Numerical and analytical investigations of physics beyond the Standard Model

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ACADEMIC DISSERTATION

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Theory is when you know everything,
but nothing works.

Practise is when everything works,
but no one knows why.

In our lab, theory and practise are combined:
nothing works and no one knows why.

– Unknown
Abstract

This thesis consists of three separate research papers, and an introductory overview concentrating on the physics relevant for these subjects. In the first paper we use classical, real-time simulations to study gravitational wave production in the early Universe. The second paper focuses on extended Technicolor corrections to the Higgs mass. The third paper uses lattice gauge field theory to study the properties of a strongly interacting SU(2) gauge model with fermions in the fundamental representation.

Many cosmological models suggest that an inflatory epoch took place in the early Universe, and it was followed by a "reheating" phase. These events might have emitted gravitational waves, which could be detectable today as a primordial gravitational wave background. Using large-scale numerical simulations we compute the gravitational wave spectrum from a tachyonic preheating transition of a Standard Model like SU(2)-like Higgs system. Comparing non-Abelian and Abelian models to their scalar-only counterparts, we find that adding the non-Abelian gauge field to a scalar-only theory reduces and the Abelian gauge field increases the gravitational wave production, but unfortunately none of these models fall in the detection region of LISA.

All known elementary particles and three of the four fundamental forces of nature are described by the Standard Model (SM). However, it leaves some phenomena unexplained, which leads us believe that the SM must be extended in some way. We focus on Technicolor (TC) theory that generates the masses of the W and Z bosons through the dynamics of the new gauge interactions. Extending the gauge dynamics of TC, effectively represented by four fermion interactions, we can generate the masses of SM fermions, but instead of the 125 GeV scalar, TC typically predicts a scalar state of the order of 1 TeV. We show that by systematically treating these four-fermion interactions, the mass of the scalar resonance can be greatly reduced while remaining in agreement with the experimental constraints of the oblique corrections.

Building a successful extended Technicolor (ETC) model has proven to be difficult, as it is challenged by experimental constraints on flavor-changing neutral currents and precision electroweak measurements. Walking TC models suggest that the gauge couplings must evolve very slowly, i.e walk instead of run, over a certain range of energy scales. Walking behaviour exhibits nearly conformal behavior that is caused by an infrared fixed point (IRFP), but a theory with an IRFP must be studied by lattice field simulations due to its non-perturbative effects. We compute the mass spectrum of SU(2) gauge theory with two, four and six massless Dirac fermions in the fundamental representation of the gauge group. We observe that $N_f = 2$ and
4 behave with the usual pattern of chiral symmetry breaking, whereas our results for $N_f = 6$ indicate the existence of an IRFP.
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List of included papers

This thesis is based on the following publications [1–3]:

I Gravitational waves from non-Abelian gauge fields at a tachyonic transition
Anders Tranberg, Sara Tähtinen and David J. Weir

II Dynamical Origin of the Electroweak Scale and the 125 GeV Scalar
Stefano Di Chiara, Roshan Foadi, Kimmo Tuominen and Sara Tähtinen

III From chiral symmetry breaking to conformality in SU(2) gauge theory.
Alessandro Amato, Viljami Leino, Kari Rummukainen, Kimmo Tuominen and Sara Tähtinen

In all of the papers the authors are listed alphabetically according to particle physics convention.

The author’s contribution

In paper I, the author carried out all of non-Abelian and scalar-only simulations and analysed all the results and made all the plots in the paper. In paper II, the author participated in all analytic loop calculations. In paper III, the author carried out most of the simulations and analysed the results.
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Chapter 1

Introduction

According to the Standard Model of cosmology, the universe had a beginning at a finite time in the past and over the first $10^{-10}$ seconds, it underwent rapid changes. In the very early stages, during inflation, the universe expanded exponentially. Inflation is often associated with at least one scalar field, the inflaton, and all the energy in the universe was stored in its potential energy. When inflation ended, the energy was transferred to particles that were created in the process of reheating. The first stage of reheating is called preheating, and it refers to the era when the inflaton decays into particles very rapidly. Inflation and tachyonic preheating are discussed at the beginning of chapter 2.

The most energetic events in the universe, including processes in the early universe, distort spacetime, i.e. they produce gravitational waves. Some of the gravitational waves could even be observed by today’s detectors and thus they give us a new tool to study events in the early universe that are otherwise invisible. In paper [1] we study the gravitational wave power spectrum from a tachyonic preheating transition with a scalar field coupled to a non-Abelian gauge field. We use classical, real-time simulations, discussed in section 2.2, to study gravitational wave production. In section 2.3 we explain how gravitational waves can be modelled on the lattice and in section 2.4 we summarise the key findings of our paper.

Compared to the energy scales in the early universe, today we live in a very low-energy stage. To study the high temperature era of the universe, one needs to use the Standard Model (SM) of particle physics. It is based on a symmetry group $SU(3)_c \times SU(2)_I \times U(1)_Y$, and the "natural" energy scale of the theory is above 100 GeV. Because the energy scale today is roughly 0.2 meV (the energy of the cosmic microwave background), many of the symmetries of the theory are broken.

In the early universe, the particles in the SM consisted of free massless fermions and massless gauge bosons mediating the interactions between the fermions. The theory also includes a particle called the Higgs boson, which plays a big role in a process called electroweak symmetry breaking (EWSB). After EWSB, the electroweak interaction separates into two interactions: the weak interaction mediated by the three massive bosons and the electromagnetic interaction mediated by one massless boson. This is discussed in section 3.1.
Even though the SM has been validated by many experimental observations, it is not considered a complete theory of fundamental interactions, as it leaves some phenomena unexplained. For example, the SM does not contain any candidate particle for dark matter; it does not account for the nonzero neutrino masses; and it does not explain the baryon asymmetry of the universe. There are many extensions to the SM that try to patch the problems of SM, but we will discuss only one of them, the technicolor (TC) theory. Note that TC has changed a lot since the earliest days, and researchers define it in different ways, but in this thesis TC refers to any new, strongly interacting SU($N$) gauge theory that dynamically breaks the electroweak sector. TC is discussed in section 3.2.

For decades researchers have known, that the original TC must be modified in some way to account for the SM fermion masses. Extended technicolor (ETC) introduces new interactions between the SM fermions and techniquarks providing masses for the fermions, and it is discussed in section 3.3. The early version of TC predicted the Higgs mass to be \( \sim 1 \) TeV, which is off by an order of magnitude. However, in paper [2] we show that by taking into account all the four-fermion interactions between quarks and techniquarks, the Higgs mass can be greatly reduced, while remaining in agreement with experimental constraint. This work is summarised in section 3.4.

In order to make ETC work, the theory must posses 'walking' behaviour which is expected to be found near the conformal window. However, TC is a strongly interacting gauge theory and perturbative calculations quickly reach their predictive limits. To produce reliable results of the non-perturbative effects, one needs to use lattice gauge theory. In paper [3] we compute the mass spectrum of SU(2) gauge theory with two, four and six Dirac fermions in the fundamental representation of the gauge group. Chapter 4 focus on this study. In sections 4.1 and 4.2 we explain the lattice action used for the bosons and fermions, respectively. Then in section 4.3 we discuss hybrid Monte Carlo simulations. Relevant details of the lattice simulations and examples of the measurements are given in the sections 4.4 and 4.5. Finally, in section 4.6 we summarise the results of our paper [3]. Note that in this thesis we use TC to motivate our lattice gauge theory calculations, but these results might also be applicable to dark matter models that are based on composite states of new fermions.

This thesis consists of an overview of the three research papers, each of them investigating different aspects of the early universe. The main components of two of these studies are numerical simulations, and one of them is a theoretical calculation. Due to the limited space, this thesis should not be considered as a comprehensive review of the subjects, but more like an overview of the used methods and the relevant physics behind them. In case the reader is interested in a more detailed analysis, the following reviews are recommended: Ref. [4] for inflation, Ref. [5] for cosmological backgrounds of gravitational waves, Refs. [6, 7] for technicolor models, and Ref. [8] for lattice field theory for beyond the SM dynamics.
Chapter 2

Gravitational waves

The earliest period of the universe is described by a cosmological model called the Big Bang Theory. The term "Big Bang" refers to a period in the early universe when the universe was in a very hot, very dense, and rapidly expanding state. Using general relativity to extrapolate the expansion of the universe backwards in time, we can find a finite time in the past when the universe had an infinite density and temperature. We set the origin of our time coordinate to zero at this time, and it can be considered as the "beginning" of the Big Bang. The times $t > 10^{-10}$ s can be described by general relativity and the Standard Model (SM) of particle physics, but for $t < 10^{-10}$ s the physics is uncertain, and up to speculation.

At the earliest stages of the Big Bang, the general assumption is that the universe was homogeneous and isotropic, expanding and cooling very rapidly with very high energy density, temperature and pressure. Within a fraction of a second, the universe grew exponentially at an accelerating rate. The behaviour of this exponential expansion in the very early universe is described by a theory known as inflation [9, 10]. It is not known for sure whether inflation occurred, but due to its success in making predictions that agree with observations (e.g. the redshifts of galaxies and the discovery of the cosmic microwave background), it has become a part of the standard model for cosmology.

The simplest models of inflation involve only a single scalar field $\sigma$, the inflaton. The inflaton had a large potential energy, and its approximately constant energy density acted as a repulsive force making the distance between any two points in space grow at exponentially large speeds. When the inflaton field reached the end of the slow-roll region of its potential, inflation ended. At the end of inflation the universe was only filled with the inflaton, and it was empty of any other particles.

After inflation, the inflaton field started to oscillate around the minimum of its potential. This mechanism "reheated" the universe and populated the universe with matter and radiation [11,12]. During reheating, all the particles in the universe were created from the decay of the inflaton field, and gradually the elementary particles started to interact with each other, eventually reaching a state of thermal equilibrium.

However, in many models the first stage of reheating, termed "preheating", is considered
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separately. In preheating particles are produced explosively by non-perturbative mechanisms (e.g. parametric resonance). Due to the Pauli Exclusion Principle, fermions cannot be created explosively, and thus preheating creates bosons more effectively than fermions. If the conversion of the energy density to particles happens very rapidly, mostly within a single oscillation, we call it tachyonic preheating [13, 14]. It occurs due to spontaneous symmetry breaking, which is caused by the negative effective mass of the inflaton.

Tachyonic preheating is a strongly nonlinear and nonperturbative process. This complicates studies of this model, as one should go beyond perturbation theory. Fortunately, lattice simulations can be used for this purpose. The bosonic fields in tachyonic preheating can be interpreted as classical waves and their dynamics can be studied by solving the classical equations of motion numerically.

There are few ways to collect observational data on the formation of the early universe shortly after the Big Bang. However, the most violent and energetic processes in the universe produce gravitational waves and the universe should be filled with gravitational waves originating from different sources, including primordial processes during and after inflation (see for instance [5,15–17]). The direct detection of gravitational waves by LIGO [18] and the launch of LISA [19] in 2034, have heightened interest in discovering events of the early universe by studying the spectrum of the gravitational waves. Even though the gravitational waves reaching the earth today are millions of times smaller and less disruptive than they were when they were born, it is well possible that some of the cosmological sources could be within the detectors’ range.

We study gravitational wave production from tachyonic preheating in an SU(2)-Higgs system. In section 2.1 we discuss how to model tachyonic preheating. In section 2.2 we explain the idea behind the classical simulations and in section 2.3 how to compute the gravitational wave spectrum on the lattice. In section 2.4 we discuss the key findings of our study.

2.1 Models of tachyonic preheating

Models of inflation always include some way of ending inflation. In hybrid inflation models [20] an inflaton $\sigma$ is coupled to another scalar $\phi$, often through a quartic portal coupling $\sim g^2 \sigma^2 \phi^2$ or through a trilinear coupling $\sim y \sigma \phi^2$. When $\sigma$ slow-rolls to a certain critical value, the effective mass of $\phi$ becomes negative. Due to this transition, the slow-roll stage ends and inflation stops. Tachyonic transition can arise from variety of settings and the specific triggering mechanism is not relevant for our study, so we will use the potential

$$V(\phi) = V_0 + \mu_{\text{eff}}^2(t) \phi^\dagger \phi + \lambda (\phi^\dagger \phi)^2$$  \hspace{1cm} (2.1)

with a time-dependent effective mass

$$\mu_{\text{eff}}^2(t) = \begin{cases} 
\mu^2 \left(1 - \frac{2t}{\tau_q}\right), & \text{when } 0 \leq t \leq 2\tau_q, \\
-\mu^2, & \text{when } t \geq 2\tau_q,
\end{cases}$$  \hspace{1cm} (2.2)
where the quench time $\tau_q$ tells how fast is the transition from $+\mu^2$ to $-\mu^2$. This is a good approximation for us, as we want to investigate the infrared (IR) properties of the gravitational wave spectrum from tachyonic transition.

Reheating at the end of inflation is typically modelled with one or more scalar fields, and previous simulations [21–24] have shown that gravitational waves are indeed produced in such models, but resulting in a slightly disappointing spectrum in the LISA-detectable region. In hybrid models of inflation, the field $\sigma$ is often assumed to be a gauge singlet, but the field $\phi$ doesn’t need to be. Motivated by the SM we are interested in scalar fields that are coupled to either Abelian $U(1)$ or non-Abelian $SU(2)$ gauge fields. Thus we refer to the field $\phi$ participating in the preheating mechanism as the Higgs field, but our results can be rescaled to higher energies. At the end we will compare the gravitational wave spectrum produced from four different models: $U(1)$-Higgs, $SU(2)$-Higgs, and scalar-only complex Higgs and doublet Higgs models.

The action for the $SU(2)$-scalar model reads

$$S = \int d^4x \left[ -\frac{1}{2} \text{Tr}(F_{\mu\nu}F_{\mu\nu}) + (D_\mu\phi)^\dagger D^\mu\phi - V(\phi) \right],$$

where the potential is defined in eq. (2.1). We use the following definitions

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - ig[A_\mu, A_\nu],$$

$$D_\mu\phi = (\partial_\mu - igA_\mu)\phi,$$

$$A_\mu = \frac{\sigma^a}{2} A^a_\mu,$$

$$\text{Tr} \left[ \frac{\sigma^a}{2} \frac{\sigma^b}{2} \right] = \frac{1}{2} \delta^{ab},$$

where $\sigma^a$ are the Pauli matrices. We set temporal gauge $A_0 = 0$. The scalar field components are

$$\phi = \frac{1}{\sqrt{2}} \begin{pmatrix} \phi_2 + i\phi_3 \\ \phi_0 + i\phi_1 \end{pmatrix}.$$  

The definitions for the reduced models can be found from our paper [1]. If the energy scale $\mu$ is small enough relative to the Planck mass, the expansion of the Universe during the time-scale of the simulation can be ignored.

### 2.2 Classical simulations

In certain problems, the fields behave classically and it is well-justified to use classical, real-time simulations. This means that the time evolution of the system is solved in a straightforward way by using classical equations of motion in a spatial cubic grid with volume $V = L^3$, where $L$ is the spatial lattice size. The lattice points are labeled by integers $n_i \in [0, L - 1]$, and positions are given by $x_i = n_i a_i$, where $a_i = a$ is the spatial lattice spacing. The discretisation on the
lattice is done for operators accordingly:

\[
\phi(x_i) \rightarrow \phi(n_i a),
\]

(2.9)

\[
\frac{\partial \phi}{\partial x_i} \rightarrow \frac{\phi(x + a \hat{\mu}) - \phi(x)}{a},
\]

(2.10)

\[
\frac{\partial^2 \phi}{\partial^2 x_i} \rightarrow \frac{\phi(x + a \hat{\mu}) + \phi(x - a \hat{\mu}) - 2\phi(x)}{a^2},
\]

(2.11)

\[
\int d^3 x \rightarrow \sum_x a^3.
\]

(2.12)

Here, \(x + a \hat{\mu}\) refers a position one lattice site away from \(x\) in the \(i\) direction. The time is evolved in discrete time steps \(t_{j+1} = t_j + \delta t\), \(\delta t\) is the timestep and \(j \in [0, \frac{t_{\text{final}}}{\delta t} - 1]\), and the field values at each time step are updated according to the classical equations of motion.

In classical simulations we use the leapfrog algorithm to numerically integrate the differential equations. It alternates coordinate and momentum updates, such that the fields \(\phi\) and their conjugate momentum \(\pi\) are updated at interleaved time points. First, the momentum is evolved for a half step, followed by the coordinate for a full step, and finally the momentum for another half step

\[
\pi_{j+\frac{1}{2}} = \pi_j + \frac{\delta t}{2} F(\phi_j),
\]

(2.13)

\[
\phi_{j+1} = \phi_j + \delta t \pi_{j+\frac{1}{2}},
\]

(2.14)

\[
\pi_{j+1} = \pi_{j+\frac{1}{2}} + \frac{\delta t}{2} F(\phi_{j+1}),
\]

(2.15)

where \(F(\phi)\) is the equation of motion for the field \(\phi\). The two momentum half steps combine into a full step, and the full momentum steps "leapfrog" over the full coordinate steps. This method is time-reversible; one can return to the starting point by first integrating \(n\) steps forward, then reversing the direction and integrating \(n\) steps backwards. Leapfrog is also a symplectic integrator, which preserves the Hamiltonian structure of the system and makes it suitable for various other numerical methods.

Every physical variable on the lattice depends on the spatial lattice spacing \(a\), that has dimension \([a] = [1/\text{mass}]\). In our simulations we set \(a = 1\), but at the end we need to fix the value of it. We will use \(a \mu = 0.17\), with Higgs mass \(m_H = \sqrt{2} \mu\). The time step \(\delta t\) has to be chosen such that it's small enough for the simulations to be reliable, but big enough that the computational cost is not too high. In our work, a good value for it was found to be \(\delta t = 0.05\), which conserves the energy very precisely, \(E_{\text{final}}/E_{\text{initial}} \sim 99.8\%\).

### 2.2.1 Gauge fields on lattice

The non-Abelian gauge fields on the lattice are defined as elements of the gauge groups \(U_\mu \in \text{SU}(N)\). These elements are called links as they are paths connecting the nearest neighbour points, the site \(x\) to \(x + a \hat{\mu}\), where \(\hat{\mu}\) is a unit vector in the \(\mu\)-direction\(^1\). Links are related to

\(^1\)We use these equations also in chapter 4, so we stick to four dimensional lattice coordinates \(x = (a n_0, a n_i)\) throughout this section.
the group algebra through
\[ U_\mu(x) = e^{-iga_\mu A_\mu(x)}, \]  
(2.16)
where \( A_\mu = \frac{a^a}{2} A^a_\mu \). The smallest closed path on the lattice is defined as a product of gauge links around a square\(^2\)
\[ U_{\mu\nu}(x) = U_\mu(x) U_\nu(x + a_\mu \hat{\mu} U^\dagger_\mu(x + a_\nu \hat{\nu}) U^\dagger_\nu(x), \]  
(2.17)
where \( U^{-1}_\mu = U^\dagger_\mu \) changes the link to opposite direction. The simplest discretisation of the gauge part of eq. (2.3) uses plaquettes, that are defined as a trace of the closed loop
\[ \text{Tr} \ U_{\mu\nu}(x) = \text{Tr} \left( U_\mu(x) U_\nu(x + a_\mu \hat{\mu} U^\dagger_\mu(x + a_\nu \hat{\nu}) U^\dagger_\nu(x) \right) \]  
(2.18)
Plaquette and gauge links are sketched in Fig. 2.1.

With noncommutative elements in eq. (2.18) one must use the Baker-Campbell-Hausdorff formula
\[ e^A e^B = e^{A + B + \frac{1}{2} [A, B] + \frac{1}{12} ([A, [A, B]] + [B, [B, A]]) + \ldots}, \]  
(2.19)
where \( A \) and \( B \) are general non-Abelian matrices. After some algebra we have
\[ U_{\mu\nu} = e^{-iga_\mu a_\nu (\partial_\mu A_\nu(x) - \partial_\nu A_\mu(x) - ig [A_\mu(x), A_\nu(x)] + O(a^3)} \]  
= \[ e^{-iga_\mu a_\nu F_{\mu\nu} + O(a^3)}, \]  
(2.20)
since
\[ A_\nu(x + \hat{\mu}) = A_\nu(x) + a_\mu \partial_\mu A_\nu(x) + \frac{1}{2} a^2_\mu \partial^2_\mu A_\nu(x) + O(a^3). \]  
(2.21)
The plaquette (2.18) becomes
\[ \text{Tr} \ U_{\mu\nu}(x) = N - \frac{g^2 a^2_\mu a^2_\nu (F_\mu^a)^2}{4} + O(a^6), \]  
(2.22)
\(^2\)No summing over indices \( \mu \) and \( \nu \) on right.
and thus
\[(F^a_{\mu\nu})^2 = \frac{4}{g^2 a^2} (N - \text{Tr} \, U_{\mu\nu}(x)) + \mathcal{O}(a^2).\] (2.23)

Now the gauge action in eq. (2.3) can be replaced with the Wilson gauge action [25]
\[S_G = \beta_L \sum_x \sum_{\mu<\nu} \left( 1 - \frac{1}{N} \text{Re} \, \text{Tr} \, U_{\mu\nu}(x) \right) + \mathcal{O}(a^2),\] (2.24)
where we have used
\[\sum_{\mu,\nu} (N - \text{Tr} \, U_{\mu\nu}) = 2 \sum_{\mu<\nu} N - \sum_{\mu<\nu} \text{Tr} \left( U_{\mu\nu}(x) + U^\dagger_{\mu\nu}(x) \right),\] (2.25)
as \text{Tr} \, U_{\mu\mu} = N \text{ and } U_{\nu\mu} = U^\dagger_{\mu\nu}. \text{ The lattice gauge coupling } \beta_L \text{ is defined as}
\[\beta_L = \frac{2N}{g^2}.\] (2.26)

We also need to obtain lattice version of the covariant derivative (2.5). The lattice derivative is defined in eq. (2.10), but the covariant derivative must take into account also the gauge field. It is easy to show that the choice
\[D_\mu \phi = \frac{1}{a_\mu} (U_\mu(x)\phi(x+\mu) - \phi(x))\] (2.27)
in the continuum limit reduces to (2.5) when using
\[U_\mu(x) = 1 - i g a_\mu \frac{\sigma^a}{2} A^a_\mu(x) + \mathcal{O}(a^2)\] (2.28)
and expanding \(\phi(x+\mu)\) in a similar way as in eq. (2.21).

For Abelian gauge fields the field tensor is
\[F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu,\] (2.29)
which changes the lattice action into
\[S_G, \text{ Abelian} = \beta_L \sum_x \sum_{\mu<\nu} (1 - \text{Re} \, \text{Tr} \, U_{\mu\nu}(x)) + \mathcal{O}(a^2),\] (2.30)
where
\[\beta_L = \frac{1}{g^2}.\] (2.31)
The link \(U_\mu \in \text{U}(1)\) becomes
\[U_\mu(x) = 1 - i g a_\mu A_\mu(x) + \mathcal{O}(a^2),\] (2.32)
which is also used in the covariant derivative (2.27). For the scalar-only theory, the covariant derivative reduces to normal lattice derivative (2.10).
2.2.2 Equations of motion

To derive the equations of motion, we follow the standard procedure of varying $\phi$ and $A_j$ as was done in Refs. [26, 27]. We introduce the conjugate momentum variables $\pi^\dagger$ and $P_j^a$ for the fields $\phi$ and $A_j$, respectively. For the Lagrangian density

$$\mathcal{L} = -\frac{1}{2} \text{Tr}(F^{\mu\nu}F_{\mu\nu}) + (D_\mu \phi)^\dagger D^\mu \phi - V(\phi),$$

setting the temporal gauge $A_0 = 0$ yields

$$\text{Tr}(F^{\mu\nu}F_{\mu\nu}) = A_i^a A_j^a + \frac{1}{2} F_{i,j}^a F_{ij}^a,$$

and thus the conjugate momenta are

$$\pi^\dagger = \frac{\partial \mathcal{L}}{\partial \dot{\phi}} = \dot{\phi}^\dagger \quad \text{and} \quad P_j^a = \frac{\partial \mathcal{L}}{\partial A_{j}^a} = -\dot{A}_j^a.$$

The Hamiltonian density on the lattice is

$$\mathcal{H} = \pi^\dagger \pi + \frac{1}{2} P_j^a P_j^a + \frac{1}{g^2 a_t^4} \sum_{i,j} [2 - \text{Tr} U_{ij}(x)] + (D_i \phi)^\dagger (D_i \phi) + V(\phi),$$

where

$$(D_i \phi)^\dagger D_i \phi = \frac{1}{a^2} \left( \phi^\dagger(t, x + a \hat{i}) \phi(t, x + a \hat{i}) + \phi^\dagger(t, x) \phi(t, x) - 2 \text{Re} \left( \phi^\dagger(t, x) U_j(t, x) \phi(t, x + a \hat{i}) \right) \right).$$

The equations of motion are defined as

$$\frac{\partial \mathcal{H}}{\partial \pi} = \frac{\partial \phi}{\partial t}, \quad \frac{\partial \mathcal{H}}{\partial \pi} = -\frac{\partial \pi}{\partial t},$$

$$\frac{\partial \mathcal{H}}{\partial P_j^a} = \frac{\partial A_{j}^a}{\partial t}, \quad \frac{\partial \mathcal{H}}{\partial P_j^a} = -\frac{\partial P_j^a}{\partial t}.$$

It is straightforward to calculate them, and they read

$$\pi(t + \frac{\delta t}{2}, x) = \pi(t - \frac{\delta t}{2}, x) + \delta t \left[ \sum_i \left( -2 \phi^\dagger(t, x) + \phi^\dagger(t, x - \hat{i}) U_i(t, x - \hat{i}) \right) + \phi^\dagger(t, x + \hat{i}) U_i^\dagger(t, x) \right] - \frac{\partial V}{\partial \phi},$$

$$\phi(t + \delta t, x) = \phi(t, x) + \delta t \pi^\dagger(t + \frac{\delta t}{2}, x),$$

$$P_j^a(t + \frac{\delta t}{2}, x) = P_j^a(t - \frac{\delta t}{2}, x) + \delta t \left[ g \text{Re} \left( i \phi^\dagger(t, x + \hat{j}) U_j^\dagger(t, x) \sigma^a \phi(t, x) \right) \right] - \frac{1}{g} \sum_i \text{Tr} \left( i \sigma^a U_i(t, x) U_j(t, x + \hat{j}) U_j^\dagger(t, x + \hat{i}) \right),$$

$$U_j(t + \delta t, x) = \exp \left( -i g a \delta t P_j(t + \frac{\delta t}{2}, x) \right) U_j(t, x).$$
Besides equations of motion we have the Gauss constraint, that implies the lattice gauge symmetry. The idea is, that we do not yet set \( A_0 = 0 \), but instead calculate

\[
\partial_{\mu} \left( \frac{\partial L}{\partial (\partial_{\mu} A_0)} \right) - \frac{\partial L}{\partial A_0} = 0,
\]

and then check that it also holds when \( A_0 \) is set to zero. The constraint we get on the lattice is

\[
G_k(x) = \sum_j \text{Tr} \left[ \sigma^k P_j(t + \frac{\delta t}{2}, x - \hat{j}) P_i(t + \frac{\delta t}{2}, x - \hat{j}) U_j(x - \hat{j}) \right] \]

\[
+ g \text{Re} \left[ i \pi \tilde{\pi} \big( t + \frac{\delta t}{2}, x \big) \sigma^k \phi(x) \right] = 0,
\]

where we have used eqs. (2.41) and (2.43).

### 2.2.3 Initial condition for the fields

In a tachyonic transition, the effective term \( \mu_{\text{eff}}^2 \phi \phi \) changes sign \( \mu_{\text{eff}}^2 = \mu^2 \rightarrow -\mu^2 \). After the transition the low momentum modes \( (|k| \leq \mu) \) of the Higgs field will grow roughly exponentially. This justifies the use of a classical approximation for the observables that are dominated by the low momentum modes. The Higgs field before the transition is set to

\[
\langle \phi_a(k) \phi_b(k) \rangle = \frac{1}{2a^3 \sqrt{\mu^2 + k^2}} \delta_{ab}
\]

\[
\langle \pi_a(k) \pi_b(k) \rangle = \frac{1}{2a^3} \sqrt{\mu^2 + k^2} \delta_{ab},
\]

with \( a \) and \( b \) denoting the scalar degrees of freedom. We only initialise the unstable, low momentum modes. All the gauge fields are initially put to zero \( A_{\mu}^a = 0 \), as well as the gauge field conjugate momenta \( P_{\mu}^a \). These initial conditions are carefully discussed in Ref. [28] and they closely related to values in Ref. [29]. However, our choice for the initial conditions will violate the Gauss law, but the per-site violation is of order \( 10^{-8} \), and thus insignificant.

### 2.3 Gravitational waves on lattice

Gravitational waves are perturbations of the metric. In the weak field approximation a small perturbation \(|h_{\mu\nu}| \ll 1\) is added over flat Minkowski metric

\[
g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}.
\]

GWs propagate following the equation of motion

\[
\ddot{h}_{ij} - \nabla^2 h_{ij} = 16\pi G \Pi_{ij},
\]

where the source of the GWs, \( \Pi_{ij} \), is the transverse and traceless (TT) part of the of the total anisotropic stress-tensor \( T_{ij} \). The terms proportional to \( g_{\mu\nu} \) in \( T_{\mu\nu} \) can be dropped as they do not affect the TT part of \( \Pi_{ij} \), and thus the relevant part of \( T_{ij} \) is

\[
T_{ij} = 2 \text{Re} \left[ (D_i \phi)^\dagger (D_j \phi) \right] + \eta^{\alpha\beta} F_{i\alpha}^a F_{j\beta}^a.
\]
The TT part of the tensor $T_{ij}$ can be easily obtained by using the projection operators $P_{ij}$ in momentum space

$$\Pi_{ij}(k) = \lambda_{ij,lm}(k)T_{lm}(k), \quad (2.51)$$

where

$$\lambda_{ij,lm}(k) = P_{il}(k)P_{jm}(k) - \frac{1}{2}P_{ij}(k)P_{lm}(k), \quad (2.52)$$

$$P_{ij}(k) = \delta_{ij} - k_i k_j |k|^2. \quad (2.53)$$

However, evolving $h_{ij}$ according to eq. (2.49) requires a Fourier transform of $T_{ij}(k)$ at each time step, which essentially reduces the scalability of the simulation. To minimise the computational cost of the simulation, we can instead evolve the unprojected equation of motion in real space \[24\]

$$\ddot{u}_{ij} - \nabla^2 u_{ij} = 16\pi G T_{ij}, \quad (2.54)$$

where $u_{ij}$ is an auxiliary tensor in position space. On the lattice this means that we evolve two tensors, $u_{ij}$ and $\dot{u}_{ij}$, using the leapfrog algorithm according to equations

$$\dot{u}_{ij}(t + \frac{\delta}{2}, x) = \dot{u}_{ij}(t - \frac{\delta}{2}, x) + \delta t (16\pi G T_{ij}(t, x) + \nabla^2 u_{ij}(t, x)), \quad (2.55)$$

$$u_{ij}(t + \delta t, x) = u_{ij}(t, x) + \delta t \dot{u}_{ij}(t + \frac{\delta}{2}, x), \quad (2.56)$$

where $T_{ij}$ is the lattice version of eq. (2.50) and $\nabla^2 u_{ij}$ is obtained by using eq. (2.11). To obtain the metric perturbations $h_{ij}$ from $u_{ij}$, one must first Fourier transform $u_{ij}(t, x) \rightarrow u_{ij}(t, k)$ and then apply the projection operator in the following way:

$$h_{ij}(t, k) = \lambda_{ij,lm}(k)u_{lm}(t, k). \quad (2.57)$$

Thus a Fourier transform is not required at each time step, but only when we wish to recover the spatial metric perturbations $h_{ij}$.

In continuum, the three dimensional Fourier transformation is

$$f(k) = \frac{1}{(2\pi)^3} \int dx \, e^{-ix \cdot k} f(x), \quad (2.58)$$

and on the lattice this turns into

$$f(k) = \frac{a^3}{(2\pi)^3} \sum_x e^{-ix \cdot k} f(x). \quad (2.59)$$

On a finite lattice $0 \leq ax_i < L$, and due to periodic boundary conditions $x_i = x_i + L$, $k$ is restricted to the Brillouin zone $-\pi < ak_{\mu} \leq \pi$, that contains all the unique $k$-vectors. The variables $p_i$ and $x_i$ should have equal number of points, and thus only momenta with $p_i = n_i \frac{2\pi}{L}$, where $n_i$ are integers between $-L/2 + 1$ and $L/2$, are allowed.
The main observables of our simulations are the total energy density and the spectrum of gravitational waves. The total energy density is defined as

$$\rho_{GW}(t) = \frac{1}{32\pi GV} \int d^3x \dot{h}^{ij}(t,x)\dot{h}^{ij}(t,x)$$

(2.60)

which can be calculated using the operator $\lambda_{ij,lm}$ on Fourier transformed $\dot{u}_{lm}$

$$\dot{h}^{ij}(t,k)\dot{h}^{ij}(t,-k) = \lambda_{ij,lm} \lambda_{ij,kn} \dot{u}_{lm}(t,k)\dot{u}_{kn}(t,-k) = \lambda_{lm,kn} \dot{u}_{lm}(t,k)\dot{u}_{kn}(t,-k).$$

(2.62)

The power spectrum is defined as the power per logarithmic frequency interval in gravitational waves, i.e.

$$\frac{d\rho_{GW}(t)}{d \ln k} = \frac{1}{32\pi GV} \frac{d}{d \ln k} \int \frac{dk}{(2\pi)^3} k^2 \int d\Omega \dot{h}^{ij}(t,k)\dot{h}^{ij}(t,-k)$$

(2.63)

where the integral is now over solid angle $\Omega$. This is typically normalised by the vacuum energy $\rho_0 = \lambda v^4/4$, where $v = \mu/\sqrt{\lambda}$.

### 2.4 Gravitational waves from non-Abelian gauge fields

Our main interest is to compute the gravitational wave spectrum from a tachyonic preheating transition of a Standard Model-like SU(2)-Higgs system. Previous simulations with scalar fields only [21–24] have shown that gravitational waves are produced, but their typical frequency today is roughly $10^6 - 10^9$ Hz, which is too high for current detectors (LISA: 0.1 mHz - 1 Hz, LIGO: 10 Hz - 1000 Hz). Also gravitational waves from tachyonic transitions with a scalar field coupled to Abelian gauge fields have been studied in Ref. [29], but non-Abelian fields self-interact strongly, which might change the results. For comparison purposes, we also compute the gravitational wave production for three other models, one with only a complex scalar field; one with a complex scalar field coupled to a U(1) gauge field; and one with only a complex doublet field. In the figures of this section, these are denoted with labels '2', '2+G' and '4', respectively, and the SU(2)-Higgs system is labelled with '4+G'.

In our main model we have a complex scalar doublet $\phi$ (four components), coupled to an SU(2) gauge field $A_\mu$. We will refer to the scalar field as the Higgs field, even though only at energy scales $\mu \approx 100$ GeV can this be identified as the Standard Model Higgs. We use the following parameter choices: the Higgs mass is defined as $m_H = \sqrt{2}\mu$, the Higgs expectation value is $v = \mu/\sqrt{\lambda}$ and the W-mass for the gauge mass is $m_W = gv/2$. We use the Standard Model values $m_H = 125$ GeV, $m_W = 88$ GeV, and $v = 246$ GeV, resulting in couplings $\lambda \simeq 0.13$ and $g \simeq 0.65$. We keep these fixed through most of our study, and only at the end briefly discuss varying $\lambda$. Our lattice spacing is set to $a\mu = 0.17$ and the lattice size is $L^3 = 384^3$. 


2.4 Gravitational waves from non-Abelian gauge fields

Figure 2.2: The energy components of the doublet scalar system (left) and the gauge-doublet system (right). Quench time is $\mu \tau_q = 0$.

Figure 2.3: The total gravitational wave energy density (left) and the final power spectrum (right) for all four models with $\mu \tau_q = 0$.

In Fig. 2.2 we investigate where the energy goes as a function of time, when the quench is instantaneous. Initially, all the energy (besides small fluctuations in the Higgs field) is in the potential $V_0$. With the doublet scalar, the potential energy is very quickly transferred to kinetic and gradient energy. The gauge field will change the situation little, as the Higgs field oscillates more and for longer, and some energy is transferred to the gauge field. Note that no energy is lost because of the quenching process with $\mu \tau_q = 0$.

Next we start comparing the four models. In Fig. 2.3 we plot the total energy density and the final gravitational wave spectrum, multiplied by a factor $M_p^2/\mu^2$ that is scaled back when $\mu$ is fixed, for our four models. The addition of a U(1) gauge field increases the total gravitational wave energy, whereas the SU(2) gauge field suppresses it. Comparing the spectrum of the U(1)-Higgs system to its scalar-only counterpart, one can see that the maximum amplitude is roughly
Gravitational waves

Figure 2.4: The final spectrum for the gauge-doublet system with different quench times (left) and the final value for the total gravitational wave energy density with different quench times and models (right).

doubled, whereas the amplitude of the scalar doublet model does not change when the gauge field is added.

We are also interested in how the quench time affects the results. On the left in Fig. 2.4 we plot the final spectrum of the SU(2)-Higgs model with five different quench times. The peak value decreases with longer quench time, and the peak amplitude for the slowest quench goes down by a factor of 8. Note that the final spectrum is taken at $\mu t_{\text{final}} = \mu \tau_q + 70$, as we have observed that the power spectra converges in shape and magnitude about at time $\mu t \simeq 60$ after the quench. On the right in Fig. 2.4 we show the final value of the total gravitational wave energy density for all the models with different quench times. For all the models, the production of gravitational waves reduces as the quench time is increased, but adding a U(1) gauge field increases and SU(2) gauge field decreases the total gravitational energy density when compared to their scalar-only counterparts.

Finally, we discuss varying $\lambda$. Thus far all of our simulations are done with the SM value for $\lambda = 0.13$, and the gauge boson and Higgs masses are very similar

$$\frac{m_H}{m_W} = \sqrt{\frac{8\lambda}{g^2}} \simeq 1.57. \quad (2.65)$$

Thus, we only see one peak at the spectrum. To disentangle the two scales, we can make $\lambda$ smaller. On the left in Fig. 2.5, we show the spectrum of the gauge-doublet model for four different $\lambda$. As $\lambda$ decreases, the peak resolves into two distinct peaks and the amplitude increases significantly, by one or two orders of magnitude. To be sure of the origin of the second peak we compute all four models with $\lambda = 0.001$. On the right in Fig. 2.5 we see that for scalar-only theories there is no trace of the second peak, and thus the gauge field is the cause of it. Such peak structure would be an ideal target for observations, but unfortunately the Higgs and W-mass scales are far from the observational range of LISA.
Figure 2.5: SU(2)-Higgs spectrum for different $\lambda$ (left) and the final power spectrum for $\lambda = 0.001$ for all four model (right), all with $\mu \tau_q = 0$. Notice the Brillouin zone edge at $k/\mu \approx 18.5$.

Now these results can be redshifted to today’s values. As long as the energy scale $\mu$ is small compared to the Planck mass, we can ignore the expansion of the universe during the time-scale of our simulations. Thus, the frequency today is

$$f = 4 \times 10^{10} \text{ Hz} \left( \frac{k}{\mu} \right) (4\lambda)^{1/4} = 3.4 \times 10^{10} \text{ Hz} \times \frac{k}{\mu},$$

(2.66)

where we used $\lambda = 0.13$. The value for $k/\mu$ can be read from the plots. Similarly, the amplitude of the spectrum is given by

$$\Omega_{gw} h^2 = 9.3 \times 10^{-6} \frac{1}{\rho} \frac{d\rho_{GW}}{d \ln k}.$$  

(2.67)

Once we have fixed the scale $\mu$, the amplitude can be read off from the figures as well.

Our maximum signal occurs at $k/\mu \simeq 1.5$, which corresponds to $5 \times 10^{10}$ Hz. The peak amplitude is roughly at

$$\Omega_{gw} h^2 = 9.3 \times 10^{-6} \left( \frac{\mu}{M_p} \right)^2,$$

(2.68)

which yields $10^{-38}$ for the electroweak and $10^{-12}$ for the GUT scale transition. With smaller $\lambda$, these can be increased by a few orders of magnitude. The presence or the absence of the gauge fields also slightly affects the results. However, overall our findings support the conclusion that tachyonic preheating will not be observable at LISA.
Chapter 3

Beyond the Standard Model

3.1 Introduction to the Standard Model

The Standard Model (SM) of particle physics is a theory of fundamental particles and how they interact. There are two classes of fundamental particles in SM: fermions and bosons. Fermions are the building blocks of matter that obey Fermi-Dirac statistics, and bosons are mediators of interactions following Bose-Einstein statistics. The SM describes three of four known fundamental interactions of the universe: the strong and weak nuclear forces and electromagnetism (EM), leaving aside gravity. Figure 3.1 illustrates the four fundamental forces of nature. Even though the SM is one of the most successful theories in physics, many conceptual and phenomenological observations hint that the SM is not yet complete and new physics exists beyond the SM.

The formulation of the SM is based on a gauge theory possessing

$$SU(3)_c \times SU(2)_I \times U(1)_Y$$

symmetry. The SU(3)$_c$ symmetry group represents quantum chromodynamics (QCD) that describes the strong interaction between colored particles, quarks and gluons. The eight bosonic gluons are the force carriers of QCD, and they interact with themselves and with six types of fermionic quarks named up, down, charm, strange, top and bottom. The SU(2)$_I \times U(1)_Y$ part of the SM gauge group represents the electroweak (EW) interactions, i.e the combination of electromagnetism and weak interactions. The letter $I$ in SU(2)$_I$ stands for the weak isospin and the letter Y in U(1)$_Y$ the weak hypercharge. The force carriers in the electroweak interactions are the three W bosons ($W^1$, $W^2$, $W^3$) for SU(2)$_I$ and $B$ boson for U(1)$_Y$, all of which are massless.

Besides gauge bosons and fermions, SM includes a scalar particle called the Higgs boson. It plays an important role in a mechanism called spontaneous electroweak symmetry breaking (EWSB). In EWSB the symmetry gets broken from SU$_I$(2)$\times$U(1)$_Y$ to U(1)$_{EM}$ and due to this, the masses for the weak gauge bosons and fermions are generated. The Higgs was the last missing piece of the SM until it was discovered at CERN in 2012 [30,31]. To understand EWSB, the SM must first be discussed a bit more carefully.
Figure 3.1: Figure illustrating the possible unification of the four forces of nature. A grand unified theory (GUT) is a hypothetical model at very high energies, where the three interactions, strong, weak and electromagnetic, are fused into one single force. Below the GUT scale, the interactions are separated into strong and electroweak forces possessing SU(3)$_c$ × SU(2)$_I$ × U(1)$_Y$ symmetry. At the EW scale, the electroweak force is spontaneously broken to electromagnetic and weak forces.

Fermions can be divided into two different categories, leptons and quarks, all of which have an anti-particle with opposite quantum numbers. These matter fields can be arranged into three generations with the first generation being the lightest. All stable matter found in nature is composed of the first generation. The fermionic particles have a property called chirality, meaning that the fields can be projected to their left and right handed components $\psi = \psi_L + \psi_R$. The handedness is defined with right- and left- handed projections:

$$\psi_L = P_L \psi = \frac{1 - \gamma_5}{2} \psi, \quad \psi_R = P_R \psi = \frac{1 + \gamma_5}{2} \psi,$$

(3.2)

where $\gamma_5$ anticommutes with $\gamma_\mu$. Chirality is important because the weak interaction couples only to left handed fermions. This means that under weak isospin SU(2) transformations the left-handed particles transform as SU(2) doublets, whereas the right-handed fermions are SU(2) singlets with zero weak isospin number. Thus, for quarks we have

$$q_{iL} = \left( \begin{array}{c} u \\ d \end{array} \right)_{L}, \quad (c, s)_{L}, \quad \left( \begin{array}{c} t \\ b \end{array} \right)_{L},$$

(3.3)

$$u_{iR} = (u_R, c_R, t_R),$$

(3.4)

$$d_{iR} = (d_R, s_R, b_R)$$

(3.5)
3.1 Introduction to the Standard Model

and for leptons
\[
\begin{align*}
\ell^i_L &= \begin{pmatrix} \nu_e \\ e \end{pmatrix}_L, \quad \begin{pmatrix} \nu_\mu \\ \mu \end{pmatrix}_L, \quad \begin{pmatrix} \nu_\tau \\ \tau \end{pmatrix}_L \\
e^i_R &= (e_R, \mu_R, \tau_R).
\end{align*}
\] (3.6)

3.1.1 Lagrangian density of Standard Model

The Lagrangian density for the SM can be written in three parts:

\[
\mathcal{L}_{\text{SM}} = \mathcal{L}_{\text{gauge}} + \mathcal{L}_{\text{kinetic}} + \mathcal{L}_{\text{Higgs}}.
\] (3.8)

The first part \(\mathcal{L}_{\text{gauge}}\) contains kinetic terms of the gauge fields. It can be defined in terms of the field strength tensors of each gauge group

\[
\mathcal{L}_{\text{gauge}} = -\frac{1}{4} B_{\mu\nu} B^{\mu\nu} - \frac{1}{4} \text{Tr}(W_{\mu\nu} W^{\mu\nu}) - \frac{1}{4} \text{Tr}(G_{\mu\nu} G^{\mu\nu}),
\] (3.9)

where the field tensors are

\[
\begin{align*}
B_{\mu\nu} &= \partial_\mu B_\nu - \partial_\nu B_\mu, \\
W^i_{\mu\nu} &= \partial_\mu W^i_\nu - \partial_\nu W^i_\mu + g_2 \epsilon_{ijk} W^j_\mu W^k_\nu, \\
G^a_{\mu\nu} &= \partial_\mu G^a_\nu - \partial_\nu G^a_\mu + g_s f_{abc} G^b_\mu G^c_\nu.
\end{align*}
\] (3.10, 3.11, 3.12)

Here \(\epsilon_{ijk}\) and \(f_{abc}\) are the structure constants of SU(2) and SU(3), respectively.

The fermion kinetic terms in the SM are

\[
\mathcal{L}_{\text{kinetic}} = \bar{q}_L (i \gamma^\mu D_\mu, L) q_L + \bar{u}_R (i \gamma^\mu D_\mu, R) u_R + \bar{d}_R (i \gamma^\mu D_\mu, R) d_R + \bar{l}_L (i \gamma^\mu D_\mu, L) l_L + \bar{e}_R (i \gamma^\mu D_\mu, R) e_R,
\] (3.13)

with covariant derivatives

\[
\begin{align*}
D_\mu, L &= \partial_\mu - i \frac{g_Y}{2} Y B_\mu, \\
D_\mu, R &= \partial_\mu - i \frac{g_2}{2} \sigma^a W^a_\mu, \\
D_\mu, L' &= \partial_\mu - i \frac{g_Y}{2} Y B_\mu - i \frac{g_2}{2} \sigma^i W^i_\mu - i \frac{g_s}{2} \lambda^a G^a_\mu, \\
D_\mu, R' &= \partial_\mu - i \frac{g_Y}{2} Y B_\mu - i \frac{g_s}{2} \lambda^a G^a_\mu.
\end{align*}
\] (3.14, 3.15, 3.16, 3.17)

The hypercharge \(Y\), the Pauli matrices \(\sigma_i, i = 1, 2, 3\), and the Gell-Mann matrices \(\lambda_a, a = 1, \ldots, 8\), are the generators of U(1), SU(2) and SU(3), respectively. The gauge couplings \(g_Y, g_2\) and \(g_s\) determine the strength of gauge interactions. The covariant derivative \(D_\mu\) acts on the fermions that feel only electroweak interaction, whereas \(D'_\mu\) is for the fermions that interact with strong force too. Notice that adding a mass term to the fermions would break the symmetry, and thus SU(3)_c × SU(2)_L × U(1)_Y gauge theory cannot accompany massive particles.
3.1.2 Electroweak symmetry breaking

The Higgs mechanism generates the masses for the gauge bosons and the SM fermions. It was formulated independently by Higgs [32], Brout and Englert [33], and Guralnik, Hagen and Kibble [34]. In the SM, the Higgs field is a SU(2) scalar doublet with field components

$$\phi = \frac{1}{\sqrt{2}} \begin{pmatrix} \phi_2 + i\phi_3 \\ \phi_0 + i\phi_1 \end{pmatrix}. \tag{3.18}$$

The Higgs Lagrangian can be written as

$$\mathcal{L}_{\text{Higgs}} = (D_\mu \phi)^\dagger(D_\mu \phi) - V(\phi) + \mathcal{L}_{\text{Yukawa}} \tag{3.19}$$

where $D_\mu$ is defined in Eq. (3.14). The scalar potential with self interaction and SU(2)$_L$ $\times$ U(1)$_Y$ symmetry is written in a form

$$V(\phi) = \mu^2 \phi_0 \phi^\dagger \phi + \lambda(\phi_1 \phi_2)^2, \tag{3.20}$$
where $\lambda$ must be greater than zero. Depending on the sign of $\mu^2$, the potential has different minima. When $\mu^2 > 0$, the potential is symmetric and the minimum is at zero, so the electroweak symmetry is still intact. However, when $\mu^2 < 0$ potential has two minima at $\text{Re}(\phi) = \pm \sqrt{-\frac{\mu^2}{2\lambda}}$ and the electroweak symmetry is spontaneously broken. Because of this, $\phi$ gets a non-zero vacuum expectation value (VEV). By choosing unitary gauge we can rewrite the Higgs field in terms of the VEV and a new scalar field $h$ with $\langle h \rangle = 0$ as

$$\phi = \frac{1}{\sqrt{2}} \left( \begin{array}{c} 0 \\ v + h(x) \end{array} \right),$$

where $v^2 = -\mu^2/\lambda$. After symmetry breaking the potential becomes

$$V(\phi) = \frac{1}{2} m_h^2 h^2 + \sqrt{\frac{\lambda}{2}} m_h h^3 + \frac{1}{4} \lambda h^4 - \frac{1}{4} v^4 \lambda,$$

where $m_h^2 = -2\mu^2$ is the Higgs boson mass. The covariant derivative of $\phi$ now reads

$$|D_\mu \phi|^2 = \frac{v^2}{8} \left( g_2^2 W_\mu^+ W^{+\mu} + (g_2^2 + g_1^2) Z_\mu Z^\mu \right) + \ldots$$

where the gauge bosons are

$$W^\pm_\mu = \frac{1}{\sqrt{2}} (W^1_\mu \mp W^2_\mu),$$

$$Z_\mu = \cos \theta_w W^3_\mu - \sin \theta_w B_\mu,$$

$$A_\mu = \sin \theta_w W^3_\mu + \cos \theta_w B_\mu,$$

with

$$\cos \theta_w = \frac{g_2}{\sqrt{g_2^2 + g_Y^2}}, \quad \sin \theta_w = \frac{g_Y}{\sqrt{g_2^2 + g_Y^2}}.$$  

Thus, the masses are

$$m_W = \frac{1}{2} g_2 v, \quad m_Z = \frac{1}{2} v \sqrt{g_2^2 + g_Y^2}, \quad \text{and} \quad m_\gamma = 0.$$  

EWSB is illustrated in Fig. 3.2.

Besides breaking the electroweak symmetry and generating masses for the weak gauge bosons, the Higgs field is responsible for generating the fermion masses. The Lagrangian density that describes the interaction between fermions and the Higgs fields yielding fermion masses is

$$\mathcal{L}_{\text{Yukawa}} = -g_{dij} \bar{q}_L^i \phi d_R^j + g_{Uij} \bar{q}_L^i i\sigma_2 \phi^* u_R^j + g_{lj} \bar{l}_L^i \phi e_R^j + \text{h.c.}$$

where $g_{ij}$ are Yukawa couplings with flavor indices $i, j$. These Yukawa couplings must be fixed such that the fermion masses, $m_f = v y_f / \sqrt{2}$, match their measured values.
3.1.3 Problems in Standard Model

Being in very good agreement with experimental data at low energies, the SM is widely accepted as a low energy effective theory valid up to some energy scale $\Lambda$. However, the SM is not a complete model and due to the shortcomings of the SM, there are reasons to believe that new physics appears at higher energy scales. Next, we discuss some limitations of the SM.

In the SM, right-handed neutrinos are absent. This leads to massless neutrinos, as we cannot write Yukawa couplings for them. However, in neutrino oscillation experiments it has been found that neutrinos actually have tiny, but non-zero, masses of order $\sim 1 \text{ eV}$ [35]. Adding right-handed neutrinos to the SM allows us to write Yukawa-type interaction terms for neutrinos as well, but due to the really tiny masses of neutrinos, the Yukawa coupling must be extremely small compared to the couplings of the leptons. This leads to a Yukawa coupling hierarchy problem as the SM does not provide any explanation for the couplings to differ by many orders of magnitude, and thus it is believed that some other model is required to explain the smallness of neutrino masses.

Cosmological measurements, such as the Planck results [36, 37], suggest that there is more dark matter than ordinary matter in the Universe. Many beyond the SM (BSM) models provide good candidates for the dark matter particles, but SM itself does not explain the existence of dark matter. Another cosmological observation that SM fails to predict is the matter-antimatter asymmetry; the Universe is made mostly out of matter rather than antimatter. Some mechanism is needed to explain the commonness of matter over anti-matter.

A theory with a fundamental scalar, such as the Higgs particle in the SM, is problematic as it raises a mass hierarchy problem. This is due to the quadratic divergences, $\delta m_h^2 \sim \Lambda^2$, appearing in the one loop scalar mass corrections, $m^2 = m_{h0}^2 + \delta m_h^2$, where $m_{h0}$ is the bare mass, after the cut-off scale is defined. If the SM would be valid up to the Planck scale, $\Lambda_P \sim 10^{19} \text{ GeV}$, and the Higgs mass is the measured $M_h = 125 \text{ GeV}$, it means that the mass parameter must be highly fine-tuned with $\delta m_h^2/M_h^2 \sim 10^{34}$. Thus the real mass of the Higgs boson can be achieved only by cancelling extremely large radiative corrections to the bare mass.

3.2 Technicolor

3.2.1 Chiral symmetry

Even without the Higgs, QCD contributes to the electroweak symmetry breaking through the chiral condensate. The Lagrangian density for massless fermions reads:

$$\mathcal{L} = \bar{\psi}(i\gamma^\mu D'_\mu)\psi = \bar{\psi}_L(i\gamma^\mu D'_{\mu,L})\psi_L + \bar{\psi}_R(i\gamma^\mu D'_{\mu,R})\psi_R, \quad (3.30)$$

where $\psi_L$ and $\psi_R$ are defined in Eq. (3.2) and $D'_{\mu,L}$ and $D'_{\mu,R}$ in Eqs. (3.16) and (3.17). The chiral symmetry gets broken when the that quark condensate becomes non-zero

$$\langle \bar{q}^a_{RL} q^b_L \rangle = v\delta^{ab} \neq 0. \quad (3.31)$$
However, this mechanism alone cannot explain the physical masses of the gauge bosons. The contribution from chiral symmetry breaking to the W boson mass is

$$m_W = \frac{g_2 F_\pi}{2} \sim 29 \text{ MeV}$$

with pion decay constant $F_\pi \simeq 93 \text{ MeV}$ \cite{38}, whereas the measured value of $m_W$ is $\sim 80 \text{ GeV}$, and thus one typically neglects it.

The idea of the early technicolor (TC) models in 1970s \cite{39, 40} was to introduce a new non-Abelian gauge field, technigauge, and new massless fermions, techniquarks. Similar to the SM, techniquarks have chirality and in the simplest case with two techniquarks this means

$$Q_L = \left( \begin{array}{c} U \\ D \end{array} \right)_L, \quad U_R, \quad \text{and} \quad D_R.$$  \hspace{1cm} (3.33)

The technicolor condensate $\langle Q\bar{Q} \rangle$ then breaks the electroweak symmetry in a similar way to the QCD chiral condensate, but at an energy scale 1000 times higher. The scale of the theory is chosen such that the pion decay constant is of order the electroweak symmetry breaking scale,

$$F_\pi^{TC} = v_{\text{weak}} \simeq 246 \text{ GeV}.$$  \hspace{1cm} (3.34)

This early version of TC raised multiple problems concerning fermion masses and flavor-changing neutral currents, so this naive 'scaled-up QCD' is nowadays discarded. In this thesis, all the models that use a new strong interaction to dynamically break the electroweak sector are called technicolor models. In general, technicolor theories must be SU($N$) gauge theories with $N_f$ techniquark flavors, but the values of $N$ and $N_f$, or the representation of the fermions are not fixed.

### 3.3 Extended and walking technicolor

While TC correctly breaks the electroweak symmetry, and the weak gauge bosons get their physical masses, it fails to generate the SM fermion masses. Removing the Higgs field also removes the Yukawa coupling between the Higgs and the fermions. Extended technicolor (ETC)\footnote{The conventions differ by factors of $\sqrt{2}$.}
models \[41, 42\] introduce new interactions providing mass terms for the fermions. In ETC a gauge boson, with a mass \( M_{ETC} \), couples SM fermions to the techniquarks. ETC gauge bosons become massive after a gauge symmetry gets broken at some high energy scale \( \Lambda_{ETC} \). Well below this energy scale, ETC couplings can be represented schematically by four fermion interactions as sketched in fig. 3.3. The schematic interactions read

\[
g_{ETC}^2 \frac{\Lambda_{ETC}^2}{\Lambda_{TC}^2} \bar{Q}^i Q^i \bar{Q}^j Q^j, \quad g_{ETC}^2 \frac{\Lambda_{ETC}^2}{\Lambda_{TC}^2} \bar{q}^i q^i \bar{q}^j q^j, \quad g_{ETC}^2 \frac{\Lambda_{ETC}^2}{\Lambda_{TC}^2} \bar{q}^i q^i \bar{q}^j q^j, \tag{3.35}
\]

where the indices \( i \) and \( j \) refer to the possibility of having different flavors in the interactions and \( g_{ETC} \) is the ETC coupling constant. The middle term in Eq. (3.35) gives fermion masses

\[
m_q \propto \langle \bar{Q}Q \rangle M_{ETC}^2, \tag{3.36}
\]

and the last term contributes to the unwanted neutral currents. Due to precision electroweak measurements that constrain flavor changing neutral currents, we must have \( M_{ETC} \gtrsim 1000 \Lambda_{EW} \).

On the other hand, in order to generate the weak boson masses correctly, the chiral condensate evaluated at the technicolor scale must be \( \langle \bar{Q}Q \rangle_{TC} \propto \Lambda_{TC}^3 \approx \Lambda_{EW}^3 \), which then leads to fermion masses that are too small. To overcome this dilemma the condensate at ETC has to be enhanced.

To calculate the scale dependence of the \( \langle \bar{Q}Q \rangle \) condensate, we can use the renormalization group equation. First, note that \( m \langle \bar{Q}Q \rangle \) is scale invariant and thus

\[
\frac{d}{d\mu} \left( m \langle \bar{Q}Q \rangle \right) = 0 \tag{3.37}
\]

\[
\Rightarrow \frac{dm}{d\mu} \langle \bar{Q}Q \rangle + m \frac{d\langle \bar{Q}Q \rangle}{d\mu} = 0. \tag{3.38}
\]

Defining the mass anomalous dimension as \( \gamma_m = -\frac{\mu}{m} \frac{dm}{d\mu} \) we get

\[
- \frac{m \gamma_m}{\mu} \langle \bar{Q}Q \rangle + m \frac{d\langle \bar{Q}Q \rangle}{d\mu} = 0, \tag{3.39}
\]

and the solution is

\[
\langle \bar{Q}Q \rangle_{ETC} = \langle \bar{Q}Q \rangle_{TC} \exp \left( \int_{\Lambda_{TC}}^{\Lambda_{ETC}} \frac{d\mu}{\mu} \gamma_m(\mu) \right). \tag{3.40}
\]
3.3 Extended and walking technicolor

Figure 3.5: Conformal window of SU($N$) gauge theory with $N_f$ fermion flavors in different representations. The representations of fermions are: fundamental ('Fund'), two-index antisymmetric ('2A'), two-index symmetric ('2S') and adjoint representations ('Adj').

Theories in which the coupling stays nearly constant over a range of scales are called walking TC [43–47]. For walking theories, one expects that $\gamma_m \approx \text{constant}$ over the range from $\Lambda_{TC}$ to $\Lambda_{ETC}$. Thus

$$\langle \bar{Q}Q \rangle_{ETC} = \langle \bar{Q}Q \rangle_{TC} \left( \frac{\Lambda_{ETC}}{\Lambda_{TC}} \right)^{\gamma_m}.$$  \hfill (3.41)

If $\gamma_m \approx 1$-2 the theory can produce realistic masses for the SM fermions. Schematic behaviour of a walking theory is represented in Fig. 3.4.

For walking theories the $\beta$-function

$$\beta = \mu \frac{dg}{d\mu}$$  \hfill (3.42)

reaches almost zero at some value of the coupling. If at some point $g^2 = g^2_c$, the theory has an infrared fixed point (IRFP) and the long distance behaviour is scale invariant. The area in $(N_c, N_f)$-plane where the theory has an IRFP is called the conformal window [48–51], sketched in Fig. 3.5. The walking behaviour is expected to occur when $N_f$ is just slightly below the conformal window. However, the lower boundary of the conformal window lies at strong coupling and thus analytic calculations are not reliable. Lattice simulations are needed to determine the properties of IRFP theories.

3.3.1 Oblique parameters

There are two ways of collecting evidence of new physics from collider experiments. First, the particles can be observed by producing them in high energy collisions and detecting the particles
directly or via their decay products. However, this can be done only if the production rate is large enough and the detected signal differs enough from the SM background. The second way to observe the new particles is to study their effect on known processes coupling to SM particles. The contributions from new physics to electroweak radiative corrections can be measured with oblique parameters. Often used oblique parameters are the Peskin-Takeuchi parameters called $S$, $T$ and $U$ [52, 53]. Especially the parameters $S$ and $T$ are relevant to the calculations in this thesis and they are defined as

$$S = \frac{16\pi}{g \tilde{g}'} \Pi_{W^3B}(0),$$

$$T = \frac{1}{\alpha M^2_W} \left[ \Pi_{W^3W^3}(0) - \Pi_{W^-W^-}(0) \right],$$

where $\alpha$ is the electromagnetic coupling evaluated at the $Z$ pole and $\Pi$ is the gauge boson vacuum polarization. In TC all sectors contribute to the $S$ parameter, and special attention must be paid in order to account for them. At the same time the parameter $T$ must agree with the measurements.

### 3.4 125 GeV scalar

TC models assume that the Higgs boson is a bound state of techniquarks, and after fixing the gauge boson and fermion masses, the expected mass $m_H$ of the composite Higgs is $\sim 1$ TeV. However, the discovery of the Higgs boson at the Large Hadron Collider (LHC) observed a far lighter particle, $m_H \simeq 125$ GeV. It is clear that the TC prediction was wrong, but on the other hand, the estimate for the mass of the Higgs in TC was result of a simple scaling argument, where the new strong dynamics was isolated from the SM fields. Even though the observed Higgs particle was not predicted by TC, such a particle can still be accommodated. In Ref. [2], we show that systematically treating the four-fermion interactions in the extended gauge sector leads to a large reduction of the mass of the scalar resonance. In addition, we show that the TC models generating a 125 GeV scalar are in agreement with experimental constraints. Additional new resonances appear at a few TeV and they might be detectable by future experiments.

The need for extended TC sectors to account for the generation of fermion masses has been known for a long time, but their effect on the loop corrections to the scalar mass is less studied. A fully dynamical model setup of simple ETC with four-fermion interactions was used in Ref. [54]. With a relatively simple model they were able to show a large reduction of the scalar mass as a result of the new strong dynamics. However, the idea of the paper was simply to illustrate the effect, and it cannot be considered as a realistic explanation for the SM masses. We continue the research in our paper [2], but we also take into account the generation of fermion masses and address the model’s viability by calculating the oblique parameters.

In our model we have one colorless weak technidoublet, $Q = (U, D)$, in the $SU(N)$ TC gauge group. To avoid the topological Witten anomaly, $N$ must be an even number, and because the
spinorial representation is not complex, we must have \( N = 4, 6, 8, \ldots \). The hypercharges are

\[
Y_{Q_L} = 0, \quad Y_{U_R} = \frac{1}{2}, \quad \text{and} \quad Y_{D_R} = -\frac{1}{2}.
\]  

(3.45)

We use a chiral fermion model, similar to the Nambu-Jona-Lasinio model (NJL), with a whole set of four-fermion operators. Lagrangian density reads

\[
\mathcal{L} = \mathcal{L}_{SM} + \mathcal{L}_{TC} + \mathcal{L}_{ETC},
\]

(3.46)

where \( \mathcal{L}_{SM} \) is the SM Lagrangian without the Higgs doublet. The second term refers to the non-pertubative TC dynamics containing both composite resonances and the techniquarks. The lightest resonances are a massless technipion triplet \( \Pi^i \), which becomes the longitudinal components of the W and Z bosons, and a scalar singlet, identified as the Higgs particle. We have

\[
\mathcal{L}_{TC} = \overline{Q}_L i \gamma \mu Q_L + U_R i \gamma \mu U_R + D_R i \gamma \mu D_R - M_Q \left( 1 + \frac{y}{M_Q} H + \cdots \right) \left( \overline{Q}_L \Sigma Q_R + \overline{Q}_R \Sigma^\dagger Q_L \right) - \frac{M^2}{2} H^2 + \cdots,
\]

(3.47)

where the covariant derivatives are with respect to electroweak gauge fields and the ellipses refer to higher-order terms in \( H \) which have negligible effect on our calculations. The field \( \Sigma \) is

\[
\Sigma \equiv \exp \frac{2 i \Pi^i \tau^i}{v},
\]

(3.48)

where \( \tau^i \) are the Pauli matrices and \( v \) is the vacuum expectation value.

The last term in Eq. (3.46) contain four-fermion operators that are obtained after heavy ETC gauge bosons are integrated out. Using the Fierz rearrangement formulas we can write

\[
\mathcal{L}_{ETC} = 2 G_{QqU} \left[ \left( \overline{Q}_L U^c \right) \left( \overline{T}_R Q^c \right) + \left( \overline{Q}_L Q^c \right) \left( \overline{U}_R Q^c \right) \right] \\
+ 2 G_{QqDb} \left[ \left( \overline{Q}_L D^c \right) \left( \overline{B}_R Q^c \right) + \left( \overline{Q}_L Q^c \right) \left( \overline{D}_R Q^c \right) \right] \\
+ 2 G_{QQUU} \left( \overline{Q}_L U^c \right) \left( \overline{U}_R Q^c \right) + 2 G_{QQDD} \left( \overline{Q}_L D^c \right) \left( \overline{D}_R Q^c \right) \\
+ 2 G_{qqtt} \left( \overline{q}_L t^c \right) \left( \overline{T}_R Q^c \right) + 2 G_{qqbb} \left( \overline{q}_L b^c \right) \left( \overline{b}_R Q^c \right) + \Delta \mathcal{L}_{ETC},
\]

(3.49)

where the couplings are defined as

\[
G_{QqU} \equiv \frac{g_{QqU}}{M^2}, \quad G_{QqDb} \equiv \frac{g_{QqDb}}{M^2}, \quad G_{QQUU} \equiv \frac{g_{QQUU}}{N_M^2}, \\
G_{QQDD} \equiv \frac{g_{QQDD}}{N_M^2}, \quad G_{qqtt} \equiv \frac{g_{qqtt}}{N_c M^2}, \quad G_{qqbb} \equiv \frac{g_{qqbb}}{N_c M^2}
\]

(3.50)

with the ETC scale \( M \). The contribution \( \Delta \mathcal{L}_{ETC} \) is more complicated and it doesn’t contribute to scalar or fermion masses. We get

\[
\Delta \mathcal{L}_{ETC} = -\frac{1}{2} \frac{g^2_{QQ}}{M^2} \left( \overline{Q}_L \gamma_\mu Q_L \right)^2 - \frac{1}{2} \frac{g^2_{Qq}}{M^2} \left( \overline{q}_L \gamma_\mu q_L \right)^2 - \frac{1}{2} \frac{g^2_{UU}}{M^2} \left( \overline{U}_R \gamma_\mu U_R \right)^2 - \frac{1}{2} \frac{g^2_{DD}}{M^2} \left( \overline{D}_R \gamma_\mu D_R \right)^2 \\
- \frac{1}{2} \frac{g^2_{tt}}{M^2} \left( \overline{T}_R \gamma_\mu t_R \right)^2 - \frac{1}{2} \frac{g^2_{bb}}{M^2} \left( \overline{b}_R \gamma_\mu b_R \right)^2 - \frac{g_{QQUU} + g_{QQDD} g_{qqtt} + g_{qqbb} g_{qqtt}/M^2}{N_c M^2} \left( \overline{Q}_L \gamma_\mu Q_L \right) \left( \overline{q}_L \gamma_\mu q_L \right).
\]
\[ -\frac{g_{QQ}g_{tt}}{M^2} \left( \overline{q}_{L} \gamma_{\mu} q_{L} \right) \left( \overline{t}_{R} \gamma^{\mu} t_{R} \right) - \frac{g_{QQ}g_{bb}}{M^2} \left( \overline{q}_{L} \gamma_{\mu} q_{L} \right) \left( \overline{b}_{R} \gamma^{\mu} b_{R} \right) \\
- \frac{g_{QQ}g_{UL}}{M^2} \left( \overline{q}_{L} \gamma_{\mu} q_{L} \right) \left( \overline{U}_{R} \gamma^{\mu} U_{R} \right) - \frac{g_{qU}g_{DD}}{M^2} \left( \overline{q}_{L} \gamma_{\mu} q_{L} \right) \left( \overline{D}_{R} \gamma^{\mu} D_{R} \right) \\
- \frac{g_{U}g_{DD}}{M^2} \left( \overline{U}_{R} \gamma_{\mu} U_{R} \right) \left( \overline{D}_{R} \gamma^{\mu} D_{R} \right) - \frac{g_{UU}g_{tb} + g_{tb}^2}{M^2} \left( \overline{U}_{R} \gamma_{\mu} U_{R} \right) \left( \overline{t}_{R} \gamma^{\mu} t_{R} \right) \\
- \frac{g_{DD}g_{tb} + g_{tb}^2}{M^2} \left( \overline{D}_{R} \gamma_{\mu} D_{R} \right) \left( \overline{t}_{R} \gamma^{\mu} t_{R} \right) - \frac{g_{U}g_{bb} + g_{tb}^2}{M^2} \left( \overline{U}_{R} \gamma_{\mu} U_{R} \right) \left( \overline{b}_{R} \gamma^{\mu} b_{R} \right) \\
- \frac{g_{D}^2}{M^2} \left( \overline{D}_{R} \gamma_{\mu} D_{R} \right) \left( \overline{D}_{R} \gamma^{\mu} D_{R} \right) - \frac{g_{D}^2}{M^2} \left( \overline{U}_{R} \gamma_{\mu} U_{R} \right) \left( \overline{b}_{R} \gamma^{\mu} b_{R} \right) \\
- \frac{g_{D}^2}{M^2} \left( \overline{U}_{R} \gamma_{\mu} U_{R} \right) \left( \overline{D}_{R} \gamma^{\mu} D_{R} \right) - \frac{g_{D}^2}{M^2} \left( \overline{U}_{R} \gamma_{\mu} U_{R} \right) \left( \overline{t}_{R} \gamma^{\mu} t_{R} \right) \\
- \frac{2g_{D}^2}{M^2} \left( \overline{Q}_{L} \gamma_{\mu} T_{u}^i q_{L} \right) \left( \overline{q}_{L} \gamma^{\mu} t_{QCD} q_{L} \right) + \frac{4g_{QQ}g_{UU}}{M^2} \left( \overline{Q}_{L} T_{\text{TC}}^A q_{L} \right) \left( \overline{U}_{R} T_{\text{TC}}^B q_{L} \right) \\
+ \frac{4g_{QQ}g_{DD}}{M^2} \left( \overline{Q}_{L} T_{\text{TC}}^A q_{L} \right) \left( \overline{D}_{R} T_{\text{TC}}^B q_{L} \right) + \frac{4g_{qq}g_{bb}}{M^2} \left( \overline{q}_{L} T_{\text{TC}}^A q_{L} \right) \left( \overline{b}_{R} T_{\text{TC}}^B q_{L} \right) \right). \tag{3.51}
\]

\[ T_{\text{TC}}^A \] are the TC generators in SU(N) and \[ T_{\text{QCD}}^A \] are the generators for the fundamental representation of SU(Nc), respectively. The generators are normalised as follows:

\[ \text{Tr} T_{i}^{\dagger} T_{j} = \frac{1}{2} \delta_{ij}, \quad \text{Tr} T_{QCD}^{a} T_{QCD}^{b} = \frac{1}{2} \delta^{ab}, \quad \text{Tr} T_{\text{TC}}^{A} T_{\text{TC}}^{B} = \frac{1}{2} \delta^{AB}. \tag{3.52} \]

It is evident from the Lagrangian that this model is complicated. Thus, we need to make approximations in order to calculate the loop corrections. We compute the observables in the large \( N \) limit, with \( N/N_c \) finite. The couplings \( G_{QQ} \) and \( G_{qq} \) in Eq. (3.49) scale like \( 1/N \), whereas \( G_{QQXX} \) and \( G_{qqXX} \) scale like \( 1/N^2 \) due to the extra factor of \( 1/N \) arising from the Fierz rearrangement. Thus \( G_{QQ} \) and \( G_{qq} \) contribute at leading order (LO) in the mass terms, and \( G_{QQXX} \) and \( G_{qqXX} \) at next-to-leading order (NLO), whereas the terms in \( \Delta \mathcal{L}_{\text{ETC}} \) do not contribute to the mass terms at LO or at NLO. The NLO computations are complicated in this model, but it is important to account for the contribution from \( G_{QQXX} \) and \( G_{qqXX} \) to the masses. Thus, we treat these couplings as quantities scaling like \( 1/N \) and compute them at LO in the large-\( N \) expansion.

There are two possible cutoffs in this model, \( \Lambda \) from TC dynamics and the ETC scale \( \mathcal{M} \). It is not clear which cutoff we should use, because we do not want to lose information from the dynamics occurring between \( \Lambda \) and \( \mathcal{M} \). Thus we use the following rules: employ the cutoff \( \Lambda \) in integrals over techniquarks, and \( \mathcal{M} \) in integrals over ordinary quarks. Also, we can retain only the logarithmically divergent part of the interaction vertices and evaluate the integrals at zero momenta. With these assumptions we can proceed to calculating the masses.

The diagrams contributing to the fermion masses in the large-\( N \) expansion is given in Fig.
3.6. The coupled equations are

\[
M_U = M_Q + 4NG_{QQU}MU_IU + 4N_cG_{qqU}MU_IU + 4NG_{QqU}M_MU_IU
\]
\[
M_t = 4NG_{QqU}MU_IU + 4N_cG_{qqU}M_MU_IU
\]
\[
M_D = M_Q + 4NG_{QQDD}MD_ID + 4N_cG_{qqD}MD_ID + 4NG_{QqD}MD_ID
\]
\[
M_b = 4NG_{QD}MD_ID + 4N_cG_{qqb}MD_ID + 4NG_{QqD}MD_ID
\]

(3.53)

and

\[
M_D = M_Q + 4NG_{QQDD}MD_ID + 4N_cG_{qqD}MD_ID + 4NG_{QqD}MD_ID
\]
\[
M_b = 4NG_{QD}MD_ID + 4N_cG_{qqb}MD_ID + 4NG_{QqD}MD_ID
\]

(3.54)

where

\[
I_X = i \int \frac{d^4k}{(2\pi)^4} \frac{1}{k^2 - M_X^2}.
\]

(3.55)

In a similar manner we can compute the weak boson masses and the mass of the TC Higgs.

The effective Yukawa vertices at leading order in the large \( N \) approximation read

\[
\mathcal{L}_{Yukawa} = -y_U \overline{U} U H - y_D \overline{D} D H - y_t \overline{t} t H - y_b \overline{b} b H.
\]

(3.56)

Calculating the loop corrections to these vertices gives us \( y_U, y_t, y_D \) and \( y_b \) with respect to the couplings in Eq. (3.50), the cutoffs \( \Lambda \) and \( \mathcal{M} \), and \( N \). In a similar way, we can calculate the couplings to weak bosons by introducing the effective Lagrangian

\[
\mathcal{L}_{HWW} = 2M_W^2 \left( \sqrt{2G_F} \right)^{1/2} a_W H W^+ W^- + M_Z^2 \left( \sqrt{2G_F} \right)^{1/2} a_Z H Z_\mu Z^\mu,
\]

(3.57)

and then use loop calculations to calculate the couplings \( a_W \) and \( a_Z \). In the numerical part of the paper, we will show that all of these couplings are close to their SM values.
Next we proceed to calculate the loop corrections to the oblique parameters, defined in equations (3.43) and (3.44). These are rather complicated, and they are given in Appendix D of Ref. [2].

The parameters of the model are at first \( g, g', y, M_Q, \Lambda, \mathcal{M} \), \( g_{QQ}, g_{qq}, g_{Qq}, g_{UU}, g_{DD}, g_{UD}, g_{tt}, g_{bb}, g_{tb}, g_{Ut}, g_{Db}, g_{Ut}, g_{Ub}, g_{Db}, \) and \( N \). The Yukawa coupling \( y \) disappears after renormalisation, \( M \) can be traded with the dynamical mass \( M_{HO} \) and the terms \( g, g', M_Q, M_{HO}, g_{Ut} \) and \( g_{Db} \) can be replaced with experimental values of \( \alpha, G_F, M_Z, M_H, M_t, \) and \( M_b, \) respectively. Thus, our free parameters are actually \( \Lambda, \mathcal{M} \), \( g_{QQ}, g_{qq}, g_{Qq}, g_{UU}, g_{DD}, g_{UD}, g_{tt}, g_{bb}, g_{tb}, g_{Dt}, g_{Ub}, \) and \( N \). The parameter \( \Lambda \) can be estimated by scaling up the corresponding quantity in QCD, giving

\[
\Lambda \simeq \begin{cases} 
2.7 \text{ TeV} & N = 4 \\
2.2 \text{ TeV} & N = 6 
\end{cases}
\] (3.58)

The ETC scale \( \mathcal{M} \) remains as a free parameter, and we perform a random scan over the couplings \( g_{QQ}, g_{qq}, g_{Qq}, g_{UU}, g_{DD}, g_{UD}, g_{tt}, g_{bb}, g_{tb}, g_{Dt}, g_{Ub} \). These couplings are restricted by observations, so the couplings have different bounds that need to be taken into account. We are interested in knowing how different couplings change the \( S \) and \( T \) parameters, and with what kind of couplings \( S \) and \( T \) can be within the experimental bounds of the measured values. Thus, for each generated point we evaluate the oblique parameters as well as the masses and the coupling to the Higgs particle. We notice that the Higgs couplings are within \( \sim 10\% \) of the required SM values.

In Fig. 3.7 we plot our results for \( N = 4 \) in the \((S,T)\)-plane (in Ref. [2] this is also shown for \( N = 6 \)). The color coding is related to different couplings: the bigger the value of \( |g_{Db}| \) the greener the point, and the bigger the value of \( |g_{Ub}| \) the redder the point, and the larger the value of \( |g_{Qq}| \) the darker the shade of the color is. For small \( \mathcal{M} \) the green points are disfavoured as the magnitude of \( T \) becomes too large. The preferred points are thus the orange ones, where \( |g_{Dt}| \approx |g_{Ub}| \). On the other hand, with the large value of \( \mathcal{M} \) it is harder to reduce the Higgs mass sufficiently as \( g_{QQ}g_{UU} \) and \( g_{QQ}g_{DD} \) get too large. Overall, to get our model to agree with experiment requires some fine-tuning as the magnitude of \( T \) varies a lot.
Figure 3.7: Result for the random scan of the model parameter space for $N = 4$ and $\Lambda = 2.7$ TeV. The four figures correspond to $M = 2.7, 3.2, 3.7$ and 5 TeV and in all points the Higgs mass is 125 GeV. The plot from [2].
Chapter 4

Lattice field theory

To simulate strongly interacting gauge theories, one must use lattice field theory. Lattice field simulations are approximations of continuum gauge field theories, in which the quantum field theory equations are described in discretized (Euclidean) space-time. The lattice points are labeled by integers $n_\mu \in \mathbb{Z}$, and the space-time positions are given by $x = (a_t n_0, n_i a)$, where $a_t$ and $a$ are the temporal and spatial lattice spacings, respectively. In principle, numerical calculations can produce results with arbitrary precision, but in practice certain errors are always present in the results as the numerical cost in several areas of the simulations quickly becomes prohibitive with today’s computers. However, with the lattice simulations all information, including non-perturbative effects, can be sampled, and this makes them a very efficient tool to use in various studies.

Lattice field theory is formulated using the Feynman path integral. The partition function for a gauge field $A$ and a fermionic field $\psi$ is defined as

$$Z = \int \mathcal{D}A \mathcal{D}\bar{\psi} \mathcal{D}\psi e^{iS(A,\psi,\bar{\psi})}. \quad (4.1)$$

The action $S$ is the classical action, defined as an integral over the space-time volume of the Lagrangian density $\mathcal{L}$

$$S(A, \psi, \bar{\psi}) = \int d^4x \mathcal{L}(A, \psi, \bar{\psi}). \quad (4.2)$$

The Lagrangian density for a SU($N$) gauge theory with $N_f$ massive fermions $\psi_f$ in continuum is written as

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu}^a F^{a,\mu\nu} + \sum_{f=1}^{N_f} \bar{\psi}_f (i \slashed{D} - m_f) \psi_f, \quad (4.3)$$

where $\slashed{D} = \gamma^\mu D_\mu$. From this on, we drop off the subscript $f$ and the explicit sum over fermion flavors.

The continuum path integral formalism given in eq. (4.1) uses the Minkowski metric, but on the lattice this leads to a highly oscillatory imaginary exponent which complicates the numerical simulation unnecessarily. We change into into the Euclidean metric by Wick rotating the time
axis. The time axis gets rotated from the real to the imaginary axis $t \rightarrow i\tau$. This means that

$$x^\mu x_\mu = (x_0)^2 - |\vec{x}|^2 \Rightarrow -x_0^2 - |\vec{x}|^2 = -|x_E|^2,$$

(4.4)

$$\mathcal{L}_E(A, \psi, \bar{\psi}) = -\frac{1}{4} F_{\mu \nu}^a F_{\mu \nu}^a + \bar{\psi} (i\slashed{D} + m) \psi,$$

(4.5)

$$S_E = -iS(t \rightarrow i\tau) = \int d\tau d^3x \mathcal{L}_E(A, \psi, \bar{\psi}),$$

(4.6)

which leads to

$$Z = \int DA D\bar{\psi} D\psi e^{-S_E(A, \psi, \bar{\psi})}.$$

(4.7)

From now on we drop off the subscript $E$. In the chiral limit the fermion mass disappears and the fermion action becomes $S_F = \bar{\psi} i\slashed{D} \psi$. Thus the chiral symmetry relation on lattice is

$$\bar{\psi} \gamma_5 + \gamma_5 \slashed{D} = 0.$$

(4.8)

To calculate the expectation value of an operator $O$, one must compute the integral

$$\langle O \rangle = \frac{1}{Z} \int DA D\bar{\psi} D\psi O(A, \bar{\psi}, \psi) e^{-S(A, \bar{\psi}, \psi)},$$

(4.9)

where $Z$ is defined in eq. (4.7). However, calculating this multidimensional integral is a massive numerical challenge. Thus the aim of numerical sampling methods is to select a sufficiently complete subset of points such that the ratio of these two integrals is estimated as well as possible. The sample points are called field configurations and each configuration designates the value of the field on all lattice points. We will use the Hybrid Monte Carlo method to generate the field configurations for our model. This method will be introduced in section 4.3. Before this we need to go through the full lattice action. It is often convenient to separate the bosonic and fermionic actions as follows

$$S = S_G + S_F.$$

(4.10)

In section 4.1 we go through the action for the gauge fields and in section 4.2 we will discuss the fermionic action.

### 4.1 Discretized gauge action

The lattice gauge part of the action is defined by using the Wilson gauge action in a similar way to chapter 2. We have

$$S_G(U) = \beta_L \sum_x \sum_{\mu < \nu} \left( 1 - \frac{1}{N} \text{Re Tr } U_{\mu \nu}(x) \right),$$

(4.11)

where $U_{\mu \nu}(x)$ is the plaquette defined already in eq. 2.18 and

$$\beta_L = \frac{2N}{g_0^2}.$$

(4.12)
4.2 Fermions on the lattice

This action takes into account only the smallest loops on lattice, the plaquettes (see Fig. 2.18), and it is called the unsmeared gauge action. We are mostly interested in theories with large bare coupling $g_0$, but the unsmeared gauge action has an unphysical bulk phase transition at strong couplings which limits our choice for the coupling. To reduce the discretization errors and in order to be able to use larger couplings on the lattice we need to add another term, a smeared gauge action. One of the most common smeared actions is the hypercubic truncated stout smearing (HEX smearing) [55]. With this action, the gauge part of eq. (4.10) now reads

$$S_G = (1 - c_s)S_G(U) + c_sS_G(V),$$  \hspace{1cm} (4.13)

where $U$ are the unsmeared and $V$ the smeared gauge links, and $c_s = 0.5$ is a mixing parameter. The smeared links are calculated in three sequential stout smearing steps

$$V_{x,y} = P \left( \sum_{\nu \neq \mu} \frac{\alpha_1}{6} \tilde{V}_{x,\nu,\mu} \tilde{V}_{x+\nu,\mu,\nu} U^\dagger_{x,\nu,\mu} \right) U_{x,\mu},$$  \hspace{1cm} (4.14)

$$\tilde{V}_{x,\nu,\mu} = P \left( \sum_{\nu \neq \mu, \nu} \frac{\alpha_2}{4} \tilde{V}_{x,\nu,\mu,\nu} \tilde{V}_{x+\nu,\mu,\nu} U^\dagger_{x,\nu,\mu,\nu} \right) U_{x,\mu},$$  \hspace{1cm} (4.15)

$$\tilde{V}_{x,\rho,\nu,\mu} = P \left( \sum_{\nu \neq \mu, \nu, \rho} U_{x,\rho} \tilde{U}_{x+\rho,\mu,\nu} U^\dagger_{x,\rho,\mu,\nu} \right) U_{x,\mu},$$  \hspace{1cm} (4.16)

where

$$P(U) = \exp \left[ U - U^\dagger - \frac{1}{2} \text{Tr}(U - U^\dagger) \right].$$  \hspace{1cm} (4.17)

The HEX smeared coefficients $\alpha_i$ for SU(2) are measured in [56] and they are $\alpha_1 = 0.78$, $\alpha_2 = 0.61$ and $\alpha_3 = 0.35$. Another common smearing methods are the stout smearing [57] and the HYP smearing [58].

4.2 Fermions on the lattice

Fermions obey Fermi-Dirac statistics, and due to the Pauli exclusion principle they must be expressed as anticommuting Grassmann variables. Grassmann variables are not numbers, so one cannot simulate them directly, but luckily they can be integrated out of the action. For Grassmann numbers it holds that

$$\int D\bar{\psi} D\psi e^{-\bar{\psi}D[U]\psi} = \det D[U],$$  \hspace{1cm} (4.18)

where $D[U]$ is some discretization of the Dirac operator. The path integral is now

$$Z = \int DUE^{-S_G} \det D[U],$$  \hspace{1cm} (4.19)

where $S_G$ is the gauge action (4.13). However, calculating the determinant of a four-dimensional matrix is very time-consuming and, as the determinant must be computed frequently, overly
expensive. The standard way to deal with this problem is to introduce a set of scalar \( \chi \) fields \( \chi \). If \( D[U] \) is positive definite, the determinant can be written as an integral over complex numbers
\[
\det D[U] = \int D\chi^* D\chi e^{-\chi^* D[U]^{-1}\chi}.
\] (4.20)

The easiest way to be sure that \( D[U] \) is positive definite is to use the square of the Dirac operator \( \gamma^{\mu} D_{\mu} = D[U]^\dagger D[U] \). This can be achieved by using an even number of fermions such that the fermions always come in degenerate pairs
\[
\int D\chi^* D\chi e^{-\chi^* D[U]^{-1}\chi} = \det (D[U]^\dagger D[U]) = \int D\chi^* D\chi e^{-\chi^* \gamma^{\mu} D_{\mu}[U]^{-1}\chi},
\] (4.21)

where we have used the determinant identity \( \det(AB) = \det(A)\det(B) \) and \( D^\dagger = \gamma_5 D\gamma_5 \) required for unbroken chirality in the limit of vanishing mass. Observe that with this method we can only simulate theories with an even number of fermion flavors.

### 4.2.1 Fermionic action

The fermionic action on the lattice reads
\[
S_F = a^4 \sum_{x,y} \bar{\psi}(x) (\not{\partial} + m\delta_{xy}) \psi(y).
\] (4.22)

However, discretizing the Dirac operator turns out to be difficult if we want to retain all the continuum properties. Naively we could write the action using the symmetric derivative
\[
\not{\partial} = \frac{1}{2} \gamma^\mu (\nabla_\mu + \nabla^\mu),
\] (4.23)
\[
\nabla_\mu \psi(x) = \frac{1}{a} \left( U_\mu(x) \psi(x + a\hat{\mu}) - \psi(x) \right),
\] (4.24)
\[
\nabla^\mu \psi(x) = \frac{1}{a} \left( \psi(x) - U_\mu^\dagger(x + a\hat{\mu}) \psi(x - a\hat{\mu}) \right).
\] (4.25)

Thus
\[
S_F^{\text{naive}} = a^4 \sum_x \left( m\bar{\psi}(x) \psi(x) + \frac{1}{2a} \bar{\psi}(x) \gamma^\mu \left( U_\mu(x) \psi(x + a\hat{\mu}) - U_\mu^\dagger(x + a\hat{\mu}) \psi(x - a\hat{\mu}) \right) \right).
\] (4.26)

There lies a problem with this approach. To point it out better we can assume the gauge fields \( U \) to be unitary for now. First, let us write the fields in momentum space
\[
\psi_x = \int \frac{d^4p}{(2\pi)^4} \psi_p e^{ip \cdot x},
\] (4.27)
\[
\bar{\psi}_x = \int \frac{d^4p}{(2\pi)^4} \bar{\psi}_p e^{-ip \cdot x}.
\] (4.28)
It follows that

\[
S_{\text{naive}}^F = \int \frac{d^4p}{(2\pi)^4} \bar{\psi}_p \left( \gamma_\mu \frac{e^{ip_\mu a} - e^{-ip_\mu a}}{2a} + m \right) \psi_p, \tag{4.29}
\]

\[
= \int \frac{d^4p}{(2\pi)^4} \bar{\psi}_p \left( i\gamma_\mu \frac{1}{a} \sin(p_\mu a) + m \right) \psi_p, \tag{4.30}
\]

\[
= \int \frac{d^4p}{(2\pi)^4} \bar{\psi}_p S^{-1}(p) \psi_p. \tag{4.31}
\]

The inverse propagator can be written as

\[
S(p) = \frac{m - i\gamma_\mu \frac{1}{a} \sin(p_\mu a)}{m^2 + \left( \frac{1}{a} \sin(p_\mu a) \right)^2}. \tag{4.32}
\]

When \( a \to 0 \) we can clearly see that it approaches the continuum result

\[
S(p) = \frac{m - i\gamma_\mu p_\mu}{m^2 + p^2 + O(a^2)}. \tag{4.33}
\]

However, the momentum can get values \(-\pi < ap_\mu \leq \pi \) due to the Brillouin zone (see eq. (2.59) and discussion following it). With \( ap_\mu = \pi \) the sine function also disappears and the continuum propagator can be restored. In fact, in four dimensions there are \( 2^4 = 16 \) regions where the sine function disappears. This is a problem, as only one of these is the naive fermion and the extra 15 are so-called doublers. We must modify the naive covariant derivative to get rid of these unwanted extra fermionic degrees of freedom.

Getting rid of the doublers is not easy though. In fact it is impossible to write a discretized covariant derivative that satisfies all the following criteria at once: there are no doublers in the theory, \( D_\mu \) is an analytic function with period \( 2\pi/a \) in momentum space, chiral symmetry \( \{ D, \gamma_5 \} = 0 \) is unbroken and \( D_\mu(p) \to i\gamma^\mu p_\mu + O(a^2) \) when \( a \to 0 \). This is known as the Nielsen-Ninomiya no-go theorem [59]. We need to violate at least one of these properties in order to simulate fermions on the lattice.

We will use Wilson fermions [25] to remove the doublers. The idea is to introduce a new term that vanishes in the continuum limit and removes the doublers, while also preserving the physical measurements of the fermions. As a downside the new term will break chiral symmetry as the Wilson-Dirac operator no longer anticommutes with \( \gamma_5 \), but that is the price we need to pay. The doubler problem can also be dealt with by other methods, such as staggered [60], domain wall [61] or overlap fermions, but as stated by the Nielsen-Ninomiya no-go theorem, none of these can take care of the doublers without violating one or more of the continuum properties.

The Wilson action uses second-order derivatives that are added to the fermionic action

\[
\mathcal{L} = \frac{1}{2} \gamma^\mu (\nabla_\mu + \nabla^\mu) - \frac{a}{2} \nabla_\mu \nabla^\mu. \tag{4.34}
\]
We get
\[
D(x, y) = m\delta_{xy} + \frac{1}{2}\gamma^\mu(\nabla_\mu + \nabla_\mu^*) - \frac{a}{2}\nabla_\mu^\star \nabla_\mu,
\]
(4.35)
and after normalising the fermion field by a factor of \(\sqrt{m+4/a}\) we have
\[
S_{\text{Wilson}} = a^4 \sum_{x,y} \bar{\psi}(x) \left[ \delta_{xy} - \kappa \left( (\gamma^\mu - 1)U_\mu(x)\delta_{x+a\hat{\mu},y} - (\gamma^\mu + 1)U_\mu(x-a\hat{\mu})\delta_{x-a\hat{\mu},y} \right) \right] \psi(y),
\]
(4.37)
where
\[
\kappa = \frac{1}{8 + 2ma}.
\]
(4.38)

The Wilson operator fixes the continuum problem at order \(O(a)\). As the gauge field has errors of order \(O(a^2)\), we wish to improve the accuracy of the fermionic fields. This is done by using the clover fermion action explained below.

### 4.2.2 Symanzik improvement programme

The errors of the gauge action eq. (4.13) and of the naive fermionic action eq. (4.26) scale like \(O(a^2)\), but adding the Wilson term to the fermion action will make it only \(O(a)\) accurate. To reduce the discretization effects we will use the Symanzik improvement programme [62, 63] that adds new terms to eliminate the unwanted effects. The new terms will vanish in the continuum limit, but they cancel discretization errors at non-zero lattice spacing. The fermionic action is now
\[
S_F = S_{\text{Wilson}}^F(V) + \delta S_{SW}(V),
\]
(4.39)
where \(\delta S_{SW}\) is the clover-improved action
\[
\delta S_{SW} = a^5 \sum_x c_{SW}\bar{\psi}(x)\left( \frac{i}{4}\sigma_{\mu\nu}\hat{F}_{\mu\nu}\psi(x) \right).
\]
(4.40)
Here $\sigma^{\mu\nu} = \frac{i}{2}[\gamma^\mu, \gamma^\nu]$ and $\hat{F}_{\mu\nu}(x)$ is the clover term. The name 'clover' comes from the shape of the lattice links sketched in Fig. 4.1 and it is defined as

$$
\hat{F}_{\mu\nu}(x) = \frac{1}{4a^2}(Q_{\mu\nu}(x) - Q_{\nu\mu}(x))
$$

(4.41)

where

$$
Q_{\mu\nu} = U_\mu(x)U_\nu(x + a\hat{\mu})U_\nu^\dagger(x + a\hat{\nu})U_\mu^\dagger(x)
+ U_\nu(x)U_\mu^\dagger(x - a\hat{\mu} + a\hat{\nu})U_\nu^\dagger(x - a\hat{\mu})
+ U_\mu^\dagger(x - a\hat{\mu})U_\nu^\dagger(x - a\hat{\mu} - a\hat{\nu})U_\mu(x - a\hat{\mu} - a\hat{\nu})U_\nu(x - a\hat{\nu})
+ U_\nu^\dagger(x - a\hat{\nu})U_\mu(x - a\hat{\nu})U_\nu(x + a\hat{\mu} - a\hat{\nu})U_\mu^\dagger(x).
$$

(4.42)

The factor $c_{SW}$ is called the Sheikholeslami-Wohlert parameter which need to be calculated for each theory separately, either by using non-pertubative simulations or perturbative analytic calculations. For the purposes of this work it is enough to use the tree level result $c_{SW} = 1$.

### 4.3 Hybrid Monte Carlo simulations

As mentioned at the beginning of this chapter, the expectation value of the observable $O$ is given by

$$
\langle O \rangle = \frac{1}{Z} \int DU O[U] e^{-SG} \det D[U].
$$

(4.43)

For a field $\psi$, the aim of the Monte Carlo method is to generate field configurations $\psi^{(k)}$ with a probability distribution

$$
P(\psi) \propto \exp[-S(\psi)].
$$

(4.44)

Then the expectation value of the observable is the average of the observable over the sample of configurations. With an infinite number of configurations the result would be exact:

$$
\langle O \rangle = \lim_{x \to \infty} \frac{1}{N} \sum_{n=1}^{N} O[U_n],
$$

(4.45)

but with a finite number of configurations the estimate is

$$
\hat{O} = \frac{1}{N} \sum_{n=1}^{N} O[U_n] = \langle O \rangle + \mathcal{O}\left(\frac{1}{\sqrt{N}}\right).
$$

(4.46)

Thus the error with this method is of order $\mathcal{O}(1/\sqrt{N})$. One should compute as many configurations as possible, but due to limited amount of computer resources we have $N \sim 100$.

As stated earlier, with an even number of fermion flavors $N_f$, we have

$$
\det D[U] = \int D\chi D\chi^\dagger e^{-\chi^\dagger M[U]^{-1}\chi},
$$

(4.47)
where \( M = D^\dagger D \). To evaluate the pseudofermions \( \chi \) we need to make a change of variable \( \chi = D^\dagger \xi \):

\[
\det D[U] = \det(D^\dagger D) \int D\xi D\xi e^{-\xi^\dagger \xi}.
\]

(4.48)

Now the field \( \xi \) follows a Gaussian distribution, and we can generate Gaussian random numbers for the field values of \( \xi \). Then the field \( \chi \) can be evaluated by calculating \( \chi = D^\dagger \xi \).

Computing a new gauge field configuration requires a bit more work. We will use a Markov process that generates a new configuration \( \psi' \) starting from a previous \( \psi \) with a transition probability \( P(\psi \to \psi') \). Then we will use molecular dynamics (MD) trajectories and a Metropolis test [64]. This whole procedure is known as the hybrid Monte Carlo (HMC) Method [65].

The MD method uses classical dynamics to update the fields. The Hamiltonian of the system with one field \( \psi \) can be written as

\[
H(\pi, \psi) = \frac{1}{2}\pi^2 + S(\psi),
\]

(4.49)

where \( \pi \) is the conjugate momenta. Introducing a fictitious Markov time \( \tau \), the Hamilton equations become

\[
\frac{d\psi}{d\tau} = \frac{\partial H}{\partial \pi}, \quad \frac{d\pi}{d\tau} = -\frac{\partial H}{\partial \psi}.
\]

(4.50)

In HMC the initial random momenta are drawn from a Gaussian distribution of zero mean and unit variance. The Hamiltonian equations must be integrated numerically by using a discrete time step \( \delta t \) in a similar way as was done in chapter 2. The series of discrete steps in Markov time is called a trajectory.

In order to follow the Feynman path integral accurately, the step size in the integrator must be very small or the integration method very sophisticated. Both options increase the computational cost greatly and in practice the integration with a finite step size will always deviate from the exact solution. Thus HMC employs a Metropolis test to ensure that large deviations from the exact result are discarded. The accept/reject decision is based on a change in the Hamiltonian \( \delta H(\psi) \). The new configuration is accepted with a probability

\[
P = \min(1, e^{-\delta H}).
\]

(4.51)

If the proposal is rejected, the old configuration is restored and the new configuration gets discarded. In our simulation the acceptance rate is high, usually \( \sim 90\% \).

Notice that for simplicity the Hamiltonian had only one field \( \psi \). For gauge theories the Hamiltonian becomes

\[
H(\pi, U, \chi) = \frac{1}{2}\pi^2 + S_G(U) + \chi^\dagger (D^\dagger U D[U])^{-1} \chi,
\]

(4.52)

where \( S_G \) is defined in eq. (4.13) and \( D[U] \) contains the Wilson fermion operator eq. (4.36) and
4.4 Simulation details

We use the usual gauge field lattice boundaries. The gauge field has periodic boundaries in all directions, whereas the fermion fields are periodic in spatial boundaries, but anti-periodic in the Euclidean time direction:

\[ U_\mu(x_0, x) = U_\mu(x_0, x + L), \quad \psi(x_0, x) = \psi(x_0, x + L), \]
\[ U_\mu(0, x) = U_\mu(T, x), \quad \psi(0, x) = -\psi(T, x). \quad (4.55) \]

Simulating the fermions on the lattice is the most time-consuming part of the simulation code. We wish to reduce the cost in some way. We employ matrix preconditioning, which uses an even-odd method \[68,69\] to divide the matrix \( D[U] \) into two parts, namely the sites with even and odd parity. On top of this we will use the Hasenbusch performance trick \[70, 71\]. Both of these methods are technical tricks that speed up the simulations, but do not affect the physical measurements. We will not go through these in any more detail.

4.5 Correlators and meson masses

So now we know how to compute lattice configurations. Next we go through how to measure observables \( \mathcal{O}_i \) from these configurations. However, it is important to mention that the approximation in eq. (4.46) can be used only if the measurements are uncorrelated, which clearly is not the case for the Markov chain. The simulation algorithm always produces a new configuration
from a previous configuration, so the new members will inherit properties of the previous configurations. In practice this is solved by skipping some number of configurations before saving a new configuration. To reduce autocorrelation, we skip the next ∼ 20 configurations after each measurement.

In lattice simulations many physical quantities are measured from two-point functions. Two-point correlator functions have the form

\[ f^{\Gamma\Gamma'}(t) = \sum_x \langle \mathcal{O}^SINK_\Gamma (x,t) \mathcal{O}^SOURCE_{\Gamma'}(0,0) \rangle, \] (4.56)

where \( \mathcal{O}^SINK_\Gamma \) is a sink operator at time \( t \) and \( \mathcal{O}^SOURCE_{\Gamma'} \) is a source operator at time \( 0 \). The choice for the sink and source operators should not have an affect on the measurements, but in practice it is known that some sinks and sources work better than others. In our work we mostly use local sinks and sources, defined as

\[ \mathcal{O}^{SOURCE}_\Gamma (x,t) = \mathcal{O}^{SINK}_\Gamma (x,t) = \bar{\psi}(x,t) \Gamma \frac{1}{2} \sigma^a \psi(x,t) \] (4.57)

with a particular choice for \( a \). For example, another often used operator is the Gaussian smearing

\[ \mathcal{O}_\Gamma = \sum_{y_1, y_2} \phi(x, y_1) \phi(x, y_2) \bar{\psi}(x,t) \Gamma \frac{1}{2} \sigma^a \psi(x,t) \] (4.58)

where

\[ \phi(x, y) = e^{-\left( \frac{|x-y|}{R} \right)^2}. \] (4.59)

The local correlator can now be written as

\[ f^{\Gamma,\Gamma'}_L(t) = \sum_x \left( \left( \bar{\psi}(x,t) \Gamma \frac{1}{2} \sigma^a \psi(x,t) \right)^\dagger \bar{\psi}(0,0) \Gamma' \frac{1}{2} \sigma^a \psi(0,0) \right). \] (4.60)

Contracting the creation and annihilation operators into quark propagators \( S(x, t; x', t') \) we can write

\[ f^{\Gamma,\Gamma'}_L(t) = - \frac{a^3}{V_s} \sum_x \text{Tr} \left[ \gamma_0 \Gamma' \gamma_0 S(x, t; 0, 0) \Gamma' \gamma_5 S(x, t; 0, 0) \right], \] (4.61)

where we used \( \gamma_5 S(x, t; x', t') \gamma_5 = S(x', t'; x, t) \). The quark propagator can be calculated by solving the equation

\[ a^4 \sum_y D(x, y) S(y, z) = I \delta_{x, z}, \] (4.62)

i.e. inverting the matrix \( D(x, y) \), where \( I \) is the identity matrix. The different choices for \( \Gamma \) and \( \Gamma' \) result in different correlators. In this section we will only go through the correlators relevant for our study.

If the sink and source operators have the same symmetries, i.e \( \Gamma = \Gamma' \), we can write \( f^{\Gamma,\Gamma'}_L = f_\Gamma \). The expectation value for the vacuum state is then

\[ f_\Gamma(t) = \sum_x \langle 0 | \mathcal{O}^{SINK}_\Gamma (x) \mathcal{O}^{SOURCE}_\Gamma (0) | 0 \rangle \] (4.63)
Figure 4.2: Example of the lattice measurements. Notice that in order to get precise measurements, the lowest energy state must dominate. This can be achieved only if the correlation time between the sink and source operators is large enough. Because the source operator is situated at \( \tau = 0 \), we must skip a couple of first and last data points.
Expanding this by using the energy eigenstates and the four-momentum operator $\mathcal{P}$ such that $\mathcal{O}(x) = e^{i\mathcal{P} \cdot x}\mathcal{O}(0)e^{-i\mathcal{P} \cdot x}$, we eventually get

$$f_\Gamma(\tau) = \sum_n \frac{1}{2E_n} e^{-E_n\tau} \langle 0|\mathcal{O}^{\delta}_{\text{SINK}}(0)|n\rangle \langle n|\mathcal{O}^{\delta}_{\text{SOURCE}}(0)|0\rangle. \quad (4.64)$$

At large Euclidean time $\tau$ the correlator is dominated by the lowest energy state $|\Gamma\rangle$

$$f_\Gamma(\tau) \to \frac{1}{2m_\Gamma} e^{-m_\Gamma\tau} \langle 0|\mathcal{O}^{\delta}_{\text{SINK}}(0)|\Gamma\rangle \langle \Gamma|\mathcal{O}^{\delta}_{\text{SOURCE}}(0)|0\rangle \equiv A_\Gamma e^{-m_\Gamma\tau}. \quad (4.65)$$

For finite lattice volume we also have to take into account an extra term due to the antiperiodic time boundaries. We have

$$f_\Gamma(\tau) \to A_\Gamma \left( e^{-m_\Gamma\tau} + e^{-m_\Gamma(L_\tau)} \right) = A_\Gamma \cosh(m_\Gamma(t - L/2)). \quad (4.66)$$

In order to measure the mass $m_\Gamma$ we have to fit a hyperbolic cosine to the corresponding correlator. By definition, for the pseudoscalar channel $\Gamma = \gamma_5$, and for the vector channel $\Gamma = \gamma_i$, $i \in 1, 2, 3$. Fig. 4.2a illustrates how the pseudoscalar mass is measured.

As previously explained, the Wilson fermions will break the chiral symmetry and thus we need an additional renormalization for the quark mass. The quark mass is defined through the partially conserved axial current (PCAC) [72]:

$$am_q(t) = \frac{1}{4} \left( \partial_t^2 + \partial_t \right) f_A(t) f_P(t) = \frac{1}{4} \frac{f_A(t + a) - f_A(t - a)}{f_P(t)}, \quad (4.67)$$

where $f_A = f_{\gamma_5\gamma_5\gamma_5}$ and $f_P = f_{\gamma_5}$. The PCAC mass is measured in Fig 4.2b for one $\kappa$. To simulate a massless theory, we need to find the critical $\kappa_c$ where the physical quark mass vanishes. An example of this extrapolation of $\kappa_c$ is shown in Fig. 4.2c.

We are also interested in the pseudoscalar decay constant $F_\pi$. It is defined via

$$F_\pi(\tau) = \frac{2m_\pi(\tau)G_\pi(\tau)}{m_\pi^2(\tau)}, \quad (4.68)$$

where

$$G_\pi(\tau) = \sqrt{\frac{2m_\pi(\tau)f_{\gamma_5}}{\cosh(m_\pi(\tau - L/2))}}. \quad (4.69)$$

An example of the $F_\pi$ measurement is plotted in Fig. 4.2d. The decay constant should form a clear plateau at around $\sim T/2$, and we get the value for the pseudoscalar decay constant by fitting a constant function to the plateau. Compared to the masses, the value for the decay constant depends a lot more on the choice of the fitting range, and with certain values of $\beta_L$ and $\kappa$ the plateau is almost nonexistent due to the short correlation times. In section 4.7 we measure the decay constant, bearing in mind that the measurements are not as accurate as they are for the meson masses.
4.6 Approaching the conformal window

As mentioned in Chapter 3, a theory needs to be close to the lower edge of the conformal window in order to be a viable candidate for walking technicolor. By definition, the theories inside the conformal window have an IRFP, whereas below it, the theory breaks chiral symmetry. The exact location of the lower edge of the conformal window is hard to locate as it lies at strong coupling, where the perturbative calculations are not accurate. The properties of theories close to the conformal window can be studied by using nonperturbative lattice simulations. The aim is to find a theory, that is just below the conformal window or barely within it.

As can be seen in Fig 3.5, different gauge groups SU($N$) and different representations of the fermions result in different boundaries for the conformal window. Often studied theories use fundamental fermions due to their relation with QCD, but as $N$ grows, the number of fermion flavors $N_f$ grows rapidly. Furthermore, the most phenomenologically interesting technicolor models are theories where the conformal window can be reached by using small number of flavours. The most common example of that is a SU(2) gauge field theory with two adjoint fermions named as 'minimal walking technicolor'.

We will study the particle spectrum of SU(2) gauge theory with $N_f = 2$, 4 and 6 massless fermions in fundamental representation in Ref. [3]. These theories have been studied by several groups [73–82], but there is no clear consensus of where the lower edge of the conformal window lies. There is clear evidence that $N_f = 2$ and 4 are far below the conformal window and the chiral symmetry gets broken in an expected pattern. The cases $N_f = 8$ and 10 are a bit more challenging, but simulations indicate the existence of an IRFP [75, 80]. The case of $N_f = 6$ is interesting, as the results of the different groups diverge [74, 75, 77, 78]. However, recent studies suggest that $N_f = 6$ is just inside the conformal window [81, 82], and we want to provide more evidence of it.

4.6.1 Particle spectrum and mass anomalous exponent

The mass spectrum can be used to determine the nature of the theory at long distances. In case of the chiral symmetry breaking, the mass of the pseudoscalar "$\pi$" should go to zero as $\propto \sqrt{m_q}$, when the quark mass $m_q \to 0$, whereas the other states remain massive in the chiral limit. For a theory with an IRFP the situation changes as all the states become massless as

$$M_\pi \propto m_q^{1-\gamma_m}$$

(4.70)

when $m_q \to 0$. The factor $\gamma_m$ is the mass anomalous exponent at the fixed point.

In Ref. [3] we study hadron masses as a function of the quark mass. This dependence should reveal the nature of the $N_f = 6$ theory, and give us more evidence for the existence of the IRFP. The quark mass is measured through PCAC relation in eq. (4.67), and we can vary it by changing the value for $\kappa$ defined in eq. (4.38). Varying the bare lattice coupling is done by changing the value for $\beta_L$ in eq. (4.12). We also measure the mass anomalous exponent from
the particle spectrum and compare it to the recent analysis Ref. [81]. In addition, we calculate the pseudoscalar decay constant $F_\pi$, and check that the behaviour is consistent with the particle spectrum behaviour in the limit $m_q \to 0$.

4.7 Particle spectrum of SU(2) gauge theory with fermions in the fundamental representation

We investigate the scaling of the meson masses as a function of the quark mass when approaching the massless limit $m_q \to 0$ in SU(2) gauge theory with $N_f = 2$, 4 and 6 massless fermions in the fundamental representation of the gauge group. First, we briefly discuss the results for two and four flavor theory, and then carry on to six flavor theory. Mostly we will use a lattice of size $L^3 \times T = 24^3 \times 48$, but when there is clear indication on finite size lattice artefacts, we increase the lattice size to $L^3 \times T = 32^3 \times 60$.

The results for $N_f = 2$ and $\beta_L = 1.0$ are easy to interpret. The pseudoscalar at small $m_q$ becomes massless as $M_\pi \propto m_q^{1/2}$ and the vector "$\rho$" mass has a finite intercept in the chiral limit. The behaviour of the decay constant is well described by a fit

$$aF_\pi(m_q) = aF_{\pi,0} + k \times am_q.$$  \hfill (4.71)

Because this case is well understood in the literature we will not repeat this for different $\beta_L$.

The four flavor theory behaves in a similar way, but reaching the chiral limit becomes more difficult. Thus, we will use three different lattice gauge couplings $\beta_L = 0.6$, 0.8 and 1.0, and for the small quark masses we also use the bigger lattice volumes for $\beta_L = 0.8$. With the strongest lattice coupling we can clearly observe $M_\pi \propto m_q^{1/2}$, whereas with bigger $\beta_L$ the finite volume effects become stronger and the square root fit is less favorable. The pseudoscalar decay constant behaves similarly to eq. (4.71). Thus, we conclude that $N_f = 4$ is also consistent with chiral symmetry breaking.

The six flavor theory behaves in a completely different way. At small quark masses the volume dependence of meson masses becomes stronger than in the other two cases. This can be seen especially for $\beta_L = 0.6$ in Fig. 4.3, where the meson masses at small quark masses stay almost constant even though they should decrease as the quark mass decreases. This levelling off also happens with the bigger lattice volume, but the bigger lattice size allows us to run towards smaller masses. Thus, we need to be really careful when choosing the region for the mass anomalous fit.

The behaviour in eq. (4.70) holds only when $m_q \to 0$. However, at small quark masses the mass measurements suffer from finite volume size affects, so we need to discard some of the points. The power law fits for the pseudoscalar and vector masses at each $\beta_L$ can be seen in Fig. 4.3 and the resulting $\gamma_m$ values for each bare coupling $\beta_L = 4/g_0^2$ can be seen in Fig. 4.4.

Next, we relate the mass anomalous exponent to physical couplings measured on the lattice. For this, we use the results from Ref. [81], where the running coupling of SU(2) with $N_f = 6$
4.7 Particle spectrum of SU(2) gauge theory with fermions in the fundamental representation

Figure 4.3: The pseudoscalar (at left) and the vector mass (at right) as a function of the quark mass for $N_f = 6$ and $\beta_L = 0.5, 0.6, 0.7$ and 0.8. The masses are multiplied by factors of 1, 2, 4 and 8, respectively. Empty markers refer to lattice size $L^3 \times T = 24^3 \times 48$ and the filled points to the lattice size $L^3 \times T = 32^3 \times 48$. The dashed line shows the power law fit.

Figure 4.4: At left: the anomalous mass exponent as a function of bare coupling $g_0^2$. At right: the shaded band is the the anomalous dimension as a function of the running coupling $g_{GF}$ measured using the gradient flow method in [81]. The five curves also show the different order perturbative results.
fermions has been measured using the Schrödinger functional method with gradient flow. The lattice action in Ref. [81] is exactly the same as in here, so we can map our \((\beta_L, L)\) results to the gradient flow couplings \(g_{GF}\) found in the Appendix A in Ref. [81].

In Fig. 4.4 we plot the mass anomalous dimension determined from the pseudoscalar and vector masses together with the result from Ref. [81] and the results from the perturbation theory at different orders [83]. The different nonperturbative methods, the step scaling and the hadron spectrum, produce results consistent with each other, and they seem to be in good agreement with the high-order perturbative result as well. However, the perturbative result is scheme-dependent, whereas the non-perturbative results are not, and thus the match between these results are partly coincidental.

We have also measured the pseudoscalar decay constant for \(N_f = 6\) and the results support our interpretation of the existence of the IRFP. We can fit a power law \(aF_\pi = a(m_q)^b\) to the data, and the exponent \(b\) gets similar values as we obtained in the mass spectrum case. However, we do not plot these values in Fig. 4.4 as the exponent measured from the decay constant is less reliable.

In conclusion, the behaviour of the pseudoscalar and vector masses of the two and four flavor theory suggest that the chiral symmetry gets broken, whereas the results for the six flavor theory shows evidence on the existence of the IRFP. Our results are fully consistent with the results in Ref. [81], and thus we can conclude with a higher certainty that the theory with six flavors is just within the conformal window.
4.7 Particle spectrum of SU(2) gauge theory with fermions in the fundamental representation
**Bibliography**


[82] Viljami Leino, Kari Rummukainen, and Kimmo Tuominen. Slope of the beta function at the fixed point of SU(2) gauge theory with six or eight flavors. 2018, 1804.02319.