Inverse problem for the wave equation: partial data and novel boundary sources

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Academic dissertation

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This thesis consists of an introduction and the following five articles:

[I] M. Lassas and L. Oksanen, Inverse problem for wave equation with sources and observations on disjoint sets. Inverse Problems 26 (2010), no. 8, 085012, 19 pp.

[II] T. Helin, M. Lassas and L. Oksanen, An inverse problem for the wave equation with one measurement and the pseudorandom source. Analysis & PDE (to appear), preprint arXiv:1011.2527.

[III] L. Oksanen, Solving an inverse problem for the wave equation by using a minimization algorithm and time-reversed measurements. Inverse Problems and Imaging 5 (2011), no. 3, pp. 731–744.

[IV] L. Oksanen, Inverse obstacle problem for the non-stationary wave equation with an unknown background. Preprint arXiv:1106.3204.

[V] M. Lassas and L. Oksanen, The inverse problem for the wave equation and two acquisition geometries. In "Proceedings of Workshop on Inverse Problems, Data and Mathematical Statistics and Ecology", LiTH-MAT-R–2011/11–SE, Linköping University, pp. 43–50.

The author had a major part in the analysis of the joint articles [I] and [V]. T. Helin, M. Lassas and the author had an equal role in the analysis of the joint article [II].

1. Introduction

The field of inverse problems studies how to convert measurements into information about a physical system. As an example, let us consider obstetric sonography in which ultrasound measurements are transformed into an image of the fetus in its mother's uterus. The creation of an image from sound is done in three steps - producing sound waves, receiving echoes, and interpreting those echoes. In our terminology, the first two steps form the measurements and the last step is the inverse problem. The solution to this problem is, ideally, the wave speed as a function of the position inside the uterus. In general, a solution to an inverse problem gives us information about a physical parameter that we can not directly observe. Moreover, the connection between measurements and their interpretation is not one-way: inverse problems research can guide the measurement design.

In the context of inverse problems, a mathematical model of the measurements is called the direct problem. To understand this term, let us consider a model for obstetric sonography. An ultrasonic scanner produces a sound wave by using a transducer, the wave propagates in the body and echoes back to the transducer that records the echo. The propagation is mathematically modelled by the wave equation which gives a connection between the wave speed (a property of the medium) and the sound wave (oscillation in the medium). If we are given the wave speed as function of the position in the body and the vibrations of the transducer as a function of time, then we can solve the wave equation for the received echoes. In other words, the wave equation gives a model for ultasound measurements. Solving this equation with a wave speed corresponding to the mixture of different tissues in the human body requires computationally demanding simulations and is indeed a problem, the direct problem.

In this thesis we consider inverse problems with the wave equation as the direct problem. To be more precise, we consider problems where sound sources and receivers are located on the surface of an object and the ideal goal is to reconstruct the wave speed inside the object. Moreover, we allow measurements of arbitrary causal waves in contrast to acoustic scattering measurements. Acoustic scattering theory is related to measurements using time harmonic waves – for a review of this theory, we refer the interested reader to [**27**]. We give a brief mathematical introduction to the wave equation in section 2.

The inverse problems for the wave equation are unstable with respect to measurement errors and also nonlinear. Hence they are hard to solve computationally, and a good theoretical understanding of the problem plays crucial role in designing practical solution methods. This is reflected in the fact that a typical sonographic device does not solve an inverse problem for the wave equation in the above sense. Instead of an image of the wave speed, it displays an image of the echo: a point in the image corresponds the location of the transducer and the time it took for the echo to return, and the shade of the point is determined by the strength of the echo [**77**, p. 4]. We believe that better understanding of inverse problems for the wave equation will eventually lead to better sonograms.

In addition to sonography, reflection seismology gives an example of an imaging method where the wave equation can be used as a model of measurements. Seismic reflection method seeks to create an image of the Earths crust from recording of echoes stimulated for example by explosions. In contrast to medical sonography, seismic reflection method reconstructs an image of the wave speed. However, the nonlinear inverse problem is linearized and this causes the image to capture only certain features of the wave speed. Typically only the singularities in the wave speed are displayed [**105**]. In this thesis we consider only the nonlinear problem.

We will next review the most important results related to inverse problems for the wave equation. We will focus on uniqueness questions and on practical reconstruction methods, and will not consider stability questions or other questions related to measurement noise. For stability results, we refer the reader to [**1, 13, 100**].

1.1. Uniqueness questions. Acoustic measurements on the boundary of an object give enough information to determine the wave speed inside the object uniquely or, put differently, the inverse problem for the wave equation has a unique solution. This is a result by Belishev [**9**] for an isotropic wave speed and by Belishev and Kurylev [**12**] for an anisotropic wave speed. The difference between the isotropic and the anisotropic case is that in the former the wave speed does not depend on the direction of the propagation whereas in the latter it does. The uniqueness proofs [**9, 12**] are based on a control theoretic approach called the boundary control (BC) method. We describe the BC method briefly in section 3 and refer to [**10**] for a review article and to [**55**] for a monograph describing the method in detail.

If the wave speed is isotropic, then the inverse problem for the wave equation, even with fixed frequency data, can be solved by using the complex geometrical optics (CGO) solutions developed by Sylvester and Uhlmann in their fundamental 1987 paper [**104**]. The first formulation of the BC method [**9**] is from the same year. However, the uniqueness proofs based on the BC method depend on the unique continuation principle for the wave equation, and it was not until 1995 that Tataru proved the principle [**106**]. In [**104**] Sylvester and Uhlmann solved the inverse problem posed by Calderón [**24**]. Calderón's problem is an inverse problem for an elliptic partial differential equation related to electrostatic measurements. Within a year, the CGO solutions were applied to solve the inverse acoustic scattering problem [**84, 87**], and they also yielded a solution to the inverse problem for the isotropic wave equation [**85**].

It should be pointed out that the proof in [**104**] gives uniqueness only in dimensions three or higher. The two dimensional Calderón's problem was first solved by Nachmann in 1996 [**86**] and is by now better understood than the higher dimensional cases. Following ideas by Sylvester [**103**], the two-dimensional anisotropic Calderón's problem can be solved, see [**4, 86, 102**]. In dimensions three and higher, the anisotropic Calderón's problem has not been solved in general. For the solved case of real analytic material parameters, see [**70, 72, 74**].

Astala and Päivärinta have solved the two-dimensional Calderón's problem assuming that the material parameters are only L^∞ functions [**3**]. In dimensions three and higher, the sharpest known smoothness result requires that material parameters have one and a half derivatives [**18, 89**]. Moreover, there are results in the case that material parameters are non-smooth along a hypersurface [**37, 63**].

The uniqueness proofs based on the BC method usually assume that the wave speed is smooth. However, in a recent article [**59**], Kirpichnikova and Kurylev consider piecewise smooth wave speeds on Riemannian polyhedra. Moreover, the stability results [**1, 13, 100**] establish uniqueness for wave speeds with a limited number of derivatives.

The inverse acoustic scattering problems and Calderón's problem are, in a sense, harder than the inverse problem for the wave equation. The CGO solutions based methods can be used to solve the latter but the BC method can not be used to solve the former. This is because the direct problems for the former problems are elliptic partial differential equations and the BC method uses the hyperbolic features of the wave equation in an essential way. This is also reflected in the fact that there is no time dimension in a measurement data set corresponding to single frequency scattering or to electrostatic measurements, whence the dimension of a typical data set is one less than that of a data set obtained by boundary measurements of arbitrary causal waves. In this thesis we employ only the BC method. For a review of CGO solutions based methods we refer the interested reader to [**108**].

1.2. Different acquisition geometries. Let us return to the two applications mentioned above. Although the direct problem is the wave equation both for sonography and for reflection seismology, the respective inverse problems are typically different. This is because in sonography the transducer acts simultenously as the sound source and as the receiver of echoes, but in reflection seismology receivers can not typically lie near the powerful sound sources. We say that the acquisition geometries differ between these two examples.

The uniqueness results based on the BC method [**9, 12**] are for the acquisition geometry that does not restrict at all the locations of the sources and receivers on the boundary. Using the BC method, Katchalov and Kurylev have proved uniqueness also in the case that the sources and receivers lie in the same arbitrarily small part of the boundary [**54**]. In the context of Calderón's problem, there is an extensive literature about results similar to [**54**] by using the CGO solutions based techniques [**21, 39, 41, 50, 49, 51, 58**]. We give a brief review of this literature in [I].

In view of reflection seismology, it is well-motivated to study acquisition geometries that restric the sources and receivers to far apart locations. For such an acquisition geometry it is an open question in general if the inverse problem for the wave equation has a unique solution. In [I] we show that the inverse problem for the wave equation has unique solution for certain acquisition geometries where the sources and receivers are not simultaneously in the same location.

1.3. Geometric inverse problems. The inverse problem for the wave equation is closely related to several inverse problems of geometric nature. By a geometric inverse problem we mean a problem to recover a Riemannian manifold from a geometric data set, that is, a data set derived from the Riemannian structure only. A widely studied example is the boundary rigidity problem for which the data set is the distances between each pair of boundary points.

The wave equation on a domain gives rise to a natural Riemannian distance between a pair of points – the distance being the shortest travel time of an acoustic wave. Moreover, it is known that the acoustic boundary measurements determine the distances in this sense between each pair of boundary points, see e.g. [**96, 107**]. Put differently, the inverse problem for the wave equation can be reduced to the purely geometric boundary rigidity problem.

The boundary rigidity problem is open in general even for simple[1] manifolds. The case of a two-dimensional simple manifold is solved

[1] We recall the definition of a simple compact manifold in section 4.

by Pestov and Uhlmann [**91**], and Burago and Ivanov have solved the problem for metrics close to the Euclidean metric [**22**]. An important earlier but partial result about boundary rigidity near the Euclidean metric was obtained in [**71**]. Moreover, Muhometov and Romanov have shown that a simple manifold is determined within a conformal class of manifolds [**81, 82, 83**]. A similar result holds also when the simplicity assumption is replaced with the so-called strong geodesic minimizing assumption [**30**]. We refer to [**31**] for a review of results concerning subdomains of certain symmetric spaces and spaces with nonpositive curvature [**14, 29, 40, 80, 88**].

For a simple manifold, the boundary rigidity problem is equivalent with the scattering rigidity problem for which the data set consists, roughly speaking, of all the travel times between each pair of boundary points – not just the shortest travel times. See [**80**] for a proof of the equivalence. Moreover, it is known that the acoustic boundary measurements on a compact manifold determine the scattering relation, see e.g. [**44, 107**].

The scattering rigidity problem has been solved for non-trapping real analytic compact manifolds by Vargo [**110**] and for compact manifolds close to a real analytic one by Stefanov and Uhlmann [**101**] under certain assumptions. Moreover, Uhlmann has conjectured that non-trapping compact manifolds are scattering rigid [**107**]. It is possible that scattering rigidity holds for even larger class of manifolds as Croke has recently given an example of a scattering rigid manifold with trapped geodesics [**32**]. In [II] we show that the echo of a single, explicitly chosen, wave source contains enough information to determine the scattering relation uniquely.

Apart from the boundary and lens rigidity problems, the inverse problem for the wave equation can also be reduced to certain purely geometic problems that have not been studied in their own right to our knowledge. Kurylev have shown that acoustic boundary measurements determine the set of so-called boundary distance functions and that this set determines again the manifold [**55, 64**]. Moreover, Kurylev, Lassas and Uhlmann have shown that also the so-called broken scattering relation determines the set of boundary distance functions and whence the manifold [**66**]. We outline the reconstruction of the set of boundary distance functions from acoustic boundary measurements in section 3.

In [III] we show that the volumes of certain sets, called the domains of influence, can be computed from acoustic boundary measurements. Moreover, we show that a simple manifold is determined by the volumes of domains of influence. A domain of influence is defined by using the distance function only and thus is a purely geometric concept.

1.4. Reconstruction methods. A uniqueness proof for an inverse problem does not necessarily give a reconstruction method for the material parameters we are interested in. In the case of acoustic measurements, a reconstruction method should answer the following two questions: How to choose the wave sources? How to compute an approximation of the wave speed from the corresponding echoes? The uniqueness proofs based on the BC method [**9, 12**] give an answer to these questions, but the resulting reconstruction method is unstable and thus vulnerable to measurement noise and hard to implement.

To our knowledge, there are two computational implementations of BC method [**11, 53**] in its original form. In addition to these two implemetations, the only numerical results related to the BC method we are aware of are in the recent article by Pestov, Bolgova and Kazarina [**90**]. The numerical results [**11, 90**] are for the two-dimensional isotropic wave equation and the result in [**53**] is for the one-dimensional wave equation.

In general, the gap between uniqueness results for the inverse problem for the wave equation and practical applications such as sonography seems to be wider than that between uniqueness results for Calderón's problem and applications such as electrical impedance tomography. Moreover, the main theoretical advances related to Calderón's problem have stimulated a large amount of computational research. For example, in the two-dimensional case, Nachman's techniques have been first exploited computationally in [**99**], the smoothness result by Brown and Uhlmann [**19**] in [**61**] and the result by Astala and Päivärinta in [**2**]. We refer to [**17**] for a review of computational results related to Calderón's problem.

1.5. Inclusion detection methods. An important class of reconstruction methods aim not to recover the wave speed as a function but only the surfaces on which it is non-smooth. We call such methods inclusion detection methods. The inverse problems to reconstruct the support of the non-smooth part of material parameters are often called also inverse obstacle problems. The inclusion detection methods are typically based on the assumption that the material parameters consist of unknown obstacles in a known background, and exploitation of this additional knowledge can lead to computationally more robust methods.

In the context of acoustic scattering theory, there is an extensive literature about inclusion detection methods. Some of these methods have also been applied to measurements of causal waves to solve inverse obstacle problems for the time domain wave equation. By time domain wave equation we mean the wave equation in the sense of section 2 as opposed to the frequency domain equations considered in the acoustic scattering theory. The methods in [**75**] and in [**79**] take Fourier transforms of time domain measurement data and solve the inverse obstacle problem by using inclusion detection methods developed for scattering problems in the frequency domain. Moreover, the method in [**23**] process the measurement data partly in the frequency domain.

The only method processing the measurement data entirely in the time domain that we are aware of is the recent sampling method by Chen, Haddar, Lechleiter and Monk [**26**]. However, the method is similar to the linear sampling method developed for inverse obstacle scattering problem by Colton and Kirsch [**28**], and the analysis of the method depends on frequency domain techniques.

A modification of the frequency domain linear sampling method by Kirsch is called the factorization method [**60**], and it can be interpreted by using localized potentials [**35**]. The enclosure method by Ikehata [**47**] is another well known inclusion detection method applicable to acoustic scattering measurements. It is the first inclusion detection method based on the CGO solutions. For later CGO based methods see [**45, 48, 109**]. We point out that, although the factorization and enclosure methods and the seismic reflection method all aim to reconstruct singularities in the material parameters, the former two methods do not invoke any kind of linearization in contrast to the latter.

The factorization method has been applied also to electrostatic measurements [**20, 42**], and the enclosure method was developed for both acoustic scattering and electrostatic measurements from the very beginning. For other solutions to inverse obstacle problems related to scattering and electrostatic measurements see the probe [**46**] and singular sources [**93**] methods, the no response test [**78**], the scattering support techniques [**43, 67, 95**] and the review article [**94**].

In [IV] we study an inverse obstacle problem for the time domain wave equation. The inclusion detection method we introduce is based on the BC method and employs only control theoretic techniques.

1.6. Summary of the articles included in the thesis. In [I] we study certain acquisition geometries where the sources and receivers are not simultaneously in the same location.

In [II] an explicit wave source is constructed so that its echo determines the scattering relation and whence the wave speed under certain geometric assumptions. This result should be contrasted with the above-mentioned uniqueness results which require measurement of echoes stimulated by a sequence of sources. Moreover, in [II] the dimension of the measurement data set is the same as the dimension of the solution to the inverse problem. Even for Calderón's problem in dimensions three and higher, the dimension of the measurement data set is greater than that of the solution.

In [III] we give a method to choose wave sources iteratively so that the volumes of the domains of influence can be easily computed from the echoes. We also show that these volumes determine the wave speed uniquely under certain geometric assumptions. A common theme between [II] and [III] is to study how to choose the wave sources so that the corresponding echoes yield geometric information about the wave speed in readily exploitable form.

In [IV] we apply the method of [III] to an inverse obstacle problem. We show that the obstacle can be located by solving a sequence of linear problems. In the conference proceedings article [V] we adapt the results of [I] to an isotropic case.

2. The direct problem

Let $n \geq 2$ and let $M \subset \mathbb{R}^n$ be an open and connected set with a smooth boundary ∂M. The wave equation on M has the form,

$$\partial_t^2 u(t,x) - c(x)^2 \Delta u(t,x) = 0, \quad (t,x) \in (0,\infty) \times M, \tag{1}$$

where c is a function on M that is pointwise bounded from above and from below by strictly positive constants. The function c gives the isotropic wave speed and a solution u is a sound wave propagating in the medium described by c. In [IV] we consider a wave speed that is discontinuous along a smooth hypersurface. In [I-III] wave speeds are smooth, that is, infinitely differentiable. For the purposes of this introduction, let us assume that c is smooth.

For (1) to have a unique solution we must equip it with initial and boundary conditions. In the context of inverse problems we typically impose vanishing initial conditions

$$u(0,x) = 0, \ \partial_t u(0,x) = 0, \quad x \in M.$$

This means that we assume that there is no wave propagating when we begin a measurement. A boundary source f is typically modelled either by Dirichlet type boundary condition,

$$u(t,x) = f(t,x), \quad (t,x) \in (0,\infty) \times \partial M,$$

or by Neumann type boundary condition,

$$\partial_\nu u(t,x) = f(t,x), \quad (t,x) \in (0,\infty) \times \partial M, \tag{2}$$

where ∂_ν is the normal derivative on the boundary ∂M. In [II] we consider a third option, where the source is modelled by replacing the right hand side of (1) with a distribution supported on the boundary.

Let us consider sources of Neumann type. Then it is natural to model the boundary measurements by using the Neumann-to-Dirichlet operator,

$$\Lambda : f \mapsto u^f|_{(0,\infty)\times\partial M}, \tag{3}$$

where $u^f = u$ is the solution of (1) with vanishing initial conditions and the boundary condition (2). This is the measurement model considered in [III] and in [IV]. By the standard regularity theory for hyperbolic equations we have for $T > 0$,

$$\Lambda : C_0^\infty((0,T)\times\partial M) \to C^\infty((0,T)\times\partial M), \tag{4}$$

see e.g. [**34, 68, 76**]. Moreover, we may consider Λ as a compact operator on $L^2((0,T)\times\partial M)$ for any $T > 0$. This follows from the regularity result,

$$\Lambda : L^2((0,T)\times\partial M) \to H^{1/5-\epsilon}((0,T)\times\partial M), \tag{5}$$

where $\epsilon > 0$ [**69**].

Note that the operators (4) and (5) contain equivalent information since $C_0^\infty((0,T)\times\partial M)$ is dense in $L^2((0,T)\times\partial M)$. In other words, the choice between these two domains does not matter from the the point of view of uniqueness proofs for inverse problems. The same is true for the choice between Neumann and Dirichlet type boundary conditions. Indeed, if the source f is infinitely smooth and vanish near time $t = 0$, then the echo Λf has also these two properties. By solving a wave equation with the Dirichlet boundary condition Λf it follows easily that Λ is invertible. Thus the Neumann-to-Dirichlet operator Λ and the Dirichlet-to-Neumann operator Λ^{-1} contain equivalent information. For a study of different boundary measurements and their equivalance as data sets we refer to [**55**]. From the point of view of practical computations, the Neumann type boundary conditions can be more convenient to work with than the Dirichlet type, since Λ is a compact operator on $L^2((0,T)\times\partial M)$ but Λ^{-1} is not.

An essential feature of the wave equation is the finite speed of propagation. The wave speed c defines a smooth Riemannian metric tensor by

$$g(x) := \frac{(dx^1)^2 + (dx^2)^2 + \cdots + (dx^n)^2}{c(x)^2}, \quad x \in M, \tag{6}$$

and, as M is connected, the metric tensor g defines the Riemannian distance function $d(x,y)$, $x,y \in M$, see e.g. [**25, 52, 73, 92**]. The finite speed of propagation for the wave equation says roughly that oscillations can not propagate from one point $x \in M$ to another point $y \in M$ in shorter time than $d(x,y)$. Let us formulate this more carefully. For an open set $U \subset M$ and $T > 0$, a solution

$$u \in C([0,2T]; H^1(M)) \cap C^1([0,2T]; L^2(M)). \tag{7}$$

of (1) vanish on the set,

$$\{(t,x) \in (0,\infty) \times M;\ d(x, M \setminus U) - |t - T| > 0\},$$

whenever $u(T,x) = \partial_t u(T,x) = 0$ for all $x \in U$, see e.g. [**34, 68**]. To get the above interpretation, we let $U \subset M \setminus \{x\}$ be a neighborhood of y.

The unique continuation principle for the wave equation can be considered as a complementary property to the finite speed of propagation. This is a deep result by Tataru [**106**] saying roughly that oscillations will propagate from $x \in M$ to $y \in M$ within time $d(x,y)$. For an earlier result to this direction, see [**98**]. Let us formulate this more carefully. For an open set $U \subset M$ and $T > 0$, a solution u of (1) with the smoothness properties (7) vanish on the set,

$$\{(t,x) \in (0,\infty) \times M;\ d(x,U) + |t - T| < T\},$$

whenever $u(t,x) = \partial_t u(t,x) = 0$ for all $x \in U$ and $t \in (0,2T)$. To get the above interpretation, we let U be a small neighborhood of y and notice that the point x can not oscillate at time $t = d(x,y)$ if we do not see any oscillations in U during the time interval $(0, 2d(x,y))$.

We say that the Riemannian distance function $d(x,y)$ gives travel time between points x and y in M. The shortest paths with respect to the travel time distance are geodesics on the Riemannian manifold (M,g) and, in fact, it is possible to construct a class of solutions of (1), the Gaussian beams, that travel along the unit speed geodesics of (M,g), see [**5, 6, 7, 97**].

It is often convenient to take one step further and consider the operator $c(x)^2\Delta$ in (1) as a weighted Laplace-Beltrami operator on (M,g),

$$c^2\Delta u = \mu^{-1}|g|^{-1/2} \sum_{j,k=1}^{n} \frac{\partial}{\partial x^j}\left(\mu |g|^{1/2} g^{jk} \frac{\partial u}{\partial x^k}\right), \tag{8}$$

where $(g^{jk}(x))_{j,k=1}^n = (c(x)^2\delta_{jk})_{j,k=1}^n$ is the inverse of g written as a matrix, $|g|$ denotes the determinant of g and $\mu(x) = c(x)^{n-2}$ is the weight function.

Let us denote by dm the weighted Riemannian volume measure,

$$dm(x) = \mu(x)dV(x) = \mu(x)|g(x)|^{1/2}dx, \quad x \in M. \tag{9}$$

Then we have for $u, v \in C_0^\infty(M)$,

$$(10) \qquad \int_M v\, c^2 \Delta u\, dm = \int_M \sum_{j,k=1}^n g^{jk} \frac{\partial v}{\partial x^j} \frac{\partial u}{\partial x^k} dm = \int_M (dv, du)_g \mu dV,$$

where dv and du denote the exterior derivatives of v and u respectively, and $(\cdot, \cdot)_g$ is the inner product given by g on the contangent bundle. We see that $c^2 \Delta$ is formally self-adjoint on the space $L^2(M, dm)$. Moreover, the last integral in (10) is clearly independent of the choice of coordinates in M, whence the variational formulation of the equation (1) has coordinate-free nature. In particular, the operator Λ is invariant with respect to diffeomorphisms $\Phi : M \to M$ fixing the boundary ∂M, that is, satisfying $\Phi(x) = x$ for all $x \in \partial M$. This suggests us to consider (M, g) as an abstract Riemannian manifold. The coordinate-free point of view was first taken by Lee and Uhlmann in the context of Calderón's inverse problem [**74**].

We may conjugate the operator $c(x)^2 \Delta$ with the multiplier operator given by the function $\kappa := c^{(n-2)/2}$ to get,

$$\kappa c^2 \Delta(\kappa^{-1} u) = (\Delta_g + q) u,$$

where Δ_g is the Laplace-Beltrami operator of (M, g) and q is a smooth function determined by c, see e.g. [**55**]. The Dirichlet-to-Neumann operator Λ_κ^{-1} corresponding to the wave equation,

$$(11) \qquad \partial_t^2 u(t, x) - (\Delta_g + q) u(t, x) = 0,$$

can be computed from Λ and $c|_{\partial M}$ by

$$\Lambda_\kappa^{-1} f = \kappa \Lambda^{-1}(\kappa^{-1} f) + (\partial_\nu \kappa) \kappa^{-1} f, \quad f \in C_0^\infty((0, \infty) \times \partial M).$$

This is the measurement model considered in [I].

From a mathematical point of view, it is natural to consider an arbitrary smooth Riemannian metric tensor g in (8), not necessarily the conformally Euclidean one given by (6). Moreover, we may consider a weighted Laplace-Beltrami operator $\Delta_{g,\mu}$ defined by

$$\int_M v\, \Delta_{g,\mu} u\, \mu dV = \int_M (dv, du)_g \mu dV, \quad u, v \in C_0^\infty(M),$$

on an arbitrary compact smooth Riemannian manifold with boundary (M, g), with an arbitrary smooth strictly positive weight function μ. The corresponding wave equation,

$$(12) \qquad \partial_t^2 u(t, x) - \Delta_{g,\mu} u(t, x) = 0, \quad (t, x) \in (0, \infty) \times M,$$

gives a model where the wave speed is anisotropic.

In [I] the direct problem is the wave equation (11), where g is an arbitrary smooth Riemannian metric tensor and the potential q is an arbitrary smooth function. Moreover, in [III] the direct problem is the anisotropic wave equation (12). In [IV] we consider a two dimensional inclusion detection problem, and the direct problem is (12) with $\mu = 1$. Note that $\mu = 1$ in (8) for an isotropic wave speed in two dimensions. In [II], the direct problem is again (12) with $\mu = 1$ but we assume that $M \subset \mathbb{R}^n$.

3. Inverse problems

As explained in the previous section there are many ways to model acoustic boundary measurements. For the purposes of this section, let us consider the direct problem (12). Moreover, to avoid some technicalities, let us assume that the Riemannian manifold $(\partial M, g|_{\partial M})$ is known and that $\mu|_{\partial M} = 1$. Let us consider the Neumann-to-Dirichlet map Λ defined by (3), where $u^f = u$ is the solution of (12) with vanishing initial conditions and the Neumann boundary condition (2). We define the normal derivative in (2) by $\partial_\nu u := (\nabla u, \nu)_g$, where the inner product, the gradient and the exterior unit normal ν are those determined by the metric tensor g.

For open and nonempty sets $\Gamma_1, \Gamma_2 \subset \partial M$ and $T > 0$ we define the restriction of the Neumann-to-Dirichlet operator

$$\Lambda^T_{\Gamma_1,\Gamma_2} : C_0^\infty((0,T) \times \Gamma_1) \to C^\infty((0,T) \times \Gamma_2),$$

by $\Lambda^T_{\Gamma_1,\Gamma_2} f := (\Lambda f)|_{(0,T)\times\Gamma_2}$. In this section we consider inverse problems of the form:

(IP) Given the operator $\Lambda^T_{\Gamma_1,\Gamma_2}$, determine the Riemannian manifold (M, g).

A physical interpretation of $\Lambda^T_{\Gamma_1,\Gamma_2} f$ is that the echo of an acoustic source f located on Γ_1 is measured for T time units with receivers on Γ_2.

We emphasize that (M, g) is considered as an abstract Riemannian manifold in (IP). As Λ is invariant with respect to changes of coordinates fixing the boundary ∂M, we can not reconstruct (M, g) in predefined coordinates unless we have some additional information. This phenomenon was first observed in the context of Calderón's inverse problem following a remark by Tartar, see [**62**]. Moreover, by using singular changes of variables it is possible to construct models of artificially structured materials that can be used to render objects invisible to certain measurements. We refer to the review paper [**36**] for this transformation optics approach to invisibility. The first use of singular changes of variables to construct examples of nondetectability was in [**38**].

The inverse problems considered in [I] and [III] are of the form (IP). In [IV], the metric tensor g is discontinuous along a smooth hypersurface and we consider the inverse problem to determine this hypersurface given the operator Λ. In [II], we consider a measurement model different from Λ. The measurement in [II] corresponds roughly to Λ evaluated for a single nonsmooth source f.

The finite speed of propagation for the wave equation (12) gives a necessary condition for T in order to (IP) to have a unique solution. If there is $x_0 \in M$ such that

$$T < d(x_0, \Gamma_1) + d(x_0, \Gamma_2),$$

then the measurements $\Lambda_{\Gamma_1,\Gamma_2}^T$ can not contain any information about $c(x_0)$. On the other hand, if

$$T > \max_{x \in M} d(x, \partial M),$$

then the operator $\Lambda_{\partial M,\partial M}^{2T}$ determines the operator $\Lambda_{\partial M,\partial M}^{\infty}$, see [**65**]. This kind of time continuation question is considered in [I] for a measurement setup where sources and receivers are not simultaneously located in the same set. In this introduction our standing assumption is that T is "large enough", and the reader may consult the references for the exact assumptions on T.

The problem (IP) with $\Lambda_{\Gamma_1,\Gamma_2}^T = \Lambda_{\partial M,\partial M}^T$ was first solved by Belishev [**9**] in the case of a conformally Euclidean metric (6) and by Belishev and Kurylev [**12**] in the case of an arbitrary metric and a constant weight function μ. Note however, that both these results precede and depend on the unique continuation result by Tataru [**106**]. For arbitrary metric, weight and $\Gamma \subset \partial M$, the problem (IP) with $\Lambda_{\Gamma_1,\Gamma_2}^T = \Lambda_{\Gamma,\Gamma}^T$ was solved by Katchalov and Kurylev [**54**]. The proofs of these three results are all based on a control theoretic tecnique called the boundary control (BC) method. It is the only known technique to show uniqueness for (IP) with an arbitrary anisotropic wave speed in dimensions $n \geq 3$.

In the case of a conformally Euclidean metric (6) the uniqueness for the problem (IP) can be shown also by using the complex geometrical optics (CGO) solutions developed in the context of Calderón's inverse problem [**104**]. See [**85**] for a proof using these techniques. The uniqueness for Calderón's problem in the case of an arbitrary metric tensor is an open question in dimensions $n \geq 3$. For two dimensional domains, the anisotropic problem can be reduced to the isotropic one by using isothermal coordinates [**103**]. See [**33, 57**] for a study on the CGO solutions in the $n \geq 3$ dimensional anisotropic case.

In this thesis we focus on anisotropic problems and use solely the BC method. Let us next explain briefly how the method works. For a detailed exposition of the method we refer to the monograph [**55**]. Let $\Gamma \subset \partial M$ be open and $\tau > 0$. Using Tataru's unique continuation we see, roughly speaking, that the presence of any oscillations at time $t = \tau$ in the domain of influence,

$$M(\Gamma, \tau) = \{x \in M;\ d(x, \Gamma) \leq \tau\},$$

can be observed in a neighborhood of the set $(0, 2\tau) \times \Gamma$. By using a control theoretic duality technique this observability implies the controllability:

(C) $\{u^f(\tau);\ f \in C_0^\infty((0, 2\tau) \times \Gamma)\} \subset L^2(M(\Gamma, \tau))$ is dense,

where $u^f = u$ is the solution of (12) with vanishing initial conditions and the Neumann boundary condition (2).

An elementary integration by parts argument yields the identity,

(B) $(u^f(T), u^h(T))_{L^2(M;dm)} = (f, Kh)_{L^2((0,2T)\times\partial M);dt\otimes dS)}$,

where $T > 0$, dm is the measure defined in (9), dS is the Riemannian volume measure of $(\partial M, g|_{\partial M})$ and $K = J\Lambda - R\Lambda RJ$, where J and R are the time integral and time reversal operators,

$$Jf(t) := \frac{1}{2}\int_0^{2T} 1_L(t, s) f(s) ds, \quad Rf(t) := f(2T - t),$$
$$L := \{(t, s) \in \mathbb{R}^2;\ t + s \leq 2T,\ s > t > 0\}.$$

The identity (B) was first proved by Blagovestchenskii [**16**].

The controllability (C) and the identity (B) are the two main ingredients of the boundary control method. They are used by all the control theoretic studies of (IP) that we are aware of. However, there are many ways to choose the sources h and f in (B) in order to get uselful information about the metric tensor g.

A typical way is to choose such a source $h \in C_0^\infty((0, 2T) \times \partial M)$ that the corresponding solution is approximately localized in a neighborhood of a point $x_0 \in M$ at time $t = T$. Such a source can be explicitly constructed, for example, as a boundary value of a Gaussian beam [**8**].

By (C) we can use (B) to test if the solution $u^h(T)$ is orthogonal to $L^2(M(\Gamma, \tau))$. If $u^h(T)$ is localized around x_0, the orthogonality is, roughly speaking, equivalent to $d(x_0, \Gamma) > \tau$. Letting Γ tend to a point $y \in \partial M$, we see that Λ determines the boundary distance function

$$r_{x_0}(y) := d(x_0, y), \quad y \in \partial M.$$

We stress that before fully solving (IP) we do not know to which point x_0 the function r_{x_0} corresponds. However, we have determined the set of functions,

$$R(M) := \{r_x \in C(\partial M);\ x \in M\},$$

and reduced (IP) to the purely geometric problem:

(GP) Given the set $R(M)$, determine the Riemannian manifold (M, g).

This problem has a unique solution as was proved by Kurylev [**64**], whence we have uniqueness also for (IP) with $\Lambda^T_{\Gamma_1,\Gamma_2} = \Lambda^T_{\partial M,\partial M}$. Moreover, the problem (GP) is stable under certain geometric conditions [**56**].

Let us now return to the question:

(Q) How to choose the boundary sources?

In the isotropic case, a modification of (B) enables us to compute $(u^f(T), \phi)_{L^2(M;dm)}$ for any smooth source f and any harmonic function ϕ. In fact, Belishev used first order polynomials in the article in which the BC method [**9**] was introduced. Localized functions, constructed by using (B) with $u^f(T)$ and $u^h(T)$ supported in different domains of influence, were also needed in [**9**].

A recent computational implementation of the BC method in the isotropic case exploits the fact that the products of pairs of harmonic functions are dense in $L^2(M)$ and thus avoids the use of localized functions [**90**]. We believe that this avoidance contributes to the good quality of the reconstructions obtained in the article. We point out that the density of the products of pairs of harmonic functions was already observed by Calderón in the article in which he formulated the inverse problem now carrying his name [**24**]. In the article, Calderón employed the density to solve a linearized version of the problem.

In the anisotropic case, in addition to the Gaussian beam approach described above, a typical approach is to employ functions localized in an intersection of domains of influence [**10, 15**]. Again such functions can be constructed by using (B) with $u^f(T)$ and $u^h(T)$ supported in different domains of influence.

In [III] we observe that a modification of (B) enables us to compute $(u^f(T), 1)_{L^2(M;dm)}$ for any smooth source f. This modification is analogous with the modification allowing the computation of the inner product with a harmonic function in the isotropic case. We show that it is possible to avoid the use of localized functions also in the anisotropic case under certain geometric assumptions.

We believe that after a careful consideration of the question (Q) the BC method can become a robust, practical reconstruction method.

4. Statement of the results

In [I] we show the following two uniqueness results.

Theorem 1. *If $\bar{\Gamma}_1 \cap \bar{\Gamma}_2 \neq \emptyset$ and $T = \infty$, then (IP) has a unique solution.*

Theorem 2. *Let $\Gamma_1, \Gamma_2, \Gamma_3 \subset \partial M$ be open and nonempty. Then the operators $\Lambda^{\infty}_{\Gamma_1,\Gamma_2}, \Lambda^{\infty}_{\Gamma_1,\Gamma_3}, \Lambda^{\infty}_{\Gamma_2,\Gamma_3}$ determine the Riemannian manifold (M, g).*

Moreover, we prove a result similar to Theorem 2 for finite measurement time T under certain geometric assumptions.

A common theme between [II] and [III] is to study the question (Q). The proof in [II] does not employ the BC method. Instead, the proof is based on analysis of propagation of singularities by using Gaussian beams. We show that the scattering relation of (M, g) can be obtained from the echo of a single source. As pointed out in section 1, the scattering relation determines the Riemannian manifold (M, g) uniquely in many classes of manifolds.

To define the scattering relation, let TM denote the tangent space of M, let $\dot{\gamma}$ denote the tangent vector of a smooth curve $\gamma : [a, b] \to M$, let $SM = \{(x, \xi) \in TM;\ \|\xi\|_g = 1\}$ denote the unit sphere bundle on M and define

$$\partial_{\pm}SM = \{(x, \xi) \in SM;\ x \in \partial M,\ \pm(\nu, \xi)_g < 0\},$$

where ν is the exterior normal vector of ∂M. Moreover, let $\tau(x, \xi)$ be the infimum of the set $\{t \in (0, \infty]; \gamma_{x,\xi}(t) \in \partial M\}$, where $\gamma_{x,\xi}$ denotes the geodesic with initial data $(x, \xi) \in TM$. The scattering relation is the map Σ,

$$\Sigma : D(\Sigma) \to \overline{\partial_+ SM} \times \mathbb{R}, \quad D(\Sigma) = \{(x, \xi) \in \partial_- SM;\ \tau(x, \xi) < \infty\}$$

defined by $\Sigma(x, \xi) = (\gamma_{x,\xi}(\tau(x, \xi)), \dot{\gamma}_{x,\xi}(\tau(x, \xi)), \tau(x, \xi))$.

The main result in [II] is the following.

Theorem 3. *Let $M \subset \mathbb{R}^n$, $n \geq 2$, be a compact set with smooth boundary and nonempty interior, let g be a smooth Riemannian metric on $\mathbb{R}^n$ and let $(x_j)_{j=1}^{\infty} \subset \partial M$ be a dense sequence of disjoint points. Suppose that g is bounded from above and from below. If $T_0 < 0$ and*

$$T > \sup_{(x,\xi) \in \partial_- SM} \tau(x, \xi) \quad or \quad T = \infty,$$

then $g|_{\mathbb{R}^n \setminus M}$ and the measurement $u|_{(T_0,T)\times\partial M}$ of the solution u of

$$(\partial_t^2 - \Delta_g)u(t,x) = \sum_{j=1}^{\infty} 2^{-2^j} \delta_{x_j}(x)\delta(t) \quad \text{in } (0,\infty)\times\mathbb{R}^n,$$
$$u|_{t=T_0} = \partial_t u|_{t=T_0} = 0,$$

determine the scattering relation Σ.

In [III] we show that the volumes of the domains of influence can be computed from the Neumann-to-Dirichlet operator. We define the domain of influence for a function $\tau : \partial M \to \mathbb{R}$ by

$$M(\tau) := \{x \in M;\ \text{there is } y \in \partial M \text{ such that } d(x,y) \le \tau(y)\}.$$

Moreover, we denote

$$C_T(\partial M) := \{\tau \in C(\partial M);\ 0 \le \tau(x) \le T \text{ for all } x \in \partial M\},$$

and recall that a compact Riemannian manifold (M,g) with boundary is simple if it is simply connected, any geodesic has no conjugate points and ∂M is strictly convex with respect to the metric g. The main result in [III] is the following.

Theorem 4. *The volume data,*

$$M(\tau), \quad \tau \in C_T(\partial M), \tag{13}$$

can be computed from Λ_{2T} by solving a sequence of linear equations on $L^2((0,2T)\times\partial M)$. Moreover, if $T > \max_{x,y\in\partial M} d(x,y)$ and (M,g) is simple, then the volume data (13) determines the Riemannian manifold (M,g) uniquely.

In [IV] we show how to use the volume data (13) to localize an inclusion in the metric. We assume that M is two dimensional, $\Sigma \subset M^{int}$ is a compact set with smooth boundary and nonempty interior. Moreover, we assume that the metric is of the form,

$$g(x) = \begin{cases} c(x)^{-2}b(x), & x \in \Sigma, \\ b(x), & x \in M \setminus \Sigma, \end{cases}$$

where b is a smooth Riemannian metric on M and c is a smooth scalar function on Σ satisfying pointwise $c > 1$. The main result in [IV] is the following.

Theorem 5. *If $T > \sup_{y\in\partial M} d(y,\Sigma)$, then the following two implications hold.*

(i) *If (M,b) is simple, then $d(y,\Sigma)$, $y \in \partial M$, can be reconstructed from the volume data (13).*

(ii) *If* $M \subset \widehat{M}$, $b = \widehat{b}|_M$, *where* $(\widehat{M}, \widehat{b})$ *is a complete smooth Riemannian manifold without boundary, and* $(\widehat{M}, \widehat{b})$ *is known, then* $\widehat{d}(y, \Sigma)$, $y \in \partial M$, *can be reconstructed from the volume data (13). Here* $\widehat{d}$ *denotes the Riemannian distance function of* $(\widehat{M}, \widehat{b})$.

The distance function $d(y, \Sigma)$, $y \in \partial M$, can be used to localize the inclusion Σ. In the case (ii) of the previous theorem, we can reconstruct a superset of Σ,

$$H_{\partial M}(\Sigma) := M \setminus \bigcup_{y \in \partial M} \widehat{B}(y, \widehat{d}(y, \Sigma)),$$

where $\widehat{B}(y, r) := \{x \in \widehat{M};\ \widehat{d}(x, y) < r\}$ for $y \in \widehat{M}$ and $r > 0$. If $(\widehat{M}, \widehat{b})$ is the plane $\mathbb{R}^2$ with the Euclidean metric, then

$$\Sigma \subset H_{\partial M}(\Sigma) \subset \mathrm{Conv}(\Sigma),$$

where $\mathrm{Conv}(\Sigma)$ is the convex hull of Σ. We believe that an inclusion detection based on Theorem 5 allows for a robust implementation.

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