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Department of Mathematics and Statistics

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Modal Logics and Bisimulation Invariance

Author
Max Sandström

Supervisor
Juha Kontinen

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Tiivistelmä — Referat — Abstract

Logiikan alalla eräs kiinnostava kysymys on minkälaisia malleja saadaan määriteltyä eri logiikoiden kaavoilla. Tätä kutsutaan logiikan ilmaisuvoimaksi. Modaalilogiikoille ilmaisuvoiman rajoja osoittavia lauseita kutsutaan van Benthemin lauseiksi. Modaalilogiikoiden ilmaisuvoimalle keskeinen piirre on bisimulaatioinvarianssi.

Tässä tutkielmassa osoitetaan van Benthemin lause kahdelle logiikalle: modaalilogiikalle ja laajennetulle modaaliselle riippuvuuslogiikalle. Työssä esitellään myös modaalilogiikan ja riippuvuuslogiikan perusteita lyhyesti ennen kunkin logiikan van Benthem lauseen todistusta.

Tässä tutkielmassa modaalilogiikan van Benthemin lause todistetaan peliteorian keinoin, mikä mahdollistaa todistuksen pelkästään äärellisiä malleja käyttäen, toisin kuin alkuperäinen malliteoreettinen todistus. Lause sanoo, että modaalilogiikka on ilmaisuvoimaltaan sama kuin ensimmäisen kertaluvun logiikan bisimulaatioinvariantti fragmentti.

Kaikille logiikoille van Benthemin lauseen suora todistaminen ei onnistu yhtä vaivattomasti. Tällöin käytetään hyväksi välituloksia. Esimerkkinä tästä toimii laajennettu modaalinen riippuvuuslogiikka, jolle lause osoitetaan todistamalla se eri logiikalle, joka puolestaan todistetaan loogisesti ekvivalentiksi laajennetun modaalisen riippuvuuslogiikan kanssa. Tässä työssä van Benthem lause osoitetaan aluksi modaalilogiikalle, jota on laajennettu intuitionistisella disjunktiolla. Tämä logiikka todistetaan loogisesti ekvivalentiksi laajennetun modaalisen riipuvuuslogiikan kanssa. Näille logiikoille van Benthemin lause rajaa logiikan kykenevän määrittelemään alaspäin suljettuja, tyhjän tiimin ominaisuuden omaavia malleja jotka ovat bisimulaation suhteen invariantteja.

Avainsanat — Nyckelord — Keywords

Modaalilogiikka, riippuvuuslogiikka, modaalinen riippuvuuslogiikka, van Benthemin lause

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Chapter 1

Introduction

This thesis work deals with questions regarding the expressive power of modal logics. The reader is assumed to be familiar with mathematical logic to the extent of having a handle on proposition logic and first order logic. The basic concepts of modal logics, particularly the semantics, are briefly explained in the work, and thus prior knowledge of modal logic is not necessary to understand the arguments presented, but probably beneficial. Similarly, dependence logic and team semantics are given a cursory introduction. However some previous experience of the use of games in mathematical logic is recommended, as some of the concepts are glanced over quickly.

In our natural languages we can, and often do, talk of necessities and possibilities. The idea of the logics of such sentences is therefore an old notion, however the formal study of such logics is much more recent. The roots of modern modal logic lie in the early critique of Bertrand Russell's and Alfred Whitehead's Principia Mathematica. Written in 1910, Principia Mathematica was the authors' attempt at finding a common and basic foundation for all mathematics. Given this goal, Clarence Irving Lewis objected to the formalisation of the implication used in the work, claiming it did not correspond to the way conditional statements are used in natural languages. The so called material implication used in Principia Mathematica is the formalisation still commonly used today in mathematical logic, while Lewis suggested also taking the contingent nature of truth values into consideration. He wished for implications to be necessarily binding: that no possible change in circumstances could make the antecedent of the conditional true, but the consequent false. He called this the strict implication, and its definition requires a modal framework for the language.[3] Lewis continued developing this modal theory into the axiomatic systems he called S1 through S5, which birthed an abundance of research in the syntax of modal logics.

The semantic basis of modal logic however, which is at the core of this work, was not formalised until much later. The lack of a rigorous meaning of statements being true caused some philosophers, Robert Quine chief among them, to doubt the validity of modal arguments especially within metaphysics. To mend this rift, Saul Kripke proposed in 1959 the semantics of modal logic conceptualised as possible worlds and simultaneously proving a completeness theorem for the logic.[3] Modal semantics are hence often called Kripke semantics, and the logical structures created by the semantics are called Kripke structures. These results eventually led to some interesting results regarding the expressive power of modal logics. Johan van Benthem proved in 1976 the van Benthem theorem, which states that modal logic is in fact exactly the bisimulation invariant fragment of first order logic.[1] In this work we will show van Benthem's result using a game theoretical approach, differing from van Benthem's original model theoretical proof. This method of proving the theorem is more straight forward in the technical details, and it broadens the scope of the proof by not requiring the use of infinite structures when showing the claim for finite ones.

The third chapter of this thesis work proves the analogous theorem for an expansion of modal logic. The logic considered in this chapter is extends modal logic by the so called dependence atom and team semantics; two terms adopted from dependence logic. Dependence logic was discovered by Jouko Väänänen in 2007 with the aim to capture the essence of different notions of dependence central to scientific language in a formal framework. Dependence logic requires team semantics to interpret it. Väänänen defined two extensions of modal logic: modal dependence logic and extended modal dependence logic.[6]

Modal logic has found applications within theoretical computer science in the field of software modelling. In software modelling, programs are often considered to be state processes, which are analogous to the possible worlds of Kripke structures thus ensuring a snug match. Applications of dependence logic range from linguistics via database theory to quantum physics.

Chapter 2

Modal Logic

Modal logic is the logic of possible worlds; the logic of possibilities and necessities. Despite the seemingly dense air of those concepts, it turns out the logic formalising them is limited in its power. Modal logic can be shown to be as expressive as the so-called bisimulation invariant part of first-order logic. This is the essence captured by the van Benthem Theorem, which will be proven in this chapter using a game theoretical approach. But first, a brief run-through of the basic concepts in modal logic. This first chapter follows the outline of Martin Otto in [7], while expanding on the arguments presented.

2.1 The Fundamentals of Modal Logic

The usual way to define modal logic in mathematical logic is as an extension of propositional logic, with the definition of structures in the following vein.

Definition 2.1 (Kripke structures)**.** Let W be a non-empty set, E be a binary relation symbol, and Φ be a collection of proposition symbols. Suppose $V\colon \Phi \to \mathcal{P}(W)$ defines a value function. Then $\mathcal{M} = (W, E^{\mathcal{M}}, V)$ is a *Kripke structure.*

The value function gives for each proposition in Φ the worlds in which the proposition is true. The rest of the semantics is defined as in classical logic with the addition of the modal operators necessity, $\Box$, and possibility, $\Diamond$. Using this definition one would denote the modal logic over the vocabulary Φ by $\mathrm{ML}(\Phi)$. However for the purposes of this approach to the van Benthem theorem we will use a definition based on predicates like in first order logic.

Definition 2.2 (Predicative Kripke structures)**.** Let W be a non-empty set, E be a binary relation symbol, and $\mathbf{P}$ be finitely many unary predicate symbols (the vocabulary of our structure). Then $\mathcal{M} = (W, E^{\mathcal{M}}, \mathbf{P}^{\mathcal{M}})$ is a *predicative Kripke structure.*

Under this definition the set W is the set of possible worlds, E is the attainability relation and the predicates $\mathbf{P}$ represent the propositions in the structure. We write the set of worlds attainable from a world $m \in W$ as $E^{\mathcal{M}}[m]$. We denote modal logic over the vocabulary $(E; \mathbf{P})$ by $\mathrm{ML}(\mathbf{P})$.

In this particular notation we refer to modal logics with conventions from first order logic. Notice, however, that constant and function symbols are disregarded, as they do not correspond to any aspect of modal logic as it is usually defined.

Definition 2.3 (Formulae). An *atomic formula* of ML is Px, where x is a variable and $P \in \mathbf{P}$. *Formulae* of ML are defined recursively as being atomic formulae or compounded from other formulae using negation, $\neg$; conjunction, $\wedge$; disjunction, $\vee$; implication, $\rightarrow$; or the modal quantifiers for necessity, $\Box$, and possibility, $\Diamond$.

Let $P \in \mathbf{P}$ be a predicate, x a variable, φ and ψ be modal logic formulae. The semantics are defined at particular worlds as follows

1. $\mathcal{M}, m \models Px$ if and only if $m \in P^{\mathcal{M}}$,
2. $\mathcal{M}, m \models \neg\varphi$ if and only if $\mathcal{M}, m \not\models \varphi$,
3. $\mathcal{M}, m \models \varphi \wedge \psi$ if and only if $\mathcal{M}, m \models \varphi$ and $\mathcal{M}, m \models \psi$,
4. $\mathcal{M}, m \models \varphi \vee \psi$ if and only if $\mathcal{M}, m \models \varphi$ or $\mathcal{M}, m \models \psi$,
5. $\mathcal{M}, m \models \varphi \rightarrow \psi$ if and only if $\mathcal{M}, m \not\models \varphi$ or $\mathcal{M}, m \models \psi$,
6. $\mathcal{M}, m \models \Box\varphi$ if and only if $\mathcal{M}, m' \models \varphi$ for all $m' \in E^{\mathcal{M}}[m]$, and
7. $\mathcal{M}, m \models \Diamond\varphi$ if and only if $\mathcal{M}, m' \models \varphi$ for some $m' \in E^{\mathcal{M}}[m]$,

where $\mathcal{M}$ is a Kripke model, m and m' are worlds of $\mathcal{M}$.

For a given modal logic formula φ we denote the class of pointed Kripke structures that φ defines by $\mathrm{Mod}(\varphi)$.

Definition 2.4 (Modal depth). Let φ be a ML-formula. The modal depth of φ, $\mathrm{md}(\varphi)$, is defined as follows:

1. if $\varphi = Px$, or $\varphi = \neg Px$, for some variable x, $\mathrm{md}(\varphi) = 0$;
2. if $\varphi = \psi \wedge \theta$, $\varphi = \psi \vee \theta$, or $\varphi = \psi \rightarrow \theta$ for some ML-formulae ψ and θ, $\mathrm{md}(\varphi) = \max(\mathrm{md}(\varphi), \mathrm{md}(\theta))$;
3. if $\varphi = \Diamond\psi$, or $\varphi = \Box\psi$, for some ML-formula ψ, $\mathrm{md}(\varphi) = \mathrm{md}(\psi) + 1$.

The modal logic restricted to formulas of modal depth $\ell \in \mathbb{N}$ is denoted by ML_ℓ.

Definition 2.5 (Standard translation to FO)**.** The *standard translation* of a ML-formula φ to first order logic, denoted $[\varphi]^*$, is as follows:

$$\begin{gathered}[P(x)]^* = P(x),\\ [\neg\psi]^* = \neg[\psi]^*,\\ [\psi \wedge \theta]^* = [\psi]^* \wedge [\theta]^*,\\ [\psi \vee \theta]^* = [\psi]^* \vee [\theta]^*,\\ [\psi \rightarrow \theta]^* = [\psi]^* \rightarrow [\theta]^*,\\ [\Box\psi]^*(x) = \forall y(Exy \rightarrow [\psi]^*(y/x)), \text{ and}\\ [\Diamond\psi]^*(x) = \exists y(Exy \wedge \neg[\psi]^*(y/x)).\end{gathered}$$

Since this translation uses only two variables, which can be cycled repeatedly in nested modal quantifiers, this definition essentially embeds modal logic into FO^2. The remainder of this chapter is used to find the precise fragment of FO that corresponds to modal logic.

Definition 2.6 (ℓ-neighbourhood)**.** Suppose $\mathcal{M}$ is a Kripke structure with a distinguished element m. Given a natural number ℓ, the *ℓ-neighbourhood* of m is the set $U^\ell(m)$ of all nodes reachable from m on E-paths of length at most ℓ.

Definition 2.7 (Tree structures)**.** A Kripke structure with a distinguished element, $\mathcal{M}, m$, is called a *tree structure* if $E^{\mathcal{M}}$ is acyclic and every other world can be reached from m by a unique path. A branch of the tree is a path, which cannot be expanded further and the worlds at the end of branches are called leaves. The depth of a tree structure is the E-length of its longest branch. A Kripke structure $\mathcal{M}, m$ is said to be ℓ-locally a tree structure if the restriction $\mathcal{M} \restriction U^\ell(m), m$ is a tree structure.

Definition 2.8 (Property of Kripke structures)**.** A *property of pointed Kripke structures* is a class of Kripke structures closed under isomorphisms.

The following concept expresses the idea that whether a structure has a property can be deduced from a restriction of the structure.

Definition 2.9 (ℓ-locality)**.** A property $\mathcal{K}$ of Kripke structures is said to be *ℓ-local*, for $\ell \in \mathbb{N}$, if for every Kripke structure $\mathcal{M}, m$

$$(\mathcal{M}, m) \in \mathcal{K} \Leftrightarrow (\mathcal{M} \restriction U^\ell(m), m) \in \mathcal{K}.$$

If the property $\mathcal{K}$ is defined by a formula φ, ℓ-locality can be expressed as the equivalence

$$\mathcal{M}, m \models \varphi \iff \mathcal{M} \restriction U^\ell(m), m \models \varphi.$$

2.2 Equivalences and Bisimulation

In order to better grasp the upcoming concepts, we will go through some more familiar relations from first order logic. The upcoming bisimulation relations are similar to equivalence, but more limited in range, giving them some unique features.

In the following definitions and lemmas we assume that $\mathcal{M} = (W, E^{\mathcal{M}}, \mathbf{P}^{\mathcal{M}})$ and $\mathcal{M}' = (W', E^{\mathcal{M}'}, \mathbf{P}^{\mathcal{M}'})$ are Kripke structures.

Definition 2.10 (Quantifier depth)**.** The *quantifier depth* of a FO-formula φ, denoted $\mathrm{qd}(\varphi)$, is defined recursively as follows:

1. if $\varphi = Px$, or $\varphi = \neg Px$, for some variable x, $\mathrm{qd}(\varphi) = 0$;
2. if $\varphi = \psi \wedge \theta$, $\varphi = \psi \vee \theta$, or $\varphi = \psi \to \theta$ for some FO-formulae ψ and θ, $\mathrm{qd}(\varphi) = \max(\mathrm{qd}(\varphi), \mathrm{qd}(\theta))$;
3. if $\varphi = \exists\psi$, or $\varphi = \forall\psi$, for some FO-formula ψ, $\mathrm{qd}(\varphi) = \mathrm{qd}(\psi) + 1$.

Definition 2.11 (FO-equivalence)**.** Two structures $\mathcal{M}, m$ and $\mathcal{M}', m'$ are FO-*equivalent*, denoted $\mathcal{M}, m \equiv^{\mathrm{FO}} \mathcal{M}', m'$, if $\mathcal{M} \models_{[m/x]} \varphi$ if and only if $\mathcal{M}' \models_{[m'/x]} \varphi$, for all FO-formulae $\varphi(x)$. The two structures are FO-*equivalent to depth* n, for some $n \in \mathbb{N}$ if $\mathcal{M} \models_{[m/x]} \varphi$ if and only if $\mathcal{M}' \models_{[m'/x]} \varphi$, for all FO-formulas φ such that $\mathrm{qd}(\varphi) \leq n$. This is denoted by $\mathcal{M}, m \equiv^{\mathrm{FO}}_n \mathcal{M}', m'$.

Definition 2.12 (Partial isomorphism)**.** A partial function $f \colon \mathcal{M} \to \mathcal{M}'$ is a *partial isomorphism* if the function $g \colon \mathcal{M} \restriction \mathrm{Dom}(f) \to \mathcal{M}' \restriction \mathrm{Rng}(f)$ defined by $g(a) = f(a)$ is an isomorphism.

Definition 2.13 (Partially isomorphic structures)**.** Let k be a natural number. Two Kripke structures $\mathcal{M}, m$ and $\mathcal{M}', m'$ are k *partially isomorphic*, written $\mathcal{M}, m \cong_k \mathcal{M}', m'$, if there exists a sequence $(I_n)_{n \leq k}$ of non-empty sets of partial isomorphisms f, for which $(m, m') \subseteq f$, between $\mathcal{M}$ and $\mathcal{M}'$, such that it has the following properties:

1. For every partial isomorphism $f \in I_{n+1}$ and $w \in W$ there is a partial isomorphism $g \in I_n$, such that $f \subseteq g$ and $w \in \mathrm{Dom}(g)$,
2. For every partial isomorphism $f \in I_{n+1}$ and $w' \in W'$ there exists a partial isomorphism $g \in I_n$, such that $f \subseteq g$ and $w' \in \mathrm{Rng}(g)$.

A classic result in model theory states that for two structures $\mathcal{M}$ and $\mathcal{M}'$, $\mathcal{M}, m \equiv_n \mathcal{M}', m'$ if and only if $\mathcal{M}, m \cong_n \mathcal{M}', m'$ and, furthermore, both claims are equivalent with the duplicator having a winning strategy in the n-round Ehrenfeucht-Fraïsse game. For more details on this topic see for example [5].

Now we shall define similar concepts for modal logic and show an equivalence result of a similar nature. This requires us to define firstly equivalence in modal logic, secondly the bisimulation relation, and thirdly a game that encapsulates the essence of this relation.

Definition 2.14 (ML-equivalence)**.** Two Kripke structures $\mathcal{M}, m$ and $\mathcal{M}', m'$ are ML-*equivalent*, denoted $\mathcal{M}, m \equiv^{\mathrm{ML}} \mathcal{M}', m'$, if $\mathcal{M}, m \models \varphi$ if and only if $\mathcal{M}', m' \models \varphi$, for all ML-formulas φ. The two Kripke structures are ML-*equivalent to depth* n, for some $n \in \mathbb{N}$ if $\mathcal{M}, m \models \varphi$ if and only if $\mathcal{M}', m' \models \varphi$, for all ML-formulas φ such that $\mathrm{md}(\varphi) \leq n$. This is denoted by $\mathcal{M}, m \equiv^{\mathrm{ML}}_n \mathcal{M}', m'$.

Definition 2.15 (Atomic correspondence)**.** Two worlds $w \in W$ and $w' \in W'$ are said to be in *atomic correspondence* if they satisfy the same atomic formulae, i.e.

$$M, w \models Px \Leftrightarrow M', w' \models Px$$

for all $P \in \mathbf{P}$.

Definition 2.16 (Bisimulation)**.** A *bisimulation* between $\mathcal{M}$ and $\mathcal{M}'$ is a binary relation $\sim \subset W \times W'$ such that, for $w \in W$ and $w' \in W'$, if $w \sim w'$, then the following hold:

1. w and w' are in atomic correspondence,
2. if $(w, v) \in E^{\mathcal{M}}$ for some world $v \in W$, then there exists a world $v' \in W'$ such that $v \sim v'$ and $(w', v') \in E^{\mathcal{M}'}$,
3. if $(w', v') \in E^{\mathcal{M}'}$ for some world $v' \in W'$, then there exists a world $v \in W$ such that $v' \sim v$ and $(w, v) \in E^{\mathcal{M}}$.

When a bisimulation exists between $\mathcal{M}, w$ and $\mathcal{M}', w'$ they are said to be *bisimilar*.

We define the bounded variant of the bisimulation by recursion. The resulting relation is similar, despite the differences in their definition.

Definition 2.17 (k-bisimulation)**.** Let $w \in W$ and $w' \in W'$ be worlds and $k \in \mathbb{N}$. *k-bisimulations* between $\mathcal{M}$ and $\mathcal{M}'$ are defined recursively as follows:

$$\begin{gathered}\mathcal{M}, w \sim_0 \mathcal{M}', w' \text{ if and only if } w \text{ and } w' \text{ are in atomic correspondence,}\\ \mathcal{M}, w \sim_{k+1} \mathcal{M}', w' \text{ if and only if the following hold:}\\ \mathcal{M}, w \sim_0 \mathcal{M}', w',\\ \text{for every } v \in E^{\mathcal{M}}[\{w\}] \text{ there exists } v' \in E^{\mathcal{M}'}[\{w'\}] \text{ such that } v \sim_k v', \text{ and}\\ \text{for every } v' \in E^{\mathcal{M}}[\{w'\}] \text{ there exists } v \in E^{\mathcal{M}}[\{w\}] \text{ such that } v \sim_k v'.\end{gathered}$$

If $\mathcal{M}, m \sim_k \mathcal{M}', m'$ they are said to be *k-bisimilar*.

2.3 Bisimulation as a Game

Bisimulation can alternatively be defined by a zero-sum game of perfect information played by two players, denoted **I** and **II** in this text. During each round of the game each player moves a pebble in one of the two Kripke structures starting from either of the distinguished elements. Player **I** moves the pebble of his choice forward along an E-edge, thereby challenging player **II**, who responds by moving the other pebble forward along an E-edge in the other model. The victory condition for player **II** is that she maintains atomic equivalence; she loses if the played worlds of a given turn do not agree on monadic predicates or she runs out of worlds connected by E.

Definition 2.18 (Bisimulation game)**.** Assume $\mathcal{M}$ and $\mathcal{M}'$ are two Kripke structures, with the same vocabulary $\mathbf{P}$ and distinguished elements m and m' respectively. The *bisimulation game* $BG(\mathcal{M}, m, \mathcal{M}', m)$ over $\mathcal{M}$ and $\mathcal{M}'$ is a game, where the winning condition consists of plays $x_0, y_0, x_1, y_1, ...$ such that player **II** has adhered to the following rules:

1. If $x_0 = m$, then $y_0 = m'$, and if $x_0 = m'$, then $y_0 = m$. In either case, m and m' have atomic correspondence.
2. If $x_n = m_n$, where $m_n \in W$ and $(m_{n-1}, m_n) \in E^{\mathcal{M}}$ for m_{n-1} from the previous round, then $y_n = m'_n$, where $m'_n \in W'$ and $(m'_{n-1}, m'_n) \in E^{\mathcal{M}'}$. Otherwise, if $x_n = m'_n$, where $m'_n \in W'$ and $(m'_{n-1}, m'_n) \in E^{\mathcal{M}'}$ for m'_{n-1} from the previous round, then $y_n = m_n$, where $m_n \in W$ and $(m_{n-1}, m_n) \in E^{\mathcal{M}}$. Additionally m'_n and m_n have atomic correspondence.

Both players continue playing worlds as long as there are E-edges from previously played worlds.

In a similar vein, k-bisimulations can be defined through a k-round bisimulation game.

Definition 2.19 (Winning strategy)**.** A *strategy* for player **II** of a bisimulation game $BG(\mathcal{M}, m, \mathcal{M}', m')$ is an infinite sequence

$$\sigma = (\sigma_0, \sigma_1, ...)$$

of functions $\sigma_i \colon (W \cup W')^i \to W \cup W'$ such that $\sigma_n(v_0, ..., v_{n-1}, w) \in W'$ for all $w \in W$ and $\sigma(v_0, ..., v_{n-1}, w') \in W$ for all $w' \in W'$, where $v_0, ..., v_{n-1} \in W \cup W'$ and $n \in \mathbb{N}$. A strategy is said to be a *winning strategy* if all possible plays $x_0, \sigma_0(x_0), x_1, \sigma_1(x_0, x_1), ...$ are contained in the winning condition of the game. In other words, a winning strategy is a way for player **II** to decide what to play that maintains atomic correspondence and an unbroken path to the distinguished worlds.

Before showing how these concepts are related to each other we shall define Hintikka formulae, which are modal logic formulae that crystallise the information about a world in a Kripke structure.

Definition 2.20 (Hintikka formula)**.** Assume $\mathbf{P} = \{P_i \mid i \in I\}$ for some finite index set I and let $J \subseteq I$ be index sets such that $\mathcal{M}, m \models P_i x$ if and only if $i \in J$. The *Hintikka formula* for $\mathcal{M}, m$ of depth n, $\chi^n_{[\mathcal{M},m]}$, for some $n \in \mathbb{N}$, is defined recursively in the following manner:

$$\chi^0_{[\mathcal{M},m]} = \bigwedge_{i \in J} P_i x \wedge \bigwedge_{i \in I \setminus J} \neg P_i x;$$

$$\chi^{n+1}_{[\mathcal{M},m]} = \chi^0_{[\mathcal{M},m]} \wedge \bigwedge_{u \in E^{\mathcal{M}}[m]} \Diamond \chi^n_{[\mathcal{M},u]} \wedge \Box \bigvee_{u \in E^{\mathcal{M}}[m]} \chi^n_{[\mathcal{M},u]}.$$

An immediate result from this definition is that, for every Kripke structure $\mathcal{M}, m$, $\mathcal{M}, m \models \chi^n_{[\mathcal{M},m]}$, for all $n \in \mathbb{N}$.

Lemma 2.21. *There are a finite number of Hintikka formulae of depth k up to equivalence, for every $k \in \mathbb{N}$.*

Proof. We will prove the claim by induction on the depth of the Hintikka formula. By simple combinatorics there are $2^{|\mathbf{P}|}$ possible combinations of the predicates in $\mathbf{P}$, which is finite since $\mathbf{P}$ is finite. Hence the number of Hintikka formulae of depth 0 is finite.

Suppose there are a finite number of Hintikka formulae of depth k up to equivalence. Now consider a Hintikka formula of depth $k+1$. As we already showed, there are a finite number of Hintikka formulae of depth 0. For the conjunction and disjunction we have a selection of Hintikka formulae of depth k, of which there is a finite number by the induction hypothesis. Multiplying these numbers together gives the amount of Hintikka formulae of depth $k+1$ which is finite.

By the induction principle, a Hintikka formula of any depth is equivalent to a finite formula and there are a finite number of Hintikka formulae of that depth. $\square$

Now we are ready to connect these concepts together: that the game actually is an alternative definition for the bisimulation relation and that they are equivalent with claims regarding modal logic equivalence and Hintikka formulae.

Lemma 2.22. *Suppose $\mathcal{M}, m$ and $\mathcal{M}', m'$ are Kripke structures, and let $\ell \in \mathbb{N}$. The following are equivalent:*

1. $\mathcal{M}, m \sim_\ell \mathcal{M}', m'$;

2. *Player **II** has a winning strategy in the ℓ-round bisimulation game over $\mathcal{M}, m$ and $\mathcal{M}', m'$;*

3. *$\mathcal{M}, m \equiv_\ell^{\mathrm{ML}} \mathcal{M}', m'$;*

4. *$\mathcal{M}', m' \models \chi^\ell_{[\mathcal{M},m]}$.*

Proof. 1. $\Rightarrow$ 2.: Assume $\mathcal{M}, m \sim_\ell \mathcal{M}', m'$. Consider the ℓ-round bisimulation game over $\mathcal{M}, m$ and $\mathcal{M}', m'$. We will show by induction on the rounds of the game that the bisimulation relation defines a winning strategy for player **II**. Basis case: in the first round player **I** plays either m or m', and player **II** plays the other one. Since the models at those worlds are bisimilar, the worlds have atomic correspondence. Hence player **II** has upheld the winning conditions.

Induction case: assume player **II** has adhered to the winning conditions for the previous $n < \ell$ rounds. Now we can assume that on round $n+1$ player **I** can play either a world from W or one from W', since otherwise the game is over and player **II** is victorious. Suppose player **I** plays a world $m_{n+1} \in W$, such that $(m_n, m_{n+1}) \in E^{\mathcal{M}}$, for a world m_n played in the previous round. Since player **II** has been playing according to the bisimulation relation the other world played during the previous round $m'_n \in W'$ is such that $\mathcal{M}, m_n \sim_{\ell-(n-1)} \mathcal{M}', m'_n$. Now, as $n+1 \leq \ell$, the bisimulation relation states there exists a world $m'_{n+1} \in W'$ such that $(m'_n, m'_{n+1}) \in E^{\mathcal{M}'}$ and the worlds m_{n+1} and m'_{n+1} have atomic correspondence. Hence player **II** can play m'_{n+1} and uphold the winning condition. An analogous argument holds for when player **I** plays a world from W', due to the third condition of the ℓ-bisimulation. Therefore by the induction principle if $\mathcal{M}, m \sim_\ell \mathcal{M}', m'$, then player **II** has a winning strategy in the ℓ-round $BG(\mathcal{M}, m, \mathcal{M}', m')$.

2. $\Rightarrow$ 3.: Assume player **II** has a winning strategy in the ℓ-round bisimulation game over $\mathcal{M}, m$ and $\mathcal{M}', m'$. Let φ be a ML_ℓ-formula. We will show by induction on the structure of φ that if player **II** has a winning strategy in the ℓ-round bisimulation game over $\mathcal{M}, w$ and $\mathcal{M}', w'$, then $\mathcal{M}, w \models \varphi$ if and only if $\mathcal{M}', w' \models \varphi$, for all worlds $w \in W$ and $w' \in W'$. Assume then w and w' are such that player **II** has her winning strategy. Base step: Suppose $\varphi = Px$ or $\varphi = \neg Px$, for some $P \in \mathbf{P}$. Now $\mathcal{M}, w \models \varphi$ if and only if $\mathcal{M}', w' \models \varphi$, as the winning condition of player **II** ensures that the initial worlds have atomic correspondence.

Induction step: This step breaks down into several cases.

1. $\varphi = \neg\psi$ for some ML_ℓ-formula ψ. Assume as the induction hypothesis that $\mathcal{M}, w \models \psi \Leftrightarrow \mathcal{M}', w' \models \psi$ if player **II** has a winning strategy. Now the induction hypothesis is equivalent with $\mathcal{M}, w \not\models \psi \Leftrightarrow \mathcal{M}', w' \not\models \psi$ if player **II** has a winning strategy, and hence $\mathcal{M}, w \models \neg\psi$ if and only if $\mathcal{M}', w' \models \neg\psi$.

2. $\varphi = \psi \wedge \theta$ for some ML_ℓ-formulae ψ and θ. For the induction hypothesis suppose that $\mathcal{M}, w \models \psi \Leftrightarrow \mathcal{M}', w' \models \psi$ and $\mathcal{M}, w \models \theta \Leftrightarrow \mathcal{M}', w' \models \theta$, if player **II** has a winning strategy. Seeing as this is the case, by the definition of a ML-formula it follows that $\mathcal{M}, w \models \psi \wedge \theta$ if and only if $\mathcal{M}', w' \models \psi \wedge \theta$.

3. $\varphi = \psi \vee \theta$ for some ML_ℓ-formulae ψ and θ. Assume for the induction hypothesis that $\mathcal{M}, w \models \psi \Leftrightarrow \mathcal{M}', w' \models \psi$ and $\mathcal{M}, w \models \theta \Leftrightarrow \mathcal{M}', w' \models \theta$, when player **II** has a winning strategy in the bisimulation game. Since we assumed so, by the definition of the disjunction it also holds that $\mathcal{M}, w \models \psi \vee \theta$ if and only if $\mathcal{M}', w' \models \psi \vee \theta$.

4. $\varphi = \psi \rightarrow \theta$ for some ML_ℓ-formulae ψ and θ. The induction hypothesis is that $\mathcal{M}, w \models \psi \Leftrightarrow \mathcal{M}', w' \models \psi$ and $\mathcal{M}, w \models \theta \Leftrightarrow \mathcal{M}', w' \models \theta$, if player **II** has a winning strategy. Once again, since we assumed the winning strategy to exist, by the definition of the semantics of ML-formulae it holds that $\mathcal{M}, w \models \psi \rightarrow \theta$ if and only if $\mathcal{M}', w' \models \psi \rightarrow \theta$.

5. $\varphi = \Diamond\psi$ for some $\mathrm{ML}_{\ell-1}$-formula ψ. As the induction hypothesis assume that if player **II** has a winning strategy in the $\ell - 1$-round bisimulation game over $\mathcal{M}, v$ and $\mathcal{M}', v'$, then $\mathcal{M}, v \models \psi \Leftrightarrow \mathcal{M}', v' \models \psi$, for all $v \in W$ and $v' \in W'$. Now assume $\mathcal{M}, w \models \Diamond\psi$. By the definition of the possibility quantifier, there exists a world $v \in E^{\mathcal{M}}[w]$, such that $\mathcal{M}, v \models \psi$. Since player **II** has a winning strategy in the ℓ-round bisimulation game over $\mathcal{M}, w$ and $\mathcal{M}', w'$, we let $v' \in E^{\mathcal{M}'}[w']$ be the world given by the winning strategy. Now player **II** has a winning strategy in the $\ell - 1$-round bisimulation game over $\mathcal{M}, v$ and $\mathcal{M}', v'$, and therefore it follows from the induction hypothesis that $\mathcal{M}', v' \models \psi$. Hence $\mathcal{M}', w' \models \Diamond\psi$, by definition. The other direction is analogous.

6. $\varphi = \Box\psi$ for some $\mathrm{ML}_{\ell-1}$-formula ψ. Suppose player **II** has a winning strategy in the $\ell - 1$-round bisimulation game over $\mathcal{M}, v$ and $\mathcal{M}', v'$, then $\mathcal{M}, v \models \psi \Leftrightarrow \mathcal{M}', v' \models \psi$. Suppose $\mathcal{M}, w \models \Box\psi$. If $E^{\mathcal{M}}[w]$ is empty we are finished, since then $E^{\mathcal{M}'}[w']$ is also empty, or otherwise player **II** would lose on the second round of the game. Therefore we assume $E^{\mathcal{M}}[w]$ is not empty. Suppose $v \in E^{\mathcal{M}}[w]$ is an arbitrary element. Now $\mathcal{M}, v \models \psi$, and due to player **II**'s winning strategy there also exists a world $v' \in E^{\mathcal{M}'}[w']$. Since v and v' can be played on the second round of the $BG(\mathcal{M}, w, \mathcal{M}', w')$, player **II** has a winning strategy in the $\ell - 1$-round bisimulation game over $\mathcal{M}, v$ and $\mathcal{M}', v'$. By the induction hypothesis then $\mathcal{M}', v' \models \psi$. This argument holds for all $v \in E^{\mathcal{M}}[w]$ and hence $\mathcal{M}', w' \models \Box\psi$. Analogously for the other direction.

By the induction principle $\mathcal{M}, w \models \varphi \Leftrightarrow \mathcal{M}', w' \models \varphi$ for all $w \in W$, $w' \in W'$, and $\varphi \in \mathrm{ML}_\ell(\mathbf{P})$, if player **II** has a winning strategy in the ℓ-round bisimulation game over

$\mathcal{M}, w$ and $\mathcal{M}', w'$. In particular $\mathcal{M}, m \models \varphi$ if and only if $\mathcal{M}', m' \models \varphi$.

3. $\Rightarrow$ 4.: Assume $\mathcal{M}, m \equiv_\ell \mathcal{M}', m'$. Since $\mathrm{md}(\chi^\ell_{[\mathcal{M},m]}) = \ell$ and $\mathcal{M}, m \models \chi^\ell_{[\mathcal{M},m]}$, by the definition of modal equivalence it follows that $\mathcal{M}', m' \models \chi^\ell_{[\mathcal{M},m]}$.

4. $\Rightarrow$ 1.: Assume $\mathcal{M}', m' \models \chi^\ell_{[\mathcal{M},m]}$. We will show that if a world $w' \in W'$ is such that $\mathcal{M}', w' \models \chi^k_{[\mathcal{M},m]}$, then $\mathcal{M}, m \sim_k \mathcal{M}', w'$, for all $k \leq \ell$. Basis case: suppose $\mathcal{M}', w' \models \chi^0_{[\mathcal{M},m]}$. Now from the construction of the Hintikka formula it follows immediately that $\mathcal{M}', w'$ satisfies exactly the same predicates as $\mathcal{M}, m$, and hence they have atomic correspondence. In other words $\mathcal{M}, m \sim_0 \mathcal{M}', w'$.

Induction case: $k = i + 1$ for some $0 \leq i < \ell$. For the induction hypothesis suppose that if $\mathcal{M}', v' \models \chi^i_{[\mathcal{M},v]}$, then $\mathcal{M}, v \sim_i \mathcal{M}', v'$, for all $v \in W$ and $v' \in W'$. Assume that $\mathcal{M}', w' \models \chi^{i+1}_{[\mathcal{M},m]}$. Now $\mathcal{M}', w' \models \chi^0_{[\mathcal{M},m]}$, since $\chi^0_{[\mathcal{M},m]}$ is one of the conjuncts of $\chi^{i+1}_{[\mathcal{M},m]}$. As explained previously this means $\mathcal{M}', w'$ and $\mathcal{M}, m$ have atomic correspondence. The second conjunct of $\chi^{i+1}_{[\mathcal{M},m]}$ is

$$\bigwedge_{u \in E^{\mathcal{M}}[m]} \Diamond \chi^i_{[\mathcal{M},u]},$$

a conjunction in its own right. Now $\mathcal{M}', w'$ satisfies this conjunction and thereby it also satisfies each of the conjuncts, i.e. $\mathcal{M}', w' \models \Diamond \chi^i_{[\mathcal{M},u]}$ for all $u \in E^{\mathcal{M}}[m]$. This means by the definition of the semantics of modal logic that there exists a world $u' \in E^{\mathcal{M}'}[w']$ such that $\mathcal{M}', u' \models \chi^i_{[\mathcal{M},u]}$, for every $u \in E^{\mathcal{M}}[m]$. It follows now from the induction hypothesis that for every $u \in E^{\mathcal{M}}[m]$ there exists a world $u' \in E^{\mathcal{M}'}[w']$ such that $\mathcal{M}, u \sim_i \mathcal{M}', u'$. The third and final conjunct of $\chi^{i+1}_{[\mathcal{M},m]}$ is a disjunction quantified by a necessity quantifier

$$\Box \bigvee_{u \in E^{\mathcal{M}}[m]} \chi^i_{[\mathcal{M},u]}.$$

That model $\mathcal{M}, m$ satisfies this formula is equivalent to $\mathcal{M}', u' \models \bigvee_{u \in E^{\mathcal{M}}[m]} \chi^i_{[\mathcal{M},u]}$ for every $u' \in E^{\mathcal{M}'}[w']$, by the definition of the modal semantics. Since the disjunction runs the gamut of worlds attainable from $\mathcal{M}, m$, the previous claim can be expressed as for every $u' \in E^{\mathcal{M}'}[w']$ there exists a world $u \in E^{\mathcal{M}}[m]$ such that $\mathcal{M}', u' \models \chi^i_{[\mathcal{M},u]}$. By the induction hypothesis this implies that for every $u' \in E^{\mathcal{M}'}[w']$ there exists a world $u \in E^{\mathcal{M}}[m]$ such that $\mathcal{M}, u \sim_i \mathcal{M}', u'$. Now we have shown all three conditions for the bounded bisimulation $\mathcal{M}, m \sim_{i+1} \mathcal{M}', w'$, hence by the induction principle $\mathcal{M}, m \sim_k \mathcal{M}', w'$ holds for all integers $0 \leq k \leq \ell$, if $\mathcal{M}', w' \models \chi^k_{[\mathcal{M},m]}$. Since we assumed that $\mathcal{M}', m' \models \chi^\ell_{[\mathcal{M},m]}$, it now follows that $\mathcal{M}, m \sim_\ell \mathcal{M}', m'$. $\Box$

One of the consequences of this result is that the two definitions for the bounded bisimulation are equivalent. The game theoretical definition will be used for the remainder of this chapter.

This result has the following corollary when one considers the bounded bisimulation as an equivalence relation. That is an apt interpretation, since the identity function defines a winning strategy for player **II** in the game $BG(\mathcal{M}, m, \mathcal{M}, m)$, the game is symmetric by definition $BG(\mathcal{M}, m, \mathcal{M}', m') = BG(\mathcal{M}', m', \mathcal{M}, m)$, and if $M, m \sim_\ell \mathcal{M}', m'$ and $\mathcal{M}', m' \sim_\ell \mathcal{M}'', m''$ then the compound function of the strategies for player **II** is a winning strategy in the game $BG(\mathcal{M}, m, \mathcal{M}'', m'')$.

Corollary 2.23. *Let $\ell \in \mathbb{N}$ and $\mathbf{P}$ be finite. Suppose the relation $\sim_\ell$ is a bounded bisimulation between Kripke structures of a similar propositional type. The index of $\sim_\ell$ is finite.*

Proof. By Lemma 2.22. all Kripke models within a equivalence class satisfy the same Hintikka formula. By Lemma 2.21. there are only a finite number of Hintikka formulae of depth ℓ. Hence the index of the equivalence relation $\sim_\ell$ is finite. □

2.4 Some Features of Bisimulation

Definition 2.24 (Disjoint sum)**.** The *disjoint sum* $\mathcal{M} + \mathcal{M}'$ is a Kripke-structure with the universe $W \sqcup W'$, where $E^{\mathcal{M}+\mathcal{M}'} = \{((w, L), (v, L)) \mid$ *if* $(w, v) \in E^L$*, where* $L = \mathcal{M}$ *or* $L = \mathcal{M}'\}$, and $P^{\mathcal{M}+\mathcal{M}'} = \{(w, L) \mid$ *if* $w \in P^L$*, where* $L = \mathcal{M}$ *or* $L = \mathcal{M}'\}$.

Lemma 2.25. *If $\mathcal{M}, \mathcal{M}'$ and $\mathcal{M}''$ are Kripke structures, then $\mathcal{M}, m \sim \mathcal{M}', m'$ if and only if $\mathcal{M} + \mathcal{M}'', (m, \mathcal{M}) \sim \mathcal{M}', m'$.*

Proof. Suppose $\mathcal{M}, \mathcal{M}'$ and $\mathcal{M}''$ are Kripke-structures, and let $m \in W$ and $m' \in W'$. For one direction of the equivalence, assume that $\mathcal{M}, m \sim \mathcal{M}', m'$ and let σ be the winning strategy of player **II**. Now consider the bisimulation game for $\mathcal{M} + \mathcal{M}'', m$ and $\mathcal{M}'', m'$. Player **II** has a winning strategy in the game, since the union of the two structures is disjoint, i.e. the same strategy σ is victorious as in the game over $\mathcal{M}, m$ and $\mathcal{M}', m'$. For example, if player **I** plays $(w, \mathcal{M})$ on round n, with a history $\bar{v} \in W^{n-1}$, the winning strategy gives $\sigma_{n-1}(\bar{v}, w) = w'$, such that w and w' have atomic correspondence and w' is attainable from a world played in the previous turn. When this is the case, w' also has atomic correspondence with $(w, \mathcal{M})$.

For the other direction of the equivalence, suppose that $\mathcal{M} + \mathcal{M}'', (m, \mathcal{M}) \sim \mathcal{M}', m'$ and let σ' be the winning strategy of player **II**. Now it is simple to construct a set of functions F consisting of functions $f_i \colon (W \cup W')^i \to ((W \sqcup W'') \cup W')^i$, and a function $g \colon (W \sqcup W'') \cup W' \to W \cup W'$, such that $f_i(\bar{v}) = \bar{v'}$, where

$$v'_j = \begin{cases} (v_j, \mathcal{M}), & \text{if } v_j \in W, \text{ or} \\ v_j, & \text{otherwise,} \end{cases}$$

and

$$g(v) = \begin{cases} u, & \text{if } v \in W \sqcup W'' \text{ and } v = (u, L), \text{ where } L = \mathcal{M} \text{ or } L = \mathcal{M}'', \text{ or} \\ v, & \text{otherwise.} \end{cases}$$

Now the set of composite functions $\{g \circ \sigma_i \circ f_i \mid i \in \mathbb{N}\}$ is a winning strategy for player **II** in the bisimulation game over $\mathcal{M}, m$ and $\mathcal{M}', m'$. Hence $\mathcal{M}, m \sim \mathcal{M}', m'$. □

Lemma 2.26. *The following holds for all Kripke-structures $\mathcal{M}$ and $\mathcal{M}'$, with fixed elements m and m' respectively.*

1. *$\mathcal{M}, m \sim_\ell \mathcal{M}', m'$ if and only if $\mathcal{M}\restriction U^\ell(m), m \sim_\ell \mathcal{M}'\restriction U^\ell(m'), m'$.*
2. *If both $\mathcal{M}, m$ and $\mathcal{M}', m'$ are tree structures of depth ℓ, then $\mathcal{M}, m \sim_\ell \mathcal{M}', m'$ if and only if $\mathcal{M}, m \sim \mathcal{M}', m'$.*

Proof. Let $\mathcal{M}$ and $\mathcal{M}'$ be Kripke structures with distinguished elements m and m' respectively, and let $\ell \in \mathbb{N}$.

For claim 1. suppose $\mathcal{M}, m \sim_\ell \mathcal{M}', m'$ and let the winning strategy of player **II** be σ. Now the restriction of the same strategy can be used victoriously in the bisimulation game over $\mathcal{M}\restriction U^\ell(m), m$ and $\mathcal{M}'\restriction U^\ell(m'), m'$, since the restriction does not affect which worlds can be played in an ℓ-round game. The game progresses linearly over the edges of the models, and hence the worlds at a distance greater than ℓ from m (or m') cannot be played even in the unrestricted game. For the other direction suppose $\mathcal{M}\restriction U^\ell(m), m \sim_\ell \mathcal{M}'\restriction U^\ell(m'), m'$ and let τ be the winning strategy, containing functions $\tau_i\colon (W \restriction U^\ell(m) \cup W' \restriction U^\ell(m'))^i \to W \restriction U^\ell(m) \cup W' \restriction U^\ell(m')$, for $i \leq \ell - 1$. Now a winning strategy for the bisimulation game over $\mathcal{M}, m$ and $\mathcal{M}', m'$, denoted by σ, can be constructed, since the values the strategy assigns for the worlds at a modal distance greater than ℓ do not matter. The strategy can be defined as follows

$$\sigma(\bar{v}, w) = \begin{cases} \tau(\bar{v}, w), & \text{if } w \in W \restriction U^\ell(m) \cup W' \restriction U^\ell(m'), \\ m, & \text{if } w \in W \text{ and } w \notin W \restriction U^\ell(m) \cup W' \restriction U^\ell(m'), \\ m', & \text{if } w \in W' \text{ and } w \notin W \restriction U^\ell(m) \cup W' \restriction U^\ell(m'). \end{cases}$$

This is a winning strategy since it assigns a world in atomic correspondence for every move player **I** makes until turn ℓ.

Now consider claim 2. Assume $\mathcal{M}, m$ and $\mathcal{M}', m'$ are tree structures of depth ℓ. Now each play of the unbounded bisimulation game over the structures consists of a branch in each of the trees, because they are tree structures and therefore have no loops in their collections of edges. These branches are at most of length ℓ, since the structures are of depth ℓ. Therefore each play of the unbounded bisimulation game takes at most ℓ-rounds.

In other words the game is bound by ℓ. Hence any strategy for the ℓ-round game is a strategy for the unbounded game and vice versa. This is true especially for a winning strategy, which proves the claim. □

Definition 2.27 (Tree unravelling)**.** The tree unravelling $\mathcal{M}^*_m$ of $\mathcal{M}$ from m is a Kripke structure constructed in the following fashion. The universe of the unravelling W^*_m is populated with all the paths from m in $E^{\mathcal{M}}$ and the world m. In turn E is interpreted so that for each path of length $k+1$, e.g. $((w_0, w_1), ..., (w_k, w_{k+1}))$, is connected to its subpath of length k, i.e. $(((w_0, w_1), ..., (w_{k-1}, w_k)), ((w_0, w_1), ..., (w_k, w_{k+1})))$, where $k \in \mathbb{N}$. The predicates in $\mathbf{P}$ are interpreted according to the last world in each path. Due to this the natural projection $\pi\colon W^*_m \to W$, which maps a path to its final world, conserves the value of the predicates.

Lemma 2.28. *Assume $\mathcal{M}$ is a Kripke structure with a distinguished world m.*

1. *The tree unravelling $\mathcal{M}^*_m$ is bisimilar to $\mathcal{M}$ via the natural projection π.*
2. *For every $\ell \in \mathbb{N}$ the restriction of the tree unravelling $\mathcal{M}^*_m$ to depth ℓ is a tree structure ℓ-bisimilar to $\mathcal{M}, m$.*
3. *Let Kripke structure $\mathcal{M}$ be finite with distinguished world m and let $\ell \in \mathbb{N}$. Now there exists a partial unravelling (to depth ℓ) that yields a finite bisimilar companion that is ℓ-locally a tree structure.*

Proof. Let $\mathcal{M}$ be a Kripke structure with the distinguished world m and let $\mathcal{M}^*_m$ be the tree unravelling of $\mathcal{M}$ from m.

For the first claim, consider the bisimulation game over $\mathcal{M}, m$ and $\mathcal{M}^*_m, m$. Initially player **I** plays the world m from either model and player **II** answers with m from the other structure. Due to the construction of the tree unravelling the worlds are in atomic correspondence. For every consequent turn player **I** plays either a path from $\mathcal{M}^*_m$ or a world from $\mathcal{M}$. If he plays a path, player **II** answers with its image under the natural projection, which is by definition in atomic correspondence with the path. The world must also be attainable from the world played in the previous turn, as there exists a path in the tree unravelling corresponding to it. If player **I** plays a world, player **II** responds with the path to the world. The path is defined to be in atomic correspondence with the world and since the world was attainable from the world played on the preceding turn, there must exist a path to it with a subpath leading to the previous world. Hence player **II** has a winning strategy and $\mathcal{M}^*_m, m$ and $\mathcal{M}, m$ are bisimilar.

Consider the second claim. As proven in the previous point, $\mathcal{M}, m$ is bisimilar to $\mathcal{M}^*_m, m$. Since a winning strategy for a longer game is also a winning strategy for a shorter game of the bisimulation game over some given structures, $\mathcal{M}, m$ and $\mathcal{M}^*_m, m$ are also

ℓ-bisimilar. According to part 1 of Lemma 2.26 $\mathcal{M}_m^* \restriction U^\ell(m), m$ and $\mathcal{M} \restriction U^\ell(m), m$ are ℓ-bisimilar. Seeing as the game is bound by ℓ, restricting the playable worlds to the modal distance ℓ is redundant in this case. Worlds at a modal distance greater than ℓ from m could not be played in an ℓ-round game anyway. Hence the same winning strategy can be used in the bisimulation game over $\mathcal{M}_m^* \restriction U^\ell(m), m$ and $\mathcal{M}, m$. Therefore the structures are ℓ-bisimilar for all $\ell \in \mathbb{N}$.

To prove claim 3., consider the restriction of the tree unravelling $\mathcal{M}_m^* \restriction U^\ell(m)$. This restriction is ℓ-locally a tree structure and it is ℓ-bisimilar to $\mathcal{M}, m$, as shown in the previous part. In order to expand the restriction in a way that maintains bisimilarity, consider the worlds of the restriction at a modal distance of ℓ from m. Let $b*$ be one such leaf. Now for each of these leaves we attach a copy of the original model $\mathcal{M}$ to the restricted tree, identifying the leaf with its image under the natural projection $b = \pi(b*)$. This means we add fresh worlds for each of the worlds in $\mathcal{M}$, call them $w*$ for each $w \in W$, at each of the leaves at a modal distance of ℓ from m, add corresponding worlds to the interpretation of E, and add the worlds to the interpretation of the predicates as in $\mathcal{M}$.

Now consider the bisimulation game over $\mathcal{M}$ and the previously described structure. A winning strategy can be devised for player **II** by first making her play for ℓ-rounds according to the winning strategy in the ℓ-round bisimulation game over $\mathcal{M}, m$ and $\mathcal{M}_m^* \restriction U^\ell(m), m$. If the game continues after this point the players have reached the copies of the original model attached at the ends of the branches, and hence player **II** can play $w*$ if player **I** plays w and vice versa. This ensures the victory, seeing as at this point the edges and the predicates in the new construction are identical to the original structure. Therefore the structures are bisimilar. As previously noted, the new structure is ℓ-locally a tree structure and finite. □

2.5 The van Benthem Theorem

Definition 2.29 (Bisimulation invariance)**.** A formula $\varphi(x) \in \mathrm{FO}[E; \mathbf{P}]$ is *bisimulation invariant* if whenever $\mathcal{M}, m \sim \mathcal{M}', m'$, then $\mathcal{M}, m \models \varphi$ if and only if $\mathcal{M}', m' \models \varphi$.

Theorem 2.30. *The following are equivalent for any $\varphi(x) \in \mathrm{FO}$ of quantifier rank q.*

i) $\varphi(x)$ is invariant under bisimulation.

ii) $\varphi(x)$ is equivalent to a formula of ML_ℓ, where $\ell = 2^q - 1$.

Proof. The proof has three stages: showing that every bisimulation invariant formula is ℓ-local; proving that any bisimulation invariant formula that is ℓ-local is also invariant under ℓ-bisimulation; all properties invariant under ℓ-bisimulation are definable in ML_ℓ.

Step 1: Suppose $\varphi(x)$ is bisimulation invariant, let $q = qr(\varphi)$, and let $\ell = 2^q - 1$. Assume $\mathcal{M}$ is a Kripke structure with distinguished element m. By the definition of ℓ-locality, we need to show that $\mathcal{M}, m \models \varphi$ if and only if $\mathcal{M} \restriction U^\ell(m), m \models \varphi$. Since φ is bisimulation invariant we may replace the restriction $\mathcal{M} \restriction U^\ell(m)$ in the claim with any bisimilar structure. To that end, consider the tree unravelling of $\mathcal{M}$. By Lemma 2.28 the tree unravelling $\mathcal{M}^*_m, m$ is bisimilar to $\mathcal{M}, m$, and hence by Lemma 2.26 the restriction of the tree unravelling $\mathcal{M}^*_m \restriction U^\ell(m), m$ is bisimilar to the restriction $\mathcal{M} \restriction U^\ell(m), m$. Now it suffices to show that $\mathcal{M}, m \models \varphi$ if and only if the $\mathcal{M}^*_m \restriction U^\ell(m), m \models \varphi$, where the latter structure is a tree structure of depth ℓ.

In order to show this equivalence we endeavour to find two structures $\mathcal{M}'$ and $\mathcal{M}''$, such that $\mathcal{M}', m' \sim \mathcal{M}, m$, $\mathcal{M}'', m'' \sim \mathcal{M}^*_m \restriction U^\ell(m), m$ and $\mathcal{M}', m' \equiv^{\mathrm{FO}}_q \mathcal{M}'', m''$, where m' and m'' are distinguished worlds of their respective structures. This is enough since then the following equivalence chain holds:

$$\begin{gathered} \mathcal{M}, m \models \varphi \text{ if and only if} \\ \mathcal{M}', m' \models \varphi \text{ if and only if} \\ \mathcal{M}'', m'' \models \varphi \text{ if and only if} \\ \mathcal{M}^*_m \restriction U^\ell(m), m \models \varphi, \end{gathered}$$

where the first and third equivalences are due to φ being bisimulation invariant and the second equivalence holds because of the partial equivalence of the structures. As the models $\mathcal{M}'$ and $\mathcal{M}''$ we choose the disjoint sum of q copies of both $\mathcal{M}, m$ and $\mathcal{M}^*_m \restriction U^\ell(m), m$ with an additional copy of $\mathcal{M}, m$ for $\mathcal{M}'$ and $\mathcal{M}^*_m \restriction U^\ell(m), m$ for $\mathcal{M}''$. The distinguished element m' is one of the elements corresponding to m in one of the copies of $\mathcal{M}$. Similarly, the distinguished world m'' is one of the copies of m in one of the duplicated restrictions.

According to Lemma 2.25, the structure $\mathcal{M}, m$ is bisimilar to $\mathcal{M}', m'$ and the restriction $\mathcal{M}^*_m \restriction U^\ell(m), m$ is bisimilar to $\mathcal{M}'', m''$, seeing as the structures $\mathcal{M}'$ and $\mathcal{M}''$ are disjoint sums with the original structures as terms. The q-equivalence is shown by finding a winning strategy for player **II** in the q-round Ehrenfeucht-Fraïsse game over $\mathcal{M}'$ and $\mathcal{M}''$. The idea behind the strategy is that she only begrudgingly plays in a copy with marked worlds in it; she plays her turn in a fresh copy whenever she can get away with it. To that end we will require the concepts of *critical distance* and *local context*.

The critical distance refers to the distance within which player **I** is still capable of checking the connection between two worlds during the remainder of the game. For each round r of the game, define the critical distance by

$$d_r = 2^{q-r}.$$

Initially the critical distance has a value of $d_1 = 2^{q-1}$, which happens to be $\ell/2$ rounded up. This then drops by a factor of $1/2$ round by round. We say that player **II** plays according to local context when she plays a world from an already occupied term in the disjoint sum. When playing in accordance with the winning strategy, she only plays according to local context when player **I** plays a world within the critical distance of a previously played world. If his move is further than the critical distance from all previously marked elements, player **II** answers with the corresponding element from a fresh copy of the structure $\mathcal{M}$ or the restriction $\mathcal{M}_m^* \upharpoonright U^\ell(m)$ respective to player **I**'s play.

The aim of this strategy is to group the played pebbles into clusters, which player **I** can check the internal coherence of during the remainder of the game. However the clusters are so far apart from each other, on any given round r the distance between two clusters is larger than d_r, in order to check whether they are in the same copy or not would take longer than the remaining game time. When playing by the winning strategy, player **II** upholds the condition that after each round r any two corresponding clusters are linked by an isomorphism that extends to all worlds within a distance d_r of the members of the clusters. See Figure 1.1, for a concrete example of how the game may progress. That this outlines a winning strategy is proven by induction over the rounds r of the EF game. In the base case $r = 0$, the correspondence of m' and m'' is checked before the first round. In this case m' belongs to the same predicates as m, since $\mathcal{M}, m$ and $\mathcal{M}', m'$ are assumed to be bisimilar, and similarly m'' belongs to the same predicates as m, as $\mathcal{M}'', m''$ is bisimilar to $\mathcal{M}_m^* \upharpoonright U^\ell(m), m$. Hence m' and m'' belong to the same predicates. The worlds also have a matching number of E-edges, due to the way the tree unravelling is constructed. These observations also apply to all the worlds at a distance of $d_1 = \ell/2$, since the restriction is of a depth ℓ. Therefore the one element clusters can be connected by an isomorphism that maps the worlds of $\mathcal{M}''$ to worlds in $\mathcal{M}'$ as given by the natural projection.

For the induction case suppose that at round $k \in \mathbb{N}$ $k \leq q$ the strategy has yielded a state where the clusters are connected by isomorphisms as per the claim. Now consider the subsequent round $r = k + 1$. At this point player **I** has the liberty to play any world from either of the two structures. Assume he plays a world within the critical distance d_{k+1} from a world in a marked cluster. That world is now considered a part of the same cluster. Now player **II** plays according to local context and due to the induction hypothesis the same isomorphism ensures the claim holds, since $d_{k+1} < d_k$. On the other hand, suppose player **I** plays a world at a distance greater than d_{k+1} from any world in any marked cluster. Now player **II** responds by playing the corresponding world in a fresh copy of either $\mathcal{M}$ or the restriction, depending on player **I**'s play, in the opposing structure. These worlds now form their own, new clusters. According to the induction hypothesis there exists a partial isomorphism from the previous round, which can now be expanded by the played worlds and their surroundings up to d_{k+1}. This expansion is still a partial isomorphism, since the worlds are picked from copies of the same structure. A contradictory assignment of values does not occur, since there are no marked worlds within the critical distance

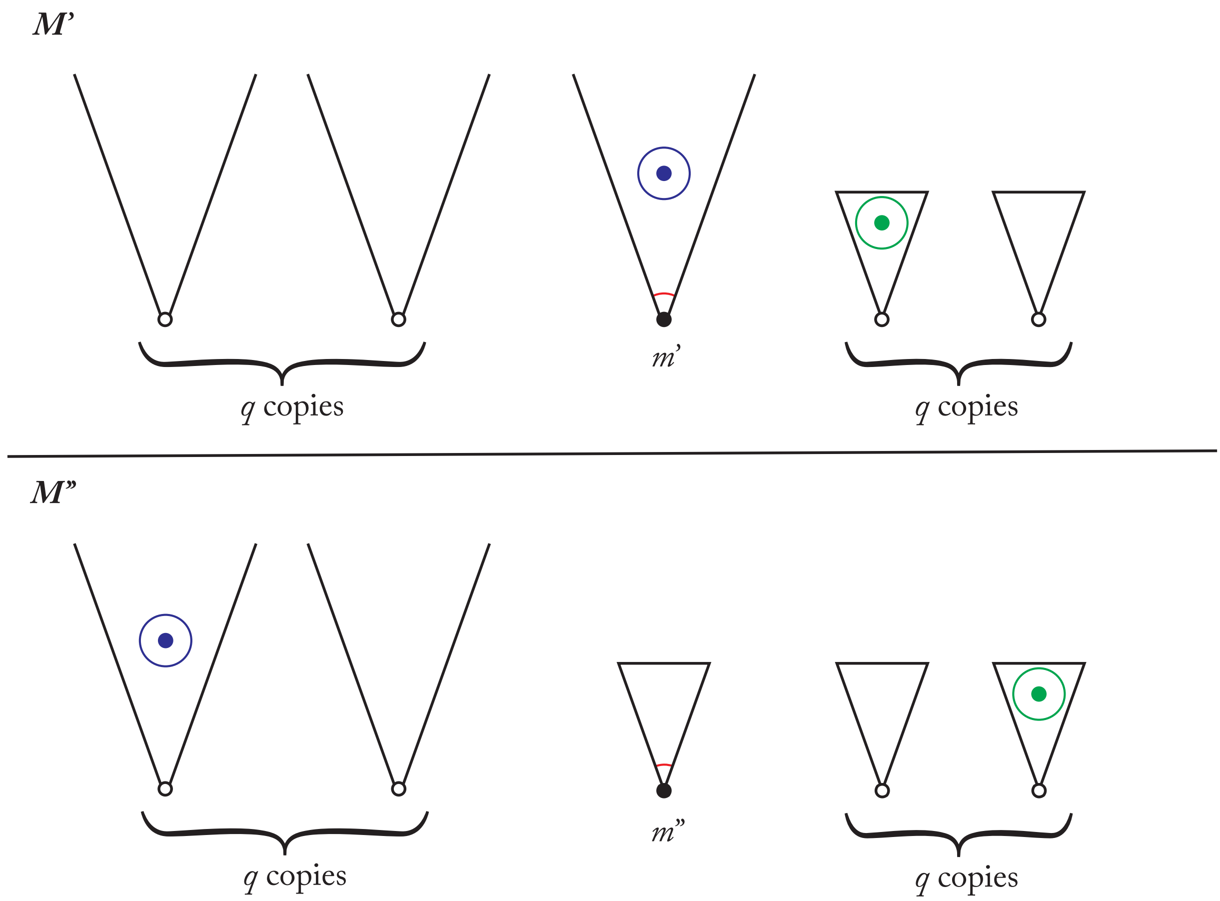

Figure 2.1: The Kripke structures $\mathcal{M}'$ and $\mathcal{M}''$ after three rounds of the EF-game. The red, blue and green refer to the first, second and third rounds, respectively. The dots represent played worlds and the outline represents the critical distance d_3.

from the played worlds. By the induction principle the winning condition is upheld for all q-rounds of the game. Hence the models $\mathcal{M}'$ and $\mathcal{M}''$ are equivalent up to q, and therefore φ is ℓ-local.

Step 2: Let $\varphi(x)$ be ℓ-local and bisimulation invariant. Suppose $\mathcal{M}$ and $\mathcal{M}'$ are Kripke structures with the distinguished elements m and m' respectively, such that $\mathcal{M}, m \sim_\ell \mathcal{M}', m'$ and $\mathcal{M}, m \models \varphi$. As previously we may assume without loss of generality $\mathcal{M}, m$ and $\mathcal{M}', m'$ are ℓ-locally tree structures. Since φ is ℓ-local, it holds that $\mathcal{M}, m \models \varphi$ if and only if $\mathcal{M} \restriction U^\ell(m), m \models \varphi$. Now by Lemma 2.26, $\mathcal{M}, m \sim_\ell \mathcal{M}', m'$ if and only if $\mathcal{M} \restriction U^\ell(m), m \sim_\ell \mathcal{M}' \restriction U^\ell(m'), m'$ and since both structures are tree structures of depth ℓ the restrictions $\mathcal{M} \restriction U^\ell(m), m$ and $\mathcal{M}' \restriction U^\ell(m'), m'$ are also bisimilar. Hence $\mathcal{M} \restriction U^\ell(m), m \models \varphi$ if and only if $\mathcal{M}' \restriction U^\ell(m'), m' \models \varphi$, and due to the ℓ-locality of φ the structure $\mathcal{M}, m$ satisfies φ if and only if $\mathcal{M}', m'$ satisfies φ. Therefore φ is also ℓ-bisimulation invariant.

Step 3: Suppose φ is ℓ-bisimulation invariant. Now according to Lemma 2.22, there exists ML_ℓ formulae that define each of the equivalence classes of the bounded bisimulation relation $\sim_\ell$, namely the Hintikka formulae. Let $\chi^\ell_{[\mathcal{M},m]}$ be the Hintikka formula characterising the equivalence class of $\mathcal{M}, m$ under $\sim_\ell$. Then we can formulate a disjunction over the equivalence classes validating φ:

$$\bigvee_{\mathcal{M},m \models \varphi} \chi^\ell_{[\mathcal{M},m]},$$

which characterises all the equivalence classes that satisfy φ. This disjunction is potentially infinite, but since the index of $\sim_\ell$ is finite the disjunction is equivalent to a finite disjunction, where only one witness per equivalence class is picked. This finite disjunction is logically equivalent to φ, since each structure that satisfies φ is in one of the equivalence classes represented in the disjunction and hence the structure satisfies one of the disjuncts. $\square$

Unlike other proofs of the van Benthem theorem, this proof holds for both finite and infinite models, whereas the original proof of the theorem relies on expanding finite models into infinite ones, as can be seen in [2]. The theorem above is equivalent with the following classical formulation of the van Benthem theorem.

Theorem 2.31. *Let $\mathcal{K}$ be a property of Kripke structures. The following claims are equivalent:*

i) $\mathcal{K} = \mathrm{Mod}(\varphi)$, *for some* $\varphi \in \mathrm{ML}$,

ii) $\mathcal{K}$ *is* k*-bisimulation invariant for some* $k \in \mathbb{N}$,

iii) $\mathcal{K} = \mathrm{Mod}(\varphi)$, *for some* $\varphi \in \mathrm{FO}$ *and* φ *is invariant under bisimulation.*

Proof. It follows from Theorem 2.30. that *i*) is equivalent with *iii*), and from Lemma 2.22. that *i*) is equivalent with *ii*). □

This theorem states that the fragment of first-order logic which corresponds to modal logic is precisely the fragment that is bisimulation invariant.

Chapter 3

Modal Dependence Logic

In this chapter we will prove similar results to the van Benthem theorem discussed in the previous chapter, only this time considering extensions of modal logic. The particular extension in focus is the so-called extended modal dependence logic. Additionally, two other extensions will be defined as they will be useful in the upcoming analysis. These logics are modal logic with Boolean disjunction and modal dependence logic. The approach used follows the work of Lauri Hella, et al. found in [6].

3.1 The Fundamentals of Modal Dependence Logic and Team Semantics

Definition 3.1 (ML($\varovee$), MDL, and EMDL)**.** *Modal logic with intuitionistic disjunction*, *modal dependence logic*, and *extended modal dependence logic*, denoted ML($\varovee$), MDL and EMDL and respectively, are all extensions of modal logic.

i) The logic ML($\varovee$) extends the syntax of modal logic by the grammar rule $\varphi = \psi \varovee \theta$, where ψ and θ are ML($\varovee$)-formulae.

ii) The logic MDL extends the syntax of modal logic by the grammar rule $\varphi == (P_1x, ..., P_nx, Qx)$, where $Q, P_i \in \mathbf{P}$ for $1 \leq i \leq n$.

iii) The logic EMDL extends the syntax of modal logic by the grammar rule $\varphi == (\psi_1, ..., \psi_n, \theta)$, where ψ_i for $1 \leq i \leq n$ and θ are ML-formulae.

The semantics of these logics are defined using team semantics. In the following definitions let $\mathcal{M} = (W, E, \mathbf{P})$ be a Kripke structure.

Definition 3.2 (Team)**.** A *team* of $\mathcal{M}$ is any subset $T \subseteq W$. We denote the team attainable from T as $E^{\mathcal{M}}[T] = \{v \in W \mid \textit{there exists } w \in T \textit{ such that } wEv\}$ and the team that T is attainable from as $E^{-1}[T] = \{w \in W \mid \textit{there exists } v \in T \textit{ such that } wEv\}$. For two teams T and S we write $T[E]S$, if $S \subseteq E^{\mathcal{M}}[T]$ and $T \subseteq E^{-1}[S]$.

Hence $T[E]S$ holds if every world in S is attainable from some world in T and every world in T is connected to some world in S. It is a way of denoting a reciprocal team attainable from T that fulfils some condition, which is useful when defining the semantics of the extensions of ML.

When speaking in terms of team semantics it is standard to assume all formulae are in negation normal form, which is a form where all negations occur before atomic formulae. This form is adopted in order to avoid confusion, since the dependence atom does not have a defined negation.

Definition 3.3 (Team semantics of modal logic)**.** Let T be a team of $\mathcal{M}$. The semantics of ML($\varovee$), MDL and EMDL are defined as follows, given that P is a predicate in $\mathbf{P}$.

$$\begin{aligned}
\mathcal{M}, T \models Px \quad &\Leftrightarrow \quad T \subseteq P^{\mathcal{M}} \\
\mathcal{M}, T \models \neg Px \quad &\Leftrightarrow \quad T \cap P^{\mathcal{M}} = \emptyset \\
\mathcal{M}, T \models \varphi \wedge \psi \quad &\Leftrightarrow \quad \mathcal{M}, T \models \varphi \text{ and } \mathcal{M}, T \models \psi \\
\mathcal{M}, T \models \varphi \vee \psi \quad &\Leftrightarrow \quad \text{there exists } S, S' \subseteq T \text{ such that } \mathcal{M}, S \models \varphi, \\
& \qquad \mathcal{M}, S' \models \psi \text{ and } S \cup S' = T \\
\mathcal{M}, T \models \Diamond\varphi \quad &\Leftrightarrow \quad \text{there exists } S \subseteq W \text{ such that } \mathcal{M}, S \models \varphi \text{ and } T[E]S \\
\mathcal{M}, T \models \Box\varphi \quad &\Leftrightarrow \quad \mathcal{M}, S \models \varphi, \text{ for } S = E^{\mathcal{M}}[T],
\end{aligned}$$

where x is a free variable, and φ and ψ are formulae of their respective extension. In addition ML($\varovee$) has the clause

$$\mathcal{M}, T \models \varphi \varovee \psi \qquad \Leftrightarrow \qquad \mathcal{M}, T \models \varphi \text{ or } \mathcal{M}, T \models \psi,$$

and both MDL and EMDL feature the dependence atom, as defined by the clause

$$\begin{aligned}
\mathcal{M}, T \models =(\varphi_1, ..., \varphi_n, \theta) \qquad \Leftrightarrow \qquad & \forall w, v \in T\colon \bigwedge_{i=1}^{n} (\mathcal{M}, w \models \varphi_i \Leftrightarrow \mathcal{M}, v \models \varphi_i) \\
& \text{implies } (\mathcal{M}, w \models \theta \Leftrightarrow \mathcal{M}, v \models \theta),
\end{aligned}$$

where φ, ψ, θ, and φ_i for $1 \leq n$ are ML-formulae in EMDL and predicates in MDL.

The following lemma encapsulates the flatness property ML has in team semantics.

Lemma 3.4. *Let $\mathcal{M}$ be a Kripke structure, T a team of $\mathcal{M}$, and φ an* ML(**P**)*-formula. Then*

$$\mathcal{M}, T \models \varphi \Leftrightarrow \mathcal{M}, w \models \varphi \text{ for every } w \in T.$$

Proof. Proof by induction over the structure of φ for all teams of $\mathcal{M}$. The base step consists of the following two cases.

1. $\varphi = Px$. The definition of modal team semantics states that $\mathcal{M}, T \models Px$ if and only if $T \subseteq P^{\mathcal{M}}$, which in turn holds if and only if $\mathcal{M}, w \models Px$ for every $w \in T$. Hence the claim holds for $\varphi = Px$.

2. $\varphi = \neg Px$. Again by definition $\mathcal{M}, T \models \neg Px$ if and only if $T \cap P^{\mathcal{M}} = \emptyset$, which in turn is true if and only if $\mathcal{M}, w \models Px$ for no $w \in T$. This is equivalent with $\mathcal{M}, w \models \neg Px$ for all $w \in T$, and thus the claim holds for $\varphi = \neg Px$.

The induction step breaks down into the following cases.

1. $\varphi = \psi \wedge \theta$, for some ML-formulae ψ and θ. Suppose for the induction hypothesis that the claim holds for ψ and θ. Now by definition $\mathcal{M}, T \models \psi \wedge \theta$ if and only if $\mathcal{M}, T \models \psi$ and $\mathcal{M}, T \models \theta$, which the induction hypothesis equates with $\mathcal{M}, w \models \psi$ and $\mathcal{M}, v \models \theta$ for all worlds $v \in T$. By the definition of conjunction in ML this holds if and only if $\mathcal{M}, w \models \psi \wedge \theta$ for all $w \in T$. Hence the claim holds for $\varphi = \psi \wedge \theta$.

2. $\varphi = \psi \vee \theta$ for some ML-formulae ψ and θ. Now the induction hypothesis states that $\mathcal{M}, R \models \psi$ if and only if $\mathcal{M}, w \models \psi$ for all $w \in R$, and $\mathcal{M}, R \models \theta$ if and only if $\mathcal{M}, v \models \theta$ for all $v \in R$ for all $R \subseteq W$. Definition 2.3. states that $\mathcal{M}, T \models \psi \vee \theta$ if and only if there exists two subteams $S, S' \subseteq T$ such that $\mathcal{M}, S \models \psi$, $\mathcal{M}, S' \models \theta$ and $S \cup S' = T$, which is equivalent with that there are $S, S' \in T$ such that $\mathcal{M}, w \models \psi$ for all $w \in S$, $\mathcal{M}, v \models \theta$ for all $v \in S'$ and $S \cup S' = T$. Now for single worlds we use the definition in modal logic, for which a disjunction is true in a world if and only if either disjunct is true. Now the previous statement is equivalent with that $\mathcal{M}, w \models \psi \vee \theta$ for all $w \in T$, since S and S' cover T and hence every world in T satisfies either ψ or θ. Therefore the claim holds true for $\varphi = \psi \vee \theta$.

3. $\varphi = \Diamond\psi$, for some ML-formula ψ. The induction hypothesis equates $\mathcal{M}, R \models \psi$ with $\mathcal{M}, w \models \psi$ for all $w \in R$, where R is a team of $\mathcal{M}$. Now the definition states that $\mathcal{M}, T \models \Diamond\psi$ if and only if there exists $S \subseteq W$ such that $\mathcal{M}, S \models \psi$ and $T[E]S$. This in turn is equivalent with the claim that there exists a team $S \subseteq W$ such that $\mathcal{M}, w \models \psi$ for all $w \in S$ and $T[E]S$, which is true if and only if for every $v \in T$ there exists $w_v \in S$ such that $\mathcal{M}, w_v \models \psi$. This coincides with the definition of the semantics of the possibility operator, and thus $\mathcal{M}, v \models \Diamond\psi$ for all $v \in T$. Thus the claim holds for $\varphi = \Diamond\psi$.

4. $\varphi = \Box\psi$ for some ML-formula ψ. Now the induction hypothesis states that $\mathcal{M}, R \models \psi$ is equivalent with $\mathcal{M}, w \models \psi$ for all $w \in R$ for any team R of $\mathcal{M}$. By definition $\mathcal{M}, T \models \Box\psi$ if and only if $\mathcal{M}, S \models \psi$ for $S = E^{\mathcal{M}}[T]$, which according to the induction hypothesis is equivalent with $\mathcal{M}, w \models \psi$ for all $w \in S$, when $S = E^{\mathcal{M}}[T]$. This is a rephrased version of the definition of the semantics of the necessity operator and hence the previous claim equates to $\mathcal{M}, v \models \Box\psi$ for all $v \in T$. Hence the claim holds for $\varphi = \Box\psi$.

Now by the induction principle $\mathcal{M}, T \models \varphi$ if and only if $\mathcal{M}, w \models \varphi$ for all worlds $w \in T$ and ML-formulae φ. $\square$

We denote the class of Kripke models with predicates from $\mathbf{P}$ and teams by $\mathcal{KT}(\mathbf{P})$.

3.2 Expressive Power and Hierarchies of Expression

Definition 3.5 (Expressive power)**.** Suppose L is one of the logics ML, MDL, EMDL or ML($\varovee$). Each formula $\varphi \in L(\mathbf{P})$ defines a class of structures with teams, which corresponds precisely to the structures and teams that satisfy φ. This is denoted

$$\|\varphi\| = \{(K, T) \in \mathcal{KT}(\mathbf{P}) \mid K, T \models \varphi\}.$$

A class $\mathcal{K} \subseteq \mathcal{KT}(\mathbf{P})$ is definable in L, if there is a formula $\varphi \in L(\mathbf{P})$ such that $\mathcal{K} = \|\varphi\|$. The expressive power of L is the collection of classes definable in L, in essence $\|\varphi\|$ for all $\varphi \in L$. An order can be defined on the expressive powers of logics. For two of the aforementioned logics L and L', the order is defined as follows

- L' is at least as expressive as L, denoted $L \leq L'$, if for all $\varphi \in L(\mathbf{P})$ there is a formula $\psi \in L'(\mathbf{P})$ such that $\|\varphi\| = \|\psi\|$.
- L is less expressive than L', written $L < L'$, if $L \leq L'$ and $L' \nleq L$.
- Lastly, L and L' are equally expressive, $L \equiv L'$, when $L \leq L'$ and $L' \leq L$.

Lemma 3.6. ML $<$ EMDL $\leq$ ML($\varovee$).

Proof. Since EMDL is an extension of ML each formula in ML is a formula in EMDL. Hence ML $\leq$ EMDL. To show that EMDL $\nleq$ ML it suffices to find a formula of EMDL that breaks the flatness property, since then by Lemma 3.4. the formula cannot have an equivalent one in ML. This is easily seen by considering the EMDL($\mathbf{P}$), $\mathbf{P} = \{P\}$, formula $\varphi == (Px)$ and the Kripke structure $\mathcal{M} = (\{w, w'\}, \emptyset, \{w\})$. Now $\mathcal{M}, \{w, w'\} \not\models \varphi$, but $\mathcal{M}, w \models \varphi$ and $\mathcal{M}, w' \models \varphi$. However there is no ML($\mathbf{P}$) formula ψ such that $\mathcal{M}, w \models \psi$, $\mathcal{M}, w' \models \psi$ and yet $\mathcal{M}, \{w, w'\} \not\models \psi$, since this would break the flatness property of ML.

To show that EMDL $\leq$ ML($\varovee$) we will construct a translation from EMDL to ML($\varovee$). Suppose $=(\varphi_1, ..., \varphi_n, \psi)$ is a EMDL-formula, where φ_i, for $1 \leq i \leq n$, and ψ are ML-formulae. Now consider

$$\sigma = \bigvee_{\mathbf{b} \in \{\bot, \top\}^n} (\varphi_1^{b_1} \wedge ... \wedge \varphi_n^{b_n} \wedge (\psi \varovee \psi^{\bot})),$$

where $\mathbf{b} = (b_1, ..., b_n)$, $b_i \in \{\bot, \top\}$ for $1 \leq i \leq n$, $\varphi^{\top} = \varphi$ and $\varphi^{\bot}$ is the formula obtained from $\neg\varphi$ by putting it in negation normal form. Now $\mathcal{M}, T \models \sigma$ if and only if T can be divided into 2^n subteams $T_{\mathbf{b}}$, $\mathbf{b} \in \{\bot, \top\}^n$, such that $T_{\mathbf{b}}$ contains the worlds w that satisfy the row of formulas given by $\mathbf{b}$. Furthermore, since $\mathcal{M}, T_{\mathbf{b}} \models \psi \varovee \psi^{\bot}$, each row $\mathbf{b}$ fixes the truth value of the formula ψ for the corresponding team $T_{\mathbf{b}}$. This is equivalent with $\mathcal{M}, T \models =(\varphi_1, ..., \varphi_n, \psi)$. □

Definition 3.7 (Downward closure). A logic L is *downward closed* if for every formula $\varphi \in L$ it holds that if $\mathcal{M}, T \models \varphi$, then $\mathcal{M}, S \models \varphi$, for all Kripke structures $\mathcal{M}$ with teams T and S, such that $S \subseteq T$.

A class $\mathcal{K}$ is *downward closed* if, given that $(\mathcal{M}, T) \in \mathcal{K}$ and $S \subseteq T$, it follows that $(\mathcal{M}, S) \in \mathcal{K}$.

Definition 3.8 (Empty team property). A logic L has the *empty team property* if for all $\varphi \in L$ it holds that $\mathcal{M}, \emptyset \models \varphi$, for all Kripke structures $\mathcal{M}$.

A class $\mathcal{K}$ has the *empty team property* if $(\mathcal{M}, \emptyset) \in \mathcal{K}$, for all Kripke structures $\mathcal{M}$.

Next we will show that the logics ML, EMDL and ML($\varovee$) are downward closed and have the empty team property. In order to do this is efficiently in one go, we will prove that a logic extending all of them, namely ML($=(...), \varovee$), has these properties. Then, seeing as every formula of ML, EMDL or ML($\varovee$) is a formula of ML($=(...), \varovee$), the properties of the parent are inherited by the children.

Lemma 3.9. *The logic* ML($=(...), \varovee$)

i) is downward closed,

ii) has the empty team property.

Proof. Let φ be a ML($=(...), \varovee$)-formula. Both claims are proven by induction over the structure of the formula φ.

i) Let $\mathcal{M}$ be a Kripke structure and let T and S be teams of $\mathcal{M}$, such that $S \subseteq T$. The basis cases are as follow:

1. Suppose $\varphi = Px$ for some predicate $P \in \mathbf{P}$. Assume $\mathcal{M}, T \models Px$. Now φ is a ML-formula and due to the flatness property of ML, $\mathcal{M}, w \models Px$ holds for all worlds $w \in T$. Since S is a subset of T, by the same flatness property as before $\mathcal{M}, S \models Px$.

2. Suppose $\varphi = \neg Px$ for some $P \in \mathbf{P}$. Assume $\mathcal{M}, T \models \neg Px$, in other words $\mathcal{M}, T \not\models Px$. As above the formula is also a ML-formula and the flatness property holds. Thereby $\mathcal{M}, w \not\models Px$ for all $w \in T$, and since $S \subseteq T$ the flatness property also yields $\mathcal{M}, S \models \neg Px$.

On to the induction cases.

1. Suppose $\varphi = \psi_1 \wedge \psi_2$ for some ML$(=(...), \varovee)$-formulae ψ_1, ψ_2. As an induction hypothesis assume that if $\mathcal{M}, T' \models \psi_i$ then $\mathcal{M}, S' \models \psi_i$ for all $i \in \{1, 2\}$ and teams T' and S', such that $S' \subseteq T'$. Now suppose $\mathcal{M}, T \models \varphi$. By the truth definition it now follows that $\mathcal{M}, T \models \psi_i$ for $i \in \{1, 2\}$ and according to the induction hypothesis it now follows that $\mathcal{M}, S \models \psi_i$ for $i \in \{1, 2\}$. Once again using the truth definition for the conjunction yields the result $\mathcal{M}, S \models \varphi$.

2. Suppose $\varphi = \psi_1 \vee \psi_2$ for some ML$(=(...), \varovee)$-formulae ψ_1, ψ_2. For the induction hypothesis assume if $\mathcal{M}, T' \models \psi_i$, then $\mathcal{M}, S' \models \psi_i$ for all teams T' and S' of $\mathcal{M}$ such that $S' \subseteq T'$ and for $i \in \{1, 2\}$. Now suppose $\mathcal{M}, T \models \varphi$. Now there exist subteams $T_1, T_2 \in T$, such that $T_1 \cup T_2 = T$ and $\mathcal{M}, T_i \models \psi_i$ for all $i \in \{1, 2\}$. Hence we have $(S \cap T_1) \cup (S \cap T_2) = S$, and by the induction hypothesis $\mathcal{M}, S \cap T_i \models \psi_i$ for all $i \in \{1, 2\}$. Thereby $\mathcal{M}, S \models \varphi$.

3. Suppose $\varphi = \Diamond\psi$ for ML$(=(...), \varovee)$-formula ψ. As induction hypotheses suppose if $\mathcal{M}, T' \models \psi$, then $\mathcal{M}, S' \models \psi$ for all teams T', S' such that $S' \subseteq T'$. Now suppose $\mathcal{M}, T \models \varphi$. Hence by the definition of the semantics of the possibility operator there exist a team $T' \subseteq W$ such that $\mathcal{M}, T' \models \psi$ and $T[E]T'$. It is now essential to note that there exists a subteam $S' \subseteq T'$ such that $S[E]S'$. By the induction hypotheses $\mathcal{M}, S' \models \psi$ holds true, and hence $\mathcal{M}, S \models \varphi$.

4. Suppose $\varphi = \Box\psi$ for some ML$(=(...), \varovee)$-formula ψ. Assume for the induction hypothesis if $\mathcal{M}, T' \models \psi$, then $\mathcal{M}, S' \models \psi$ for all teams T', S' such that $S' \subseteq T'$. Suppose then $\mathcal{M}, T \models \varphi$. Now by the truth definition $\mathcal{M}, E^{\mathcal{M}}[T] \models \psi$. Note now that $E^{\mathcal{M}}[S] \subseteq E^{\mathcal{M}}[T]$, and hence by the induction hypothesis $\mathcal{M}, E^{\mathcal{M}}[S] \models \psi$. By definition then $\mathcal{M}, S \models \varphi$.

5. Suppose $\varphi == (\psi_1, ..., \psi_n)$ for some ML$(=(...), \varovee)$-formulae ψ_i, $1 \leq i \leq n$. As the induction hypothesis assume if $\mathcal{M}, T' \models \theta$, then $\mathcal{M}, S' \models \theta$ for all teams T' and

S', such that $S' \subseteq T'$, and for $\theta \in \{\psi_1, ..., \psi_n\}$. Now suppose $\mathcal{M}, T \models \varphi$. The truth definition states that for every pair of worlds in T if they agree on all of the ψ_i formulae, where $1 \leq i \leq n-1$, then they agree on the ψ_n formula. Since this agreement is determined pairwise and without referring to the rest of the team, the claim holds for all subteams of T. Hence $\mathcal{M}, S \models \varphi$.

6. Suppose $\varphi = \psi' \otimes \psi''$ for some ML$(= (...), \otimes)$-formulae ψ' and ψ''. As the induction hypothesis assume if $\mathcal{M}, T' \models \theta$, then $\mathcal{M}, S' \models \theta$ for all teams T' and S', such that $S' \subseteq T'$, and for $\theta \in \{\psi', \psi''\}$. We notice now that the truth definition breaks the claim $\mathcal{M}, T \models \varphi$ down into two for our purposes symmetrical cases; either $\mathcal{M}, T \models \psi'$ or $\mathcal{M}, T \models \psi''$. Suppose $\mathcal{M}, T \models \psi'$. Now according to the induction hypothesis $\mathcal{M}, S \models \psi'$, and hence $\mathcal{M}, S \models \varphi$. Due to symmetry the same argument works when $\mathcal{M}, T \models \psi''$.

By the induction principle if $\mathcal{M}, T \models \varphi$, then $\mathcal{M}, S \models \varphi$ for all $ML(= (...), \otimes)$-formulae φ. Therefore the logic $ML(= (...), \otimes)$ is downward closed.

ii) Let $\mathcal{M}$ be a Kripke structure. The basis case follows from technicalities relating to the empty set:

1. Suppose $\varphi = Px$. Now $\mathcal{M}, \emptyset \models \varphi$, since for all $w \in \emptyset$ it holds that $\mathcal{M}, w \models \varphi$.

2. Suppose $\varphi = \neg Px$. Now $\mathcal{M}, \emptyset \models \varphi$, since $\mathcal{M}, w \not\models \varphi$ holds true for all $w \in \emptyset$.

For the induction case

1. Suppose $\varphi = \psi \wedge \theta$ for some ML$(= (...), \otimes)$-formulae ψ and θ. As an induction hypothesis assume that $\mathcal{M}, \emptyset \models \psi$ and $\mathcal{M}, \emptyset \models \theta$. Now by the truth definition $\mathcal{M}, \emptyset \models \varphi$ follows immediately from the induction hypothesis.

2. Suppose $\varphi = \psi \vee \theta$ for some ML$(= (...), \otimes)$-formulae ψ and θ. Assume for the induction hypothesis that $\mathcal{M}, \emptyset \models \psi$ and $\mathcal{M}, \emptyset \models \theta$. Now since $\emptyset \subseteq \emptyset$ and $\emptyset \cup \emptyset = \emptyset$, it follows from the induction hypothesis that $\mathcal{M}, \emptyset \models \varphi$.

3. Suppose $\varphi = \Diamond\psi$ for some ML$(= (...), \otimes)$-formula ψ. For the induction hypothesis assume $\mathcal{M}, \emptyset \models \psi$. Now notice $\emptyset[E]\emptyset = \emptyset$ and hence it follows from the induction hypothesis that $\mathcal{M}, \emptyset \models \varphi$.

4. Suppose $\varphi = \Box\psi$ for some ML$(= (...), \otimes)$-formula ψ. The induction hypothesis is $\mathcal{M}, \emptyset \models \psi$. In a similar sense to the above this follows from the induction hypothesis, when one considers that $E^{\mathcal{M}}[\emptyset] = \emptyset$.

5. Suppose $\varphi == (\psi_1, ..., \psi_n)$ for some ML$(= (...), \varovee)$-formulae ψ_i, where $1 \leq i \leq n$. The induction hypothesis is that $\mathcal{M}, \emptyset \models \psi_i$ for $1 \leq i \leq n$. Now since every world in the empty team satisfies $\psi_1, ..., \psi_{n-1}$ and ψ_n, it follows that $\mathcal{M}, \emptyset \models \varphi$.

6. Suppose $\varphi = \psi \varovee \theta$ for some ML$(= (...), \varovee)$-formulae ψ and θ. Assume as the induction hypothesis that $\mathcal{M}, \emptyset \models \psi$ and $\mathcal{M}, \emptyset \models \theta$. Then, by the truth definition, $\mathcal{M}, \emptyset \models \psi \varovee \theta$.

Thus $\mathcal{M}, \emptyset \models \varphi$ for all ML$(= (...), \varovee)$-formulae φ, by the induction principle. Since the structure $\mathcal{M}$ was arbitrarily chosen, it therefore follows that ML$(= (...), \varovee)$ has the empty team property. $\square$

3.3 Team Bisimulation

Definition 3.10 (Team k-bisimulation)**.** Let $(\mathcal{M}, T), (\mathcal{M}', T') \in \mathcal{KT}(\mathbf{P})$ and $k \in \mathbb{N}$. The structures $(\mathcal{M}, T)$ and $(\mathcal{M}', T')$ are *team k-bisimilar*, denoted $\mathcal{M}, T[\sim_k]\mathcal{M}', T'$, if

1. for each world $w \in T$, there exists a world $w' \in T'$ such that $\mathcal{M}, w \sim_k \mathcal{M}', w'$,

2. for each world $w' \in T'$, there exists a world $w \in T$ such that $\mathcal{M}, w \sim_k \mathcal{M}', w'$.

Lemma 3.11. *Let $k \in \mathbb{N}$, and assume that $(\mathcal{M}, T), (\mathcal{M}', T') \in \mathcal{KT}(\mathbf{P})$ are such that $\mathcal{M}, T[\sim_{k+1}]\mathcal{M}', T'$. Then*

i) for every team S such that $T[E]S$ there exists a team S' such that $T'[E]S'$ and $\mathcal{M}, S[\sim_k]\mathcal{M}', S'$,

ii) for every team S' such that $T'[E']S'$ there exists a team S such that $T[E]S$ and $\mathcal{M}, S[\sim_k]\mathcal{M}', S'$,

iii) $\mathcal{M}, S[\sim_k]\mathcal{M}', S'$ for $S = E^{\mathcal{M}}[T]$ and $S' = E'^{\mathcal{M}'}[T']$,

iv) for all $T_1, T_2 \subseteq T$ such that $T_1 \cup T_2 = T$ there exists teams $T_1', T_2' \subseteq T'$ such that $T_1' \cup T_2' = T'$ and $\mathcal{M}, T_i[\sim_{k+1}]\mathcal{M}', T_i'$ for $i \in \{1, 2\}$.

Proof. i) Suppose that $T[E]S$. We define the team

$$S' = \{w' \in E'^{\mathcal{M}}[T'] \mid \exists w \in S \text{ such that } \mathcal{M}, w \sim_k \mathcal{M}', w'\}.$$

Now we need to show that this S' is such that $T'[E]S'$ and $\mathcal{M}, S[\sim_k]\mathcal{M}', S'$. First of, by the definition of S' it holds that $S' \subseteq E'^{\mathcal{M}}[T']$. Let $w' \in T'$. Now since $\mathcal{M}, T[\sim_{k+1}]\mathcal{M}', T'$, there exists a world $w \in T$ such that $\mathcal{M}, w \sim_{k+1} \mathcal{M}', w'$. As $T[E]S$, there exists a $v \in S$

such that wEv holds, and because $\mathcal{M}, w$ and $\mathcal{M}', w'$ are $k+1$-bisimilar, there exists a $v' \in T'[E'^{\mathcal{M}}]$ such that $w'E'v'$ and $\mathcal{M}, v \sim_k \mathcal{M}', v'$. Therefore v' fulfils the condition of S' and hence $T' \subseteq E^{-1}[S']$.

In order to show that $\mathcal{M}, S[\sim_k]\mathcal{M}', S'$ we need find a k-bisimilar companion to any arbitrary world in each team. By definition each world in S' is k-bisimilar to a world in S, hence it suffices to show that for each world in S there exists a world in S' bisimilar to it. To that end suppose $v \in S$. Since $T[E]S$ there exists a world $w \in T$ such that wEv, and because $\mathcal{M}, T$ is team $k+1$-bisimilar to $\mathcal{M}, T'$ there exists a world $w' \in T'$ such that $\mathcal{M}, w \sim_{k+1} \mathcal{M}', w'$. By the definition of $k+1$-bisimilarity it now follows that there exists a world $v' \in W'$ such that $w'E'v'$ and $\mathcal{M}, v \sim_k \mathcal{M}', v'$. Now $v' \in S'$ by the definition of S', and therefore $\mathcal{M}, S[\sim_k]\mathcal{M}', S'$.

ii) This proof is analogous to part *i)* above.

iii) Suppose $v \in S$. Since $S = E^{\mathcal{M}}[T]$ there exists a world $w \in T$ such that wEv, and due to $\mathcal{M}, T[\sim_{k+1}]\mathcal{M}', T'$ there exists a world $w' \in T'$ such that $\mathcal{M}, w \sim_{k+1} \mathcal{M}', w'$. Now the definition of $k+1$-bisimilarity states that there exists a world $v' \in W'$ such that $w'E'v'$ and $\mathcal{M}, v \sim_k \mathcal{M}', v'$. By definition $v' \in S'$.

Now suppose $v' \in S'$. By our assumption there exists a world $w' \in T'$ such that $w'E'v'$, and since $\mathcal{M}, T[\sim_{k+1}]\mathcal{M}', T'$ there exists a world $w \in T$ such that $\mathcal{M}, w \sim_{k+1} \mathcal{M}', w'$. Now by the definition of $k+1$-bisimilarity, there exists a world $v \in W$ such that wEv and $\mathcal{M}, v \sim_k \mathcal{M}', v'$. Due to the assumption that $S = E^{\mathcal{M}}[T]$, it follows that $v \in S$. Hence $\mathcal{M}, S[\sim_k]\mathcal{M}', S'$.

iv) Let $T_1, T_2 \subseteq T$ such that $T_1 \cup T_2 = T$. Next we define the teams T_1' and T_2' as follows:

$$T_i' = \{w' \in T' \mid \exists w \in T_i \text{ such that } \mathcal{M}, w \sim_{k+1} \mathcal{M}', w'\},$$

where $i \in \{1, 2\}$. This definition guarantees that $T_1' \cup T_2' = T'$, since each world in T' is $k+1$-bisimilar to some world in either T_1 or T_2. Furthermore the definition specifies that each world in either T_1' or T_2' has a $k+1$-bisimilar counterpart in T_1 or T_2 respectively. Hence it suffices to show that for each world w in T_1 or T_2 there exists a world w' in T_1' or T_2' such that $\mathcal{M}, w \sim_{k+1} \mathcal{M}', w'$. To that end let $w \in T_i$, where $i = 1$ or $i = 2$. Now $w \in T$, and hence it has a corresponding world w' in T', such that $\mathcal{M}, w \sim_{k+1} \mathcal{M}', w'$. Since the world w' has a $k+1$-bisimilar world in T_i, it by definition belongs to T_i'. Therefore $\mathcal{M}, T_i[\sim_{k+1}]\mathcal{M}', T_i'$, for $i \in \{1, 2\}$. $\square$

3.4 A van Benthem theorem for $\mathrm{ML}(\varovee)$

Theorem 3.12. *The class $\mathcal{K} \subseteq \mathcal{KT}(\mathbf{P})$ is definable in $\mathrm{ML}(\varovee)$ if and only if $\mathcal{K}$ is downward closed, $\mathcal{K}$ has the empty team property, and there exists $k \in \mathbb{N}$ such that $\mathcal{K}$ is invariant under team k-bisimulations.*

Proof. First we will show that for all classes definable in ML($\oslash$) there exists a $k \in \mathbb{N}$ such that the class is invariant under k-bisimulations. This is sufficient since Lemma 3.9 shows ML($\oslash$) is downward closed and has the empty team property. Let $\varphi \in$ ML($\oslash$) and $k = \mathrm{md}(\varphi)$. We will prove by induction over the structure of φ that the class $\|\varphi\|$ is closed under k-bisimulations. Only one direction of the equivalences in the arguments of closure under k-bisimulation will be described, since the other direction is analogous. The basis cases where $k = 0$ are proven as follows:

1. Suppose $\varphi = Px$ for some $P \in \mathbf{P}$ and assume $\mathcal{M}, T \models \varphi$ and $\mathcal{M}, T[\sim_0]\mathcal{M}', T'$. Now $\mathcal{M}, w \models Px$ for all $w \in T$, and hence by the definition of team k-bisimulation it follows that $\mathcal{M}', w' \models Px$ for all $w' \in T'$. Thus $\mathcal{M}', T' \models \varphi$.

2. Let $\varphi = \neg Px$ for some $P \in \mathbf{P}$ and assume that $\mathcal{M}, T \models \varphi$ and $\mathcal{M}, T[\sim_0]\mathcal{M}', T'$. By the definition $\mathcal{M}, w \not\models Px$ for all $w \in T$, and therefore $\mathcal{M}', w' \not\models Px$ for all $w' \in T'$. Hence $\mathcal{M}', T' \models \neg Px$.

In the induction cases we suppose the claim holds for the subformulae and aim to show that it holds for their superformula.

Suppose $\varphi = \psi_1 \wedge \psi_2$ for some ML($\oslash$)-formulae ψ_1 and ψ_2, where

$$k = \mathrm{md}(\varphi) = \max(\mathrm{md}(\psi_1), \mathrm{md}(\psi_2)).$$

As the induction hypothesis assume if $\mathcal{M}, R[\sim_k]\mathcal{M}', R'$ and $\mathcal{M}, R \models \psi_i$ then $\mathcal{M}', R' \models \psi_i$ for all teams $R \subseteq W$, $R' \subseteq W'$ and indices $i \in \{1, 2\}$. Now assume $\mathcal{M}, T \models \varphi$ and $\mathcal{M}, T[\sim_k]\mathcal{M}', T'$. In other words $\mathcal{M}, T \models \psi_1$ and $\mathcal{M}, T \models \psi_2$. Hence by the induction hypothesis $\mathcal{M}', T' \models \psi_i$ for $i \in \{1, 2\}$, and therefore $\mathcal{M}', T' \models \varphi$.

Suppose $\varphi = \psi_1 \vee \psi_2$ for some ML($\oslash$)-formulae ψ_1 and ψ_2, where

$$k = \mathrm{md}(\varphi) = \max(\mathrm{md}(\psi_1), \mathrm{md}(\psi_2)).$$

In this case the induction hypothesis is if $\mathcal{M}, R \models \psi_i$ and $\mathcal{M}, R[\sim_k]\mathcal{M}', R'$, then $\mathcal{M}', R' \models \psi_i$ for all teams $R \subseteq W$, $R' \subseteq W'$ and indices $i \in \{1, 2\}$. Assume then $\mathcal{M}, T \models \varphi$ and $\mathcal{M}, T[\sim_k]\mathcal{M}', T'$. The definition of team Kripke semantics states that there exists subteams $T_1, T_2 \subseteq T$, such that $T_1 \cup T_2 = T$ and $\mathcal{M}, T_i \models \psi_i$ for $i \in \{1, 2\}$. Now by Lemma 3.11. there exists subteams $T_1', T_2' \subseteq T'$, such that $T_1' \cup T_2' = T'$, $\mathcal{M}, T_1[\sim_m]\mathcal{M}', T_1'$ and $\mathcal{M}, T_2[\sim_n]\mathcal{M}, T_2'$, where $m, n \in \mathbb{N}$ and $m, n \leq k$. Consequently it follows from the induction hypothesis that $\mathcal{M}', T_i' \models \psi_i$ for $i \in \{1, 2\}$. Therefore $\mathcal{M}', T' \models \varphi$.

Let $\varphi = \psi_1 \oslash \psi_2$ for some ML($\oslash$)-formulae ψ_1 and ψ_2, where

$$k = \mathrm{md}(\varphi) = \max(\mathrm{md}(\psi_1), \mathrm{md}(\psi_2)).$$

Suppose for the induction hypothesis that if $\mathcal{M}, R \sim_k \mathcal{M}', R'$ and $\mathcal{M}, R \models \psi_i$ then $\mathcal{M}', R' \models \psi_i$ for all teams $R \subseteq W$, $R' \subseteq W'$ and indices $i \in \{1, 2\}$. Assume $\mathcal{M}, T \models \varphi$ and $\mathcal{M}, T[\sim_k]\mathcal{M}', T'$. Now $\mathcal{M}, T \models \psi_i$ for either $i \in \{1, 2\}$. By the induction hypothesis $\mathcal{M}', T' \models \psi_i$ for either $i = 1$ or $i = 2$. Hence $\mathcal{M}', T' \models \varphi$.

Suppose $\varphi = \Diamond\psi$ for some ML($\varovee$)-formula ψ, where $k = \mathrm{md}(\psi) + 1$. In this case the induction hypothesis is if $\mathcal{M}, R[\sim_{k-1}]\mathcal{M}', R'$ and $\mathcal{M}, R \models \psi$ then $\mathcal{M}', R' \models \psi$ for all teams $R \subseteq W$ and $R' \subseteq W'$. Assume $\mathcal{M}, T \models \varphi$ and $\mathcal{M}, T[\sim_k]\mathcal{M}', T'$. Now there exists a team $S \subseteq W$ such that $T[E]S$ and $\mathcal{M}, S \models \psi$, and by Lemma 3.11. there exists a team $S' \subseteq W'$ such that $T'[E']S'$ and $\mathcal{M}, S[\sim_{k-1}]\mathcal{M}', S'$. Now by the induction hypothesis $\mathcal{M}', S' \models \psi$, and hence $\mathcal{M}', T' \models \varphi$.

Suppose $\varphi = \Box\psi$ for some ML($\varovee$)-formula ψ, for which $k = \mathrm{md}(\psi) + 1$. As in the previous case, the induction hypothesis is if $\mathcal{M}, R[\sim_{k-1}]\mathcal{M}', R'$ and $\mathcal{M}, R \models \psi$ then $\mathcal{M}', R' \models \psi$, for all teams $R \subseteq W$ and $R' \subseteq W'$. Assume $\mathcal{M}, T \models \varphi$ and $\mathcal{M}, T[\sim_k]\mathcal{M}', T'$. Now we denote $S = E^{\mathcal{M}}[T]$ and $S' = E'^{\mathcal{M}'}[T']$, and notice that by Lemma 3.11. $\mathcal{M}, S[\sim_{k-1}]\mathcal{M}', S'$. Hence by the induction hypothesis $\mathcal{M}', S' \models \psi$, and therefore $\mathcal{M}', T' \models \varphi$.

By the induction principle all classes defined by ML($\varovee$)-formulae are closed under team k-bisimulation, where k is the modal depth of the defining formula.

Next we will show the other direction; if a class of Kripke structures is downward closed, has the empty team property, and is closed under k-bisimulation for some k, then the class is ML($\varovee$)-definable. To that end, assume $\mathcal{K}$ is a downward closed class of Kripke structures closed under team k-bisimulations for some $k \in \mathbb{N}$. Now let

$$\psi = \bigovee_{(\mathcal{M},T)\in\mathcal{K}} \bigvee_{w\in T} \chi^k_{\mathcal{M},w}.$$

As discussed in the proof of Theorem 2.29. there are a finite amount of Hintikka formulae up to equivalence, and hence there is only a finite number of pointed Kripke structures up to bisimulation. Additionally, since $\mathcal{K}$ is closed up to team k-bisimulations, it consists of combinations of equivalence classes of Kripke structures. Therefore the number of unique elements in $\mathcal{K}$ is at most the size of the power set of the equivalence classes, and since there is a finite number of equivalence classes we see that ψ is equivalent to a finite formula, which is a restriction of ψ to a single representative per equivalence class. Let φ be that restricted formula, for which it holds that $\varphi \in$ ML($\varovee$).

In order to show that φ defines $\mathcal{K}$ we need to show that every pointed Kripke structure in $\mathcal{K}$ satisfies φ and if a pointed Kripke structure satisfies φ, then it belongs to $\mathcal{K}$. For the first direction, assume that $(\mathcal{M}_0, T_0) \in \mathcal{K}$. Now $\mathcal{M}_0, \{v\} \models \chi^k_{\mathcal{M}_0,v}$ for every $v \in T_0$. Thus $\mathcal{M}_0, T_0 \models \bigvee_{w\in T_0} \chi^k_{\mathcal{M}_0,w}$, and therefore $\mathcal{M}_0, T_0 \models \psi$. Now $\mathcal{M}_0, T_0 \models \varphi$, since ψ and φ are equivalent.

Now consider the other direction; suppose $\mathcal{M}_0, T_0 \models \varphi$. Equivalently then $\mathcal{M}_0, T_0 \models \psi$, and hence there exists a Kripke structure and team $(\mathcal{M}_1, T_1) \in \mathcal{K}$ such that $\mathcal{M}_0, T_0 \models \bigvee_{w \in T_1} \chi^k_{\mathcal{M}_1,w}$. Thereby there are subteams T_w of T_0, such that $\bigcup_{w \in T_1} T_w = T_0$ and $\mathcal{M}_0, T_w \models \chi^k_{\mathcal{M}_1,w}$. Now we shall construct a subteam of T_1, containing precisely the worlds corresponding to the worlds of T_0. Let T' be the subteam of T_1 such that $T' = \{w \in T \mid T_w \neq \emptyset\}$. Seeing as $\mathcal{K}$ is downward closed, $(\mathcal{M}_1, T_1) \in \mathcal{K}$ and $T' \subseteq T_1$, it follows that $(\mathcal{M}_1, T') \in \mathcal{K}$. The subteam T' now contains exactly the worlds corresponding to the worlds of T_0, since for every world $v \in T_0$ there exists a world $w \in T'$ such that $\mathcal{M}_0, v \models \chi^k_{\mathcal{M}_1,w}$ and for every world $w \in T'$ there is at least one world $v \in T_0$ such that $\mathcal{M}_0, v \models \chi^k_{\mathcal{M}_1,w}$. Consequently by Lemma 2.21. for each world $v \in T_0$ there is a $w \in T'$ such that $\mathcal{M}_0, v \sim_k \mathcal{M}_1, w$, and for every world $w \in T'$ there is a $v \in T_0$ such that $\mathcal{M}_0, v \sim_k \mathcal{M}_1, w$. Therefore $\mathcal{M}_0, T_0[\sim_k]\mathcal{M}_1, T'$, and since $\mathcal{K}$ is closed under team k-bisimulations we now have $(\mathcal{M}_0, T_0) \in \mathcal{K}$. □

3.5 EMDL $\equiv$ ML($\varovee$)

We have now proven a van Benthem-theorem for the logic ML($\varovee$), however that was not our goal. Next we will go through some definitions and intermediary results leading us to the realisation that EMDL and ML($\varovee$) have the same expressive power. In other words we will show that EMDL and ML($\varovee$) define the same classes of Kripke structures and hence the previous result also holds for EMDL. With that in mind, we are on to the first stepping stone.

Definition 3.13 (Type)**.** Let Φ be a finite set of ML($\mathbf{P}$)-formulae, and let $\mathcal{M}$ be a Kripke structure with the distinguished world w. The Φ-*type* of w in $\mathcal{M}$ is defined

$$\mathrm{tp}_\Phi(\mathcal{M}, w) = \{\varphi \in \Phi \mid \mathcal{M}, w \models \varphi\}.$$

For a team T of $\mathcal{M}$ the Φ-type is defined by way of the types of its constituent worlds:

$$\mathrm{Tp}_\Phi(\mathcal{M}, T) = \{\mathrm{tp}_\Phi(\mathcal{M}, w) \mid w \in T\}.$$

Lemma 3.14. *Suppose Φ is a finite set of* ML*-formulae and $\Gamma \subseteq \Phi$. Let*

$$\theta_\Gamma = \bigwedge_{\varphi \in \Gamma} \varphi \wedge \bigwedge_{\varphi \in \Phi \setminus \Gamma} \varphi^\neg,$$

where $\varphi^\neg$ is the negation of φ with the negations pushed to the atomic level. Then $\mathrm{tp}_\Phi(\mathcal{M}, w) = \Gamma$ *if and only if* $\mathcal{M}, w \models \theta_\Gamma$.

Proof. Suppose $\mathrm{tp}_\Phi(\mathcal{M}, w) = \Gamma$. By definition this is true if and only if $\mathcal{M}, w \models \varphi$ for all $\varphi \in \Gamma$ and $\mathcal{M}, w \not\models \psi$ for all $\psi \in \Phi \backslash \Gamma$, and hence $\mathcal{M}, w \models \bigwedge_{\varphi \in \Gamma} \varphi$ and $\mathcal{M}, w \models \bigwedge_{\varphi \in \Phi \backslash \Gamma} \varphi^{\neg}$. Therefore equivalently $\mathcal{M}, w \models \theta_\Gamma$. □

Lemma 3.15. *Assume $(\mathcal{M}, T), (\mathcal{M}', T') \in \mathcal{KT}(\mathbf{P})$, and let Φ be a finite set of* ML-*formulae.*

1. *For each $\varphi \in \Phi$, $\mathcal{M}, T \models \varphi$ if and only if $\varphi \in \bigcap \mathrm{Tp}_\Phi(\mathcal{M}, T)$.*
2. *If $\mathcal{M}, T \models \bigcirc\!\!\!\!\bigvee \Phi$ and $\mathrm{Tp}_\Phi(\mathcal{M}', T') \subseteq \mathrm{Tp}_\Phi(\mathcal{M}, T)$, then $\mathcal{M}', T' \models \bigcirc\!\!\!\!\bigvee \Phi$.*

Proof. 1) Suppose $\mathcal{M}, T \models \varphi$. Now by the flatness property of ML it holds that $\mathcal{M}, w \models \varphi$ for all $w \in T$, and hence $\varphi \in \mathrm{tp}_\Phi(\mathcal{M}, w)$ for every $w \in T$. Thereby $\varphi \in \bigcap \mathrm{Tp}_\Phi(\mathcal{M}, T)$. For the other direction suppose $\varphi \in \bigcap \mathrm{Tp}_\Phi(\mathcal{M}, T)$. Now $\varphi \in \mathrm{tp}_\Phi(\mathcal{M}, w)$ for all $w \in T$, and consequently $\mathcal{M}, w \models \varphi$ for all $w \in T$. By the definition of team Kripke semantics $\mathcal{M}, T \models \varphi$.

2) Assume $\mathcal{M}, T \models \bigcirc\!\!\!\!\bigvee \Phi$ and $\mathrm{Tp}_\Phi(\mathcal{M}', T') \subseteq \mathrm{Tp}_\Phi(\mathcal{M}, T)$. Now $\mathcal{M}, T \models \varphi$ for some $\varphi \in \Phi$, and by the previous part $\varphi \in \bigcap \mathrm{Tp}_\Phi(\mathcal{M}, T)$. Seeing as φ is in all of the constituent types of $\mathrm{Tp}_\Phi(\mathcal{M}, T)$, it is especially holds that $\varphi \in \bigcap \mathrm{Tp}_\Phi(\mathcal{M}', T')$, since $\mathrm{Tp}_\Phi(\mathcal{M}', T') \subseteq \mathrm{Tp}_\Phi(\mathcal{M}, T)$. Again by the above $\mathcal{M}', T' \models \varphi$, and thus $\mathcal{M}', T' \models \bigcirc\!\!\!\!\bigvee \Phi$. □

Definition 3.16 (γ-formulae)**.** Suppose Φ is a finite set of ML-formulae. The γ-formula of Φ is an EMDL-formula, defined by

$$\gamma^\Phi = \bigwedge_{\varphi \in \Phi} = (\varphi).$$

We define k-γ-formulae recursively through $\gamma_0 = Px \wedge \neg Px$ and $\gamma_{k+1} = (\gamma_k \vee \gamma^\Phi)$.

Basically what the γ^Φ-formula says is that a team is undivided in its interpretation of the formulae of Φ. It is easy to see that $\mathcal{M}, T \models \gamma^\Phi$ if and only if $|\mathrm{Tp}_\Phi(\mathcal{M}, T)| \leq 1$, since T satisfies γ^Φ only if all the worlds of T satisfy the same formulae of Φ, hence being of the same type, or if T is empty.

Lemma 3.17. *For all $k \in \mathbb{N}$, $\mathcal{M}, T \models \gamma_k$ if and only if $|\mathrm{Tp}_\Phi(\mathcal{M}, T)| \leq k$.*

Proof. The claim is proved by induction over k. For the basis case suppose $k = 0$. Suppose $\mathcal{M}, T \models \gamma_0$. Since γ_0 is a contradiction, and no possible world can satisfy both Px and $\neg Px$. Thus $T = \emptyset$. Now, since $\mathrm{Tp}_\Phi(\mathcal{M}, T)$ is not populated with any types, $\mathrm{Tp}_\Phi(\mathcal{M}, T) = \emptyset$. Therefore $|\mathrm{Tp}_\Phi(\mathcal{M}, T)| = 0$.

Suppose for the other direction that $|\mathrm{Tp}_\Phi(\mathcal{M}, T)| = 0$. Since the set is empty, there are no worlds in T providing it with types. An empty team satisfies all ML-formulae, and hence $\mathcal{M}, T \models \gamma_0$.

For the induction case suppose $\mathcal{M}, T \models \gamma_k$ if and only if $|\mathrm{Tp}_\Phi(\mathcal{M}, T)| \leq k$. Assume $\mathcal{M}, T \models \gamma_{k+1}$, whence there exists subteams $T_1, T_2 \subseteq T$ such that $T_1 \cup T_2 = T$ and $\mathcal{M}, T_1 \models \gamma_k$ and $\mathcal{M}, T_2 \models \gamma^\Phi$. Now $|\mathrm{Tp}_\Phi(\mathcal{M}, T_2)| \leq 1$ and by the induction hypothesis $|\mathrm{Tp}_\Phi(\mathcal{M}, T_1)| \leq k$. The type $\mathrm{Tp}_\Phi(\mathcal{M}, T)$ is the union of the two aforementioned types, and hence

$$|\mathrm{Tp}_\Phi(\mathcal{M}, T)| \leq |\mathrm{Tp}_\Phi(\mathcal{M}, T_1)| + |\mathrm{Tp}_\Phi(\mathcal{M}, T_2)| \leq k + 1.$$

For the other direction suppose $|\mathrm{Tp}_\Phi(\mathcal{M}, T)| \leq k + 1$. Now, since every Φ-type of worlds corresponds to at least one world, there exists subteams $T_1, T_2 \subseteq T$ such that $T_1 \cup T_2 = T$, $|\mathrm{Tp}_\Phi(\mathcal{M}, T_1)| \leq k$ and $|\mathrm{Tp}_\Phi(\mathcal{M}, T_2) \leq 1$. By the induction hypothesis and the definition of γ-formulae $\mathcal{M}, T_1 \models \gamma_k$ and $\mathcal{M}, T_2 \models \gamma$. Hence by the definition of team Kripke semantics $\mathcal{M}, T \models \gamma_{k+1}$. By the induction principle $\mathcal{M}, T \models \gamma_n$ if and only if $|\mathrm{Tp}_\Phi(\mathcal{M}, T)| \leq n$, for all $n \in \mathbb{N}$. □

Next we will construct an EMDL-formula that captures the notion that a given team contains worlds of a type not represented in another team. We will do this by using the properties of the disjunction in team semantics to divide the compared team into two pieces. These pieces either consist of worlds of some type foreign to the team under scrutiny or are of a size smaller than the considered team. This distinction will be formalised through the use of the previously introduced formulae θ and γ_k.

Lemma 3.18. *Let Φ be a finite set of* ML*-formulae. If $(\mathcal{M}, T) \in \mathcal{KT}$, $T \neq \emptyset$, then there is a formula $\xi_{\mathcal{M},T} \in$* EMDL *such that for every $(\mathcal{M}', T') \in \mathcal{KT}$*

$$\mathcal{M}', T' \models \xi_{\mathcal{M},T} \Leftrightarrow \mathrm{Tp}_\Phi(\mathcal{M}, T) \nsubseteq \mathrm{Tp}_\Phi(\mathcal{M}', T').$$

Proof. Suppose $|\mathrm{Tp}_\Phi(\mathcal{M}, T)| = k + 1$. We want to define a formula which separates a team satisfying it into components that either contain types not in $Tp_\Phi(\mathcal{M}, T)$ or are of a known size. To that end we define

$$\xi_{\mathcal{M},T} = (\bigvee_{\Gamma \in X} \theta_\Gamma) \vee \gamma_k,$$

where $X = \mathcal{P}(\Phi) \setminus \mathrm{Tp}_\Phi(\mathcal{M}, T)$. Let $(\mathcal{M}', T') \in \mathcal{KT}$ be an arbitrary pair. Now by the definition of team Kripke semantics and Lemma 3.17. $\mathcal{M}', T' \models \xi_{\mathcal{M},T}$ if and only if there are $T_1, T_2 \subseteq T'$ such that $T_1 \cup T_2 = T'$ and $\mathrm{Tp}_\Phi(\mathcal{M}', T_1) \subseteq X$ and $|\mathrm{Tp}_\Phi(\mathcal{M}', T_2)| \leq k$. The latter claim is equivalent with $|\mathrm{Tp}_\Phi(\mathcal{M}, T) \cap \mathrm{Tp}_\Phi(\mathcal{M}', T')| \leq k$, since T_1 is in essence the part of T' which is different from T and T_2 is the part common to both T and T'. This holds if and only if there exists some type in $\mathrm{Tp}_\Phi(\mathcal{M}, T)$ that is not in $\mathrm{Tp}_\Phi(\mathcal{M}', T')$, or in other words $\mathrm{Tp}_\Phi(\mathcal{M}, T) \nsubseteq \mathrm{Tp}_\Phi(\mathcal{M}', T')$. □

Theorem 3.19. $\mathrm{ML}(\varovee) \equiv \mathrm{EMDL}$.

Proof. We showed in Lemma 3.6. that $\mathrm{EMDL} \leq \mathrm{ML}(\varovee)$, and thus it now suffices to show that $\mathrm{ML}(\varovee) \leq \mathrm{EMDL}$. Let φ be a $\mathrm{ML}(\varovee)$-formula. By Theorem 3.12. φ can be written in the form $\bigovee \Phi$, for a finite set of ML-formulae Φ. We want to construct an EMDL-formula η such that $\|\varphi\| = \|\eta\|$. In order to reach this goal it suffices to find a formula that defines the Φ-types of the teams in $\|\varphi\|$. To this end we define

$$\eta = \bigwedge_{(\mathcal{M},T)\in\overline{\|\varphi\|}} \xi_{\mathcal{M},T},$$

where $\overline{\|\varphi\|} = \mathcal{KT} \setminus \|\varphi\|$ and $\xi_{\mathcal{M},T}$ is defined as in the previous proof. Now there is a finite number of unique $\xi_{\mathcal{M},T}$-formulae, since they are distinguished by the subsets Γ from $\mathcal{P}(\Phi) \setminus \mathrm{Tp}_{\Phi}(\mathcal{M},T)$, which is finite seeing as Φ is finite. Hence η is finite and therefore $\eta \in \mathrm{EMDL}$.

To show that η is the formula we are looking for, i.e. $\|\eta\| = \|\varphi\|$, let $(\mathcal{M}_0,T_0) \in \mathcal{KT}$. Suppose $(\mathcal{M}_0,T_0) \in \|\varphi\|$ and consider an arbitrary pair $(\mathcal{M},T) \in \overline{\|\varphi\|}$. Now $\mathcal{M}_0,T_0 \models \varphi$, $\mathcal{M},T \not\models \varphi$ and $\varphi = \bigovee \Phi$, hence if $\mathrm{Tp}_{\Phi}(\mathcal{M},T) \subseteq \mathrm{Tp}_{\Phi}(\mathcal{M}_0,T_0)$ this would break Lemma 3.15. Therefore $\mathrm{Tp}_{\Phi}(\mathcal{M},T) \nsubseteq \mathrm{Tp}_{\Phi}(\mathcal{M}_0,T_0)$, which by Lemma 3.18. implies $\mathcal{M}_0,T_0 \models \xi_{\mathcal{M},T}$. Thus $(\mathcal{M}_0,T_0) \in \|\eta\|$.

For the other direction assume that $(\mathcal{M}_0,T_0) \notin \|\varphi\|$. Now it holds trivially that $\mathrm{Tp}_{\Phi}(\mathcal{M}_0,T_0) = \mathrm{Tp}_{\Phi}(\mathcal{M}_0,T_0)$ and thus, by Lemma 3.18, $\mathcal{M}_0,T_0 \not\models \xi_{\mathcal{M}_0,T_0}$. Since by our assumption $(\mathcal{M}_0,T_0) \in \overline{\|\varphi\|}$, $\xi_{\mathcal{M}_0,T_0}$ is one of the conjuncts in η and therefore $\mathcal{M}_0,T_0 \not\models \eta$. Thus $(\mathcal{M}_0,T_0) \notin \|\eta\|$. $\square$

Now to recapitulate the preceding argument: we proved a van Benthem theorem for the logic $\mathrm{ML}(\varovee)$ and then proceeded to show that $\mathrm{ML}(\varovee) \equiv \mathrm{EMDL}$. In other words, by proving the van Benthem result for $\mathrm{ML}(\varovee)$, we also showed that it holds for EMDL. Hence we are ready to state a van Benthem theorem for the logic EMDL in the following corollary.

Corollary 3.20. *A class $\mathcal{K} \subseteq \mathcal{KT}$ is definable in* EMDL *if and only if $\mathcal{K}$ is downward closed, $\mathcal{K}$ has the empty team property, and there is a $k \in \mathbb{N}$ such that $\mathcal{K}$ is closed under k-bisimulation.*

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