

Boundedness of Bergman projections and Toeplitz operators

Paula Mannersalo

Academic dissertation

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Theorem. Let x be a PhD candidate in math. Assume that x has a good supervisor, a loving spouse, a cheerful daughter, supportive parents and friends who stand by x year after year. Moreover, assume that x has curiosity and eagerness towards math, patience to deal with frustration and on top of that a little bit of skill as well. Then, there is a chance, or a risk, depending on the perspective, that the dream of a PhD may come true. *Proof.* We leave the details for the interested reader.

I thank my advisor, Dr. Jari Taskinen, for the supportive supervision, good co-operation and patience over the years. Moreover, I am grateful for his pedagogical skills and that he always took into account my expectations towards this project. I wish to thank Professor Jouni Rättyä and Associate Professor Dragan Vukotić for the pre-examination of my thesis and for the suggestions for improving the manuscript. I would also like to thank Professor Mikael Lindström for acting as the opponent of my dissertation. In addition, I am delighted to have Associate Professor Pekka Pankka as the custos.

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Espoo, January 2019

Paula Mannersalo née Erkkilä

List of included articles

This dissertation consists of an introductory part and the following three articles:

- [A] P. Erkkilä and J. Taskinen. Sup-norm estimates for Bergman-projections on regulated domains. *Mathematica Scandinavica*, Volume 102, Number 1, 111–130, (2008).
- [B] P. Mannersalo. Toeplitz operators with locally integrable symbols on Bergman spaces of bounded simply connected domains. *Complex Variables and Elliptic Equations*, Volume 61, Issue 6, 854–874, (2016). DOI: <https://doi.org/10.1080/17476933.2015.1120293>
- [C] P. Mannersalo. Toeplitz operators on Bergman spaces of polygonal domains. *Proceedings of the Edinburgh Mathematical Society* (to appear). Preprint arXiv:1808.03308 (2018).

The joint article [A] contains a significant contribution by the author. The articles [B] and [C] consist of author's independent research.

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1 Introduction

The thesis deals with Bergman spaces, Bergman projections and Toeplitz operators on Bergman spaces. The theory of these integral operators is a combination of complex analysis, operator theory and functional analysis. The first systematic study of Bergman spaces was presented in 1950 by Stefan Bergman in [2]. The Bergman space $A^2(\Omega)$ over a bounded domain $\Omega \subset \mathbb{C}$ consists of all analytic functions $f : \Omega \rightarrow \mathbb{C}$ which are square-integrable with respect to the Lebesgue area measure. The Bergman space $A^2(\Omega)$ is a closed linear subspace of the Hilbert space $L^2(\Omega)$ and thus a Hilbert space on its own. The Bergman projection P_Ω is the orthogonal projection from $L^2(\Omega)$ onto $A^2(\Omega)$ or, in other words, the integral operator induced by the Bergman kernel function.

In the beginning of the 20th century Otto Toeplitz studied bi-infinite matrices, with constant diagonals, on the space l^2 . The matrix was named as the Toeplitz matrix and the integral operator related to it as the Toeplitz operator. In the original setting, Toeplitz operators act on the Hardy space H^2 of the unit circle. Later, the properties of Toeplitz operators on the Bergman spaces were explored as well – the papers [13, 26, 35, 54] are among the first studies with this approach. Toeplitz operator with a symbol a (in the Bergman space setting) is defined by $T_a(f) = P_\Omega(af)$, where f is an analytic function on Ω and P_Ω is the Bergman projection. In this thesis we consider Toeplitz operators acting on the Bergman spaces $A^p(\Omega)$, $1 < p < \infty$, where $\Omega \subset \mathbb{C}$ is a bounded simply connected domain with certain restrictions for the boundary. Moreover, we deal with generalized Bergman projections on the weighted space $L_v^\infty(\Omega)$, where $\Omega \subset \mathbb{C}$ is a quite general domain, so called regulated domain.

During the last thirty years the knowledge of analytic function spaces and operators acting on them has increased significantly. Bergman spaces A^p and Bergman and Toeplitz operators have been studied extensively because of their importance in pure and applied mathematics and in other sciences. The applications include for example the Berezin-Toeplitz quantization (quantum physics) [17], the Korteweg-de Vries-equation [20] and the connection of the Bergman kernel with partial differential equations [3]. Despite the intense study, the results concerning boundedness (and compactness) are only partial and the following fundamental questions remain open: *For which domains $\Omega \subset \mathbb{C}$ and for which $1 < p < \infty$ the Bergman projection defines a continuous mapping $L^p(\Omega) \rightarrow A^p(\Omega)$ [23]? What is a sufficient and necessary condition for Toeplitz operator to be bounded on the Bergman space A^2 ?* Let $\mathbb{D} \subset \mathbb{C}$ be the unit disk and $1 < p < \infty$. Then, if the symbol a is bounded, it is clear that the Toeplitz operator $T_a : A^p(\mathbb{D}) \rightarrow A^p(\mathbb{D})$ is bounded due to the boundedness of the Bergman projection. In the case of unbounded symbols, there are still open problems, and conditions for the boundedness and compactness in terms of the symbol are known only in some special cases.

The boundedness results in the literature deal with various classes of symbols and domains and different weights on the spaces L^p and A^p . In Section 2 we present some of these results. For further reading on the Bergman and Toeplitz operators, we refer to [15, 24, 40, 59], which give an overview of the general theory and of the research and advances from the 1980's till the 2000's.

The properties of the operators that we focus on are boundedness and compactness. In the classical setting the Bergman projection acts on the Hilbert space $L^2(\mathbb{D})$ and the Toeplitz operator T_a on the Bergman space $A^2(\mathbb{D})$. The main contribution of this thesis is that the boundedness results concern more general bounded simply connected planar domains. Loosely speaking, the more complicated the boundary of the domain is, the more challenging it is to consider the boundedness because of the singularities of the Bergman kernel at the boundary. Moreover, we study Toeplitz operators with quite general symbols, i.e., locally integrable symbols. We also generalize the concepts of Bergman and Toeplitz operators by using more general definitions for them.

The thesis consists of an introductory part and three published articles. In section 2 we

give a general overview of the theory and literature of Bergman spaces and projections and Toeplitz operators. Some necessary concepts and results are also presented. Section 3 includes the main results of the article [A], where we obtain boundedness results for generalized Bergman projections on the growth space $L_v^\infty(\Omega)$. In article [B] we show sufficient conditions for the boundedness and compactness of generalized Toeplitz operators on the Bergman spaces $A^p(\Omega)$, where Ω is a bounded simply connected domain with C^4 -smooth boundary. The main results of [B] are introduced in Section 4. Section 5 consists of the main results of the article [C], which is a direct continuation of the article [B]. In [C] the consideration is extended from smoothly bounded domains to polygons. In the last section we propose a further study by presenting some open questions.

2 Preliminaries

2.1 Bergman spaces

Let Ω be a bounded simply connected domain in the complex plane. The Bergman space $A^p(\Omega)$ consists of all the analytic functions that belong to the space $L^p(\Omega)$ of measurable, complex-valued p -th power integrable functions. More precisely, let $1 \leq p < \infty$,

$$A^p(\Omega) := \left\{ f : \Omega \rightarrow \mathbb{C} \text{ analytic} \mid \|f\|_p := \left(\int_{\Omega} |f(z)|^p dA(z) \right)^{1/p} < \infty \right\},$$

where $dA(z) = \pi^{-1} dx dy$ (with $z = x + iy$) is the Lebesgue area measure in the plane, scaled such that $\int_{\mathbb{D}} dA(z) = 1$. Respectively, the Bergman space $H^\infty(\Omega)$ consists of bounded analytic functions, i.e.

$$H^\infty(\Omega) := \left\{ f : \Omega \rightarrow \mathbb{C} \text{ analytic} \mid \|f\|_\infty := \sup_{z \in \Omega} |f(z)| < \infty \right\}.$$

The notation $H^\infty(\Omega)$ is used because $H^\infty(\Omega)$ is also a Hardy space. Bergman spaces $A^p(\Omega)$ and $H^\infty(\Omega)$ are Banach spaces since $A^p(\Omega)$ (resp. $H^\infty(\Omega)$) is a closed linear subspace of $L^p(\Omega)$ (resp. $L^\infty(\Omega)$). In particular, the space $A^2(\Omega)$ is a Hilbert space with the inner product $\langle f, g \rangle = \int_{\Omega} f(z) \overline{g(z)} dA(z)$, $f, g \in A^2(\Omega)$.

Remark 2.1. The domain Ω could be a more general planar domain or a domain in higher dimensional complex plane \mathbb{C}^n . However, in this thesis we focus on bounded simply connected planar domains.

Bergman spaces have the following fundamental properties, see [15, 24, 59] for details.

(i) Let $1 \leq p < \infty$. For all $f \in A^p(\Omega)$

$$|f(z)| \leq \text{dist}(z, \partial\Omega)^{-\frac{2}{p}} \|f\|_p, \quad \text{for all } z \in \Omega.$$

(ii) Let $K \subset \Omega$ be compact. There exists $C_K > 0$ such that $\sup_{z \in K} |f(z)| \leq C_K \|f\|_p$ for all $f \in A^p(\Omega)$. This follows from (i). In particular, for a fixed $z \in \Omega$, the point-evaluation $f \mapsto f(z)$ is a bounded linear functional on $A^p(\Omega)$.

(iii) Let $1 \leq p < \infty$, $f \in A^p(\mathbb{D})$ and $0 < r < 1$. Let f_r be the dilation of f , i.e. $f_r(z) := f(rz)$ for all $z \in \mathbb{D}$. Then $\|f - f_r\|_p \rightarrow 0$ as $r \rightarrow 1_-$.

(iv) The functions of $A^p(\mathbb{D})$, $1 < p < \infty$, can be approximated by Taylor polynomials with respect to the $\|\cdot\|_p$ -norm. More precisely, let $f(z) = \sum_{k=0}^{\infty} a_k z^k$ be the Taylor series of $f \in A^p(\mathbb{D})$. Then $\lim_{n \rightarrow \infty} \|f - \sum_{k=0}^n a_k z^k\|_p = 0$. For polynomial approximation in Bergman spaces of more general planar domains, see [7].

Let $1 < p < \infty$ and $1/p + 1/q = 1$. It is well known that $A^p(\mathbb{D})^*$, the dual space of $A^p(\mathbb{D})$, is isomorphic to $A^q(\mathbb{D})$: for each functional $F \in A^p(\mathbb{D})^*$ there exists a unique $g \in A^q(\mathbb{D})$ such that $F(f) = \int_{\mathbb{D}} f(z)\overline{g(z)}dA(z)$ for all $f \in A^p(\mathbb{D})$ and under the same integral pairing, each function $g \in A^q(\mathbb{D})$ induces a bounded linear functional F_g on $A^p(\mathbb{D})$. If $p \neq 2$, this isomorphism is not isometric [15, p.35]. The dual spaces of more general Bergman spaces (for example $A^p(\Omega)$ and $A_v^p(\mathbb{D})$) have been studied in [24, 31, 43, 44], where it can be seen that the representation of the dual space is closely connected with the boundedness of the Bergman projection.

We also consider weighted Bergman spaces $A_v^p(\Omega)$ and $H_v^\infty(\Omega)$, where a measurable function $v : \Omega \rightarrow \mathbb{R}_+$ is a weight function. Bergman space $A_v^p(\Omega)$ is defined as $A^p(\Omega)$, the norm $\|\cdot\|_p$ replaced by the weighted norm

$$\|f\|_{p,v} := \left(\int_{\Omega} |f(z)|^p v(z) dA(z) \right)^{1/p}.$$

Respectively, $H_v^\infty(\Omega)$ is defined as $H^\infty(\Omega)$, with the weighted norm

$$\|f\|_{\infty,v} := \operatorname{ess\,sup}_{z \in \Omega} v(z)|f(z)|.$$

If the weight v is continuous, then the space $A_v^p(\Omega)$ is a Banach space. Especially, $A_v^2(\Omega)$ is a Hilbert space with the inner product

$$\langle f, g \rangle_v = \int_{\Omega} f(z)\overline{g(z)}v(z)dA(z) \quad \text{for all } f, g \in A_v^2(\Omega),$$

and with the norm $\|f\|_{2,v} = \langle f, f \rangle_v^{1/2}$.

Remark 2.2. Continuous weights belong to a more general class of weights, admissible weights, which are important in the theory of Bergman spaces. A weight v (on $A^2(\Omega)$) is admissible if for every compact $K \subset \Omega$, there exists $C_K > 0$ such that $\sup_{z \in K} |f(z)| \leq C_K \|f\|_{2,v}$ for all $f \in A_v^2(\Omega)$ [39]. If the weight $v : \Omega \rightarrow \mathbb{R}_+$ is admissible, there exists a unique Bergman type kernel, i.e. reproducing kernel of $A_v^2(\Omega)$.

In this work, we consider weights of the natural form, i.e. a power of the boundary distance

$$\omega_\alpha(z) := (\operatorname{dist}(z, \partial\Omega))^\alpha, \quad z \in \Omega, \quad \alpha > -1, \quad (2.1)$$

and equivalent weights

$$v_\alpha(z) := (1 - |\varphi(z)|^2)^\alpha |\varphi'(z)|^{-\alpha}, \quad z \in \Omega, \quad \alpha > -1, \quad (2.2)$$

where $\varphi : \Omega \rightarrow \mathbb{D}$ is a Riemann conformal mapping. The weights (2.1) and (2.2) are equivalent due to the Koebe distortion theorem [41, p.9]:

$$\frac{1}{4}(1 - |\varphi(z)|^2)|\varphi'(z)|^{-1} \leq \operatorname{dist}(z, \partial\Omega) \leq (1 - |\varphi(z)|^2)|\varphi'(z)|^{-1} \quad \text{for all } z \in \Omega.$$

The equivalence $\omega_\alpha \sim v_\alpha$ is essential since it enables the consideration of the unit disk instead of a (more general) bounded simply connected domain Ω .

Let us now take a closer look at the Hilbert space $A^2(\Omega)$, where $\Omega \subset \mathbb{C}$ is an arbitrary, bounded simply connected domain (with no restrictions for the boundary). Let us fix $z \in \Omega$. Since the point-evaluation $f \mapsto f(z)$ is a bounded linear functional on $A^2(\Omega)$, according to the Riesz representation theorem, there exists a unique function $K_{z,\Omega} \in A^2(\Omega)$ such that

$$f(z) = \langle f, K_{z,\Omega} \rangle = \int_{\Omega} f(w)\overline{K_{z,\Omega}(w)}dA(w) \quad (2.3)$$

for all $f \in A^2(\Omega)$. The function $K_\Omega(z, w) := \overline{K_{z, \Omega}(w)}$ is known as the Bergman kernel function of the domain Ω or as the *reproducing kernel* because of the property (2.3). The Bergman kernel of Ω can be represented as

$$K_\Omega(z, w) = \sum_{n=1}^{\infty} \varphi_n(z) \overline{\varphi_n(w)}, \quad z, w \in \Omega, \quad (2.4)$$

where $\{\varphi_n\}_n$ is an arbitrary orthonormal basis of $A^2(\Omega)$ [15, 18]. The series in (2.4) converges uniformly and absolutely on compact subsets of $\Omega \times \Omega$. The kernel function K_Ω and the Riemann conformal mapping $\varphi : \Omega \rightarrow \mathbb{D}$ are related as follows [18]

$$K_\Omega(z, w) = \varphi'(z) K_{\mathbb{D}}(\varphi(z), \varphi(w)) \overline{\varphi'(w)} = \frac{\varphi'(z) \overline{\varphi'(w)}}{(1 - \varphi(z) \overline{\varphi(w)})^2}, \quad (2.5)$$

where $K_{\mathbb{D}}(\lambda, \xi) = \frac{1}{(1 - \lambda \overline{\xi})^2}$ is the kernel function of $A^2(\mathbb{D})$. The Bergman kernel is conjugate symmetric, i.e. $K_\Omega(z, w) = \overline{K_\Omega(w, z)}$. For further reading on Bergman kernels of various domains and on the boundary behavior of the kernels, see [29].

2.2 Bergman projections

First, we recall basic definitions of projection operators. Let X be a Banach space. Linear operator $P : X \rightarrow X$ is called a projection if $P^2 = P$, i.e., if $P(Px) = Px$ for all $x \in X$. In particular, on a Hilbert space a projection can be orthogonal: Let H be a Hilbert space equipped with an inner product $\langle \cdot, \cdot \rangle$ and let $P : H \rightarrow H$ be a projection. If $\text{Ker } P \perp \text{Im } P$, i.e., $\langle x, y \rangle = 0$ for all $x \in \text{Ker } P$, $y \in \text{Im } P$, then P is called an orthogonal projection. Equivalently, a bounded linear operator P on a Hilbert space H is an orthogonal projection if and only if $P^2 = P = P^*$, where P^* is the adjoint of P .

Let $M \subset H$ be a closed linear subspace of Hilbert space H and let $M^\perp := \{x \in H \mid \langle x, y \rangle = 0 \text{ for all } y \in M\}$ be the orthogonal complement of M . Then, there exists a unique orthogonal projection $P : H \rightarrow M$ such that $\text{Im } P = M$ and $\text{Ker } P = M^\perp$. This projection is called the orthogonal projection from H onto M , and obviously $P|_M$ is the identity operator on M .

We introduce now the Bergman projections used in this thesis.

- (i) The orthogonal projection from $L^2(\mathbb{D})$ onto $A^2(\mathbb{D})$ is called the standard Bergman projection P , and it has the integral representation

$$(Pf)(z) = \int_{\mathbb{D}} \frac{f(w)}{(1 - z\overline{w})^2} dA(w), \quad \text{for } f \in L^2(\mathbb{D}), \quad z \in \mathbb{D}. \quad (2.6)$$

- (ii) Let us denote $L_\alpha^2(\mathbb{D}) := L_{(1-|z|^2)^\alpha}^2(\mathbb{D})$ and $A_\alpha^2(\mathbb{D}) := A_{(1-|z|^2)^\alpha}^2(\mathbb{D})$, where $\alpha > -1$. Let P_α be the orthogonal projection from $L_\alpha^2(\mathbb{D})$ onto $A_\alpha^2(\mathbb{D})$. The projection P_α is called the weighted Bergman projection and it can be expressed as

$$(P_\alpha f)(z) = (1 + \alpha) \int_{\mathbb{D}} \frac{(1 - |w|^2)^\alpha}{(1 - z\overline{w})^{2+\alpha}} f(w) dA(w). \quad (2.7)$$

Especially, if $\alpha = 0$, we have $P_\alpha = P$. Even though the Bergman projection is originally defined on the Hilbert space $L_\alpha^2(\mathbb{D})$, the integral formula (2.7) extends the domain of P_α to $L_\alpha^1(\mathbb{D})$ since the integral (2.7) converges when $f \in L_\alpha^1(\mathbb{D})$. Thus, we can apply the operator P_α to $L_\alpha^p(\mathbb{D})$ for all $1 \leq p < \infty$ [24, p.6]. In particular, if $f \in A_\alpha^1(\mathbb{D})$, then P_α reproduces f , i.e., $P_\alpha f = f$.

- (iii) Let $\Omega \subset \mathbb{C}$ be a bounded, simply connected domain. The Bergman projection P_Ω is the orthogonal projection from $L^2(\Omega)$ onto $A^2(\Omega)$, and it has the integral formula

$$(P_\Omega f)(z) = \int_\Omega \frac{\varphi'(z)\overline{\varphi'(w)}}{(1 - \varphi(z)\overline{\varphi(w)})^2} f(w) dA(w), \quad z \in \Omega, \quad (2.8)$$

where $\varphi : \Omega \rightarrow \mathbb{D}$ is a Riemann conformal mapping. The projection P_Ω has the reproducing property on $A^p(\Omega)$, $p \geq 1$, provided that the integral in (2.8) converges for all $f \in A^p(\Omega)$ and all $z \in \Omega$. This is the case at least if $K_\Omega(\cdot, w) \in A^q(\Omega)$ ($1/p + 1/q = 1$), i.e. if $|\varphi'|^q$ is integrable over Ω (respectively, $|\psi'|^{2-q}$ integrable over \mathbb{D}), see the proof of Theorem 2 in [43].

- (iv) Generalized Bergman projections (or Bergman type projections) $P_{\varphi, \alpha, \eta}$, $\alpha > -1$, $\eta \in \mathbb{R}$, are defined by

$$(P_{\varphi, \alpha, \eta} f)(z) := (1 + \alpha) \int_\Omega \frac{\varphi'(z)\overline{\varphi'(w)}(1 - |\varphi(w)|^2)^\alpha}{(1 - \varphi(z)\overline{\varphi(w)})^{2+\alpha}} \left(\frac{\varphi'(z)}{\varphi'(w)} \right)^\eta f(w) dA(w), \quad (2.9)$$

where $\varphi : \Omega \rightarrow \mathbb{D}$ is a Riemann conformal mapping and $f : \Omega \rightarrow \mathbb{C}$ is a measurable function. These operators were first introduced in [50, p.66] in the case $\eta \in \mathbb{Z}$ and acting on $L_{\omega_\sigma}^p(\Omega)$, where $\omega_\sigma(z) = (\text{dist}(z, \partial\Omega))^\sigma$, $\sigma > -1$ and $1 < p < \infty$. If $\Omega = \mathbb{D}$ and $\eta = 0$, then $P_{\varphi, \alpha, \eta} = P_\alpha$ (see (2.7)). When $\eta = \alpha/2$, we have the orthogonal projection $P_{\varphi, \alpha, \eta} = P_{\varphi, \alpha, \alpha/2}$ from $L_{v_\alpha}^2(\Omega)$ onto $A_{v_\alpha}^2(\Omega)$, where v_α is as in (2.2) (for the proof, see [A, Lemma 3.1]). Recall that $v_\alpha(z) \sim (\text{dist}(z, \partial\Omega))^\alpha$. It follows from an easy change of variables formula that the relation between $P_{\varphi, \alpha, \eta}$ and P_α can be expressed as

$$P_{\varphi, \alpha, \eta} = T_{\varphi, \eta} \circ P_\alpha \circ T_{\psi, \eta}, \quad (2.10)$$

where $T_{\varphi, \eta}(f) = (f \circ \varphi)(\varphi')^{1+\eta}$ and $T_{\psi, \eta}(f) = (f \circ \psi)(\psi')^{1+\eta}$. If the integral in (2.9) converges for all $f \in L_{\omega_\sigma}^p(\Omega)$ and if the operator $P_{\varphi, \alpha, \eta}$ is bounded on $L_{\omega_\sigma}^p(\Omega)$, then $P_{\varphi, \alpha, \eta}$ projects $L_{\omega_\sigma}^p(\Omega)$ onto $A_{\omega_\sigma}^p(\Omega)$: obviously the image $P_{\varphi, \alpha, \eta} f$ is always analytic and moreover, $P_{\varphi, \alpha, \eta}$ reproduces analytic functions, see (2.10). In [A], the main theorem deals with the boundedness of $P_{\varphi, \alpha, \eta}$ on the weighted space $L_\sigma^\infty(\Omega) := L_{\omega_\sigma}^\infty(\Omega)$, where $\omega_\sigma(z) = (\text{dist}(z, \partial\Omega))^\sigma$, $\sigma > 0$ and $\Omega \subset \mathbb{C}$ is a regulated domain.

- (v) The maximal Bergman projection P_Ω^+ (not actually a projection or a linear operator despite the name) is given by

$$(P_\Omega^+ f)(z) := \int_\Omega \frac{|\varphi'(z)| |\overline{\varphi'(w)}|}{|1 - \varphi(z)\overline{\varphi(w)}|^2} |f(w)| dA(w), \quad z \in \Omega, \quad (2.11)$$

for $f \in L^p(\Omega)$. The operator P_Ω^+ is useful in our study (in [B] and [C]) because it is bounded on $L^p(\Omega)$, $1 < p < \infty$, if and only if the Bergman projection P_Ω is bounded on $L^p(\Omega)$ [30].

We can consider all the preceding operators acting on the Banach spaces $L^p(\Omega)$ (or $L_v^p(\Omega)$), where $1 \leq p < \infty$. In the case $p = \infty$ the situation is different because there is no bounded projection from L^∞ onto H^∞ [25]. For example, the Bergman projection P_α , $\alpha > -1$, does not even map $L^\infty(\mathbb{D})$ into $H^\infty(\mathbb{D})$: Let $f(z) := \log(1 - z)$, $z \in \mathbb{D}$. It can be shown that $f \in \text{Im } P_\alpha$ but $f \notin H^\infty(\mathbb{D})$ [59]. The projection P_α maps $L^\infty(\mathbb{D})$ boundedly onto the Bloch space \mathcal{B} , which consists of analytic functions on \mathbb{D} with $\sup_{z \in \mathbb{D}} (1 - |z|^2) |f'(z)| < \infty$ and which includes the space $H^\infty(\mathbb{D})$ [59]. The Bloch space \mathcal{B} is isomorphic to $A_\alpha^1(\mathbb{D})^*$, $\alpha > -1$ [59, p.103]. In our study, we want that the image spaces are Bergman spaces. Thus, in the case $p = \infty$ we consider Bergman projections acting on weighted spaces $L_v^\infty(\Omega)$ (see [A]).

2.3 Geometric preliminaries

In each article [A], [B] and [C] we consider bounded simply connected planar domains Ω with locally connected boundaries (for the definition of a locally connected set, see [41, p.19]). One of our main methods is, whenever useful, to transfer the situation from Ω to the unit disk using the Riemann conformal mapping $\varphi : \Omega \rightarrow \mathbb{D}$ (and its inverse $\psi : \mathbb{D} \rightarrow \Omega$). Therefore, the behavior of the function φ (and ψ) and its derivatives are essential for our study.

A closed Jordan curve γ is of class C^n if it has a parametrization $\gamma : w(\tau)$, $0 \leq \tau \leq 2\pi$, that is n times continuously differentiable and satisfies $w'(\tau) \neq 0$ for all τ . We need the following facts related to the Riemann conformal mapping.

Theorem 2.3. [28, 41] *Let $\Omega \subset \mathbb{C}$ be a bounded simply connected domain and let $\psi : \mathbb{D} \rightarrow \Omega$ be a Riemann conformal mapping.*

- (i) *The mapping ψ has a continuous extension $\psi : \overline{\mathbb{D}} \rightarrow \overline{\Omega}$ if and only if the boundary $\partial\Omega$ is locally connected.*
- (ii) *(Osgood-Taylor-Carathéodory extension theorem) The mapping ψ has a continuous injective extension $\psi : \overline{\mathbb{D}} \rightarrow \overline{\Omega}$, which is a homeomorphism, if and only if the boundary $\partial\Omega$ is a closed Jordan curve.*
- (iii) *If the boundary is a closed C^{n+1} -smooth ($n \geq 0$) Jordan curve, then ψ has a continuous injective extension $\psi : \overline{\mathbb{D}} \rightarrow \overline{\Omega}$ and the derivative $\psi^{(k)}$, $k = 1, \dots, n$ (when $n \geq 1$), has a continuous extension $\psi^{(k)} : \overline{\mathbb{D}} \rightarrow \overline{\Omega}$. Likewise, the inverse function $\varphi : \Omega \rightarrow \mathbb{D}$ has the corresponding properties. Thus, the functions $|\psi^{(k)}|$ and $|\varphi^{(k)}|$, $k = 0, \dots, n$, are bounded above.*

Note that C^1 -smoothness of $\partial\Omega$ does not imply that ψ' has a continuous extension to $\overline{\mathbb{D}}$, an example of this is presented in [41, p.45].

When studying the properties of the Bergman or Toeplitz operators on bounded simply connected domains, the geometry of the boundary is crucial for the study. The main reason is that the boundary behavior of the function φ' , which appears in the expression of the Bergman kernel K_Ω , depends on the geometry of the boundary.

Since, in our study, the boundary of Ω is always locally connected, the mapping $\psi : \mathbb{D} \rightarrow \Omega$ has a continuous extension $\psi : \overline{\mathbb{D}} \rightarrow \overline{\Omega}$. Thus, we can define the curve $w(t) = \psi(e^{it})$, $0 \leq t \leq 2\pi$, which we extend to 2π -periodic function. By an argument function $\arg(\psi(z) - \psi(e^{it}))$, $z \in \mathbb{D}$ (t fixed), we mean the imaginary part of the function $\log(\psi(z) - \psi(e^{it}))$ (t fixed). The branch of the logarithm, for each $t \in [0, 2\pi[$, is chosen so that $(z, t) \mapsto \arg(\psi(z) - \psi(e^{it}))$ is a continuous mapping $\mathbb{D} \times [0, 2\pi[\rightarrow \mathbb{R}$. If $\theta, t \in [0, 2\pi[$, $\theta \neq t$ and $\psi(e^{i\theta}) \neq \psi(e^{it})$, we write $\arg(\psi(e^{i\theta}) - \psi(e^{it})) := \lim_{z \rightarrow e^{i\theta}} \arg(\psi(z) - \psi(e^{it}))$, where $z \in \mathbb{D}$ (see the definition for the argument function in [11, p.297]). The definition is extended by an agreement $\arg(w(\theta + n2\pi) - w(t + m2\pi)) := \arg(w(\theta) - w(t))$ for all $n, m \in \mathbb{Z}$. Moreover, we define (see [41, p.51])

$$\pi\alpha := \lim_{\theta \rightarrow t_-} \arg(w(\theta) - w(t)) - \lim_{\theta \rightarrow t_+} \arg(w(\theta) - w(t)),$$

if the limits exist. The number $\pi\alpha$ ($0 \leq \alpha \leq 2$) is called an inner angle at the boundary point $w(t) = \psi(e^{it})$ (corresponding to t). If $0 < \alpha < 2$ and $\alpha \neq 1$, $\pi\alpha$ is a corner at the point $w(t) \in \partial\Omega$. More precisely, if $0 < \alpha < 1$, the corner is outward and if $1 < \alpha < 2$, the corner is inward. If $\alpha = 1$, there is a tangent of the boundary curve at the point $w(t) = \psi(e^{it}) \in \partial\Omega$. In the case $\alpha = 0$, we have an outward-pointing cusp; in the case $\alpha = 2$, we have an inward-pointing cusp.

For $0 \leq t < 2\pi$, we define the function $\beta(t) := \lim_{\theta \rightarrow t_+} \arg(w(\theta) - w(t))$, which we call the direction angle of the forward half-tangent of $\partial\Omega$ at $w(t) \in \partial\Omega$. The function β is extended to

mapping $\beta : \mathbb{R} \rightarrow \mathbb{R}$ by $\beta(t + n2\pi) := \beta(t) + n2\pi$, where $t \in [0, 2\pi[$ and $n \in \mathbb{Z}$. Moreover, we define

$$\pi \geq \delta_1 := \sup_t \lim_{\theta \rightarrow 0_+} (\beta(t + \theta) - \beta(t - \theta)) \geq 0, \quad (2.12)$$

$$-\pi \leq \delta_2 := \inf_t \lim_{\theta \rightarrow 0_+} (\beta(t + \theta) - \beta(t - \theta)) \leq 0. \quad (2.13)$$

The number $0 \leq \delta_1 \leq \pi$ corresponds to the sharpest outward corner such that $\pi - \delta_1 = \inf_t(\alpha_t\pi)$, where $0 \leq \alpha_t\pi \leq 2\pi$ is the inner angle at the boundary point $w(t)$ (for clarity, we use the notation α_t instead of α in this context). The number $-\pi \leq \delta_2 \leq 0$ corresponds to the biggest inward corner such that $\pi + |\delta_2| = \sup_t(\alpha_t\pi)$. If $\delta_1 = \pi$, there is an outward cusp at the boundary. If $\delta_2 = -\pi$, there is an inward cusp at the boundary.

Our assumptions for the boundary $\partial\Omega$, specified for each article, read as follows.

- (i) In [B] we have the simplest case regarding to the Riemann conformal mapping: We assume that the boundary $\partial\Omega$ is C^4 -smooth. This guarantees the following: $1/C \leq |\varphi(z)| \leq C$, $1/C' \leq |\varphi'(z)| \leq C'$, $|\varphi''(z)| \leq C''$ and $|\varphi'''(z)| \leq C'''$ for all $z \in \bar{\Omega}$.
- (ii) In [C] we consider polygonal domains with a finite number of corners. Let $\Omega \subset \mathbb{C}$ be an n -sided polygon with inner angles $\alpha_1\pi, \dots, \alpha_n\pi$ ($0 < \alpha_k < 2$, $\alpha_k \neq 1$ for all k), and let $w_1, \dots, w_n \in \partial\Omega$ be the corresponding vertices. It follows from the Schwarz-Christoffel formula that (see [41, p.42])

$$\psi'(z) = A \prod_{k=1}^n (1 - \bar{z}_k z)^{\alpha_k - 1}, \quad z \in \mathbb{D}, \quad (2.14)$$

where A is a constant and $z_k = \varphi(w_k) \in \partial\mathbb{D}$ for all k . Respectively,

$$\varphi'(w) = A^{-1} \prod_{k=1}^n (1 - \overline{\varphi(w_k)} \varphi(w))^{1 - \alpha_k}, \quad w \in \Omega, \quad (2.15)$$

where $w_k = \psi(z_k) \in \partial\Omega$ are the vertices.

- (iii) The most complicated case takes place in [A], where Ω is a bounded regulated domain. Then, ψ' has the representation (see [41, p.62])

$$\log \psi'(z) = \log |\psi'(0)| + \frac{i}{2\pi} \int_0^{2\pi} \frac{e^{it} + z}{e^{it} - z} \left(\beta(t) - t - \frac{\pi}{2} \right) dt, \quad z \in \mathbb{D}, \quad (2.16)$$

where $\beta(t)$ is the direction angle of the forward half-tangent of the boundary point $\psi(e^{it}) \in \partial\Omega$.

2.4 Boundedness of Bergman projections, classical and recent results

Let us first recall a few basic facts on boundedness. An orthogonal projection is always bounded, and the norm of it equals to 1 if the image of the projection is non-trivial. On the other hand, a general projection need not be bounded. For example, let $P : l^\infty \rightarrow l^\infty$ be such that $P(x) = (x_1 - 2x_2, 0, x_3 - 3x_4, 0, x_5 - 4x_6, 0, \dots)$ for all $x = (x_1, x_2, x_3, \dots) \in l^\infty$. Then P is a projection, but it is unbounded.

Suppose that X and Y are Banach spaces and $T : X \rightarrow Y$ is a bounded linear operator. Then T induces a linear operator $T^* : Y^* \rightarrow X^*$, the adjoint of T , such that $(T^*f)(x) := f(T(x))$, where $x \in X$ and $f \in Y^*$. If T is bounded, then T^* is bounded as well and $\|T^*\| = \|T\|$.

The boundedness of the Bergman projection P_Ω is ‘‘conjugate symmetric’’: Suppose that for all $w \in \Omega$, $K_\Omega(\cdot, w) \in A^p(\Omega) \cap A^q(\Omega)$, which guarantees the convergence of the integrals in the

formulas $P_\Omega f$ and $P_\Omega g$, $f \in L^p(\Omega)$, $g \in L^q(\Omega)$. Then, by the duality, P_Ω is bounded on $L^p(\Omega)$ if and only if it is bounded on $L^q(\Omega)$, where $1 < p < \infty$ and $1/p + 1/q = 1$ [23, 43].

In the study of Bergman projections the following well-known integral estimates [24, 48] are often useful : Let $\alpha > -1$, $\beta \in \mathbb{R}$ and

$$I_{\alpha,\beta}(z) := \int_{\mathbb{D}} \frac{(1 - |w|^2)^\alpha}{|1 - z\bar{w}|^{2+\alpha+\beta}} dA(w) \quad \text{and} \quad J_\beta(z) := \int_0^{2\pi} \frac{d\theta}{|1 - ze^{-i\theta}|^{1+\beta}}, \quad z \in \mathbb{D}.$$

Then

$$I_{\alpha,\beta}(z) \sim J_\beta(z) \sim \begin{cases} 1 & \text{if } \beta < 0, \\ \log \frac{1}{1-|z|^2} & \text{if } \beta = 0, \\ \frac{1}{(1-|z|^2)^\beta} & \text{if } \beta > 0, \end{cases} \quad (2.17)$$

as $|z| \rightarrow 1_-$. The estimates in (2.17) are called Forelli-Rudin estimates.

Let $\Omega \subset \mathbb{C}$ be a bounded simply connected domain. Then, all polynomials belong to $A^p(\Omega)$ and $A^p(\Omega)$ is infinite-dimensional. Moreover, assume that Ω is such that the Bergman projection is well-defined on $L^p(\Omega)$, i.e., that the corresponding integral converges for all $f \in L^p(\Omega)$ and all $z \in \Omega$. Then, the Bergman projection restricted to $A^p(\Omega)$ is the identity map on the infinite-dimensional space $A^p(\Omega)$, so it can not be compact. Thus, we only consider the boundedness of the Bergman projections.

Although the boundedness of the Bergman projection P_Ω on $L^p(\Omega)$ has not been determined completely yet, several partial results, with some smoothness assumptions on $\partial\Omega$, are known. For the boundedness results on various planar domains, we refer to [4, 5, 6, 8, 9, 23, 30, 43, 44, 46, 50]. In the rest of the subsection we give a brief overview of this topic. Some of the results deal with the weighted Bergman spaces.

Let $1 \leq p < \infty$. The Bergman projection P_α , $\alpha > -1$ (see (2.7)), defines a bounded projection on $L_\beta^p(\mathbb{D}) := L_{(1-|z|^2)^\beta}^p(\mathbb{D})$, $\beta > -1$, if and only if $p(1 + \alpha) > 1 + \beta$, see [31, 59]. Thus, P_α is a bounded projection from $L^p(\mathbb{D})$ onto $A^p(\mathbb{D})$ if and only if $p(1 + \alpha) > 1$. In particular, the standard Bergman projection P is a bounded projection $P : L^p(\mathbb{D}) \rightarrow A^p(\mathbb{D})$ for all $1 < p < \infty$ but unbounded on $L^1(\mathbb{D})$ and on $L^\infty(\mathbb{D})$. As an example of the use of the Forelli-Rudin estimates we show the unboundedness on $L^1(\mathbb{D})$ (see [15, pp.37–38]): Let us consider the adjoint $P^* = P$ on $L^\infty(\mathbb{D}) \cong (L^1(\mathbb{D}))^*$. Let $f_z(w) := (1 - z\bar{w})^2 |1 - z\bar{w}|^{-2}$, $z, w \in \mathbb{D}$. Now $\|f_z\|_\infty = 1$ for all $z \in \mathbb{D}$. It follows from the Forelli-Rudin estimates that $(P^* f_z)(z) = (P f_z)(z) \sim \log \frac{1}{1-|z|^2}$ as $|z| \rightarrow 1_-$. Thus $\|P^* f_z\|_\infty \rightarrow \infty$ as $|z| \rightarrow 1_-$, which implies the unboundedness of P^* on $L^\infty(\mathbb{D})$ and the unboundedness of P on $L^1(\mathbb{D})$.

The projection P_α is not bounded on $L^\infty(\mathbb{D})$; as mentioned earlier, there is no bounded projection from $L^\infty(\mathbb{D})$ onto $H^\infty(\mathbb{D})$. However, it is possible to define bounded projections from $L_v^\infty(\mathbb{D})$ onto $H_v^\infty(\mathbb{D})$: for example, P_α is bounded on $L_v^\infty(\mathbb{D})$, where $v(z) = (1 - |z|^2)^\beta$ and $\alpha + 1 > \beta > 0$ [33].

Shikhvatov, see [43, 44], studied the question when the Bergman spaces $A^p(\Omega)$ and $A^q(\Omega)$ ($1 < p < \infty$, $1/p + 1/q = 1$) are mutually conjugate (i.e. when $A^p(\Omega)^*$ and $A^q(\Omega)$ are isomorphic) in the case where the boundary of a simply connected domain Ω is a piecewise analytic curve (an analytic Jordan curve except for a finite number points where the boundary has a corner or a cusp). His research included boundedness results of P_Ω in terms of the largest inner angle of the boundary, see [43, Theorem 2, Theorem 3] and [44, Theorem 1].

The research of Solov'ev included convex or piecewise C^1 domains [45, 46]. Békollé characterized the boundedness of the Bergman projection in terms of the $B_p(\mathbb{D})$ -condition (described below) and considered the boundedness of P_Ω in Lavrentiev domains [4, 5]. The projection $P_{\mathbb{D}} = P : L_\omega^p(\mathbb{D}) \rightarrow A_\omega^p(\mathbb{D})$ ($1 < p < \infty$) is bounded if and only if the locally integrable positive weight ω satisfies the Békollé condition $B_p(\mathbb{D})$: there exists a constant $C_p > 0$ such that for all

Carleson squares Q in \mathbb{D}

$$\left(\frac{1}{|Q|} \int_Q \omega\right) \left(\frac{1}{|Q|} \int_Q \omega^{-1/(p-1)}\right)^{p-1} \leq C_p,$$

where $|Q|$ denotes the normalized area of Q [4]. Therefore, P_Ω is a bounded projection from $L^p(\Omega)$ onto $A^p(\Omega)$ if and only if $\omega = |\psi'|^{2-p}$ satisfies the condition $B_p(\mathbb{D})$ ($\psi : \mathbb{D} \rightarrow \Omega$ is the inverse of the Riemann mapping φ in (2.8)). Necessary and sufficient conditions for the boundedness of P_Ω on Lavrentiev domains were shown in [5, Theoreme 2.1], and this result was generalized to the case of regulated domains by Taskinen [50, 51]. Burbea's study [8] deals with multiply connected planar domains with some smoothness requirements for the boundary. The following theorem is a result of Hedenmalm [23].

Theorem 2.4. [23, Theorem 1.1] *Let Ω be a simply connected domain in \mathbb{C} , other than the plane itself. There exists a universal constant p_0 , $\frac{4}{3} \leq p_0 < 2$, independent of Ω , such that the Bergman projection $P_\Omega : L^2(\Omega) \rightarrow A^2(\Omega)$ extends to a bounded projection $L^p(\Omega) \rightarrow A^p(\Omega)$ for all p in the interval $p_0 < p < p'_0$, where $1/p_0 + 1/p'_0 = 1$. Moreover, the boundedness of the Bergman projection fails in general outside the closed interval $[p_0, p'_0]$.*

Note that there are no restrictions for the boundary $\partial\Omega$ in the previous theorem and that according to the famous Brennan's conjecture, which is still an open problem, $p_0 = 4/3$. For further boundedness results on non-smooth, simply connected planar domains we refer to [30, Theorem 3.2].

2.5 Toeplitz operators on Bergman spaces, classical and recent results

Given a measurable $a : \Omega \rightarrow \mathbb{C}$, the classical Toeplitz operator T_a on the Bergman space $A^p(\Omega)$ is defined by $T_a(f) = P_\Omega(af)$ if the integral in the formula $P_\Omega(af)(z)$ converges for all $z \in \Omega$. The function a is called the symbol of T_a . In other words, Toeplitz operator is a composition of the multiplication operator $M_a(f) := af$ (restricted to the Bergman space) and the Bergman projection P_Ω . If $\Omega = \mathbb{D}$, then $T_a(f) = P(af)$, where P is the standard Bergman projection (2.6). Toeplitz operator $T_a : A^p(\Omega) \rightarrow A^p(\Omega)$ has the following integral representation

$$(T_a f)(z) = \int_\Omega \frac{\varphi'(z)\overline{\varphi'(w)}a(w)f(w)dA(w)}{(1 - \varphi(z)\overline{\varphi(w)})^2}, \quad (2.18)$$

if the integral converges.

The problem of characterizing the boundedness of T_a on the Bergman spaces is still open. However, several partial results in some special cases are known (regarding to symbols or domains). If $a \in L^\infty(\mathbb{D})$, T_a is clearly bounded on $A^p(\mathbb{D})$, $1 < p < \infty$, due to the boundedness of the Bergman projection P . Toeplitz operator is not bounded for all $a \in L^1(\mathbb{D})$, and the challenge is the following: to determine when Toeplitz operators with unbounded symbols are bounded. In general, it seems that the behavior of a certain average of the symbol near the boundary is more essential than the behavior of the symbol itself when considering the boundedness or compactness of the Toeplitz operator. If the symbol behaves badly near the boundary, it is still possible that Toeplitz operator is bounded or even compact, for example, due to a suitable oscillation of the symbol [21]. As an example, we consider Toeplitz operators with two kinds of radial symbols ($a(z) = a(|z|)$) [21]: The Toeplitz operator T_a with the symbol $a(r) = (1 - r^2)^\alpha$ ($\alpha > -1$) is bounded on $A^2(\mathbb{D})$ if and only if $\alpha \geq 0$ and compact if and only if $\alpha > 0$. On the other hand, the Toeplitz operator T_a with the oscillating symbol $a(r) = (1 - r^2)^{-\beta} \sin(1 - r^2)^{-\alpha}$, $\alpha > 0$, $0 < \beta < 1$, is bounded on $A^2(\mathbb{D})$ if $\alpha \geq \beta$ and compact if $\alpha > \beta$. Examples of bounded and compact Toeplitz operators on $A^2(\mathbb{D})$ with unbounded symbols can be found also in [12], where symbols are essentially unbounded only on specific types of sets of positive measure.

In the study of Toeplitz operators, the Berezin transform of a symbol is often useful: the behavior of the Berezin transform gives information about the properties of the operator. The Berezin transform \tilde{a} of symbol a (and operator T_a) is a function on Ω defined as

$$\tilde{a}(z) = \widetilde{T_a}(z) := \langle T_a k_z, k_z \rangle = \langle P_\Omega(a k_z), k_z \rangle = \langle a k_z, P_\Omega k_z \rangle = \langle a k_z, k_z \rangle,$$

where $k_z(w) := K_{z,\Omega}(w) / \|K_{z,\Omega}\|_2 = \frac{\overline{\varphi'(z)}\varphi'(w)(1-|\varphi(z)|^2)}{|\varphi'(z)|(1-\overline{\varphi(z)}\varphi(w))^2}$ is the normalized Bergman kernel of $A^2(\Omega)$. Thus, the Berezin transform has the integral formula

$$\tilde{a}(z) = \widetilde{T_a}(z) = \int_\Omega a(w) |k_z(w)|^2 dA(w) = (1 - |\varphi(z)|^2)^2 \int_\Omega \frac{a(w) |\varphi'(w)|^2}{|1 - \overline{\varphi(z)}\varphi(w)|^4} dA(w).$$

If T_a is bounded on $A^2(\Omega)$ (where Ω is a bounded simply connected planar domain), then the Berezin transform \tilde{a} is bounded on Ω :

$$|\tilde{a}(z)| = |\langle T_a k_z, k_z \rangle| \leq \|T_a\| \quad \text{for all } z \in \Omega.$$

If T_a is compact on $A^2(\Omega)$, then the Berezin transform \tilde{a} vanishes at the boundary: It follows from [56, Lemma 2.3] (by changing variables) that $k_z \rightarrow 0$ weakly in $A^2(\Omega)$ as $z \rightarrow \partial\Omega$. Thus, $\tilde{a}(z) \rightarrow 0$ as $z \rightarrow \partial\Omega$ if T_a is compact. These facts hold also in $A^p(\Omega)$ ($1 < p < \infty$) if $\Omega = \mathbb{D}$ (see [14], [34, Lemma 2.3] and the proof of Theorem 1.1 in [36]) or if Ω is a bounded simply connected domain with C^2 -smooth boundary. The latter claim can be shown by similar arguments as in the case $p = 2$: consider the functions $k_{z,p} := K_{z,\Omega} / \|K_{z,\Omega}\|_p$, $k_{z,q} := K_{z,\Omega} / \|K_{z,\Omega}\|_q$ ($1/p + 1/q = 1$), where $\|K_{z,\Omega}\|_p \sim |\varphi'(z)|^{-1}(1 - |\varphi(z)|^2)^{2/q}$ and $\|K_{z,\Omega}\|_q \sim |\varphi'(z)|^{-1}(1 - |\varphi(z)|^2)^{2/p}$ by the Forelli-Rudin estimates, and note that $|\varphi'|$ is bounded (above and below away from zero) because of the boundary smoothness.

We collect here some results on different classes of symbols.

- The boundedness of Toeplitz operators has been characterized for **positive** $L^1(\mathbb{D})$ -symbols. Let $a \geq 0$ and $a \in L^1(\mathbb{D})$. Let $B(z, r)$ be a Bergman disk (hyperbolic disk) with a center $z \in \mathbb{D}$ and a radius $r > 0$. The averaging function of a in the Bergman metric is denoted by $\hat{a}_r(z)$ and defined as

$$\hat{a}_r(z) := \frac{1}{|B(z, r)|} \int_{B(z, r)} a(w) dA(w),$$

where $|B(z, r)|$ is the Euclidean area of $B(z, r)$. The following statements are equivalent, see [36, 52, 58, 59]:

- (i) The Toeplitz operator $T_a : A^p(\mathbb{D}) \rightarrow A^p(\mathbb{D})$ is bounded;
- (ii) The Berezin transform $\tilde{a}(z)$ is bounded on \mathbb{D} ;
- (iii) The averaging function $\hat{a}_r(z)$ is bounded on \mathbb{D} .

An analogous result is known for the compactness [52, 58, 59]: Let $a \geq 0$ and $a \in L^1(\mathbb{D})$. The following statements are equivalent:

- (i) The Toeplitz operator $T_a : A^p(\mathbb{D}) \rightarrow A^p(\mathbb{D})$ is compact;
- (ii) The Berezin transform $\tilde{a}(z) \rightarrow 0$ as $z \rightarrow \partial\mathbb{D}$;
- (iii) The averaging function $\hat{a}_r(z) \rightarrow 0$ as $z \rightarrow \partial\mathbb{D}$.

Moreover, for bounded positive symbols, a characterization of the compactness of T_a on $A^2(\mathbb{D})$ in terms of Carleson type measures was presented in [35]. For positive L^1 -symbols the boundedness and compactness criteria in terms of Carleson measures were shown in [32].

- Let a be a **continuous** function on $\overline{\mathbb{D}}$. Then the Toeplitz operator T_a on $A^p(\mathbb{D})$ is compact if and only if a vanishes at the boundary [35, 56].
- The characterization of compact Toeplitz operators on $A^2(\mathbb{D})$ with **bounded** symbols was presented in [49] (in fact, this result concerns the open unit ball and polydisks in \mathbb{C}^n as well). In [1, Corollary 2.5] it was proved that if $a \in L^\infty(\mathbb{D})$, then T_a is compact on $A^2(\mathbb{D})$ if and only if $\tilde{a}(z) \rightarrow 0$ as $z \rightarrow \partial\mathbb{D}$. This result was extended in [36, Theorem 1.3] to the case $1 < p < \infty$ and to the class $\text{BT} := \{a \in L^1(\mathbb{D}) : \sup_{z \in \mathbb{D}} |\tilde{a}(z)| < \infty\}$, where $L^\infty(\mathbb{D}) \subset \text{BT}$. Let $a \in \text{BT}$. Then T_a is compact on $A^p(\mathbb{D})$ if and only if $\tilde{a}(z) \rightarrow 0$ as $z \rightarrow \partial\mathbb{D}$. The generalization of [1] to the class of smoothly bounded multiply-connected planar domains was presented in [42].
- The boundedness and compactness of Toeplitz operators on $A^2(\mathbb{D})$ with **radial** symbols were characterized in [21, Corollary 2.6]: Let $a(z) = a(r)$, $r = |z|$. Then T_a is bounded if and only if $\sup_{m \in \mathbb{Z}_+} (m+1) |\int_0^1 a(\sqrt{r}) r^m dr| \leq C$ and compact if and only if $\int_0^1 a(\sqrt{r}) r^m dr \rightarrow 0$ as $m \rightarrow \infty$. Moreover, T_a is bounded on $A^2(\mathbb{D})$ if $\sup_r \frac{1}{1-r} |\int_r^1 a(t) dt| \leq C$ and compact if $\frac{1}{1-r} \int_r^1 a(t) dt \rightarrow 0$ as $r \rightarrow 1_-$. In [27] it was shown that if, in particular, symbol a is a *bounded* radial function on \mathbb{D} , then T_a is compact on $A^2(\mathbb{D})$ *if and only if* $\frac{1}{1-r} \int_r^1 a(t) dt \rightarrow 0$ as $r \rightarrow 1_-$. For further study on radial symbols, we refer to [34, 60].
- Toeplitz operators with **BMO-symbols** (bounded mean oscillation) have been studied for example in [61]: Let $a \in \text{BMO}(\mathbb{D}) \subset L^1(\mathbb{D})$. Then T_a is bounded on $A^2(\mathbb{D})$ if and only if \tilde{a} is bounded on \mathbb{D} and T_a is compact on $A^2(\mathbb{D})$ if and only if $\tilde{a}(z) \rightarrow 0$ as $z \rightarrow \partial\mathbb{D}$. The class of $\text{BMO}(\mathbb{D})$ contains positive and bounded symbols. The results of Zorboska [61] were extended to general $L^1(\mathbb{D})$ -symbols in [14].
- Toeplitz operators with **locally integrable symbols** have been studied in [52, 53], [B] and [C]. We consider these results in detail in Sections 4 and 5.

For examples of other type of settings studied so far, we refer to [19, 22, 57] (the unit ball), [16] (bounded symmetric domain in \mathbb{C}^n), [10] (multiply connected domain), [55] (a minimal bounded homogeneous domain).

3 Boundedness of Bergman projections

3.1 Background

In article [A] we continue the study of Taskinen [50, 51] on Bergman type projections, see (4.7) in [50]. The projections of [50] are as the operators $P_{\varphi, \alpha, \eta}$ in (2.9), restricted to the case $\eta \in \mathbb{Z}$ (substitute $\eta = n - 1$ into (2.9) to get (4.7) of [50]). Taskinen considered the boundedness of the Bergman type projections on the weighted spaces $L_{\omega_\sigma}^p(\Omega)$, where Ω is a bounded regulated planar domain and $\omega_\sigma(z) = (\text{dist}(z, \partial\Omega))^\sigma$, $\sigma > -1$ (note that for clarity, we use the notation ω_σ instead of ω , the notation of [50]). However, only the cases $1 < p < \infty$ were studied. In article [A] we generalize the notion of the Bergman type projections (see (2.9)) and deal with the case $p = \infty$. We give necessary and sufficient conditions for the boundedness of the Bergman type projections $P_{\varphi, \alpha, \eta}$ on the space $L_{\omega_\sigma}^\infty(\Omega)$, $\sigma > 0$. In most cases the conditions depend on the geometry of the domain.

A (bounded) regulated domain is defined as follows (for details, see [41]): Let $\Omega \subset \mathbb{C}$ be a bounded simply connected domain with a locally connected boundary $\partial\Omega$. Let $\psi : \overline{\mathbb{D}} \rightarrow \overline{\Omega}$ be the continuous extension of $\psi : \mathbb{D} \rightarrow \Omega$. The domain Ω is called regulated if

- (i) each point of $w(t) := \psi(e^{it}) \in \partial\Omega$ is attained only finitely often by ψ , and

(ii) $\beta(t) := \lim_{\theta \rightarrow t_+} \arg(w(\theta) - w(t))$ exists for all $t \in [0, 2\pi]$ (see the comments preceding the definition (2.12)), and

(iii) the function $\beta : [0, 2\pi] \rightarrow \mathbb{R}$ can be uniformly approximated by finitely many step functions: for every $\epsilon > 0$ there exist $0 = t_0 < t_1 < \dots < t_n = 2\pi$ and constants $\gamma_1, \dots, \gamma_n$ such that $|\beta(t) - \gamma_k| < \epsilon$ for all $t_{k-1} < t < t_k$, all $k = 1, \dots, n$.

Since β is continuous except for countably many jumps (which correspond to the corners and cusps), domain is regulated if there are only countably many corners or cusps at the boundary and otherwise continuously varying tangent. Note that the boundary need not be a Jordan curve. In particular, a domain with a piecewise smooth boundary is regulated.

3.2 Main results of Article [A]

The following integral estimate was proved in article [A]. The method of the proof was to divide the domain of the integration into parts and to apply Forelli-Rudin estimates. Similar results in n -dimensional case can be found for example in [37, Lemmas 3.4 and 3.5] and [38, Lemma 2.5].

Lemma 3.1. [A, Lemma 2.2] *If $\alpha > a + \max(b, 1) - 2$, $a > \max(0, -b)$ and $\lambda \in [0, 2\pi]$, then there exists a constant $C > 0$ such that*

$$\int_{\mathbb{D}} \frac{(1 - |\zeta|)^{\alpha-a}}{|1 - e^{-i\lambda\zeta}|^b |1 - z\bar{\zeta}|^{2+\alpha}} dA(\zeta) \leq \frac{C}{(1 - |z|)^a |1 - e^{-i\lambda z}|^b} \quad \text{for all } z \in \mathbb{D}.$$

The weight in the next lemma corresponds to the case where Ω is a polygon (or homeomorphic to a polygon). See [A, Lemma 3.2] and the formula for $\psi'(z)$ in (2.14). The Bergman projection P_α is as defined in (2.7).

Lemma 3.2. [A, Prop. 2.1, Cor. 2.4] *Let $n \in \mathbb{N}$ and let the real numbers a and b_j , $j = 1, \dots, n$, satisfy $a > \max_{j=1, \dots, n} (0, -b_j)$. Let the n different numbers $\theta_j \in [0, 2\pi]$ be given, and let v be the weight*

$$v(z) = (1 - |z|)^a \prod_{j=1}^n |1 - e^{-i\theta_j} z|^{b_j}, \quad z \in \mathbb{D}.$$

The Bergman projection $P_\alpha : L_v^\infty(\mathbb{D}) \rightarrow H_v^\infty(\mathbb{D})$ is bounded if and only if $\alpha > a + \max_{j=1, \dots, n} (1, b_j) - 2$.

The next theorem is a continuation of [50, Theorem 4.5]. Theorem 4.5 in [50] deals with the case $1 < p < \infty$ and the Bergman projection $P_{\varphi, \alpha, \eta}$, where $\eta = n - 1$ and $n \in \mathbb{Z}$.

Theorem 3.3. [A, Thm. 4.1] *Let $\Omega \subset \mathbb{C}$ be a bounded regulated domain, $\omega_\sigma(z) := (\text{dist}(z, \partial\Omega))^\sigma$, $\sigma > 0$, $\eta \in \mathbb{R}$, $\alpha > -1$ and $P_{\varphi, \alpha, \eta}$ as defined in (2.9). Let δ_1 and δ_2 be as in (2.12) and (2.13).*

(i) *Assume that $1 + \eta - \sigma \geq 0$ and $\sigma(1 + \frac{|\delta_2|}{\pi}) > (1 + \eta) \frac{|\delta_1|}{\pi}$. The Bergman projection $P_{\varphi, \alpha, \eta}$ is a bounded projection from $L_\sigma^\infty(\Omega) := L_{\omega_\sigma}^\infty(\Omega)$ onto $H_\sigma^\infty(\Omega) := H_{\omega_\sigma}^\infty(\Omega)$, if*

$$\alpha > \sigma + \max \left\{ (1 + \eta - \sigma) \frac{\delta_1}{\pi}, 1 \right\} - 2. \quad (3.19)$$

Conversely, if

$$\alpha < \sigma + \max \left\{ (1 + \eta - \sigma) \frac{\delta_1}{\pi}, 1 \right\} - 2, \quad (3.20)$$

then $P_{\varphi, \alpha, \eta}$ is not bounded on $L_\sigma^\infty(\Omega)$.

(ii) Assume that $1 + \eta - \sigma \leq 0$ and $\sigma(1 + \frac{\delta_1}{\pi}) > (1 + \eta)\frac{\delta_1}{\pi}$. Then the result of (i) holds with $\delta_1 \geq 0$ replaced by $\delta_2 \leq 0$.

Example 3.4. It is difficult to find any simple geometric characterization of the boundedness according to Theorem 3.3. However, we give a few concrete examples when Theorem 3.3 leads to a geometric condition for the boundary $\partial\Omega$:

- (a) If, in particular, $\eta = 1$, $\sigma \geq 6$ and $\alpha = \sigma$, then it follows from Theorem 3.3 (ii) that $P_{\varphi, \alpha, \eta}$ is bounded if $|\delta_2| < \pi/2$ (and unbounded if $|\delta_2| > \pi/2$), i.e., if the maximum inner angle of $\partial\Omega$ is less than $3\pi/2$.
- (b) Let Ω be such that $\delta_2 = 0$, i.e. Ω does not have any inward corners or cusps. Moreover, assume that $\alpha = \sigma$ and $\eta = \sigma + 2$. Then we have a restriction for the outward angles: Theorem 3.3 (i) implies that $P_{\varphi, \alpha, \eta}$ is bounded if $\delta_1 < 2\pi/3$, i.e., if the minimum inner angle of $\partial\Omega$ is greater than $\pi/3$ (and if $\delta_1 > 2\pi/3$ then $P_{\varphi, \alpha, \eta}$ is unbounded).
- (c) Let us consider especially the (orthogonal) projection $P_{\varphi, \alpha, \alpha/2}$. Theorem 3.3 implies that if $\sigma > 1/3$, then $P_{\varphi, \alpha, \alpha/2}$ is bounded on $L_{\omega_\sigma}^\infty(\Omega)$ if $\sigma - 1 < \alpha < 2\sigma(\min\{\frac{\pi}{\delta_1}, \frac{\pi}{|\delta_2|}\} + 1) - 2$.

Remark 3.5. There are the following misprints in [A]. On the last line of (2.15) the sign \leq should be replaced by \geq . On the line following (2.20) $e^{i\theta_j}$ should be corrected to $e^{-i\theta_j}$. On page 121, instead of $\eta \geq 0$ there should be $\eta \in \mathbb{R}$. In the end of the page 124, the correct reference is (4.8)–(4.11) instead of (4.8)–(4.1). In the last integral of (4.33) on page 127 there should be $-\frac{1}{t+\tau}$ instead of $-\frac{1}{t}$. The reference number (4.2) should be corrected to (4.31) on pages 127–128. Moreover, there is a condition $0 < \epsilon < \frac{1}{100}\min(1, \sigma)$ on page 121 above (4.5); this must be replaced by a stricter condition $0 < \epsilon < \frac{1}{100}\min(1, \sigma, \frac{1}{C})$, where the constant $C > 0$ is the constant appearing in the inequalities (4.19) and (4.31) in the exponents (the constant does not depend on ϵ). This corrected condition is needed on pages 128–129, where we then have the assumptions (4.5), (4.6) and $0 < \epsilon < \frac{1}{100}\min(1, \sigma, \frac{1}{C})$ for ϵ . In addition, the word “polyhedron” should be replaced by the word “polygon” on pages 112–113. The above changes do not affect the calculations.

In the rest of the section we explain briefly the main ideas of the proof of Theorem 3.3. First of all, the consideration is transferred from the regulated domain to the unit disk via the Riemann map: The projection $P_{\varphi, \alpha, \eta}$ is bounded on $L_{\omega_\sigma}^\infty(\Omega)$ if and only if the projection P_α (see (2.7)) is bounded on $L_w^\infty(\mathbb{D})$, where $w(z) = (1 - |z|)^\sigma |\psi'(z)|^{\sigma-1-\eta}$, $z \in \mathbb{D}$ (see [A, Lemma 3.2]).

Let $|K_\alpha(z, \zeta)| = (1 + \alpha)(1 - |\zeta|^2)^\alpha |1 - z\bar{\zeta}|^{-2-\alpha}$, $z, \zeta \in \mathbb{D}$, $\alpha > -1$ and $C > 0$. The proof of the part (i) of Theorem 3.3 is based on the fact: If the weight $v : \mathbb{D} \rightarrow \mathbb{R}_+$ satisfies the inequality

$$\int_{\mathbb{D}} \frac{|K_\alpha(z, \zeta)|}{v(\zeta)} dA(\zeta) \leq \frac{C}{v(z)} \quad \text{for all } z \in \mathbb{D}, \quad (3.21)$$

then $P_\alpha : L_v^\infty(\mathbb{D}) \rightarrow H_v^\infty(\mathbb{D})$ is bounded. Indeed, let $f \in L_v^\infty(\mathbb{D})$ be such that $\|f\|_{\infty, v} \leq 1$. Then for all $z \in \mathbb{D}$

$$|P_\alpha f(z)| \leq \int_{\mathbb{D}} |K_\alpha(z, \zeta)| |f(\zeta)| dA(\zeta) \leq \int_{\mathbb{D}} \frac{|K_\alpha(z, \zeta)|}{v(\zeta)} dA(\zeta) \leq \frac{C}{v(z)}.$$

The proof of (i) proceeds by the notation $|\psi'|^{\sigma-1-\eta} = C\tilde{\nu}_1\tilde{\nu}_2$, where the functions $\tilde{\nu}_j : \mathbb{D} \rightarrow \mathbb{R}_+$ have the formulas

$$\tilde{\nu}_j(z) := \exp\left(-\frac{\sigma-1-\eta}{2\pi} \operatorname{Im} \int_0^{2\pi} \frac{e^{it} + z}{e^{it} - z} \beta_j(t) dt\right), \quad (3.22)$$

where $\beta_1 = \sum_{j=1}^n \gamma_j \chi_j$ and $\beta_2 = \beta - t - \beta_1$, see (2.16) and [A, p.122]. Thus, the weight $w(z) = (1 - |z|)^\sigma |\psi'(z)|^{\sigma-1-\eta}$ can be written as $w(z) = C(1 - |z|^2)^\sigma \tilde{\nu}_1(z) \tilde{\nu}_2(z)$. The factor $\tilde{\nu}_1$ has the expression [A, pp.123–124]

$$\tilde{\nu}_1(z) = \prod_{j=0}^{n-1} |1 - ze^{-i\theta_j}|^{-\frac{(\sigma-1-\eta)(\gamma_{j+1}-\gamma_j)}{\pi}}, \quad (3.23)$$

for the definition of the constants $\theta_j \in [0, 2\pi]$ and γ_j , $j = 0, \dots, n$, see [A, p.122]. Thus $(1 - |z|^2)^\sigma \tilde{\nu}_1$ is of the same form as the weight in Lemma 3.2. Moreover, for the use of Hölder's inequality in [A, p.123], the auxiliary functions ν_1 and ν_2 are defined, see (4.12) in [A]. Then, it is shown that the inequality (3.21) holds for ν_1 (this follows from [A, Corollary 2.4, Remark 2.5]). In addition, it is proved that $\tilde{\nu}_2$ satisfies the estimates (4.19) and (4.31) in [A]. These estimates are used when showing that the conditions of [6, Lemma 1] hold for ν_2 (the conditions are related to the specific partition of the unit disk, for details see [6, pp.78–80]). Therefore, also ν_2 satisfies (3.21), see the proof of [6, Lemma 1]. Since both ν_1 and ν_2 satisfy (3.21), it follows from the Hölder's inequality that the weight w satisfies (3.21) as well (see (4.15) in [A]). Thus, the projection P_α is bounded on $L_w^\infty(\mathbb{D})$.

The part (ii) in Theorem 3.3 is proved by defining suitable functions ($z \in \mathbb{D}$ fixed) $f_{z,M} \in L_w^\infty(\mathbb{D})$ such that $\|f_{z,M}\|_{\infty,w} \leq C$, where C is independent of z and M and $M > 4$ is arbitrary (the idea of the definition of $f_{z,M}$ can be seen in the proof of [A, Proposition 2.1]). It is shown that $|P_\alpha f_{z,M}(z)| \geq C' M^t$, where $t > 0$ and the constant $C' > 0$ does not depend on M . This proves the unboundedness of P_α on $L_w^\infty(\mathbb{D})$.

4 Boundedness and compactness of Toeplitz operators

4.1 Background

Sufficient conditions for the boundedness and compactness of generalized Toeplitz operators on $A^p(\mathbb{D})$ ($1 < p < \infty$) with locally integrable symbols are given in [52]. The boundedness condition requires that certain averages of symbol a over annular sectors (rectangles in polar coordinates) are bounded above, see Definition 2.2 and Theorem 2.3 in [52]. In [B] we consider generalized Toeplitz operators $T_{a,\Omega}$ with locally integrable symbols and extend the result of [52]. More specifically, we deal with bounded simply connected domains $\Omega \subset \mathbb{C}$ with smooth enough boundary. Our approach in [B] is partly similar to that of [52, 53], but instead of annular sectors we use certain Cartesian squares in the condition for the symbol and in the partition of Ω . The method of [52] depends essentially on the radial symmetry on \mathbb{D} and the use of polar coordinates, whereas our approach applies to domains with quite arbitrary geometries. We give weak sufficient conditions for the boundedness [B, Theorem 1.5] and the compactness [B, Theorem 1.6] of $T_{a,\Omega}$ in terms of “averages” of symbol a over Cartesian squares whose side lengths are comparable to the boundary distance. The main tools in the proofs are integration by parts and a Whitney decomposition of Ω . The properties of the Whitney decomposition are essential in our study: we can form a partition of the domain Ω into countably many closed squares S_n with mutually disjoint interiors so that the side length ρ_n of each square is comparable to the boundary distance $\text{dist}(S_n, \partial\Omega)$, i.e. $\rho_n \sim \text{dist}(S_n, \partial\Omega)$ for all n . More precisely, Whitney squares have the following properties [47]:

- (i) $\Omega = \bigcup_n S_n$,
- (ii) S_n are mutually disjoint (interiors are disjoint),
- (iii) $\text{diam} S_n \leq \text{dist}(S_n, \partial\Omega) \leq 4 \text{diam} S_n$,

(iv) there are at most 144 squares S_k ($k \neq n$) that intersect with a square S_n .

We define the generalized Toeplitz operator as follows.

Definition 4.1. [B, C] Let $\{S_n\}_n$ be a Whitney decomposition of Ω and let $a \in L^1_{loc}(\Omega)$. Given $f \in A^p(\Omega)$, we define a generalized Toeplitz operator

$$(T_{a,\Omega}f)(z) := \sum_{n=1}^{\infty} \int_{S_n} K_{\Omega}(z, w) a(w) f(w) dA(w), \quad (4.24)$$

if the sum converges for all $z \in \Omega$.

The choice of the Whitney decomposition for our definition is “natural” in the sense that with it we get a partition of Ω which has properties similar to the Bergman disk decompositions of the unit disk. Note also that the definition (4.24) coincides with the classical one (2.18) whenever the latter makes sense (see [C, Remark 1.3]): If $f \in A^p(\Omega)$ is such that $\varphi' a f \in L^1(\Omega)$, then $T_a f = T_{a,\Omega} f$, i.e. the usual (2.18) and the generalized (4.24) definitions of the Toeplitz operator coincide. In the case of the unit disk, the analogous definition presented in [53, Theorem 1.2] coincides with a radial limit of the classical Toeplitz operator.

4.2 Main results of Article [B]

Let $\Omega \subset \mathbb{C}$ be a bounded simply connected domain. For $u + iv \in \Omega$, we denote Cartesian squares by

$$S := S(u + iv, \rho) := \{x + iy \in \Omega \mid u \leq x \leq u + \rho, v \leq y \leq v + \rho\},$$

where $\rho > 0$ is the side length of S . Here ρ is always so small that $S(u + iv, \rho) \subset \Omega$. The area of S is denoted by $|S| := \rho^2$, and the distance of S to the boundary $\partial\Omega$ is denoted by $\text{dist}(S, \partial\Omega) := \inf_{w \in S, z \in \partial\Omega} |w - z|$. We have the following condition for the symbol $a \in L^1_{loc}(\Omega)$ in [B] and [C]: There exists a constant $C > 0$ such that

$$|\hat{a}_S(z')| = \frac{1}{|S|} \left| \int_v^{y'} \int_u^{x'} a(x + iy) dx dy \right| \leq C \quad (4.25)$$

for all $z' = x' + iy' \in S := S(u + iv, \rho)$ and for all squares $S \subset \Omega$ with $\sqrt{2}\rho \leq \text{dist}(S, \partial\Omega) \leq 4\sqrt{2}\rho$.

It is crucial in the condition (4.25) that the modulus is outside the integral. This, together with the definition of the generalized Toeplitz operator, enables the cancellation phenomena impact on the boundedness. We also point out that the condition (4.25) is “weak” compared to the known sufficient condition for the boundedness in the case of the unit disk (see [58, Theorem 8], [36, Lemma 2.2, Lemma 2.3]): Let $a \in L^1(\mathbb{D})$, $r > 0$ (fixed) and assume that there is $C > 0$ such that

$$\frac{1}{|B(z, r)|} \int_{B(z, r)} |a(w)| dA(w) \leq C$$

for all Bergman (hyperbolic) disks $B(z, r) \subset \mathbb{D}$.

Our main theorems in [B] read as follows:

Theorem 4.2. [B, Thm. 1.5] *Let $\Omega \subset \mathbb{C}$ be a bounded simply connected domain with C^4 smooth boundary. Assume that $a \in L^1_{loc}(\Omega)$ and that the condition (4.25) holds. Then the sum in (4.24) converges absolutely for all $z \in \Omega$ and the generalized Toeplitz operator $T_{a,\Omega}$, defined by (4.24), is a bounded operator from $A^p(\Omega)$ into $A^p(\Omega)$ for all $1 < p < \infty$.*

Theorem 4.3. [B, Thm. 1.6] *Let $\Omega \subset \mathbb{C}$ be a bounded simply connected domain with C^4 smooth boundary. Assume that $a \in L^1_{loc}(\Omega)$ and that*

$$\lim_{\text{dist}(S, \partial\Omega) \rightarrow 0} \sup_{z \in S} |\hat{a}_S(z)| = 0, \quad (4.26)$$

where $S \subset \Omega$ denotes an arbitrary square with $\sqrt{2}\rho \leq \text{dist}(S, \partial\Omega) \leq 4\sqrt{2}\rho$. Then the Toeplitz operator $T_{a,\Omega}$, defined by (4.24), is a compact operator on $A^p(\Omega)$ for all $1 < p < \infty$.

We also show in [B] that the given conditions are necessary if a is nonnegative. This is proved by using the Berezin transform of the symbol a .

Proposition 4.4. [B, Prop. 4.2, Prop. 4.3] *Let $1 < p < \infty$. Let the symbol a be nonnegative and assume that the sum (4.24) converges. Then,*

- (i) *The Toeplitz operator $T_{a,\Omega} : A^p(\Omega) \rightarrow A^p(\Omega)$ is bounded only if the condition (4.25) holds true.*
- (ii) *The Toeplitz operator $T_{a,\Omega} : A^p(\Omega) \rightarrow A^p(\Omega)$ is compact only if the condition (4.26) holds true.*

It was proved in [B] that the definition of $T_{a,\Omega}$ is independent of the choice of the Whitney decomposition of Ω :

Theorem 4.5. [B, Thm. 5.1] *Let $\{S_n\}_n$ and $\{Q_n\}_n$ be two different Whitney decompositions of Ω . Let $a \in L^1_{loc}(\Omega)$ and assume that the condition (4.25) holds for a . Let $f \in A^p(\Omega)$. Then*

$$\sum_{n=1}^{\infty} \int_{S_n} K_{\Omega}(z, w) a(w) f(w) dA(w) = \sum_{n=1}^{\infty} \int_{Q_n} K_{\Omega}(z, w) a(w) f(w) dA(w) \quad (4.27)$$

for all $z \in \Omega$.

The usefulness of the condition (4.25) is shown by [B, Example 5.3]. We construct in [B, Example 5.3] a locally integrable symbol which has the property (4.25) but which does not belong to $L^1(\Omega)$: Let $\Omega \subset [0, 1] \times [0, 1] \subset \mathbb{R}^2 \cong \mathbb{C}$ be a domain with C^4 boundary such that $] \frac{1}{2}, 1[\times] \frac{1}{5}, \frac{4}{5}[\subset \Omega$ and $\Omega_0 :=] \frac{9}{10}, 1[\times] \frac{2}{5}, \frac{3}{5}[\subset \Omega$. Let $a : \Omega \rightarrow \mathbb{R}$ be such that

$$a(x + iy) := \begin{cases} 1, & \text{if } x + iy \in \Omega \setminus \Omega_0, \\ \frac{1}{1-x} \sin\left(\frac{1}{1-x}\right), & \text{if } x + iy \in \Omega_0. \end{cases}$$

We remark that, with this a , $T_{a,\Omega}$ is bounded on $A^p(\Omega)$ whereas $T_{|a|,\Omega}$ is not. In other words, the boundedness of $T_{a,\Omega}$ in this case is due to the cancellation phenomena of the symbol.

5 Boundedness of Toeplitz operators on polygonal domains

5.1 Main results of Article [C]

In article [C] we continue the study of generalized Toeplitz operators with locally integrable symbols. We improve the boundedness results of [B] so that instead of smoothly bounded domains we consider polygons $\Omega \subset \mathbb{C}$ with a finite number of corners. The main difference compared to [B] is that the derivative of the Riemann conformal mapping φ' (or ψ') is not always bounded. The main results rely on the boundedness of the Bergman projection which depends on the index p and on the size of the maximum corner $\alpha_m := \max_k(\alpha_k)$ of $\partial\Omega$ (to be precise, α_m is the factor of the maximum inner angle). The following theorem is a special case of more general results presented in [5, 45, 46, 52, 53].

Theorem 5.1. *Let $\Omega \subset \mathbb{C}$ be a polygon with the maximum inner angle $\alpha_m\pi$. The Bergman projection P_{Ω} defines a bounded projection from $L^p(\Omega)$ onto $A^p(\Omega)$ if and only if*

$$(2-p)(\alpha_m - 1) < 2(p-1) \text{ in the case } p \leq 2, \text{ or, } (p-2)(\alpha_m - 1) < 2 \text{ in the case } p \geq 2.$$

The main result reads as follows.

Theorem 5.2. [C, Thm. 1.5] *Let $1 < p < \infty$. Let $\Omega \subset \mathbb{C}$ be a polygon with corners $\alpha_1\pi, \dots, \alpha_n\pi$ at vertices $w_1, \dots, w_n \in \partial\Omega$ and $\alpha_m = \max_k(\alpha_k)$. Suppose that $\alpha_m < 1 + \frac{2}{p-2}$ if $p > 4$ and $\alpha_m < 1 + \frac{2(p-1)}{2-p}$ if $1 < p < 4/3$. Let $a \in L^1_{loc}(\Omega)$ and assume that symbol a satisfies the condition (4.25). Then the generalized Toeplitz operator $T_{a,\Omega}$, defined as*

$$(T_{a,\Omega}f)(z) = \lim_{m \rightarrow \infty} T_{a,\Omega}^{(m)}f(z) = \sum_{n=1}^{\infty} \int_{S_n} K_{\Omega}(z, w)a(w)f(w)dA(w), \quad (5.28)$$

is a bounded operator from $A^p(\Omega)$ into $A^p(\Omega)$ and the sum in (5.28) converges pointwise, absolutely for all $z \in \Omega$. Moreover, $T_{a,\Omega}^{(m)} \rightarrow T_{a,\Omega}$ strongly, as $m \rightarrow \infty$.

The assumptions of Theorem 5.2 mean that if $p > 4$ or $1 < p < 4/3$, then the size of the maximum inward corner is restricted. There are no restrictions for Ω when $4/3 \leq p \leq 4$.

If $p > 4$ or $p < 4/3$ and polygon Ω has big enough inward corners ($1 < \alpha < 2$), then the Bergman projection P_{Ω} is unbounded according to Theorem 5.1. We have considered these cases in [C, Propositions 4.1 and 4.2]. We prove a boundedness result in a weighted Bergman space $A^p_{\omega}(\Omega)$, $p > 4$, with the weight $\omega(w) = |1 - \overline{\varphi(\mathbf{w}_m)}\varphi(w)|^t$, where \mathbf{w}_m denotes the vertex of $\partial\Omega$ with the maximum angle α_m (Proposition 4.1). In the case $p < 4/3$, we show a sufficient condition for the boundedness of generalized Toeplitz operator: we strengthen the condition (4.25) by requiring the ‘‘average’’ of the symbol, $|\hat{a}_S(z')|$, to converge to zero (at pace $|1 - \overline{\varphi(\mathbf{w}_m)}\varphi(z')|^t$) when approaching the vertex \mathbf{w}_m (Proposition 4.2). The formula $|1 - \overline{\varphi(\mathbf{w}_m)}\varphi(z')|^t$ corresponds to that factor of φ' (see (2.15)) which causes the unboundedness of the Bergman projection P_{Ω} .

Proposition 5.3. [C, Prop. 4.1] *Let $p > 4$ and let $\Omega \subset \mathbb{C}$ be a polygon with corners $\alpha_1\pi, \dots, \alpha_n\pi$ ($0 < \alpha_k < 2$, $\alpha_k \neq 1$) at vertices $w_1, \dots, w_n \in \partial\Omega$. Suppose that $\alpha_m \geq 1 + \frac{2}{p-2}$ and $\alpha_k < 1 + \frac{2}{p-2}$ for all $k \neq m$. Let $a \in L^1_{loc}(\Omega)$ and assume that the condition (4.25) holds. Then the sum (5.28) converges absolutely for all $z \in \Omega$ and the generalized Toeplitz operator $T_{a,\Omega}$ is a bounded operator from $A^p_{\omega}(\Omega)$ into $A^p_{\omega}(\Omega)$, where $\omega(w) = |1 - \overline{\varphi(\mathbf{w}_m)}\varphi(w)|^t$ and $t > (p-2)(\alpha_m - 1) - 2$.*

Proposition 5.4. [C, Prop. 4.2] *Let $1 < p < 4/3$ and let $\Omega \subset \mathbb{C}$ be a polygon with corners $\alpha_1\pi, \dots, \alpha_n\pi$ ($0 < \alpha_k < 2$, $\alpha_k \neq 1$) at vertices $w_1, \dots, w_n \in \partial\Omega$. Suppose that $\alpha_m \geq 1 + \frac{2(p-1)}{2-p}$ and $\alpha_k < 1 + \frac{2(p-1)}{2-p}$ for all $k \neq m$. Let $a \in L^1_{loc}(\Omega)$ and assume that $t > (2-p)(\alpha_m - 1) - 2(p-1)$ and*

$$|\hat{a}_S(z')| \leq C|1 - \overline{\varphi(\mathbf{w}_m)}\varphi(z')|^t \quad (5.29)$$

for all $z' \in S$ and all squares $S \subset \Omega$ with $\sqrt{2}\rho \leq \text{dist}(S, \partial\Omega) \leq 4\sqrt{2}\rho$. Then the sum (5.28) converges absolutely for all $z \in \Omega$ and the generalized Toeplitz operator $T_{a,\Omega}$ is a bounded operator from $A^p(\Omega)$ into $A^p(\Omega)$.

In the last section of article [C] we consider the classical Toeplitz operator T_a on $A^p(\Omega)$ ($p > 4$ or $1 < p < 4/3$) in the case where the polygon Ω has such a big inward corner that the Bergman projection is unbounded. We give examples of some special symbols that guarantee the boundedness of the Toeplitz operator. For example, we show the following (see [C, Example 5.4]): Let $p > 4$ and let $\alpha_m = \max_k(\alpha_k) > 1 + \frac{2}{p-2}$. Moreover, assume that $0 < \alpha_k < 1$ for all $k \neq m$, i.e. except for the maximum angle the angles of the polygon are outward. If we define symbol $a : \Omega \rightarrow \mathbb{C}$ such that

$$a(z) := (\varphi'(z))^{1-2/p}(1 - |\varphi(z)|^2) \left(\frac{\varphi(z)}{|\varphi(z)|} - |\varphi(z)| \right)^m, \quad z \in \Omega,$$

where $m \in \{2, 3, \dots\}$, then $T_a : A^p(\Omega) \rightarrow A^p(\Omega)$ is bounded.

6 Concluding remarks

Our study gives rise to the following problems and open questions:

- (i) Article [A]: We could not handle the limiting case, i.e. the equation in Theorem 3.3. What happens in the limit of (3.19) and (3.20)? Moreover, is it possible to characterize the boundedness of the Bergman projection on L_v^∞ in terms of the weight v , possibly with a Muckenhoupt type condition? This is a profound question and might be a challenging task to find out.
- (ii) Article [B]: Is our definition of the generalized Toeplitz operator independent of the geometric form of the partition of Ω ? In other words, could we choose the disjoint sets arbitrarily, provided that the sets form a partition and the area of each set is comparable to the square of the boundary distance of the set? If the answer is affirmative to the previous question, are the weak sufficient conditions (4.25) and (4.26) also necessary?
- (iii) Article [C]: Is it possible to use our method for more general bounded simply connected domains, for example, for domains with cusps or countable many corners at the boundary? Related to the last section of [C], the consideration in the case of the unbounded Bergman projection could be improved by finding a reasonably general, sufficient condition for the boundedness of Toeplitz operators.

References

- [1] S. Axler and D. Zheng. Compact operators via the Berezin transform. *Indiana Univ. Math. J.* 47(2):387–400, 1998.
- [2] S. Bergman. *The Kernel Function and Conformal Mapping*. Mathematical Surveys, No. 5, American Mathematical Society, New York, 1950.
- [3] S. Bergman and J.G. Herriot. Application of the method of the kernel function for solving boundary-value problems. *Numer. Math.* 3:209–225, 1961.
- [4] D. Békollé. Inégalités à poids pour le projecteur de Bergman dans la boule unité de \mathbb{C}^n . *Studia Math.* 71(3):305–323, 1981/1982.
- [5] D. Békollé. Projections sur des espaces de fonctions holomorphes dans des domaines plans. *Canad. J. Math.* 38(1):127–157, 1986.
- [6] J. Bonet, M. Engliš, and J. Taskinen. Weighted L^∞ -estimates for Bergman projections. *Studia Math.* 171(1):67–92, 2005.
- [7] P.S. Bourdon. Density of the polynomials in Bergman spaces. *Pac. J. Math.* 130(2):215–221, 1987.
- [8] J. Burbea. Projections on Bergman spaces over plane domains. *Canad. J. Math.* 31(6):1269–1280, 1979.
- [9] J. Burbea. The Bergman projection over plane regions. *Ark. Mat.* 18(2):207–221, 1980.
- [10] G.F. Cao. Toeplitz operators on Bergman spaces of multiply connected domains. (Chinese. English, Chinese summary) *Chinese Ann. Math. Ser. A* 29(5):671–678, 2008.
- [11] J.J. Carmona and Ch. Pommerenke. On the argument oscillation of conformal maps. *Michigan Math. J.* 46(2):297–312, 1999.
- [12] J.A. Cima and Ž. Čučković. Compact Toeplitz operators with unbounded symbols. *J. Operator Theory* 53(2):431–440, 2005.
- [13] L.A. Coburn. Singular integral operators and Toeplitz operators on odd spheres. *Indiana Univ. Math. J.* 23(5):433–439, 1973/1974.
- [14] A. Dieudonne and E. Tchoundja. Toeplitz operators with L^1 symbols on Bergman spaces in the unit ball of \mathbb{C}^n . *Adv. Pure Appl. Math.* 2(1):65–88, 2011.
- [15] P. Duren and A. Schuster. *Bergman Spaces*. American Mathematical Society, Providence, RI, 2004.
- [16] M. Engliš. Compact Toeplitz operators via the Berezin transform on bounded symmetric domains. *Integral Equations Operator Theory* 33(4):426–455, 1999.
- [17] M. Engliš. An excursion into Berezin-Toeplitz quantization and related topics. *Oper. Theory Adv. Appl.* 251:69–115, 2016.
- [18] R.E. Greene and S.G. Krantz. *Function Theory of One Complex Variable*. John Wiley & Sons, New York, 1997.
- [19] S. Grudsky, A. Karapetyants, and N. Vasilevski. Toeplitz operators on the unit ball in \mathbb{C}^n with radial symbols. *J. Operator Theory* 49(2):325–346, 2003.

- [20] S. Grudsky and A. Rybkin. On Toeplitz and Hankel operators with oscillatory symbols containing Blaschke products and applications to the KdV equation. *Oper. Theory Adv. Appl.* 228:127–150, 2013.
- [21] S. Grudsky and N. Vasilevski. Bergman-Toeplitz operators: radial component influence. *Integral Equations Operator Theory* 40(1):16–33, 2001.
- [22] Z. He and G. Cao. Toeplitz operators with L^1 symbols on the weighted Bergman spaces. *Int. J. Math. Anal. (Ruse)* 7:291–300, 2013.
- [23] H. Hedenmalm. The dual of a Bergman space on simply connected domains. *J. Anal. Math.* 88:311–335, 2002.
- [24] H. Hedenmalm, B. Korenblum, and K. Zhu. *Theory of Bergman Spaces*. Springer-Verlag, New York, 2000.
- [25] K. Hoffman. *Banach Spaces of Analytic Functions*. Prentice-Hall, Englewood Cliffs, 1962.
- [26] N.P. Jewell. Toeplitz operators on the Bergman spaces and in several complex variables. *Proc. London Math. Soc. (3)* 41(2):193–216, 1980.
- [27] B. Korenblum and K. Zhu. An application of Tauberian theorems to Toeplitz operators. *J. Operator Theory* 33(2):353–361, 1995.
- [28] S.G. Krantz. *Geometric Function Theory: Explorations in Complex Analysis*. Birkhäuser Boston, Boston, MA, 2006.
- [29] S.G. Krantz. A tale of three kernels. *Complex Var. Elliptic Equ.* 53(11):1059–1082, 2008.
- [30] L. Lanzani and E.M. Stein. Szegő and Bergman projections on non-smooth planar domains. *J. Geom. Anal.* 14(1):63–86, 2004.
- [31] D.H. Luecking. Representation and duality in weighted spaces of analytic functions. *Indiana Univ. Math. J.* 34(2):319–336, 1985.
- [32] D.H. Luecking. Trace ideal criteria for Toeplitz operators. *J. Funct. Anal.* 73(2):345–368, 1987.
- [33] W. Lusky and J. Taskinen. Bounded holomorphic projections for exponentially decreasing weights. *J. Funct. Spaces Appl.* 6(1):59–70, 2008.
- [34] W. Lusky and J. Taskinen. Toeplitz operators on Bergman spaces and Hardy multipliers. *Studia Math.* 204(2):137–154, 2011.
- [35] G. McDonald and C. Sundberg. Toeplitz operators on the disk. *Indiana Univ. Math. J.* 28(4):595–611, 1979.
- [36] J. Miao and D. Zheng. Compact operators on Bergman spaces. *Integral Equations Operator Theory* 48(1):61–79, 2004.
- [37] J.M. Ortega and J. Fàbrega. Corona type decompositions in some Besov spaces. *Math. Scand.* 78(1):93–111, 1996.
- [38] J.M. Ortega and J. Fàbrega. Pointwise multipliers and corona type decompositions in BMOA. *Ann. Inst. Fourier* 46(1):111–137, 1996.
- [39] Z. Pasternak-Winiarski. On weights which admit the reproducing kernel of Bergman type. *Int. J. Math. Math. Sci.* 15(1):1–14, 1992.

- [40] A. Perälä, J. Taskinen, and J.A. Virtanen. New results and open problems on Toeplitz operators in Bergman spaces. *New York J. Math.* 17A:147–164, 2011.
- [41] Ch. Pommerenke. *Boundary Behaviour of Conformal Maps*. Springer-Verlag, Berlin, 1992.
- [42] R. Raimondo. Compact operators on the Bergman space of multiply-connected domains. *Proc. Amer. Math. Soc.* 129(3):739–747, 2001.
- [43] A.M. Shikhvatov. Spaces of analytic functions in a region with an angle. *Math. Notes* (translated from Mat. Zametki) 18:833–839, 1975.
- [44] A.M. Shikhvatov. L_p -spaces of functions analytic in a region with piecewise analytic boundary. *Math. Notes* (translated from Mat. Zametki) 20(4):858–864, 1976.
- [45] A.A. Solov’ev. L^p -estimates of integral operators associated with spaces of analytic and harmonic functions. *Soviet Math. Dokl.* 19:764–768, 1978.
- [46] A.A. Solov’ev. On the continuity of an integral operator with Bergman kernel in the L^p space (Russian). *Vest. Leningrad Universiteta* 19:77–80, 1978.
- [47] E. Stein. *Singular integrals and differentiability properties of functions*. Princeton University Press, Princeton, N.J., 1970.
- [48] K. Stroethoff. Generalizations of the Forelli-Rudin Estimates. *J. Math. Anal. Appl.* 252(2):936–950, 2000.
- [49] K. Stroethoff and D. Zheng. Toeplitz and Hankel operators on Bergman spaces. *Trans. Amer. Math. Soc.* 329(2):773–794, 1992.
- [50] J. Taskinen. Regulated domains and Bergman type projections. *Ann. Acad. Sci. Fenn. Math.* 28(1):55–68, 2003.
- [51] J. Taskinen. Note on the paper ”Regulated domains and Bergman type projections”. *J. Funct. Spaces Appl.* 2(2):97–106, 2004.
- [52] J. Taskinen and J. Virtanen. Toeplitz operators on Bergman spaces with locally integrable symbols. *Rev. Mat. Iberoam.* 26(2):693–706, 2010.
- [53] J. Taskinen and J. Virtanen. On generalized Toeplitz and little Hankel operators on Bergman spaces. *Arch. Math.* 110(2):155–166, 2018.
- [54] U. Venugopalkrishna. Fredholm operators associated with strongly pseudoconvex domains on \mathbb{C}^n . *J. Funct. Anal.* 9:349–373, 1972.
- [55] S. Yamaji. Positive Toeplitz operators on the Bergman space of a minimal bounded homogeneous domain. *Hokkaido Math. J.* 41:257–274, 2012.
- [56] X. Zeng. Toeplitz operators on Bergman spaces. *Houston J. Math.* 18(3):387–407, 1992.
- [57] K. Zhang, C.M. Liu, and Y.F. Lu. Toeplitz operators with BMO symbols on the weighted Bergman space of the unit ball. *Acta Math. Sin. (Engl. Ser.)* 27(11):2129–2142, 2011.
- [58] K. Zhu. Positive Toeplitz operators on weighted Bergman spaces of bounded symmetric domains. *J. Operator Theory* 20(2):329–357, 1988.
- [59] K. Zhu. *Operator theory in function spaces*. Marcel Dekker, New York, 1990.
- [60] N. Zorboska. The Berezin transform and radial operators. *Proc. Amer. Math. Soc.* 131(3):793–800, 2003.

- [61] N. Zorboska. Toeplitz operators with BMO symbols and the Berezin transform. *Int. J. Math. Math. Sci.* 2003(46):2929–2945, 2003.