_n$.

Proof. Let $\varepsilon > 0$, $\varphi\colon \mathbb{D} \to \mathbb{D}$ analytic, $a \in \mathbb{D}$, $n \in \mathbb{N}$ and $f \in BMOA(\mathbb{D}, X)$. Let $0 < \delta < \frac{1}{64}$ be a number such that

$$8e\delta^{1/2} + 16\delta^{1/4} \log \frac{1}{16\delta} < \varepsilon,$$

and assume that (4.1) holds for this δ. Define the analytic function $g\colon \mathbb{D} \to X$ by $g = (S_n f) \circ \sigma_{\varphi(a)} - (S_n f)(\varphi(a))$, so that

$$\begin{aligned} \|g\|_{H^1(X)} &\le \sup_{b\in\varphi(\mathbb{D})} \|(S_n f) \circ \sigma_b - (S_n f)(b)\|_{H^1(X)} \le \|S_n f\|_{**,X} \\ &\le \|f\|_{**,X} + \|V_n f\|_{**,X} \le 4\|f\|_{**,X}, \end{aligned} \tag{4.2}$$

by Proposition 4.5 (i) and (2.4). Since $\varphi_a(0) = (\sigma_{\varphi(a)} \circ \varphi \circ \sigma_a)(0) = 0$, we have also that $\|g(\varphi_a(0))\|_X = 0$. By part (3.3) of Lemma 3.4, we obtain

$$\begin{aligned} \|C_{\varphi\circ\sigma_a}(S_n f) - (S_n f)(\varphi(a))\|_{H^1(X)} &= \|g \circ \varphi_a\|_{H^1(X)} \\ &= \frac{1}{2\pi} \int_{\mathbb{D}} N(\varphi_a, w) d[\triangle \|g\|_X](w) \\ &= \frac{1}{2\pi} \int_{\delta^{1/4}\le|w|<1} N(\varphi_a, w) d[\triangle \|g\|_X](w) \\ &\quad + \frac{1}{2\pi} \int_{|w|<\delta^{1/4}} N(\varphi_a, w) d[\triangle \|g\|_X](w) \\ &=: A + B. \end{aligned} \tag{4.3}$$

According to the assumption (4.1) and the fact $\delta < \frac{1}{64} < e^{-2}$, we may apply Lemma 4.10 to obtain $N(\varphi_a, w) \leq 2e\delta^{1/2}\log(1/|w|)$, for $\delta^{1/4} < |w| < 1$. Thus by identity (3.4) of Lemma 3.4 and (4.2), we get that

$$\begin{aligned}
A &\leq \frac{2e\delta^{1/2}}{2\pi}\int_{\delta^{1/4}<|w|<1}\log\left(\frac{1}{|w|}\right)d[\triangle\|g\|_X](w)\\
&\leq 2e\delta^{1/2}\|g\|_{H^1(X)}\\
&\leq 8e\delta^{1/2}\|f\|_{**,X}.
\end{aligned}$$

To estimate the term B, we notice that $N(\varphi_a, w) \leq \log(1/|w|)$ for $w \neq \varphi_a(0) = 0$, by Lemma 4.10. Moreover, from $|w| < \delta^{1/4} < \frac{1}{4}$ we calculate that $2\log\frac{2\delta^{1/4}}{|w|} \geq 1$ and $2\log\frac{1}{2\delta^{1/4}} \geq 1$, for $|w| < \delta^{1/4}$. Thus,

$$\begin{aligned}
N(\varphi_a, w) &\leq \log\frac{1}{|w|} = \log\frac{2\delta^{1/4}}{|w|} + \log\frac{1}{2\delta^{1/4}}\\
&\leq 2\log\frac{1}{2\delta^{1/4}}\log\frac{2\delta^{1/4}}{|w|} + 2\log\frac{1}{2\delta^{1/4}}\log\frac{2\delta^{1/4}}{|w|} = \log\frac{1}{16\delta}\log\frac{2\delta^{1/4}}{|w|},
\end{aligned}$$

for $|w| < \delta^{1/4}$. Applying this to the term B, gives

$$\begin{aligned}
B &\leq \frac{1}{2\pi}\log\frac{1}{16\delta}\int_{\delta^{1/4}\mathbb{D}}\log\frac{2\delta^{1/4}}{|w|}d[\triangle\|g\|_X](w)\\
&\leq \frac{1}{2\pi}\log\frac{1}{16\delta}\int_{2\delta^{1/4}\mathbb{D}}\log\frac{2\delta^{1/4}}{|w|}d[\triangle\|g\|_X](w)\\
&= \frac{1}{2\pi}\log\frac{1}{16\delta}\int_{\mathbb{D}}\log^+\frac{2\delta^{1/4}}{|w|}d[\triangle\|g\|_X](w)\\
&= \frac{1}{2\pi}\log\frac{1}{16\delta}\int_0^{2\pi}\|g(2\delta^{1/4}e^{i\theta}) - g(0)\|_X d\theta\\
&\leq \log\frac{1}{16\delta}\frac{2\delta^{1/4}}{1-2\delta^{1/4}}\|g\|_{H^1(X)} \leq 4\delta^{1/4}\log\frac{1}{16\delta}\|g\|_{H^1(X)},
\end{aligned}$$

where we used the positivity of the measure $\Delta\|g\|_X$, part (3.4) of Lemma 3.4, and Proposition 2.9. Thus,

$$B \leq 16\delta^{1/4}\log\frac{1}{16\delta}\|f\|_{**,X},$$

by (4.2). Combining the estimates of the terms A and B in (4.3) together with the choice of δ, implies now that

$$\|C_{\varphi\circ\sigma_a}(S_n f) - (S_n f)(\varphi(a))\|_{H^1(X)} \leq \varepsilon\|f\|_{**,X}.$$

□

Lemma 4.12. *Let* $\varphi\colon \mathbb{D} \to \mathbb{D}$ *be analytic and* $E_a(\varphi, t) = \{e^{i\theta} : |(\varphi\circ\sigma_a)^*(e^{i\theta})| > t\}$, *for* $t \in (0, 1)$. *Suppose that* $\varepsilon > 0$, $R > 0$, $0 < t < 1$ *and* $a \in \mathbb{D}$ *are such that* $|\varphi(a)| \leq R$ *and*

$$m(E_a(\varphi, t)) < \varepsilon^2. \tag{4.4}$$

Then there exists a number $n_\varepsilon \in \mathbb{N}$ *and a number* $K > 0$ *such that*

$$\|C_{\varphi\circ\sigma_a}(S_n f) - (S_n f)(\varphi(a))\|_{H^1(X)} \leq K\varepsilon\|f\|_{**,X},$$

for every $n \geq n_\varepsilon$ *and* $f \in BMOA(\mathbb{D}, X)$, *where* $S_n = I - V_n$.

Proof. Let $\varepsilon > 0$, $R > 0$, $t \in (0, 1)$, and $a \in \mathbb{D}$ be as above. We will abbreviate $E_a = E_a(\varphi, t)$ for simplicity. Since $f \in BMOA(\mathbb{D}, X)$, the radial boundary function f^* is defined almost everywhere on $\mathbb{T}$. Since the operators S_n and $C_{\varphi\circ\sigma_a}$ are bounded in $BMOA(\mathbb{D}, X)$, also the analytic function $h\colon \mathbb{D} \to X$ defined by $h = C_{\varphi\circ\sigma_a}(S_n f) - (S_n f)(\varphi(a))$ has a boundary function h^* defined almost everywhere on $\mathbb{T}$. Moreover, $h \in H^1(\mathbb{D}, X)$ with $\|h\|_{H^1(X)} = \|h^*\|_{L^1(X)}$ and it suffices to show that

$$\|h^*\|_{L^1(X)} \leq K'\varepsilon\|f\|_{**,X}, \tag{4.5}$$

for $n \geq n_\varepsilon$ where K' is some positive constant. We estimate $\|h^*\|_{L^1(X)}$ in two parts,

$$\begin{aligned}\|h^*\|_{L^1(X)} &= \frac{1}{2\pi}\int_{\mathbb{T}\setminus E_a} \|h^*(e^{i\theta})\|_X d\theta + \frac{1}{2\pi}\int_{E_a} \|h^*(e^{i\theta})\|_X d\theta \\ &=: G + H.\end{aligned} \tag{4.6}$$

Consider first the term G. If $e^{i\theta} \in \mathbb{T}\setminus E_a$, then $|(\varphi\circ\sigma_a)^*(e^{i\theta})| \leq t < 1$. Note that by the continuity of $S_n f$ on $t\overline{\mathbb{D}}$,

$$\begin{aligned}[(S_n f)\circ\varphi\circ\sigma_a]^*(e^{i\theta}) &= \lim_{r\to 1^-} (S_n f)((\varphi\circ\sigma_a)(re^{i\theta})) \\ = (S_n f)(\lim_{r\to 1^-}(\varphi\circ\sigma_a)(re^{i\theta})) &= [(S_n f)\circ(\varphi\circ\sigma_a)^*](e^{i\theta}),\end{aligned}$$

for every $e^{i\theta} \in \mathbb{T}\setminus E_a$ (alternatively use Theorem 3.7). Hence,

$$\begin{aligned}G &= \frac{1}{2\pi}\int_{\mathbb{T}\setminus E_a} \|[(S_n f)\circ(\varphi\circ\sigma_a) - (S_n f)(\varphi(a))]^*(e^{i\theta})\|_X d\theta \\ &= \frac{1}{2\pi}\int_{\mathbb{T}\setminus E_a} \|[(S_n f)\circ(\varphi\circ\sigma_a)^*](e^{i\theta}) - (S_n f)(\varphi(a))\|_X d\theta \\ &\leq \sup_{e^{i\theta}\in\mathbb{T}\setminus E_a} \|(S_n f)((\varphi\circ\sigma_a)^*(e^{i\theta}))\|_X + \|(S_n f)(\varphi(a))\|_X \\ &\leq \sup_{|z|\leq t} \|(S_n f)(z)\|_X + \|(S_n f)(\varphi(a))\|_X.\end{aligned}$$

Since $|\varphi(a)| \le R$, there is a number $n_\varepsilon \in \mathbb{N}$ such that for every $n \ge n_\varepsilon$ and $|z| \le t$ we have

$$\max\{\|(S_n f)(z)\|_X, \|(S_n f)(\varphi(a))\|_X\} \le \varepsilon \|f\|_{**,X},$$

by Proposition 4.5. It follows that $G \le 2\varepsilon\|f\|_{**,X}$ for $n \ge n_\varepsilon$. To estimate the term H suppose that $e^{i\theta} \in E_a$. Then

$$\begin{aligned} H &\le m(E_a)^{\frac{1}{2}} \left(\frac{1}{2\pi} \int_{\mathbb{T}} \|[(S_n f) \circ \varphi \circ \sigma_a - (S_n f)(\varphi(a))]^*(e^{i\theta})\|_X^2 d\theta \right)^{\frac{1}{2}} \\ &\le \varepsilon \|(S_n f) \circ \varphi \circ \sigma_a - (S_n f)(\varphi(a))\|_{H^2(X)} \\ &\le \varepsilon(\|f \circ \varphi \circ \sigma_a - f(\varphi(a)))\|_{H^2(X)} + \ \|(V_n f) \circ \varphi \circ \sigma_a - (V_n f)(\varphi(a)))\|_{H^2(X)}) \\ &\le \varepsilon(\|f \circ \varphi\|_{**,2,X} + \|(V_n f) \circ \varphi\|_{**,2,X}), \end{aligned}$$

where we used the Schwarz' inequality and (4.4), and the last step followed by taking the supremum over $a \in \mathbb{D}$ and using the $p = 2$ version of the $BMOA(\mathbb{D}, X)$ seminorm. It follows from Theorem 2.13, Theorem 3.3 (i), and Proposition 4.5 (i), that there exists a number K such that

$$\begin{aligned} H &\le K\varepsilon(\|f \circ \varphi\|_{**,X} + \|(V_n f) \circ \varphi\|_{**,X}) \\ &\le K\varepsilon(\|f\|_{**,X} + \|V_n f\|_{**,X}) \le 4K\varepsilon\|f\|_{**,X}. \end{aligned}$$

Combining the estimates of the terms G and H in (4.6) yields a number $K' > 0$ such that (4.5) holds for every $n \ge n_\varepsilon$. ☐

Proof of Proposition 4.9. Assume that φ satisfies property (S) and let $\varepsilon > 0$. We will show that there exist numbers $n_\varepsilon \in \mathbb{N}$ and $K > 0$ such that

$$\|C_\varphi(S_n f)\|_{\mathcal{BMOA}(\mathbb{D},X)} \le K\varepsilon\|f\|_{**,X},$$

for every $f \in BMOA(\mathbb{D}, X)$ and $n \ge n_\varepsilon$, where $S_n = I - V_n$. Note first that $\|(C_\varphi(S_n f))(0)\|_X = \|(S_n f)(\varphi(0))\|_X \le \varepsilon\|f\|_{**,X}$ for n large enough, by Proposition 4.5. Hence

$$\|C_\varphi(S_n f)\|_{\mathcal{BMOA}(\mathbb{D},X)} \le \|C_\varphi(S_n f)\|_{**,X} + \varepsilon\|f\|_{**,X},$$

for such n. Since $(C_\varphi(S_n f)) \circ \sigma_a = C_{\varphi\circ\sigma_a}(S_n f)$, we may write $\|C_\varphi(S_n f)\|_{**,X} = \sup_{a\in\mathbb{D}} \|C_{\varphi\circ\sigma_a}(S_n f) - (S_n f)(\varphi(a))\|_{H^1(X)}$. Hence it suffices to show that

$$\|C_{\varphi\circ\sigma_a}(S_n f) - (S_n f)(\varphi(a))\|_{H^1(X)} \le K\varepsilon\|f\|_{**,X}, \tag{4.7}$$

for all $f \in BMOA(\mathbb{D}, X)$ and $a \in \mathbb{D}$ once n is large enough. Since φ satisfies property (S) (see Definition 4.7), there exists a number $R = R(\varepsilon) \in (0, 1)$ such that

$$\sup_{0<|w|<1} |w|^2 N(\sigma_{\varphi(a)} \circ \varphi \circ \sigma_a, w) < \delta, \quad \text{for every } a \in \{z\colon |\varphi(z)| > R\},$$

where δ is such that Lemma 4.11 holds. Moreover, there exists a number $t_0 = t_0(\varepsilon) \in (0,1)$ such that

$$m((E_a(\varphi, t_0)) < \varepsilon^2, \qquad \text{for every } a \in \{z\colon |\varphi(z)| \leq R\}.$$

The claim (4.7) follows now for all $a \in \mathbb{D}$ by combining Lemmas 4.11 and 4.12. □

We are now ready to complete the proof of Theorem 4.6. Recall that by Theorem 4.8, the composition operator C_φ is compact on $BMOA(\mathbb{D}, \mathbb{C})$ if and only if φ satisfies property (S).

Proof of Theorem 4.6. Assume that X is a reflexive Banach space and C_φ is compact on $BMOA(\mathbb{D}, \mathbb{C})$. Since X is reflexive, Proposition 4.5 implies that V_n is weakly compact. Recall that since the weakly compact operators form a closed operator ideal, the operator $C_\varphi V_n$ is weakly compact for every $n \geq 1$. Thus it suffices to show that $\|C_\varphi - C_\varphi V_n\|_{\mathcal{L}(BMOA(\mathbb{D},X))} \to 0$ as $n \to \infty$. But, since φ satisfies property (S), the proof follows from Proposition 4.9. □

As a corollary we obtain an analogoue of Theorem 4.6 for $VMOA(\mathbb{D}, X)$. Recall that by Theorem 4.8, the composition operator C_φ is compact on the space $VMOA(\mathbb{D}, \mathbb{C})$ if and only if $\varphi \in VMOA(\mathbb{D}, \mathbb{C})$ and φ satisfies property (S).

Corollary 4.13. *Let X be a reflexive Banach space and let φ be an analytic function that induces a compact composition operator on $VMOA(\mathbb{D}, \mathbb{C})$. Then C_φ is weakly compact on $VMOA(\mathbb{D}, X)$*

Proof. Assume that X is reflexive and C_φ is compact on $VMOA(\mathbb{D}, \mathbb{C})$. Then $\varphi \in VMOA(\mathbb{D}, \mathbb{C})$ and C_φ is also compact on $BMOA(\mathbb{D}, \mathbb{C})$, by Theorem 4.8. Thus C_φ is weakly compact on $BMOA(\mathbb{D}, X)$, by Theorem 4.6. Let (f_n) be a bounded sequence in $VMOA(\mathbb{D}, X)$. Then $(f_n \circ \varphi)$ has a weakly converging subsequence $(f_{n_k} \circ \varphi)$ in $BMOA(\mathbb{D}, X)$. By Theorem 3.5, the subsequence belongs to $VMOA(\mathbb{D}, X)$ and hence converges weakly in $VMOA(\mathbb{D}, X)$ to a function $g \in BMOA(\mathbb{D}, X)$. Since $VMOA(\mathbb{D}, X)$ is a closed subspace of $BMOA(\mathbb{D}, X)$, it is weakly closed, by Mazur's theorem [Woj96, Theorem II.A.4]. Hence the weak limit g belongs to $VMOA(\mathbb{D}, X)$. □

4.3 Characterization of weakly compact composition operators

In the light of Corollary 4.3 and Theorem 4.6 the characterization of weakly compact composition operators on $BMOA(\mathbb{D}, X)$ depends on the question whether all weakly compact composition operators on $BMOA(\mathbb{D}, X)$ are compact or not. Unfortunately the answer to this question is not known for arbitrary composition operators C_φ. The question remains one of the most important open problems in the theory of analytic composition operators at the present moment (see [Tja96, p. 64], [BCM99, p. 2195], and [CM02]).

Open problem 4.14. *Are all weakly compact composition operators on the spaces $BMOA(\mathbb{D},\mathbb{C})$ and $VMOA(\mathbb{D},\mathbb{C})$ compact?*

It is worth noting that there exists a weakly compact linear operator on $VMOA(\mathbb{D},\mathbb{C})$ that is not compact [CM02, §4]. The purpose of this section is to recall some partial positive results from the literature. In particular, combined with Theorem 4.6 this allows one to characterize the weakly compact composition operators C_φ on $BMOA(\mathbb{D},X)$ whose symbol φ is univalent, or maps $\mathbb{D}$ into a polygon inscribed in the unit circle and belongs to $VMOA(\mathbb{D},\mathbb{C})$.

We call an analytic map $\varphi\colon \mathbb{D} \to \mathbb{D}$ *univalent* if it is one-to-one onto its image $\varphi(\mathbb{D})$. We use the fact that all bounded univalent functions belong to $VMOA(\mathbb{D},\mathbb{C})$. Indeed, if $\varphi\colon \mathbb{D} \to \mathbb{D}$ is univalent, then

$$\int_{\mathbb{D}} |\varphi'(z)|^2 dA(z) = \int_{\varphi(\mathbb{D})} dA(z) = A(\varphi(\mathbb{D})) < \infty,$$

and it is not difficult to see that all analytic functions $\varphi\colon \mathbb{D} \to \mathbb{C}$ satisfying $\int_{\mathbb{D}} |\varphi'(z)|^2 dA(z) < \infty$ belong $VMOA(\mathbb{D},\mathbb{C})$ (see e.g. [Dan99, Theorem 10]). By combining this fact with [Smi99, Theorem 4.1] and [CM02, Theorem 1], one obtains the following result (see p. 940 of [CM02]).

Proposition 4.15 ([Smi99], [CM02]). *Let φ be a univalent self-map of the unit disk.*

(*i*) *If the composition operator C_φ is weakly compact on $BMOA(\mathbb{D},\mathbb{C})$, then C_φ is compact on $BMOA(\mathbb{D},\mathbb{C})$.*

(*ii*) *If the composition operator C_φ is weakly compact on $VMOA(\mathbb{D},\mathbb{C})$, then C_φ is compact on $VMOA(\mathbb{D},\mathbb{C})$.*

Recall that a composition operator C_φ is compact on $BMOA(\mathbb{D},\mathbb{C})$ if and only if φ satisfies property (S). We may now state our characterization of the weakly compact composition operators $BMOA(\mathbb{D},X)$ whose symbol is univalent.

Theorem A. *Let $\varphi\colon \mathbb{D} \to \mathbb{D}$ be univalent. Then*

(*i*) *C_φ is weakly compact on $BMOA(\mathbb{D},X)$ if and only if X is reflexive and φ induces a compact composition operator on $BMOA(\mathbb{D},\mathbb{C})$.*

(*ii*) *C_φ is weakly compact on $VMOA(\mathbb{D},X)$ if and only if X is reflexive and φ induces a compact composition operator on $VMOA(\mathbb{D},\mathbb{C})$.*

Proof. (i) Assume that $\varphi\colon \mathbb{D} \to \mathbb{D}$ is a univalent map such that C_φ is weakly compact on $BMOA(\mathbb{D}, X)$. Then C_φ is weakly compact on $BMOA(\mathbb{D}, \mathbb{C})$ and X is reflexive, by Corollary 4.3. Thus C_φ is compact on $BMOA(\mathbb{D}, \mathbb{C})$, by Proposition 4.15. The converse follows from Theorem 4.6.

(ii) Follows from Corollary 4.3, Proposition 4.15, and Corollary 4.13 similarly as (i). □

We recall next the case of analytic self-maps φ of the unit disk such that $\varphi(\mathbb{D})$ lies inside a polygon inscribed in the unit circle, that is, there exists a finite set of points $\{w_j\}_{j=1}^n \subset \overline{\mathbb{D}}$ such that $\overline{\varphi(\mathbb{D})} \subset P \subset \overline{\mathbb{D}}$, where P denotes the convex hull of $\{w_n\}$. Note that there exist analytic functions $\varphi\colon \mathbb{D} \to \mathbb{D}$ such that $\varphi(\mathbb{D})$ lies inside a polygon inscribed in the unit circle, but $\varphi \notin VMOA(\mathbb{D}, \mathbb{C})$ (see e.g. [Tja96, p. 56]). By inspecting the proof of [Tja96, Theorem 3.15] (alternatively see [MT00, Corollary 5.4]) we obtain the following result.

Proposition 4.16 ([Tja96]). *Let $\varphi\colon \mathbb{D} \to \mathbb{D}$ be an analytic map such that $\varphi(\mathbb{D})$ lies inside a polygon inscribed in the unit circle and $\varphi \in VMOA(\mathbb{D}, \mathbb{C})$.*

- *(i) If the composition operator C_φ is weakly compact on $BMOA(\mathbb{D}, \mathbb{C})$, then C_φ is compact on $BMOA(\mathbb{D}, \mathbb{C})$.*
- *(ii) If the composition operator C_φ is weakly compact on $VMOA(\mathbb{D}, \mathbb{C})$, then C_φ is compact on $VMOA(\mathbb{D}, \mathbb{C})$.*

By replacing Proposition 4.15 by Proposition 4.16 in the proof of Theorem A, we obtain the characterization of the weakly compact composition operators on $BMOA(\mathbb{D}, X)$ and $VMOA(\mathbb{D}, X)$ whose symbol φ maps the unit disk inside a polygon inscribed in the unit circle and belongs to $VMOA(\mathbb{D}, \mathbb{C})$.

Theorem B. *Let $\varphi\colon \mathbb{D} \to \mathbb{D}$ be an analytic map such that $\varphi(\mathbb{D})$ lies inside a polygon inscribed in the unit circle and $\varphi \in VMOA(\mathbb{D}, \mathbb{C})$. Then*

- *(i) C_φ is weakly compact on $BMOA(\mathbb{D}, X)$ if and only if X is reflexive and φ induces a compact composition operator on $BMOA(\mathbb{D}, \mathbb{C})$.*
- *(ii) C_φ is weakly compact on $VMOA(\mathbb{D}, X)$ if and only if X is reflexive and φ induces a compact composition operator on $VMOA(\mathbb{D}, \mathbb{C})$.*

4.4 Rosenthal composition operators

Recall that a linear operator $T\colon E \to F$ is called a *Rosenthal operator* if every bounded sequence (x_n) in E contains a subsequence (x_{n_k}) such that the sequence (Tx_{n_k}) is weakly Cauchy, i.e., the sequence $(\langle Tx_{n_k}, x^*\rangle)$ converges for every $x^* \in E^*$. Rosenthal operators are often also called *weakly conditionally compact* operators. By Rosenthal's l^1 theorem, T is a Rosenthal operator if and only if $T(E)$ does not have a subspace isomorphic to l^1 [LT77, Theorem 2.e.5]. Since Rosenthal operators form a closed operator ideal [Pie80, p. 61, 65], Lemma 4.1 has the following corollary (c.f. Corollary 4.3).

Corollary 4.17. *If C_φ is a Rosenthal operator on $\mathcal{BMOA}(\mathbb{D}, X)$, $BMOA(\mathbb{D}, X)$, or $VMOA(\mathbb{D}, X)$, then X does not have a subspace isomorphic to l^1 and φ induces a Rosenthal composition operator on $BMOA(\mathbb{D}, \mathbb{C})$ or $VMOA(\mathbb{D}, \mathbb{C})$, respectively.*

Note that if none of the subspaces of X are isomorphic to l^1, then X may not have the ARNP. For example, we may take $X = c_0$ (that c_0 does not contain a subspace isomorphic to l^1 is for example a consequence of [LT77, Proposition 2.a.2]; the fact that c_0 does not have the ARNP has been shown in Example 2.6). In particular, there are Banach spaces X not having subspaces isomorphic to l^1 such that $BMOA(\mathbb{D}, X) \neq \mathcal{BMOA}(\mathbb{D}, X)$.

We prove a version of Theorem 4.6 for Rosenthal operators on $BMOA(\mathbb{D}, X)$. The key to the proof is the fact that we have also a version of Proposition 4.5 for Rosenthal operators:

Lemma 4.18. *Let $V_n \colon BMOA(\mathbb{D}, X) \to VMOA(\mathbb{D}, X)$ be the operator from Proposition 4.5, where $n \in \mathbb{N}$. If X does not contain a subspace isomorphic to l^1, then V_n is a Rosenthal operator.*

Proof. Assume that X does not have a subspace isomorphic to l^1. We show first that if X does not have a subspace isomorphic to l^1, then the identity operator of the space $Y = l^2_{2n}(X)$ is a Rosenthal operator, for any $n \in \mathbb{N}$. Assume that $(y_j)_{j=1}^\infty = (x_1^j, \dots, x_{2n}^j)_{j=1}^\infty$ is a bounded sequence in Y. Then $(x_k^j)_{j=1}^\infty$ is a bounded sequence in X, for every $1 \leq k \leq 2n$. Since the identity operator of X is a Rosenthal operator, we may choose a subsequence $(y_{j_r})_r = (x_1^{j_r}, \dots, x_{2n}^{j_r})_r$ of $(y_j)_j$ such that $(\langle x_k^{j_r}, x^* \rangle)_r$ converges for every $x^* \in X$ and $1 \leq k \leq 2n$. Let now $y^* \in Y^* = l^2_{2n}(X^*)$. Then $y^* = (x_1^*, \dots, x_{2n}^*)$ where $x_k^* \in X^*$ for $1 \leq k \leq 2n$. Consequently the sequence $(\langle y_{j_r}, y^* \rangle)_r = (\sum_{k=1}^{2n} \langle x_k^{j_r}, x_k^* \rangle)_r = \sum_{k=1}^{2n} (\langle x_k^{j_r}, x_k^* \rangle)_r$ converges and the identity operator of Y is a Rosenthal operator.

Since the identity operator of Y is a Rosenthal operator, all bounded operators from $BMOA(\mathbb{D}, X)$ to Y are Rosenthal operators. As in the proof of Proposition 4.5, the operator V_n is of the form $V_n = T_n R_n$ where $T_n \colon BMOA(\mathbb{D}, X) \to Y$ and $R_n \colon Y \to BMOA(\mathbb{D}, X)$ are bounded. Hence V_n is a Rosenthal operator, for $n \in \mathbb{N}$. □

Theorem 4.19. *Let X be a complex Banach space that does not have a subspace isomorphic to l^1 and let φ be such that the composition operator C_φ is compact on $BMOA(\mathbb{D}, \mathbb{C})$. Then C_φ is a Rosenthal operator on $BMOA(\mathbb{D}, X)$.*

Moreover, if the map φ is such that the composition operator C_φ is compact on $VMOA(\mathbb{D}, \mathbb{C})$, then C_φ is a Rosenthal operator on $VMOA(\mathbb{D}, X)$.

Proof. Assume that X does not have a subspace isomorphic to l^1 and C_φ is compact on $BMOA(\mathbb{D}, X)$. Since the Rosenthal operators form a closed operator ideal, the operator $C_\varphi V_n$ is Rosenthal for every $n \geq 1$, by Lemma 4.18. The proof follows now from Theorem 4.8 and Proposition 4.9 as in the proof of Theorem 4.6.

Assume next that X does not have a subspace isomorphic to l^1 and C_φ is compact on $VMOA(\mathbb{D}, X)$. Then φ belongs to $VMOA(\mathbb{D}, \mathbb{C})$ and satisfies property (S), by Theorem 4.8. By Lemma 4.18, the operators $V_n \colon VMOA(\mathbb{D}, X) \to VMOA(\mathbb{D}, X)$ are Rosenthal operators. Since C_φ maps $VMOA(\mathbb{D}, X)$ into itself (Theorem 3.5), the operators $C_\varphi S_n$ are Rosenthal operators on $VMOA(\mathbb{D}, X)$. Hence the proof follows again from Proposition 4.9. □

Remark. The proof of Theorem 4.19 can not be extended to the larger space $\mathcal{BMOA}(\mathbb{D}, X)$. In the proof we use Proposition 4.9 that relies on Lemma 4.12. Since the proof of Lemma 4.12 requires the existence of the boundary functions of all functions involved, the argument is not valid for $\mathcal{BMOA}(\mathbb{D}, X)$ functions.

We do not have a version of Theorems A and B for Rosenthal operators. The problem is that we do not know if Rosenthal composition operators on $BMOA(\mathbb{D}, \mathbb{C})$ are compact. We state this as an open problem.

Open problem 4.20. *Are Rosenthal composition operators on $BMOA(\mathbb{D}, \mathbb{C})$ (weakly) compact?*

Since $VMOA(\mathbb{D}, \mathbb{C})$ itself does not contain a subspace isomorphic to l^1, every bounded operator on $VMOA(\mathbb{D}, \mathbb{C})$ is a Rosenthal operator (this follows from the fact that the dual space $H^1(\mathbb{D}, \mathbb{C})$ of $VMOA(\mathbb{D}, \mathbb{C})$ is separable). Hence the analogue of 4.20 for $VMOA(\mathbb{D}, \mathbb{C})$ would be meaningless. Instead, we have the following question.

Open problem 4.21. *Is C_φ a Rosenthal operator on $VMOA(\mathbb{D}, X)$ if and only if X does not have a subspace isomorphic to l^1? Or more generally, if $VMOA(\mathbb{D}, X)$ contains a subspace isomorphic to l^1, does it follow that X must contain a subspace isomorphic to l^1?*

Notes

Our approach to weakly compact and Rosenthal composition operators is similar to the approach in [LST98] and [BDL01] in context of composition operators on various spaces of vector-valued analytic functions (apart from $BMOA(\mathbb{D}, X)$ and $VMOA(\mathbb{D}, X)$). Recently the weak compactness of the composition operator on spaces of analytic functions taking values in a locally convex space has been studied in [BF02].

Compact composition operators on the complex-valued spaces $BMOA(\mathbb{D}, \mathbb{C})$ and $VMOA(\mathbb{D}, \mathbb{C})$ have been completely characterized by Smith [Smi99], and

Bourdon, Cima, and Matheson [BCM99]. In the proof of Theorem 4.6 we used Smith's characterization since the characterization given in [BCM99] is more implicit involving the Banach space norm of $BMOA(\mathbb{D}, \mathbb{C})$. Compact composition operators on $VMOA(\mathbb{D}, \mathbb{C})$ have also been characterized by [Tja96].

Appendix A

Vector-valued functions of bounded mean oscillation

A.1 Vector-valued integration

In this section we give basic definitions and results from the vector-valued integration theory. We restrict ourselves only to the facts needed in the sequel.

Let $\Omega = (\Omega, \Sigma, \mu)$ be a finite measure space and X a Banach space. A function $f\colon \Omega \to X$ is called *simple* if there exist points $x_1, x_2, ..., x_n \in X$ and disjoint sets $E_1, E_2, ..., E_n \in \Sigma$ such that $f = \sum_{i=1}^{n} x_i \chi_{E_i}$, where χ_{E_i} is the characteristic function of E_i. A function $f\colon \Omega \to X$ is (strongly) *measurable* if there exists a sequence of simple functions (f_n) with $\lim_{n\to\infty} \|f_n - f\|_X = 0$ μ-almost everywhere on Ω. It can be verified that the usual facts from the real analysis, regarding the stability of measurable functions under sums, scalar multiples and pointwise (almost everywhere) limits hold (see [HP57, Theorem 3.5.4]).

We are now ready to define the vector-valued analogue of the Lebesgue integral.

Definition A.1. If $f\colon \Omega \to X$ is simple, say $f = \sum_{i=1}^{n} x_i \chi_{E_i}$, then for any $E \in \Sigma$, define

$$\int_E f d\mu = \sum_{i=1}^{n} \mu(E \cap E_i) x_i.$$

A measurable function $f\colon \Omega \to X$ is *Bochner integrable* if there exists a sequence of simple functions (f_n) such that

$$\lim_{n\to\infty} \int_\Omega \|f_n - f\|_X d\mu = 0. \tag{A.1}$$

In this case $\int_E f d\mu$ is defined for each $E \in \Sigma$ by $\int_E f d\mu = \lim_{n\to\infty} \int_E f_n d\mu$. It can be shown that the limit is independent of the sequence satisfying (A.1).

A fundamental theorem by Bochner (see [DU77, p. 45],[HP57, Theorem 3.7.4]) states that a measurable function $f\colon \Omega \to X$ is Bochner integrable if and only if $\int_\Omega \|f\|_X d\mu < \infty$ (see [DU77, p. 45],[HP57, Theorem 3.7.4]). Thus f is Bochner integrable whenever the real-valued function $\|f\|_X$ is Lebesgue integrable. Many properties of the scalar-valued integrals have straightforward generalizations to the vector-valued case. The following inequality will be particularly useful. If f is Bochner integrable X-valued function on Ω, then

$$\left\| \int_E f d\mu \right\|_X \leq \int_E \|f\|_X d\mu,$$

for every $E \in \Sigma$. The triangle inequality establishes the inequality for simple functions; the general case is obtained by approximation (see [DU77, p. 46]).

As usual, we will use the notation $L^p(\Omega, X)$ $(1 \leq p < \infty)$ for the space of all (equivalence classes of) Bochner integrable functions $f\colon \Omega \to X$ with $\int_\Omega \|f\|_X^p d\mu < \infty$ ([DS58], [DU77]). The norm in $L^p(\Omega, X)$ is defined by

$$\|f\|_{L^p(\Omega,X)} = \left(\int_\Omega \|f\|_X^p d\mu \right)^{\frac{1}{p}}.$$

As in the scalar case, one sees that $L^p(\Omega, X)$ $(1 \leq p < \infty)$ is a Banach space. In addition, simple functions are dense in $L^p(\Omega, X)$. The symbol $L^\infty(\Omega, X)$ stands for the space of all (equivalence classes of) measurable functions $f\colon \Omega \to X$ that are essentially bounded; $\|f\|_{L^\infty(\Omega,X)} = \operatorname{ess\,sup}\{\|f(\omega)\|_X \colon \omega \in \Omega\} < \infty$. This space is also a Banach space under the norm $\|\cdot\|_{L^\infty(\Omega,X)}$, but the simple functions are not dense in $L^\infty(\Omega, X)$.

If $\Omega = \mathbb{T}$ and $\mu = m/2\pi$, where m denotes the Lebesgue measure on $\mathbb{T}$, we will denote the spaces $L^p(\mathbb{T}, X)$ also by $L^p(X)$. If $f \in L^1(\mathbb{T}, X)$, we define the t-translate $T_t f \in L^1(\mathbb{T}, X)$ of f by $T_t f(e^{i\theta}) = f(e^{i(\theta - t)})$ for every $e^{i\theta} \in \mathbb{T}$. We will need the following vector-valued version of a theorem by Lebesgue (see [HP57, Theorem 3.8.3]):

Theorem A.2. *If $f \in L^1(\mathbb{T}, X)$, then*

$$\lim_{t \to 0} \|T_t f - f\|_{L^1(X)} = 0.$$

A.2 Functions of bounded mean oscillation

In the following sections we extend the classical scalar-valued theory of functions of bounded mean oscillation (BMO) and vanishing mean oscillation (VMO) to the vector-valued case. Once the vector-valued spaces have been defined, we have no great difficulty in generalizing the classical results. In fact, in the literature it is often customary to refer to the classical results and leave the generalizing to the reader. However, since a systematic treatment of vector-valued BMO has not appeared before (although some results are implicit in papers such as [Bou86], [Bla88b], [Bla97]), we want to discuss it here in detail. Moreover, the vector-valued VMO does not seem to have been considered explicitely before this. We will follow the classical theory very closely. For the scalar theory we refer to [Gar81], [Tor86], and [Gir01].

For the rest of the chapter I will denote a subinterval of $\mathbb{T}$. If $I \subset \mathbb{T}$, we may write also $\int_I f(e^{it})dt$ instead of $\int_I f dm$. In this section X will stand for an arbitrary Banach space. We define the vector-valued BMO functions analogously to the usual scalar-valued definition:

Definition A.3. The space $BMO(\mathbb{T}, X)$ is the set of the functions $f \in L^1(\mathbb{T}, X)$ satisfying

$$\|f\|_{*,X} := \sup \frac{1}{m(I)} \int_I \|f(e^{it}) - f_I\|_X dt < \infty,$$

where

$$f_I = \frac{1}{m(I)} \int_I f(e^{it}) dt$$

and the supremum is taken over all subintervals of $\mathbb{T}$.

The quantity $\|f\|_{*,X}$ is the *mean oscillation* of f and $BMO(\mathbb{T}, X)$ is the space of functions with *bounded mean oscillation*. Note that $\|f\|_{*,X} = 0$ if and only if f is constant. However, it is easy to see that $\|f\|_{BMO(\mathbb{T},X)} = \|f_{\mathbb{T}}\|_X + \|f\|_{*,X}$ is a norm on $BMO(\mathbb{T}, X)$. Note that

$$\|f\|_{L^1(X)} \leq \|f_{\mathbb{T}}\|_X + \|f - f_{\mathbb{T}}\|_{L^1(X)} \leq \|f_{\mathbb{T}}\|_X + \|f\|_{*,X} = \|f\|_{BMO(\mathbb{T},X)}, \quad \text{(A.2)}$$

so that $BMO(\mathbb{T}, X)$ embeds continuously into $L^1(\mathbb{T}, X)$. By scalar-valued BMO we mean always the space $BMO(\mathbb{T}, \mathbb{C})$.

Proposition A.4. *$(BMO(\mathbb{T}, X), \|\cdot\|_{BMO(\mathbb{T},X)})$ is a Banach space.*

Proof. Let (f_n) be a Cauchy sequence in $BMO(\mathbb{T}, X)$ and let $\varepsilon > 0$. Then there exists $n_\varepsilon > 0$ such that for every $n, k \geq n_\varepsilon$ and for every interval $I \subset \mathbb{T}$,

$$\frac{1}{m(I)} \int_I \|f_n - f_k\|_X dm \leq \|f_n - f_k\|_{BMO(\mathbb{T},X)} < \varepsilon. \quad \text{(A.3)}$$

Hence for every I equipped with the normalized Lebesgue measure $m/m(I)$, (f_n) is a Cauchy sequence in the space $L^1(I, X)$. Hence (f_n) converges to a function f^I in $L^1(I, X)$. It is easy to see that for every interval $I \subset \mathbb{T}$, we have $f^I = f^{\mathbb{T}}|_I$, almost everywhere on I, so that we may speak of the single limit function $f = f^{\mathbb{T}} \in L^1(\mathbb{T}, X)$ in each of the spaces $L^1(I, X)$. Thus there exists $n_I \geq n_\varepsilon$ such that for $k \geq n_I$,

$$\frac{1}{m(I)} \int_I \|f_k - f\|_X dm < \varepsilon.$$

Let now $n \geq n_\varepsilon$. Then for every interval $I \subset \mathbb{T}$,

$$\begin{aligned}
&\frac{1}{m(I)}\int_I \|f_n - f - (f_n - f)_I\|_X dm \\
&\qquad \leq \frac{1}{m(I)}\int_I \|f_n - f\|_X dm + \left\|\frac{1}{m(I)}\int_I (f_n - f) dm\right\|_X \\
&\qquad \leq \frac{2}{m(I)}\int_I \|f_n - f\|_X dm \\
&\qquad \leq 2\varepsilon + 2\sup_{k \geq n_I} \frac{1}{m(I)}\int_I \|f_n - f_k\|_X dm \\
&\qquad \leq 2\varepsilon + 2\sup_{k \geq n_\varepsilon} \|f_n - f_k\|_{*,X} \leq 4\varepsilon,
\end{aligned}$$

by (A.3). Taking the supremum over intervals $I \subset \mathbb{T}$, gives $\|f_n - f\|_{*,X} \leq 4\varepsilon$. Hence $f \in BMO(\mathbb{T}, X)$. Since $\|f_\mathbb{T} - (f_n)_\mathbb{T}\|_X \leq \|f - f_n\|_{L^1(X)} \to 0$, as $n \to \infty$, and X is a Banach space, we have that $(f_n)_\mathbb{T}$ converges to $f_\mathbb{T}$ in X. Consequently,

$$\lim_{n\to\infty} \|f_n - f\|_{BMO(\mathbb{T},X)} = 0.$$

□

A.3 Vector-valued John-Nirenberg theorem

The John-Nirenberg theorem is one of the most fundamental result in the BMO theory, since it implies the reverse Hölder's inequality.

Theorem A.5 (Vector-valued John-Nirenberg theorem). *Assume that $f \in BMO(\mathbb{T}, X)$ is not constant. Then there exist positive constants C and c such that for every $\lambda > 0$,*

$$m(\{\zeta \in I : \|f(\zeta) - f_I\|_X > \lambda\}) \leq Ce^{-c\lambda/\|f\|_{*,X}} m(I),$$

for all intervals $I \subset \mathbb{T}$.

For the proof of the theorem we need a vector-valued Calderón-Zygmund decomposition for functions integrable on an interval of the unit circle:

Lemma A.6. *Let I be a subinterval of $\mathbb{T}$, let $f \in L^1(I, X)$, and suppose that $\alpha > 0$ is such that*

$$\frac{1}{m(I)}\int_I \|f\|_X dt < \alpha.$$

Then there is a (possibly finite) sequence $(I_i)_{i\in\mathbb{N}}$ of pairwise disjoint open subintervals of I such that

$$(i)\quad \|f\|_X \le \alpha \qquad \textit{almost everywhere on } I\setminus\bigcup_i I_i,$$

$$(ii)\quad \alpha \le \frac{1}{m(I_i)}\int_{I_i}\|f\|_X dt < 2\alpha, \qquad \textit{for every } i\in\mathbb{N}, \textit{ and}$$

$$(iii)\quad \sum_i m(I_i) \le \frac{1}{\alpha}\int_I \|f\|_X dt, \qquad \textit{for every } i\in\mathbb{N}.$$

Proof. Let f and α be as above and let I be any subinterval of $\mathbb{T}$. Partition I into two disjoint open intervals J_1 and J_2 of length $m(I)/2$. Then for each J_i $(i=1,2)$, either

$$\text{Case 1: } \frac{1}{m(J_i)}\int_{J_i}\|f\|_X dt < \alpha,$$

or

$$\text{Case 2: } \frac{1}{m(J_i)}\int_{J_i}\|f\|_X dt \ge \alpha,$$

and we call J_i Case 1 or Case 2 interval respectively. For both $i=1$ and $i=2$ we check if J_i $(i=1,2)$ is Case 2 interval (by assumption at most one of them can be Case 2!), and if it is, we make J_i the next element in the sequence (I_j). Then we check for both $i=1,2$ if J_i is Case 1 interval, and if it is, we partition J_i into two disjoint open intervals J_{i_j} $(j=1,2)$ of length $m(J_{i_j}) = m(J_i)/2$. Each of these new intervals J_{i_j} is again a Case 1 or Case 2 interval. When we get a Case 1 interval we repeat the partition process. When we reach a Case 2 interval J we stop and put J in the sequence (I_j) and do not partition it. None of the intervals in (I_j) are partitioned and thus are pairwise disjoint.

If $\xi \in I\setminus\bigcup_j I_j$, then every interval containing ξ is a Case 1 interval. Denote by $D(r,\xi)$ the set of dyadic intervals J with $\xi\in J\subset I\setminus\bigcup_j I_j$ and $m(J)\le r$. Since every $J\in D(r,\xi)$ is a Case 1 interval, it follows by an application of Lebesgue's differentation theorem [Rud87, p. 138] to the real-valued function $\zeta\mapsto\|f(\zeta)\|_X$, that

$$\|f(\xi)\|_X = \lim_{r\to 0}\sup_{J\in D(r,\xi)}\frac{1}{m(J)}\int_J \|f(\zeta)\|_X d\zeta \le \alpha,$$

for almost every $\xi\in I\setminus\bigcup_j I_j$, and so (i) holds.

Each selected Case 2 interval $J\in(I_j)$ is contained in a unique dyadic interval K with $m(K)=2m(J)$. The larger interval K was not selected and is thus a Case 1 interval. Therefore

$$2\alpha > \frac{2}{m(K)}\int_K \|f\|_X dt \ge \frac{1}{m(J)}\int_I \|f\|_X dt.$$

Since J is a Case 2 interval, (ii) follows.

Since the intervals (I_j) are pairwise disjoint, we have

$$\sum_{j\in\mathbb{N}} m(I_j) \leq \sum_{j\in\mathbb{N}} \frac{1}{\alpha}\int_{I_j} \|f\|_X dt \leq \frac{1}{\alpha}\int_I \|f\|_X dt$$

and (iii) holds. □

Proof of Theorem A.5. Let $I \subset \mathbb{T}$ be a subinterval and let $f \in BMO(\mathbb{T}, X)$ be nonconstant. By replacing f by $f/\|f\|_{*,X}$ if necessary, we may assume that $\|f\|_{*,X} = 1$. Thus $\frac{1}{m(I)}\int_I \|f - f_I\|_X dt \leq \|f\|_{*,X} = 1$, and we may apply Lemma A.6 to the function $f - f_I$ on I with $\alpha = 2$. We obtain open disjoint intervals $(I_j^1)_{j\in\mathbb{N}}$ such that

(i) $\|f - f_I\|_X \leq 2$ almost everywhere on $I \setminus \bigcup_j I_j^1$,

(ii) $\|f_{I_j^1} - f_I\|_X = \|\frac{1}{m(I_j^1)}\int_{I_j^1}(f - f_I)dt\|_X \leq \frac{1}{m(I_j^1)}\int_{I_j^1}\|f - f_I\|_X dt \leq 4$, and

(iii) $\sum_j m(I_j^1) \leq \frac{1}{2}\int_I \|f - f_I\|_X dt \leq \frac{1}{2}m(I)$, for every $j \in \mathbb{N}$.

Fix $k \in \mathbb{N}$ and consider $I_k^1 \in (I_j^1)$. Then $\frac{1}{m(I)}\int_I \|f - f_{I_k^1}\|_X dt \leq 1$ again, and we apply Lemma A.6 to the function $f - f_{I_k^1}$ on I_k^1 to obtain intervals $(I_{k,i}^2)$ such that each $I_{k,i}^2$ is contained in I_k^1 and

(i) $\|f - f_{I_k^1}\|_X \leq 2$ almost everywhere on $I_k^1 \setminus \bigcup_i I_{k,i}^2$,

(ii) $\|f_{I_{k,i}^2} - f_{I_k^1}\|_X \leq 4$, and

(iii) $\sum_i m(I_{k,i}^2) \leq \frac{1}{2}\sum_i \int_{I_{k,i}^2}\|f - f_{I_k^1}\|_X dt \leq \frac{1}{2}m(I_k^1)$, for every $k \in \mathbb{N}$.

Now $\|f - f_I\|_X \leq \|f - f_{I_k^1}\|_X + \|f_{I_k^1} - f_I\|_X \leq 2 + 4 \leq 8$ almost everywhere on $I_k^1 \setminus \bigcup_i I_{k,i}^2$ and the same is true for every $k \in \mathbb{N}$. Since $\|f - f_I\|_X \leq 2$ almost everywhere on $I \setminus \bigcup_j I_j^1$ it follows that

(i') $\|f - f_I\|_X \leq 8$ a.e. on $(I \setminus \bigcup_j I_j^1) \cup \left(\bigcup_j (I_j^1 \setminus \bigcup_i I_{j,i}^2)\right) = I \setminus \bigcup_{j,i} I_{j,i}^2$, and

(ii') $\sum_{j,i} m(I_{j,i}^2) \leq \frac{1}{2}\sum_j m(I_j^1) \leq (\frac{1}{2})^2 m(I)$.

By repeating this process, we get at stage n a family of open disjoint intervals (I_l^n) of I such that

(i'') $\|f - f_I\|_X \leq 4n$ almost everywhere on $I \setminus \bigcup_l I_l^n$, and

(ii'') $\sum_l m(I_l^n) \leq (\frac{1}{2})^n m(I)$.

Consider first $\lambda > 4$. Then there is an integer $n \geq 1$ such that $4n < \lambda < 4(n+1)$. Since $(\frac{1}{2})^n = e^{-n \log 2}$ and $\lambda \leq 8n$, we note that $2^{-n} \leq e^{-c\lambda}$, where $c = (\log 2)/8$. By (i''), we have $\{\zeta \in I \colon \|f - f_I\|_X > \lambda\} \subset \{\zeta \in I \colon \|f - f_I\|_X > 4n\} \subset \bigcup_l I_l^n$ and thus by (ii''),

$$m(\{\zeta \in I : \|f(\zeta) - f_I\|_X > \lambda\}) \leq \sum_l m(I_l^n) \leq m(I)e^{-c\lambda}.$$

On the other hand, if $0 < \lambda \leq 4$, then

$$m(\{\zeta \in I : \|f(\zeta) - f_I\|_X > \lambda\}) \leq m(I) \leq Ce^{-c\lambda}m(I),$$

where $C = \sqrt{2}$ ensuring that $Ce^{-c\lambda} \geq 1$ for $\lambda \leq 4$, and we obtain the claim for all $\lambda > 0$. □

As a corollary one obtains the reverse Hölder's inequality which we state in the following form:

Corollary A.7 (Reverse Hölder's inequality). *Let $f \in BMO(\mathbb{T}, X)$ and let $0 < p < \infty$. Then*

$$\|f\|_{*,p,X} := \sup_I \left(\frac{1}{m(I)} \int_I \|f(e^{it}) - f_I\|_X^p dt \right)^{\frac{1}{p}} \leq C_p \|f\|_{*,X},$$

where C_p depends only on p.

Proof. If f is a constant, the assertion follows trivially. Hence assume that $f \in BMO(\mathbb{T}, X)$ is not a constant. Fix $I \subset \mathbb{T}$ and denote by

$$\Delta(\lambda) = m(\{\zeta \in I : \|f(\zeta) - f_I\|_X > \lambda\})$$

the distribution function of $\|f - f_I\|_X$. By the well-known formula for distribution functions (see [Rud87, Theorem 8.16]), we have

$$\frac{1}{m(I)} \int_I \|f(e^{it}) - f_I\|_X^p dt = \frac{p}{m(I)} \int_0^\infty \lambda^{p-1} \Delta(\lambda) d\lambda.$$

By the John-Nirenberg theorem, there are positive constants c and C such that $\Delta(\lambda) \leq Ce^{-c\lambda/\|f\|_{*,X}} m(I)$. Thus,

$$\frac{1}{m(I)} \int_I \|f - f_I\|_X^p dt \leq Cp \int_0^\infty \lambda^{p-1} e^{-c\lambda/\|f\|_{*,X}} d\lambda = \frac{Cp\Gamma(p)}{c^p} \|f\|_{*,1}^p,$$

where $\Gamma(p) = \int_0^\infty x^{p-1}e^{-x}dx$ was obtained by the substitution $x = c\lambda/\|f\|_{*,X}$. Since the number $Cp\Gamma(p)/c^p$ is finite for all $p > 0$, the proof is completed by taking the supremum over intervals $I \subset \mathbb{T}$ in the inequality. □

Corollary A.8. *Let $f \in L^1(\mathbb{T}, X)$ and let $1 \le p < \infty$. Then*

$$\|f\|_{*,p,X} \approx \|f\|_{*,X},$$

where the relation means that there are positive numbers C_1 and C_2, depending only on p, such that $\|f\|_{*,p,X} \le C_1 \|f\|_{*,X} \le C_2 \|f\|_{*,p,X}$, for every $f \in L^1(\mathbb{T}, X)$.

Proof. One direction comes from Corollary A.7. The other comes from Hölder's inequality. □

A.4 Garsia type seminorms

Sometimes the classical BMO seminorm is difficult to estimate, so that it is convenient to introduce other equivalent seminorms on BMO. We introduce next the vector-valued version of the so called Garsia norm.

Let $1 \le p < \infty$ and define the function $\|\cdot\|_{P,p,X} : L^1(\mathbb{T}, X) \to \mathbb{R}^+$ by

$$\|f\|_{P,p,X} = \sup_{a \in \mathbb{D}} \left(\frac{1}{2\pi} \int_{\mathbb{T}} \|f(e^{it}) - P[f](a)\|_X^p P_a(t) dt \right)^{1/p},$$

where $P_a(t) = (1 - |a|^2)/|e^{it} - a|^2$ is the Poisson kernel and $P[f]$ denotes the Poisson integral $P[f](z) = \frac{1}{2\pi} \int_0^{2\pi} f(e^{it}) P_z(t) dt$. For $1 \le p < \infty$, the quantity $\|\cdot\|_{P,p,X}$ defines a seminorm on $L^1(\mathbb{T}, X)$. In the literature the introduced seminorm is sometimes called the Garsia seminorm. We will prove the following theorem.

Theorem A.9. *Let $f \in L^1(\mathbb{T}, X)$ and let $1 \le p < \infty$. Then*

$$\|f\|_{*,p,X} \approx \|f\|_{P,p,X}.$$

Before giving the proof we need some auxiliary results that are again vector-valued generalizations of well-known scalar results.

Lemma A.10. *Let $f \in L^p(\mathbb{T}, X)$ for some $1 \le p < \infty$, let $a \in \mathbb{D}$, $x \in X$ and $I \subset \mathbb{T}$ a subinterval. Then*

$$(i) \quad \int_I \|f(e^{it}) - f_I\|_X^p dt \le 2 \int_I \|f(e^{it}) - x\|_X^p dt, \text{ and}$$

$$(ii) \quad \int_0^{2\pi} \|f(e^{it}) - P[f](a)\|_X^p P_a(t) dt \le 2 \int_0^{2\pi} \|f(e^{it}) - x\|_X^p P_a(t) dt.$$

Proof. We prove only the second assertion, since the proof of the first one is similar. By Minkowski's inequality,

$$\left(\int_0^{2\pi} \|f(e^{it}) - P[f](a)\|_X^p P_a(t)dt\right)^{\frac{1}{p}}$$
$$\leq \left(\int_0^{2\pi} (\|f(e^{it}) - x\|_X + \|P[f](a) - x\|_X)^p P_a(t)dt\right)^{\frac{1}{p}}$$
$$\leq \left(\int_0^{2\pi} \|f(e^{it}) - x\|_X^p P_a(t)dt\right)^{\frac{1}{p}} + \|P[f](a) - x\|_X,$$

where the last term is

$$\left\|\int_0^{2\pi} (f(e^{it}) - x)P_a(t)dt\right\|_X$$
$$\leq \int_0^{2\pi} \|f(e^{it}) - x\|_X P_a(t)dt \leq \left(\int_0^{2\pi} \|f(e^{it}) - x\|_X^p P_a(t)dt\right)^{\frac{1}{p}},$$

by Hölder's inequality. □

If $I \subset \mathbb{T}$ is an interval, denote by αI the interval with the same centre as I and α times the length of I, that is, $m(\alpha I) = \alpha m(I)$ (where α is such that $\alpha \leq 2\pi/m(I)$).

Lemma A.11. *Let $f \in BMO(\mathbb{T}, X)$ and $I \subset \mathbb{T}$ an interval. Then*

$$\|f_{2I} - f_I\|_X \leq 2\|f\|_{*,X}$$

and

$$\|f_{2^k I} - f_I\|_X \leq 2k\|f\|_{*,X},$$

for $k \in \mathbb{N},\ k \geq 2$.

Proof. The second inequality follows from the first one on the account of the triangle inequality:

$$\|f_{2^k I} - f_I\|_X \leq \sum_{i=1}^{k} \|f_{2^i I} - f_{2^{i-1} I}\|_X,$$

and the first one is

$$\|f_{2I} - f_I\|_X = \frac{1}{m(I)}\left\|\int_I (f - f_{2I})dt\right\|_X$$
$$\leq \frac{2}{m(2I)} \int_{2I} \|f - f_{2I}\|_X dt \leq 2\|f\|_{*,X}.$$

□

For every interval $I \subset \mathbb{T}$ there exists a unique point $a \in \mathbb{D}$ such that $a/|a|$ is the centre of I and $1 - |a| = m(I)/2\pi$. Namely, if $I = \{e^{it} \colon \theta_1 \leq t \leq \theta_2\}$ then $a = (1 - (\theta_2 - \theta_1)/2\pi)e^{i(\theta_1+\theta_2)/2}$. We need the following lemma to approximate the Poisson kernels $P_a(t)$.

Lemma A.12. *Let a and I be as above and $\alpha \leq 2\pi/m(I)$. Then,*

$$\frac{m(I)}{2\pi} \leq |\zeta - a| \leq (1+\pi)\frac{m(I)}{2\pi} \tag{A.4}$$

for every $\zeta \in I$, and

$$\frac{2\alpha - 1}{2\pi} m(I) \leq |\zeta - a|, \tag{A.5}$$

for every $\zeta \in \mathbb{T} \setminus \alpha I$.

Proof. We prove first (A.4). The first inequality in (A.4) is clear, since the fact that of the points of I the point $a/|a|$ is the closest one to a, implies that for every $\zeta \in I$, we have $|\zeta - a| \geq |a/|a| - a| = 1 - |a| = m(I)/2\pi$. The second inequality in (A.4) comes from the easily verified fact that for every $\zeta \in I$, we have $|a/|a| - \zeta| \leq m(I)/2$. Namely, this implies that for every $\zeta \in I$, $|a - \zeta| \leq |a - a/|a|| + |a/|a| - \zeta| \leq (1+\pi)m(I)/2\pi$. Thus the chain of inequalities (A.4) holds hor $\zeta \in I$.

Let $\alpha I = \{e^{it} \in \mathbb{T} \colon \theta_1 \leq t \leq \theta_2\}$ and put $\xi_1 = e^{i\theta_1}$. Then (A.5) holds clearly for every $\zeta \in \mathbb{T} \setminus \alpha I$, if it holds in one point $\zeta = \xi_1$. It is easy to see that $(\pi/2)|\xi_1 - a/|a|| \geq m(\alpha I)/2$, since $a/|a|$ is the centre of αI. Thus by the triangle inequality, we get

$$|\xi_1 - a| \geq \left|\xi_1 - \frac{a}{|a|}\right| - \left|a - \frac{a}{|a|}\right| \geq \frac{m(\alpha I)}{\pi} - \frac{m(I)}{2\pi} = \frac{2\alpha - 1}{2\pi} m(I),$$

and (A.5) holds for every $\zeta \in \mathbb{T} \setminus \alpha I$. □

The following proposition gives one half of Theorem A.9. We need the result also in the next section.

Proposition A.13. *Let $f \in L^1(\mathbb{T}, X)$ and let $1 \leq p < \infty$. Let $I \subset \mathbb{T}$ be an interval and let $a \in \mathbb{D}$ such that $a/|a|$ is the centre of I and $1 - |a| = m(I)/2\pi$. Then there exists a positive constant C such that*

$$\frac{1}{m(I)} \int_I \|f - f_I\|_X^p dt \leq \frac{C}{2\pi} \int_0^{2\pi} \|f - P[f](a)\|_X^p P_a(t) dt.$$

Proof. Let $1 \leq p < \infty$ and let $f \in L^1(\mathbb{T}, X)$. Let $I \subset \mathbb{T}$ be a subinterval and let $a \in \mathbb{D}$ such that $a/|a|$ is the centre of I and $1 - |a| = m(I)/2\pi$. By part (A.4) of Lemma A.12 there is a constant $C < \infty$ such that $|e^{it} - a| \leq C^{1/2} m(I)/2\pi$

for $e^{it} \in I$. Since $1-|a|^2 = (1-|a|)(1+|a|) \geq 1-|a| = m(I)/2\pi$, we have for $e^{it} \in I$, that

$$P_a(t) = \frac{1-|a|^2}{|e^{it}-a|^2} \geq \frac{m(I)/2\pi}{Cm(I)^2/(2\pi)^2} = \frac{2\pi}{Cm(I)}.$$

Thus we obtain

$$\begin{aligned}\frac{1}{m(I)}\int_I \|f-P[f](a)\|_X^p dt &\leq \frac{C}{2\pi}\int_I \|f-P[f](a)\|_X^p P_a(t)dt \\ &\leq \frac{C}{2\pi}\int_0^{2\pi} \|f-P[f](a)\|_X^p P_a(t)dt.\end{aligned}$$

The proof follows by choosing $x = P[f](a)$ in Lemma A.10 (i). □

We are now ready to prove Theorem A.9.

Proof of Theorem A.9. By Proposition A.13, there exists a positive constant such that

$$\frac{1}{m(I)}\int_I \|f-f_I\|_X^p dt \leq \frac{C}{2\pi}\int_0^{2\pi} \|f-P[f](a)\|_X^p P_a(t)dt \leq C\|f\|_{P,p,X}^p,$$

where $a/|a|$ is the centre of I and $1-|a| = m(I)/2\pi$. By taking the supremum over subintervals $I \subset \mathbb{T}$ in the inequality we obtain $\|f\|_{*,p,X} \leq C^{1/p}\|f\|_{P,p,X}$.

We need yet to show that there exists a number C (depending only on p) such that

$$\|f\|_{P,p,X} \leq C\|f\|_{*,p,X}.$$

Let $1 \leq p < \infty$ and let $f \in L^1(\mathbb{T}, X)$ be such that $\|f\|_{*,p,X} < \infty$. Given $a \in \mathbb{D}$, choose I related to a as in Proposition A.13. Denote by N the largest integer with $m(2^N I) \leq 2\pi$ and put $2^{N+1}I := \mathbb{T}$. Then $1-|a| = m(I)/2\pi = 2^{-k}m(2^k I)/2\pi$ for $k = 0, 1, \ldots, N$ and $m(I) \geq 2^{-(N+1)}m(2^{N+1}I)/2\pi$. Since we may write $\mathbb{T}$ as a union

$$\mathbb{T} = I \cup \left(\bigcup_{k=1}^{N+1}(2^k I \setminus 2^{k-1}I)\right)$$

of disjoint sets, we have

$$\int_{\mathbb{T}} g(t)P_a(t)dt = \int_I g(t)P_a(t)dt + \sum_{k=1}^{N+1}\int_{2^k I\setminus 2^{k-1}I} g(t)P_a(t)dt, \qquad \text{(A.6)}$$

where $g(t) = \|f(e^{it}) - f_I\|_X^p$.

If $e^{it} \in I$, then $|e^{it}-a| \geq m(I)/2\pi$, by part (A.4) of Lemma A.12. Thus,

$$P_a(t) = \frac{1-|a|^2}{|e^{it}-a|^2} \leq \frac{(1-|a|)(1+|a|)}{(m(I)/2\pi)^2} \leq \frac{2m(I)/2\pi}{(m(I)/2\pi)^2} = \frac{4\pi}{m(I)},$$

for $e^{it} \in I$. If $e^{it} \in 2^k I \backslash 2^{k-1} I$ ($k = 1, \ldots, N+1$), then $|e^{it} - a| \geq (2^k - 1)m(I)/(2\pi)$, by part (A.5) of Lemma A.12. Thus we get

$$P_a(t) = \frac{1 - |a|^2}{|e^{it} - a|^2} \leq \frac{4\pi}{(2^k - 1)^2 m(I)} \leq \frac{4\pi}{2^{2k} m(I)} = \frac{4\pi}{2^k m(2^k I)},$$

for $e^{it} \in 2^k I \setminus 2^{k-1} I$. By combining these facts to (A.6) it follows that

$$\int_{\mathbb{T}} g(t) P_a(t) dt \leq \frac{4\pi}{m(I)} \int_I g(t) dt + \sum_{k=1}^{N+1} \frac{4\pi}{2^k m(2^k I)} \int_{2^k I \backslash 2^{k-1} I} g(t) dt. \tag{A.7}$$

Note that the first summand on the right hand side of the inequality is bounded by $4\pi \|f\|_{*,p,X}^p$. We obtain $\|f_{2^k I} - f_I\|_X \leq 2k \|f\|_{*,X} \leq 2k \|f\|_{*,p,X}$, for $k = 1, \ldots, N+1$, by Lemma A.11 and Corollary A.8. Since $2^k I \setminus 2^{k-1} I \subset 2^k I$, it follows from Minkowski's inequality that

$$\begin{aligned}
\left(\frac{1}{m(2^k I)} \int_{2^k I \backslash 2^{k-1} I} g(t) dt \right)^{\frac{1}{p}} &\leq \left(\frac{1}{m(2^k I)} \int_{2^k I} \|f(e^{it}) - f_{2^k I}\|_X^p dt \right)^{\frac{1}{p}} \\
&\leq \left(\frac{1}{m(2^k I)} \int_{2^k I} \|f(e^{it}) - f_{2^k I}\|_X^p dt \right)^{\frac{1}{p}} \\
&\quad + \|f_{2^k I} - f_I\|_X \\
&\leq (1 + 2k) \|f\|_{*,p,X},
\end{aligned}$$

for every $k = 1, \ldots, N+1$. Hence

$$\begin{aligned}
\int_{\mathbb{T}} \|f(e^{it}) - P[f](a)\|_X^p P_a(t) dt &\leq 2 \int_{\mathbb{T}} \|f(e^{it}) - f_I\|_X^p P_a(t) dt \\
&\leq 8\pi \left(1 + \sum_{k=1}^{N+1} \frac{1 + 2k}{2^k} \right) \|f\|_{*,p,X}^p,
\end{aligned}$$

by (A.7) and Lemma A.10. Since $\sum_{k=1}^{\infty} k 2^{-k} \leq M < \infty$, taking the pth root and supremum over $a \in \mathbb{D}$ gives the desired inequality $\|f\|_{P,p,X} \leq C \|f\|_{*,p,X}$. □

For any $a \in \mathbb{D}$ define the mapping σ_a by $\sigma_a(w) = (a - w)/(1 - \bar{a} w)$, for $w \in \overline{\mathbb{D}}$. It is easy to see that σ_a maps $\overline{\mathbb{D}}$ onto $\overline{\mathbb{D}}$ with $(\sigma_a \circ \sigma_a)(w) = w$. Moreover, $|\sigma_a'(e^{it})| = (1 - |a|^2)/|1 - a e^{-it}|^2 = P_a(t)$, for $t \in [0, 2\pi)$.

Lemma A.14. *Let $f \in L^1(\mathbb{T}, X)$ and $a \in \mathbb{D}$. Then $P[f] \circ \sigma_a = P[f \circ \sigma_a]$.*

Proof. Let $z \in \mathbb{D}$. Then

$$(P[g] \circ \sigma_a)(z) = P[g](\sigma_a(z)) = \frac{1}{2\pi} \int_0^{2\pi} g(e^{it}) \frac{1 - |\sigma_a(z)|^2}{|e^{it} - \sigma_a(z)|^2} dt.$$

On the other hand, by the change of variables $e^{it} \mapsto \sigma_a(e^{it})$, we get

$$P[g\circ\sigma_a](z) = \frac{1}{2\pi}\int_0^{2\pi}(g\circ\sigma_a)(e^{it})\frac{1-|z|^2}{|e^{it}-z|^2}dt$$
$$= \frac{1}{2\pi}\int_0^{2\pi} g(e^{it})\frac{1-|z|^2}{|\sigma_a(e^{it})-z|^2}\frac{1-|a|^2}{|1-\overline{a}e^{it}|^2}dt.$$

That $(P[g]\circ\sigma_a)(z) = P[g\circ\sigma_a](z)$ follows now from the well-known identity

$$1-|\sigma_a(z)|^2 = \frac{(1-|z|^2)(1-|a|^2)}{|1-\overline{a}z|^2}$$

and the calculation

$$|e^{it}-\sigma_a(z)|^2|1-\overline{a}z|^2 = |e^{it}(1-\overline{a}z)-a+z|^2$$
$$= |a-e^{it}-z(1-\overline{a}e^{it})|^2 = |\sigma_a(e^{it})-z|^2|1-\overline{a}e^{it}|^2.$$

□

In particular, we may apply Lemma A.14 to the complex-valued function g to obtain

$$\int_0^{2\pi} g(e^{it})P_a(t)dt = (P[g]\circ\sigma_a)(0) = P[g\circ\sigma_a](0) = \int_0^{2\pi}(g\circ\sigma_a)(e^{it})dt.$$

By choosing $g = \|f-P[f](a)\|_X^p$, Corollary A.8 yields the following result.

Corollary A.15. *Let $f \in L^1(\mathbb{T}, X)$ and let $1 \le p < \infty$. Then*

$$\int_0^{2\pi}\|f(e^{it})-P[f](a)\|_X^p P_a(t)dt = \int_0^{2\pi}\|(f\circ\sigma_a)(e^{it})-P[f](a)\|_X^p dt$$

for every $a \in \mathbb{D}$, and

$$\|f\|_{*,X} \approx \sup_{a\in\mathbb{D}}\|f\circ\sigma_a - P[f](a)\|_{L^p(X)}.$$

Summary A.16. Let us summarize some of the results obtained in the chapter so far. For $1 \le p < \infty$, the following conditions are equivalent:

(i) $f \in BMO(\mathbb{T}, X)$.

(ii) $\|f\|_{*,p,X} = \sup_I \left(\frac{1}{m(I)}\int_I \|f(e^{it})-f_I\|_X^p dt\right)^{\frac{1}{p}} < \infty.$

(iii) $\|f\|_{P,p,X} = \sup_{a\in\mathbb{D}}\left(\frac{1}{2\pi}\int_{\mathbb{T}}\|f(e^{it})-P[f](a)\|_X^p P_a(t)dt\right)^{1/p} < \infty.$

(iv) $\sup_{a\in\mathbb{D}}\|f\circ\sigma_a - P[f](a)\|_{L^p(X)} < \infty.$

Moreover, all quantities $(ii)-(iv)$ define equivalent seminorms on $BMO(\mathbb{T}, X)$.

A.5 Vanishing mean oscillation

In this section we generalize the scalar valued theory of functions of vanishing mean oscillation to the vector-valued case. Let X denote again an arbitrary Banach space.

Definition A.17. A function $f \in BMO(\mathbb{T}, X)$ is said to have *vanishing mean oscillation* if

$$\lim_{m(I)\to 0} \frac{1}{m(I)} \int_I \|f(e^{it}) - f_I\|_X dt = 0.$$

We equip the space of functions of vanishing mean oscillation with the norm from $BMO(\mathbb{T}, X)$ and denote the space obtained by $VMO(\mathbb{T}, X)$.

Proposition A.18. *$VMO(\mathbb{T}, X)$ is a closed subspace of $BMO(\mathbb{T}, X)$.*

Proof. Let $f \in BMO(\mathbb{T}, X)$ belong to the norm closure of $VMO(\mathbb{T}, X)$ and let $\varepsilon > 0$. Then there exists a function $g \in VMO(\mathbb{T}, X)$ and a number $\delta > 0$, such that $\|f - g\|_{*,X} < \varepsilon/2$, and

$$\frac{1}{m(I)} \int_I \|g - g_I\|_X dm < \varepsilon/2,$$

for every interval $I \subset \mathbb{T}$ with $m(I) < \delta$. Let I be such an interval. Then

$$\frac{1}{m(I)} \int_I \|f - f_I\|_X dm \leq \frac{1}{m(I)} \int_I \|g - g_I\|_X dm + \|f - g\|_{*,X} \leq \varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, we have $f \in VMO(\mathbb{T}, X)$. □

From Propositions A.4 and A.18 we obtain the following corollary.

Corollary A.19. *The space $(VMO(\mathbb{T}, X), \|\cdot\|_{BMO(\mathbb{T},X)})$ is a Banach space.*

We complete the basic theory of vector-valued VMO functions by giving several characterizations of $VMO(\mathbb{T}, X)$ that are vector-valued analogues of the most fundamental scalar characterizations (for the scalar theory see e.g. [Sar78, p. 49-50]). The functions $g\colon \mathbb{T} \to X$ of the form $g(e^{it}) = \sum_{k=-n}^{n} a_k e^{ikt}$ where $a_k \in X$, are called trigonometric polynomials. If $f \in L^1(\mathbb{T}, X)$, denote by $T_t f$ the t-translate $T_t f(e^{i\theta}) = f(e^{i(\theta - t)})$ of f. For $0 < r < 1$, we write g_r for the function $g_r(e^{i\theta}) = g(re^{i\theta})$ defined on $\mathbb{T}$.

Theorem A.20. *Let $f \in BMO(\mathbb{T}, X)$ and let $1 \leq p < \infty$. Then the following conditions are equivalent:*

(i) $f \in VMO(\mathbb{T}, X)$.

(ii) $\lim_{t \to 0} \|T_t f - f\|_{BMO(\mathbb{T},X)} = 0$.

(iii) $\lim_{r \to 1} \|P[f]_r - f\|_{BMO(\mathbb{T},X)} = 0$.

(iv) *f belongs to the closure of the trigonometric polynomials in $BMO(\mathbb{T}, X)$.*

(v) $\lim_{|a| \to 1} \left(\frac{1}{2\pi} \int_0^{2\pi} \|f(e^{it}) - P[f](a)\|_X^p P_a(t) dt \right)^{\frac{1}{p}} = 0$.

(vi) $\lim_{|a| \to 1} \|f \circ \sigma_a - P[f](a)\|_{L^p(X)} = 0$.

(vii) $\lim_{m(I) \to 0} \left(\frac{1}{m(I)} \int_I \|f(e^{it}) - f_I\|_X^p dt \right)^{\frac{1}{p}} = 0$.

We need some auxiliary results to prove the theorem. The first result is just a reformulation of a theorem in [Koo80]. The proof follows from the basic properties of the Poisson kernel.

Proposition A.21 ([Koo80, p. 11]). *Let $f \in L^1(\mathbb{T}, \mathbb{C})$ and suppose that f is continuous at $\zeta \in \mathbb{T}$. Then $P[f](z) \to f(\zeta)$, as $z \to \zeta$.*

The following lemma implies in particular that the continuous functions $\mathbb{T} \to X$ belong to $VMO(\mathbb{T}, X)$:

Lemma A.22. *If $f \in C(\mathbb{T}, X)$ and $1 \leq p < \infty$, then*

$$\lim_{m(I) \to 0} \left(\frac{1}{m(I)} \int_I \|f - f_I\|_X^p dt \right)^{\frac{1}{p}} = 0.$$

Proof. Let f be continuous, and hence uniformly continuous, on $\mathbb{T}$ and let $\varepsilon > 0$. Then there is $\delta > 0$ such that $|t - \theta| < \delta$ implies $\|f(e^{it}) - f(e^{i\theta})\|_X < \varepsilon$, for every $e^{it}, e^{i\theta} \in \mathbb{T}$. Hence

$$\begin{aligned} \frac{1}{m(I)} \int_I \|f(e^{it}) - f_I\|_X^p dt &= \frac{1}{m(I)} \int_I \left\| \frac{1}{m(I)} \int_I \left(f(e^{it}) - f(e^{i\theta}) \right) d\theta \right\|_X^p dt \\ &\leq \frac{1}{m(I)} \int_I \left(\frac{1}{m(I)} \int_I \|f(e^{it}) - f(e^{i\theta})\|_X d\theta \right)^p dt \\ &\leq \varepsilon, \end{aligned}$$

for every interval $I \subset \mathbb{T}$ such that $m(I) \leq |t - \theta| < \delta$. □

Now we are ready to prove Theorem A.20. We will proceed as follows: $(i) \Rightarrow (ii) \Rightarrow (iii) \Rightarrow (iv) \Rightarrow (i) \Rightarrow (v) \Rightarrow (vi) \Rightarrow (v) \Rightarrow (vii) \Rightarrow (i)$.

Proof of Theorem A.20. $(i) \Rightarrow (ii)$. Assume that $f \in VMO(\mathbb{T}, X)$. Then for any $\varepsilon > 0$ there is a positive number $0 < \delta < \varepsilon$ such that

$$\sup_{m(I) \leq \delta} \frac{1}{m(I)} \int_I \|f(e^{i\theta}) - f_I\|_X d\theta < \frac{\varepsilon}{2}. \tag{A.8}$$

Let $F_t = T_t f - f$. We need to show that

$$\frac{1}{m(I)} \int_I \|F_t(e^{i\theta}) - (F_t)_I\|_X d\theta \to 0$$

uniformly in I as $|t| \to 0$. By Theorem A.2, there exists α such that $\|F_t\|_{L^1(X)} < \delta^2/4\pi$ for $|t| < \alpha$. Let I be a subinterval of $\mathbb{T}$ and $|t| < \alpha$. If $m(I) \geq \delta$, then by Lemma A.10, we have

$$\begin{aligned}\frac{1}{m(I)} \int_I \|F_t(e^{i\theta}) - (F_t)_I\|_X d\theta &\leq \frac{2}{m(I)} \int_I \|F_t(e^{i\theta})\|_X d\theta \\ &\leq \frac{4\pi}{m(I)} \|F_t\|_{L^1(X)} \leq \frac{\delta^2}{m(I)} < \varepsilon.\end{aligned}$$

If $m(I) < \delta$, then

$$\begin{aligned}\frac{1}{m(I)} \int_I \|F_t(e^{i\theta}) - (F_t)_I\|_X d\theta &\leq \frac{1}{m(I)} \int_I \|f(e^{i\theta}) - f_I\|_X d\theta \\ &\quad + \frac{1}{m(I)} \int_I \|T_t f(e^{i\theta}) - (T_t f)_I\|_X d\theta \\ &= \frac{2}{m(I)} \int_I \|f(e^{i\theta}) - f_I\|_X d\theta < \varepsilon,\end{aligned}$$

by (A.8). Thus $\|T_t f - f\|_{*,X} \to 0$ as $t \to 0$ and (ii) follows since we notice that $\int_0^{2\pi} (T_t f - f) d\theta = 0$.

$(ii) \Rightarrow (iii)$. Assume that $f \in BMO(\mathbb{T}, X)$ and $\|T_t f - f\|_{*,X} \to 0$ as $t \to 0$. It follows from the properties of the Poisson kernel that if $0 < r < 1$, then

$$\begin{aligned}P[f]_r(e^{i\theta}) = P[f](re^{i\theta}) &= \frac{1}{2\pi} \int_0^{2\pi} f(e^{is}) P_{re^{i\theta}}(s) ds \\ &= \frac{1}{2\pi} \int_0^{2\pi} f(e^{is}) P_r(s - \theta) ds = \frac{1}{2\pi} \int_0^{2\pi} f(e^{i(\theta+t)}) P_r(t) dt,\end{aligned}$$

where the last step followed by the change of variables $t = s - \theta$. It follows that

$$P[f]_r(e^{i\theta}) = \frac{1}{2\pi} \int_0^{2\pi} T_{-t} f(e^{i\theta}) P_r(t) dt.$$

If I is any interval in $\mathbb{T}$, then

$$\begin{aligned}
&\frac{1}{2\pi}\int_0^{2\pi}(T_{-t}f-f)_I P_r(t)dt\\
&\qquad=\frac{1}{2\pi}\int_0^{2\pi}\left(\frac{1}{m(I)}\int_I (T_{-t}f-f)(e^{i\theta})d\theta\right)P_r(t)dt\\
&\qquad=\frac{1}{m(I)}\int_I\left(\frac{1}{2\pi}\int_0^{2\pi}T_{-t}f(e^{i\theta})P_r(t)dt-f(e^{i\theta})\right)d\theta\\
&\qquad=\frac{1}{m(I)}\int_I(P[f]_r(e^{i\theta})-f(e^{i\theta}))d\theta\\
&\qquad=(P[f]_r-f)_I.
\end{aligned}$$

Since $(T_tf-f)_{\mathbb{T}}=0$, it follows that $(P[f]_r-f)_{\mathbb{T}}=0$. Moreover,

$$\begin{aligned}
&\frac{1}{m(I)}\int_I\|(P[f]_r-f)(e^{i\theta})-(P[f]_r-f)_I\|_X\ d\theta\\
&\quad=\frac{1}{m(I)}\int_I\left\|\frac{1}{2\pi}\int_0^{2\pi}[(T_{-t}f-f)(e^{i\theta})-(T_{-t}f-f)_I]P_r(t)dt\right\|_X d\theta\\
&\quad\leq\frac{1}{2\pi}\int_0^{2\pi}\left(\frac{1}{m(I)}\int_I\|(T_{-t}f-f)(e^{i\theta})-(T_{-t}f-f)_I\|_X\ d\theta\right)P_r(t)dt\\
&\quad\leq\frac{1}{2\pi}\int_0^{2\pi}\|T_{-t}f-f\|_{*,X}P_r(t)dt\\
&\quad=P[\|T_{-t}f-f\|_{*,X}](r).
\end{aligned}$$

It is clear that $\|T_{-t}f-f\|_{*,X}=\|T_tf-f\|_{*,X}$. Since $f\in L^1(\mathbb{T},X)$, it is also clear that the function φ defined by $\varphi(t)=\|T_tf-f\|_{*,X}$ belongs to $L^1(\mathbb{T},\mathbb{C})$. By assumption φ is continuous at 0. Hence by Proposition A.21,

$$\|P[f]_r-f\|_{*,X}\leq P[\varphi](r)\to\varphi(0)=0,$$

as $r\to 1^-$, and (iii) follows since $(P[f]_r-f)_{\mathbb{T}}=0$ for $0<r<1$.

$(iii)\Rightarrow(iv)$. The claim follows since $P[f]_r$ is continuous as the uniform limit of the functions $g_n(e^{it})=\sum_{k=-n}^{n}\hat{f}(n)r^{|n|}e^{int}$ (c.f. [Hen86, Lemma 1.2]).

$(iv)\Rightarrow(i)$. The claim follows from Lemma A.22 (for $p=1$) and Proposition A.18.

$(i)\Rightarrow(v)$. Let $1\leq p<\infty$, suppose that $f\in VMO(\mathbb{T},X)$ and take $\varepsilon>0$. Since $VMO(\mathbb{T},X)$ is the $BMO(\mathbb{T},X)$ closure of $C(\mathbb{T},X)$ (Theorem A.20), we can write f in the form $f=f_1+f_2$, where $f_1\in C(\mathbb{T},X)$ and $f_2\in BMO(\mathbb{T},X)$ with $\|f_2\|_{*,X}<\varepsilon$. By Corollary A.8 and Theorem A.9, we obtain a constant C such that $\|f_2\|_{P,p,X}\leq C\varepsilon$.

We will show that if g is a continuous X-valued function on $\mathbb{T}$, then there is $\delta>0$ such that $1-|a|<\delta$ implies

$$\left(\frac{1}{2\pi}\int_0^{2\pi}\|g(e^{it})-P[g](a)\|_X^p P_a(t)dt\right)^{\frac{1}{p}}<C'\varepsilon, \tag{A.9}$$

where C' depends only on p. Once this has been proved, the implication follows, since

$$\begin{aligned}&\left(\frac{1}{2\pi}\int_0^{2\pi}\|f(e^{it})-P[f](a)\|_X^pP_a(t)dt\right)^{\frac{1}{p}}\\&\leq\left(\frac{1}{2\pi}\int_0^{2\pi}\|f_1(e^{it})-P[f_1](a)\|_X^pP_a(t)dt\right)^{\frac{1}{p}}+C\|f_2\|_{P,p,X}<(C'+C)\varepsilon,\end{aligned}$$

for some $\delta>0$, by Minkowski's inequality.

Let g be continuous on $\mathbb{T}$. Then the function $\varphi(e^{it},e^{is})=\|g(e^{it})-g(e^{is})\|_X^p$ is continuous, and hence uniformly continuous on $\mathbb{T}\times\mathbb{T}$. It follows from Proposition A.21 that there is $\delta>0$ such that if $e^{it_0}\in\mathbb{T}$ is any point, then

$$\left|\frac{1}{2\pi}\int_0^{2\pi}\varphi(e^{it},e^{is})P_a(s)ds-\varphi(e^{it},e^{it_0})\right|<\varepsilon^p,$$

for every e^{it} and $a\in\mathbb{D}$ such that $|a-e^{it_0}|<\delta$. Moreover,

$$\left|\frac{1}{2\pi}\int_0^{2\pi}\varphi(e^{it},e^{it_0})P_a(t)dt-\varphi(e^{it_0},e^{it_0})\right|<\varepsilon^p.$$

It follows (since φ is positive) that

$$\begin{aligned}\frac{1}{(2\pi)^2}\int_0^{2\pi}\int_0^{2\pi}\varphi(e^{it},e^{is})P_a(s)dsP_a(t)dt&\leq\frac{1}{2\pi}\int_0^{2\pi}\varphi(e^{it},e^{it_0})P_a(t)dt+\varepsilon^p\\&\leq\varphi(e^{it_0},e^{it_0})+2\varepsilon^p=2\varepsilon^p,\end{aligned}$$

for every $a\in\mathbb{D}$ such that $|a-e^{it_0}|<\delta$. Since δ was chosen independently of e^{it_0}, we obtain by Hölder's inequality that

$$\begin{aligned}&\frac{1}{2\pi}\int_0^{2\pi}\|g(e^{it})-P[g](a)\|_X^pP_a(t)dt\\&\quad=\frac{1}{2\pi}\int_0^{2\pi}\left\|\frac{1}{2\pi}\int_0^{2\pi}\left(g(e^{it})-g(e^{is})\right)P_a(s)ds\right\|_X^pP_a(t)dt\\&\quad\leq\frac{1}{(2\pi)^2}\int_0^{2\pi}\int_0^{2\pi}\varphi(e^{it},e^{is})P_a(s)dsP_a(t)dt\\&\quad\leq 2\varepsilon^p,\end{aligned}$$

for $a\in\mathbb{D}$ such that $1-|a|<\delta$. Thus (A.9) follows for $C'=2^{1/p}$.

$(v)\Leftrightarrow(vi)$. This is a consequence of Corollary A.15.

$(v)\Rightarrow(vii)$. Fix $\varepsilon>0$. Assume that there is $\delta>0$ such that if $1-|a|<\delta$, then

$$\frac{1}{2\pi}\int_0^{2\pi}\|f-P[f](a)\|_X^pP_a(t)dt<\varepsilon.$$

Let $I \subset \mathbb{T}$ be a subinterval with $m(I) < 2\pi\delta$. Then there is a point $a \in \mathbb{D}$ such that $a/|a|$ is the centre of I and $1 - |a| = m(I)/2\pi < \delta$. By Proposition A.13, there is a positive constant C such that

$$\frac{1}{m(I)} \int_I \|f - f_I\|_X^p dt \leq \frac{C}{2\pi} \int_0^{2\pi} \|f - P[f](a)\|_X^p P_a(t) dt < C\varepsilon,$$

for every $I \subset \mathbb{D}$ such that $m(I) < 2\pi\delta$ and (vii) follows.

$(vii) \Rightarrow (i)$. The claim follows from Hölder's inequality. □

Notes

Our approach to the vector-valued BMO and VMO theory is similar to the treatment of the classical theory as presented in [Sar75], [Sar78], [Gar81], [Tor86], and [Gir01]. Some vector-valued arguments used in the proof of of Theorem A.20 have appeared in [Hen86] in connection with the vector-valued Hardy spaces.

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