A Note on Gentzen's Ordinal Assignment*

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Abstract

Gentzen's height measure of the 1938 consistency proof is a cumulative complexity measure for sequents that is measured bottom-up in a derivation. By a factorisation of the ordinal assignment a top-down ordinal assignment can be given that does not depend on information occurring below the sequent to which the ordinal is assigned. Furthermore, an ordinal collapsing function is defined in order to collapse the top-down ordinal to the one assigned by Gentzen's own ordinal assignment. A direct definition of the factorised assignment follows as a corollary.

This extraction of an ordinal collapsing function hopes to provide a formal or conceptual clarification of Gentzen's ordinal assignment and its height-line argument.

Keywords: Relative consistency proof (03F25), Normalization (03F05), Ordinal notations (03F15)

1 Introduction: The height-line argument

In this paper it is assumed that the reader is familiar with the basics of Gentzen's 1938 consistency proof including an understanding of Gentzen's assignment of ordinals to derivations as defined in (Takeuti, 1987, Definition 12.6). I refer to (Takeuti, 1987, Ch. 2) for further reading on the topic.

The construction of an adequate ordinal assignment is complicated by structural properties of formal derivations. A duplication of subderivations may be necessary for the reduction of a cluster of rules. This transformation may increase the absolute number of sequents and inferences in the derivation. Therefore, a measure of complexity for derivations must be more complex than an absolute count of the number of sequents. Gentzen's solution was a construction known as the height-line argument. The argument gives a method for rearranging the structure of derivations based on a bottom-up cumulative complexity measure, i.e. height, of sequents. The height measure in turn affects the

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ordinal assignment. This shows a clear order of dependency: The ordinal of a derivation depends on the height of sequents, the height measure depends on the given derivation. The fact that derivations are defined top-down and the height is measured bottom-up leads to the requirement that for the determining of an ordinal of a subderivation information about the subderivation itself, but also about what occurs below the derivation, has to be given.

The notion of a height-line is mentioned in (Gentzen, 1969, p. 271). Lines can be drawn in an abstract proof-tree to visualise a reorganisation of the proof-tree. The height-lines introduce levels in the derivation that correspond to levels of exponentiation in the ordinal. The levels that the height-lines introduce create a new possibility for reductions of derivations. If the height-lines are permuted up in the derivation while the structure of the derivation otherwise remains unchanged, then the ordinal measure of two copies of the new derivation can be less than one copy of the old derivation. This property of redistribution of height-lines is the crucial solution known as the height-line argument.

In this paper the height-line argument is separated from the ordinal assignment by a factorisation of Gentzen's ordinal assignment. This leads to a top-down ordinal assignment that does not depend on information occurring below the sequent to which the ordinal is assigned. Furthermore, an ordinal collapsing function is defined in order to collapse the top-down ordinal to the one assigned by Gentzen's own ordinal assignment (Theorem 4.2). A direct definition for this factorised ordinal assignment follows as a corollary (Corollary 5.2).

2 Gentzen's Ordinal Assignment

The ordinals that are assigned have an upper bound $\varepsilon_0 = \lim_{i\to\infty}\omega_i(0)$, where $\omega_0(\alpha) = \alpha$ and $\omega_{n+1}(\alpha) = \omega^{\omega_n(\alpha)}$ for some $n \in \mathbb{N}$ and α an ordinal.

The following complexity measures are defined as in (Takeuti, 1987, Definition 12.4).

Definition 2.1. 1. The *grade of a formula*, denoted $g(A)$, is the number of logical symbols in the formula A.

2. The *grade of a cut* inference $\mathcal{I}$ [resp. induction] is the grade of the cut formula [resp. induction formula], denoted $g(\mathcal{I})$.

3. The *height of a sequent* in a derivation, denoted $h(S, \Pi)$, is the maximum grade of the cuts and inductions below the sequent S in the derivation Π.

Definition 2.2 (Gentzen's ordinal assignment)**.** 1. $\Pi^S :=$ the subderivation of Π ending with the sequent S.

2. $o(S, \Pi) :=$ the ordinal assigned to the sequent S in the derivation Π as defined in (Takeuti, 1987, Definition 12.6). Let $\mathcal{I}$ be the last inference of Π and $\Pi_1, \ldots, \Pi_r (r \leqslant 2)$ the immediate subderivations of Π.

$$o(S,\Pi) := \begin{cases} 1 & \text{if } r = 0. \\ o(S_1,\Pi) + 1 & \text{if } r = 1 \text{ and } \mathcal{I} \notin \mathit{Str} \cup \mathit{Ind}. \\ o(S_1,\Pi)\#o(S_2,\Pi) & \text{if } r = 2 \text{ and } \mathcal{I} \notin \mathit{Cut}. \\ o(S_1,\Pi) & \text{if } r = 1 \text{ and } \mathcal{I} \in \mathit{Str}. \\ \omega_{l \dot{-} k}(o(S_1,\Pi)\#o(S_2,\Pi)) & \text{if } \mathcal{I} \in \mathit{Cut} \text{ with } l := g(\mathcal{I}) \text{ and } k := h(S,\Pi). \\ \omega_{l \dot{-} k}(o(S_1,\Pi) \cdot \omega) & \text{if } \mathcal{I} \in \mathit{Ind} \text{ with } l := g(\mathcal{I}) \text{ and } k := h(S,\Pi). \end{cases}$$

Here Str (Cut, Ind, resp.) denotes the set of all weak structural inferences (cuts, induction inferences, resp.)

3. $o(\Pi)$:= the ordinal assigned to the derivation Π, which is the ordinal assigned to the endsequent of the derivation.

By the reduction strategy of Gentzen's proof for each given derivation of the empty sequent, Π, a reduced derivation, Π', with a lower ordinal is produced. The proof found in (Takeuti, 1987, Lemma 12.8) is omitted here.

Lemma 2.3. *If Π is a derivation of the empty sequent, then there is another derivation, Π', with the same conclusion, but a lower ordinal.*

$$o(\Pi) > o(\Pi')$$

3 An integrated ordinal collapse

As it turns out, the chain of dependencies of the central concepts in the consistency proof is not random. The height of a sequent is measured bottom-up. The ordinal of a sequent is measured top-down, but requires that the height of the sequent has been determined. Therefore, the process of assigning an ordinal to a sequent is both bottom-up and top-down. However, this also suggests that another kind of ordinal assignment could be defined.

Definition 3.1 (A top-down ordinal assignment)**.** An ordinal assignment is *top-down* if the ordinal assigned to the conclusion of an inference only depends on information extractable from the sequent, the rule and the ordinals assigned to its premises.

By Definition 3.1 Gentzen's ordinal assignment is not top-down. A main difference between Gentzen's ordinal assignment and a top-down ordinal assignment is that for the latter the ordinal of a subderivation is fixed regardless of how the derivation is extended. If a top-down ordinal assignment can be defined, then it can be given by induction on the length of the derivation. Thus, because both the ordinal assignment and the derivation are defined top-down they could be given simultaneously.

Because the bottom-up element of Gentzen's ordinal assignment are the height-lines the key to constructing a top-down ordinal assignment is a separation of this concept from the concept of ordinal assignment. A clue for how to achieve the separation is found when considering the utilisation of the height-lines for the ordinal assignment. The level of exponentiation of Gentzen's ordinal assignment depends on whether the inference is a cut or an induction as well as on the grade of the cut [resp. induction] formula. This information can be coded into the top-down ordinal assignment and is enough for factorising the ordinal assignment into a top-down ordinal assignment and an ordinal collapse.

Definition 3.2 (The top-down ordinal assignment)**.** Let $\mathcal{I}$ be the last inference of Π and $\Pi_1, \ldots, \Pi_r$ the immediate subderivations of Π.

$$\hat{o}(\Pi) := \begin{cases} 1 & \text{if } r = 0. \\ \hat{o}(\Pi_1) + 1 & \text{if } r = 1 \text{ and } \mathcal{I} \notin Str \cup Ind. \\ \hat{o}(\Pi_1) \# \hat{o}(\Pi_2) & \text{if } r = 2 \text{ and } \mathcal{I} \notin Cut. \\ \hat{o}(\Pi_1) & \text{if } r = 1 \text{ and } \mathcal{I} \in Str. \\ \omega_{2l}(\hat{o}(\Pi_1) \# \hat{o}(\Pi_2)) & \text{if } \mathcal{I} \in Cut \text{ with } l := g(\mathcal{I}). \\ \omega_{2l+1}(\hat{o}(\Pi_1) + 1) & \text{if } \mathcal{I} \in Ind \text{ with } l := g(\mathcal{I}). \end{cases}$$

4 A factorization of Gentzen's ordinal assignment

Definition 4.1 (The ordinal collapsing functions f_k)**.** The ordinal collapsing functions, $f_k : \varepsilon_0 \to \varepsilon_0$, are defined in the following way:

1. $f_k(\alpha) := \alpha$ if none of case 2.-4. applies;
2. $f_k(\alpha \# \beta) := f_k(\alpha) \# f_k(\beta)$ if $\alpha, \beta \neq 0$;
3. If $\alpha \notin \{\omega^\gamma : \gamma \in On\}$ and $l > 0$, then
$$f_k(\omega_{2l}(\alpha)) := \omega_{l \dot{-} k}(f_{max(l,k)}(\alpha));$$
4. If $\alpha_0 \neq 0$ then
$$f_k(\omega_{2l+1}(\alpha_0 + 1)) := \omega_{l \dot{-} k}(f_{max(l,k)}(\alpha_0) \cdot \omega)$$

Theorem 4.2 shows that the functions f_k, in fact, collapse the ordinal of the top-down assignment, Definition 3.2, to the ordinal of Gentzen's assignment, Definition 2.2.

Theorem 4.2. *Assume that a derivation* Π *is given. Then*

$$o(S, \Pi) = f_{h(S,\Pi)}(\hat{o}(\Pi^S))$$

Proof. The proof is by induction on the length of the derivation Π. Let S be a sequent in the derivation derived with the inference $\mathcal{I}$ with the immediate subderivations ending in the respective sequents $S_1, \ldots, S_r$.

1. If $r = 0$, then $o(S, \Pi) \overset{\text{def}}{=} 1 \overset{\text{def}}{=} f_{h(S,\Pi)}(1) \overset{\text{def}}{=} f_{h(S,\Pi)}(\hat{o}(\Pi^S))$.

2. If $r = 1$ and $\mathcal{I} \notin Str \cup Ind$, then $o(S,\Pi) \stackrel{\text{def}}{=} o(S_1,\Pi)+1 \stackrel{IH}{=} f_{h(S_1,\Pi)}(\hat{o}(\Pi^{S_1}))+1 \stackrel{\text{def}}{=} f_{h(S_1,\Pi)}(\hat{o}(\Pi^{S_1})+1) \stackrel{\text{def}}{=} f_{h(S_1,\Pi)}(\hat{o}(\Pi)) = f_{h(S,\Pi)}(\hat{o}(\Pi))$ because $h(S_1,\Pi) = h(S,\Pi)$.

3. If $r = 2$ and $\mathcal{I} \notin Cut$, then $o(S,\Pi) \stackrel{\text{def}}{=} o(S_1,\Pi)\#o(S_2,\Pi) \stackrel{IH}{=} f_{h(S_1,\Pi)}(\hat{o}(\Pi^{S_1}))\#f_{h(S_2,\Pi)}(\hat{o}(\Pi^{S_2})) \stackrel{\text{def}}{=} f_{h(S,\Pi)}(\hat{o}(\Pi^{S_1})\#\hat{o}(\Pi^{S_2})) \stackrel{\text{def}}{=} f_{h(S,\Pi)}(\hat{o}(\Pi))$ because $h(S_i,\Pi) = h(S,\Pi)$.

4. If $r = 1$ and $\mathcal{I} \in Str$, then $o(S,\Pi) \stackrel{\text{def}}{=} o(S_1,\Pi) \stackrel{IH}{=} f_{h(S_1,\Pi)}(\hat{o}(\Pi^{S_1})) \stackrel{\text{def}}{=} f_{h(S,\Pi)}(\hat{o}(\Pi^{S}))$.

5. If $\mathcal{I} \in Cut$ let $l := g(\mathcal{I})$, $k := h(S,\Pi)$, $n := l \dot{-} k$ and $m := max(l,k)$. Then using (†) $m = h(S_i,\Pi)$ we prove $o(S,\Pi) \stackrel{\text{def}}{=} \omega_n(o(S_1,\Pi)\#o(S_2,\Pi)) \stackrel{IH+(\dagger)}{=} \omega_n(f_m(\hat{o}(\Pi^{S_1}))\#f_m(\hat{o}(\Pi^{S_2}))) \stackrel{\text{def}}{=} \omega_n(f_m(\hat{o}(\Pi^{S_1})\#\hat{o}(\Pi^{S_2}))) \stackrel{\text{def}}{=} f_k(\omega_{2l}(\hat{o}(\Pi^{S_1})\#\hat{o}(\Pi^{S_2}))) \stackrel{\text{def}}{=} f_{h(S,\Pi)}(\hat{o}(\Pi^{S}))$.

6. If $\mathcal{I} \in Ind$ let l,k,n and m be defined as in the previous case. Then $o(S,\Pi) \stackrel{\text{def}}{=} \omega_n(o(S_1,\Pi)\cdot\omega) \stackrel{IH+(\dagger)}{=} \omega_n(f_m(\hat{o}(\Pi^{S_1}))\cdot\omega) \stackrel{\text{def}}{=} f_k(\omega_{2l+1}(\hat{o}(\Pi^{S_1}+1)) \stackrel{\text{def}}{=} f_{h(S,\Pi)}(\hat{o}(\Pi^{S}))$.

□

5 An alternative definition of Gentzen's ordinal assignment

In (Mints, 1992, Definition 1, p. 80) a top-down ordinal assignment (depending on a parameter $k \in \mathbb{N}$) $O_k(d)$ is given. Adapting this to the present context leads to the following assignment $o_k(\Pi)$.

Definition 5.1. Let $\mathcal{I}$ be the last inference of Π and $\Pi_1,\ldots,\Pi_r$ the immediate subderivations of Π. Then

$$o_k(\Pi) := \begin{cases} \omega_{l\dot{-}k}(o_{max(l,k)}(\Pi_1)\#o_{max(l,k)}(\Pi_2)) & \text{if } \mathcal{I} \in Cut \text{ and } l := g(\mathcal{I}). \\ \omega_{l\dot{-}k}(o_{max(l,k)}(\Pi_1)\cdot\omega) & \text{if } \mathcal{I} \in Ind \text{ and } l := g(\mathcal{I}). \\ \text{analogous to Definition 3.2} & \text{in all other cases.} \end{cases}$$

With a proof analogous to that of Theorem 4.2 it follows that:

Corollary 5.2.

$$o(S,\Pi) = o_{h(S,\Pi)}(\Pi^S)$$

Moreover, one easily can see that $o_k = f_k \circ \hat{o}$.

Note 5.3. The author is indebted to an anonymous referee for several improvements of this paper. The improvements include, but are not limited to, a concise formulation of the proof of Theorem 4.2 and the direct definition of o_k in Section 5.

References

G. Gentzen, *Neue Fassung des Widerspruchsfreiheitsbeweises für die reine Zahlentheorie*, ***Forschungen zur Logik und zur Grundlegung der exakten Wissenschaften***, vol. 4 (1938), pp. 19–44.

G. Mints, ***Selected Papers in Proof Theory***, Bibliopolis, Napoli, 1992.

M. E. Szabo, editor. ***The Collected Papers of Gerhard Gentzen***, North-Holland, Amsterdam, 1969.

G. Takeuti, ***Proof Theory: Second Edition***, Dover Books on Mathematics, Dover Publications, 2013. First ed. 1987.