

Probing Gravitational Degrees of Freedom through Cosmic Inflation

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ACADEMIC DISSERTATION

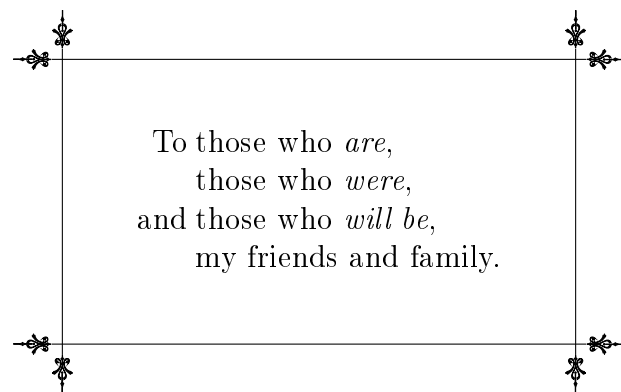
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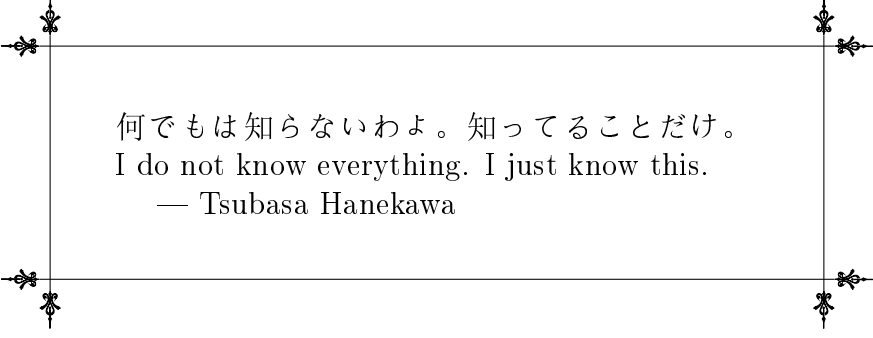
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To those who *are*,
those who *were*,
and those who *will be*,
my friends and family.



何でもは知らないわよ。知ってることだけ。
I do not know everything. I just know this.
— Tsubasa Hanekawa

Abstract

Among all models of inflation, Higgs inflation stands out in its minimalistic approach. In Higgs inflation, the Standard Model Higgs boson drives the expansion of the spacetime. The properties of the Higgs boson are known from collider experiments, and the only new ingredient is a non-minimal coupling of the Higgs boson to gravity. There is no need to add any new particles, and the non-minimal coupling is the only free parameter of the model.

While the predictions of Higgs inflation agree with observations at classical level, loop corrections to the Higgs self-potential and gravitational action complicate the picture. From the renormalisation group equations of the Standard Model it is known that the Higgs self-coupling decreases when the energy scale increases. Significant running at the scale of inflation can foil the flat plateau of tree-level inflation. It is also known that loop corrections to gravity will destabilise pure Higgs inflation.

There is also another fundamental source of uncertainty: the gravitational degrees of freedom. In Higgs inflation, the spacetime metric is usually taken to be the only gravitational degree of freedom, but this need not be the case. In the Palatini formulation of General Relativity both the metric and the connection are independent degrees of freedom. In the case of Higgs inflation, these two approaches lead to physically inequivalent theories.

This thesis focuses on the differences of Higgs inflation in metric and the Palatini formulation. First we show that the metric perturbations must be quantised, if the Higgs boson is the inflaton. Then we consider loop corrections to the Higgs self-coupling, and find that the tensor-to-scalar ratio is smaller in the Palatini formulation. We also consider dimension four correction terms in the gravitational action and find a similar effect on the tensor-to-scalar ratio.

There is no clear theoretical indication of how to choose the gravitational degrees of freedom. Hence it is important to be able to differentiate between different choices by observations. We find that the metric and Palatini formulation of General Relativity have distinct cosmological signatures, which can be tested with next generation experiments. If a non-zero tensor-to-scalar ratio is detected, we can rule out Higgs inflation in the Palatini formulation.

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April 2019 in Helsinki
Lumi-Pyry Wahlman



List of Publications

This thesis is based on the following publications [1–4]

- I** *Inflation without quantum gravity*
T. Markkanen, S. Räsänen, and P. Wahlman
Phys. Rev. D **91** (8 2015), 084064.
- II** *Higgs inflation with loop corrections in the Palatini formulation*
S. Räsänen and P. Wahlman
JCAP **1711.11** (2017), 047.
- III** *Inflation with R^2 term in the Palatini formulation*
V.-M. Enckell, K. Enqvist, S. Räsänen, and L.-P. Wahlman
JCAP **1902.02** (2019), 022.
- IV** *Higgs- R^2 inflation – full slow-roll study at tree-level*
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Submitted to JCAP.

The Contribution of the Author

Article I: The author made the analytical calculations together with Syksy Räsänen, and contributed to the article by supplementing the draft and creating table 1.

Article II: The author made the analytical calculations together with Syksy Räsänen, wrote the code used in the numerical analysis, created all figures, and wrote the section on numerical results in the article.

Article III: The author made the analytical calculations together with Syksy Räsänen and wrote the first draft of the article, after which it was jointly worked on.

Article IV: The author made the analytical calculations and numerical analysis in co-operation with Vera-Maria Enckell, and contributed to the writing of the article. The article is also featured in the PhD thesis of Vera-Maria Enckell.

Units and Conventions

We use the Einstein summation convention for repeated indices in tensors. We also use commas to denote partial derivatives

$$g_{\mu\nu,\kappa} \equiv \partial_\kappa g_{\mu\nu} \quad (1)$$

and semicolons to denote covariant derivatives

$$R_{\mu\nu;\kappa} \equiv \nabla_\kappa R_{\mu\nu} , \quad (2)$$

where \equiv signifies a *definition*. We use Greek alphabet for spacetime indices running from zero to three, lowercase Latin alphabet for spacetime indices running from one to three, and uppercase Latin alphabet for field space indices. For indices to be summed over we pick characters from the beginning of the alphabet (e.g. α, β, \dots), while the characters denoting different components are chosen from the middle of the alphabet (e.g. μ, ν, \dots).

We use the $(+++)$ convention for quantities of General Relativity and denote the Minkowski metric with

$$\eta_{\mu\nu} \equiv \text{diag}(-1, +1, +1, +1) . \quad (3)$$

We use natural units, in which the speed of light and reduced Planck constant are equal to one

$$c = \hbar = 1 \quad (4)$$

We also often set the reduced Planck mass to unity

$$M_{\text{Pl}} = \sqrt{\frac{\hbar c}{8\pi G}} = 1 . \quad (5)$$

We use a bar over a quantity to denote background quantities e.g. $\bar{\varphi}$ is the background field. In the case of $F(R)$ inflation in Chapter 6 we also denote quantities of the usual Higgs inflation with a bar.

We denote quantities calculated in the Einstein frame with a tilde e.g. $\tilde{g}_{\mu\nu}$ is the Einstein frame metric, while a hat is used to denote operators e.g. $\hat{\phi}$ is the field operator.

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1 Introduction

現実は大体一番詰まらない答えが真実だ。

In reality, the most boring answer is usually the right one.

— Tet, the God of Games



Inflation can be described as an accelerating expansion of the early universe. Such a period explains cosmological observations of flatness, homogeneity, and isotropy of the universe at large scales, while also solving a multitude of other problems like the origin of perturbations, the horizon problem, and the relic problem [5]. However, inflation is not a precise model of the evolution of space-time – it is a concept. Inflation can be realised in many ways, and it can be estimated that there are currently over a hundred different models of inflation [6].

While there are many models of inflation, one of them, the Higgs inflation, stands out in its minimalistic approach. In Higgs inflation the expansion of spacetime is driven by the Standard Model Higgs boson [7]. This model requires no new particles, and the properties of the Higgs boson are known from the measurements conducted at the Large Hadron Collider (LHC). The only new ingredient is a non-minimal coupling of the Higgs boson to gravity ξ , which is also the only free parameter in the model.

At classical level, Higgs inflation is a success: it predicts a spectral index $n_s = 0.96$ and a tensor-to-scalar ratio $r = 5 \times 10^{-3}$, which agree well with the current observations of the Cosmic Microwave Background (CMB) measured by the Planck satellite [8]. However, loop corrections to the Higgs potential and the gravitational action complicate the picture. From the renormalisation group equations of the Standard Model we know that the value of the Higgs self-coupling at high energy scales can drastically differ from the value measured at the LHC [9]. In particular, such a strong running can foil the flat plateau of tree-level Higgs inflation. In addition, loop corrections to gravity are also known to destabilise pure Higgs inflation [10].

There is, however, another fundamental source of uncertainty: the choice of the gravitational degrees of freedom. Higgs inflation is usually considered in the metric formulation of General Relativity, in which the metric is the only degree of freedom, but this need not necessarily be the case. One could instead take the metric and the connection as independent variables, which is the approach in the Palatini formulation [11]. For Higgs inflation, these two formulations yield

different physical theories: in the Palatini formulation the tensor-to-scalar ratio is suppressed compared to the result of the metric formulation [12].

This thesis focuses on probing the gravitational degrees of freedom through inflation, when Higgs is assumed to be the inflaton. In Chapter 2 we review the approaches to General Relativity in both formulations, and in Chapter 3 we outline the concepts of slow-roll inflation and Higgs inflation. Chapter 4 focuses on the quantum origin of perturbations and is based on [1]. In Chapter 5 we investigate the effects of loop corrections on the shape of the Higgs self-potential and review the numerical results of [2]. Chapter 6 is based on [3] and [4], and focuses on the gravitational loop corrections parameterised as a dimension four correction to the Einstein-Hilbert action. In Chapter 7 we summarise our findings and comment briefly on next generation experiments.



2 General Relativity

時間があつとやまに流れていく。今だけはEinsteinに文句を言いたい気分。
Time is flowing so fast. Right now I want to complain to Einstein.

— Kurisu Makise



Let us begin by reviewing some key concepts of General Relativity. We focus on the gravitational degrees of freedom, which differ between the metric and Palatini formulation. We also demonstrate the difference of these two approaches in the simple case of a non-minimally coupled scalar field.

2.1 Einstein-Hilbert Action

The theory of General Relativity can be constructed from the simple Einstein-Hilbert action [13]

$$S = \int dx \sqrt{-g} \frac{1}{2} M_{\text{Pl}}^2 g^{\alpha\beta} R_{\alpha\beta} , \quad (2.1)$$

where M_{Pl} is the reduced Planck mass (5), $g_{\mu\nu}$ is the spacetime metric, g its determinant $\det(g_{\mu\nu})$, and $R_{\mu\nu}$ is the Ricci tensor

$$R_{\mu\nu} \equiv R^{\alpha}{}_{\mu\alpha\nu} . \quad (2.2)$$

The Ricci tensor is a contraction of the Riemann tensor, which is defined in terms of the connection $\Gamma_{\mu\nu}^{\kappa}$

$$R^{\kappa}{}_{\lambda\mu\nu} \equiv \partial_{\mu}\Gamma_{\nu\lambda}^{\kappa} - \partial_{\nu}\Gamma_{\mu\lambda}^{\kappa} + \Gamma_{\alpha\mu}^{\kappa}\Gamma_{\lambda\nu}^{\alpha} - \Gamma_{\alpha\nu}^{\kappa}\Gamma_{\lambda\mu}^{\alpha} . \quad (2.3)$$

The spacetime geometry is described by specifying the metric and the connection. In the metric formulation of General Relativity the connection is *a priori* chosen to be the Levi-Civita connection [12, 13]

$$\bar{\Gamma}_{\mu\nu}^{\kappa} \equiv \frac{1}{2} g^{\kappa\alpha} (g_{\alpha\mu,\nu} + g_{\alpha\nu,\mu} - g_{\mu\nu,\alpha}) , \quad (2.4)$$

which is the unique symmetric ($\bar{\Gamma}_{\mu\nu}^{\kappa} = \bar{\Gamma}_{\nu\mu}^{\kappa}$) and metric compatible ($g_{\mu\nu;\kappa} = 0$) connection. The metric is then the only degree of freedom in the action (2.1).

To inspect the dynamics of the action (2.1) let us take the variation of it with respect to the metric, ignore the boundary terms¹, and set the resulting tensor equal to zero. This yields the vacuum Einstein equation [13]

$$G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 0 , \quad (2.5)$$

where we have defined the Einstein tensor $G_{\mu\nu}$ and R denotes the Ricci scalar

$$R \equiv g^{\alpha\beta}R_{\alpha\beta} . \quad (2.6)$$

The Einstein equation (2.5) is in fact a system of ten linearly independent non-linear second order partial differential equations with four constraint equations from the Bianchi identities, which means that solving it is in general difficult.

2.2 Palatini Formulation

While the metric formulation of General Relativity is the most widely known one, there are other equally possible formulations. We focus on the Palatini formulation, which was first proposed by Albert Einstein [11], but is often attributed to Attilio Palatini. In the Palatini formulation, both the metric and the connection are taken to be independent degrees of freedom. The variation with respect to the metric then yields the Einstein equation, while the variation with respect to the connection yields an equation of motion, from which the connection is to be solved. This is in contrast to the metric formulation where we assume the form (2.4) for the connection.

In the case of Einstein-Hilbert action (2.1) the variation with respect to the metric yields the same Einstein equation as in the metric formulation. However, the details differ slightly. Because the action does not contain explicit derivatives of the metric in the Palatini formulation there are no cumbersome boundary terms. Let us then consider the variation with respect to the connection. We assume a symmetric connection² for which the variation simplifies to the constraint [12]

$$g^{\mu\nu}{}_{;\kappa} = 0 . \quad (2.7)$$

¹A careful reader will notice that the Einstein-Hilbert action has a non-vanishing boundary term that can be cancelled by including the York-Gibbon-Hawkins-term in the action [14]. We do not explicitly write such boundary terms in our actions, but we implicitly assume them to be present, while taking variations in the metric formulation.

²Strictly speaking the connection need not be symmetric. However, a non-symmetric connection introduces torsion, which complicates the equations. For simplicity we consider only torsionless connections.

This is nothing else, but the constraint for a metric compatible connection. Hence, the Palatini formulation leads to the exactly same physical theory as the metric formulation in the case of the Einstein-Hilbert action (2.1).

While the two approaches are physically equivalent for the Einstein-Hilbert action, this is not the case in general. If the gravitational sector is more complicated, for example because of a non-minimal coupling of a matter field to gravity, the connection will no longer have the Levi-Civita form in the Palatini formulation. In particular, this is the case for Higgs inflation, in which the Higgs field is non-minimally coupled to gravity through an interaction term between the field and the Ricci scalar. We will demonstrate this in Section 3.3.

2.3 Einstein Frame

Thus far we have considered only an empty spacetime. Let us now add matter to the action in the form of a scalar field ϕ and couple it non-minimally to gravity. The action then reads

$$S = \int dx \sqrt{-g} \left[\frac{M_{\text{Pl}}^2 + \xi\phi^2}{2} g^{\alpha\beta} R_{\alpha\beta} - \frac{1}{2} g^{\alpha\beta} \phi_{,\alpha} \phi_{,\beta} - V(\phi) \right], \quad (2.8)$$

where the constant ξ is called the non-minimal coupling and $V(\phi)$ is the self-interaction potential of the scalar field. The action (2.8) is defined in the Jordan frame, where the field has a canonical kinetic term and the gravitational sector is non-trivial.

By taking the variation of (2.8) with respect to the connection we find [12]

$$\nabla_\kappa [(M_{\text{Pl}}^2 + \xi\phi^2)g^{\mu\nu}] = 0. \quad (2.9)$$

As we mentioned in the previous section, Levi-Civita connection is no longer a solution to this equation, and therefore the metric and Palatini formulation are no longer physically equivalent. However, we can write the connection in terms of a Levi-Civita connection of a scaled metric

$$\tilde{g}_{\mu\nu} \equiv \Omega^2 g_{\mu\nu} = (M_{\text{Pl}} + \xi\phi^2) g_{\mu\nu}. \quad (2.10)$$

In fact, we can express the action (2.8) in terms of the new metric $\tilde{g}_{\mu\nu}$. Such a transformation is called a conformal transformation, and the resulting action reads [12, 13, 15, 16]

$$\tilde{S} = \int dx \sqrt{-\tilde{g}} \left[\frac{1}{2} \tilde{g}^{\alpha\beta} \tilde{R}_{\alpha\beta} - \frac{1}{2} \frac{1 + \alpha\xi\phi^2}{(1 + \xi\phi^2)^2} \tilde{g}^{\alpha\beta} \phi_{,\alpha} \phi_{,\beta} - U(\phi) \right], \quad (2.11)$$

where we have now set the reduced Planck mass to unity and defined the Einstein frame potential $U(\phi) \equiv \Omega^{-4}V(\phi)$. The details of the transformation differ between the two formulations and are encoded in the coefficient α , which is one in the Palatini formulation and $1 + 6\xi$ in the metric formulation.

The action (2.11) is now represented in the Einstein frame, where the gravitational sector is of the Einstein-Hilbert form (2.1), but the kinetic term of the field is non-trivial. The kinetic term of the scalar field can be made canonical by transforming the field according to the differential equation

$$\frac{d\phi}{d\chi} = \sqrt{\frac{(1 + \xi\phi^2)^2}{1 + \alpha\xi\phi^2}}, \quad (2.12)$$

where χ is the Einstein frame field. The action then becomes

$$\tilde{S} = \int dx \sqrt{-\tilde{g}} \left[\frac{1}{2} \tilde{g}^{\alpha\beta} \tilde{R}_{\alpha\beta} - \frac{1}{2} \tilde{g}^{\alpha\beta} \chi_{,\alpha} \chi_{,\beta} - U(\chi) \right]. \quad (2.13)$$

The difference between the two approaches and the effects of the non-minimal coupling ξ are now neatly encoded in the Einstein frame potential U and the field transformation (2.12), while the Einstein equation has the standard form of a minimally coupled scalar field with a canonical kinetic term.



Summary. In this chapter we have reviewed the action based approach to General Relativity. We also introduced the Palatini formulation, in which the metric and connection are independent degrees of freedom. Finally, we considered the difference between the metric and Palatini formulation, when a scalar field is non-minimally coupled to gravity. In particular, this is the situation in Higgs inflation, which we consider in Section 3.3.



3 Inflation

宇宙には始まりはあるが、終わりはない。無限。
Space has a beginning but not an end. – Infinite.
— Rintarou Okabe



Inflation can be described as a period of accelerating expansion of space. Such a phase explains a multitude of cosmological observations, including the origin of observed CMB perturbations. We are especially interested in Higgs inflation and review the classical level predictions of it in Section 3.3.

3.1 Background and Perturbations

If we assume that inflation has lasted long enough, the universe will be isotropic, homogeneous, and flat [5, 17–20]. It can then be described by the flat Friedman-Lemaître-Robertson-Walker (FLRW) metric

$$ds^2 = -dt^2 + a(t)^2 \delta_{ab} dx^a dx^b, \quad (3.1)$$

where $a(t)$ is the scale factor, which grows exponentially during inflation. For convenience, it is useful to define the conformal time η that satisfies

$$\frac{d\eta}{dt} \equiv \frac{1}{a(t)}. \quad (3.2)$$

In terms of the conformal time the metric reads

$$ds^2 = a(\eta)^2 [-d\eta^2 + \delta_{ab} dx^a dx^b] = a^2 \eta_{\alpha\beta} dx^\alpha dx^\beta. \quad (3.3)$$

We denote derivatives with respect to the coordinate time with a dot (e.g. $\partial_t a = \dot{a}$), while the derivatives with respect to conformal time are denoted with a prime (e.g. $\partial_\eta a = a'$), unless otherwise stated.

Let us now consider a minimally coupled scalar field ϕ in a FLRW background. The action then reads

$$S = \int d^4x \left[\frac{1}{2} R - g^{\alpha\beta} \phi_{,\alpha} \phi_{,\beta} - V(\phi) \right]. \quad (3.4)$$

We also divide the field into an isotropic and homogeneous background φ and a small perturbation $\delta\phi$

$$\phi = \varphi + \delta\phi . \quad (3.5)$$

Let us first consider the evolution of the background field. The Einstein equations are to zeroth order in perturbations [21]

$$3H^2 = \frac{1}{2}\dot{\varphi}^2 + V(\varphi) \quad (3.6)$$

$$2\dot{H} + 3H^2 = -\frac{1}{2}\dot{\varphi}^2 + V(\varphi) , \quad (3.7)$$

where H is the Hubble parameter

$$H \equiv \frac{\dot{a}}{a} \quad (3.8)$$

and a dot denotes derivative with respect to the coordinate time t . By taking the difference of the two equations we obtain the momentum equation

$$2\dot{H} = -\dot{\varphi}^2 , \quad (3.9)$$

which allows us to write the background equations in terms of the Hubble slow-roll parameters ϵ_H , η_H , and ζ_H given in Appendix A.

The equation of motion for the background field is [21]

$$V'(\varphi) = -\ddot{\varphi} - 3H\dot{\varphi} = -(3 + \epsilon_H - \eta_H)H\dot{\varphi} , \quad (3.10)$$

where a prime denotes a derivative of the potential with respect to the background field φ . Differentiating with respect to the coordinate time we find

$$V''(\varphi) = (3\eta_H + \zeta_H)H^2 , \quad (3.11)$$

which yields the effective mass of perturbations.

Let us then consider perturbations. In addition to the field perturbation $\delta\phi$, we also need to consider perturbations of the FLRW metric

$$ds^2 = a^2 [\eta_{\alpha\beta} + h_{\alpha\beta}] dx^\alpha dx^\beta , \quad (3.12)$$

where $\eta_{\mu\nu}$ is the Minkowski metric and

$$h_{\mu\nu} = \begin{pmatrix} -2A & B_{,i} \\ B_{,i} & -2\psi\delta_{ij} + 2E_{,ij} \end{pmatrix} \quad (3.13)$$

is a small scalar perturbation. The scalars A , B , E , and ψ are functions of both time and spatial coordinates. There are four degrees of freedom in the perturbed metric, but the perturbed Einstein equations determine only two of these. The remaining two degrees of freedom can be chosen arbitrarily, which introduces a gauge redundancy. The details of details of gauge transformations are covered in Appendix B.

Eventually we want to quantise the perturbations. However, neither the field perturbation nor the metric perturbations alone satisfy the canonical commutation relations. In fact, the Sasaki-Mukhanov variable

$$v = \delta\phi + \frac{\dot{\phi}}{H}\psi \quad (3.14)$$

is the only combination of perturbations that satisfies the canonical commutation relations [21]. It is therefore this quantity, that we use when quantising the perturbations. For convenience, we use the spatially flat gauge, where $\psi = E = 0$ and the Sasaki-Mukhanov variable corresponds to the field perturbation. The equation of motion for the field perturbation can then be written to first order in the perturbations as [1]

$$\delta\phi''_{\vec{k}} + 2\mathcal{H}\delta\phi'_{\vec{k}} + [k^2 - 6\epsilon_H + 3\eta_H - 4\epsilon_H^2 + 4\epsilon_H\eta_H + \zeta_H] \delta\phi_{\vec{k}} = 0, \quad (3.15)$$

where we have made a Fourier transformation to momentum space, prime denotes a derivative with respect to the conformal time, and the conformal Hubble parameter is $\mathcal{H} \equiv a'/a$.

3.2 Slow-roll Inflation

Thus far we have written all equations in terms of the Hubble slow-roll parameters without any approximations. Let us now make the assumption, that the scalar field rolls slowly down its potential. We can then use the slow-roll approximation [22, 23]

$$|\ddot{\phi}| \ll |\dot{\phi}H| \quad \text{and} \quad \dot{\phi}^2 \ll V(\phi). \quad (3.16)$$

Under the slow-roll conditions the Einstein equations and the equation of motion for the field become approximately

$$3H^2 \approx V \quad (3.17)$$

$$\frac{V'}{V} \approx -\frac{\dot{\phi}}{H}. \quad (3.18)$$

The Hubble slow-roll parameters are then related to the potential slow-roll parameters ϵ_V , η_V , and ζ_V , defined in Appendix A. In particular $\epsilon_H \approx \epsilon_V$ and $\eta_H \approx \eta_V$ in our notation. When the slow-roll conditions are satisfied, the potential slow-roll parameters are small

$$\epsilon_V \ll 1, \quad |\eta_V| \ll 1. \quad (3.19)$$

However, while the above conditions are necessary, they are not sufficient to satisfy the slow-roll assumptions (3.16).

During slow-roll the equation of motion for the field perturbation (3.15) simplifies to

$$\delta\phi_k'' + 2\mathcal{H}\delta\phi_k' + k^2\delta\phi_k = 0. \quad (3.20)$$

Let us now quantise the perturbations by replacing the field perturbation with the operator

$$\hat{\delta\phi} = \sum_k \left[\hat{a}_{\vec{k}} u_{\vec{k}} + \hat{a}_{\vec{k}}^\dagger u_{\vec{k}}^* \right], \quad (3.21)$$

where $\hat{a}_{\vec{k}}$ and $\hat{a}_{\vec{k}}^\dagger$ are the annihilation and creation operator respectively, and $u_{\vec{k}}$ are the corresponding mode functions. Assuming the Bunch-Davies vacuum, the correctly normalised mode functions are then [5, 21]

$$u_{\vec{k}} = \frac{1}{(2\pi)^{\frac{3}{2}}} \frac{H}{\sqrt{2k^3}} \left(i + \frac{k}{aH} \right) \exp\left(i \frac{k}{aH} \right). \quad (3.22)$$

Eventually we would like to compare the theoretical predictions of a model to observations. We cannot predict the amplitudes of the individual wave-modes observed in CMB, but we can extract the power spectrum of the perturbations from the observations [8]. We define the power spectrum of the field perturbations to be

$$\mathcal{P}_\phi \equiv \mathcal{P}_{\delta\phi} = (2\pi)^3 \frac{k^3}{2\pi^2} |u_{\vec{k}}|^2. \quad (3.23)$$

By substituting the modes (3.22) and taking the super-Hubble limit $k \ll aH$, we find the power spectrum to be

$$\mathcal{P}_\phi = \left(\frac{H}{2\pi} \right)^2. \quad (3.24)$$

However, we do not observe the power spectrum of the field directly. What we can extract from the CMB is the spectrum of the comoving curvature perturbation, defined in the spatially flat gauge as

$$\mathcal{R} \equiv -\frac{H}{\dot{\phi}} \delta\phi. \quad (3.25)$$

The spectrum of the comoving curvature perturbation can finally be expressed as

$$\mathcal{P}_{\mathcal{R}} = \left(\frac{\mathcal{H}}{\dot{\phi}}\right)^2 \mathcal{P}_{\phi} = \left(\frac{H^2}{2\pi\dot{\phi}}\right)^2 = \frac{1}{24\pi} \frac{V(\varphi)}{\epsilon_V} . \quad (3.26)$$

While we have focused on scalar perturbations, a similar treatment is possible for tensor perturbations as well. The resulting tensor spectrum can be expressed in terms of the field spectrum as

$$\mathcal{P}_{\tau} = 8\mathcal{P}_{\delta\phi} . \quad (3.27)$$

An important observable quantity is then the tensor-to-scalar ratio

$$r \equiv \frac{\mathcal{P}_{\tau}}{\mathcal{P}_{\mathcal{R}}} = 8 \left(\frac{\dot{\phi}}{H}\right)^2 = 16\epsilon_V , \quad (3.28)$$

which can be used to differentiate between the metric and Palatini formulation, as we will demonstrate in the next section.

The next ingredient we need is the time at which the perturbations were generated. To this end, we define the number of e-folds

$$N = - \int_{a_{\text{end}}}^a d \ln a = - \int_{\varphi_{\text{end}}}^{\varphi} d\varphi \frac{H}{\dot{\phi}} \approx \int_{\varphi_{\text{end}}}^{\varphi} d\phi \frac{V}{V'} , \quad (3.29)$$

where the last approximation is valid under the slow-roll conditions. By definition, the number of e-folds measures the expansion of space in logarithmic scale. The subscript end refers to end of inflation, which we define to be the point when either ϵ_V or $|\eta_V|$ becomes of the order of unity, whichever occurs first. A positive N then means that we are still in the slow-roll phase.

The observed CMB perturbations were generated around the time when the pivot scale $k_* = 0.05 \text{ Mpc}^{-1}$ exited the horizon. In terms of e-folds the pivot scale corresponds to [2, 22]

$$\begin{aligned} N_* &= 61 - \Delta N_{\text{reh}} + \frac{1}{4} \ln \frac{V_*}{V_{\text{end}}} + \frac{1}{4} \ln V_* \\ &\approx 52 + \frac{1}{4} \ln \frac{r_*}{0.079} , \end{aligned} \quad (3.30)$$

where ΔN_{reh} is the number of e-folds accumulated during preheating. On the second line we have taken $\Delta N_{\text{reh}} = 4$, which is the result for Higgs inflation in the metric formulation with the Standard Model particle content [24–26]. We have also neglected the change in the potential between the pivot scale and the end of inflation, written $V_* = \frac{3\pi^2}{2} A_s r_*$, and substituted the observed value

$24\pi^2 A_s = 5 \times 10^{-7}$. Finally, we have inserted the maximum value allowed by observations $r_* = 0.079$ as the point of comparison in the logarithm.

Now we have an expression for the spectrum of perturbations at the time the observed CMB perturbations were generated. However, it would be tedious to compare our theoretical predictions with the observations by fitting the theoretical spectrum to the observational data. Fortunately, the CMB spectrum is almost scale-invariant [8], which allows us to expand the power spectrum of the curvature perturbation as

$$\ln \mathcal{P}_{\mathcal{R}} = \ln A_s + (n_s - 1) \ln \frac{k}{k_*} + \frac{1}{2} \alpha_s \ln^2 \frac{k}{k_*} + \mathcal{O} \left(\ln^3 \frac{k}{k_*} \right), \quad (3.31)$$

where A_s is the amplitude of perturbations, n_s is the spectral index, and α_s is the running of the spectral index. We also denote $\ln^n x = (\ln x)^n$. In the slow-roll approximation we can express these quantities to leading order in the potential slow-roll parameters as [23]

$$A_s = \frac{1}{24\pi} \frac{V}{\epsilon_V} \quad (3.32)$$

$$n_s = 1 - 6\epsilon_V + 2\eta_V \quad (3.33)$$

$$\alpha_s = -24\epsilon_V^2 + 16\epsilon_V\eta_V - 2\zeta_V, \quad (3.34)$$

which makes the comparison between predictions and observations simple. Thus far we have been laying down the general framework for slow-roll inflation – let us next demonstrate our machinery in action.

3.3 Higgs Inflation

Higgs inflation is an appealing model due to its minimalistic nature and a straightforward connection to collider physics [7, 27]. As the name suggests, in Higgs inflation the Higgs boson, the unique scalar field in the Standard Model of particle physics, drives the inflation. The only extension to the Standard Model is the non-minimal coupling to gravity ξ , which is required in order to achieve the correct amplitude for the power spectrum.

The relevant part of the Standard Model action is

$$S = \int dx \sqrt{-g} \left[\frac{M^2 + \xi h^2}{2} R - g^{\alpha\beta} h_{,\alpha} h_{,\beta} - V(h) \right], \quad (3.35)$$

where h is the radial mode of the Higgs doublet (which dominates during inflation). The Higgs self-interaction potential is

$$V(h) = \frac{1}{4} \lambda (h^2 - \nu^2)^2, \quad (3.36)$$

where λ is the Higgs self-coupling and $\nu = 246$ GeV the vacuum expectation value of the Higgs boson [7]. During inflation, the vacuum expectation value is negligible compared to the field value, $\nu \ll h$.

After transforming the action to Einstein frame, we find the potential to be

$$U(h) = \frac{\lambda}{4\xi^2} \left(\frac{\xi h^2}{1 + \xi h^2} \right)^2, \quad (3.37)$$

which is flat at large values of h . Such a flat region of the potential is a prime candidate for slow-roll inflation, because the derivatives of the potential are small. In the case $\xi h^2 \gg 1$, the field transformation (2.12) yields [2]

$$\sqrt{\xi}h \simeq \frac{1}{2\sqrt{\alpha}} \left(\frac{1 - \beta}{1 + \beta} \right)^{-\frac{1}{2}\beta} \exp \left(\sqrt{\frac{\xi}{\alpha}} \chi \right), \quad (3.38)$$

where $\beta \equiv \sqrt{\alpha - 1}\alpha$. At large field values the suppression of the potential is thus exponential i.e. Higgs inflation belongs to the class of plateau inflation. In general we will not use approximate field transformations, such as (3.38), but will instead write everything in terms of the Jordan frame fields, and use the exact transformation (2.12) to replace derivatives with respect to the Einstein frame field.

From (3.29) the number of e-folds at the pivot scale is

$$\begin{aligned} N_* &= \int_{h_{\text{end}}}^{h_*} dh \left(\frac{d\chi}{dh} \right)^2 \frac{U}{U'} = \int_{h_{\text{end}}}^{h_*} dh \frac{h}{4} \frac{1 + \alpha\xi h^2}{1 + \xi h^2} \\ &\approx \int_{h_{\text{end}}}^{h_*} dh \frac{\alpha h}{4} \\ &\approx \frac{1}{2} \alpha h_*^2, \end{aligned} \quad (3.39)$$

where we have on the first line transformed the integral measure and the derivative of the potential, on the second line approximated $\xi h^2 \gg 1$ to leading order, and on the third line dropped the h_{end} term, which is typically small. The last line also justifies the $\xi h^2 \gg 1$ approximation as $N_* \sim 50$. We can now use (3.39) to replace the field value at the pivot scale for the number of e-folds. The slow-roll parameters then read to leading order in $1/N_*$

$$\epsilon_V = \frac{\alpha}{8\xi N_*^2} \quad (3.40)$$

$$\eta_V = -\frac{1}{N_*} \quad (3.41)$$

$$\zeta_V = \frac{1}{N_*^2}. \quad (3.42)$$

The observables can now be expressed as

$$24\pi^2 A_s = \frac{2\lambda}{\alpha\xi} N_*^2 \quad (3.43)$$

$$r = \frac{2\alpha}{\xi N_*^2} \quad (3.44)$$

$$n_s - 1 = -\frac{2}{N_*} \quad (3.45)$$

$$\alpha_s = \frac{2}{N_*^2}. \quad (3.46)$$

The difference between the metric and Palatini formulation is the magnitude of ξ and the suppression of the tensor-to-scalar ratio by a factor of ξ . Let us now take $N_* = 50$ and $\lambda = 0.1$, which is the value of the coupling at the Electro-Weak minimum. In the metric formulation the correct amplitude is then reproduced for $\xi = 10^4$ while the tensor-to-scalar ratio is $r = 5 \times 10^{-3}$. In the Palatini formulation the non-minimal coupling has to be larger, $\xi = 10^9$, while the tensor-to-scalar ratio is smaller $r = 8 \times 10^{-13}$. In both cases the spectral index and its running are $n_s = 0.96$ and $\alpha_s = 8 \times 10^{-4}$. These predictions agree with the observed values listed in Appendix C.



Summary. In this chapter we have reviewed the key concepts of inflation, focusing on slow-roll inflation. We derived the slow-roll formulae for the tensor-to-scalar ratio r , spectral index n_s , and the running of the spectral index α_s , which are the main observables extracted from the observed CMB perturbations. We then applied our machinery to the case of Higgs inflation, in which the metric and Palatini formulation can be distinguished by the magnitude of the tensor-to-scalar ratio.



4 Semiclassical Approximation

信じるな、疑え。

Have suspicions, not faith.

— Deishuu Kaiki



While inflation is usually considered in an approximation, where both the metric and inflaton perturbations are quantised, this need not be the case. We could instead consider quantum perturbations of the inflaton in a classical spacetime, as is done in curved space field theory. We do not know *a priori*, which approximation of the complete quantum theory (with quantised gravity) is accurate at the energy scale of inflation. Therefore the only possibility to differentiate between them is by comparing the predicted spectrum of perturbations to CMB observations.

4.1 Classical Background

Let us now outline the framework used in [1], where we consider a quantised scalar field on a classical background. At first, the field is in a superposition state and the spacetime is described exactly by the flat FLRW metric (i.e. it is completely homogeneous and isotropic). At some point the field collapses to a definite value, which breaks the symmetry of the initial state leading to metric perturbations. For simplicity, we assume an instantaneous global collapse of all wave-modes to a definite state on some spacelike hypersurface.

Let us first consider the classical case. For the action (2.8) variation with respect to the metric yields $G_{\mu\nu} = T_{\mu\nu}$. The stress-energy tensor $T_{\mu\nu}$ is in the metric case [1, 28]

$$T_{\mu\nu} = S_{\mu\nu} - \xi [G_{\mu\nu} - \nabla_\mu \nabla_\nu + g_{\mu\nu} \square] \phi^2, \quad (4.1)$$

where the square denotes the d'Alembertian $\square \equiv g^{\alpha\beta} \nabla_\alpha \nabla_\beta$ and

$$S_{\mu\nu} \equiv \phi_{,\mu} \phi_{,\nu} - g_{\mu\nu} \left[\frac{1}{2} g^{\alpha\beta} \phi_{,\alpha} \phi_{,\beta} + V(\phi) \right]. \quad (4.2)$$

In the Palatini formulation, the stress-energy tensor differs in that there are no derivative terms in the square brackets of (4.1). For cosmological purposes, it

is easier to use the Einstein equation to solve $G_{\mu\nu}$ and define the stress-energy tensor as

$$T_{\mu\nu} \equiv \frac{1}{1 + \xi\phi^2} [S_{\mu\nu} + \xi(\nabla_\mu \nabla_\nu - g_{\mu\nu} \square \phi^2)] , \quad (4.3)$$

With this definition we recover the usual Einstein equation for the standard cosmology (i.e. the stress-energy tensor is independent of the Einstein tensor $G_{\mu\nu}$, which is set equal to (4.3)).

4.2 Coherent State

Before the collapse, the field and metric are connected through the semiclassical Einstein equation

$$G_{\mu\nu} = \langle T_{\mu\nu} \rangle , \quad (4.4)$$

where $G_{\mu\nu}$ is the classical Einstein tensor calculated from the FLRW metric and $\langle T_{\mu\nu} \rangle$ is the expectation value of the stress-energy tensor (4.3) in some quantum state. Let us now divide the field to a classical background φ and a quantised perturbation $\delta\hat{\phi}$. By taking the variation of (2.8) with respect to the field we find (for super-Hubble wavelengths)

$$\varphi'' + 2\mathcal{H}\varphi' - 6\xi(\mathcal{H}' + \mathcal{H}^2)\varphi + a^2V'(\varphi) = 0 \quad (4.5)$$

$$u_{\vec{k}}'' + 2\mathcal{H}u_{\vec{k}}' - 6\xi(\mathcal{H}' + \mathcal{H}^2)u_{\vec{k}} + a^2V''(\varphi)u_{\vec{k}} = 0 , \quad (4.6)$$

for the background and wave-modes. It is important to note, that under the slow-roll conditions the mode equation is to leading order the same as in the usual framework of inflation (3.22). This need not be the case in general.

Our next ingredient is the quantum state. As discussed in [1], we can assume the expectation value of the field to be zero in the Bunch-Davies vacuum and choose a coherent state for the field. Such a state is defined as

$$\hat{U}|0\rangle \equiv e^{N\hat{a}_0 + N^*\hat{a}_0^\dagger}|0\rangle , \quad (4.7)$$

where $|0\rangle$ denotes the vacuum state. The function N , not to be confused with the number of e-folds, is fixed by hand so that the expectation value of the field follows the background evolution $\langle \hat{\phi} \rangle = \varphi$, where $\hat{\phi} \equiv \varphi + \delta\hat{\phi}$ is the field operator. We can then calculate the expectation value of the stress-energy tensor (4.3) in the state (4.7) by replacing the classical field ϕ with the operator $\hat{\phi}$. As argued in [1], it reduces to the classical form (4.3) with the full field replaced by the background, when the power spectrum of the perturbation $\delta\hat{\phi}$ is small.

Constant	Curvature perturbation \mathcal{R}
$C = 0$	$-(1 - \chi_1)\mathcal{H}\frac{\delta\phi_{\vec{k}}}{\varphi'} - \frac{\delta\phi'_{\vec{k}}}{\varphi'}$
θ	$-3\epsilon_H\frac{\mathcal{H}^2}{k^2}\left[(3 + \chi_1)\mathcal{H}\frac{\delta\phi_{\vec{k}}}{\varphi'} + \frac{\delta\phi'_{\vec{k}}}{\varphi'}\right]$
$\sigma = 0$	$-2(1 - \chi_1)\mathcal{H}\frac{\delta\phi_{\vec{k}}}{\varphi'} - 2\frac{\delta\phi'_{\vec{k}}}{\varphi'}$
$\dot{\varphi}$	$-\frac{\mathcal{H}^2}{k^2}\left[(3\chi_1 + \chi_2)\mathcal{H}\frac{\delta\phi_{\vec{k}}}{\varphi'} + \chi_1\frac{\delta\phi'_{\vec{k}}}{\varphi'}\right]$
ρ	$3\frac{\mathcal{H}^2}{k^2}\left[(3 + \chi_1)\mathcal{H}\frac{\delta\phi_{\vec{k}}}{\varphi'} + \frac{\delta\phi'_{\vec{k}}}{\varphi'}\right]$
p	$-\frac{\mathcal{H}^2}{k^2}\left[(9 + 9\chi_1 + 2\chi_2)\mathcal{H}\frac{\delta\phi_{\vec{k}}}{\varphi'} + (3 + 2\chi_1)\frac{\delta\phi'_{\vec{k}}}{\varphi'}\right]$
R	$-3\frac{\mathcal{H}^2}{k^2}\left[(6 + 5\chi_1 + \chi_2)\mathcal{H}\frac{\delta\phi_{\vec{k}}}{\varphi'} + (2 + \chi_1)\frac{\delta\phi'_{\vec{k}}}{\varphi'}\right]$
${}^{(3)}R = 0$	$-2\mathcal{H}\frac{\delta\phi_{\vec{k}}}{\varphi'}$

Table 4.1: Curvature perturbation for different choices of what is kept constant (in the case of C , σ and ${}^{(3)}R$, kept zero) on the hypersurface of collapse. All quantities are evaluated at the time of collapse. We have also taken $\xi = 0$ and denoted $\chi_1 \equiv \epsilon_H - \eta_H$ and $\chi_2 \equiv \epsilon_H\chi_1 - \eta_{H2}$. The table is from [1].

4.3 Hypersurface of Collapse

Let us then focus on the actual collapse. When the field collapses to a definite value, the stress-energy tensor acquires perturbations in a discontinuous jump. These perturbations seed scalar metric perturbations, and the metric can be written in the perturbed form (3.12). The evolution equations for the background and perturbations are then derived in the Jordan frame according to the procedure of Section 3.1. The full equations are given in [1].

An important difference to the usual treatment is that the perturbations are not continuous across the collapse. Before the collapse, the metric is homogeneous and isotropic and the field has quantum perturbations. After the collapse, both the metric and the field have classical perturbations, which we need to match with the quantum perturbations at the time of collapse. While the field itself is discontinuous at the collapse, its power spectrum is continuous. We therefore equate the pre-collapse power spectrum of the quantised

field perturbations with the post-collapse power spectrum of the classical field perturbations on some spacelike hypersurface of collapse. A similar matching is possible for the power spectrum of the first time derivative of the field, but not for higher order derivatives, which are always discontinuous.

Unlike in the usual treatment, the power spectrum of the curvature perturbation at horizon exit now depends on the hypersurface of collapse. There is no theoretical argument for choosing one hypersurface over another and it is possible to obtain an arbitrary power spectrum by choosing a suitable hypersurface. However, some choices are arguably more 'natural'. In Table 4.1 we list the curvature perturbation in terms of the field perturbation for some reasonable choices of the hypersurface. In each case we have chosen a quantity, which is kept constant. These are the expansion rate θ , shear σ , energy density ρ , pressure p , Ricci scalar R , and spatial curvature scalar ${}^{(3)}R$. The case $C \equiv B - E' = 0$ corresponds to the conformal-Newtonian gauge.

There are two kinds of modifications to the amplitude: either it is multiplied by a factor of $(k/\mathcal{H})^{-4}$ or there is a uniform change. In the former case, the spectrum is not close to scale invariant and matching on these hypersurfaces is thus excluded by observations. In the latter case (i.e. for the hypersurfaces with $C = 0$, $\sigma = 0$, and ${}^{(3)}R = 0$), the amplitude depends on when the collapse happens, which makes it model dependent.

Minimal coupling. When the field is minimally coupled, the background equations are the same as in the usual case up to slow-roll suppressed corrections. Therefore the spectrum of the field is essentially the same as in the usual case, if the collapse happens during the slow-roll phase. In the case $C = 0$, the spectrum differs from the usual one only by slow-roll suppressed corrections. In the $\sigma = 0$ and ${}^{(3)}R = 0$ cases, the amplitude is multiplied by a factor of two, but other observables remain the same (up to slow-roll suppressed corrections). If the collapse happens after inflation, for example during preheating, the amplitude can change by an arbitrary amount (although non-Gaussianity imposes some constraints as discussed in [1]).

Non-minimal coupling. When the field is non-minimally coupled to gravity, the equation of motion for the mode functions can be expressed as [1]

$$\begin{aligned} \frac{d^2 u_{\vec{k}}}{dN^2} + 3 \frac{du_{\vec{k}}}{dN} + (3\eta_H + \zeta_H)u_{\vec{k}} + 6 \frac{\xi \bar{\varphi}^2}{1 + \xi \bar{\varphi}^2} \left(\xi [4 - 2\epsilon_H + 3\eta_H - \zeta_H] \right. \\ \left. - \frac{\bar{\varphi}'}{\mathcal{H}\bar{\varphi}} [2 + 3\xi + (1 + 3\xi)(\epsilon_H - \eta_H)] \right) u_{\vec{k}} = 0 . \end{aligned} \quad (4.8)$$

In order to obtain nearly scale invariant spectrum, the effective mass of the field must be small i.e. the last term in (4.8) must vanish. This is only possible, if either the non-minimal coupling has negligible effect on the dynamics, or $\xi \ll 1$ and the background field is super-Planckian $\bar{\varphi} \gg 1$ (so that $\xi\bar{\varphi}^2 \gtrsim 1$) [1]. Therefore we do not obtain a scale invariant spectrum for the curvature perturbation (at least not for sub-Planckian field values). In the Palatini formulation the situation is similar – the mode equation (4.8) would only differ in terms proportional to $\bar{\varphi}'$ that are anyway negligible during slow-roll. In particular this implies, that the metric perturbations must be quantised in Higgs inflation.



Summary. We have shown in [1] that it is possible to obtain a scale invariant scalar power spectrum without quantising the metric perturbations, as long as the inflaton is minimally coupled to gravity. However, the tensor power spectrum is negligible in such a model, because the scalar perturbations seed tensor perturbations only at second order. A detection of primordial tensor amplitude could therefore be used as a proof of the quantisation of the metric perturbations at the time of inflation.

On the other hand we have also shown in [1] that the power spectrum is not close to scale invariant, if the inflaton is coupled non-minimally to gravity (as is the case in Higgs inflation). Thus if the Higgs boson is assumed to be the inflaton, we must quantise the metric perturbations.



5 Loop Corrections

疑い致せば切がないか。

I guess once you start doubting, there is no end to it.

— Batou



While Higgs inflation predicts a almost scale-invariant spectrum with a spectral index and a tensor-to-scalar ratio that agree with the observations, there are some dark clouds in a perfect sky. From the renormalisation group running of the Standard Model, it is known that the Higgs self-coupling λ decreases, when the energy scale increases. Because the energy scale is large during inflation, a strong running could foil slow-roll inflation.

5.1 Loop Corrections and the Effective Potential

Let us now consider the effect of loop-corrections on the predictions of Higgs inflation. The numerical solution to the renormalisation group equations of the Standard Model shows that the Higgs self-coupling has a minimum around the $10^{17} \dots 10^{18}$ GeV scale [9, 29]. The precise location of the minimum and the value of the coupling at the minimum are sensitive to the masses of the Higgs boson and the top quark, which are measured at low energy scale in collider experiments [9, 29–32]. For the current experimental and theoretical errors the Higgs self-coupling could run all the way to negative values.

Furthermore, the Standard Model becomes non-perturbative between the low energy scale and the scale of inflation. Because there can be threshold corrections at this intermediate scale (i.e. corrections from new physics), the uncertainty in the couplings at the scale of inflation is even larger, which makes them essentially free parameters.

The Higgs self-coupling can be approximated around its minimum as [30]

$$\lambda(h) = \lambda_0 + \frac{b}{4} \ln^2 \left(\frac{\xi h^2}{\kappa^2 (1 + \xi h^2)} \right), \quad (5.1)$$

where λ_0 is the value of the coupling at the minimum, κ is related to the location of the minimum, and the constant $b = 2.3 \times 10^{-5}$ is determined from the Standard Model running. As discussed in the previous paragraph, κ and λ_0

can be treated as free parameters because of the threshold corrections. There are two different conventions for κ in the literature: one in which κ is of order one [9, 30, 31] and one in which $\kappa = \sqrt{\xi}\tilde{\kappa}$ with $\tilde{\kappa}$ of order one [29, 32]. We consider both cases. To guarantee the stability of the electro-weak vacuum, we only allow positive values for λ_0 .

The effective Einstein frame potential is obtained by replacing λ in (3.37) with the function (5.1). The resulting potential then reads [2]

$$\begin{aligned} U(h) &= \frac{1}{4} \left[\lambda_0 + \frac{b}{4} \ln^2 \left(\frac{\xi h^2}{\kappa^2(1 + \xi h^2)} \right) \right] \left(\frac{\xi h^2}{1 + \xi h^2} \right)^2 \\ &\equiv \frac{\lambda_0 \kappa^4}{4\xi^2} (1 + c \ln^2 x) x^2 \end{aligned} \quad (5.2)$$

where $c \equiv b/(4\lambda_0)$, $\ln^2 x = (\ln x)^2$, and

$$x \equiv \frac{\xi h^2}{\kappa^2(1 + \xi h^2)}. \quad (5.3)$$

Note that x runs from zero to one when h runs from zero to infinity. The approximation (5.1) is good as long as the logarithm of x is less than one, which is the case during inflation (as long as κ is close to one).

5.2 Different Scenarios for Inflation

Depending on the value of c , the shape of the potential can change drastically. In the case $c < 4$, the potential is monotonic and there is a plateau at large values of the field. For $c = 4$ (or equivalently $\lambda_0 = b/16$), an inflection point emerges below the plateau at $\ln x = -1/2$. If $c < 4$, the potential can have a local maximum (a hilltop) and a second minimum, in addition to the usual electro-weak vacuum. These three cases are illustrated in Figure 5.1. Note that the seemingly steep slope near $x \sim 1$ is stretched to a plateau, when the potential is plotted as a function of the field h .

For the case $c \ll 4$, illustrated with a solid green line in Figure 5.1, we can have the usual plateau inflation discussed in Section 3.3. However, when $c \lesssim 4$ the situation changes. Then we have a second flat region below the plateau, as shown in Figure 5.1 with a dashed blue line, and a significant number of e-folds can be accumulated there. Such a setup is called critical point inflation and it has been studied in the metric formulation in [29–35]. The predictions of critical point inflation can differ significantly from the plateau case, because the field can start in the steeper part of the potential and then roll to the flat

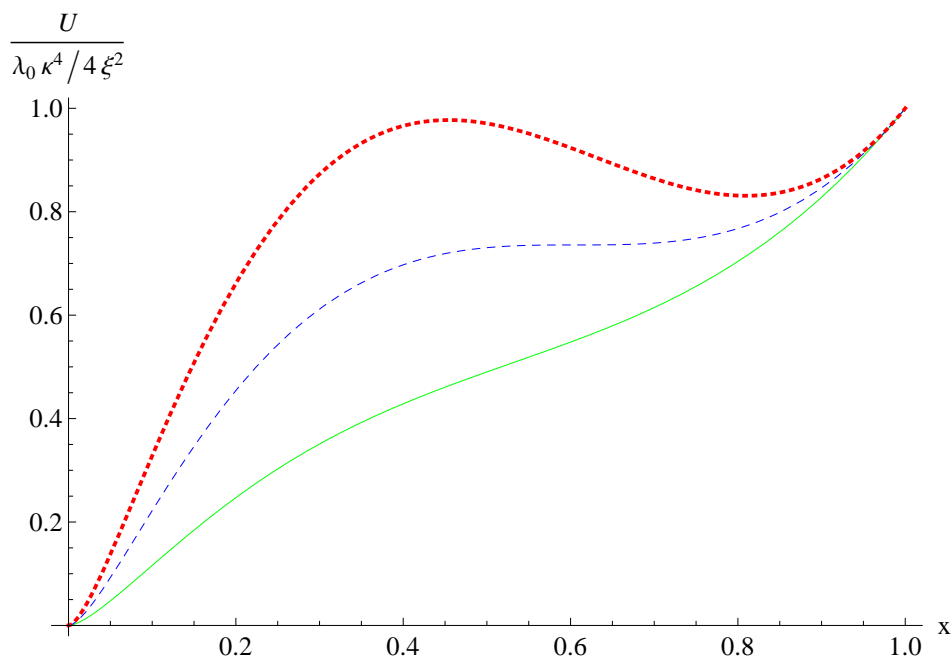


Figure 5.1: The potential as a function of x for $c = 2, 4, 6$ (solid green, dashed blue, dotted red). The top curve is relevant for hilltop and false vacuum inflation, the middle curve for critical point inflation and the bottom curve for plateau inflation. The figure is from [2].

region around the inflection point, where most of the e-folds are accumulated. In particular, this can lead to a large tensor-to-scalar ratio [29, 30].

When $c > 4$, we can have two flat regions, where inflation can occur: a hilltop and a local minimum. This situation is illustrated in Figure 5.1 with a dotted red line. The hilltop exists for $\ln \kappa < \frac{1}{4} \dots \frac{1}{2}$ (depending on the value of c) and inflation on top of it is fine-tuned, because the inflaton must somehow be on the top of the hill at the beginning of the slow-roll phase, and roll to the electroweak vacuum (instead of the false vacuum at the second minimum). Such a model has been considered in [34, 36]. Another possibility is false vacuum inflation at the second minimum. In that case we need to add another scalar field to the model, which makes the setup less minimal. The predictions are also the same (up to slow-roll suppressed corrections) in both the metric and Palatini formulation [2], which makes this model less interesting from our point of view.

Let us now focus on critical point inflation and hilltop inflation, which are the two most interesting cases. The exact expressions for the slow-roll paramet-

ers are complicated and the analytical approximations in [2] proved inaccurate. Hence, we used numerical methods to explore the full parameter space.

5.3 Numerical Results

5.3.1 Setup

In critical point inflation, we have five parameters ξ , κ , λ_0 , h_* and h_{end} . There are three constraints: the normalisation condition on the amplitude, the number of e-folds at the pivot scale, and the condition that either $\epsilon_V = 1$ or $|\eta_V| = 1$ at the end of inflation, whichever occurs first. We therefore have two free parameters that we choose to be ξ and κ . For comparison with observations and previous work on the subject, we present the results in (n_s, r) -plane. We also impose the bounds $0.1 < \kappa < 10$ or $0.1 < \tilde{\kappa} < 10$.

We determine h_* from the normalisation condition $24\pi^2 A_* = U/\epsilon_V$. In some cases this condition is degenerate with up to three different solutions. For plateau inflation we would have only one solution, but the flat region around the inflection point leads to a peak in the amplitude, because ϵ_V is small. Therefore we can also have solutions around the inflection point in addition to the solution at the plateau. The derivative of the amplitude with respect to the Einstein frame field χ can be expressed as

$$\frac{dA_s}{d\chi} = (1 - n_s) \frac{A_s}{\sqrt{2\epsilon}}, \quad (5.4)$$

which implies, that the spectrum is blue (red) when the derivative is negative (positive). Because the derivative must change its sign between solutions, we then have alternating blue and red solutions. We have confirmed, that the lowest field value does not yield enough e-folds, and the middle solution has a blue spectrum. We thus choose the largest field value whenever the normalisation condition is degenerate.

The field value at the end of inflation h_{end} is solved from the condition $\epsilon_V = 1$ or $|\eta_V| = 1$. As in the case of the normalisation condition, there can again be multiple solutions. This implies, that in some cases we have multiple disjoint slow-roll regions. In particular, we can have a fast-roll region between the inflection point and the plateau. We disregard such fast-roll regions and choose the smallest field value to be the end point of inflation. Finally, we solve λ_0 by requiring the number of e-folds (3.29) calculated at the pivot scale to be N_* given by (3.30). We also impose the lower bound $\lambda_0 > b/16$, which guarantees a monotonic potential.

In the case of hilltop inflation, the field has to be below the local maximum of the potential and on the side of the electroweak vacuum. Because of this

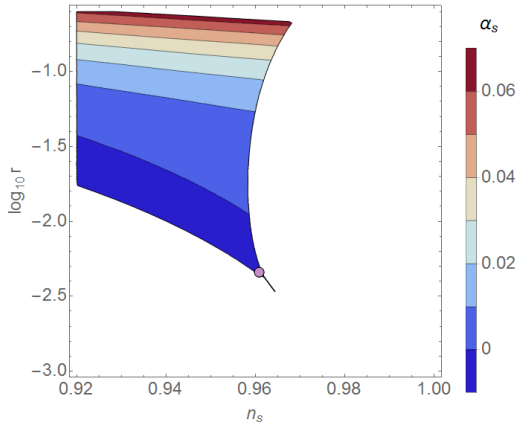


Figure 5.2: Running of the spectral index α_s in the metric formulation for critical point inflation. The purple dot corresponds to plateau inflation of Section 3.3. The figure is from [2].

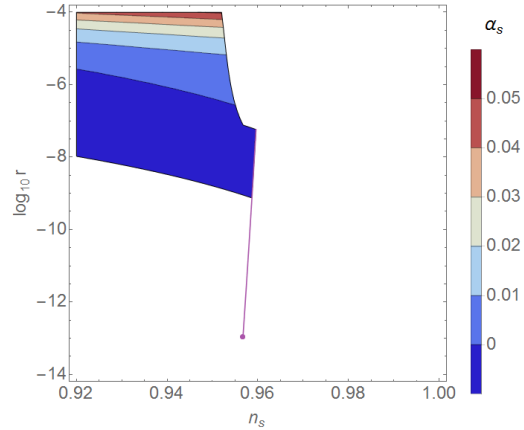


Figure 5.3: Running of the spectral index α_s in the Palatini formulation for critical point inflation with κ . The purple line is the plateau case of Section 3.3, λ being one at the dot. The figure is from [2].

constraint, there is no degeneracy on the normalisation condition or the end of inflation. We then use the same procedure as in the critical point case to determine h_* , h_{end} , and λ_0 , with the exception that $0 < \lambda_0 < b/16$. We also use the same limits for κ and $\tilde{\kappa}$.

5.3.2 Critical Point Inflation

Metric formulation. Our results for critical point inflation in the metric formulation are illustrated in Figure 5.2 where we show the running of the spectral index α_s as a function of n_s and r . The purple dot corresponds to the result of plateau inflation and there is no difference between constraining κ or $\tilde{\kappa}$ to be close to unity. The boundary on the right comes from the number of e-folds at the pivot scale, while the bottom boundary corresponds to the limit $\xi h^2 \gg 1$.

Our results show, that the tensor-to-scalar ratio can be larger than in the plateau case, while the spectral index can be smaller. The tensor-to-scalar ratio is bounded from above $r < 4 \times 10^{-2}$ by the observational constraint on the running of the spectral index (C.4). These results agree with previous studies [29–35]. The main difference is the boundary on the right, which cuts off large values of n_s .

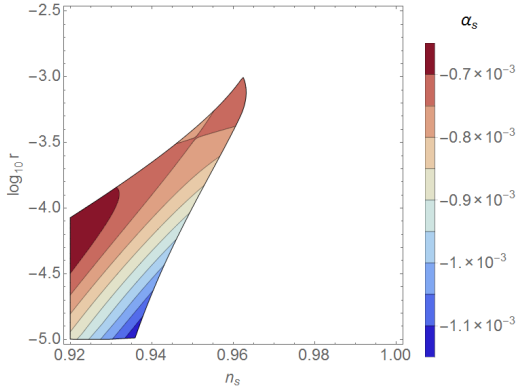


Figure 5.4: Running of the spectral index α_s in the metric formulation for hilltop inflation with κ . The figure is from [2].

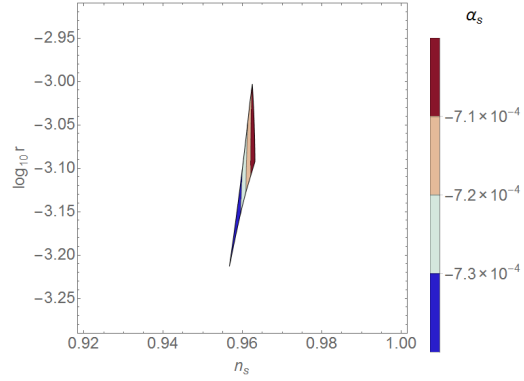


Figure 5.5: Running of the spectral index α_s in the metric formulation for hilltop inflation with $\tilde{\kappa}$. The figure is from [2].

Palatini formulation. Figure 5.3 shows our results in the Palatini case. The purple line corresponds to plateau inflation, the point indicating the result in the case $\lambda = 1$. As in the metric case, the right side boundary comes from the number of e-folds. However, the bottom boundary is related to the cut-off $\kappa > 0.1$ and would be lower for a lower cut-off. The results also depend strongly on the chosen convention: if $\tilde{\kappa}$ is close to one, critical point inflation is no longer possible and the results reduce to the plateau case.

Comparing to the metric case, the tensor-to-scalar ratio is smaller in the Palatini formulation. The observational upper bound on the running of the spectral index is more important, because the tensor-to-scalar ratio is suppressed. When we take into account the constraint on the spectral index, the tensor-to-scalar ratio is constrained from above to $r < 3 \times 10^{-7}$. This upper limit is sensitive to the observational limits of the spectral index (C.3) and the details of reheating: if reheating lasts longer than four e-folds, the upper bound on r decreases further.

5.3.3 Hilltop Inflation

Metric formulation. Figures 5.4 and 5.5 show our results for hilltop inflation in the metric case. In Figure 5.4 the boundary on the left side comes from the constraint $\lambda_0 > 0$. The boundary on the right side comes from the constraint $\lambda_0 < b/16$ for $r > 5 \times 10^{-4}$ and from the number of e-folds for $r < 5 \times 10^{-4}$. In the case of $\tilde{\kappa}$ the cut-off $\tilde{\kappa} > 0.1$ reduces the allowed region to the tip of the

region in Figure 5.4, as shown in Figure 5.5 (the bottom right boundary comes from the cut-off).

Comparing to plateau inflation, the tensor-to-scalar ratio is reduced and the running of the spectral index is small and negative. Taking into account the observational constraints on the spectral index (C.3), the tensor-to-scalar ratio is $2 \times 10^{-4} < r < 1 \times 10^{-3}$. These results agree with the follow-up study [36].

Palatini formulation. The results in the Palatini case are illustrated in Figure 5.6. Because the running of the spectral index is not single-valued, we show only the allowed region. The boundary on the left side comes from the fact that n_s has a minimum for a given value of r [2]. The right side boundary comes from the upper bound on λ_0 for $r > 5 \times 10^{-10}$ and from the constraint on e-folds for $r < 5 \times 10^{-10}$. In the case of $\tilde{\kappa}$ close to unity, hilltop inflation is not allowed.

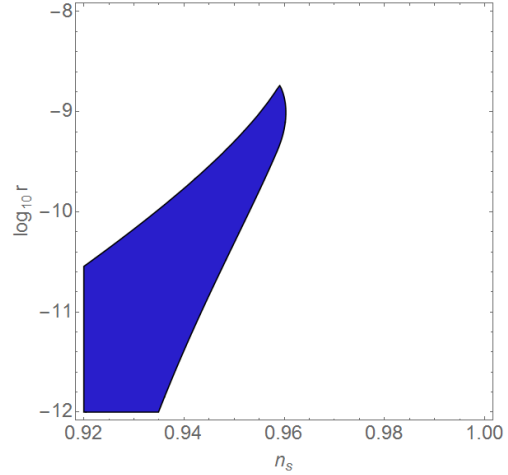


Figure 5.6: Allowed parameter region on the (n_s, r) plane in the Palatini formulation for hilltop inflation with κ . The running is not single-valued on this plane. The figure is from [2].

Comparing to the metric formulation, the tensor-to-scalar ratio is again reduced. For the current observational constraints it is bounded as $2 \times 10^{-10} < r < 2 \times 10^{-9}$. These results are also in agreement with the follow-up study [36].



Summary. In [2] we have shown that the tensor-to-scalar ratio is suppressed in the Palatini formulation in various different inflation scenarios, when compared to the metric formulation. When the loop corrections to the Higgs self-coupling are taken into account, the combined range for the tensor-to-scalar ratio is in the metric formulation $2 \times 10^{-4} < r < 4 \times 10^{-2}$ and in the Palatini formulation $1 \times 10^{-13} < r < 3 \times 10^{-7}$. We could thus rule out the Palatini formulation if a tensor-to-scalar ratio above 3×10^{-7} is detected by future experiments; conversely, if nothing is detected above 2×10^{-4} we can rule out the metric formulation.



6 Corrections to Gravity

人を恐怖される条件は三つだ。一つ、怪物は言葉を喋ってはいけない。二つ、怪物は正体不明でなければいけない。三つ、怪物は不死身なければ意味がない。

There are three requirements to terrify someone. First, the monster may not speak. Second, the monster has to be unidentifiable. Third, there is no point if the monster can die.

— Touko Aozaki



While loop corrections to the Higgs self-potential can affect the predictions of Higgs inflation, there is another source of uncertainty. In general, there are also corrections to the gravitational part of the action. When General Relativity is treated as an effective field theory, loop corrections generate higher order curvature terms. The lowest order correction to the Einstein-Hilbert action is then the R^2 term, which can drive inflation on its own in the metric formulation.

6.1 $F(R)$ Gravity as a Scalar-Tensor Theory

While General Relativity is not a renormalisable theory, we can still treat it as an effective theory up to some cut-off. The loop corrections then generate higher order curvature terms [37]. Let us therefore consider an action of the form

$$S = \int dx \sqrt{-g} \left[\frac{1}{2} F(R) + \frac{1}{2} G(h) R - \frac{1}{2} g^{\alpha\beta} h_{,\alpha} h_{,\beta} - V(h) \right], \quad (6.1)$$

where $F(R)$ is an arbitrary function of the Ricci scalar and $G(h)$ is a generalised coupling to gravity (an arbitrary function of the Higgs field). The usual Higgs inflation then corresponds to the choice

$$F(R) = R, \quad G(h) = \xi h^2, \quad V(h) = \frac{\lambda}{4} (h^2 - \nu^2)^2. \quad (6.2)$$

If the function F is not linear in R , it introduces a new scalar degree of freedom to the action [38]. We can extract this new scalar from the function F

by expressing the action (6.1) as [38]

$$S = \int dx \sqrt{-g} \left[\frac{1}{2} G(h) R + \frac{1}{2} \{F(\phi) + F'(\phi)(R - \phi)\} - \frac{1}{2} g^{\alpha\beta} h_{,\alpha} h_{,\beta} - V(h) \right], \quad (6.3)$$

where ϕ is an auxiliary scalar field and prime denotes a derivative with respect to ϕ . A variation with respect to this new field yields the constraint $F''(\phi)[\phi - R] = 0$, which has the solution $\phi = R$. By substituting this back to (6.3), we recover the original action (6.1). We can further simplify the action (6.3) by redefining the scalar field as $\varphi \equiv F'(\phi)$. In terms of the redefined field the action reads [38]

$$S = \int dx \sqrt{-g} \left[\frac{1}{2} [\varphi + G(h)] R - W(\varphi) - \frac{1}{2} g^{\alpha\beta} h_{,\alpha} h_{,\beta} - V(h) \right], \quad (6.4)$$

where we have defined the potential of the new field φ to be

$$W(\varphi) \equiv \frac{1}{2} \{ \phi(\varphi) \varphi - F[\phi(\varphi)] \}. \quad (6.5)$$

We can now reach the Einstein frame by making a conformal transformation

$$\tilde{g}_{\mu\nu} = \Omega^2 g_{\mu\nu} = [\varphi + G(h)] g_{\mu\nu}. \quad (6.6)$$

The Einstein frame action then becomes [4]

$$S = \int dx \sqrt{-g} \left[\frac{1}{2} R - \frac{3\mu}{4\Omega^4} g^{\alpha\beta} (\varphi_{,\alpha} \varphi_{,\beta} + 2\varphi_{,\alpha} G_{,\beta} + G_{,\alpha} G_{,\beta}) - \frac{1}{2\Omega^2} g^{\alpha\beta} h_{,\alpha} h_{,\beta} - U(h, \varphi) \right], \quad (6.7)$$

where μ is one in the metric formulation and zero in the Palatini formulation, and the Einstein frame potential reads

$$U(h, \varphi) = \frac{V(h) + W(\varphi)}{[\varphi + G(h)]^2}. \quad (6.8)$$

An important difference between the two formulations is the role of the auxiliary field φ . In the metric formulation, the action contains derivatives of φ and a variation of the action with respect to it yields a dynamical equation of motion. In the Palatini formulation, the action has no derivatives of φ and a variation with respect to it yields a constraint equation. Because of this fundamental difference, we consider these cases separately.

6.2 Two-Field Inflation in Metric Formulation

Let us first consider the metric formulation, in which we have two dynamical scalar fields. If we restrict ourselves to dimension four terms, the most general case is

$$F(R) = R + \alpha R^2, \quad G(h) = \xi h^2, \quad V(h) = \frac{\lambda}{4}(h^2 - \nu^2)^2, \quad (6.9)$$

where α and ξ are constants. The action (6.7) can then be written as

$$S = \int dx \left[\frac{1}{2}R - \frac{1}{2}g^{\alpha\beta}G_{AB}\phi_{,\alpha}^A\phi_{,\beta}^B - U(\phi^A) \right], \quad (6.10)$$

where we have defined the field space vector $\phi^A \equiv (\varphi, h)$ and the field space metric

$$G_{AB} \equiv \frac{1}{\Omega^4} \begin{pmatrix} 3/2 & 3\xi h \\ 3\xi h & \Omega^2 + 6\xi^2 h^2 \end{pmatrix}. \quad (6.11)$$

With the choice (6.9) the Einstein frame potential reads

$$U = \frac{\lambda}{4} \frac{(h^2 - \nu^2)^2}{\Omega^4} + \frac{1}{8\alpha} \frac{(\varphi - 1)^2}{\Omega^4}. \quad (6.12)$$

In the metric formulation the field φ can act alone as the inflaton. If we set $\lambda = \xi = 0$, the model corresponds to Starobinsky inflation, which was first proposed in [39]. In terms of e-folds, Starobinsky inflation yields exactly the same spectral index (3.45) and tensor-to-scalar ratio (3.44) as Higgs inflation, the only difference is in the formula for the amplitude. If λ and ξ are non-zero, we can have mixed Higgs-Starobinsky inflation, which has been studied previously in [10, 40–44].

Because the field space has curvature, the fields h and φ cannot have a canonical kinetic term at the same time. Therefore we use the field covariant formalism, formulated and described in detail in [45], to calculate the slow-roll parameters and the observables³. We have listed the lowest order slow-roll parameters (ϵ_{fc} , η_{fc} , ζ_{fc}) in Appendix A for reference. The difference to the usual slow-roll expansion is, that these parameters are invariant with respect to both frame and field transformations.

Let us now consider the Higgs inflation limit. Naively we would expect to recover the usual Higgs inflation by setting $\varphi = 1$. However, approximating

³As noted in [4], there is a sign issue in [45]. We use the sign convention of [4], in which η_{fc} reduces to $2\eta_V - 4\epsilon_V$ in the single field case.

$h \gg 1$ we find

$$\epsilon_{\text{fc}} = \frac{4}{3}(1 + 6\xi) + \mathcal{O}\left(\frac{1}{h^2}\right) \quad (6.13)$$

$$\eta_{\text{fc}} = \frac{4\xi}{3\lambda\alpha} [-6\lambda\alpha + \xi(1 + 6\xi)] + \mathcal{O}\left(\frac{1}{h^2}\right), \quad (6.14)$$

which is not the result of the usual Higgs inflation. Furthermore, ϵ_{fc} is never smaller than one for a positive ξ and η_{fc} diverges as α goes to zero. Therefore we do not recover Higgs inflation in the $\alpha \rightarrow 0$ limit either. That is because the field tends to roll in the direction of the new degree of freedom, which completely destabilises pure Higgs inflation. In contrast, if we set $h = 0$ and approximate $\varphi \gg 1$, we find

$$\epsilon_{\text{fc}} = \frac{4}{3\varphi^2} + \mathcal{O}\left(\frac{1}{\varphi^3}\right) \quad (6.15)$$

$$\eta_{\text{fc}} = -\frac{8}{3\varphi} + \mathcal{O}\left(\frac{1}{\varphi^2}\right), \quad (6.16)$$

which is the result of Starobinsky inflation [39].

Let us now focus on the attractors of the model. When ξ is positive, we have an attractor approximately at the parabola [4]

$$\varphi = 1 + b + c h^2, \quad (6.17)$$

where the parameters b and c read

$$\begin{aligned} b &\equiv -\frac{2\lambda\alpha}{12\lambda\alpha\xi + \xi^2(1 + 6\xi)} = \frac{c}{(c + d)(1 - 6d)} \\ c &\equiv \frac{2\lambda\alpha}{\xi}, \end{aligned} \quad (6.18)$$

and we have also defined $d \equiv \xi + \frac{1}{6}$. The solutions at the parabola (6.17) are Higgs-like, if ξ and λ determine the amplitude of perturbations. If α determines the amplitude of perturbations, they are Starobinsky-like instead. The former case corresponds to $c \ll d$ and the latter to $c \gg d$. If ξ is negative, the Starobinsky solution at the line $h = 0$ becomes an attractor and there are no Higgs-like solutions.

In order to study the full two-field effects, including isocurvature perturbations, we used numerical methods. Overall there are six parameters (α , ξ , λ , N , and two field coordinates) and three constraints (the normalisation condition, the constraint on e-folds, and the condition $|\epsilon_{\text{fc}}| = 1$ or $|\eta_{\text{fc}}| = 1$ at the end

of inflation⁴). When written in terms of the parameters c and d , the slow-roll parameters and the number of e-folds are independent of λ while the amplitude depends on it linearly. Hence we choose the three free parameters to be c , d , and one of the field coordinates at the pivot scale.

Our results are summarised in Figure 6.1. The red points denote attractor solutions and the black star corresponds to the result of Higgs inflation (which coincides with the result of Starobinsky inflation). We have taken into account all current observational constraints of Appendix C. The upper boundary on the transfer angle constrains the solutions to be near the attractor in field space, which restricts the spectral index and tensor-to-scalar ratio to be close to the attractor value. Comparing to the usual Higgs inflation the tensor-to-scalar ratio can be larger, reaching the current observational upper bound $r < 0.079$. We also find, that Higgs-like inflation with $d \gg c$ is possible on the attractor for any value of λ . These results agree with previous work [10, 40–44].

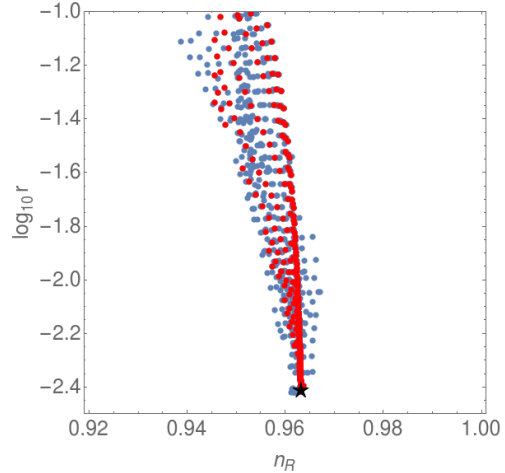


Figure 6.1: Numerical results for spectral index n_R and tensor-to-scalar ratio r . The red points are attractor solutions and the black star denotes the usual Higgs inflation result. The figure is from [4].

6.3 Single-Field Inflation in Palatini Formulation

Let us next consider the Palatini formulation. The coupling $G(h)$ of the action (6.7) can be kept arbitrary without making the equations overly complicated, but we restrict the function $F(R)$ to terms up to dimension four

$$F(R) = R + \alpha R^2 . \tag{6.19}$$

The Einstein frame potential then reads

$$U(h, \varphi) = \frac{1}{[\varphi + G(h)]^2} \left[V(h) + \frac{(\varphi - 1)^2}{8\alpha} \right] . \tag{6.20}$$

⁴Note that η_{fc} differs from η_V by a factor of two in the single field case. Therefore inflation will end slightly later, if we impose $|\eta_{fc}| = 1$ instead of $|\eta_V| = 1$. The difference in e-folds is small and comparable to the effect of reheating [2].

By taking the variation with respect to the field φ we obtain a constraint equation with the solution [3]

$$\varphi = \frac{1 + G(h) + 8\alpha V(h) + 2\alpha G(h)h^\alpha h_{,\alpha}}{1 + G(h) - 2\alpha h^\alpha h_{,\alpha}}. \quad (6.21)$$

By redefining the Higgs field through the differential equation

$$\frac{dh}{d\chi} = \pm \sqrt{1 + G + 8\alpha \frac{V}{1 + G}}, \quad (6.22)$$

and substituting the solution (6.21) into the action (6.7) we find [3]

$$S = \int dx \sqrt{-g} \left[\frac{1}{2}R - \frac{1}{2}\chi^\alpha \chi_{,\alpha} + \frac{1}{2}\alpha(1 + 8\alpha\bar{U})(\chi^\alpha \chi_{,\alpha})^2 - U \right]. \quad (6.23)$$

The field χ has now a canonical kinetic term plus a higher order kinetic term, which is subleading under the slow-roll conditions [3]. To avoid negative kinetic energy, we must have $\alpha > 0$.

From the perspective of slow-roll inflation, the effects of modified gravity are encoded in the Einstein frame potential, which can be expressed as

$$U = \frac{\bar{U}}{1 + 8\alpha\bar{U}}, \quad (6.24)$$

where

$$\bar{U}(h) \equiv \frac{V(h)}{[1 + G(h)]^2} \quad (6.25)$$

is the potential in the case $\alpha = 0$. We can now proceed as in Section 3.2 and calculate the slow-roll parameters from the potential U .

The predictions for the observables are then [3]

$$\begin{aligned} 24\pi^2 A_s &= \frac{\bar{U}}{\bar{\epsilon}} \\ n_s - 1 &= 2\bar{\eta} - 6\bar{\epsilon} \\ \alpha_s &= 16\bar{\epsilon}\bar{\eta} - 24\bar{\epsilon}^2 - 2\bar{\zeta} \\ r &= \frac{16}{1 + 8\alpha\bar{U}}\bar{\epsilon}, \end{aligned} \quad (6.26)$$

where a bar denotes the slow-roll parameters in the case $\alpha = 0$. The amplitude, the spectral index, and the running of the spectral index are unaffected by the R^2 term, as is also the number of e-folds. The only difference is that the tensor-to-scalar ratio can be decreased by an arbitrary amount by tuning α .

The reason for the result (6.26) is that the scalar power spectrum itself is unchanged [3]

$$\mathcal{P}_{\mathcal{R}} = \frac{U}{24\pi^2\epsilon} = \frac{\bar{U}}{24\pi^2\bar{\epsilon}}. \quad (6.27)$$

However, the tensor power spectrum becomes [3]

$$\mathcal{P}_T = \frac{2U}{3\pi^2} = \frac{2\bar{U}}{3\pi^2(1 + 8\alpha\bar{U})}, \quad (6.28)$$

and hence the tensor-to-scalar ratio and the tensor spectral index are suppressed by a factor of $1/(1 + 8\alpha\bar{U})$.



Summary. We have shown in [3, 4] that the tensor-to-scalar ratio is enhanced in the metric formulation and suppressed in the Palatini formulation, when corrections to gravity are taken into account. Furthermore, we have shown that pure Higgs inflation is unstable in the metric formulation (although Higgs-like inflation is possible), when the Higgs field and the new scalar degree of freedom are mixed. In contrast, Higgs inflation is stable in the Palatini formulation, where the addition of higher order corrections do not introduce new degrees of freedom to the model.



7 Conclusions

真実を探してるだけだ。

I am only searching for truth.

— Shinjuurou Yuuki



In this thesis we have investigated the possibility to probe the gravitational degrees of freedom through CMB observations. From a theoretical point of view, there is no clear indication of how we should choose the gravitational degrees of freedom, and hence it is important to have a way to test the different possibilities. We have shown that the two formulations of General Relativity have distinct cosmological signatures, which could be used to differentiate between them.

In Chapter 4 we found that the detection of primordial gravitational waves is required to prove that gravity is quantised during inflation. We also found that Higgs inflation will not yield a nearly scale invariant spectrum, unless the metric perturbations are quantised. If we assume the Higgs boson to be the inflaton, we then must quantise the metric perturbations in order to obtain the observed power spectrum of perturbations. This conclusion holds equally for metric and Palatini formulation.

In Chapter 5 we considered loop corrections of the Higgs self-potential. The shape of the potential is sensitive to the masses of the Higgs boson and the top quark measured at the low energy scale. We considered multiple models of which most prominent were critical point inflation and hilltop inflation. In the former case we found that the tensor-to-scalar ratio can be enhanced from the plateau result, while in the latter case it is suppressed. We also found that despite these modifications Palatini formulation yields a lower tensor-to-scalar ratio than metric formulation.

In Chapter 6 we considered the effects of loop corrections of the gravitational action by including a higher order R^2 term in the action. The two formulations then lead to fundamentally different models of inflation: in metric formulation we have a new scalar degree of freedom and two-field inflation, while in Palatini formulation we have no new degrees of freedom and single-field inflation. A detailed analysis showed that the tensor-to-scalar ratio is enhanced in the metric case and reduced in the Palatini case. We also demonstrated that in metric formulation pure Higgs inflation is unstable as opposed to Palatini formulation.

By taking into account all the effects discussed, we conclude that metric and Palatini formulation have distinct cosmological signatures (at least when the Higgs boson is assumed to be the inflaton). If a tensor-to-scalar ratio above 3×10^{-7} is detected, we can rule out Palatini formulation. Conversely if nothing is detected above 2×10^{-4} we can rule out metric formulation. One should note, that these limits on the tensor-to-scalar ratio are sensitive to changes in the lower bound for n_s as discussed in Chapter 5.

Next generation experiments should be able to detect a tensor-to-scalar ratio above 10^{-3} . If it is indeed detected, we can rule out Palatini formulation. If nothing is detected, we can rule out at least plateau, critical point, and Higgs-Starobinsky inflation in the metric formulation (hilltop inflation could still be an option). We should thus be able to probe the gravitational degrees of freedom through inflation with the next generation experiments.



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A Slow-roll Parameters

In literature there are various different definitions for the slow-roll parameters. The Hubble slow-roll parameters are defined in terms of the Hubble parameter as [22, 23]

$$\epsilon_H = 2 \left(\frac{H_\varphi}{H} \right)^2 \quad (\text{A.1})$$

$$\sigma_H^{(n)} = 2^n \frac{H_\varphi^{n-1}}{H^n} \frac{d^{n+1}}{d\varphi^{n+1}} H, \quad (\text{A.2})$$

where the subscript φ denotes derivative with respect to the background field. For convenience we also define

$$\eta_H \equiv 2 \frac{H_{\varphi\varphi}}{H} + \epsilon_H = \sigma_H^{(1)} + \epsilon_H, \quad (\text{A.3})$$

and denote $\zeta_H \equiv \sigma_H^{(2)}$.

For a field minimally coupled to gravity the momentum equation reads

$$2\dot{H} = -\dot{\varphi}^2, \quad (\text{A.4})$$

and we can write ϵ_H and η_H in terms of the derivatives of the background field

$$\epsilon_H = -\frac{\dot{H}}{H^2} = \frac{1}{2} \left(\frac{\dot{\varphi}}{H} \right)^2 \quad (\text{A.5})$$

$$\eta_H = -\frac{\ddot{\varphi}}{H\dot{\varphi}} + \epsilon_H. \quad (\text{A.6})$$

During slow-roll the Hubble slow-roll parameters form a hierarchy, in which the magnitude of the higher order parameters is ever decreasing.

We can also define another set of slow-roll parameters called the potential slow-roll parameters as [22, 23]

$$\epsilon_V = \frac{1}{2} \left(\frac{V'}{V} \right)^2 \quad (\text{A.7})$$

$$\sigma_V^{(n)} = \frac{1}{V} \left(\frac{V'}{V} \right)^{n-1} \frac{d^{n+1}}{d\varphi^{n+1}} V, \quad (\text{A.8})$$

and denote

$$\eta_V \equiv \sigma_V^{(1)} = \frac{V''}{V} \quad (\text{A.9})$$

$$\zeta_V \equiv \sigma_V^{(2)} = \frac{V' V'''}{V^2} . \quad (\text{A.10})$$

When the slow-roll conditions are satisfied, these parameters also form a hierarchy of decreasing magnitude. During slow-roll the lowest order parameters are approximately equal, $\epsilon_H \approx \epsilon_V$ and $\eta_H \approx \eta_V$.

Thus far we have considered a single-field case. In multi-field inflation, one has to take into account also the curvature of the field space, in order to ensure the frame invariance of the slow-roll expansion. Frame-covariant slow-roll parameters can be defined as [45]

$$\epsilon_{\text{fc}} = \frac{1}{2} \frac{G^{AB} V_{,A} V_{,B}}{V^2} \quad (\text{A.11})$$

$$\sigma_{\text{fc}}^{(n+1)} = G^{AB} \frac{V_{,A} (\sigma_{\text{fc}}^n)_{,B}}{V \sigma_{\text{fc}}^n} , \quad (\text{A.12})$$

and we again denote

$$\eta_{\text{fc}} \equiv \sigma_{\text{fc}}^{(1)} = G^{AB} \frac{V_{,A} (\epsilon_{\text{fc}})_{,B}}{V \epsilon_{\text{fc}}} \quad (\text{A.13})$$

$$\zeta_{\text{fc}} \equiv \sigma_{\text{fc}}^{(2)} = G^{AB} \frac{V_{,A} (\eta_{\text{fc}})_{,B}}{V \eta_{\text{fc}}} . \quad (\text{A.14})$$

Here A and B are field space indices and G_{AB} is the field space metric, the field space vector being $\phi^A = (\phi_1, \phi_2, \dots)$. For a single field the lowest order slow-roll parameters reduce to

$$\epsilon_{\text{fc}} = \frac{1}{2} \left(\frac{V'}{V} \right)^2 = \epsilon_V \quad (\text{A.15})$$

$$\eta_{\text{fc}} = 2 \frac{V''}{V} - 2 \left(\frac{V'}{V} \right)^2 = 2\eta_V - 4\epsilon_V , \quad (\text{A.16})$$

meaning that η_{fc} is not equal to η_V of single-field inflation.



B Gauge transformations

Gauge transformations are completely specified by two functions θ^0 and θ as

$$\begin{aligned}\tilde{\eta} &= \eta + \theta^0(x) \\ \tilde{x}^i &= x^i - \delta^{ia}\theta_{,a}(x) .\end{aligned}\tag{B.1}$$

Under such a coordinate transformation the metric perturbations transform as

$$\begin{aligned}\tilde{A} &= A - \theta^{0'} - \mathcal{H}\theta^0 \\ \tilde{B} &= B + \theta' + \theta^0 \\ \tilde{\psi} &= \psi + \mathcal{H}\theta^0 \\ \tilde{E} &= E + \theta ,\end{aligned}\tag{B.2}$$

while the field perturbation transforms as

$$\delta\tilde{\phi} = \delta\phi - \varphi'\theta^0 .\tag{B.3}$$

In our notation changes in θ^0 correspond to changes in the slicing of the spacetime into spacelike hypersurfaces, while θ changes the threading into timelike curves. It is readily seen that any change in the slicing will change the field perturbation, while changes in the threading leave it invariant.

Let us now consider the spatially flat gauge as an example of a gauge choice. In this gauge $\tilde{\psi}$ is equal to zero, which leads to $\theta^0 = -\mathcal{H}^{-1}\psi$. The field perturbation then becomes

$$\delta\tilde{\phi} = \delta\phi - \frac{\varphi'}{\mathcal{H}}\psi ,\tag{B.4}$$

which is equal to the Sasaki-Mukhanov variable. By using (B.2) and the Einstein equations we can express the equation of motion for the field perturbation as

$$\delta\tilde{\phi}'' + 2\mathcal{H}\delta\tilde{\phi}' - \square\delta\tilde{\phi} + a^2V''(\varphi)\delta\tilde{\phi} = \mathcal{H}^2\left(\frac{\varphi'}{\mathcal{H}}\right)^2\left[2 - \frac{\mathcal{H}'}{\mathcal{H}^2} + 2\frac{\varphi''}{\varphi'\mathcal{H}}\right]\delta\tilde{\phi} .\tag{B.5}$$

We have not specified the threading of the spacetime, because it is not relevant for the field perturbations, but often a convenient choice is $\tilde{E} = 0$.

We can also view the gauge choice $\tilde{\psi} = 0$ as a slicing of the spacetime in which the spatial perturbation $\tilde{\psi}$ is a constant. By choosing a suitable

gauge we can make any quantity a constant on spacelike hypersurfaces. Simple examples are the uniform energy density gauge, in which the energy density is kept constant, and the conformal-Newtonian gauge, in which the shear of the metric vanishes $\tilde{B} = \tilde{E} = 0$.



C Observations

The 2018 results of the Planck satellite indicate a scale-invariant, slightly red tilted spectrum with a small running. We quote here the 95% limits and upper bounds for the observables. The mean value for the amplitude of scalar perturbations is [8]

$$24\pi^2 A_* = \frac{V_*}{\epsilon_* \cos^2 \Theta_*} = 4.97 \times 10^{-7} , \quad (\text{C.1})$$

and the errors are negligible for our purposes. The tensor-to-scalar ratio is constrained from above as [8]

$$r_* < 0.079 , \quad (\text{C.2})$$

while the spectral index is bounded as

$$0.9554 < n_s < 0.9726 . \quad (\text{C.3})$$

The limits on the running of the spectral index read [8]

$$-0.0207 < \alpha_s < 0.0065 . \quad (\text{C.4})$$

Note that we have reanalysed the data of [2] with these limits.

In the case of two-field inflation of Section 6.2, we also need to consider the constraints on isocurvature perturbations and non-Gaussianity. The isocurvature fraction is bounded from above as [8]

$$\beta_{\text{iso}} < 0.38 . \quad (\text{C.5})$$

The transfer angle between the entropy and isocurvature perturbations is constrained as [8]

$$-0.25 < \sin \Theta < 0.23 , \quad (\text{C.6})$$

and the 2015 bounds for non-Gaussianity read [46]

$$-9.2 < f_{NL} < 10.8 . \quad (\text{C.7})$$

At the time of writing the final results for non-Gaussianity were not yet published.

