Monte Carlo Computation of Optimal Portfolio Choice with Habit Formation

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Abstract

This paper considers optimal consumption and portfolio choice of an investor with habit formation in preferences. Monte Carlo covariation method has been used for optimal portfolio selection when an investor’s preferences are time-separable. This paper works on the method so that it is applicable in the case of more general utilities. As an example, I solve the optimal portfolio problem in the case where the interest rate adheres to Cox-Ingersoll-Ross dynamics and the stock prices mean reversion using the method and compare results to time-separable case.

Keywords: Habit formation, Optimal consumption and portfolio, Monte Carlo methods

JEL classification: C15, D91, G11
1 Introduction

Time separability of utilities in consumption is an usual assumption in the theory of financial economics. Empirical studies have implied problems with this assumption. Sometimes these problems are solvable by applying more general utility formulation.

Applying time separable utilities, rational expectation models often generate results which are empirically valid only if we assume a very risk-aversive investor. If the risk aversion coefficient is plausible, the representative investor in the models puts much more money in the risky investment than empirically happens. Historically the average return on equity in the U.S’s stock market was seven percent and the average yield on short-term debt was less than one percent in the period 1889-1978. Mehra and Prescott (1985) have shown that the common general equilibrium model with separable utilities cannot explain why the first rate is so low and the second rate so high. That is so-called equity premium puzzle. The equity premium puzzle is possible to solve using more general utility function form. In this paper I reject time-separable assumption and assume that an agent’s utilities adhere to more general function habit utility function.

Merton (1971) examines the continuous-time consumption-portfolio problem for an individual whose income is generated by capital gains on investments in assets with prices assumed to satisfy the geometric Brownian motion hypothesis. For the solution of an individual’s optimization problem Merton uses Ito’s lemma and stochastic analysis. There are a few papers that have studied the consumption and investment problem of an agent with habit utilities either in the general equilibrium or in the partial equilibrium model (e.g. Sundaresan (1989), Constantinides (1990), Ingersoll (1992), Munk (2008)).

Constantinides (1990) and Sundaresan (1989) present a solution to the equity premium puzzle applying habit utilities. Constantinides’s (1990) reason for using habit function form is just to find theoretical model which can explain equity premium puzzle. But usually intuition for habit formation has been also given in the literature. There are temporal dependence in the
sense that utility in period $t$ depends on not just consumption in same period but also the level of consumption in the previous periods. An individual who consumes a lot in period $(t-1)$ will get used to that high level of consumption, and will more strongly desire consumption in period $t$ (Kocherlakota(1996)).

If the assumption of time separability has been rejected, there is possibility of two kind of effect: intertemporal substitution or intertemporal complementarity. In the case of intertemporal substitutes a consumer buys a durable good in period $t$, but get the utility of this good in periods $t+i, i > 0$ without any money spending.

Ferson and Constantinides (1991) study empirically habit persistence in preferences and the durability of consumption goods which both imply the time-nonseparability of the derived utility for consumption expenditures. They study which effect does dominate and find evidence in monthly, quarterly, and annual data that habit persistence dominates the effect of durability. Obviously, nondurables are "more habit" than durables. Detemple and Zapatero (1992) and Egglezos (2007) solve optimal consumption when an investor has habit utilities, but they do not find precise solution of optimal portfolio choice.

Munk (2008) finds a closed-form solution of the optimal consumption and portfolio choice with habit utilities and mean-reverting stock returns. He also solves numerically the problem in the habit case when interest rate is stochastic and stock prices are mean reverting. Munk uses Monte Carlo simulation to solve the PDE.

Cvitanic et al. (2003) propose the numerical method for optimal portfolio choice in the case where the interest rate adheres to Cox-Ingersoll-Ross dynamics and the stock prices mean reversion. That method is very flexible and exploiting it, it is possible solve optimal portfolio problem in habit case making different kind of assumptions about financial assets. Only requirements are that markets have to be complete and the expanded opportunity set has to be Markovian i.e. that all parameters of market processes depend on the $n$-dimensional Brownian motion process that describes the uncertainty
in economy. I extend that method for the problem of an investor with habit utilities.

The rest of the paper is structured in the following way. Chapter 2 gives some set-ups and defines utilities. Chapters 3 and 4 consider the assumptions related to financial markets and define precise optimization problem. Chapter 5 shows how to find optimal consumption in the case of habit utilities using martingale method solution. Chapter 6 presents the extension of Cvitanic’s (2003) Monte Carlo covariation method in habit case. Chapter 7 shows the results for optimal portfolio choice problem and finally, chapter 8 is for conclusion and for proposing some ideas of further research.

2 Utilities

Before the review of the agent’s utilities and the behavior of financial market, it is useful to consider some definitions in the probability theory. We have a probability space \((\Omega, \mathcal{F}, P)\) with \(\mathcal{F}\) denoting the \(\sigma\)-algebra of subsets of \(\Omega\). \(P\) is a probability measure which assigns to any event \(A\) the probability \(P(A)\). Random variables \(X\) are \((\mathcal{F}, \mathbb{R})\) measurable functions \(X : \Omega \to \mathbb{R}\). At each time \(t\), a \(\sigma\)-algebra \(\mathcal{F}_t \subset \mathcal{F}\) denotes the set of events corresponding to the information available at time \(t\). Then growing collection of \(\sigma\)-algebra describes how the information accumulates. A filtration on the measurable space \((\Omega, \mathcal{F})\) is an increasing family \((\mathcal{F}_t)_{t \geq 0}\) of sub-\(\sigma\)-algebra \(\mathcal{F}_t\) and information is never forgotten i.e. \(\mathcal{F}_t \subset \mathcal{F}_s\), whenever \(0 \leq t \leq s \leq T\). A stochastic process \(\{X_t, t \geq 0\}\) is adapted to filtration \(\{\mathcal{F}_t | t \geq 0\}\) i.e. \(\mathcal{F}_t\) if \(X_t\) is \(\mathcal{F}_t\) measurable all \(t \geq 0\). \(X_t\) is also called to be progressively measurable.

We consider an investor who maximizes utility by choosing a consumption path \(c = (c_t)\) and optimal portfolio path \(\pi = (\pi_t)\). Usually in the consumption portfolio problem lifetime utility is assumed to be time-separable. The utility function of lifetime consumption can be expressed as a sum of felicity functions in the different periods. In the literature (e.g. Constantinides
(1990)), the habit consumption utility has been formulated:

\[ U(h; \pi, c) = E\left[ \int_0^T e^{-\rho t} u(t, c(t) - h(t; c)) dt \bigg| \tilde{\mathcal{F}}_0 \right] \]  

where

\[ h(t) = h_0 e^{-\int_0^t b(s) ds} + a_t \int_0^t e^{-bt} \int_s^t b(s) c(s) ds \]  

where \( E[\tilde{\mathcal{F}}_0] \) is expectations at time 0, \( \rho \) is subjective discount rate and \( \gamma \) is parameter for the degree of risk aversion. Equation (2.2) describes the standard of living. It satisfies the differential equation \( dh(t) = (b t c_t - a_t h_t) dt \). The initial value \( h_0 \) measures the effect of past consumption on current fe-

licity. It can be interpreted as an inherited standard of living corresponding to consumption experience during youth. An other interpretation is that \( h_0 \) is a reference level corresponding to standard of living of other people.

It is easy to see that if \( b > 0 \) in (2.2), we have intertemporal com-

plementary effect i.e. habit formation and if \( b < 0 \), we have intertemporal substitu-

tion effect i.e. durability. If the consumption is complementary over time it means that a consumer does not like consume less than his living standard amount of consumption. During this paper holds a standard assumption that instantaneous utility adheres to power utility form: \( u(\cdot) = \frac{1}{\gamma} (c - h)^{\gamma} \). In the numerical solutions the habit coefficients \( a \) and \( b \) are assumed to be constant.

I consider so-called linear habit formation i.e. \( u(c_t, t) = v(c - h) \) for \( c \geq h \) and \(-\infty \) for \( c < h \). The first term on the right-hand side of the equation (2.2) is a weighted average of past consumption and gives the proportion of this average that is compared to current consumption to arrive at the level of services today. \( b \) is a scaling parameter which determines how strongly past consumption affects to consumption today. \( a \) is persistence parameter and it determines how fast the effect of previous consumption to the habit term vanishes (Egglezos, 2007). It is easy to see that the standard separable utility function is a special case of this function when \( h_0 = a = b = 0 \). If an agent increases consumption today his current utility increases all future utilities decreases through higher standard of living.
3 Financial Assets

We suppose that there are $m$ non-redundant which dynamics satisfies differential equation securities.

$$dS_{it} = (S_{it})[\alpha_i(S, t)dt + \sigma_i(S, t)dB_t]$$

(3.1)

where $\alpha_i(S, t)$ and $\sigma_i^2(S, t)$ are the instantaneous conditional percentage change price per unit time of the stock $i$ and the instantaneous conditional variance per unit time of the stock $i$. $B_t$ is a standard Brownian motion on a probability space $(\Omega, F, P)$. All uncertainty in the economy is given by realizations of the $m$-dimensional Brownian motion process. The markets are assumed to be complete.

The price of risk (Sharpe ratio) i.e. relative risk process vector, $\lambda_t$, is defined

$$\lambda_t = \frac{\alpha_t - r_t}{\sigma_t}$$

(3.2)

The interest process $r$ and processes $\lambda$ and $\sigma$ are assumed to have continuous paths and to be adapted to the information filtration which has been defined in the previous subsection. Zero-coupon can be defined by

$$\beta_t^s = E_t^{\frac{\zeta_s}{\zeta_t}} = E_t^Q[e^{-\int_t^s r_u du}]$$

(3.3)

where $\zeta_t$ is the unique state-price deflator.

Since the price of consumption can be calculated in terms of the given state-price density, the problem can be reduced to a simple static optimization problem. Harrison and Kreps (1979) have shown using Girsanov’s theorem that state-price density $\zeta^\lambda$ can be defined by

$$\zeta_t = \xi_t e^{-\int_0^t r_s ds}$$

(3.4)

where

$$\xi_t = e^{-\int_0^t \lambda_s dB_s - \frac{1}{2} \int_0^t \|\lambda_s\|^2 ds}$$

(3.5)

and the dynamics of $\xi_t$ process is

$$d\xi_t = -\xi_t \lambda_t dB_t,$$

(3.6)
The density process $\xi_t$ for $Q$ is the martingale defined by

$$\xi_{t,u} = e^{-\int_t^u r_s ds} \frac{\zeta_u}{\zeta_t}, \quad u > t$$  \hspace{1cm} (3.7)

$\xi_t = \frac{dQ}{dP}$ defines the unique equivalent martingale measure. It is possible solve optimal portfolio choice in the habit utility case using different kind of assumption. It is necessary only assume that all uncertainty in economy depends on m-dimensional Brownian motion, markets are complete, and the expanded opportunity set is Markovian.

In the numerical solution of chapter 7, I consider one particular case and assume that the interest rate follows the Cox-Ingersoll-Ross dynamics and the market-price-of-risk process follows mean reverting process. So, the interest rate dynamics adheres to differential equation

$$dr_t = \kappa_r (\bar{r} - r_t) dt + \sigma_r \sqrt{r_t} dB_t$$  \hspace{1cm} (3.8)

and the market-price-of-risk process follows differential equation

$$d\lambda_t = \kappa_\lambda (\bar{\lambda} - \lambda_t) dt + \sigma_\lambda dB_t. \hspace{1cm} (3.9)$$

Wachter (2002) find the closed form solution to the optimal portfolio choice problem for an investor with time separable utilities under mean-reverting returns, but in case with habit utilities closed form solution does not exist.

4 Problem

In the seminal article of consumption/investment decision problem for a single agent, Merton (1971) applies dynamic programming technique to a continuous-time problem. He assumes that an investor’s income is generated by capital gains in assets with prices satisfying the geometric Brownian motion. Merton finds a closed form solution for a case where stock market returns are log-normally distributed and the consumer’s utilities adhere to HARA utilities. Merton (1971) considers "a small investor" which does not
have power to influence on markets. The utilities in the original paper is assumed to be time-separable.

In this paper, I consider an agent whose consumption period is finite and whose instantaneous utilities adhere to power utilities and he does not get utility from bequest. He maximizes utility function

$$E \left[ \int_0^T u(t, c(t) - h(t; c)) \, dt | \mathcal{F}_0 \right],$$

choosing the optimal consumption path the optimal proportion of wealth $w_t$ invested in the $i$th security. In this paper has been assumed that marginal utilities have property $\lim_{c \to h} u'(c-h) = \infty$ i.e. $c_t - h(t, c) > 0, \quad \forall 0 \leq t \leq T$. This assumption presents an addiction pattern. (Detemple and Karatzas (2003) consider non-addictive habits.) If the agent increases his consumption today then the living standard index increases and he has to consume more in the later periods to get same utility level.

The consumer/investor is endowed with some initial wealth $w_0$. He can either consume wealth or invest it in any of $m$ assets. There are $m - 1$ risky stocks and 1 lower risky interest rate with an instantaneous rate of return of $r_t$. The agent invests the proportion $\sum_{i=1}^{m-1} \pi_i(t) = \pi$ of wealth $w_t$ in the $i$th stock ($1 \leq i \leq m - 1$) and remaining proportion $[1 - \sum_{i=1}^{m-1} \pi_i(t) = 1 - \pi]$ in the bond. Merton (1971) has shown that when asset prices are generated by a geometric Brownian motion, we can work with the two-asset case without loss of generality. The pair of investor consumption/investment strategy $c$ and $\pi$ must be based on available information as was formulated in the previous section. I follow Merton and assume that the agent’s income is generated by capital gains on investments in assets and the agent has not got any other income.

The process corresponding to the portfolio/consumption pair $(\pi, c)$ and initial wealth $w_0$ is the solution of the linear stochastic differential equation:

$$dw_t = \pi_t w_t (\alpha_t dt + \sigma_t dB_t) + (1 - \pi_t) r_t dt - c_t dt$$

$$= (rw_t - c_t) dt + w_t \pi_t \sigma_t d\tilde{B}_t,$$

(4.2)
where the second equivalence holds when we change probability measure and use \( \tilde{B}_t = B_t + \int_0^t \lambda_s ds \) (Egglezos(2007)). Then wealth process is admissible if \( w_t(w_0, c, \pi) \geq 0, \forall t \in [0, T] \). The wealth constraint is satisfied when \( E(\int_0^T \zeta(s)c(s)ds) \leq w_0 \) i.e. the current market value of consumption is non-negative and is equal to its initial value \( w_0 \), plus any gains from security trade less the cumulative consumption to date. If a wealth process \( w_t \) is admissible for some trading strategy \((c_t, \pi_t)\), then the strategy is budget-feasible.

5 The Optimal Consumption

Karatzas, Lehoczky and Shreve (1987) and Cox and Huang (1989) derived the method to solve optimal consumption by using a martingale representation technology. If the markets are assumed complete i.e. the number of source of uncertainty equals the number of stocks, \( k = m - 1 \), the dynamic optimization problem becomes simple static problem. Then policy \((c^*, \pi^*)\) is optimal only if the static problem \( \max_c u(\cdot) \), subject to \( E^Q \int_0^T e^{-\int_0^t \tau w dc} dt \leq w_0 \)

Detemple and Zapatero (1992) solve optimal consumption in (4.1) Then

Lagrange function is

\[
L = E\left[\int_0^T u(t, c(t) - h(t; c))dt\right] + y\left[w_0 - E\left(\int_0^T \zeta(s)c(s)ds\right)\right].
\] (5.1)

where \( y \) Lagrange multiplier and Kuhn-Tucker first-order conditions for the optimality of a consumption-rate process \( c(\cdot) \) are

\[
u_c(t, c(t) - h(t, c)) + b_t E_t \left(\int_t^T e^{-\int_s^t \tau w dc} u_h(s, c(s) - h(s; c))ds\right) = \zeta(t)y
\] \quad \forall t \in [0, T],

(5.2)

\[
E\left[\int_0^T \zeta(t)c(t)dt\right] = w_0.
\] (5.3)

Using optimality conditions we formulate an inverse function of marginal utility

\[
I(t, y\phi_t) = c^*(t) - h^*(t, c^*).
\] (5.4)
where
\[ \phi_t = \zeta_t(1 + bE[\int_t^T e^{\int_s^t (-r(v) - b + a)ds} ds]) \]

Equation 5.4 defines recursive linear stochastic equation, which describes relationship between state price density in separable case and state price density \( \hat{\zeta}_t \) in habit case

\[ \hat{\zeta}_t = \zeta_t + bE_t\left(\int_t^T e^{\int_s^t (b(v) - a(v))dv} \zeta_s ds\right) \tag{5.5} \]

where \( bE_t(\int_t^T e^{\int_s^t (b(v) - a(v))dv} \zeta_s ds) \) shows the effect of habit presence to state price density.

In the case of power utilities Detemple and Zapatero (1992) have solved optimal consumption

\[ c(y^*)_t = h_0 e^{-\int_0^t (a-b)du} + (y^*)^{1/\rho-1} [\phi_t^{1/\rho-1} + \int_0^t be^{-\int_s^t (a-b)du} \phi_s^{1/\rho-1} ds] \tag{5.6} \]

where
\[ y = [x - h_0 E \int_0^T e^{\int_0^t (b(v) - a(v))dv} dt]^{\rho-1} \left[ E \int_0^T e^{\int_0^t r_u du} \right]^{1-\rho} \]
\[ \cdot [\phi_t^{1/\rho-1} + \int_0^t be^{-\int_s^t (a-b)du} \phi_s^{1/\rho-1} ds] dt]^{1-\rho} \tag{5.7} \]

\( y \) is Lagrange coefficient. The wealth process is

\[ w(y^*) = E[\int_0^T e^{-\int_0^t r_u du} [h_0 e^{-(a-b)t} + I(t, y\phi(t)) + \int_0^T e^{-(a-b)t} I(t, y\phi(t)) ds] dt] \bigg| \mathcal{F}_0. \tag{5.8} \]

It is not possible to define the precise solution of portfolio choice without numerical method. In the next chapter, I consider numerical method for solving optimal portfolio.

6 The Simulation Method

A large number of research papers have applied Monte Carlo simulation to financial problems, mostly to asset pricing problems (option pricing). There are also some, quite new applications which use Monte Carlo simulation to
solve optimal consumption and investment problem. Detemple et al. (2003) exploit Malliavin calculus and Monte Carlo simulation to solve optimal portfolio choice. Cvitanic et al. (2001, 2003) has developed more straightforward method which use Monte Carlo simulation to solve the volatility of wealth process. Using the volatility can be determined also optimal investment choice. Cvitanic et al. (2003) restricts his analysis only to time separable case, but the problem with habit utilities is possible to solve if Monte Carlo covariation method has been devised somewhat. To use Cvitanic etc al. method it is necessary to accept the assumptions about complete markets and Markovian.

In this chapter and next chapter, I solve optimal portfolio choice in the habit case, when interest rate is assumed to follow Cox-Ingersoll-Ross dynamics and stock prices is assumed to be mean reverting. That is just one example of the use of the method, the flexibility of the method would enable us to apply a lot of different kind of dynamics.

6.1 The Method

Cvitanic etc.(2003) start considering an expression

\[ C_t = E\left[\int_t^T f(r_s, \lambda_s, B_s)ds|\mathcal{F}_t\right]. \]  

(6.1)

where \( r_s, \lambda_s \) and \( B_s \) are as before. \( C_t \) satisfies a stochastic differential equation of the type

\[ dC_t = \varphi_t dt + v_t dB_t \]  

(6.2)

where \( \varphi_t \) is the drift and \( v_t \) is diffusion coefficient. Because the diffusion terms of (6.2) and (4.2) equal, holds

\[ v_t = \pi^*_t \sigma_t \iff \pi^*_t = (\sigma_t)^{-1} v_t \]  

(6.3)

The parameter \( v \) can be obtained from the quadratic variation of the \( C_t \). So, if we can solve the volatility of the wealth process, \( v_t \) by simulation, we can also solve its linear transformation, optimal portfolio choice, \( \pi^* \).
The limit

\[ v_t = \lim_{\Delta t \to 0} E\left[ \frac{(C_{t+\Delta t} - C_t)^2}{\Delta t} \right| \tilde{\mathcal{F}}_t \right), \quad (6.4) \]

is the foundation of approximation and estimate of \( v_t \) can be computed by

\[ \hat{v}_t = \frac{1}{K} \sum_{i=1}^{K} \left[ \frac{(C_{i,t+\Delta t} - C_{i,t})(B_{i,t+\Delta t} - B_{i,t})}{\Delta t} \right] = \frac{1}{K} \sum_{i=1}^{K} \left[ \frac{(w_{i,t+\Delta t} - w_{i,t})z_i^t}{\Delta t} \right], \quad (6.5) \]

where \( z_t \) is standard normal random variable and \( K \) the total number of simulated paths. The covariation between the optimal wealth process and the uncertainty shocks provides expression for the optimal portfolio. I use 2-tier simulation in the sense that I solve the optimal path of consumption in the habit case and then use that path for solving the volatility of wealth process.

### 6.2 Optimal Portfolio

It is possible to use the method of this paper assuming different kind of behavior financial assets. Next, I compute the path of wealth process \((w_t)\) and then use Cvitanic et al. (2003)’s method in the case of intertemporal consumption to solve the volatility of wealth process.

#### 6.2.1 The computation of Lagrange multiplier

At the first step of solution, I numerically solve Lagrange coefficient. To do that equation (5.7) is expressed by

\[ y = [x - h_0 E \int_0^T e^{\int_0^v (b(v)-a(v))dv} dt]^{\rho-1} \left[ E \int_0^T e^{(-\int_0^v r_a du)} \right]^{1-\rho} \]

\[ \ast \left[ (\zeta_\theta \eta_t)^{1/\rho-1} + \int_0^T b e^{-\int_0^v (a-b)du} (\zeta_\theta \eta_t)^{1/\rho-1} ds \right] dt]^{1-\rho} \quad (6.6) \]

where

\[ \eta_t = 1 + bE \left( \int_0^T e^{\int_0^v (-r(v)-b+a)dv} ds \right) \quad (6.7) \]

It is easy to exploit simulation to solve expectations. In the every step, \( \xi_t \) process develops following (like in (3.6)):

\[ \xi_{t+\Delta t}(z^t) - \xi_t = -\xi_t \lambda t z^t, \quad (6.8) \]
where \( z^i \) is pseudo-random number with distribution \( N(0, \Delta t) \). Updated value of \( r_t \) and \( \lambda_t \) are obtained using Euler discretization of (3.8) and (3.9).

### 6.2.2 The Computation of Wealth Process

In the next step, I use algorithm which, at first, calculates the optimal path of consumption (5.6)

\[
c_t = h_0 e^{-\int_0^t (a-b)du} + (y^*)^{1/\rho-1}[(\xi_t\eta_t)^{1/\rho-1} + \int_0^t be^{-\int_s^t (a-b)du}(\xi_s\eta_s)^{1/\rho-1}ds]
\]  

(6.9)

and then the value of wealth process at time \( t + \Delta t \)

\[
w_{t+\Delta t} = E[\int_{t+\Delta t}^T e^{-\int_{t+\Delta t}^u r_u du}w_u \xi_u \eta_u ds]\]

(6.10)

\[
w_{t+\Delta t} = E[\int_{t+\Delta t}^T e^{-\int_t^u r_u du}[h_0e^{-(a-b)t} + I(t, y\eta) + \int_t^u e^{-(a-b)s}I(t, y\eta)ds]du]\]

(6.11)

Using Monte Carlo simulation the numerical values of the wealth process at \( t + \Delta t \) can be solved exactly same way as in Cvitanic et al. (2003). At first an estimate for \( w_{t+\Delta t}(z^i_1) \) is calculated by

\[
w_{t+\Delta t}(z^i_1) = \frac{1}{M} \sum_{j=1}^M \int_{t+\Delta t}^T \xi_{t+\Delta t,s}c^j_s ds.
\]  

(6.12)

In the final stage of simulation, the volatility of wealth process is solved using

\[
\hat{v}_t = \frac{1}{K} \sum_{j=1}^K \frac{(w_{t+\Delta t}(z^i_1) - w_t)(z^i_1)}{\Delta t}.
\]  

(6.13)

Using big enough number of rounds, \( K \) we can obtain reasonable precise values of \( v_t \).

### 7 A Numerical Solution

In this example, I follow Cvitanic et al. (2003) and Detemple et al. (1999) and assume same values of constants than they do: \( \rho = 0, \tau = 0.06, \sigma_r = \ldots \)
0.0364, $\kappa_r = 0.0824$, $\kappa_\theta = 0.6950$, $\bar{\theta} = 0.0871$, $\sigma_\theta = 0.21$, $\sigma_t = 0.2$, $r_0 = 0.06$, $\theta_0 = 0.1$. So called inherited standard of living $h_0$ is set to 0.04.

"Habit parameters" $a$ and $b$ are assumed to be constants. In table (1) is shown optimal portfolio for some values of parameters $a$ and $b$ when time horizon is 1. Table (2) and table (3) express optimal portfolio choice for same value of parameters in the longer time horizons. When we consider the time-separable case and set habit parameters $a$ and $b$ to equal 0, the method gives same values in Cvitanic et al. (2003).

The usual problem with Monte Carlo simulation is computational inefficiency. Cvitanic et al. (2003) use $K = 10000$ and $M = 50$ and obtain standard deviation around 0.002. The algorithm for habit case is slightly more complicated as seen in chapter 6. Using $K = 50000$ and $M = 50$ I get quite similar size standard deviation. Using MATLAB program on standard desktop PCs the computational times are from 8 minutes ($T=1$) to about 1 and half hour ($T=10$) and are not substantially longer than in Cvitanic et al. (2003).

Table 1: Optimal portfolio for different parameters $a$ and $b$ and for different values of risk aversion when time horizon $T=1$.

<table>
<thead>
<tr>
<th>$\pi$</th>
<th>$\gamma=-1$</th>
<th>$\gamma=-2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a=0$ &amp; $b=0$</td>
<td>0.243</td>
<td>0.174</td>
</tr>
<tr>
<td>$a=0.1$ &amp; $b=0.2$</td>
<td>0.209</td>
<td>0.138</td>
</tr>
<tr>
<td>$a=0.1$ &amp; $b=0.3$</td>
<td>0.220</td>
<td>0.153</td>
</tr>
<tr>
<td>$a=0.2$ &amp; $b=0.3$</td>
<td>0.205</td>
<td>0.142</td>
</tr>
<tr>
<td>$a=0.2$ &amp; $b=0.4$</td>
<td>0.215</td>
<td>0.134</td>
</tr>
<tr>
<td>$a=0.4$ &amp; $b=0.5$</td>
<td>0.199</td>
<td>0.161</td>
</tr>
</tbody>
</table>

8 Conclusion

In this paper, the assumption of time-separable utility function has been rejected and the consumer/investor has been assumed to have habit utility.
Table 2: Optimal portfolio for different parameters a and b and for different values of risk aversion when time horizon T=5.

<table>
<thead>
<tr>
<th>π</th>
<th>γ=-1</th>
<th>γ=-2</th>
</tr>
</thead>
<tbody>
<tr>
<td>a=0 &amp; b=0</td>
<td>0.297</td>
<td>0.238</td>
</tr>
<tr>
<td>a=0.1 &amp; b=0.2</td>
<td>0.247</td>
<td>0.199</td>
</tr>
<tr>
<td>a=0.1 &amp; b=0.3</td>
<td>0.262</td>
<td>0.212</td>
</tr>
<tr>
<td>a=0.2 &amp; b=0.3</td>
<td>0.246</td>
<td>0.197</td>
</tr>
<tr>
<td>a=0.2 &amp; b=0.4</td>
<td>0.252</td>
<td>0.190</td>
</tr>
<tr>
<td>a=0.4 &amp; b=0.5</td>
<td>0.240</td>
<td>0.213</td>
</tr>
</tbody>
</table>

Table 3: Optimal portfolio for different parameters a and b and for different values of risk aversion when time horizon T=10.

<table>
<thead>
<tr>
<th>π</th>
<th>γ=-1</th>
<th>γ=-2</th>
</tr>
</thead>
<tbody>
<tr>
<td>a=0 &amp; b=0</td>
<td>0.251</td>
<td>0.174</td>
</tr>
<tr>
<td>a=0.1 &amp; b=0.2</td>
<td>0.209</td>
<td>0.138</td>
</tr>
<tr>
<td>a=0.1 &amp; b=0.3</td>
<td>0.220</td>
<td>0.153</td>
</tr>
<tr>
<td>a=0.2 &amp; b=0.3</td>
<td>0.205</td>
<td>0.142</td>
</tr>
<tr>
<td>a=0.2 &amp; b=0.4</td>
<td>0.215</td>
<td>0.134</td>
</tr>
<tr>
<td>a=0.4 &amp; b=0.5</td>
<td>0.199</td>
<td>0.161</td>
</tr>
</tbody>
</table>

In this paper, Monte Carlo covariation method by Cvitanic at. (2003) has been extended so that it can be used in habit case. I have solved numerically an optimal portfolio allocation of the consumer/investor with habit utilities when interest rates are assumed be stochastic and stock returns are mean-reverting. In that case closed form solution is not possible to find. In the literature, Munk (2008) has solved the problem with more restrictive assumptions about interest rate and stock prices dynamics. His method is slightly computationally more efficient than mine. On the other hand my method is more flexible in sense that it is possible change the assumption about the behavior of financial assets.

Using the method of this paper, it is possible solve optimal portfolio
problem in habit case making different kind of assumptions about financial assets. Only requirements are that markets have to be complete and the expanded opportunity set has to be Markovian.
References


