Monte Carlo Computation of Optimal Portfolio Choice with Habit Formation

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Abstract

This paper considers optimal consumption and portfolio choice of an investor with habit formation in preferences. The Monte Carlo covariation (MCC) method has been used for optimal portfolio selection when an investor's preferences are time-separable. This paper works on the method so that it is applicable in the case of more general utilities. As an example, I solve the optimal portfolio problem in the case where the interest rate adheres to Cox-Ingersoll-Ross dynamics and the stock prices mean reversion using the method and compare results to the time-separable case.

JEL classification: C15, D91, G11

Keywords: habit formation, optimal consumption and portfolio, Monte Carlo methods.

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1 Introduction

Time separability of consumption utilities is a common usual assumption in the theory of financial economics. Empirical studies have implied problems with this assumption. Sometimes these problems are solvable by applying more general utility formulation.

Applying time separable utilities, rational expectation models often generate results which are empirically valid only if we assume a very risk-averse investor. If the risk aversion coefficient is plausible, the representative investor in the models puts much more money in the risky investment than empirically happens. Historically the average return on equity in the U.S stock market was seven percent and the average yield on short-term debt was less than one percent in the period between 1889-1978. Mehra and Prescott (1985) have shown that the common general equilibrium model with separable utilities cannot explain why the first rate is so low and the second rate is so high. That is the so-called equity premium puzzle. The equity premium puzzle is possible to solve using a more general form of the utility function. In this paper, I reject the time-separable assumption and assume that an agent’s utilities adhere to a more general function, the habit utility function.

Merton (1971) examines the continuous-time consumption-portfolio problem for an individual whose income is generated by capital gains on investments in assets with prices assumed to satisfy the geometric Brownian motion hypothesis. For the solution of an individual’s optimization problem, Merton uses Ito’s lemma and stochastic analysis. There are a few papers that have studied the consumption and investment problem of an agent with habit utilities either in the general equilibrium or in the partial equilibrium model (e.g. Sundaresan (1989), Constantinides (1990), Ingersoll (1992), Munk (2008)).

Constantinides (1990) and Sundaresan (1989) present a solution to the equity premium puzzle applying habit utilities. Constantinides’s (1990) reason for using a habit function form is just to find theoretical model which can explain the equity premium puzzle. In literature a intuitive interpretation
also is given for the habit utility function. There are temporal dependencies in the sense that utility in period $t$ depends on not just consumption in the same period but also the level of consumption in previous periods. An individual who consumes a lot in period $(t-1)$ will get used to that high level of consumption, and will more strongly desire consumption in period $t$ (Kocherlakota(1996)).

If the assumption of time separability has been rejected, two kinds of effects are possible: intertemporal substitution or intertemporal complementarity. In the case of intertemporal substitutes a consumer buys a durable good in period $t$, but gets the utility of this good in periods $t+i, i > 0$ without any money spending.

Ferson and Constantinides (1991) study empirically habit persistence in preferences and the durability of consumption goods which both imply the time-nonseparability of the derived utility for consumption expenditures. They study which effect dominates and find evidence in monthly, quarterly and annual data that habit persistence dominates the effect of durability. Obviously, nondurables are "more habit" than durables. Detemple and Zapatero (1992) and Egglezos (2007) solve the optimal consumption when an investor has habit utilities, but they do not find a precise solution of optimal portfolio choice.

Munk (2008) finds a closed-form solution of the optimal consumption and portfolio choice with habit utilities and mean-reverting stock returns. He also solves numerically the problem in the habit case when the interest rate is stochastic and stock prices are mean reverting. Munk uses Monte Carlo simulation to solve the partial differential equation.

Cvitanic et al. (2003) propose the numerical method for optimal portfolio choice in the case where the interest rate adheres to Cox-Ingersoll-Ross dynamics and the stock prices adhere to mean-reversion. This is a very flexible method and by exploiting it, it is possible to solve the optimal portfolio problem in the habit case making different kinds of assumptions about financial assets. The only requirements are that markets have to be complete and the
expanded opportunity set has to be Markovian i.e. all parameters of market processes depend on the n-dimensional Brownian motion process which describes the uncertainty in economy. In this paper, I extend this method for the problem of an investor with habit utilities.

The rest of the paper is structured in the following way. Chapter 2 gives some set-ups and defines utilities. Chapters 3 and 4 consider the assumptions related to financial markets and define a precise optimization problem. Chapter 5 shows how to find optimal consumption in the case of habit utilities using the martingale method solution. Chapter 6 presents the extension of Cvitanic’s (2003) Monte Carlo covariation method in the habit case. Chapter 7 shows the results for the optimal portfolio choice problem and finally, chapter 8 is for conclusion.

2 Financial Assets

Consider a complete market with $m$ non-redundant securities whose price is supposed to satisfy the following dynamics

$$dS_{it} = (S_{it})[\alpha_i(S_{it}, t)dt + \sigma_i(S_{it}, t)dB_{it}]$$

(2.1)

where $\alpha_i(S_{it}, t)$ is the instantaneous conditional percentage change in price per unit time of stock $i$ and and $\sigma_i(S_{it}, t)$ is the instantaneous conditional volatility per unit time of stock $i$. $B_i$:s are standard Brownian motions on a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})$.

Another type of security is a risk-free asset, the "bank account" earning an instantaneous continuously compound interest. Zero-coupon can be defined by

$$\beta_t^* = E_t\left[\frac{\zeta_t}{\mathcal{G}_t}\right] = E_t^Q[e^{-\int_t^r r_u du}]$$

(2.2)

where $\zeta_t$ is the unique state-price deflator.

All uncertainty in the economy is given by realizations of the $m$-dimensional Brownian motion process and the markets are assumed to be complete.
The price of risk (Sharpe ratio) i.e. relative risk process vector, $\lambda_t$, is defined by
$$\lambda_t = \frac{\alpha_t - r_t}{\sigma_t}. \quad (2.3)$$
The interest process $r_t$ and processes $\lambda_t$ and $\sigma_t$ are assumed to have continuous paths and to be adapted to the information filtration.

Since the price of consumption can be calculated in terms of the given state-price density, the problem can be reduced to a simple static optimization problem. Harrison and Kreps (1979) have shown using Girsanov’s theorem that state-price density $\zeta_t$ can be defined by
$$\zeta_t = \xi_t e^{-\int_0^t r_s ds} \quad (2.4)$$
where
$$\xi_t = e^{-\int_0^t \lambda_s dB_s - \frac{1}{2} \int_0^t \|\lambda_s\|^2 ds} \quad (2.5)$$
and the dynamics of $\xi_t$ process is
$$d\xi_t = -\xi_t \lambda_t dB_t. \quad (2.6)$$
The density process $\xi_t$ for $Q$ is the martingale defined by
$$\xi_{t,u} = e^{-\int_0^u r_s ds} \frac{\zeta_u}{\zeta_t}, \quad u > t \quad (2.7)$$
and $\xi_t = \frac{dQ}{dP}$ defines the unique equivalent martingale measure. It is possible to solve the optimal portfolio choice in the habit utility case using a different kind of assumption. It is only necessary to assume that all uncertainty in the economy depends on a m-dimensional Brownian motion, markets are complete, and the expanded opportunity set is Markovian.

## 3 Utilities

The majority of papers studying an agent’s optimal consumption portfolio choice problem follow Merton (1971) and assume a time-separable utility function which means that the lifetime consumption can be expressed as a
sum of felicity functions in the different periods. The unrealistic assumption is rejected in a few studies e.g. Constantinides (1990) and the utility function is assumed to adhere habit formation:

\[ U(h, c) = E\left[ \int_0^T e^{-\rho t} u(t, c(t)) dt \right| \mathcal{F}_0 \], \tag{3.1} \]

where \( h(t) = h_0 e^{-\int_0^t b_s ds} + a_t \int_0^t e^{-b_t} f_s ds \). \tag{3.2} \]

where \( E[\mathcal{F}_0] \) is the expectation at time 0, \( \rho \) is subjective discount rate. Equation (3.1) describes the standard of living. It satisfies a differential equation \( dh_t = (b_t c_t - a_t h_t) dt \). The initial value \( h_0 \) measures the effect of past consumption on current felicity. It can be interpreted as an inherited standard of living corresponding to consumption experience during youth. An alternative interpretation is that \( h_0 \) is a reference level corresponding to standard of living of other people.

It is easy to see that if \( b > 0 \) in (3.1), we have intertemporal complementary effect i.e. habit formation and if \( b < 0 \), we have intertemporal substitution effect i.e. durability. If the consumption is complementary over time it means that a consumer does not like consume less than his living standard amount of consumption.

During this paper holds a standard assumption that instantaneous utility adheres to power utility form:

\[ u(c, h) = \left( \frac{c - h}{\gamma} \right)^\gamma \tag{3.3} \]

where \( \gamma \) is parameter for the degree of risk aversion. In the numerical solutions the habit coefficients \( a \) and \( b \) are assumed to be constant.

I consider the so-called linear habit formation i.e. \( u(c_t, t) = v(c - h) \) for \( c \geq h \) and \(-\infty \) for \( c < h \). The first term on the right-hand side of the equation (3.2) is a weighted average of past consumption and gives the proportion of this average that is compared to current consumption to arrive at the level of services today. \( b \) is a scaling parameter which determines how strongly past consumption affects to consumption today. \( a \) is persistence parameter
and it determines how fast the effect of previous consumption to the habit term vanishes (Egglezos, 2007). It is easy to see that the standard separable utility function is a special case of this function when \( h_0 = a = b = 0 \). If an agent increases consumption today his current utility increases all future utilities decreases through higher standard of living.

4 The Problem

In the seminal article of consumption/investment decision problem Merton (1971) applies a dynamic programming technique to a continuous-time problem. He assumes that an investor’s income is generated by capital gains in assets with prices satisfying the geometric Brownian motion. Merton finds a closed form solution for a case where stock market returns are log-normally distributed and the consumer’s utilities adhere to HARA utilities. Merton (1971) considers "a small investor" who does not have power to influence the markets. The utilities in the original paper are assumed to be time-separable.

This paper consider an investor who maximizes utility by choosing a consumption path \( c = (c_t) \) and an optimal portfolio path \( \pi = (\pi_t) \). I consider an agent whose consumption period is finite and whose instantaneous utilities adhere to power utilities and he does not get utility from bequest. Thus, his optimization problem is

\[
\max_{c,\pi} U(h, c) = E\left[ \int_0^T u(t, c(t) - h(t; c)) dt \mid \mathcal{F}_0 \right],
\]

[4.1]

\( h \) is defined in ??, by choosing the optimal consumption path and the optimal proportion of wealth \( w_i \) invested in the ith security. In this paper, it has been assumed that marginal utilities have the property \( \lim_{c \to h} u'(c - h) = \infty \) i.e. \( c_t - h(t, c) > 0, \ \forall 0 \leq t \leq T \). This assumption presents an addiction pattern. (Detemple and Karatzas (2003) consider non-addictive habits.) If the agent increases his consumption today then the living standard index increases and he has to consume more in the later periods to get same utility level.
The consumer/investor is endowed with some initial wealth $w_0$. He can either consume wealth or invest it in any of $m$ assets. There are $m - 1$ risky stocks and 1 lower risky interest rate with an instantaneous rate of return of $r_t$. The agent invests the proportion $\left[\sum_{i=1}^{m-1} \pi_i(t) = \pi\right]$ of wealth $w_t$ in the $i$th stock ($1 \leq i \leq m - 1$) and remaining proportion $[1 - \sum_{i=1}^{m-1} \pi_i(t) = 1 - \pi]$ in the bond. Merton (1971) has shown that when asset prices are generated by a geometric Brownian motion, we can work with the two-asset case without loss of generality. The pair of investor consumption/investment strategy $c$ and $\pi$ must be based on available information as was formulated in the previous section. I follow Merton and assume that the agent’s income is generated by capital gains on investments in assets and the agent has no other income.

The process corresponding to the portfolio/consumption pair $(c, \pi)$ and initial wealth $w_0$ is the solution of the linear stochastic differential equation:

$$dw_t = \pi_t w_t (\alpha_t dt + \sigma_t dB_t) + (1 - \pi_t) r_t dt - c_t dt$$

$$= (r w_t - c_t) dt + w_t \pi_t \sigma_t d\tilde{B}_t,$$  \hspace{1cm} (4.2)

where the second equivalence holds when we change the probability measure and use $\tilde{B}_t = B_t + \int_0^t \lambda_s ds$ (Egglezos(2007)). Then the wealth process is admissible if $w_t(w_0, c, \pi) \geq 0, \forall t \in [0, T]$. The wealth constraint is satisfied when $E(\int_0^T \zeta(s) c(s) ds) \leq w_0$ i.e. the current market value of consumption is non-negative and is equal to its initial value $w_0$, plus any gains from security trade less the cumulative consumption to date. If a wealth process $w_t$ is admissible for some trading strategy $(c_t, \pi_t)$, then the strategy is budget-feasible.

5 The Optimal Consumption

Karatzas, Lehoczky and Shreve (1987) and Cox and Huang (1989) derived the method to solve optimal consumption by using a martingale representation technology. If the markets are assumed complete i.e. the number of source of uncertainty equals the number of stocks, $k = m - 1$, the dynamic optimization
problem becomes a simple static problem. Then policy \((c^*, \pi^*)\) is optimal only if the static problem \(\max_c u(\cdot)\), subject to \(\mathbb{E}^Q \int_0^T e^{-\int_0^t r_u du} c_t dt \leq w_0\).

Detemple and Zapatero (1992) solve the optimal consumption in (??) Then the Lagrangian function is

\[
L = \mathbb{E} \left[ \int_0^T u(t, c(t) - h(t, c)) dt \right] + y \left[ w_0 - \mathbb{E} \left( \int_0^T \zeta(s) c(s) ds \right) \right].
\]

where \(y\) is the Lagrangian multiplier and Kuhn-Tucker first-order conditions for the optimality of a consumption-rate process \(c(\cdot)\) are

\[
u'(t, c(t) - h(t, c)) + b_t \mathbb{E}_t \left( \int_t^T e^{-\int_s^t (a(v) - b(v)) dv} u'(s, c(s) - h(s, c)) ds \right)
= y \zeta(t) \quad \forall t \in [0, T],
\]

\[
\mathbb{E} \left[ \int_0^T \zeta(t) c(t) dt \right] = w_0.
\]

Using optimality conditions we formulate an inverse function of marginal utility

\[
c^*(t) = I(t, y \phi_t).
\]

where

\[
\phi_t = \zeta_t (1 + b \mathbb{E}_t \left[ \int_t^T e^{f_t''(v) a(v) - b(v) + a(v) ds} ds \right]).
\]

Equation ?? defines a recursive linear stochastic equation, which describes relationship between state price density in the separable case and state price density \(\hat{\zeta}_t\) in the habit case

\[
\hat{\zeta}_t = \zeta_t + b_t \mathbb{E}_t \left( \int_t^T e^{f_t''(b(v) - a(v)) ds} \zeta_s ds \right).
\]

\(b \mathbb{E}_t \left( \int_t^T e^{f_t''(b(v) - a(v)) ds} \zeta_s ds \right)\) shows the effect of habit presence to state price density.

Detemple and Zapatero (1992) have found the following solution for optimal consumption
Theorem 1. Consider an agents whose utilities are defined by the functions (??), (??) and (??) and the financial asset are as in section ??, the agent’s optimal consumption is

\[ c^*_t = h_0 e^{-\int_0^t (a-b) dv} + (y^*)^{\frac{1}{\rho-1}} \left[ \phi_t^{\frac{1}{\rho-1}} + \int_0^t b e^{-\int_s^t (a-b) du} \phi_s^{\frac{1}{\rho-1}} ds \right] \tag{5.6} \]

where

\[ y = [x - h_0 E \int_0^T e^{\int_0^t (b(v) - a(v))dv} dt]^{\rho-1} \left[ E \int_0^T e^{-\int_0^t r(u) du} \right]^{\rho-1} \tag{5.7} \]

y is the Lagrangian coefficient.

The wealth process is

\[ w_t(y) = E[\int_0^T e^{-\int_0^t r(u) du} [h_0 e^{\int_0^t (b(v) - a(v)) dv} + I(t, y\phi(t)) + \int_0^t e^{\int_0^s (b(v) - a(v)) dv} I(t, y\phi_t) ds] dt] \mathcal{F}_t]. \tag{5.9} \]

It is not possible to define the precise solution of portfolio choice without numerical methods. In the next chapter, I consider a numerical method for solving the optimal portfolio.

6 The Simulation Method

A large number of research papers have applied Monte Carlo simulation to financial problems, mostly to asset pricing problems (option pricing). There are also some, quite new applications which use Monte Carlo simulation to solve the optimal consumption and investment choice. Detemple et al. (2003) exploit Malliavin calculus and Monte Carlo simulation to solve the optimal portfolio choice. Cvitanic et al. (2001, 2003) has developed a more straightforward method which uses Monte Carlo simulation to solve the volatility of the wealth process. Volatility can also be used to determine the optimal investment choice. Cvitanic et al. (2003) restricts his analysis only to time separable case, but it is possible to solve the problem with habit utilities if
the Monte Carlo covariation method has been somewhat devised. To use the
method of Cvitanic etc al. it is necessary to accept the assumptions about
complete markets and Markovian opportunity set.

In this and next chapter, I solve the optimal portfolio choice in the habit
case, when interest rate is assumed to follow Cox-Ingersoll-Ross dynamics
and stock prices are assumed to be mean reverting. This is just one example
of the use of the method. The flexibility of the method would enable us to
apply a lot of different kinds of dynamics.

6.1 The Method

Cvitanic et al.(2003) start considering an expression

\[ D_t = E[ \int_t^T f(r_s, \lambda_s, B_s) ds | \mathcal{F}_t]. \] (6.1)

where \( r_s, \lambda_s \) and \( B_s \) are as before. \( D_t \) satisfies a stochastic differential equa-
tion of the type

\[ dD_t = \varphi_t dt + v_t dB_t \] (6.2)

where \( \varphi_t \) is the drift and \( v_t \) is diffusion coefficient. The parameter \( v_t \) can
be obtained from the quadratic variation of the \( D_t \). Cvitanic et al.(2003)
use relation between the wealth process in ?? and equation ?? to define the
following limit

\[ v_t = \lim_{\Delta t \to 0} E[ \frac{(D_{t+\Delta t} - D_t)^2}{\Delta t} | \mathcal{F}_t], \] (6.3)

and to get the optimal portfolio by a linear transformation of the volatility of
the wealth process:

\[ \pi^*_t = (\sigma_t)^{-1} v_t. \] (6.4)

So, if we can solve \( v_t \) by simulation we can also solve its linear transformation,
the optimal portfolio choice \( \pi^* \).

The estimate of \( v_t \) can be computed by

\[ \hat{v}_t = \frac{1}{K} \sum_{i=1}^K \frac{(D_{t+\Delta t}^i - D_t)(B_{t+\Delta t}^i - B_t)}{\Delta t} = \frac{1}{K} \sum_{i=1}^K \frac{(w_{t+\Delta t}^i - w_t)z_t^i}{\Delta t}. \] (6.5)
where $z_t$ is a standard normal random variable and $K$ the total number of simulated paths. Finally the optimal portfolio choice is gained by equation ??

The covariation between the optimal wealth process and the uncertainty shocks provides expression for the optimal portfolio.

6.2 Optimal Portfolio

It is possible to use the method of this paper assuming different kind of behavior of financial assets. I use a 2-tier simulation in the sense that I solve the optimal path of consumption in the habit case and then use that path for solving the volatility of the wealth process, I compute the path of wealth process $(w_t)$ and then use the Monte Carlo covariation method in the case of intertemporal consumption to solve the volatility of the wealth process. The MCC method of chapter ?? is possible to expand to the habit case.

**Theorem 2.** If markets are complete and expanded opportunity set is Markovian, there are the following limit

$$w_t \pi^*_t \sigma_t = \lim_{h \to 0} \frac{1}{h} E_t \left[ \frac{(B_{t+h} - B_t)w_t}{\xi_{t,t+h}} \right]$$

(6.6)

and the optimal portfolio is possible to get by a linear transformation of the volatility of the wealth process:

$$\pi^*_t = (\sigma_t)^{-1} v_t.$$  

(6.7)

6.2.1 The computation of Lagrange multiplier

At first, I solve the Lagrangian coefficient numerically. In order to do this, the equation (??) is expressed by

$$y = [x - h_0 E \int_0^T e^{\int_0^t (b(v) - a(v)) dv} dt]^{\rho-1} [E \int_0^T e^{(-\int_0^t r_u du)} \left( \frac{1}{\rho} \right) \left( \frac{1}{\rho} \right) \left( \frac{1}{\rho} \right) ds]^{1-\rho}$$

(6.8)
where
\[ \eta_t = 1 + bE(\int_t^T e^{\int_t^s (-r_s - b + a)ds} ds). \] (6.9)

It is easy to exploit simulation to solve expectations. In every step, \( \xi_t \) process develops as follows (as in (??)):
\[ \xi_{t+\Delta t}(z^t) - \xi_t = -\xi_t \lambda_t z^t, \] (6.10)
where \( z^t \) is pseudo-random number with distribution \( N(0, \Delta t) \). Updated value of \( r_t \) and \( \lambda_t \) are obtained using Euler discretization of (??) and (??).

### 6.2.2 The Computation of the Wealth Process

In the next step, I use an algorithm which, at first, calculates the optimal path of consumption (??)
\[ c_t = h_0 e^{-\int_t^0 (a-b)dv} + (y^*)^{1/\rho-1}[(\xi_t \eta_t)^{1/\rho-1} + \int_0^t be^{-\int_s^t (a-b)du}(\xi_s \eta_s)^{1/\rho-1}ds] \] (6.11)
and then the value of the wealth process at time \( t + \Delta t \)
\[ w_{t+\Delta t} = E[\int_{t+\Delta t}^T e^{-\int_{t+\Delta t}^s r_u du}[c_s] ds | \mathcal{F}_{t+\Delta t}] \] (6.12)
\[ = E[\int_{t+\Delta t}^T e^{-\int_{t+\Delta t}^s r_u du}[h_0 e^{-(a-b)t}] + I(t, y\eta_t) + \int_{t+\Delta t}^T e^{-(a-b)t}I(t, y\eta_t)ds | \mathcal{F}_{t+\Delta t}] \] (6.13)

Using Monte Carlo simulation the numerical values of the wealth process at \( t + \Delta t \) can be solved exactly in the same way as in Cvitanic et al. (2003). At first, an estimate for \( w^*_{t+\Delta t}(z^t_1) \) is calculated by
\[ w^*_{t+\Delta t}(z^t_1) = \frac{1}{M} \sum_{j=1}^M \xi_t + \Delta t, c^z_s ds. \] (6.14)

In the final stage of simulation, the volatility of the wealth process is solved by using
\[ \hat{\sigma} = \frac{1}{K} \sum_{j=1}^K \frac{\left( w_{t+\Delta t}(z^t_1) - w_t(z^t_1) \right)^2}{\Delta t} . \] (6.15)
Using big enough number of rounds, K we can obtain reasonable precise values of \( v_t \).

7 A Numerical Example

Next, I consider one particular case and assume that the interest rate follows Cox-Ingersoll-Ross dynamics and the market-price-of-risk is a mean reverting process. So, the interest rate dynamics adheres to a differential equation

\[
dr_t = \kappa_r (\bar{r} - r_t) dt + \sigma_r \sqrt{r_t} dB_t
\]

and the market-price-of-risk process follows a differential equation

\[
d\lambda_t = \kappa_\lambda (\bar{x} - \lambda_t) dt + \sigma_\lambda dB_t.
\]

Wachter (2002) finds a closed form solution of the optimal portfolio choice problem for an investor with time separable utilities under mean-reverting returns, but in case with habit utilities closed form solution does not exist.

In this example, I follow Cvitanic et al. (2003) and Detemple et al. (1999) and assume same values of constants as they do: \( \rho = 0, \bar{r} = 0.06, \sigma_r = 0.0364, \kappa_r = 0.0824, \kappa_\theta = 0.6950, \bar{\theta} = 0.0871, \sigma_\theta = 0.21, \sigma_t = 0.2, r_0 = 0.06, \theta_0 = 0.1 \). The so called inherited standard of living \( h_0 \) is set to 0.04.

"Habit parameters" \( a \) and \( b \) are assumed to be constants. In table (??) is shown the optimal portfolio for some values of parameters \( a \) and \( b \) when the time horizon is 1. Tables (??) and (??) express the optimal portfolio choice for the same value of parameters in longer time horizons. When we consider the time-separable case and set habit parameters \( a \) and \( b \) equal to 0, the method gives the same values as in Cvitanic et al. (2003).

The common problem with Monte Carlo simulation is computational inefficiency. Cvitanic et al. (2003) use \( K = 10000 \) and \( M = 50 \) and obtain a standard deviation of around 0.002. The algorithm for the habit case is slightly more complicated as seen in chapter 6. Using \( K = 50000 \) and \( M = 50 \)
I get quite a similar size standard deviation. When using MATLAB software on a 3.0-GHz Intel Core PC, the computational times are from 8 minutes (T=1) to about 1 and half hours (T=10) and are not substantially longer than in Cvitanic et al. (2003).

<table>
<thead>
<tr>
<th>$\pi$</th>
<th>$\gamma=-1$</th>
<th>$\gamma=-2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>a=0 &amp; b=0</td>
<td>0.243</td>
<td>0.174</td>
</tr>
<tr>
<td>a=0.1 &amp; b=0.2</td>
<td>0.209</td>
<td>0.138</td>
</tr>
<tr>
<td>a=0.1 &amp; b=0.3</td>
<td>0.220</td>
<td>0.153</td>
</tr>
<tr>
<td>a=0.2 &amp; b=0.3</td>
<td>0.205</td>
<td>0.142</td>
</tr>
<tr>
<td>a=0.2 &amp; b=0.4</td>
<td>0.215</td>
<td>0.134</td>
</tr>
<tr>
<td>a=0.4 &amp; b=0.5</td>
<td>0.199</td>
<td>0.161</td>
</tr>
</tbody>
</table>

Table 2: The optimal portfolio for different parameter values of a and b and for different values of risk aversion when time horizon $T=5$.

<table>
<thead>
<tr>
<th>$\pi$</th>
<th>$\gamma=-1$</th>
<th>$\gamma=-2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>a=0 &amp; b=0</td>
<td>0.297</td>
<td>0.238</td>
</tr>
<tr>
<td>a=0.1 &amp; b=0.2</td>
<td>0.247</td>
<td>0.199</td>
</tr>
<tr>
<td>a=0.1 &amp; b=0.3</td>
<td>0.262</td>
<td>0.212</td>
</tr>
<tr>
<td>a=0.2 &amp; b=0.3</td>
<td>0.246</td>
<td>0.197</td>
</tr>
<tr>
<td>a=0.2 &amp; b=0.4</td>
<td>0.252</td>
<td>0.190</td>
</tr>
<tr>
<td>a=0.4 &amp; b=0.5</td>
<td>0.240</td>
<td>0.213</td>
</tr>
</tbody>
</table>

8 Conclusion

In this paper, the time-separable utility function of a consumer/investor has been replaced with a more general form of utility function, the habit utility. The Monte Carlo covariation method by Cvitanic et al. (2003) has been extended so that it can be used in the habit case. I have solved numerically
Table 3: Optimal portfolio for different parameters $a$ and $b$ and for different values of risk aversion when time horizon $T=10$.

<table>
<thead>
<tr>
<th>$\pi$</th>
<th>$\gamma=-1$</th>
<th>$\gamma=-2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a=0 \ &amp; b=0$</td>
<td>0.251</td>
<td>0.174</td>
</tr>
<tr>
<td>$a=0.1 \ &amp; b=0.2$</td>
<td>0.209</td>
<td>0.138</td>
</tr>
<tr>
<td>$a=0.1 \ &amp; b=0.3$</td>
<td>0.220</td>
<td>0.153</td>
</tr>
<tr>
<td>$a=0.2 \ &amp; b=0.3$</td>
<td>0.205</td>
<td>0.142</td>
</tr>
<tr>
<td>$a=0.2 \ &amp; b=0.4$</td>
<td>0.215</td>
<td>0.134</td>
</tr>
<tr>
<td>$a=0.4 \ &amp; b=0.5$</td>
<td>0.199</td>
<td>0.161</td>
</tr>
</tbody>
</table>

An optimal portfolio allocation of the consumer/investor with habit utilities when interest rates are assumed be stochastic and stock returns are mean-reverting. In such a case, it is not possible to find a closed form solution. In literature, Munk (2008) has solved the problem with more restrictive assumptions about interest rate and stock prices dynamics. His method is slightly computationally more efficient than mine. On the other hand, my method is more flexible in sense that it is possible change the assumption about the behavior of financial assets.

Using the method of this paper, it is possible solve the optimal portfolio problem in the habit case making different kind of assumptions about financial assets. The only requirements are that markets have to be complete and the expanded opportunity set has to be Markovian.
References


