

Tiedekunta/Osasto — Fakultet/Sektion — Faculty		Laitos — Institution — Department	
Matemaattis-luonnontieteellinen		Matematiikan ja tilastotieteen laitos	
Tekijä — Författare — Author			
Natalia Timofeeva			
Työn nimi — Arbetets titel — Title			
Pricing and hedging exotic options using Monte Carlo method			
Oppiaine — Läroämne — Subject			
Matematiikka			
Työn laji — Arbetets art — Level		Aika — Datum — Month and year	
Pro gradu -tutkielma		Toukokuu 2019	
		Sivumäärä — Sidoantal — Number of pages	
		57 s.	
Tiivistelmä — Referat — Abstract			
<p>This thesis attempts to show the advantages of Monte Carlo method in pricing and hedging exotic options. The popularity of exotic options increased recently, mostly due to their almost unlimited flexibility and adaptability to any circumstances. The negative side of exotic options is their complexity. Due to that many exotic options does not have analytic solutions. As a result numerical solutions are a necessity.</p> <p>The Monte Carlo method of simulations is very common method in computational finance. Monte Carlo method is based on the analogy between probability and volume. Starting point in pricing and hedging options with Monte Carlo method is stochastic differential equation based on Brownian motion in the Black-Scholes world. The fair option value in the Black-Scholes world is the present value of the expected payoff at expiry under a risk-neutral assumptions.</p> <p>The analysis start from the case of the simple European options and continue with introducing different kinds of exotic options. The dynamic hedging idea is used to derive the Black-Scholes Partial Differential Equation. The numerical approximation of the stochastic differential equation is derived through the lognormal asset price model. The Monte Carlo algorithms are constructed for pricing and delta hedging and then implemented to MATLAB. For generating Monte Carlo simulations is used $N(0, 1)$ pseudo-random generator.</p> <p>The analysis is limited to the cases of simple Barrier options, which are one of the most known and used type of the exotic options. Barrier options are path dependent options, which implies that the payoff depends on the path followed by the price of the underlying asset, meaning that barrier options prices are especially sensitive to volatility. That is why, we also introduce the variance reduction techniques by antithetic variates. For hedging barrier options were chosen the dynamic delta-hedging and static hedging strategies. All calculations and figures in the examples were made in MATLAB.</p>			
Avainsanat — Nyckelord — Keywords			
Exotic Options, Monte Carlo, MATLAB, Pricing, Hedging			
Säilytyspaikka — Förvaringsställe — Where deposited			
Kumpulan tiedekirjasto			
Muita tietoja — Övriga uppgifter — Additional information			

UNIVERSITY OF HELSINKI

MASTER THESIS IN MATHEMATICS

**Pricing and hedging exotic
options using Monte Carlo
method**

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May 5, 2019

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Chapter 1

Introduction

1.1 Introduction to financial instruments

There are many different types of financial instruments on the market, such as funds, bonds, stocks, options. In general financial instruments are assets which can be traded. They are also called ‘securities’. A security is a financial instrument that represents an ownership position in a publicly-traded corporation (through a stock), a creditor relationship with governmental body or a corporation (by owning an entity’s bond) or rights to ownership represented by an option. Options provide various opportunities. They give you a chance to adapt to any possible situation. Options trading always involves risk, in other words there is no possibility of arbitrage i.e. risk-free profit. [4] [3]

1.1.1 History of option trade

The earliest option trade on record in Western literature was a bet on future crop by Thales of Miletus (Greek philosopher, mathematician and astronomer from Miletus in Asia Minor, 624 BC - 546 BC). Thales put a deposit on every olive press in the vicinity of Miletus, to benefit from a better than expected olive crop. He did not trade olives, which he would have had to sell short, instead he chose to buy the equivalent of a call option on the olive presses. Thales used the first derivative instrument, which was an option on the future.¹ [12]

1.1.2 Options

Options are also known as *financial derivatives*. A *Derivative* is a security which price ultimately depends on that of another asset

¹For more, see Russell (1945)

(known as an underlying asset). There are different categories of derivatives, from something simple as a future to something complex as an exotic option. Often options are also called *contingent claims* i.e. the value of the option is contingent on the evolution of the exchange rate. [3]

In this thesis the term *asset* is used to describe any financial object, which value is known at the present moment but it is liable to change in the future (e.g. shares in a company, commodities such as gold or oil, currencies such as the value of 1000£ English pounds in euros etc.).

Now we will introduce some basic definitions which will help us understand the main questions of this thesis that will be represented later in this section.

Definition 1.1.1. (Option) An option is a contract that gives the buyer the right, but not the obligation, to buy or sell an underlying asset or instrument at a specific price on a specified date or before a certain date, which way depends on the type of the option.

Definition 1.1.2. (Call Option) A call gives the holder the right (not the obligation) to buy a prescribed asset or instrument at a certain price at a specific period of time. Calls are similar to having a long position on a stock. Buyers of calls hope that the stock will increase before the option expires.

Definition 1.1.3. (Put Option) A put gives the holder the right (not the obligation) to sell a prescribed asset or instrument at a certain price at a specific period of time. Puts are very similar to having a short position on a stock. Buyers of puts hope that the price of the stock will fall before the option expires.

Example 1.1.4. Let us consider an Finnish building company X that builds houses and a Russian company Y that buys those houses. They signed a contract at ($t = 0$) by which X will build 10 houses for Y in six months from now ($t = T$). At the ending day Y will pay 100 million rubles to X. Assume that currency rate today is 60.00 rubles per one euro. This kind of a contract involves currency risk. Since Y does not know the rate after 6 months from now, it does not know how much they will need to pay in euros. The rate can be higher (70 rubles per one euro) or lower (50 rubles per one euro). That is why Y wants to insure itself against currency risk by buying a contract. It wants a contract which insure against a high exchange rate at $t = T$ but still allowing to take advantage of a low exchange rate at $t = T$. Such kind of contracts in fact do exist. They are called *European call options*.

Figure 1.1: Payoff diagram for a European Call Option



Definition 1.1.5. (European Option) A European Option is a contract of the form $F = f(S_T)$ for some function f of the asset price at maturity. It can only be exercised at the established time T .

Definition 1.1.6. (European Call Option) A European call option on an asset with strike price or exercise price K and exercise or maturity date T is a contract written at $t = 0$ with the following properties

1. The holder of the option has (exactly at the time $t = T$), the right to buy the prescribed asset at the price K from the writer.
2. The holder has no obligation to buy the prescribed asset.

The value of the European call option at the expiry date is

$$C(T) = \max(S(T) - K, 0) \quad (1.1)$$

or we can write it as

$$C(T) = (S(T) - K)^+, \quad (1.2)$$

where $S(T)$ is the asset price at the expiry date and K is the strike price or exercise price. We can see a simple payoff diagram of a European call option with a strike price $K = 2$ at expiry in Figure 1.1.

Definition 1.1.7. (European Put Option) The same conditions as for European Call but the holder has the right (not the obligation) to *SELL* to the writer a prescribed asset at the exercise price K at maturity date T .

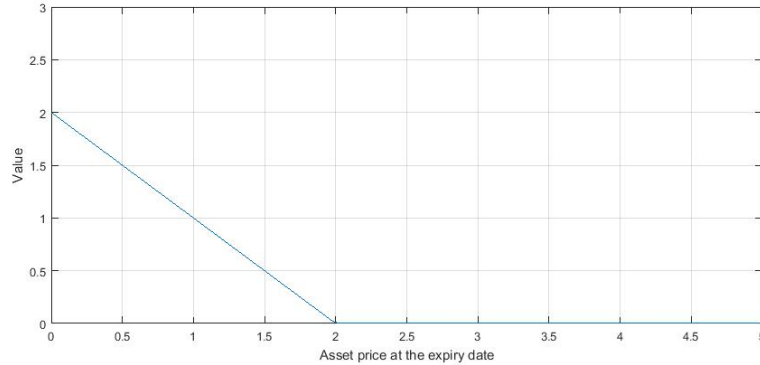
The value of the European put option at the expiry date T is

$$P(T) = \max(K - S(T), 0) \quad (1.3)$$

or

$$P(T) = (S(T) - K)^-. \quad (1.4)$$

Figure 1.2: Payoff diagram for a European Put option



We can see a payoff diagram for a European put option with the strike $K = 2$ in Figure 1.2.

Definition 1.1.8. (In-, out- and at-the-money option) At time t , a European call option is said to be

- (i) *in-the-money* if $S(t) > K$
- (ii) *out-the-money* if $S(t) < K$ and
- (iii) *at-the-money* if $S(t) = K$.

Call and put options are also called *standard* or *vanilla* options. There are more complex types of options called exotic options. Vanilla options are often used for hedging exotic options. We will go back to exotic options in the Chapter 4. Now we will show a very useful parity between call and put options, which we will use later.

Lemma 1.1.9. (Call-Put Parity) The following call-put parity holds

$$C(t) - P(t) = S(t) - Ke^{-r(T-t)}, \quad (1.5)$$

for all $t < T$.

Proof. We can consider two cases

1. $C(t) - P(t) - S(t) + Ke^{-r(T-t)} > 0 \iff Ke^{-r(T-t)} > P(t) - C(t) + S(t) \iff K > e^{r(T-t)}(P(t) - C(t) + S(t)) \iff Ke^{-r(T-t)} > P(t) - C(t) + S(t)$
2. $K < e^{r(T-t)}(P(t) - C(t) + S(t))$.

Denote $Y(t) = Ke^{-r(T-t)} - (P(t) - C(t) + S(t))$. In the first case we can borrow $Ke^{-r(T-t)} - Y(t) = P(t) - C(t) + S(t)$ at time t to buy a portfolio: buy a put $P(t)$, sell a call $C(t)$ and buy a share of the stock $S(t)$. At the time of maturity, we have two possibilities. If $S_T > K$, then the call is exercised. If $S_T < K$, the put is exercised. We pay our debt $e^{r(T-t)}(P(t) - C(t) + S(t))$ and get a profit K (debt is smaller than the gain). This means that $K - e^{r(T-t)}(P(t) - C(t) + S(t)) > 0$. This is an arbitrage opportunity, which is not possible, since we assumed the absence of arbitrage. So we get a contradiction.

In the second case, we can sell the portfolio and lend money to a bank. At the time of maturity, similarly, the profit is again positive. Hence $K < e^{r(T-t)}(P(t) - C(t) + S(t))$ is also impossible. Therefore the only possibility is that $K = e^{r(T-t)}(P(t) - C(t) + S(t))$. \square

Theorem 1.1.10. (Central Limit Theorem) Let Y_1, Y_2, \dots, Y_n be i.i.d. random variables with mean μ and finite variance $\sigma^2 > 0$. We denote $\hat{\mu}_n = \frac{1}{n} \sum_{i=1}^n Y_i$. Then for all $z \in \mathbb{R}$

$$\mathbb{P}\left(\sqrt{n} \frac{\hat{\mu}_n - \mu}{\sigma} \leq z\right) \rightarrow \Phi(z), \quad (1.6)$$

as $n \rightarrow \infty$ and where $\Phi(z)$ is the cumulative distribution function of the standard normal distribution.

Proof. Proof can be found for example in *A course in probability theory* of Chung K.-L.(1974). \square

Before going deeper to the main questions of this thesis, we need to make some basic assumptions. On the market there exists a non-risky asset with constant rate $r > 0$, exercise time T and price at each time t , $0 \leq t \leq T$, $S_t^0 = e^{rt}$. The main questions of this thesis are

1. How to evaluate at time $t = 0$ the price of an option (premium)? In other words, what is the fair price for an option?
2. How can we produce the value of an option at the maturity from the premium? How can we protect or hedge ourselves against the risk?

We will try to show, how useful are numerical methods when dealing with exotic options. We will use the Monte Carlo method to show that. Moreover, we will compare Monte Carlo results to Black-Scholes results, when it is possible.

In order to answer these questions, we need to assume that there are not any arbitrage possibilities on the market, in other words it

is not possible to obtain benefits without taking risks. From now on we will use following notations

- (i) $S(t)$, $0 \leq t \leq T$, is the observed price on the market of the underlying asset at every instant,
- (ii) $C(t)$, $0 \leq t \leq T$, and $P(t)$, $0 \leq t \leq T$, are the values of the call and put options at every instant.

1.2 Dynamic Strategy

In this section we will introduce the main strategy we will use to value and hedge options in this thesis.

Fischer Black, Myron Scholes and Robert Merton in 1973 have developed the idea according to which it is possible for an option seller to deliver the contract at maturity without incurring any residual risk by using a *dynamic trading strategy* on the underlying asset. In other words, the main assumption of the dynamic theory of pricing and hedging options is that the owner of an option can guarantee a receivable of $h(S(T))$ at the maturity. The strategy is to use the premium to buy a portfolio of stocks with price equal to the one of the option. The portfolio is called *hedging portfolio* and the strategy is called **dynamic strategy**. [13]

The value of the hedging portfolio is $V(t)$, $0 \leq t \leq T$ and we can write the absence of arbitrage as $V(0) = 0$, $V(T) \geq 0$ and $P(V(T) > 0) > 0$. Let β_t be the number of actions that the owner of the option has bought at time $0 \leq t \leq T$ and α_t the number of non-risky assets that he owns at time $0 \leq t \leq T$.

The portfolio strategy is *self-financing* i.e. we do not want to add or remove money beyond time $t = 0$. In that case at the small time period $[t, t + dt]$, the variation of the value of the portfolio only depends on the variation of the value of the option and the interest obtained on the invested cash at the bank, which is $V(t) - \beta_t S(t) = \alpha_t e^{rt}$. A market where for any $h(S(T))$ exists a self-financing replicating portfolio is called a *complete* market. Therefore we have

$$dV(t) = \beta_t dS(t) + (V(t) - \beta_t S(t)) r dt = rV(t) dt + \beta_t (dS(t) - rS(t) dt). \quad (1.7)$$

Our intention is to find a self-financing portfolio strategy which could replicate the value at maturity that $h(S_T) = v(T, S_T)$ and at each instant covers the derivative product. In other words, we

want to find two functions $v(t, x)$ and $\beta(t, x)$ for which holds that

$$\begin{cases} dv(t, S(t)) = v(t, S(t))r dt + \beta(t, S_t)(dS_t - rS(t)dt), \\ v(T, S(T)) = h(S(T)), \end{cases} \quad (1.8)$$

where $\beta(t, S(t))$ is called the *hedging portfolio* of the derivative $h(S(T))$. [18]

There are many models to describe $S(t)$ and the result of the equation above depends on the model. We will solve the equation under *the Black-Scholes model* (see next chapter).

Example 1.2.1. Let us consider a market with two assets: a non-risky asset with $r = 0$ and a risky asset with initial price $S(0) = 9$ such that $P\{S(1) = 18\} = p$, $P\{S(1) = 6\} = 1 - p$. Also, consider a put with $K = 13$ and $T = 1$, where the time in this case is assumed to be discrete (for example days). We would like to obtain a replicating portfolio. For that, we need to find the initial capital $V(0)$ and the value of the portfolio (α_1, β_1) . We know that under the self-financing assumption, we have $V(0) = 9\beta_1 + \alpha_1$. In addition, the portfolio needs to replicate the derivative

$$V(1) = \begin{cases} 5 & \text{if } S_1 = 18, \\ 0 & \text{if } S_1 = 6. \end{cases}$$

Hence

$$18\beta_1 + \alpha_1 = 5$$

and

$$6\beta_1 + \alpha_1 = 0.$$

As a result $\beta_1 = 5/12$, $\alpha_1 = -2.5$ and $V(0) = 1.25$, so the option price is 1.25. To cover the option with a capital of 1.25, we can ask for a credit of 2.5 and invest 3.75 in actions. At the time $T = 1$, we have

1. If $S(1) = 18$, the option is exercised with a cost of 5. We sell the actions and win $5/12 \cdot 18 = 7.5$ and with this we can pay back the credit and the cost of the option.
2. $S(1) = 6$, the option is not exercised. We sell the actions and we win $5/12 \cdot 6 = 2.5$ and pay back the credit.

We will show how the dynamic strategy works in more details in the following chapters.

Chapter 2

The Black-Scholes model

In this chapter we will explain in details the Black-Scholes model we mentioned before. In addition, we will derive a discrete and continuous asset models that describe how the underlying asset behaves and which we will use in the option valuation and hedging later.

As we mentioned in Chapter 1, the Black-Scholes model is a mathematical model of a financial market. It is used to determine the fair value of an option. In the Black-Scholes model it is assumed that the price of the underlying asset observed in the market follows *stochastic differential equation (SDE)*

$$\begin{cases} dS(t) = S(t)(\mu(t, S(t))dt + \sigma(t, S(t))dW(t)), \\ S(0) = S_0, \end{cases} \quad (2.1)$$

where μ is called the *drift*, which represents the expected annual rate of return of the asset, σ is the *volatility*, which measures the risk and depends on the nature of the underlying asset, and $W(t), t \in [0, T]$, is a *Brownian motion*, defined on probability space (Ω, \mathcal{F}, P) . Stochastic analysis are used to understand price fluctuations on the financial markets. [7, 13]

Definition 2.0.1. (Brownian motion) Brownian motion models the trajectory of asset prices. Brownian motion is a Gaussian process with independent and stationary increments. For $0 \leq s \leq t$, the increment $W(t) - W(s)$ follows a Gaussian centred distribution with variance $t - s$.

The SDE equation can be understood as as an ordinary differential equation. In the case of the Black-Scholes model, the SDE is solved in a closed form. If this is not the case it needs to be discretised. The discretisation of SDE is a research topic which is very active nowadays since there are lots of potential applications

in finance. Techniques which aim at accelerating the convergence in the Monte Carlo methods are also developed, for example *reduction of variance*-technique, to which we will come back in Chapter 5.

Numerical methods based on stochastic approaches are being developed in various directions. As the mathematical expectation of a random variable depending on the whole trajectory may be seen as an integral on the (infinite dimensional) space of trajectories, techniques of numerical integration in infinite dimension begin to appear.

Recent progress often relates on theoretical advances of stochastic analysis. A very good example is the use of the stochastic calculus variations due to *Paul Malliavin* (Malliavin calculus), which may be seen as a differential calculus acting upon the set of Brownian motion trajectories. In the framework of his theory, there is a very useful integrations by parts formula. Especially, it is useful in the computation of sensitivities with respect to the parameters of options prices via Monte Carlo methods that we will see later.

The solution of the SDE is given by the process

$$S(t) = S_0 e^{(\mu - \frac{\sigma^2}{2})t + \sigma W(t)}. \quad (2.2)$$

We will derive the result in the next sections.

2.1 Discrete asset model

In this section we will derive a discrete asset price model. We need the model to derive the continuous asset model in the next section. We want to have a process that could describe the asset price $S(t)$ for all times $0 \leq t \leq T$, assuming that price at time 0, S_0 , is given. The asset price fluctuations are quite unpredictable that is why $S(t)$ is a random variable. We will set up an expression for the relative change over a small time interval δt and after we will let δt go to zero so we can get an expression that is valid for continuous t .

The change in the value of a risk-free investment over a small time interval δt can be modelled as

$$D(t + \delta t) = D(t) + r\delta t D(t), \quad (2.3)$$

where r is the interest rate.[8]

We will add a ‘random fluctuation increment’ to the interest rate equation and make these increments independent for different sub-intervals for unpredictable changes in the asset price. Thus, let us define $t_i = i\delta t$ and the asset prices are determined at discrete points

$\{t_i\}$. So our discrete time model, which is also regarded as a *numerical approximation* to the SDE formulation is

$$S(t_{i+1}) = S(t_i) + \mu\delta t S(t_i) + \sigma\sqrt{\delta t}Y_i S(t_i), \quad (2.4)$$

where Y_0, Y_1, Y_2, \dots are i.i.d. $N(0, 1)$, $\mu\delta t S(t_i)$ represents a general upward drift of the asset price ($\mu > 0$ is a constant and plays the same role as the interest rate in (2.3)). We need to point out that in the model the returns $(S(t_{i+1}) - S(t_i))/S(t_i)$ are independent, identically and normally distributed random variables. Typical values of the drift are between 0.01 and 0.1. $\sigma \geq 0$ is also a constant that determines the strength of the random fluctuations and which is called *volatility*. We need to note that the model is statistically the same even if σ would be replaced by $-\sigma$ and the typical values are between 0.05 and 0.5, i.e. 5% and 50% volatility. We also need to note few points

- (i) A $N(0, 1)$ random variable is symmetric about the origin, hence the fluctuation factor $\sigma\sqrt{\delta t}Y_i$ is positive or negative with the same likelihood. Moreover, the probability that lies in an interval $[a, b]$ is the same as the probability that lies in the interval $[-b, -a]$.
- (ii) The presence of the factor $\sqrt{\delta t}$ is necessary for the existence of a sensible continuous-time limit.
- (iii) The normal distribution for Y_i is chosen because of the *Central Limit Theorem* (Theorem 1.1.10). If we just assumed that $\{Y_i\}_{i \geq 0}$ were i.i.d. with mean 0 and unit variance, we would arrive at the same continuous-time model for $S(t)$. [8]

2.2 Continuous asset model

Now we can derive the continuous asset model for the asset price. We consider the time interval $[0, t]$ with $t = L\delta t$. We know that $S(0) = S_0$ and the discrete model in (2.4) gives us expressions for $S(\delta t), S(2\delta t), \dots, S(L\delta t = t)$. Thus, we can let $\delta t \rightarrow 0$ and let $L \rightarrow \infty$ that we will get a limiting expression for $S(t)$. By the discrete model in (2.4), over each δt time interval the asset price gets multiplied by a factor $1 + \mu\delta t + \sigma\sqrt{\delta t}Y_i$, so

$$S(t) = S_0 \prod_{i=0}^{L-1} (1 + \mu\delta t + \sigma\sqrt{\delta t}Y_i).$$

Dividing by S_0 and taking logarithm, we get

$$\log\left(\frac{S(t)}{S_0}\right) = \sum_{i=0}^{L-1} \log(1 + \mu\delta t + \sigma\sqrt{\delta t}Y_i).$$

We are interested in the limit $\delta t \rightarrow 0$, so we want to use the approximation $\log(1 + \epsilon) \approx \epsilon - \epsilon^2/2 + \dots$, for small ϵ . The quantity Y_i in (2.4) is a random variable but it can be shown that what we want to do is totally fine since $\mathbb{E}(Y_i^2)$ is finite. log-expansion remains valid and we can get

$$\log\left(\frac{S(t)}{S_0}\right) \approx \sum_{i=0}^{L-1} (\mu\delta t + \sigma\sqrt{\delta t}Y_i - \frac{1}{2}\sigma^2\delta tY_i^2), \quad (2.5)$$

where we ignored terms that involve the power $\delta t^{3/2}$ or higher. The Central Limit Theorem (Theorem 1.1.10) suggests that $\log(S(t)/S_0)$ in (2.5) will behave like a normal random variable with mean $L(\mu\delta t - \frac{1}{2}\sigma^2\delta t) = (\mu - \frac{1}{2}\sigma^2)t$ and variance $L\sigma^2\delta t = \sigma^2t$, hence approximately we get

$$\log\left(\frac{S(t)}{S_0}\right) \sim N\left(\left(\mu - \frac{1}{2}\sigma^2\right)t, \sigma^2t\right)$$

That implies the limiting continuous-time expression for the asset price at time t becomes

$$S(t) = S_0 e^{(\mu - \frac{1}{2}\sigma^2)t + \sigma\sqrt{t}Z}, \quad (2.6)$$

$Z \sim N(0, 1)$. This is the solution to the SDE and $S(t)$ is the *geometric Brownian motion*. $S(t)$ is lognormally distributed. Since $S_0 > 0$, then $S(t)$ is guaranteed to be positive at any time and $\mathbb{P}(S(t) > 0) = 1$, for any $t > 0$. As a result we can see that $S(t) \in (0, \infty)$. The density function for $S(t)$ is

$$f(x) = \frac{1}{x\sigma\sqrt{2\pi t}} \cdot e^{-\frac{(\log(x/S_0) - (\mu - \frac{1}{2}\sigma^2)t)^2}{2\sigma^2t}}, \quad (2.7)$$

for $x > 0$ and with $f(x) = 0$ for $x \leq 0$. The expected value is

$$\mathbb{E}(S(t)) = S_0 e^{\mu t},$$

the second moment is

$$\mathbb{E}(S(t)^2) = S_0^2 e^{(2\mu + \sigma^2)t}$$

and the variance is

$$\text{Var}(S(t)) = S_0^2 e^{2\mu t} (e^{\sigma^2 t} - 1).$$

In (2.6) derivation the asset price starts from time 0. We can also calculate the asset price from time t_1 to t_2 , $t_2 > t_1$

$$\log \left(\frac{S(t_2)}{S(t_1)} \right) \sim N \left(\left(\mu - \frac{1}{2} \sigma^2 \right) (t_2 - t_1), \sigma (t_2 - t_1) \right).$$

The most important is that across non-overlapping time intervals, the normal random variables that describe these changes are independent. This follows from the fact that Y_i in (2.4) are i.i.d. Hence for $t_3 > t_2 > t_1$ hold

$$\log \left(\frac{S(t_3)}{S(t_2)} \right) \sim N \left(\left(\mu - \frac{1}{2} \sigma^2 \right) (t_3 - t_2), \sigma (t_3 - t_2) \right)$$

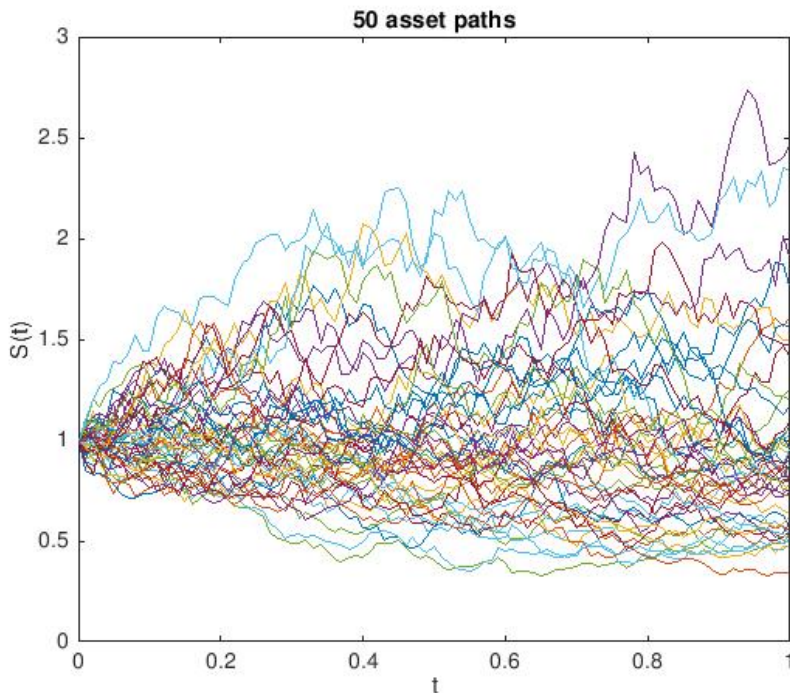
and it is independent of $\log \left(\frac{S(t_2)}{S(t_1)} \right)$.

The evolution of the asset over any sequence of time points $\{t_i\}_{i=0}^K$, $0 = t_0 < t_1 < t_2 < t_3 < \dots < t_M$ might look like

$$S(t_{i+1}) = S(t_i) e^{(\mu - \frac{1}{2} \sigma^2)(t_{i+1} - t_i) + \sigma \sqrt{t_{i+1} - t_i} Z_i}, \quad (2.8)$$

for i.i.d. $Z_i \sim N(0, 1)$, which is also the solution to the SDE. Now

Figure 2.1: 50 simulated discrete paths



we can use this solution to generate computer simulations of asset prices using $N(0, 1)$ pseudo-random numbers. We can simulate the evolution of $S(t)$ at certain points $\{t_i\}_{i=0}^K$ with $0 = t_0 < t_1 < t_2 < \dots < t_K = T$. We are computing values according to

$$S_{i+1} = S_i e^{(\mu - \frac{1}{2}\sigma^2)(t_{i+1} - t_i) + \sigma\sqrt{t_{i+1} - t_i}\xi_i}, \quad (2.9)$$

where $\xi_i \sim N(0, 1)$ are so called pseudo-random numbers. The resulting points (t_i, S_i) form a discrete asset path, see Figure 2.1 (the Matlab code used for creating Figure 2.1 can be found in the section Matlab codes).[8]

2.3 Pricing and hedging in the Black-Scholes model

Up until this moment, we have introduced the basic definitions, results related to options and developed a discrete and continuous model for the asset price. Now we can start answering to our first question (introduced on page 7): *What is the fair price for an option?*

Our aim is to determine a fair value of the option at $t = 0$ with asset price $S(0) = S_0$. In other words, we are looking for a function $V(S, t)$ that gives the option value for any asset price $S \geq 0$ at any time of the interval $0 \leq t \leq T$. Moreover, we may assume that the option may be bought or sold at this value in the market at any time point of the interval $0 \leq t \leq T$. In this scenario we will have $V(S_0, 0)$ as the time-zero option value. Furthermore, the derivatives w.r.t. these variables of the $V(S, t)$ should exist, meaning that the function needs to be smooth in both variables. Our next step is defining the *Black-Scholes partial differential equation (PDE)* for the function $V(S, t)$. But first, we need to introduce the concept of the sum-of-square increments for asset price that we will need for the derivation of the PDE.

2.3.1 Sum-of-square increments for asset price

First of all we need to define two timescales: a small timescale Δt and a very small timescale $\delta t = \Delta t/L$, where L is a large integer. In addition, we consider some general time

$$t \in [0, T],$$

and general asset price

$$S(t) \geq 0.$$

The main focus will be on the small time interval

$$[t, t + \Delta t],$$

which is divided to ‘very small’ subintervals of length δt , giving $[t_0, t_1], [t_1, t_2], \dots, [t_{L-1}, t_L]$ with $t_0 = t, t_L = t + \Delta t$ and $t_i = t + i\delta t$. We set

$$\delta S_i = S(t_{i+1}) - S(t_i)$$

to denote the change in asset price over a very small time interval. We need to analyze the sum-of-square increments $\sum_{i=0}^{L-1} \delta S_i^2$ to derive the Black-Scholes PDE. Using the discrete model from the previous chapter in (2.4) $S(t_{i+1}) = S(t_i) + \mu\delta t S(t_i) + \sigma\sqrt{\delta t} Y_i S(t_i)$, we get

$$\delta S_i = S(t_i)(\mu\delta t + \sigma\sqrt{\delta t} Y_i),$$

where Y_i are i.i.d. $N(0, 1)$. Now we have

$$\sum_{i=0}^{L-1} \delta S_i^2 = \sum_{i=0}^{L-1} S(t_i)^2 (\mu^2 \delta t^2 + 2\mu\sigma\delta t^{3/2} Y_i + \sigma^2 \delta t Y_i^2). \quad (2.10)$$

Let us replace each $S(t_i)$ by $S(t)$, then we get

$$\sum_{i=0}^{L-1} \delta S_i^2 \approx S(t)^2 \sum_{i=0}^{L-1} (\mu^2 \delta t^2 + 2\mu\sigma\delta t^{3/2} Y_i + \sigma^2 \delta t Y_i^2). \quad (2.11)$$

Using the mean and the variance of the random variables inside the sum and the Central Limit Theorem, we get

$$\sum_{i=0}^{L-1} \delta S_i^2 \sim S(t)^2 N(\sigma^2 L\delta t, 2\sigma^4 L\delta t^2) = S(t)^2 N(\sigma^2 \Delta t, 2\sigma^4 \Delta t \delta t). \quad (2.12)$$

Since the δt is very small, the variance of that final expression is tiny, leading us to conclude that the sum-of-square increments is approximately a constant multiple of $S(t)^2$

$$\sum_{i=0}^{L-1} \delta S_i^2 \approx S(t)^2 \sigma^2 \Delta t. \quad (2.13)$$

The solution (2.8) shows that

$$S(t_i) = S(t) e^{(\mu - \frac{1}{2}\sigma^2)i\delta t + \sigma\sqrt{i\delta t}Z}, \quad (2.14)$$

for some $Z \sim N(0, 1)$. Using $e^x \approx 1 + x$ for small x , we get

$$S(t_i) \approx S(t)(1 + \sigma\sqrt{i\delta t}Z) \quad (2.15)$$

and since $i\delta t \leq L\delta t = \Delta t$, we can write

$$S(t_i) - S(t) = O(\sqrt{\Delta t}). \quad (2.16)$$

We see that the approximation of $S(t_i)$ by $S(t)$ gives us an error that is roughly proportional to $\sqrt{\Delta t}$. So, we can say that replacing each $S(t_i)$ in (2.10) with $S(t)$ will not affect the leading term in the approximation (2.13). We will use the result (2.13) in the next section where we will derive the PDE and the replicating portfolio.

2.3.2 The PDE and replicating portfolio

In this section we will try to find a fair option value by setting up the replicating portfolio of asset and cash. For that first we will derive the PDE, mentioned in previous chapter. In other words, we will try to find a combination of asset and cash that has precisely the same risk as the option at all time.

Before going any further, let us recall some basic assumptions we need to remember

1. there are no transaction costs,
2. the asset can be bought/sold in arbitrary units,
3. short selling is allowed,
4. no dividends are paid,
5. the interest rate r is constant,
6. trading of the asset (and option) takes place in continuous time.

The strategy for the portfolio $\Pi(S, t) = A(S, t)S + D(S, t)$ (where S is asset price, $A(S, t)$ asset holding and $D(S, t)$ cash deposit) is to keep the amount of asset constant over very small timestep of length δt .

From that follows that the change in the value of the portfolio has two sources

1. the asset price fluctuations i.e. the change $\delta S_i = S(t_{i+1}) - S(t_i)$ produces a change $A_i \delta S_i$ in the portfolio value and
2. interest accrued on the cash deposit (in the discrete version is $r D_i \delta t$).

Altogether this means that

$$\delta \Pi_i = \Pi(S(t_{i+1}), t_{i+1}) - \Pi(S(t_i), t_i) = A_i \delta S_i + r D_i \delta t. \quad (2.17)$$

As we already discussed in the Chapter 1, our purpose is to make the portfolio $\Pi = \Pi(S(t), t)$ *self-financing*. This can be achieved by using the cash account for rebalancing the asset, which we can see in

the change from D_i to D_{i+1} . The idea of continuously fine-tuning the portfolio for reducing or removing risk, as we already know, is called *hedging*. For the next result, we need a *Taylor serious expansion*. Let us introduce it first. [7]

Theorem 2.3.1. (Taylor's Theorem) If a function f has n continuous derivatives on the interval $[x, x + h]$, then there exists some point $x + \lambda h$ for some $\lambda \in [0, 1]$ in the interval, such that

$$\begin{aligned} f(x + h) &= f(x) + hf'(x) + \frac{h^2}{2}f''(x) + \dots \\ &\dots + \frac{h^{(n-1)}}{(n-1)!}f^{(n-1)}(x) + \frac{h^n}{n!}f^n(x + \lambda h). \end{aligned} \quad (2.18)$$

If f is the function of which the derivatives of all orders exist, then we may consider increasing the value of n indefinitely. That being the case should hold

$$\lim_{n \rightarrow \infty} \frac{h^n}{n!}f^n(x) = 0. \quad (2.19)$$

Then the terms of the series converge to zero as their order increases. That implies that an infinite-order Taylor expansion is available in the form of [21]

$$f(x + h) = \sum_{j=0}^{\infty} \frac{h^j}{j!}f^j(x). \quad (2.20)$$

The function $V = V(S(t), t)$ is assumed to be smooth of S and t , thus we can use the *Taylor series expansion* and it gives us

$$\delta V_i = V(S(t_{i+1}, t_{i+1}) - V(S(t_i), t_i) \approx \frac{\partial V_i}{\partial t} \delta t + \frac{\partial V_i}{\partial S} \delta S_i + \frac{1}{2} \frac{\partial^2 V_i}{\partial S^2} \delta S_i^2. \quad (2.21)$$

Using a notation $\delta(V - \Pi)_i = \delta V_i - \delta \Pi_i$ and subtracting (2.17) from (2.21) we get

$$\delta(V - \Pi)_i = \partial V_i - \partial \Pi_i \approx \left(\frac{\partial V_i}{\partial t} - r D_i \right) \delta t + \left(\frac{\partial V_i}{\partial S} - A_i \right) \delta S_i + \frac{1}{2} \frac{\partial^2 V_i}{\partial S^2} \delta S_i^2. \quad (2.22)$$

We want the portfolio replicate the option, so that the difference between them is predictable. We can set $A_i = \frac{\partial V_i}{\partial S}$, so the unpredictable term δS_i is eliminated and we get

$$\delta(V - \Pi)_i \approx \left(\frac{\partial V_i}{\partial t} - r D_i \right) \delta t + \frac{1}{2} \frac{\partial^2 V_i}{\partial S^2} \delta S_i^2. \quad (2.23)$$

Now we can add these differences over $0 \leq i \leq L - 1$ and exploit $\sum_{i=0}^{L-1} \delta S_i^2 \approx S(t)^2 \sigma^2 \Delta t$ (this shows that the sum of the δS_i^2 terms is not random). If we are able to find the required function V , then we can differentiate it w.r.t. S , so that we can specify our strategy for updating portfolio. At the end of the step from t_i to t_{i+1} we rebalance our asset to $A_{i+1} = \partial V_{i+1} / \partial S$. This can mean selling or buying some amount of the asset.

Let us denote $\Delta(V - \Pi)$ as the change in $V - \Pi$ from time t to $t + \Delta t$, thus

$$\begin{aligned} \Delta(V - \Pi) &= V(S(t + \Delta t), t + \Delta t) - \Pi(S(t + \Delta t), t + \Delta t) \\ &\quad - (V(S(t), t) - \Pi(S(t), t)). \end{aligned} \quad (2.24)$$

Summing (2.23), we get

$$\Delta(V - \Pi) \approx \sum_{i=0}^{L-1} \left(\frac{\partial V_i}{\partial t} - rD_i \right) \delta t + \frac{1}{2} \sum_{i=0}^{L-1} \frac{\partial^2 V_i}{\partial S^2} \delta S_i^2. \quad (2.25)$$

Since V and D are smooth functions, we can replace the arguments $S(t_i)$, t_i in $\partial V_i / \partial t$ and $\partial^2 V_i / \partial S^2$ by $S(t)$, t and also using $L\delta t = \Delta t$, we get

$$\Delta(V - \Pi) \approx \left(\frac{\partial V}{\partial t} - rD \right) \Delta t + \frac{1}{2} \frac{\partial^2 V}{\partial S^2} \sum_{i=0}^{L-1} \delta S_i^2. \quad (2.26)$$

Now, using an approximation $\sum_{i=0}^{L-1} \delta S_i^2 \approx S(t)^2 \sigma^2 \Delta t$ and assuming that all approximations are exact in the limit $\delta t \rightarrow 0$

$$\Delta(V - \Pi) = \left(\frac{\partial V}{\partial t} - rD + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} \right) \Delta t. \quad (2.27)$$

By the assumptions there is no arbitrage i.e. free profit is not possible. Thus should hold

$$\Delta(V - \Pi) = r\Delta t(V - \Pi). \quad (2.28)$$

If $\Delta(V - \Pi) > r\Delta t(V - \Pi)$ or $\Delta(V - \Pi) < r\Delta t(V - \Pi)$, we can get a free profit (similarly like in call-put parity proof). By combining the results, we get

$$\frac{\partial V}{\partial t} - rD + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} = r(V - AS - D). \quad (2.29)$$

By using $A = \frac{\partial V}{\partial S}$ and rearranging a bit, we finally get the famous **Black-Scholes partial differential equation(PDE)**

$$\frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS \frac{\partial V}{\partial S} - rV = 0. \quad (2.30)$$

The formula provides a relationship between V , S , t and some partial derivatives of V . The first component in (5.14) $\frac{\partial V}{\partial t}$ in the equation is option's *theta* or θ , the second one $\frac{\partial^2 V}{\partial S^2}$ is the option's *gamma* or γ and the third one $\frac{\partial V}{\partial S}$ is the option's *delta* or δ . In Chapter 6 we will have a closer look on option's Greeks.

The PDE should work for any option on S , which value can be expressed as some smooth function $V(S, t)$. Also, we note that there is not any drift parameter μ like in the asset model. From the PDE and boundary conditions of a European call options, we get a solution (similarly as we will do for Barrier options later)

$$C(S, t) = SN(d_1) - Ke^{-r(T-t)}N(d_2). \quad (2.31)$$

Similarly for a European put

$$P(S, t) = Ke^{-r(T-t)}N(-d_2) - SN(-d_1), \quad (2.32)$$

where $N(\cdot)$ is the cumulative distribution function for a standardized normal random variable given by

$$N(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^x e^{-\frac{1}{2}y^2} dy$$

and d_1 and d_2 defined as

$$d_1 = \frac{\log(S/K) + (r + \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}} \quad (2.33)$$

$$d_2 = \frac{\log(S/K) + (r - \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}. \quad (2.34)$$

We will use the formula when we will calculate the price of European call option, using Monte Carlo method in the next chapter, where we will introduce the *Monte Carlo Method* in details.

Chapter 3

Introduction to Monte Carlo Method

Recently, with the complexity of financial derivatives there is need of accurate and fast numerical and analytical solutions. One of the most popular tool to resolve this kind of problems is the *Monte Carlo* method (*MCM*). The MCM can be used for many different purposes, such as valuation of securities, estimation of their sensitivities, hedging, risk analysis and many more.

Monte Carlo method is based on the analogy between probability and volume. The mathematics of measure associate an event with a set of outcomes and define the probability of the event to be its volume or measure relative to that of a universe of possible outcomes. In Monte Carlo it is opposite. MC calculates the volume of a set by interpreting the volume as a probability. In a simple example it means to sample randomly from a universe of possible outcomes and then take the fraction of random draws that fall in a given set as an estimate of the set's volume.

As we already know, *the option pricing theory's* idea is that the price of a derivative security is given by the expected value of discounted payoffs (with the respect to a risk neutral probability measure). MC approach is an efficient application of this theory. It is based on the *Law of Large Numbers* and *Central Limit Theorem*(see Theorem 1.1.10).

The Law of Large numbers guarantees that the estimate converges to the right value as the number of draws increases. In the option pricing theory it means that the estimate converges to the right price of the option.

The Central Limit Theorem makes sure that the standard error of the estimate tends to 0 with a rate of convergence of $\frac{1}{\sqrt{N}}$ (N is the

number of simulations). This rate is based on the assumption that the random variables are generated with the use of *pseudo-random* numbers. They are collections of numbers which are produced by a deterministic algorithm but still seem to be random because they have appropriate statistical properties. In the examples simulated in Matlab, we are using functions 'rand', which produces $U(0, 1)$ -distributed random variables and 'randn' $N(0, 1)$ -distributed.

MCM may be applied as soon as somebody knows how to simulate the random variable X . MCM has three special properties

- (i) *The Law of Large Numbers* holds under very general conditions. It is enough that the random variable is integrable, and there are no regularity conditions analogous to those necessitated by deterministic methods of numerical integration.
- (ii) The error is σ/\sqrt{N} , where σ^2 is the variance of the random variable X . Thus the convergence is slow. In order to divide the error by 10, one needs to multiply the number of trials by 100. That means that sometimes to get some good results, one should perform a large number of simulations.
- (iii) The method is efficient for high dimensional problems, that is when the random variable depends on a large number of independent sources of randomness, in which case the deterministic methods become useless.[1] [2][15]

3.1 The MCM for the computation of option prices

We can summarize MCM in 4 steps

1. Simulate a path of the underlying asset over the desired time period under the risk neutrality condition.
2. Discount corresponding payoff to the path at the risk-free interest rate.
3. Repeat this process for a high number of simulated sample paths.
4. Average discounted cash flows over the number of paths to obtain the value of option.

In the simple cases, such as European options, the random variable X is a function of the value of the underlying at date T : $X = f(S_T)$. In the case of exotic options, X may depend on all values

taken by underlying between the times 0 and T i.e. $X = \Phi(S_t, 0 \leq t \leq T)$.

The function f or the functional Φ is known explicitly but the direct simulation of $S(T)$ is not always possible. For most models, the process S is defined as the solution of the SDE in (2.8). [1][8]

Example 3.1.1. Valuation of European call option using Monte Carlo VS Black-Scholes Value. We will use Monte Carlo method to value a European call option

$$\Lambda(S(T)) = \max(S(T) - K, 0). \quad (3.1)$$

Function Λ is the payoff of the asset price at expiry. Continuous-time model for the asset price at expiry time with $t = T$

$$S(t) = S_0 e^{(\mu - \frac{1}{2}\sigma^2)t + \sigma\sqrt{t}Z}, \quad (3.2)$$

where Z is $N(0,1)$ -distributed.

We are using the risk neutrality approach i.e. the time-zero option value can be computed by taking $\mu = r$ in $e^{-rT}\mathbb{E}(\Lambda(S(T)))$. By putting all together we would like to find the expected value of the random variable

$$e^{-rT}\Lambda(S_0 \exp[(r - \frac{1}{2}\sigma^2)T + \sigma\sqrt{T}Z]), \quad (3.3)$$

where Z is $N(0,1)$ -distributed.

Summarized Monte Carlo algorithm may look as follows

```

for i = 1 to M
  generate a sample  $\xi_i \sim N(0, 1)$ 
  set  $S_i = S_0 e^{(r - \frac{1}{2}\sigma^2)T + \sigma\sqrt{T}\xi_i}$ 
  set  $V_i = e^{(-rT)}\Lambda(S_i)$ 
end

```

```

set  $a_M = (\frac{1}{M}) \sum_{i=1}^M V_i$ 
set  $b_M^2 = (\frac{1}{(M-1)}) \sum_{i=1}^M (V_i - a_M)^2$ 

```

By using directly the Black-Scholes formula for the value of a European call (2.31)

$$C(S, t) = SN(d_1) - Ee^{-r(T-t)}N(d_2), \quad (3.4)$$

where $N(\cdot)$ is the $N(0, 1)$ the cumulative distribution function of the standard normal distribution function

$$N(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^x e^{-\frac{s^2}{2}} \quad (3.5)$$

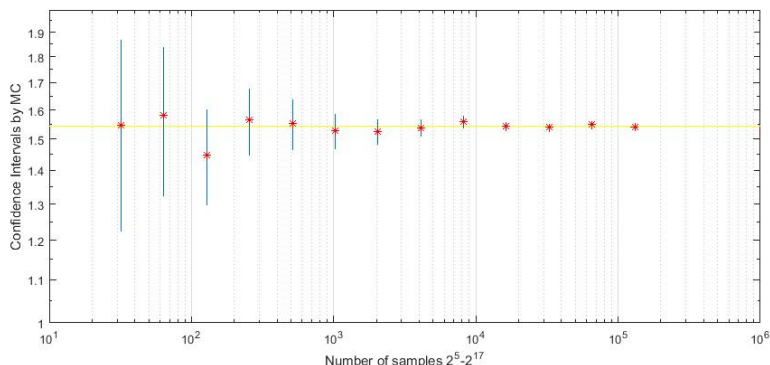
and

$$d_1 = \frac{\log(S/E) + (r + \frac{1}{2}\sigma^2)(T - t)}{\sigma\sqrt{T - t}} \quad (3.6)$$

$$d_2 = \frac{\log(S/E) + (r - \frac{1}{2}\sigma^2)(T - t)}{\sigma\sqrt{T - t}}. \quad (3.7)$$

We are able to compute the exact value and compare to Monte Carlo confidence interval values with different amount of simulations. So for example with $M = 2^5$ amount of simulations MC confidence interval is $[1.2081, 1.8777]$ and with $M = 2^{17}$ MC confidence interval is $[1.5377, 1.5482]$, while Black-Scholes-value is 1.5429. The codes used for calculating the Black-Scholes and Monte Carlo values can be found in the section Matlab codes.

Figure 3.1: Monte Carlo approximation VS Black-Scholes value of a European call option with $S_0 = 10$, $K = 9$ (strike price), $\sigma = 0.1$, $r = 0.06$, $T = 1$, $M = 2^5, \dots, 2^{17}$ sample sizes



In the figure 3.1 red crosses are the Monte Carlo mean approximations a_M , vertical lines are Monte Carlo 95% confidence intervals and horizontal line is the Black-Scholes value for a European call. We can see that the more simulations we make, the better results the Monte Carlo simulation give. This is just a simple example to illustrate how well Monte Carlo method works. In general there is no need to use Monte Carlo in such cases, since Black-Scholes gives the exact solution.

Chapter 4

Exotic options and their pricing

4.1 Introduction to Exotic Options

In this chapter we will see different types of exotic options and develop a Monte Carlo algorithm for exotic options. An exotic option is an option that differs in a structure from common European options in terms of the underlying asset, or the calculation of how or when the investor receives a certain payoff. Exotic options are generally much more complex.

Exotic options have a long history. Some of them have existed already since 1973. However, because of the thin trading, exotic options started to be more interesting for investors and more popular among financial communities only from late 1980's and the early 1990's. Their trading became more active in the over-the-counter (OTC) market. Recently, a few of exotic options have been listed also in exchange market. For example, the American Stock Exchange trades quanto options.

Today's exotic options are taking huge interest since the trading volumes are high, various types of investors and global financial markets got very complex. So basically there is demand for options with a tailored term structure. Investment strategies can be difficult, costly achieved with traditional options. The use of exotic options is increasing all the time and they are getting listed on different exchanges. Nowadays, there are numerous types of this kind of options for different functionalities, pay-off functions and term structures. Exotic options can be called as second generation options since each of them can serve a special purpose, which standard options cannot.[\[14\]](#)[\[17\]](#)

Each type of an exotic option is distinguished by

- (i) the nature of path dependency,
- (ii) whether early exercise is allowed.

Often the exact expression for the option value is not available, hence there is need of an approximation.

4.2 Path dependent options

As we can understand by the name, path dependent options are options, which payoff depends upon the asset path $S(t)$ for $0 \leq t \leq T$ i.e. on the past history of the underlying asset price and on the spot price at that moment. Path-dependence can be strong, for example Asian options or weak, for example Barrier options. Strong dependence means that we should keep track of an additional variable besides the asset level at every observation and time. The most popular kinds of path-dependent options are lookback options, barrier options, forward-start options and Asian options.

4.2.1 Lookback options

Lookback options have payoffs that depend on the realized minimum or maximum of the underlying asset over a specified period of time. There are many different types of lookback options: *American lookback*, *fixed strike lookback*, *floating strike lookback options* etc and two broad categories: fixed and floating strikes options.

A floating strike lookback call option has a payoff given by the difference between the settlement price and the minimum price achieved by the stock during the specified period

$$C(T)^{FloatCall} = \max(S(T) - S_{\min}, 0),$$

where $S(T)$ is a stock price at expiry date and S_{\min} is a minimum observed price at $[0, T]$. The payoff of a *put*, is the difference between the maximum and the settlement price

$$P(T)^{FloatPut} = \max(S^{\max} - S(T), 0),$$

where S^{\max} is a maximum price observed at $[0, T]$.

Fixed strike lookback options are similar to normal European options. The difference for a call is that the stock price at expiration is replaced by the maximum price observed (during the option's life) and for a put the minimum.

$$C(T)^{FixedCall} = \max(S^{\max} - K, 0),$$

$$P(T)^{FixedPut} = \max(K - S_{\min}, 0).$$

Lookback options have the payoff at least as good as the corresponding Europeans and many other advantages because of that are more valuable than the Europeans. They allow investors ‘look back’ at the underlying prices observed during the life of the option and exercise based on the optimal value. With floating strike options investors can be sure to buy at the low and sell at the high. Obviously advantages are balanced by high premiums.[8] [14]

4.2.2 Barrier options

Barrier options are the oldest and one of the most popular types of exotic options. One reason why they are so popular, is because the payoff opportunities are more limited than the payoff opportunities of the European options, so they are cheaper to buy. Their payoff depends on whether or not the underlying asset has reached or exceeded a predetermined price. The underlying asset price should stay in some predefined region for option to be exercised. There are many types of Barrier options. We will restrict our analysis to the standard ones. There are four standard types of barrier options

1. *down-and-in*
2. *down-and-out*
3. *up-and-in*
4. *up-and-out*.

Each of them, obviously, can be either call or put. If an option is *in* or *knock-in*, it means the option has zero value until the underlying asset reaches a certain price. *Out* or *knock-out* respectively means that if the underlying asset exceeds a certain price, the option will expire worthless. *Up* is when the barrier $B > S_0$ and *down* respectively is when $B < S_0$. For better understanding we will check the calls with more details. Let us denote first

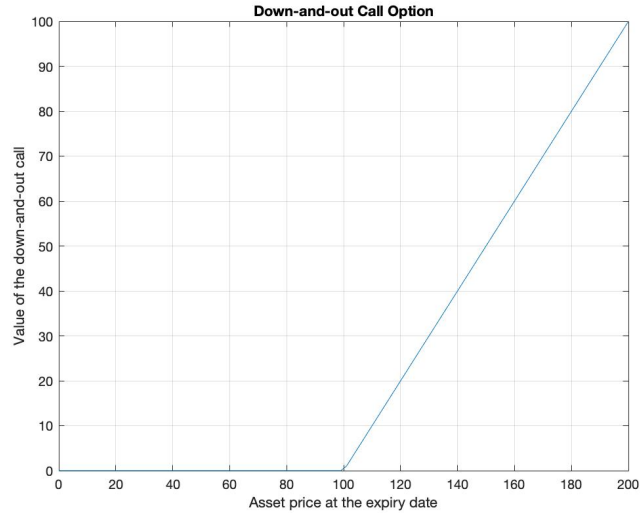
$$M_T := \max\{S(t), 0 \leq t \leq T\}, m_T := \min\{S(t) : 0 \leq t \leq T\}.$$

and

$$\mathbf{1}_{\{M_t < B\}}, \mathbf{1}_{\{m_t > B\}}, \mathbf{1}_{\{M_t \geq B\}}, \mathbf{1}_{\{m_t \leq B\}}.$$

1. A payoff of a *down-and-out call* is zero if the asset crosses a predetermined barrier $B < S_0$ at the interval $[0, T]$. If not the

Figure 4.1: Down-and-out call option with barrier 90 and strike 100



payoff becomes that of a European call $\max(S(T) - K, 0)$. In other words,

$$C^{\text{down-and-out}} = (S(T) - K)^+ \mathbf{1}_{\{m_T > B\}}. \quad (4.1)$$

If the barrier has not been hit before expiration, the terminal payoff of a down-and-out call option will have different features depending on whether the barrier is below or above the strike price K . When $K > B$, should hold

$$C(S, T)^{\text{down-and-out}} = \max(S - K, 0).$$

As S becomes large the probability that the barrier will be activated becomes very small, so

$$C(S, t)^{\text{down-and-out}} \rightarrow S \text{ as } S \rightarrow \infty.$$

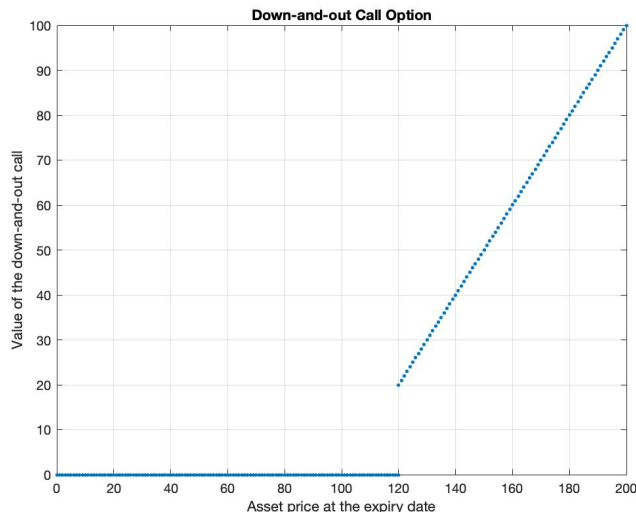
The last boundary condition is at the time when the barrier is hit, the option becomes worthless, so

$$C(B, t)^{\text{down-and-out}} = 0.$$

In the figure 4.1 we can see the situation when the barrier was not crossed and the strike 100 is bigger than the barrier 90. In the figure 4.2 we see the situation when the barrier was not crossed but the strike 100 is smaller than the barrier 120.

2. A payoff of a *down-and-in call* is zero unless the asset crosses a predetermined barrier $B < S_0$ at $[0, T]$. If the barrier is crossed,

Figure 4.2: Down-and-out call option with barrier 120 and strike 100



the payoff becomes that of a European call. In other words,

$$C^{down-and-in} = (S(T) - K)^+ \mathbf{1}_{\{m_T \leq B\}}. \quad (4.2)$$

In the case when the barrier has yet to be crossed, the option is worthless as $S \rightarrow \infty$. Since the larger S is, the less probably it will fall through the barrier before expiry and activate the option. So,

$$C(S, t)^{down-and-in} \rightarrow 0 \text{ as } S \rightarrow \infty.$$

and

$$C(S, T)^{down-and-in} = 0 \text{ for } S > B.$$

The last boundary condition obviously is

$$C(B, t)^{down-and-in} = C(B, t).$$

3. A payoff *up-and-in* is zero unless the asset crosses a predetermined barrier $B > S_0$ at $[0, T]$. If the barrier is crossed, the payoff becomes that of a European call. In other words,

$$C^{up-and-in} = (S(T) - K)^+ \mathbf{1}_{\{M_T \geq B\}}. \quad (4.3)$$

4. A payoff of *up-and-out call* is zero if the asset crosses a predetermined barrier $B > S_0$ at $[0, T]$. Otherwise, the payoff becomes that of a European call.

$$C^{up-and-out} = (S(T) - K)^+ \mathbf{1}_{\{M_t < B\}}. \quad (4.4)$$

The boundary conditions are

$$C(B, t)^{up-and-out} = 0,$$

$$C(S, T)^{up-and-out} = C(S, T)$$

and $C(S, t)^{up-and-out} \rightarrow S$ as $S \rightarrow \infty$.

Moreover, we need to note some important equivalent principles that we will use later

(i) up-and-out call + up-and-in call = vanilla call

(ii) down-and-out call + down-and-in call = vanilla call.

4.2.3 Asian options

Asian options are options with the payoff depending on some average of the underlying assets prices, indices or rates over pre-specified period of time before expiry. The averages can be defined in various ways. Asian options can be arithmetic or geometric, weighted or unweighted, discrete or continuous. The most important difference between *arithmetic* and *geometric* Asian options is that geometric averages are lognormally distributed when the underlying asset prices are lognormally distributed whereas arithmetic averages are not lognormally distributed even if the underlying asset prices are.

Asian options are less sensible to possible spot manipulation at settlement. In addition, their payoffs are less volatile than that of vanilla options. Because of that, Asian options have attracted much attention in the OTC market. Wiklund in his paper *Asian Option Pricing and Volatility* (2008) presents the Black-Scholes formula for the geometric and arithmetic Asian options.

The payoff for a European-style average price Asian call option is the difference between the average and pre-defined strike price.[8] The payoff at the expiry date T of an *average price Asian call option* is defined as

$$\max\left(\frac{1}{T} \int_0^T S(\tau) d\tau - K, 0\right),$$

where K is the strike.

An *average price Asian put* option has the payoff at the expiry date T is defined as:

$$\max\left(K - \frac{1}{T} \int_0^T S(\tau) d\tau, 0\right).$$

If we replace the strike by $S(T)$, we will get an *average strike Asian call and put*. Other Asian options can be defined by replacing the continuous average $\frac{1}{T} \int_0^T S(\tau) d\tau$ by an arithmetic average

$\frac{1}{n} \sum_{i=1}^n S(t_i)$ or geometric average $\left(\prod_{i=1}^n S(t_i)\right)^{\frac{1}{n}}$. As a result the payoff of the *geometric average call* is

$$C(T)^{GeoAvCall} = \max \left(\left(\prod_{i=1}^n S(t_i) \right)^{\frac{1}{n}} - K, 0 \right).$$

4.2.4 Forward-start options

Forward-start options exist in the interest-rate markets in which investors can use them to bet on interest-rate fluctuations. Forward-start options are put or call options purchased in advance with strike prices determined later, usually when the option becomes active at a specified date in the future. The premium however is paid in advance. In other words, forward-start options are options with up-front premium payments, they start in pre-specified time in the future with strike prices equal to the starting underlying asset prices.[8][14][12]

4.3 Correlation options

Correlation options are options which payoff is affected by more than one underlying asset. The underlying asset can be of the same or different asset class: equity, stock, bond, currency, commodity, indices, etc. It is not hard to understand that correlation among these assets will have a major part in the pricing and hedging of the instruments. The correlation is even more unstable than the variance, so the problems affected by correlation can be even more complicated. For example correlation options are *exchange options*, *foreign-equity options*, *quanto options*, *spread options* etc.

4.3.1 Exchange options

Exchange options are basic correlation options. Exchange options give the right to exchange one underlying asset for another. In other words, the value of one asset is paid while the value of the other asset is received at maturity. They can be used to construct many others exotic options.

The payoff of the exchange option to receive the first asset and pay the value of the second is

$$\max(S(T_1) - S(T_2), 0) \tag{4.5}$$

and the payoff of the exchange option to pay the second asset and receive the first is

$$\max(S(T_2) - S(T_1), 0). \quad (4.6)$$

There are many contracts, which can be thought as exchange options, such as *a performance incentive fee, the stand-by commitment (a put on a forward contract in mortgage notes), a margin account* etc.

4.4 Other exotic options

There are other exotic options on the OTC market, which cannot be classified in any of the previous classes, but anyway are very common, such as *digital options, compound options, chooser options, contingent premium options, hybrid options, American, Bermudan and shout options* etc. [12, 14, 19]

4.5 Monte Carlo for exotics

The Monte Carlo method described in the previous chapter handles easily path-dependency. The only required extra step is to set up a grid of points $t_j = j\Delta t$ for $0 \leq j \leq N$, where N is a large number and $\Delta t = \frac{T}{N}$. Assuming $S(0) = S_0$ and using the risk neutrality assumption $\mu = r$, we get an asset price $S(t_{j+1})$ in terms of $S(t_j)$

$$S(t_{j+1}) = S(t_j)e^{(r-\frac{1}{2}\sigma^2)\Delta t + \sigma\sqrt{\Delta t}Z_j}, \quad (4.7)$$

for i.i.d Z_j distributed $N(0, 1)$.

We can have an example of an up-and-out call barrier option. The simulation algorithm of an up-and-out call barrier options may be as follows

```

for  $i = 1$  to  $M$ 
  for  $j = 0$  to  $N - 1$ 
    generate  $N(0, 1)$  sample  $\xi_j$ 
    set  $S_{j+1} = S_j e^{(r-\frac{1}{2}\sigma^2)\Delta t + \sigma\sqrt{\Delta t}\xi_j}$ 
  end
  set  $S_i^{\max} = \max_{0 \leq j \leq N} S_j$ 
  if  $S_i^{\max} < B$ 
    set  $V_i = e^{-rT} \max(S_N - K, 0)$ 
  else set  $V_i = 0$ 
end
end

```

$$\begin{aligned} \text{set } a_M &= \left(\frac{1}{M}\right) \sum_{i=1}^M V_i \text{ (mean)} \\ \text{set } b_M^2 &= \frac{1}{(M-1)} \sum_{i=1}^M (V_i - a_M)^2 \text{ (standard deviation)} \end{aligned}$$

So a_M is approximated option price. We will use this algorithm in the next chapter in Example 5.1.1.

Chapter 5

Valuation of barrier options

5.1 Black-Scholes for Barrier options

In this chapter we will take a better look on the barrier options, how to value them. We start from the case of a down-and-out call option. As we already mentioned in the previous chapter the payoff at expiry of a down-and-out call option is $\max(S - K)$, if S never falls below the barrier. If S hit the barrier, the option would become worthless.

We consider only the case when $K > B$. As long as $S > B$, the value of the option satisfies the Black-Scholes equation. As we already noted in the previous chapter the final condition is that at maturity the value of the option is $\max(S - K)$. Also, the higher the S , the smaller is the probability that the barrier will be hit.

At this point, the problem is pretty much the same as for a European call. However, the boundary condition when the value of the option is zero, for a vanilla would be at $S = 0$ but for a down-and-out at $S = B$.

The price for down-and-out call option can be derived by reducing the boundary problem in the Black-Scholes PDE to a more simplified form, so called diffusion equation or heat equation and resolving it, using change of variables and method of images. We will not do it here, we will just show that the formula satisfies boundary conditions and the PDE. The formula for a down-and out call is

$$F_{call}^{down-and-out}(S, t) = C(S, t) - \left(\frac{S}{B}\right)^{1-\frac{2r}{\sigma^2}} C\left(\frac{B^2}{S}, t\right), \quad (5.1)$$

where $C\left(\frac{B^2}{S}, t\right)$ is calculated using equation (5.15) and $B \leq S$.

Explicitly

$$\begin{aligned}
& F_{call}^{down-and-out}(S, t) \\
&= C(S, t) - \left(\frac{S}{B}\right)^{1-\frac{2r}{\sigma^2}} C\left(\frac{B^2}{S}, T-t\right) \\
&= SN\left(\frac{\log(S/K) + (r + \sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}\right) \\
&\quad - Ke^{-r(T-t)} N\left(\frac{\log(S/K) + (r - \sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}\right) \\
&\quad - B\left(\frac{S^{-2r\sigma^{-2}}}{B}\right) N\left(\frac{\log(B^2/SK) + (r + \sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}\right) \\
&\quad - \left(\frac{S^{1-2r\sigma^{-2}}}{B} Ke^{-r(T-t)}\right) N\left(\frac{\log(B^2/SK) + (r - \sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}\right).
\end{aligned} \tag{5.2}$$

First we show that formula satisfies the Black-Scholes PDE. For that we consider the function $V(S, t)$. Suppose that the function $V(S, t)$ satisfies the Black-Scholes PDE (2.30). Let us set

$$\hat{V}(S, t) = S^{1-\frac{2r}{\sigma^2}} V\left(\frac{B}{S}, t\right), \tag{5.4}$$

then we can calculate partial derivatives of the function $\hat{V}(S, t)$

$$\frac{\partial \hat{V}}{\partial t} = S^{1-\frac{2r}{\sigma^2}} \frac{\partial V}{\partial t}\left(\frac{B}{S}, t\right),$$

$$\frac{\partial \hat{V}}{\partial S} = \left(1 - \frac{2r}{\sigma^2}\right) S^{-\frac{2r}{\sigma^2}} V\left(\frac{B}{S}, t\right) - BS^{-1-\frac{2r}{\sigma^2}} \frac{\partial V}{\partial S}\left(\frac{B}{S}, t\right),$$

$$\begin{aligned}
\frac{\partial^2 \hat{V}}{\partial S^2} &= \left(1 - \frac{2r}{\sigma^2}\right) \left(\frac{-2r}{\sigma^2}\right) S^{-1-\frac{2r}{\sigma^2}} V\left(\frac{B}{S}, t\right) \\
&\quad + \frac{\partial V}{\partial S}\left(\frac{B}{S}, t\right) \left(\frac{4Br}{\sigma^2}\right) S^{-2-\frac{2r}{\sigma^2}} \\
&\quad + B^2 S^{-3-\frac{2r}{\sigma^2}} \frac{\partial^2 V}{\partial S^2}\left(\frac{B}{S}, t\right).
\end{aligned}$$

Now the left side of the partial differential equation (5.4) becomes

$$\begin{aligned}
& \frac{\partial \hat{V}}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 \hat{V}}{\partial S^2} + rS \frac{\partial \hat{V}}{\partial S} - r\hat{V} \\
&= S^{1-\frac{2r}{\sigma^2}} \frac{\partial V}{\partial t} \left(\frac{B}{S}, t \right) + \left(1 - \frac{2r}{\sigma^2} \right) (-r) S^2 S^{-1-\frac{2r}{\sigma^2}} V \left(\frac{B}{S}, t \right) \\
&\quad + \frac{\partial V}{\partial S} \left(\frac{B}{S}, t \right) 2BrS^2 S^{-2-\frac{2r}{\sigma^2}} + \frac{1}{2} \sigma^2 S^2 B^2 S^{-3-\frac{2r}{\sigma^2}} \frac{\partial^2 V}{\partial S^2} \left(\frac{B}{S}, t \right) \\
&\quad + rS \left(1 - \frac{2r}{\sigma^2} \right) S^{\frac{-2r}{\sigma^2}} V \left(\frac{B}{S}, t \right) - rSB S^{-1-\frac{2r}{\sigma^2}} \frac{\partial V}{\partial S} \left(\frac{B}{S}, t \right) \\
&\quad - rS^{1-\frac{2r}{\sigma^2}} V \left(\frac{B}{S}, t \right) \\
&= S^{1-\frac{2r}{\sigma^2}} \frac{\partial V}{\partial t} \left(\frac{B}{S}, t \right) + \frac{\partial V}{\partial S} \left(\frac{B}{S}, t \right) BrS^{-\frac{2r}{\sigma^2}} \\
&\quad + \frac{1}{2} \sigma^2 B^2 S^{-1-\frac{2r}{\sigma^2}} \frac{\partial^2 V}{\partial S^2} \left(\frac{B}{S}, t \right) - rS^{1-\frac{2r}{\sigma^2}} V \left(\frac{B}{S}, t \right) \\
&= S^{1-\frac{2r}{\sigma^2}} \\
&\quad \cdot \left[\frac{\partial V}{\partial t} \left(\frac{B}{S}, t \right) + \frac{1}{2} \sigma^2 \left(\frac{B}{S} \right)^2 \frac{\partial^2 V}{\partial S^2} \left(\frac{B}{S}, t \right) + r \frac{B}{S} \frac{\partial V}{\partial S} \left(\frac{B}{S}, t \right) \right. \\
&\quad \left. - rV \left(\frac{B}{S}, t \right) \right].
\end{aligned}$$

We see that the term inside the big brackets is actually zero because V satisfies the Black-Scholes PDE, so \hat{V} solves the Black-Scholes PDE. From the above derivation and assuming that $B < S_0$, we see that the formula for the down-and-out call option (5.1) also satisfies the Black-Scholes PDE. Furthermore, it satisfies the boundary conditions (defined in the previous chapter) when $B < K$

- (i) the option is worthless when $B \leq S$

$$F_{call}^{down-and-out}(B, t) = C(B, t) - C(B, t) = 0$$

for $0 \leq t \leq T$

- (ii) at the expiry if the barrier was not crossed $S > B$ for $0 \leq t \leq T$, it would become the European call option

$$\begin{aligned}
F_{call}^{down-and-out}(S, T) &= (S(T) - K)^+ \\
&\quad - \left(\frac{S(T)}{B} \right)^{1-\frac{2r}{\sigma^2}} \left(\frac{B^2}{S(T)} - K \right)^+ \\
&= (S(T) - K)^+.
\end{aligned}$$

The value of the down-and-in call can be calculated using the in-out-parity

$$C(S, t) = F_{call}^{down-and-in}(S, t) + F_{call}^{down-and-out}(S, t) \quad (5.5)$$

$$\begin{aligned} \implies F_{call}^{down-and-in}(S, t) &= C(S, t) - F_{call}^{down-and-out}(S, t) \\ &= \left(\frac{S}{B}\right)^{1-\frac{2r}{\sigma^2}} C\left(\frac{B^2}{S}, T-t\right). \end{aligned}$$

Similarly for the up-and-out call

$$\begin{aligned} &F_{call}^{up-and-out}(S, t) \\ &= S \left(N(d_1) - N(e_1) - \left(\frac{B}{S}\right)^{1+2r/\sigma^2} (N(f_2) - N(g_2)) \right) \\ &- Ke^{-r(T-t)} \left(N(d_2) - N(e_2) - \left(\frac{B}{S}\right)^{-1+2r/\sigma^2} (N(f_1) - N(g_1)) \right), \end{aligned} \quad (5.6)$$

where

$$e_1 = \frac{\log(S/B) + (r + \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{(T-t)}}, \quad (5.7)$$

$$e_2 = \frac{\log(S/B) + (r - \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{(T-t)}}, \quad (5.8)$$

$$f_1 = \frac{\log(S/B) - (r - \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{(T-t)}}, \quad (5.9)$$

$$f_2 = \frac{\log(S/B) - (r + \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{(T-t)}}, \quad (5.10)$$

$$g_1 = \frac{\log(S/B) - (r - \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{(T-t)}}, \quad (5.11)$$

$$g_2 = \frac{\log(S/B) - (r + \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{(T-t)}}. \quad (5.12)$$

Example 5.1.1. Let us consider an up-and-out call option. We compare the Black-Scholes value versus the Monte Carlo confidence interval. Using algorithm from previous chapter for the Monte Carlo value, (5.6)-(5.12) for the Black-Scholes value and input parameters: $S = 5$ (asset price), $K = 6$ (strike), $\sigma = 0.25$ (volatility), $r = 0.05$

(interest rate), $T = 1$; $B = 9$, $Dt = 10^{-3}$, $N = T/Dt$, $M = 10^4$ (number of simulations), we get that the Black-Scholes value is 0.18 and the Monte-Carlo confidence interval is $[0.176, 1.193]$.

Example 5.1.2. Let us consider another example of a down-and-out call option, where we will again compare Black-Scholes result versus Monte Carlo confidence interval. Using input parameters: $S = 100$, $K = 100$, $\sigma = 0.3$, $r = 0.1$, $T = 0.2$, $B = 85$, $Dt = 1e-3$, $N = T/Dt$, $M = 1e4$, we get that the Black-Scholes value is 6.3076 and the Monte Carlo value is $[6.1756, 6.5296]$. The function used to calculate the Black-Scholes value for the down-and-out call can be found in the Matlab codes section.

5.1.1 Variance reduction by antithetic variates

Even though the Monte Carlo method is a simple and flexible method, it has some efficiency issues. For getting efficient results in some cases we need to make thousands of simulations. That is because there are some approaches which attempt to improve efficiency. One of them is a *antithetic variates idea*. The idea is based on the concept of covariance between random variables.

The Monte Carlo method to approximate the expected value of random variable X uses the average of independent random variables with the same distribution as X . So, $\mathbb{E}(X_i) = \mathbb{E}(X)$ and the width of the corresponding confidence interval is inversely proportional to \sqrt{M} . Thus for improving the approximation we need to take more samples. Moreover, the confidence interval width scales with $\sqrt{\text{var}(X_i)}$. It gives as an idea to replace X_i with another sequence of i.i.d. random variables with the same mean but with smaller variance. [8]

The problem here is that variance reduction techniques require using random variables, which are not independent. Basically independence means that if we know the value of one random variable, it does not give us any other additional information about the value of the other i.e. $\mathbb{E}(XY) = \mathbb{E}(X)\mathbb{E}(Y)$. For measuring dependence we can use the covariance: $\text{cov}(X, Y) = \mathbb{E}[(X - \mathbb{E}(X))(Y - \mathbb{E}(Y))]$ \iff $\text{cov}(X, Y) = \mathbb{E}(XY) - \mathbb{E}(X)\mathbb{E}(Y)$.

We can consider the case of the normally distributed random variables. The general estimate is

$$I = \mathbb{E}(f(U)), \text{ where } U \sim N(0, 1), \quad (5.13)$$

the normal Monte Carlo estimate is

$$I_M = \frac{1}{M} \sum_{i=1}^M f(U_i), \text{ i.i.d. } U_i \sim N(0, 1) \quad (5.14)$$

and the antithetic estimate is

$$\hat{I} = \frac{1}{M} \sum_{i=1}^M \frac{f(U_i) + f(-U_i)}{2}, \text{ i.i.d. } U_i \sim N(0, 1). \quad (5.15)$$

Since the $N(0,1)$ distribution is symmetric about the origin, the antithetic estimate is $-U_i$, rather than $1 - U_i$. We get that

$$\text{var}\left(\frac{f(U_i) + f(-U_i)}{2}\right) \leq \frac{1}{2} \text{var}(f(U_i)), \text{ f is monotonic.} \quad (5.16)$$

Because of (5.13),(5.14)(see below) and the fact that

$$\text{cov}(f(X), f(-X)) \geq 0$$

(since if f is monotonic increasing or decreasing, then also $f(-x)$ and applying the Lemma 5.1.3).

$$\begin{aligned} & \text{var}\left(\frac{f(U_i) + f(-U_i)}{2}\right) \\ &= \frac{1}{4}(\text{var}(f(U_i)) - \text{var}(f(-U_i)) + 2\text{cov}(f(U_i), f(-U_i))) \quad (5.17) \\ &= \frac{1}{2}(\text{var}(f(U_i)) + \text{cov}(f(U_i), f(-U_i))), \end{aligned}$$

knowing that $\text{var}(X + Y) = \text{var}(X) + \text{var}(Y) + 2\text{cov}(X, Y)$.

Lemma 5.1.3. If functions f and g are monotonic increasing or monotonic decreasing, then for any random variable X

$$\text{cov}(f(X), g(X)) \geq 0. \quad (5.18)$$

Proof. Let Y and X be independent random variables with the same distribution function. f is monotonic increasing means that $x_1 \leq x_2 \Rightarrow f(x_1) \leq f(x_2)$, similarly f is monotonic decreasing $x_1 \leq x_2 \Rightarrow f(x_2) \leq f(x_1)$. From these definitions follows that if both functions are monotonic increasing or monotonic decreasing, then

$$(f(x) - f(y))(g(x) - g(y)) \geq 0 \text{ for any } x \text{ and } y. \quad (5.19)$$

From that follows

$$\begin{aligned} & 0 \leq \mathbb{E}[(f(X) - f(Y))(g(X) - g(Y))] \\ &= \mathbb{E}[f(X)g(X)] - \mathbb{E}[f(X)g(Y)] - \mathbb{E}[f(Y)g(X)] + \mathbb{E}[f(Y)g(Y)] \\ & \quad \text{Since also } X \text{ and } Y \text{ are i.i.d. follows that} \\ &= 2\mathbb{E}[f(X)g(X)] - 2\mathbb{E}[f(X)]\mathbb{E}[g(X)] = 2\text{cov}(f(X), g(X)). \end{aligned}$$

□

Now we can apply the theory on estimation of path-dependent exotic options, in our case of up-and-in call options. We need to discretise the time interval $[0, T]$ and compute risk-neutral asset prices at $\{t_i\}_{i=1}^N$, $t_i = i\Delta t$, $N\Delta t = T$. We already know from the chapter 4 that on each increment the price update uses an $N(0,1)$ random variable Z_j , $j = 0, \dots, N - 1$. The antithetic strategy is to take the average payoff from one path with $\{Z_0, \dots, Z_{N-1}\}$ and another $\{-Z_0, -Z_1, \dots, -Z_{N-1}\}$. We can summarize the algorithm as follows

```

for  $i = 1$  to  $M$ 
  for  $j = 0$  to  $N - 1$ 
    compute an  $N(0, 1)$   $\xi_j$ 
    set  $S_{j+1} = S_j e^{(r - \frac{1}{2}\sigma^2)\Delta t + \sigma\sqrt{\Delta t}\xi_j}$ 
    set  $\bar{S}_{j+1} = \bar{S}_j e^{(r - \frac{1}{2}\sigma^2)\Delta t + \sigma\sqrt{\Delta t}\xi_j}$ 
  end

  set  $S_i^{\max} = \max_{0 \leq j \leq N} S_j$ 
  set  $\bar{S}_i^{\max} = \max_{0 \leq j \leq N} \bar{S}_j$ 

  if  $S_i^{\max} > B$ 
    set  $V_i = e^{-rT} \max(S_N - K, 0)$ 
  else  $V_i = 0$ 

  if  $\bar{S}_i^{\max} > B$ 
    set  $\bar{V}_i = e^{-rT} \max(\bar{S}_N - K, 0)$ 
  else  $\bar{V}_i = 0$ 

  set  $\hat{V}_i = \frac{1}{2}(V_i + \bar{V}_i)$ 

end
set  $a_M = \frac{1}{M} \sum_{i=1}^M \hat{V}_i$  and
set  $b_M^2 = \frac{1}{M-1} \sum_{i=1}^M (\hat{V}_i - a_M)^2$ 

```

We can calculate the Monte Carlo confidence intervals with and without antithetic variates for up-and-out call option. Using problem parameters: $S = 100$, $K = 120$, $\sigma = 0.3$, $r = 0.1$, $T = 0.2$, $B = 150$, $Dt = 0.001$, $N = T/Dt$, $M = 10000$, we get that normally computed confidence interval is $(0.6460, 0.7566)$, using antithetic variates is $(0.6612, 0.7376)$, while the Black-Scholes value is 0.7030 . Thus, we see that antithetic variates indeed improved the results.

Chapter 6

Hedging

6.1 Greeks and dynamic hedging strategy

In this chapter we will try to answer to the second question presented in Chapter 1

2) *Suppose that we sold an option, such as a call option. How can we protect or hedge ourselves against the risk?*

We will start from the case of simple European options and then continue with the case of more complicated barrier options.

We have already learned that to find a fair value of an option, first we need to set up a replicating portfolio of asset and cash like in (1.7) or we can write it as

$$\Pi(S, t) = A(S, t)S + D(S, t), \quad (6.1)$$

where D is a cash deposit and A is a number of units of asset.

As we already know, *the dynamic trading strategy* consists of units invested in the risk-free asset and units in the underlying asset of the derivative that the payoff of the option is replicated. The main idea is that a correctly hedged position should earn the risk-free rate. This strategy involves holding a "delta-neutral" portfolio. Delta measures option price sensitivity w.r.t underlying price. In other words, delta is a ratio comparing the change in the price of an asset to the corresponding change in the price of its derivative

$$\Delta = \frac{\partial V}{\partial S}. \quad (6.2)$$

For example, if a stock option has a delta value of 0.70, this means that if underlying stock increases in price by 1 euro per share, the option on it will rise by 0.70 euro per share, everything else being equal.

Obviously, delta changes from one period to another, so the investor needs to rebalance his portfolio by borrowing (or bond trading) periodically that the portfolio stays hedged. As we already know, the name of this kind of the strategy is *self-financing*. Cumulative cost of the strategy is zero, taken into account that the initial premium of the option is also invested. Delta is one of the option Greeks. They measure the sensitivity of option price w.r.t different inputs. Greeks are used extensively to measure risk exposure and hedging. A key assumption is that the only one parameter is changed at a time and the rest are held constant.

Other popular Greeks are

$$\Gamma = \frac{\partial^2 C}{\partial S^2},$$

where C is the value of the European call option, (gamma: change in delta w.r.t underlying price)

$$\rho = \frac{\partial C}{\partial r},$$

(rho: option price sensitivity w.r.t risk-free interest rate)

$$\Theta = \frac{\partial C}{\partial t},$$

(theta: option price sensitivity w.r.t time to maturity)

$$vega = \frac{\partial C}{\partial \sigma}.$$

(vega: option price sensitivity w.r.t volatility).[8]

For example, delta of a European call is

$$\Delta = \frac{\partial P}{\partial S} = N(d_1) \tag{6.3}$$

and delta of a European put is

$$\Delta = \frac{\partial C}{\partial S} = N(d_1) - 1, \tag{6.4}$$

where $N(\cdot)$ is the $N(0, 1)$ distribution function

$$N(x) = \frac{1}{2\pi} \int_{-\infty}^x e^{-\frac{s^2}{2}}$$

and

$$d_1 = \frac{\log \frac{S}{E} + (r + \frac{1}{2}\sigma^2)(T - t)}{\sigma\sqrt{T - t}}$$

as it was defined in Chapter 3.

Example 6.1.1. (Computing delta of a European call and put by Black-Scholes) Let us compute the delta for a European call and put with parameters as follows S = asset price at time $t = 2$, K = exercise price at maturity = 2.5, r = interest rate = 0.03, sigma σ = volatility = 0.25, tau τ = time to expiry $(T - t) = 1$. The results are that the value of the European call = 0.0691, call delta = 0.2586, the put = 0.4953 and the put delta = -0.7414.

6.2 Discrete hedging algorithm

In this section we will develop the process into an algorithm that can be illustrated numerically. As we already know the value of the portfolio Π_{i+1} satisfies

$$\Pi_{i+1} = A_i S_{i+1} + (1 + r\delta t) D_i.$$

The asset holding is rebalanced to A_{i+1} and in order to compensate, the cash is altered to D_{i+1} . Since no money enters or leaves the system, the new portfolio value $A_{i+1} S_{i+1} + D_{i+1}$ must equal to Π_{i+1} accordingly

$$D_{i+1} = (1 + r\delta t) D_i + (A_i - A_{i+1}) S_{i+1}. \quad (6.5)$$

We may summarize the overall hedging strategy as follows

```

set  $A_0 = \partial V_0 / \partial S$ ,  $D_0 = 1$  (arbitrary),
 $\Pi_0 = A_0 S_0 + D_0$ ,

for each new time  $t = (i + 1)\delta t$ 
  observe new asset price  $S_{i+1}$ 
  compute new portfolio value  $\Pi_{i+1}$ ,
  compute  $A_{i+1} = \frac{\partial V_{i+1}}{\partial S}$ ,
  compute new cash holding  $D_{i+1}$ 
end
new portfolio value is  $A_{i+1} S_{i+1} + D_{i+1}$ .
```

The strategy is called *discrete hedging*, because the rebalancing should be done at times $i\delta t$. We cannot let $\delta t \rightarrow 0$, otherwise there will be some error in the risk elimination.

It is possible to simulate an asset path and implement discrete hedging. We are using ξ_i to denote samples from an $N(0, 1)$ pseudo-random number generator that is used in simulating the asset path, and we let $\delta t = T/N$

```

set  $A_0 = \partial V_0 / \partial S$ ,  $D_0 = 1$  (arbitrary),
 $\Pi_0 = A_0 S_0 + D_0$ 
```

```

for  $i = 0$  to  $N - 1$ 
  compute  $S_{i+1} = S_i e^{(\mu - \frac{1}{2}\sigma^2)\delta t + \sqrt{\delta t}\sigma\xi_i}$ 
  set  $\Pi_{i+1} = A_i S_{i+1} + (1 + r\delta t)D_i$ 
  compute  $A_{i+1} = \frac{\partial V_{i+1}}{\partial S}$ 
  set  $D_{i+1} = (1 + r\delta t)D_i + (A_i - A_{i+1})S_{i+1}$ 
end.

```

The discrete hedging is successful when we have a large number of sample paths. Also, if we simulate a payoff of an option on a large scale with different values for the drift parameter μ , we will see that the option value is independent of μ in the asset price model.

Discrete delta-based strategy performs well, only when the risk measure is market risk or when the options hedged are long-term in-the-money (stock price is higher than strike price). Otherwise, the hedge ratios should be modified that the variance would be diminished. Another even bigger problem is the existence of transaction costs. The total costs of a hedging strategy is a function of the frequency of rebalancing. It is almost impossible to know from the beginning how large the costs will be.

Example 6.2.1. (The basic case of a European Call Option) In this example we will illustrate delta hedging by computing an approximate replicating portfolio for a European call option. Portfolio includes ‘asset’ units of asset and an amount ‘cash’ of cash. The results how the dynamic hedging looks like we can see in Figure 6.1.

6.3 Hedging exotic options

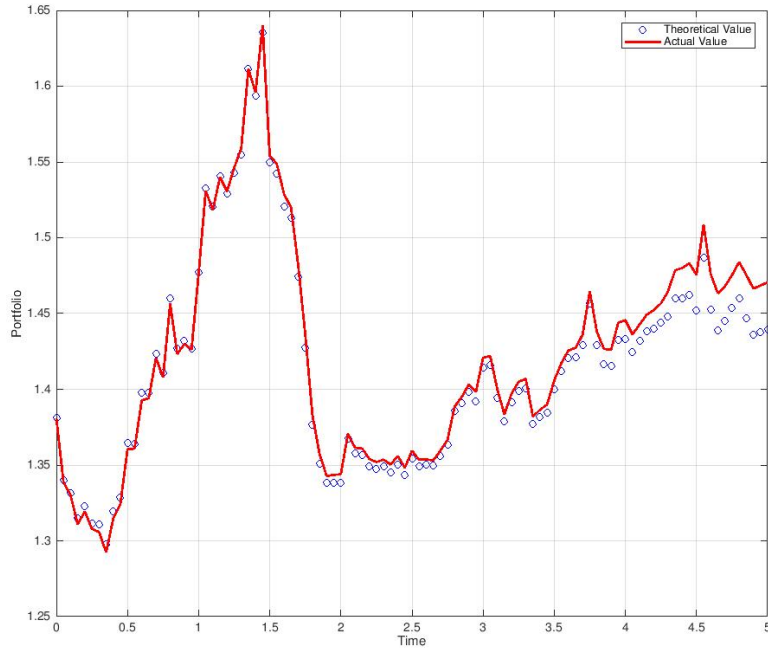
There are two general approaches to hedge exotic options

1. *static hedging* and
2. *dynamic hedging*.

Choosing which method is more efficient depends on the type of the exotic option.

1. *Static strategy* requires the construction of a portfolio of standard options (or other products) and the portfolio is maintained either until the expiration of the exotic claim (if European) or until some event occurs prior to expiration (in the case of barrier options being knock-in or -out). This approach avoids transaction costs from rehedging and this is why it is

Figure 6.1: Dynamic hedging of a European Call with parameters $S_0 = 1$, $\sigma = 0.35$, $r = 0.03$, $\mu = 0.02$, $T = 5$, $E = 2$, $Dt = 1e - 2$, $N = T/Dt$, $t = [0 : Dt : T]$



used preferably in illiquid markets. Since market makers generally try to hedge positions of exotic options using plain vanilla options, which can be of different times to expiry than the exotic options. Thus gamma and vega risk exposure is passed from the very illiquid to the more liquid plain vanilla options book. Remaining risk exposure, which cannot be hedged, is kept on the exotic options book by holding the contract to maturity. The negative part of this strategy is that it requires a relatively high amount of income from the trade to be spent paying bid/offer spreads and markets are often illiquid.

2. *Dynamic hedging* (in this thesis we are discussing only dynamic delta-hedging strategy) has been described above for a call option and for the exotic options the idea is similar. We just need to follow the lead of Black-Scholes [13], who proposed that the derivative can be exactly replicated by the construction of a portfolio of the underlying asset and riskless bond. Thus by maintaining a continuously rebalanced position in the underlying asset (equal to the delta), a riskless hedge can be obtained.

This idea is actually the foundation of option pricing in general. However the dynamic hedging is more complicated for some exotic options. It refers to all Greeks: delta was already described before, gamma, rho, theta and vega. It starts with the ‘delta hedging’ (basically that means buying and selling delta(s) of the underlying(s) against directional price movements of the underlying asset(s)) and continues with other Greeks. With other Greeks the situation gets more complicated but we will not go deeper than with the case of delta. It is logical that $\Delta > 0$ up to expiry. Since an increase in the asset price increases the likely profit at the time of expiry.

The dynamic hedging relies on several very important assumptions

- (i) rebalancing of the hedged portfolio happens continuously,
- (ii) markets must be complete, without taxes and other transaction costs and
- (iii) constant volatility.

For the first assumption, Hull [11] examined the impact of discretely rebalancing the hedge portfolio by simulations and reported that the delta-hedging performance becomes weaker as the time between the hedge rebalancing increases.

For different options different hedging strategy is more efficient. For example, for exchange options the dynamic hedging is more efficient but for path-dependent options, for example Asian options, the static hedging might be less difficult, because they have high gammas.

Example 6.3.1. We already have seen in the previous chapter that barrier options have a payoff that switches on or off depending on whether the asset crosses a level established in the contract. There are many types of such options, but we will show only cases of the down-and out call and down-and-in call options. There are two already known methods to hedge this kind of options *dynamic* and *static*. The approaches are compared in the article written by Tompkins [17]. We will relax the assumptions of continuous and frictionless financial markets in the dynamic approach and will access how the cost of the hedging deviates from the theoretical values are derived under perfect (continuous) financial markets.

1. *Dynamic hedging*

As we already have seen the delta is a measure of sensitivity of

the changing option value with respect to the changing underlying stock price. The objective of delta hedging is to reduce the delta to zero by continuous rebalancing. We can derive the delta of a down-and-in call by differentiating equation in (5.5) with respect to the underlying stock price S

$$\begin{aligned}
& \Delta^{down-and-in}(S, K, B) \\
&= \left(1 - \frac{2r}{\sigma^2}\right) S^{-\frac{2r}{\sigma^2}} B^{\frac{2r}{\sigma^2}-1} C\left(\frac{B^2}{S}, T-t\right) \\
&\quad - \left(\frac{B^2}{S^2}\right) N\left(d_1\left(\frac{B^2}{S}, T-t\right)\right) \left(\frac{S}{B}\right)^{1-\frac{2r}{\sigma^2}} \\
&= \frac{B^{\frac{2r}{\sigma^2}-1}}{S} \left(- \left(\frac{B^2}{S^2}\right) N\left(d_1\left(\frac{B^2}{S}, T-t\right)\right) \right. \\
&\quad \left. - \frac{2r - \sigma^2}{\sigma^2 S} C\left(\frac{B^2}{S}, T-t\right) \right).
\end{aligned}$$

In the delta-hedging it is possible to

- (i) buy an option and sell delta-amount of shares of stock and invest the rest of the money in a risk-free asset,
- (ii) sell an option and use the premium to buy delta-amount of shares and invest the rest in the risk free asset.

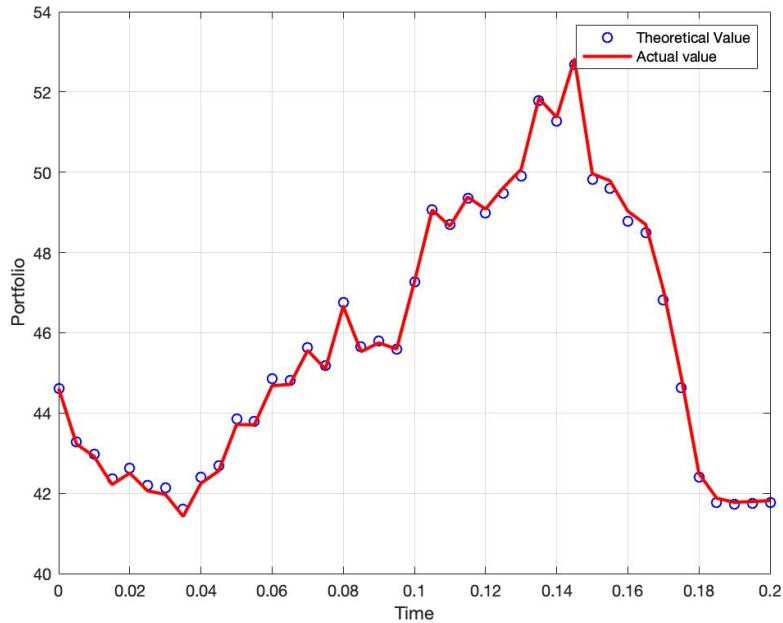
For example, an investor sells 5 calls worth 10€ with the underlying stock price 100€, then he earns a premium of $5 \cdot 10 = 50$ and has to buy stock worth $5 \cdot 100 = 500$. To hedge the position, he can buy $\delta \cdot 500 = 0.4 \cdot 500 = 200$ shares, making the delta position on option -200, on stock +200 and total hedged position 0.

We can derive the formula for a down-and-out call delta using put-call parity

$$\begin{aligned}
\Delta^{down-and-out} &= \Delta^{vanilla} - \Delta^{down-and-in} \\
&= N(d_1) - \frac{B^{\frac{2r}{\sigma^2}-1}}{S} \left(- \left(\frac{B^2}{S^2}\right) \right. \\
&\quad \left. \cdot N\left(d_1\left(\frac{B^2}{S}, T-t\right)\right) - \frac{2r - \sigma^2}{\sigma^2 S} C\left(\frac{B^2}{S}, T-t\right) \right)
\end{aligned}$$

Using the same input parameters as in Example 5.1.2 $S = 95$, $K = 100$, $\sigma = 0.3$, $r = 0.1$, $T = 0.2$, $B = 85$, $Dt = 1e - 3$,

Figure 6.2: Hedging a down-and-out call



$N = T/Dt$, $M = 1e4$, we get for the delta of down-and-out call 0.4591 and the price of the option is 3.6772. In the figure 6.2 below we can see the dynamic hedging of a down-and-out call in action.

2. *Static hedging*

Static hedging involves constructing a portfolio containing the underlying call option (or put) and other options (calls and/or puts) with different expiry dates, same strikes and with fixed weights. The portfolio needs to replicate the value of the option for any underlying stock price for a wide period of time before maturity, without any need of rebalancing. When portfolio is constructed at time 0, the trading cannot happen anymore, only at maturity or when the underlying hits the barrier. As we already know the down-and-in call option is worthless unless the asset crosses some predefined barrier, in that case the payoff becomes that of a European call $\max(S(T) - K, 0)$.

Similarly, the down-an-out call option is wothless if the barrier hits before expiration. If not, the payoff is equal to the payoff of a vanilla call option with the same strike price.

The different strategies of static hedging for down-and-in and

down-and-out options are discussed in many books and articles. The approach which is presented here was discussed in the article written by R.G.Tompkins [17]. Put-call symmetry property was first discussed by P.Carr [19] and implemented by B.Thomas [20]. For down-and-in call with the strike price 100€, barrier 90€ and 180 days to expiration to hedge, should be purchased the amount of 1.1 out-of-the-money put with the strike price 81€. The cost of the put was met by the sale of the down-and-in call. In the approach there was used the zero interest rates assumption. The idea is to simulate the underlying asset and if barrier is crossed to sell the put and to buy a 100€ call. If barrier is not crossed, nothing changes, no actions.

For the down-and-out call is used the parity relationship between European call and European barrier options that was mentioned in Chapter 4

down-and-out = European Call - down-and-in

with the same strike price and barrier. As a result for hedging down-and-out call, we need to purchase 100€ European call and to sell 1.1 amount of 81€ put. When the barrier was crossed, 100€ call is sold and 81€ put is purchased. From the put-call symmetry, these two options should have the same value and the expected hedge cost (at any time) should be zero.

In his paper R.G.Tompkins [17] compared dynamic hedging versus static hedging. He noticed that overall the dynamic hedging approach is suitable for out-options, where the underlying option is out-of-the-money when the barrier is breached. For the out-options, where the underlying option is in-the-money when the barrier is breached is very dangerous. For in-options, where the underlying option is out-of-the-money dynamic hedging is more problematic because of the discontinuity at the barrier level.

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Some Matlab codes

1. Figure 2.1.

```
%  
%Plot discrete sample paths  
%Using the cumulative product function "cumprod",  
%to produce an array of asset paths and "randn"  
%to generate pseudo-random numbers.  
  
randn('state',100);  
clf  
  
%%%Problem parameters%%%  
S=1;mu =0.05;sigma=0.5;L=1e2;T=1;dt=T/L;M=50;  
  
tvals = [0:dt:T];  
Svals = S*cumprod(exp((mu-0.5*sigma^2)*dt+sigma*sqrt(dt)...  
*randn(M,L)),2);  
Svals = [S*ones(M,1) Svals];  
%add initial asset price  
plot(tvals,Svals)  
title('50 asset paths')  
xlabel('t'),ylabel('S(t)')
```

2. Illustrates computing Black-Scholes values for a European call, European call delta, European put, European put delta

```
%BSC-function  
%delta, put and put delta  
%Input arguments:  
%S=asset price at time t  
%K=exercise price  
%r=interest rate  
%sigma=volatility  
%tau=time to expiry (T-t)  
%
```

```

%Output arguments: C=call value,Cdelta=delta value of call
%                  P=Put Value, Pdelta=delta value of put
%
%
function [C, Cdelta, P, Pdelta] = BSC(S,K,r,sigma,tau)
if tau > 0
    d1 = (log(S/K)+(r+0.5*sigma^2)*(tau))/(sigma*sqrt(tau));
    d2=d1-sigma*sqrt(tau);
    N1=0.5*(1+erf(d1/sqrt(2)));
    N2=0.5*(1+erf(d2/sqrt(2)));
    C=S*N1-K*exp(-r*(tau))*N2
    Cdelta=N1
    P=C+K*exp(-r*(tau))-S
    Pdelta=Cdelta-1
else
    C=max(S-K,0);
    Cdelta=0.5*(sign(S-K)+1);
    P=max(K-S,0);
    Pdelta=Cdelta-1;
end

randn('state',100)
S=10; K=9; sigma=0.1; r=0.06; T=1;
Dt=1e-3; N=T/Dt; M=2^17;h=10^(-4);tau=1;
BSC(S,K,r,sigma,tau)

```

3. Illustrates Monte Carlo simulation for a European Call option

```

%Monte Carlo for a European call

randn('state', 100)

%Problem and method parameters
S=10; K=9; sigma=0.1; r=0.06; T=1;
Dt=1e-3; N=T/Dt; M=2^17;

V = zeros(M,1);
for i=1:M
    Sfinal = S*exp((r-0.5*sigma^2)*T+sigma*sqrt(T)*randn);
    V(i)=exp(-r*T)*max(Sfinal-K,0);
end
aM=mean(V); bM=std(V);
conf=[aM-1.96*bM/sqrt(M),aM+1.96*bM/sqrt(M)]

```

4. Illustrates calculation of the delta for a down-and-out call barrier option:

```

%delta for down-and-out
%
function [Cdao, Cdao_delta]= downAndOutV(S,E,r,sigma,B,T)
    %tau > 0
    tau=T;
    power1 = -1+(2*r)/(sigma^2);
    power2 = 1+(2*r)/(sigma^2);
    d1 = (log(S/E)+(r+0.5*sigma^2)*(tau))/(sigma*sqrt(T));
    d2 = d1-sigma*sqrt(T);
    d3 = (log(S/B)+(r+0.5*sigma^2)*(T))/(sigma*sqrt(T));
    d4 = (log(S/B)+(r-0.5*sigma^2)*(T))/(sigma*sqrt(T));
    d5 = (log(S/B)-(r-0.5*sigma^2)*(T))/(sigma*sqrt(T));
    d6 = (log(S/B)-(r+0.5*sigma^2)*(T))/(sigma*sqrt(T));
    d7 = (log(E*S/(B^2))-(r-0.5*sigma^2)...
    *(tau))/(sigma*sqrt(tau));
    d8 = (log(E*S/(B^2))-(r+0.5*sigma^2)...
    *(tau))/(sigma*sqrt(tau));
    d9 = (log(B^2/(S*E))+(r+0.5*sigma^2)...
    *(tau))/(sigma*sqrt(tau));
    Nd1 = 0.5*(1+erf(d1/sqrt(2)));
    Nd2 = 0.5*(1+erf(d2/sqrt(2)));
    Nd3 = 0.5*(1+erf(d3/sqrt(2)));
    Nd4 = 0.5*(1+erf(d4/sqrt(2)));
    Nd5 = 0.5*(1+erf(d5/sqrt(2)));
    Nd6 = 0.5*(1+erf(d6/sqrt(2)));
    Nd7 = 0.5*(1+erf(d7/sqrt(2)));
    Nd8 = 0.5*(1+erf(d8/sqrt(2)));
    Nd9 = 0.5*(1+erf(d9/sqrt(2)));
    a = (B/S)^power1;
    b = (B/S)^power2;
    if (E > B)
        Cdao = S*(Nd1-b*(1-Nd8))-E*exp(-r*T)*(Nd2-a*(1-Nd7));
        Cdao_delta = Nd1 - ((B/S)^(2*r/sigma^2 - 1))...
        *(-(B^2/S^2)*Nd9 - ((2*r-sigma^2)/(sigma^2*S))...
        *(S*(b*(1-Nd8))-E*exp(-r*T)*(a*(1-Nd7))));
    else
        Cdao = S*(Nd3-b*(1-Nd6))-E*exp(-r*T)*(Nd4-a*(1-Nd5));
        Cdao_delta = Nd1 - ((B/S)^(2*r/sigma^2 - 1))...
        *(-(B^2/S^2)*Nd9 - ((2*r-sigma^2)/(sigma^2*S))...
        *(S*(b*(1-Nd6))-E*exp(-r*T)*(a*(1-Nd5))));
    end

```



```

    end
end

```

5. Illustrates delta hedging by computing an approximate replicating portfolio for a down-and-out barrier call:

```

%
%Illustrates delta hedging by computing an approximate
%replicating potfolio for a Barrier call
%
%Portfolio is "asset2 units of asset and an amount "cash"
%of cash
%Plot actual and theoretical portfolio values

randn('state',100)
clf

%Problem parameters
Szero=95; E=100; sigma=0.3; r=0.1; T=0.2; B=85;
Dt=1e-3; N=T/Dt; M=1e4; t = [0:Dt:T];

[Cdao, Cdao_delta]= downAndOutV(Szero,E,r,sigma,B,T);

S = zeros(N+1,1);
asset = zeros(N+1,1);
cash = zeros(N+1,1);
portfolio = zeros(N+1,1);
Value = zeros(N+1,1);

S(1)=Szero;
asset(1)=Cdao_delta;
Value(1)=Cdao;
cash(1)=1;
portfolio(1)=asset(1)*S(1)+cash(1);

for i=1:N
    S(i+1) = S(i)*cumprod(exp((r-0.5*sigma.^2)*Dt...
    +sigma*sqrt(Dt)*randn));
    portfolio(i+1)=asset(i)*S(i+1)+cash(i)*(1+r*Dt);
    [Cdao, Cdao_delta]= downAndOutV(S(i+1),E,r,sigma,B,T-t(i+1));

```

```

    asset(i+1)=Cdao_delta;
    cash(i+1)=cash(i)*(1+r*Dt)-S(i+1)*(asset(i+1)-asset(i));
    Value(i+1)=Cdao;
end

Vplot=Value-(Value(1)-portfolio(1))*exp(r*t)';
plot(t(1:5:end),Vplot(1:5:end),'bo')
hold on
plot(t(1:5:end),portfolio(1:5:end),'r-','LineWidth',2)
xlabel('Time'),ylabel('Portfolio')
legend('Theoretical Value','Actual value')
grid on

```

Abbreviations

1. **i.i.d.** is a shortcut of *independent and identically distributed*, which means that some random variables X_i , $i = 1, 2, \dots$
 - in the discrete time have the same values x_1, x_2, \dots and probabilities p_1, p_2, \dots or in the continuous time X_i have the same density function $f(x)$, and
 - knowing the values of any subset of X_i does not tell us anything about the values of the remaining X_i
 - particularly, if X_1, X_2, \dots are i.i.d., then they are pairwise independent and hence $\mathbb{E}(X_i X_j) = \mathbb{E}(X_i)\mathbb{E}(X_j)$, for $i \neq j$.
2. **i.e.** is an abbreviation of *id est*, which comes from Latin language and means *that is*
3. **e.g.** is also a Latin abbreviation of the expression *exempli gratia* and means *for example*.
4. **etc.** stands for *et cetera*, which also comes from Latin language and means *so on and so forth*.
5. **w.r.t.** is an abbreviation for *with respect to*