The Extension Theory of Hermann Grassmann and the Creation of Vector Algebra

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Tiivistelmä – Referat – Abstract

Yleisesti Sir William Rowan Hamiltonia pidetään vektorialgebran luojana. Hän oli 1800- luvun alussa syntynyt arvostettu matemaatikko. Yritettyään vuosia löytää tapaa yleistää kompleksilaskenta kolmeen ulottuvuuteen, hän lopulta vuonna vuonna 1843 keksi kvaterniot, kompleksilukujen laajennuksen neliulotteiseksi luvuiksi. Hamiltonin kvaternihin pohjautuva järjestelmä on ensimmäinen julkaistu vektorialgebra. Hamilton oli jo aikanaan kuuluisa ja vaikutusvaltainen, mutta siitä huolimatta kului yli kaksikymmentä vuotta, ennen kuin hänen järjestelmäänsä alettiin käyttää laajemmin. Vuosisadan vaiheessa tämän järjestelmän pohjalta luotiin moderni vektorialgebra.

Mutta modernin järjestelmän pohjaksi harkittiin myös toista järjestelmää. Tämän oli luonut saksalainen kouluopettaja Hermann Grassmann (1809-1877). Päinvastoin kuin Hamilton, Grassmann ei ollut aikalaisilleen tunnettu saavutuksistaan matematiikan alalla. Itseasiassa, hän teki ensimmäinen matemaattinen julkaisunsa yli 30 vuotiaana. Grassmannia oli jo vuosia kiehtonut idea suorien yhteenlaskusta ja vuonna 1844 hän vihdoin julkaisi kirjan, joka lähti tuosta yksinkertaisesta ideasta ja päätyi yhdeksi ensimmäisistä oikeista vektorialgebrojen esityksestä ja ensimmäiseksi lineaarialgebran esitykseksi.

Valitettavasti hänen tuntemattomuutensa ja se että hänen aikalaisensa pitivät kirjaa pahamaineisen vaikealukuisena, varmisti, että kirja jäi ilman suurempaa huomiota. 15 vuotta myöhemmin hän päätti yrittää uudelleen ja alkoi kirjoittaa järjestelmästään parannettua versiota. Tämän hän lopulta julkaisi vuonna 1863 nimellä “Ausdehnungslehre” tai englanniksi “Extension Theory”. Tämäkin kirja jäi vaille suurempaa huomiota hänen elinaikanansa, mutta noin sata vuotta myöhemmin Grassmannin työt nousivat uuteen arvoon. Tämä johtui siitä, että “Extension Theory” on paljon muita aikalaisia järjestelmiä abstraktimpi ja määrittelee kaiken n-ulotteisesti, sen sijaan että asiat määriteltäisiin lähinnä kolmessa ulottuvuudessa (tai neljässä kvaternioiden tapauksessa). Tässä järjestelmässä ristitulon sijasta, meillä on ulkotulo ja sisätulo määritellään tämän ulkotulon kautta ja on paljon monimutkaisempi kuin normaalissa vektorialgebrassa. Tieteen kehittyessä tarpeet käsitellä asioita n-ulotteisesti kasvoivat ja asiat joidenka takia Grassmannin aikailaiset eivät hänen töitään ymmärtäneet, toivat hänen työnsä takaisin parrasvaloihin.

Avainsanat – Nyckelord – Keywords
Hermann Grassmann, Vektorialgebra, Lineaarialgebra, Matematiikan historia, William Rowan Hamilton, Extension Theory

Säilytyspaikka – Förvaringställe – Where deposited

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Chapter 1

Introduction

Sir William Rowan Hamilton is widely considered to be the father of vector calculus due to his discovery of quaternions in 1843. But in 1840 a German school teacher named Hermann Grassmann completed a 200-page essay called "Theorie der Ebbe und Flut" (Theory of Ebb and Flow), which is the first known instance of the notion of vector space and also the first use of linear algebra. Despite repeated attempts, Grassmann's ideas did not truly catch on during his lifetime. In the 1890s, there was a brief struggle for what vector algebra would be used and his work was mostly forgotten in favour of the modern system of vector algebra and to a lesser extent the system of quaternions it was based on. Yet in the last 60 years there has been a resurgence of interest in Grassmann and his work and arguably, he is now more relevant than ever before.

Part I of this essay will describe first the life of Sir William Rowan Hamilton as his story illustrates well the difficulties and challenges of creating and spreading a brand new branch of mathematics. His life and story are also an interesting parallel to the main topic of this essay, the life and work of Hermann Grassmann. Details are then given on Grassmann's life and especially the two of the three major works of vector (and linear) calculus written by him, "Theorie der Ebbe und Flut" and "Ausdehnungslehre" of 1843 (or A1). In Part II we will cover the basics of Grassmann's third and final major work, "Ausdehnungslehre" of 1862 (or A2) in greater detail. .

Chapter 2

Sir William Rowan Hamilton

At the turn of the 19th century, many mathematicians were slowly approaching the development of a system of vector calculus. Some like Caspar Wessel (1745-1818) in 1799, approached this by defining geometric ways to represent complex numbers. Others like August Ferdinand Möbius (1790-1868) in 1827 with "Der Barycentrische Calkul" approached it by the way of analytical geometry. But it was to be Sir William Rowan Hamilton and Hermann Grassmann, who would go on to write the first known major works of vector calculus.

By most accounts, Hamilton was one of the great mathematicians of the 19th century. By his own admission, he was not a physicist, but his work was immensely influential in the field. From his development of Hamiltonian mechanics (as it is now known) to his work on optics, his historical importance should not be underestimated. But in the field of mathematics, it is the quaternions that he is best known for and it is what will be mainly focused on here.
(O'Connor & Robertson 2015)

2.1 Before Quaternions

Between 3th and 4th of August 1805 at midnight, William Rowan Hamilton was born in Dublin, Ireland. From an early age he proved himself to be a child prodigy, under the tutelage of his uncle, a clergyman he had lived with since age three. By the time he was thirteen, he was at least partially versed in Arabic, French, German, Greek, Hebrew, Hindostanee, Italian, Latin, Malay, Persian, Sanskrit, Spanish and Syriac. He also studied astronomy, geography, literature, mathematics and religion and found an error in "Mécanique Céleste" by Laplace when he was sixteen.

In 1823 Hamilton received the highest marks on the entrance exam of Trinity College of Dublin. Then, during his first year in college, he was awarded an optime in Classics, a grade greater than the best grade which was only usually awarded once in 20 years. In 1826 he presented his now famous "THEORY OF SYSTEMS OF RAYS", in which he gave the characteristic function for optics. The same year, he also received an optime in both Classics and science. This was unheard of and Hamilton soon became a celebrity in the intellectual circles of Dublin. In 1827 he was hired as Andrew's Professor of Astronomy at the University of Dublin and the Royal Astronomer of Ireland. He was still an undergraduate at college at the time.

All in all Hamilton was exceedingly accomplished in multiple fields. His name was well known in scientific circles and he was a man from whom a lot was expected. Carl Gustav Jacob Jacobi (1804-1851) even called him the Lagrange of England and as a mathematician, he seemed to do no wrong.

But there was one problem that plagued Hamilton and gave him pause. From at least as far back as 1827 he tried to find three dimensional complex numbers or "triplets" as they were called. Wessel had discovered the geometrical representation of complex numbers in a plane at the end of the 18th century. This was also discovered/studied at least by five other mathematicians, Jean-Robert Argand (1768-1822), Abbé Buée (1746-1826), Carl Friedrich Gauss (1777-1855), C. V. Mourey (1791-1830) and John Warren (1796-1852). But Hamilton (and others) wanted to extend this concept into three dimensions. He thought that this would be very useful in geometrical calculations. He also wanted to find a way to calculate space and time in geometry. Negative numbers were somewhat frowned upon in many fields. Doubly so in geometry, which was deeply rooted in the real. How can one measure a negative length? Hamilton's reasoning for tor the use of negative numbers was time. One could not indeed measure a negative length, but one could measure negative time (i.e. the past). But regardless of how much he thought on it, a three dimensional complex calculus seemed to elude him.
(Crowe, p. 17-26; O'Connor & Robertson 1998; Wilkins n.d.)

2.2 Quaternions

Years later, on October 16 1843, Hamilton was walking to the Royal Irish Academy with his wife. While crossing the Brougham bridge, he had an epiphany and the equation

$$\boldsymbol{i}^2 = \boldsymbol{j}^2 = \boldsymbol{k}^2 = \boldsymbol{ijk} = -1$$

suddenly came to him. He couldn't make the system work for three variables, but he could with four. He was so exited by this idea, that he actually stopped at the bridge to carve the equation on it. The original carving has disappeared a long time ago, but to this day, the equation is written down on a commemorative plaque on the bridge. For this was the discovery of the quaternions, of hypercomplex numbers of the form

$$w + \boldsymbol{i}x + \boldsymbol{j}y + \boldsymbol{k}z,$$

where w, x, y and z are real numbers and $\boldsymbol{i}$, $\boldsymbol{j}$ and $\boldsymbol{k}$ are unit vectors that follow the laws

$$\begin{array}{ccc} \boldsymbol{ij} = \boldsymbol{k} & \boldsymbol{jk} = \boldsymbol{i} & \boldsymbol{ki} = \boldsymbol{j} \\ \boldsymbol{ji} = -\boldsymbol{k} & \boldsymbol{kj} = -\boldsymbol{i} & \boldsymbol{ik} = -\boldsymbol{j} \end{array}$$
$$\boldsymbol{ii} = \boldsymbol{jj} = \boldsymbol{kk} = -1$$

What Hamilton had found and later created in more formal manner was the first well-known significant number system that did not obey the normal arithmetic laws (Grassmann's "Theorie der Ebbe und Flut" only getting published in 1911, well after his death). This geometric interpretation of quaternions had significant differences to the geometric interpretation of normal complex numbers. While the real and imaginary part of a complex number could be thought of as their x- and y-coordinate in a plane, the same did not work for quaternions in three dimensions thanks to their four dimensional nature. Hamilton's interpretation was that the number w represented time and x, y and z represent the forces affecting an object in time. Thus Hamilton avoided thinking about quaternions as "truly" four dimensional. This was important for in geometry, just as there couldn't be negative length, there also couldn't be more than three dimensions.

Although he'd had a breakthrough, Hamilton knew that it would take a lot of work to develop his idea. According to him, he felt that it would be something worth spending ten or fifteen years on. He read his first paper on the matter four weeks after his epiphany. And four years later, he had published 34 papers on the subject.

And so Hamilton, a rising star in the scientific circles made his grand discovery. His status was such, that he did not originally have many detractors, even though his system was very revolutionary. In actuality, most simply did not understand it, and just assumed that Hamilton's genius was "above" them. One of the most telling examples of the reaction quaternions had at the time comes from an article published in the North American Review in 1857. The article in question is a review of "Lectures on Quaternions" of 1853. The review states that quaternions will be regarded as the great discovery of 19th century, but yet in America, less than 50 men have seen it and less than five have read it. It goes on to describe Hamilton in more and more grandiose manner with lines like:

> "And if the world should stand for twenty-three hundred years longer, the name Hamilton will be found, like that of Pythagoras, made immortal by its connection with the eternal truth first revealed through him."

Yet only moments later, the writer states that he has not in fact read the 800-page book in question. And so, even receiving rave reviews, Hamilton's system failed to initially catch

on. Hamilton had failed to write his system in a manner that was understandable to his peers. And so his system did not see widespread use during his lifetime. He continued to work on quaternions until his death in 1865.
(Crowe, p. 27-42; O'Connor & Robertson 1998; Wilkins n.d.)

2.3 Hamilton's Legacy

Even though Hamilton failed in spreading his system far, there was one specific group of people that he did manage to pass it on to, his students. As he was a college professor for most of his life, Hamilton taught many aspiring young mathematicians and physicists. Some of them were able to continue his work. And as many of them, like Peter Guthrie Tait, were physicists, they saw more the practical applications of the system. Physics, in fields like thermodynamics, had a growing need for more advanced mathematical systems. And in 1867 Tait wrote the textbook "ELEMENTARY TREATISE ON QUATERNIONS", the first of many publications on the matter. In addition to being a sort of beginners guide to quaternions (something that was sorely needed), the work also gave practical applications to physics. This finally saw the more widespread use of quaternions as it also gave people a relatively easily explained practical use for the system. The use of the system would continue to grow until Josiah Willard Gibbs and Oliver Heaviside helped create the modern vector calculus in the 1880s and 1890s by selectively altering and removing parts from the quaternion system .
(Crowe, p. 27-42; O'Connor & Robertson 1998; Wilkins n.d.)

Chapter 3

Hermann Grassmann

Unlike Hamilton, the influence of Grassmann, was not truly felt in the mathematical circles of the 19th century. While he did manage to win some acclaim during his life, this was mostly due to his work in other fields like linguistics and botany. Truly, if one were to write about the history of vector algebra in the 1950s, it would not be difficult to not mention Hermann Grassmann at all. This is because even though he was decades ahead of his time in many aspects, it was not until nearly a century after his death with authors like David Hestenes, that Grassmann's work in vector calculus became valued.

3.1 Early Life

On 15th of April 1809 in the town of Stettin, Prussia (now Szczecin, Poland), Johanne Luise Friederike Medenwald gave birth to her third child, Hermann Günther Grassmann. Hermann's father was Justus Günther Grassmann, who previously worked as an ordained minister, but had later become a mathematics and physics teacher at Stettin Gymnasium (effectively the high school of Stettin). He was also the author of many books on mathematics and physics. Johanne, who would go on to have nine more children, was a daughter of a minister and well educated and so taught Hermann from a young age. But unlike Hamilton, young Hermann did not distinguish himself in any particular way and even did badly enough that at one point his father told him that he might want to become a craftsman or a gardener instead of a mathematician. But through hard work, he managed to become the second ranked student at the Stettin Gymnasium by his third year.

Somewhat surprisingly, Grassmann did not focus his studies on mathematics. In 1827, he went to study theology at the University of Berlin. His courses included classical languages, literature, philosophy and theology, but no courses on either mathematics or physics. This was doubly strange as after he had completed his studies in 1830, he became determined to become a teacher of mathematics, presumably because of his father. For the next year, he immersed himself in mathematical studies, so that he could prepare himself for the examinations for a teaching license. In keeping with the fact that he had only studied mathematics seriously for a year, the exam did not go as one might have wanted. In December 1831, he took the exam but received only a level one teaching license, a license to teach at the lower levels (below university level), and was told that he would need a much better grasp of mathematics to receive a level two license (which gave the right to teach at all levels). As one of his goals in life was to ultimately become a professor at a university, this was not great news.

And so, in 1832 Grassmann went to work at the Stettin Gymnasium as an assistant teacher. He states in the foreword of "AUSDEHNUNGSLEHRE" of 1844 that this was the period, when he had his first mathematical discoveries that made him write the book in question. Following in his fathers footsteps, in 1834 Grassmann passed theology examinations at level one that allowed him to follow a career as a minister in the Lutheran Church. Despite this he instead continued his work as a teacher. He mostly worked in Stettin, but also in a few different schools in Berlin. He had not given up on teaching on a university level and after passing the level two test for theology by 1839, he set his sights on an examination that would give him the similar right to teach chemistry, mathematics, mineralogy and physics. The assignment for that particular examination was that he had to write an essay on the theory of tides.
(Crowe, p. 44-60; O'Connor & Robertson 2005; Petsche 2011, p. 3-6)

3.2 Theorie der Ebbe und Flut

For his essay Grassmann wrote "THEORIE DER EBBE UND FLUT", one of (if not the) first works of vector algebra. As stated earlier, Grassmann's father Justus, was a mathematician who had published multiple books. Two of them, "RAUMLEHRE" of 1811 (Theory of Space) and "TRIGONOMETRIE" (Trigonometry) contained an idea that a rectangle is a geometric product of its base and height and that this geometric product behaved similarly to normal arithmetic multiplication. This had led to Hermann thinking about concepts of geometric product and directed line. Interestingly the basics of his theory is based on Pierre-Simon Laplace's (1749-1827) "MÉCANIQUE CÉLESTE", the same book Hamilton had famously found an error in as a teenager. In his later writings he explains how he came up with the basic idea for "THEORIE DER EBBE UND FLUT".

Grassmann started by thinking about negatives in geometry. He was accustomed (most likely because of his father) to thinking that, if A and B are points, then displacements AB and BA were opposite magnitudes and so $AB + BA = 0$. Now if C is a point on the same line as A and B, then

$$AB + BC = AC,$$

even if C is situated between A and B. And, if that is the case, the AB and BC aren't just lengths, but are instead magnitudes with direction and length. This means that the sum of lengths is different from the sum of these magnitudes and that there was a need to clearly establish the rules by which such magnitudes were summed. And the easiest way to do this was to set that $AB + BC = AC$ was always true, even if the points A, B and C are not on the same straight line.

As his father had thought of the triangle as a product, Hermann realized that the same could be thought of parallelograms. They could be thought of as a product of two of their sides, if those sides were thought of as directed magnitudes. By joining the two ideas together, he came to the equation

$$A\,[B + C] = AB + AC$$

where A, B and C are now directed lines (i.e. vectors). Grassmann named this type of product the *geometric product.* This product was similar to the cross product of modern vector algebra, but it defined a directed area in the plane of the directed lines (perpendicular to the cross product). After this he also defined a linear product identical to the modern dot product.

To prepare himself for the examination, Grassmann applied these new methods to Lagrange's "Mécanique Analytique" and to his delight found that they made most calculations in the book considerably simpler. His new calculations were actually often more than ten times shorter than Lagrange's. Naturally he was greatly encouraged by this and so wrote "Theorie der Ebbe und Flut". The work was extremely revolutionary for its time, containing a large part of vector analysis. It included vector addition and subtraction, the two major vector products, vector differentiation and linear functions of linear algebra.

"Theorie der Ebbe und Flut" was the first important system of vector analysis and also the first major work in linear algebra. According to Michael J. Crowe it could be compared to non-Euclidean geometry in how revolutionary it was. Interestingly Crowe also states that Grassmann's later insistence on proving everything in n-dimensions could be thought of as a discovery of non-Euclidean geometry.

After submitting the essay it was given to its chief reader on the 26th of April 1840. The chief reader was mathematician Carl Ludwig Conrad and he returned the essay five days later with barely a comment. Grassmann did receive his level two teachers license allowing him to teach his chosen subjects at all levels. But it is safe to say that Conrad did not realize the importance of the work he had reviewed. "Theorie der Ebbe und Flut" was only published in 1911, 72 years after its writing.
(Crowe, p. 60-63; O'Connor & Robertson 2005)

3.3 AUSDEHNUNGSLEHRE of 1844

Grassmann was kept very busy by his career as a teacher, a career that he took very seriously. During the years he wrote many textbooks to be used at schools and two of them were published in 1842. But he did not forget about the ideas he had expressed in "THEORIE DER EBBE UND FLUT" and he became increasingly intrigued by a system that incorporate the ideas of that essay, but in a more general form, not restricted to three dimensions. And so, in the spring of 1842 Grassmann started working on "AUSDEHNUNGSLEHRE" (or A1 as it is sometimes known). The work was completed by fall of 1843 under the full title of "DIE LINEALE AUSDEHNUNGSLEHRE, EIN NEUER ZWEIG DER MATHEMATIK DARGESTELLT UND DURCH ANWENDUNGEN AUF DIE ÜBRIGEN ZWEIGE DER MATHEMATIK, WIE AUCH AUF DIE STATIK, MECHANIK, DIE LEHRE VOM MAGNETISMUS UND DIE KRYSTALLONOMIE ERLÄUTERT" (which translates roughly to The Linear Extension Theory, a new branch of mathematics presented and explained through application to other branches of mathematics, which include statics, mechanics, magnetics and crystallography) and was published in 1844.
(Crowe, p. 63-77; O'Connor & Robertson 2005)

3.3.1 Contents

In the foreword, in addition to explaining how he came to write "THEORIE DER EBBE UND FLUT" he also regales on his reasons and goals in writing "AUSDEHNUNGSLEHRE". Grassmann had for a long time thought that geometry should not be viewed as a branch of mathematics in the same way as arithmetics. This was because geometry was defined by real world limitations, in this case the limitation of three dimensions. He thought that there had to be a completely abstract system of calculus that's laws coincided with those of geometry. The system he had created for "THEORIE DER EBBE UND FLUT" was his first glimpse of this new region of science. And A1 was his attempt at formulating that system on a purely abstract level, without any assumptions from the real world.

So in A1 Grassmann created a purely formal system that was above and independent of geometry and also of all mathematics known to the time. Included here, rewritten and in an abbreviated form, is the first example in A1.

Let us say that a and b are contentless forms. We shall now define *connections* between these forms. A connection called a *synthetic connection* is symbolized by $\frown$ and it is defined by

$$a \frown b = b \frown a$$

and

$$(a \frown b) \frown c = a \frown (b \frown c) = a \frown b \frown c.$$

So, in a synthetic connection the parentheses can always be omitted without changing the result. On the other hand, an *analytical connection* is symbolized by $\smile$ and it is defined by

$$(a \smile b) \frown b = a,$$

$$a \smile b \smile c = a \smile c \smile b = a \smile (b \frown c)$$

and

$$a \smile (b \smile c) = a \smile b \frown c.$$

It should also be noted that these analytical connections are unique, so if

$$a \smile b = c,$$

then there does not exist a d such that

$$a \smile b = d \neq c.$$

At this point in the book Grassmann explained that synthetic connections could be called addition and analytical connections subtraction. He then went on to define an indifferent form (i.e. zero or null) and an analytic form $\smile b$ (i.e. $-b$). And finally two additional types of connection. The first was multiplication, a type of synthetic connection and the second was division, an analytical connection.

The work also contained many writings that were mainly philosophical in their content. As Grassmann was also highly interested in philosophy, which was also an extremely respected field of science at the time, this was meant to better explain the reasons for this abstract system to readers. Even though A1 contained many incredibly revolutionary results and definitions worthy of mention, we will only shortly go over some of them, as many will be gone over in greater detail as we go through A2 (as ““AUSDEHNUNGSLEHRE” of 1862 is sometimes known) later in this essay. The linear and geometric product had their names changed to inner and outer product respectively. Later in the book Grassmann also in practice defined the free linear space generated by “units” $e_1, e_2, \cdots$ and all of their linear combinations $a_1e_1 + a_2 + e_2 + \cdots$, where $a_1, a_2, \cdots$ are numbers. He then went on to define the addition of such linear combinations as well as multiplication with numbers. These were defined similarly as in modern vector algebra. The linear space properties of the operations was also formally proven. A1 also had definitions for (with different names) dimension, independence, subspace, join and meet of subspaces and span and

projections of elements onto subspaces. And notably, in the book Grassmann also invents exterior algebra (as it is now known). It also had very many other proofs and identities, for example the proof of the Steinitz Exchange Theorem, which received its name from German mathematician Ernst Steinitz (1871-1928) who proved it in 1913.
(Crowe, p. 63-77; O'Connor & Robertson 2005)

3.3.2 Reception of A1

It might not be terribly surprising to learn that readers at the time did not really understand what Grassmann was trying to achieve. The extreme abstractness added to the philosophical writings made the book infamously difficult for people to read, let alone understand. Even though philosophy was prevalent at the time, most mathematicians were still not actually philosophers and vice versa. As an effort to both promote his work and to get feedback, Grassmann sent copies of A1 to numerous eminent mathematicians to little effect. One was sent to Gauss, who sent a reply letter dated the 14th of December 1844. Gauss was cordial, but also said that he had worked on similar concepts 50 years earlier. He had also concluded that to truly understand Grassmann's book, he would have to spend a considerable amount of time acclimating himself to his peculiar terminology, and unfortunately he was far too busy to do so at the time. Gauss was most likely referring to his work on representing complex numbers geographically, when talking about having worked on similar concepts. Gauss apparently never did get the time to familiarize himself to A1, which was very unfortunate for Grassmann as Gauss was more qualified than most qualified to understand it.

Another mathematician who had A1 sent to the and also had the potential to understand it was Möbius, thanks of his work on Barycentric Calculus. Barycentric Calculus was similar to vector calculus, but it did not deal with lines, only with points. Yet having attempted to read it, even Möbius stated that A1 was simply unreadable. Grassmann actually still asked Möbius to write a review of A1, but Möbius, unsurprisingly, declined. He did go on to have a correspondence with Grassmann to help out his countryman. First Möbius tried to get Moritz Wilhelm Drobisch (1802-1896) to review the book, as Drobisch was a philosopher as well as a mathematician. Unfortunately he declined and so Möbius suggested to Grassmann that he himself should write a review of A1. This Grassmann went on to do and it was the only review that A1 ever received.

As a continued attempt at helping Grassmann, Möbius told him about a contest held by the Fürstliche Jablonowski'schen Gesellschaft der Wissenschaft (The First Jablonowski Society of Science). The entries had to solve a problem originally presented by Gottfried Wilhelm Leibniz (1646-1716), to establish geometric characteristic without using metric properties. As Grassmann had already done as much, this seemed like a good way to get his name and system known. And so Grassmann wrote "Die Geometrische Analyse

GEKNÜPFT UND DIE VON LEIBNIZ CHARACTERISTIK" (Geometric analysis and the Leibniz's characteristic, DIE GEOMETRISCHE ANALYSE for short), which did win the prize in 1846. This was less impressive than it sounds as his was the only entry. His work was also heavily criticized for being too abstract and unintuitive by one of the judges, Möbius.

By 1852, Hamilton had heard about a system similar to his own and so decided to read it. As did others, he also found the work extremely hard to understand, but he also thought that Grassmann was original and brilliant. He thought that, if only Grassmann had thought to combine his inner and outer products, then he too could have discovered quaternions. And so Hamilton became perhaps the first mathematician to actually appreciate Grassmann's work, but he also naturally thought that quaternions were superior and so concentrated on them.

Almost ten years after its publication in 1853, Möbius wrote in a letter that Carl Anton Bretschnider (1808-1878) was the only mathematician he knew that had read A1 in its entirety. Heinrich Richard Baltzer (1818-1887) said that trying to read A1 made him dizzy and Hamilton, wrote to Augustus De Morgan (1806-1871) that one should learn to smoke if one wanted to be able to read Grassmann.

During the 1850's an Italian mathematician named Giusto Bellavitis (1803-1880) learned of Grassmann. As his work on the Calculus of Equipollence (a more limited study of lines and points) was actually one of Grassmann's influences for writing A1, Bellavitis had more appreciation for his system than most. But Luigi Cremona (1830-1903) seemed to have been the only mathematician actually inspired by A1, during this period. He actually used methods from A1 to solve mathematical problems in an journal called "NOUVELLE ANNALES DE MATHÉMATIQUES", published in 1860 and credited him accordingly. He would go on to use Grassmann's ideas in at least one of his books.

To sum up the books reception, here is a quote from Grassmann's biography, written by Friedrich Engel (1861-1941) :

> "Thus Grassmann experienced what must be the most painful experience for the author of a new work: his book nowhere received attention; the public was completely silent about it; there was no one who discussed it or even publicly found fault with it."

And even more telling is the letter Grassmann received from his books publisher in 1876:

> "Your book Die Ausdehnungslehre has been out of print for some time. Since your work hardly sold at all, roughly 600 copies were in 1864 used as waste paper and the remainder, a few odd copies, have now been sold with the exception of one copy that remains in our library."

(Crowe, p. 77-89; O'Connor & Robertson 2005)

3.3.3 Priority Claim

In 1845 Adhémar Jean Claude Barré de Saint-Venant (1797-1886), better known as Comte de Saint-Venant, wrote a paper in which he defined, among other things discovered by Grassmann, the "produit géométrique" (the geometric product). After first hearing about this and then traveling 150 km to Berlin just to be able to read the paper, Grassmann decided to send a letter and a copy of A1 to Saint-Venant. As he did not know his address, he instead sent them to Augustin Cauchy (1789-1857), and asked him to forward them to Saint-Venant. Unfortunately Cauchy only forwarded the letter and not the book. Saint-Venant attempted to find a copy A1, but was unable to find a copy as the French Institute had the book, but mistakenly classified.

In 1853 Cauchy and Saint-Venant wrote papers on subjects Grassmann had already discovered. The following year, after Grassmann had had time to actually read them, Grassmann decided to make a priority claim on the paper Cauchy wrote, the "COMPTES RENDUS". The French Academy formed a committee of three mathematicians to deliberate on the matter. One of the members was Cauchy. The committee never made a decision before (or after) Cauchy died in 1857.
(Crowe, p. 77-89; O'Connor & Robertson 2005)

3.4 AUSDEHNUNGSLEHRE of 1862

In 1847 Grassmann became an Oberlehrer (senior teacher) at his school and decided to finally apply for a position at a university. He was not happy about still being a gymnasium teacher, even though he had written a mathematical work that he felt, was of great import. This was partly because, as a gymnasium teacher, he could not teach his his system to his students as it was too advanced. So he wrote an application to the Prussian Ministry of Education. They felt that they were unable to accurately gauge if Grassmann was worthy of a post and so asked the mathematician Eduard Kummer (1810-1893) to give his input on the matter. Kummer read through Grassmann's essay "DIE GEOMETRISCHE ANALYSE", but felt that it had good material, but just presented in a deficient form. This ended up permanently ending Grassmann's dreams of ever becoming a university professor.

As Grassmann became increasingly frustrated with people ignoring his work, he decided to rewrite A1. Originally he intended to write a sequel to it, but because of A1's bad reception thought that reworking the title would yield better results. In 1862, it was completed under the full title of "AUSDEHNUNGSLEHRE: VOLLSTÄNDING UND IN STENGER FORM BEARBEITET" (Linear extension theory: Fully processed in strict form), known as "EXTENSION THEORY" in English. Grassmann had 300 copies of A2 printed and the book very different from A1. Gone were all the philosophical musings and the book (which was a third longer than A1) was structured more like a modern textbook. This was Grassmann's attempt to make the book more readable, and this partially worked. But unfortunately A2 also had almost no physical applications or practical examples. The book had many new and exiting results, like the solution to the "Pfaffian Problem" fifteen years ahead of its time, but it was also, once again, exceedingly difficult for the people of the time to read.

As much the contents of the book will be gone over in length later, it suffices to go over its reception. A2 actually received even less attention than A1. Less copies of Grassmann A2 were printed and not even Grassmann wrote a review of the book this time. And the only reaction he received for the books he sent to other mathematicians was one letter of thanks. Engel describes its reception thus:

> "As in the first Ausdehnungslehre so in the second: matters which Grassmann had published in it were later independently rediscovered by others, and only much later was it realized that Grassmann had discovered them earlier."

(Crowe, p. 80-96; O'Connor & Robertson 2005)

3.5 Later Years

For the rest of his life Hermann Grassmann stayed very active. He published multiple language and mathematics textbooks as well as scientific papers. He was an editor for a political magazine and published materials for the evangelization of China. He was an active teacher, married and had 11 children. He made many contributions to linguistics, foremost of which was his translation of "RIG-VEDA" (Sacred text of the Hindu), for which he was given a honorary doctorate from the University of Tübingen. He also published writings on botany, music and religion. Grassmann did not completely stop writing mathematical works, but it was not his focus.

Gradually, after 1865, a few mathematicians started to pay attention to Grassmann. Victor Schlegel (1843-1905) was a fellow teacher at Grassmann's school who became interested in his work. He was very enthusiastic about Grassmann's calculus and even wrote papers using his methods. Unfortunately, he was not a gifted mathematician and his papers had practically no effect. The biography he later went on to write in 1878 for Grassmann was more influential, thanks to its extreme praising of the man.

In a cruel twist of fate, four men who might have had more success all died within a few years of becoming interested in Grassmann. They were Herman Henkel (1839-1873), Alfred Clebsch (1833-1872), William Klingdon Clifford (1845-1879) and Hermann Noth (1840-1882). Clebsch had actually learned of Grassmann's work while studying with his son Justus. All of them were inspired by Grassmann's work and wrote papers on the matter, though Clifford is the most influential of them. This slowly started to make Grassmann's name more known and actually this was enough for A1 to be called for a second printing shortly before Grassmann's death. On September 26, 1877 Hermann Grassmann died. Here is an excerpt from his biography, written by his son Justus Grassmann:

> "Only a few days before his death, Prof. Burmeister in Dresden informed him that he was thinking of giving lec[tures] on the Ausdehnugslehre in the upcoming winter semester; and with joy he could now leave to others that which an unfavourable fate had denied to himself."

(Crowe, p. 80-96; O'Connor & Robertson 2005)

Chapter 4

AUSDEHNUGSLEHRE of 1862 or EXTENSION THEORY

The 1862 "AUSDEHNUNGSLEHRE" is the most complete form of Grassmann's vector algebra, "extension theory". It has many similarities with the modern vector algebra, but it's closest spiritual successor is geometric algebra popularized by David Hestenes in the 1960s, which was based on work done by Clifford in the 1880's, which in turn was based on the work of Grassmann and Hamilton. This essay aims to go over the basics of extension theory, hopefully allowing people to better understand this (for its time) revolutionary system and at the same time shed some light into why the system was and is so hard for people to understand.

For the most part, the symbols and terms used by Grassmann might not be familiar for the modern reader and so an attempt will be made to "translate" terms to their vector calculus equivalents when appropriate. When possible, terms and symbols have been kept the same as to avoid confusion as one of the aims of this essay is to aid in reading the book.

One of the key difficulties in learning the system, is the abstractness of it. Grassmann was motivated by a desire to completely define an abstract algebra that encompassed geometry. Unfortunately because of this, his book is very sparse on practical examples on the applications of the system. Geometric applications are investigated only well after the beginning, over a hundred pages in. Coordinates are never used. One can only assume that Grassmann was worried about limiting the potential readers understanding of the system and thus stayed purposefully abstract. This might have been a relevant concern for the time, considering the entire concept of a vector algebra was foreign to almost all mathematicians of the time. Luckily modern readers are more cognizant to the possibilities of vectors and so some basic parallels to vector calculus will be included here and there. But it is good to remember that one should not fully equate such parallels because things in extension theory tend to be more general in nature than those in basic vector calculus.

An added difficulty is that the system is generated from a few definitions which are iteratively built upon, layer by layer. Because Grassmann had to do the book practically by himself, it does not have the sort of editing as one is accustomed to these days. Meaning, among other things, that it can be difficult to gauge the importance of any one theorem for the understanding of the system as a whole..

We will now go over the basics of extension theory hopefully giving readers an elementary understanding of it and its similarities and differences to modern vector algebra. As such, most proofs will be skipped, as most readers have already read similar proofs, and there will be periodically inserted clarifying texts. After explaining the basics of the system, we will mostly be looking into the product structure in Extension Theory as that is perhaps the most striking difference of the system compared to modern vector calculus.

4.1 Basics

Extensive Magnitude

Extension theory is based on *extensive magnitudes* (also called *magnitudes* for short). It is to be taken as a primitive concept that can represent anything that can be given a numerical amount. If a vector has a direction and magnitude, an extensive magnitude only needs to have the magnitude. They can represent anything that can have a numeric representation of magnitude. So for example $5a$ could be five apples, a cube with a volume of $5m^3$ or a vector a multiplied by the scalar 5. A *numerical magnitude* simply means a number.

Numerical Relation

A magnitude a is *derived* from magnitudes $b, c, ...$ if

$$a = \beta b + \gamma c + ...$$

where $\beta, \gamma, ...$ are real or irrational numbers. An assembly of magnitudes (i.e. a set) $a, b, c, \cdots$ is subject to *numerical relation* if one of them can be derived from the rest. If an assembly only has two magnitudes, neither of which is zero, it is symbolized by

$$a \equiv b$$

and say that a is *congruent* to b. From this we get that two real numbers are always congruent and that any assembly where one magnitude is 0 is subject to numerical relation.

Unit

We then say that a *unit* is any magnitude we use to derive other magnitudes from (similar to a basis vector). The unit of numbers is one and it is called the *absolute* unit. A *system of units* is an assembly of magnitudes that are not in any numerical relation to each other. (Grassmann 1862, p. 3)
Side comments:
It is notable that, as is proved later in A2, there is really no difference between a non-zero magnitude and a unit. This is because (similarly as with basis vectors) any non-zero magnitude may be taken as a unit. To a modern reader, accustomed to vector and linear algebra, these concepts would most likely feel very familiar. One could easily find similar theorems from modern vector calculus, if one were to change the term magnitude with vector and numerical relation with linear dependence. Or replace a system of units with basis. And so one would have some idea of the reason for why Grassmann describes these

things. These were not luxuries that any of Grassmann's peers had in the middle of 19th-century. And so, calling the system obtuse, would be an understatement for them.

But one noteworthy thing here is that one should not just equate magnitudes with vectors (at least, not in the Euclidean sense). All vectors are magnitudes but not vice versa. They are closer to the concept of the multivector in geometric algebra. In geometric algebra a 1-vector is a normal vector (a line segment), but a 2-vector (or bivector) is a plane segment and so forth. Even this is not really an accurate comparison because, for one, points can also be magnitudes. The sheer abstractness and generalness of the definitions in the system makes many things relatively simple in vector calculus, like the inner product, considerably more difficult. Of course, this should not be taken to mean that everything is harder to do in Extension Theory.

Derivation Numbers

Numbers used to derive a magnitude from a system of units are called the *derivation numbers* of that magnitude. For example, if

$$a = \alpha_1 e_1 + \alpha_2 e_2 + \cdots$$

where $\alpha_1, \alpha_2, \cdots$ are real numbers and $e_1, e_2, \cdots$ form a system of units, then a is a magnitude and $\alpha_1, \alpha_2, \cdots$ its derivation numbers. The magnitude can also be marked as

$$\sum \alpha e,$$

$$\sum \alpha_r e_r$$

or

$$\overset{[}{r} = 1]n \sum \alpha_r e_r.$$

Side Comment: It should be noted that $\sum \alpha e$ is the most common notation used by Grassmann in his book as shorthand notations for sums were the norm at the time. In this text, mainly the $\sum \alpha_r e_r$ form will be used, as it is more explicit. $\overset{[}{r} = 1]n \sum \alpha_r e_r$ will be used when required for clarity.

Basic Operations

If two extensive magnitudes are derived from the same system of units, their *addition* is simply adding together the derivation numbers with the same unit, meaning

$$\sum \alpha_r e_r + \sum \beta_r e_r = \sum (\alpha_r + \beta_r)\, e_r. \tag{4.1}$$

Subtraction is defined in the same way. To *multiply* an extensive magnitude by a number, you multiply all of its derivation numbers by the number in question, meaning

$$\sum \alpha_r e_r \cdot \beta = \sum (\beta \alpha_r)\, e_r. \tag{4.2}$$

Division is similarly defined, as long as $\beta \neq 0$.

Basic Properties

From (1) we get the following if a, b and c are extensive magnitudes:

1. $a + b = b + a$
2. $(a + b) + c = a + (b + c)$
3. $a + b - b = a$
4. $a + b - a = b$

(Grassmann 1862, p. 4-5)

Domain

For any given assembly of magnitudes $a_1, a_2, \ldots, a_n$, their *domain* is the collection of all magnitudes derivable from them. So if A is the domain of the magnitudes a_r $(1 \leq r \leq n)$, then $a \in A$ if

$$a = \sum \alpha_r a_r$$

for some α_r. If the domain cannot be derived from less than n such magnitudes, it is called a domain of nth *order*. A domain is of *zeroth* order if it only includes the magnitude zero.

Domain Relations

Domains$A$ and B are *identical* if $a \in A \Longleftrightarrow a \in B$
If $A \subset B$, then A is *subordinate* to B and B *superordinate* to A.
A and B are *incident*, if $A \subset B$ *or* $B \subset A$.
$A \cap B$ is the *common* domain of A and B.
$A + B$ is the *covering* domain of A and B.

Example

If domain A is derivable from units e_1, e_2, e_3 and domain B is derivable from units e_2, e_3, e_4, then their common domain is derivable from e_2, e_3 and their covering domain is derivable from e_1, e_2, e_3, e_4.
(Grassmann 1862, p. 8)
Side comments:
At this point one would likely wonder if a domain is a vector space. So we shall check for a domain A, (here $\alpha_r, \beta_r, \beta, \gamma$ are numbers):

1. Numbers (real and irrational) are a field, and the sum of two magnitudes in A is still in A (by definition). Same for scalar multiplication.

2. Associativity and commutativity were already noted.

3. If magnitude a is in A then so is $0a = 0$, thus 0 is in A.

4. If $\sum \alpha_r e_r$is in A, then so is $-\sum \alpha_r e_r = \sum (-\alpha_r) e_r$, so the additive inverse of all magnitudes are included in their domains.

5. $\beta (\gamma \sum \alpha_r e_r) = \beta (\sum (\gamma \alpha_r) e_r) = \sum (\beta\gamma\alpha_r) e_r = \beta\gamma (\sum \alpha_r e_r)$.

6. $1 (\sum \alpha_r e_r) = \sum \alpha_r e_r$.

7. $\gamma (\sum \alpha_r e_r + \sum \beta_r e_r) = \sum (\gamma\alpha_r + \gamma\beta_r) e_r = \sum (\gamma\alpha_r) e_r + \sum (\gamma\beta_r) e_r$
$= \gamma (\sum \alpha_r e_r) + \gamma (\sum \beta_r e_r)$.

8. $(\beta + \gamma) \sum \alpha_r e_r = \sum ((\beta + \gamma) \alpha_r) e_r = \sum (\beta\alpha_r + \gamma\alpha_r) e_r$
$= \beta (\sum \alpha_r e_r) + \gamma (\sum \alpha_r e_r)$.

And so, domains are vector spaces and a subordinate domain is a vector subspace.

Properties of Magnitudes and Domains

Next we will go through several theorems about the properties of assemblies of magnitudes, most of which should be very familiar to modern readers.

- Numerical relation between magnitudes $a_1, \cdots, a_n$ can also be given in a more familiar form,

$$\alpha_1 a_1 + \cdots + \alpha_n a_n = 0$$

  if $\alpha_1, \cdots, \alpha_n$ are not all simultaneously zero.

- In the previous case, it is always possible to separate from the n magnitudes, an assembly of magnitudes less than n, from which the rest can be derived.

- If we have a magnitude a_1, that can be derivable from an assembly of magnitudes $b_1, b_2, \cdots, b_n$, where $b_1 \neq 0$, then the domain of $b_1, b_2, \cdots, b_n$ is identical to that of $a_1, b_2 \cdots, b_n$.

- If magnitudes $a_1, \cdots, a_n$ stand in no numerical relation and can be derived from magnitudes $b_1, \cdots, b_n$, then their domains are identical.

- If the domain of an assembly of magnitudes $a_1, \cdots, a_n$ is derivable from less than n magnitudes $b_1, \cdots, b_m$, then $a_1, \cdots, a_n$ always stand in numerical relation.

- All domains of nth order can be derived from any n of its magnitudes that stand in no numerical relation to one another.

- If m and n are orders of the domains M and N, r the order of their common domain, and v of their covering domain, then

$$m + n = r + v.$$

  In vector algebra, this would be in the form of

$$dim\,(M) + dim\,(N) = dim\,(M \cap N) + dim\,(M + N)\,.$$

- If domains A and B, of orders α and β respectively, are in a domain N, with an order n, they must have a domain of $(\alpha + \beta - n)$th order in common if $\alpha + \beta > n$. Or in vector algebra, if

$$dim\,(A) + dim\,(B) > dim\,(N)$$

  and $A, B \subset N$, then

$$dim\,(A \cap B) = dim\,(A) + dim\,(B) - dim\,(N)\,.$$

- If $a = \sum \alpha_r e_r$ and $b = \sum \beta_r e_r$ are magnitudes in a domain of nth order $a = b$ only if $\alpha_r = \beta_r$ for all $1 \leq r \leq n$.

(Grassmann 1862, p. 8-13)

Shadow

Let $a_1, a_2, \cdots, a_n$ be magnitudes from which you can derive a domain of nth order A. If $a \in A$, then there are n real numbers $\alpha_1, \alpha_2, \cdots, \alpha_n$ so that

$$a = \alpha_1 a_1 + \alpha_2 a_2 + \cdots + \alpha_n a_n.$$

Now if $m < n$ then we call the magnitude

$$\alpha_1 a_1 + \alpha_2 a_2 + \cdots + \alpha_m a_m,$$

the "*shadow* of the magnitude a on the domain of $a_1, a_2, \cdots, a_m$, excluding the domain of $a_{m+1}, a_{m+2}, + \cdots + a_n$." We say that shadows of several magnitudes are taken *in the same sense* if the magnitudes are shadowed on the same domain and exclude the same domain. So, for example, if $a, c \in \mathbb{R}^3$ and $a = (1, 1, 1) = e_1 + e_2 + e_3$, $c = (2, 2, 2)$. Now if b is the shadow of a on to the domain of e_1, e_2 (i.e. the xy-plane), excluding the domain of the e_3 (the z-axis), then $b = e_1 + e_2$. And if d is c:s shadow in the same sense then $d = 2e_1 + 2e_2$.
(Grassmann 1862, p. 15-16)

Side comments: We have now gone over the first chapter or 18 pages of A2. In those pages Grassmann has goes through the basics of his system, and although different in many of the terms, they are also strikingly similar to many parts of modern vector calculus. It is noteworthy that nowhere in those 18 pages was there references to three dimensions or to geometry and the system is generated almost wholly in the abstract. There will also never be any real use of coordinates. This is in some ways very laudable, as overt focus on real applications has often stymied mathematical progress (as in the cases of 0, negative numbers, i or π). But it was hardly conducive to motivating people to keep reading, as they would not have been able to understand the point of it all.

From the first chapter one could easily think that extension theory is mostly similar to modern vector calculus and so far it has been. But that all changes rather drastically when we start investigating the part of A2 that is perhaps the most influential, the product structure.

4.2 Product Structure

Product

Let $a = [r = 1]n \sum \alpha_r e_r$ and $b = [s = 1]m \sum \beta_s e_s$ be extensive magnitudes. Now choose a domain of nth degree A, such that $a, b \in A$. Then $a = [r = 1]n \sum \alpha_r e_r$ and $b = [s = 1]n \sum \beta_s e_s$. Then their *product* $[ab]$, which is always a magnitude, is defined as

$$[ab] = \left[\left([r = 1]n \sum \alpha_r e_r \right) \left([s = 1]n \sum \beta_s e_s \right) \right] = [r = 1]n \sum [s = 1]n \sum \alpha_r \beta_s \left[e_r e_s \right]. \tag{4.3}$$

It is also said that a and b are *factors* of the product $[ab]$.

Example

If

$$P = [ab] = \left[\left(\alpha_1 e_1 + \alpha_2 e_2 \right) \left(\beta_1 e_1 + \beta_2 e_2 \right) \right],$$

then this would equal

$$\alpha_1 \beta_1 \left[e_1 e_1 \right] + \alpha_1 \beta_2 \left[e_1 e_2 \right] + \alpha_2 \beta_1 \left[e_2 e_1 \right] + \alpha_2 \beta_2 \left[e_2 e_2 \right].$$

As of yet, it is not defined what type of magnitudes the four products $[e_r e_s]$ are. Here are some examples of different ways they could be defined, forming different product structures.

1. We set all of the four products as different units that one can derive P from. Then $\alpha_1 \beta_1$, $\alpha_1 \beta_2$, $\alpha_2 \beta_1$ and $\alpha_2 \beta_2$ are its derivation numbers and then no establishing equations are necessary for the product structure.

2. By setting
$$e_1 e_2 = e_2 e_1,$$
you would only have three units $[e_1 e_1]$, $[e_1 e_2]$ and $[e_2 e_2]$ and their respective derivation numbers $\alpha_1 \beta_1$, $\alpha_1 \beta_2 + \alpha_2 \beta_1$ and $\alpha_2 \beta_2$. This product structure would be defined by its commutative property.

3. If
$$[e_1 e_1] = [e_2 e_2] = 0$$
and
$$[e_1 e_2] = - [e_2 e_1],$$

then, there would be only one unit for the product P and there would be only the derivation number $\alpha_1\beta_2 - \alpha_2\beta_1$.This product structure we call *combinatorial.*

4. If one were to set
$$[e_1e_1] = [e_2e_2] = 1$$
and
$$[e_1e_2] = [e_2e_1] = 0,$$
then there would also be only one unit for the product P and it is a number. From this we get that
$$P = \alpha_1\beta_1 + \alpha_2\beta_2.$$
This product structure we call *inner.*

(Grassmann 1862, p. 19-20)

General Product Identities

For now we will only look at the product structure in its most general form. From equation (3) we get a number of identities for product structures, in these a, b, $\cdots$, a_r, b_s and p are magnitudes and α,β,$\cdots$ are numbers:

1. $[(a + b + \cdots) p] = [ap] + [bp] + \cdots$
2. $[p (a + b + \cdots)] = [pa] + [pb] + \cdots$
3. $[ab] = [(\sum \alpha_r e_r) b] = \sum \alpha_r [e_r b]$
4. $[(\alpha a) b] = [a (\alpha b)] = \alpha [ab]$
5. $[(\alpha a + \beta b + \cdots) p] = \alpha [ap] + \beta [bp] + \cdots$
6. If $a = \sum \alpha_r a_r$ and $b = \sum \beta_s b_s$, then $[ab] = \overset{[}{r}= 1]n\sum \overset{[}{s}= 1]m\sum \alpha_r\beta_s [a_r b_s]$

(Grassmann 1862, p. 20-21)

These formulas also apply if the product has more than two magnitudes, for example, if a, b,$\cdots$ are magnitudes that all are in the same domain of nth order, then

(4.4)
$$[ab\cdots] = \left[\left(\overset{[}{r}= 1]n\sum \alpha_r e_r\right)\left(\overset{[}{s}= 1]n\sum \beta_s e_s\right)\cdots\right] = \overset{[}{r}= 1]n\sum \overset{[}{s}= 1]n\sum \cdots (\alpha_r\beta_s\cdots[e_r e_s\cdots]) .$$

We use P_a as a way to mark a product which includes a as one of its factors. If $a, b, \cdots$ are factors and $\alpha, \beta, \cdots$ numbers, then

$$P_{\alpha a+\beta b+\cdots} = \alpha P_a + \beta P_b + \cdots .$$

Side comment: From these properties we can first prove that the product is multilinear. Let $a, b, p \in U$ be magnitudes, α a number and$U$ a domain, then

$$B : U \times U \longmapsto V$$

is the map of the product (V is also a domain). By 1., 2. and 4. we get that

$$B(a+b,p) = B(a,p) + B(b,p) = [ap] + [bp],$$

$$B(a,b+p) = B(a,b) + B(a,p) = [ab] + [ap]$$

and

$$B(\alpha a,b) = B(a,\beta b) = \alpha B(a,b) = \alpha [ab].$$

Thus the product is bilinear. And then because of (4.4) it is actually multilinear. (Grassmann 1862, p. 22-23)

Defining Equations

In a product structure, *defining equations* are established numerical relations between the products of the units. If an assembly of defining equations includes all the numerical relations of the products of the units and none of the assembly can be derived from the others, we call it a *system of defining equations.* If we have a system of m defining equations between n unit products $E_1, E_2, \cdots, E_n$, then $n-m$ of those unit products form a system of units from which all products of this product structure are derivable. The defining equations also tell how to derive the remaining m unit products from the $n-m$ unit products.

Example

If in a domain of 2nd order, if we have a system of three defining equations,

$$E_1 = [e_1 e_1] = 0,$$

$$E_2 = [e_2 e_2] = 0$$

and

$$E_3 = [e_1 e_2] = -[e_2 e_1] = E_4,$$

then $n = 4$, $m = 3$ and all products E_1, E_2, E_3 and E_4 can be derived from E_3.

Linear Product Structure

If all defining equations of a product structure remain true, even if you replace the units in the equation with arbitrary magnitudes, we call it a *linear* product structure.

Side comment: This can be very misleading name for modern readers as the linear product property is stronger than bilinearity or multilinearity. For example, if you have a product with three defining equations,

$$[e_1 e_1] = [e_2 e_1] = [e_2 e_2] = 0,$$

then, if $a = \alpha_1 e_1 + \alpha_2 e_2$ and $b = \beta_1 e_1 + \beta_2 e_2$

$$[ab] = [(\alpha_1 e_1 + \alpha_2 e_2)(\beta_1 e_1 + \beta_2 e_2)] = \alpha_1 \beta_2 [e_1 e_2],$$

which is bilinear (as shown before), but it is not a linear product structure. This is because

$$[e_1 e_1] = 0,$$

but

$$[aa] = \alpha_1 \alpha_2 [e_1 e_2] \neq 0,$$

when $\alpha_1 \neq 0 \neq \alpha_2$.
(Grassmann 1862, p. 24)

Theorem 51

Because of the importance of the next theorem, we will also go through its proof.

In addition to a product structure with no defining equations and one with all products being zero, there are only two different types of linear product structure for two factors:

$$[e_r e_s] + [e_s e_r] = 0 \quad (4.5)$$

and

$$[e_r e_s] = [e_s e_r]\,, \tag{4.6}$$

for all unitse_r and e_s from which the two factors are derived.

Proof: Any defining equation of two magnitudes $[ab]$ can be written in the form of

$$\overset{[}{r}= 1]n\sum \overset{[}{s}= 1]n\sum \alpha_{r,s}\,[e_r e_s] = 0, \tag{4.7}$$

where $\alpha_{r,s} = \alpha_r \beta_s$ cannot all be zero. Now if we assume the product structure to be linear, then we can replace the units by arbitrary magnitudes derived from them. So we shall replace all $e_r$with

$$\sum_u \gamma_{r,u} e_u$$

and then all e_s with

$$\sum_v \gamma_{s,v} e_v.$$

Now the derivation numbers $\gamma_{r,u}$ and $\gamma_{s,v}$ can be any chosen numbers. We get

$$\sum_{r,s} \alpha_{r,s}\left[\left(\sum_u \gamma_{r,u} e_u\right)\left(\sum_v \gamma_{s,v} e_v\right)\right]$$
$$= \sum_{r,s,u,v} \alpha_{r,s}\gamma_{r,u}\gamma_{s,v}\,[e_u e_v] = 0.$$

Because all the indexes have been chosen arbitrarily, we can exchange r with s and u with v and we get

$$\sum_{s,r,v,u} \alpha_{s,r}\gamma_{s,v}\gamma_{r,u}\,[e_v e_u] = 0$$

and if we add these two equations together we get

$$\sum_{r,s,u,v} \alpha_{r,s}\gamma_{r,u}\gamma_{s,v}\,[e_u e_v] + \sum_{s,r,v,u} \alpha_{s,r}\gamma_{s,v}\gamma_{r,u}\,[e_v e_u]$$
$$= \sum_{r,s,u,v} \gamma_{r,u}\gamma_{s,v}\left(\alpha_{r,s}\,[e_u e_v] + \alpha_{s,r}\,[e_v e_u]\right) = 0, \tag{4.8}$$

which must hold for all values $\gamma_{r,u}$ and $\gamma_{s,v}$. So if we construct an equation from (4.8) where we set one of the values $\gamma_{a,c} = 1$ and then another one where its value is -1, but keep all other values of $\gamma_{r,u}$ the same in both. Both of these equations equal zero and are equal in their terms except for terms with $\gamma_{a,c}$ once (if both $\gamma_{r,u}$ and $\gamma_{s,v}$ equal $\gamma_{a,c}$, then $1^2 = (-1)^2$). So we subtract the second from the first and divide the result by two and get

$$\sum_{s,v} \gamma_{s,v} \left(\alpha_{a,s} \left[e_c e_v \right] + \alpha_{s,a} \left[e_v e_c \right] \right) - 2\alpha_{a,a} \left[e_c e_c \right] = 0. \tag{4.9}$$

Then we do the same for (4.9) for values $\gamma_{b,d}$ and get that

$$\alpha_{a,b} \left[e_c e_d \right] + \alpha_{b,a} \left[e_d e_c \right] = 0, \tag{4.10}$$

with $a \neq b$ if $c = d$ and $c \neq d$ if $a = b$. Applying this to (4.8) we have

$$\sum_{r,u} \gamma_{r,u} \gamma_{r,u} \alpha_{r,r} \left[e_u e_u \right] = 0.$$

Now if we set $\gamma_{r,u} = 0$ for all values except $\gamma_{a,c} = 1$,then

$$\alpha_{a,a} \left[e_c e_c \right] = 0$$

and so (4.10) applies for all a, b, c, d and thus follows from (4.7), if (4.7) is a defining equation.

If for (4.10) we set $c = d$ it is changed into

$$\left(\alpha_{a,b} + \alpha_{b,a} \right) \left[e_c e_c \right] = 0 \tag{4.11}$$

and if instead we set $a = b$, it is changed to

$$\alpha_{a,a} \left(\left[e_c e_d \right] + \left[e_d e_c \right] \right) = 0. \tag{4.12}$$

In either case, it must be that either the number or magnitude is zero.

Let us first investigate (4.11) and assume that $\left[e_c e_c \right] \neq 0$, then

$$\alpha_{a,b} + \alpha_{b,a} = 0,$$

meaning

$$\alpha_{a,b} = -\alpha_{b,a},$$

for all a, b. And by this (4.10) transforms into

$$\alpha_{a,b}\left([e_c e_d] - [e_d e_c]\right) = 0,$$

where either $\alpha_{a,b} = 0$ for all a, b (which is contrary to our assumption) or

$$[e_c e_d] = [e_d e_c]\,,$$

which is the same defining equation as (4.6).

Now let us assume that $[e_c e_c] = 0$ or equivalently that $[e_a e_a] = 0$. This can now be taken as a defining equation where the derivation number $\alpha_{a,a} = 1$ and by applying it to (4.12) you have

$$[e_c e_d] + [e_d e_c] = 0,$$

which is the same as equation (4.5).

Finally, if equations (4.5) and (4.6) were to both apply, then $[e_c e_d] = 0$ for all c, d, which cannot be true by assumption. And thus they are the only possible linear product structures.

Side comment: There are a few important things to note here. Firstly, both of these linear product structures have important modern analogues. (4.5) is the outer product (similar to the cross product) and (4.6) is the inner product. Thus their formulation is, at the very least, interesting as well as a part of the justification of why Grassmann devotes so much time on defining both later in the book.

Also of import is to note that Grassmann only thought about the product of two factors and did not try to generalize the system for additional factors. For example, the multiplication of quaternions has a defining equation with a product of three factors $\boldsymbol{ijk} = -1$.

(Grassmann 1862, p. 24-27)

4.3 Combinatorial and Outer Product

Combinatorial Product

Firstly, let us study products of the type

$$[e_c e_d] + [e_d e_c] = 0.$$

Let P be a product for which:

1. All the factors are derived from a system of units.
2. For all $a = \sum \alpha_r e_r$ and $b = \sum \beta_s e_s$ that are factors of$P$, if

   $$\alpha_r \neq 0 \Longrightarrow \beta_r = 0$$

   then $P \neq 0$.
3. For an arbitrary series of units A and factors b, c

   $$[Abc] + [Acb] = 0.$$

   And thus also

   $$[bc] + [cb] = 0.$$

Then we call P a *combinatorial product* and its factors its *elementary factors*. One can now iteratively prove increasingly strong statements about the combinatorial product. The generalizations will be listed here in order:

- $[Abc] + [Acb] = 0$ continues to hold if A is an arbitrary series of factors.
- $[AbcD] + [AcbD] = 0$ if D is also an arbitrary series of factors.
- $P_{a,b} + P_{b,a} = 0$, if $P_{a,b}$ is an arbitrary combinatorial product which includes elementary factors a and b and $P_{b,a}$ is an otherwise identical combinatorial product where you have exchanged a and b.

Combinatorial Product Properties

- If $a_1, a_2, \cdots, a_m$ are elementary factors, then

  $$[a_1 a_2 \cdots a_m] = \pm [a_j \cdots a_k],$$

  where $[a_j \cdots a_k]$ is an arbitrary rearrangement of the elementary factors of the first product. The latter product is positive if you can get it from the first one by doing an even amount of exchanges and negative if the amount is odd.

- It also follows that

$$P_{a,a} = 0,$$

meaning that a combinatorial product is zero if two of its elementary factors are the same. And then, if $a_1, a_2, \cdots, a_m$ are numerically related, then

$$[a_1 a_2 \cdots a_m] = 0.$$

- The converse also holds, meaning that if a combinatorial product is zero, then there is a numerical relation between its elementary factors. And

$$P_{a,b+\alpha a} = P_{a,b} + \alpha P_{a,a} = P_{a,b}, \tag{4.13}$$

meaning that you can add an arbitrary multiple one of its elementary factors to one of its other elementary factors without changing the product.

Side comment: It should be noted that the modern version of this type of product, the outer product, is marked with a $\wedge$ symbol. So for example, if magnitudes a and b are linearly independent vectors, then

$$[ab] = a \wedge b$$

and this is a bivector (i.e. a plane segment). This plane segment is orthogonal to the cross product

$$a \times b$$

and if $a = \alpha e_1$ and $b = \beta e_2$, then

$$a \times b = \alpha\beta\,(e_1 \times e_2)$$

and

$$a \wedge b = \alpha\beta\,(e_1 \wedge e_2)\,.$$

(Grassmann 1862, p. 29-35)

Combinatorial Product as a Magnitude

It has been mentioned before that products, and thus combinatorial products as well, are magnitudes. Let us now investigate more fully what types of magnitudes they are. When are they numerically related and how?

Let us say that $A, B, \cdots$, are all the different combinatorial products of the magnitudes $a_1, a_2, \cdots, a_n$ that stand in no numerical order. For example, if $n = 2$, then $A = [a_1]$, $B = [a_2]$ and $C = [a_1 a_2]$. Now if $\alpha, \beta, \cdots$ are numbers, then the equation

$$\alpha A + \beta B + \cdots = 0,$$

can be replaced by the equation

$$\alpha = \beta = \cdots = 0.$$

This is because, if for example $A = [a_1]$ and $A_1 = [a_2 \cdots a_n]$ and we multiply the entire equation with A_1, we get

$$\alpha A A_1 + \beta B A_1 + \cdots = 0.$$

Now $AA_1 = [a_1 a_2 \cdots a_n] \neq 0$, but $BA_1 = 0$ for all possible values of B (if for example $B = [a_1 a_2]$, then $BA_1 = [a_1 a_2 a_2 \cdots a_n] = 0$ and thus we get

$$\alpha A A_1 = 0,$$

meaning

$$\alpha = 0.$$

And by doing this for all the products we get that all the derivation numbers are zero. An interesting result of this is that if A and B are combinatorial products, then

$$A \equiv B$$

if and only if the elementary factors of A and B define the same domains.

Side comment: The modern equivalent to this would be saying that

$$a_1 \wedge a_2 \wedge \cdots \wedge a_n = \alpha \left(b_1 \wedge b_2 \wedge \cdots \wedge b_n \right),$$

for some scalar α if and only if

$$\langle a_1, a_2, \cdots, a_n \rangle = \langle b_1, b_2, \cdots, b_n \rangle .$$

Where $a_1 \wedge a_2 \wedge \cdots \wedge a_n$ and $b_1 \wedge b_2 \wedge \cdots \wedge b_n$ are m-vectors and $\langle a_1, a_2, \cdots, a_n \rangle$ and $\langle b_1, b_2, \cdots, b_n \rangle$ are vector subspaces formed of the vectors $a_1, a_2, \cdots, a_n$ and $b_1, b_2, \cdots, b_n$ respectively.

For example, if $a_1 = (1, 0, 0)$ and $a_2 = (0, 1, 0)$, then if

$$a_1 \wedge a_2 = \beta \left(b_1 \wedge b_2 \right),$$

for some number β, then b_1 and b_2 are both linearly independent vectors in the xy-plane.

One should note that these results also mean that the combinatorial product is clearly different from the cross product (which is its closest parallel in normal vector calculus). This is because, if a_1 and a_2 are as previously and $a_3 = (0, 0, 1)$,then

$$a_1 \times a_2 = a_3,$$

but

$$[a_1 a_2] \neq \alpha a_3,$$

for all numbers α.
(Grassmann 1862, p. 36-37)

Linear evolution

In a series of magnitudes, if you add an arbitrary multiple of a neighboring magnitude to a magnitude, we call it a *linear evolution*. The new series has been arrived by an *elementary linear evolution* from the original. If you linearly evolve the new series again, then it has been arrived by a *multiple linear evolution* from the first series. So, if a series of magnitudes has the magnitudes p and q as neighboring magnitudes,

$$\cdots, p, q, \cdots$$

then

$$\cdots, p + \alpha q, q, \cdots$$

where α is a any number, would be an (elementary) linear evolution of the first series. The same can be done by multiple linear evolution, even if p and q are not neighbors. Meaning that you can get from

$$\cdots, p, \cdots, q, \cdots$$

to

$$\cdots, p + \alpha q, \cdots, q, \cdots$$

by multiple linear evolution.

Example

One of the important things about linear evolution is that any combinatorial product is unchanged after a linear evolution, as shown by (4.13). So

$$[a_1 a_2 \cdots a_n]$$

can be changed to

$$[a_1 (a_2 + \alpha a_n) \cdots a_n]$$
$$= [a_1 a_2 \cdots a_n] + \alpha [a_1 a_n \cdots a_n] = [a_1 a_2 \cdots a_n],$$

where $a_1, a_2, \cdots, a_n$ are magnitudes and α is a number. We have the converse, so that if

$$[a_1 a_2 \cdots a_n] = [b_1 b_2 \cdots b_n],$$

where $a_1, a_2, \cdots, a_n$ and $b_1, b_2, \cdots, b_n$ are magnitudes, then $b_1, b_2, \cdots, b_n$ can be derived from $a_1, a_2, \cdots, a_n$ by linear evolution.
(Grassmann 1862, p. 38-44)

Order of Magnitudes

Original units and other magnitudes that cannot be numerically derived from combinatorial products of multiple magnitudes are called magnitudes of first order. A magnitude a that is only numerically derivable by a combinatorial product of n magnitudes of first order is called a magnitude of nth order. Magnitudes and units of first or more degree can also be called of *higher order.*

Elementary and Compound Magnitude

The magnitude A is *elementary* if it can be represented by a single combinatorial product, meaning

$$A = [a_1 \cdots a_n]$$

where $a_1 \cdots a_n$ are magnitudes of first order, is an elementary magnitude of nth order. If a magnitude A is not elementary and

$$A = \alpha [a_1 \cdots a_n] + \beta [b_1 \cdots b_n] + \cdots,$$

where $a_1, \cdots, a_n$, $b_1, \cdots, b_n$, $\cdots$ are magnitudes of first order and $\alpha, \beta, \cdots$ are numbers, then A is a *compound* magnitude of nth order. As an example, a vector (and thus a sum of vectors) and a point are both elementary magnitudes of first order and a trivector would be an elementary magnitude of third order.

Side comment: We will mostly be looking at products with elementary magnitudes, as the scope of this essay is rather limited and compound magnitudes can make many of the theorems very messy.

Magnitude Relations

If we have a magnitude $A = [a_1 a_2 \cdots a_n]$, where $a_1, \cdots, a_n$ are elementary factors, then the domain derivable from those factors is the domain *belonging* to the magnitude A. The magnitude A is also *superordinate*, *subordinate* or *incident* to other magnitudes as their domains would be. So if $B = [a_1]$ and $A == [a_1 a_2 \cdots a_n]$, then A and B are incident, B is subordinate to A and A is superordinate to B. As shown previously, the combinatorial product of two incident magnitudes is always zero.

Side comment: It should also be noted that there is a natural inclination to equate the word *order* with the word *dimension*, but this should be avoided as they are not the same and the meaning of order differs according to context (meaning it is different for magnitudes than for domains). In modern vector calculus space or $\mathbb{R}^3$ has three dimensions. Any vector in $\mathbb{R}^3$ can be defined by a linear combination of the three base vectors $\boldsymbol{e_j}$ ($\boldsymbol{j}$ = 1,2 or 3), so if we have a vector $\boldsymbol{a}$, then $\boldsymbol{a} = (x_1, x_2, x_3) = x_1\boldsymbol{e_1} + x_2\boldsymbol{e_2} + x_3\boldsymbol{e_3}$for some real numbers x_1, x_2, x_3. In Extension Theory, you can also define any displacement (as vectors are called in Extension Theory) in space (i.e. $\mathbb{R}^3$) with three unit displacements. But points are also magnitudes of first order, and with only displacements, you can never form a point (When you add a point and a displacement, you make the displacement starting from that point and get the end point. So point $(1,1)$ added to vector $(1,1)$ gives point $(2,2)$). And so you need four units e_1, e_2, e_3, e_4 to form space, for example, the origo and the three base vectors. And so the domain belonging to them, (i.e. space or $\mathbb{R}^3$) is a domain of fourth order and a magnitude a in space would be $a = \alpha_1 e_1 + \alpha_2 e_2 + \alpha_3 e_3 + \alpha_4 e_4$.

For example, as stated, the collection of all magnitudes of first order in space is a domain of fourth order, with e_1, e_2, e_3 and e_4 its units of first order. And the collection of all magnitudes of second order in space is a domain of sixth order, with $[e_1e_2]$, $[e_1e_3]$, $[e_1e_4]$, $[e_2e_3]$, $[e_2e_4]$ and $[e_3e_4]$ as its units of second order.
(Grassmann 1862, p. 44-45)

Outer Product

If you have elementary magnitudes $A = a_1 a_2 \cdots a_m$ and $B = a_{m+1} \cdots a_n$, then to *outer multiply* them is to combinatorially multiply their elementary factors in order without changing their position, meaning

$$[AB] = [(a_1 a_2 \cdots a_m)(a_{m+1} \cdots a_n)] = [a_1 a_2 \cdots a_n].$$

Side comment: Grassmann's reasoning for calling it outer multiplication was that $[AB] \neq 0$ only if A was "outside" of the domain defined by B. This also made it the opposite of inner multiplication.

Outer Product Properties

Here is a short list of some of the properties of the outer product.

- If A and B are nonzero elementary magnitudes, and A is subordinate to B, then B can be presented as the outer product of A and some elementary magnitude C so that
$$B = [AC] \, .$$
- Outer multiplication is also associative and so for magnitudes A, B and C
$$[A\,(BC)] = [ABC] \, .$$
- Let elementary magnitudes $a_1, a_2, \cdots, a_n, b_1, b_2, \cdots, b_m$ stand in no numerical relation. Now if A is formed of $a_1, a_2, \cdots, a_n$ by summation and/or multiplication (as an example, $A = [a_1 a_2 + a_3 \cdots a_n]$) and B similarly from $b_1, b_2, \cdots, b_m$. Now if the outer product
$$[AB] = 0,$$
then either $A = 0$ or $B = 0$. Also if $S = A_1 + A_2 + \cdots$ is a sum of elementary magnitudes (which we will call A_j), and the outer product
$$[aS] = 0,$$
for some magnitude of first order a, then a is incident to all the magnitudes A_j and there exists a magnitude P such that
$$S = [aP] \, .$$
If instead
$$0 = [a_1 S] = [a_2 S] = \cdots$$
for m outer products and magnitudes of first order not standing in a numerical relation, $a_1, a_2, \cdots, a_m$ then
$$S = [a_1 a_2 \cdots a_m P] \, ,$$
for some magnitude P.

(Grassmann 1862, p. 45-48)

Principal Domain

We call the domain of all the magnitudes under consideration as the *principal domain.* If the sum of the orders of two elementary magnitudes A and B is more than the order of the principal domain by γ, then

$$A = [CA_1]$$

and

$$B = [CB_1]\,,$$

for some elementary magnitudes A_1, B_1 and C and C has order γ.

Supplement

Let the principal domain be of nth order and $e_1, e_2, \cdots, e_n$ be its original units and E a unit of arbitrary order (meaning that$E$ can be one of the original units or a combinatorial product of some of them). Now, if we set

$$[e_1 e_2 \cdots e_n] = 1 \tag{4.14}$$

and E'is the combinatorial product of the units not incident to E, then

$$[EE'] = \pm 1.$$

Then the *supplement* of $E,$ marked as$\mid E$ is $\pm E'$ so that

$$[E \mid E] = 1$$

or said in another way

$$\mid E = [EE']\, E'.$$

This means, naturally, that the supplement can only be defined in a principal domain.

Properties of the Supplement

Here is a short list of some of the properties of the supplement.

- The supplement of a number is set as the number, so $\mid \alpha = \alpha$ if α is a number.

- The supplement of an arbitrary magnitude $A = \alpha_1 E_1 + \alpha_2 E_2 + \cdots$ derived from units $E_1, E_2, \cdots$ by numbers $\alpha_1, \alpha_2, \cdots$ is defined as

$$\mid A = \mid (\alpha_1 E_1 + \alpha_2 E_2 + \cdots) = \alpha_1 \mid E_1 + \alpha_2 \mid E_2 + \cdots .$$

  This means that in a principal order of nth order, the supplement of a magnitude with order m, has order $n - m$.

- If q is the order of a magnitude A and r is the order of its supplement, then

$$\mid\mid A = (-1)^{qr} A.$$

  If the principal domain is odd, then $\mid\mid A = A$, otherwise $\mid\mid A = (-1)^q A$.

Side comment: Definition (4.14) changes how the outer product works considerably. For example, in a normal outer product

$$[(e_1 e_2)(e_2 e_3)] = 0,$$

where e_1, e_2 and e_3 are original units. But if the principal domain is of third order and we set $[e_1 e_2 e_3] = 1$,

$$[(e_1 e_2)(e_2 e_3)] = -[e_1 e_2 e_3 e_2] = -e_2.$$

And so, when definition (4.14) is enforced, Grassmann's outer product works somewhat differently than its modern counterpart.
(Grassmann 1862, p. 49-52)

Progressive and Regressive Product

Let E_1 and E_2 are units of orders α and β respectively in a principal domain of order n. If $\alpha + \beta \leq n$, then their *progressive product* is the outer product as long as we take the progressive product of all original units to be 1. If on the other hand $\alpha + \beta > n$, then their *regressive product* is the magnitude that is the supplement to the progressive product of their supplements. Both progressive and regressive products are products *relative to* a principal domain. And so in a relative product we always take the progressive product of the original units to equal one.

Example

If $n = 3$, $E_1 = e_1 e_2$ and $E_2 = e_2 e_3$, then the regressive product of E_1 and E_2 is

$$[E_1 E_2] = \mid [\mid E_1 \mid E_2] = \mid [\mid (e_1 e_2) \mid (e_2 e_3)] = \mid [e_3 e_1] = -e_2$$

which is the same as

$$[E_1E_2] = [(e_1e_2)\,(e_2e_3)] = -\,[e_1e_2e_3e_2] = -e_2.$$

(Grassmann 1862, p. 52-53)

4.4 Inner Product

Lastly we will be taking a quick look at the inner product in Extension Theory. In normal vector calculus, the basic calculations using inner product are relatively simple operations, while calculations using the cross product are more difficult. In Extension Theory, it is the opposite, at least as far as definitions go. This is partly because in Extension Theory, the inner product is defined using the outer product. It is also a fact that the inner product of two magnitudes is considerably simpler if you assume the magnitudes to be of the same order, otherwise the inner product is not commutative. This section will be severely condensed with many parts of the original book skipped as it is only meant to give a basic grasp of the basics as anything more would be outside the scope of this essay.

Inner Product

Let A and B be magnitudes of arbitrary order. Their *inner product* is defined as the relative product

$$[A \mid B]\,.$$

Side comment: It is very interesting, that in Extension Theory, there is no separate symbol for the inner product. It is somewhat less general, as it is always relative in nature. In his previous work Grassmann had a separate symbol for it, but as he found a way to represent the product with the supplement, he omitted it from A2. Amusingly, he originally chose the modern symbol for cross product ($\times$), which would no doubt confuse modern readers.

Inner Product Properties

We will now go through some of the basic properties of the inner product as a quick glance to the similarities and differences to modern vector calculus.

- Let A be and B be magnitudes of ordersα and β respectively, in a principal domain of order n. Then their inner product has order $\alpha - \beta$ if $\alpha \geq \beta$ and order $n + \alpha - \beta$ otherwise. This means that if A and B are of the same order, then their inner product is of order zero and so a number and also commutative.

- The inner product of a unit with the same unit is one

$$[E_r \mid E_r] = 1.$$

- Otherwise the inner product of units of the same order is zero . Meaning that for units of the same order E_r and E_s order

$$[E_r \mid E_s] = 0,$$

when $r \neq s$.

- If E_r and E_s are units of arbitrary order and

$$[E_r \mid E_s] \neq 0,$$

then they are incident.

Example

Let $e_1, \cdots, e_n$ be the original units of a principal domain of order 3. Now

$$[(e_1e_2) \mid (e_2e_1)] = -[(e_1e_2)(e_3)] = -1$$

and

$$[(e_2e_1) \mid (e_1e_2)] = [(e_2e_1)(e_3)] == -[(e_1e_2)(e_3)] - 1$$

But,

$$[(e_1e_2) \mid (e_2)] = [(e_1e_2)(e_1e_3)] = -e_1$$

and

$$[(e_2) \mid (e_1e_2)] = [e_2e_3].$$

Normal and Completely Normal

If their inner product is zero, E_r and E_s are *normal* to one another. Two domains are *completely normal* to one another if the inner product of all magnitudes of the first order from one of the domains is normal to all similar magnitudes from another.

Inner Square and Numerical Value We say that

$$[A \mid A] = A^{\underline{2}}$$

is called the *inner square* of A and if $A = \alpha_1 E_1 + \cdots + \alpha_n E_n$, then

$$A^{\underline{2}} = \alpha_1^2 + \cdots + \alpha_n^2.$$

We also say that

$$\sqrt{A^2} = \sqrt{\alpha_1^2 + \cdots + \alpha_n^2}$$

is the *numerical value* of A.
(Grassmann 1862, p. 93-98)

The Angle If magnitudes A and B are of the same order, which is not zero, then $\angle AB$ (the angle AB) is defined so that

$$\cos \angle AB = \frac{[A \mid B]}{\alpha\beta},$$

where $0 \leq \angle AB \leq \pi$ and α and β are the numerical values of A and B. This gives us the more recognizable form

$$[A \mid B] = \alpha\beta \cos \angle AB.$$

And it is also true that if a and b are magnitudes of first order, then

$$[ab]^2 = (\alpha\beta \sin \angle AB)^2,$$

where α and β are the numerical values of a and b.
(Grassmann 1862, p. 117-119)

Chapter 5

Conclusions

The comparison of Grassmann to Hamilton is hard to avoid as they both came from exceedingly different backgrounds. As an example, the title page of Hamilton's "Lectures on Quaternions" read:

> "SIR WILLIAM ROWAN HAMILTON, LL.D., M. R. I. A., FELLOW OF THE AMERICAN SOCIETY OF ARTS FOR SCOTLAND; OF THE ROYAL ASTRONOMICAL SOCIETY OF LONDON; AND OF THE ROYAL NORTHERN SOCIETY OF ANTIQUARIES AT COPENHAGEN; CORRESPONDING MEMBER OF THE INSTITUTE OF FRANCE; HONORARY OR CORRESPONDING MEMBER OF THE IMPERIAL OR ROYAL ACADEMIES OF ST. PETERSBURGH, BERLIN, AND TURIN; OF THE ROYAL SOCIETIES OF EDINBURGH AND DUBLIN; OF THE CAMBRIDGE PHILOSOPHICAL SOCIETY; THE NEW YORK HISTORICAL SOCIETY; THE SOCIETY OF NATURAL SCIENCES AT LAUSANNE; AND OF OTHER SCIENTIFIC SOCIETIES IN BRITISH AND FOREIGN COUNTRIES; ANDREWS PROFESSOR OF ASTRONOMY IN THE UNIVERSITY OF DUBLIN; AND ROYAL ASTRONOMER OF IRELAND."

While the title page of "Ausdehnungslehre" of 1844 read:

> "Hermann Graßmann
> Lehrer an der Friedrich-Wilhelms-Schule zu Stettin"

And the title page of "Ausdehnungslehre" of 1862 read:

> "Hermann Graßmann
> Professor am Gymnasium zu Stettin"

One was a prodigy, immensely respected who everyone thought was destined for greatness. And the other was a school teacher who came in to the profession of mathematics relatively late and who had no fame to speak of. Hamilton was immensely respected when he created the quaternions. He was a man people expected to revolutionize mathematics. Yet he was unable to present his work in a form that others could understand, at least in writing. It was only after his students started writing about quaternions that the system truly became used.

In contrast, no one expected a mathematical revolution from Hermann Grassmann. As Hamilton, Grassmann failed to present his work in a way understandable to his peers, but even more unfortunately, he lacked the position and fame necessary to inspire others. Strangely the defining difference may be that Grassmann never could teach his system to students. As he never received a post at a university he could never propagate his system personally. But it is also likely that many of his peers, like Gauss, might have tried to understand his writing more, had he more fame.

Fortunately in the last 50 years, there has been a growing appreciation of Grassmann and his work as uses for a more general form of vector calculus have become apparent. And so I will end with a quote from the foreword of $A2$, written by Grassmann himself:

> "I remain completely confident that the labour I have expended on the science presented here and which has demanded a significant part of my life as well as the most strenuous application of my powers, will not be lost. It is true that I am aware that the form which I have given the science is imperfect and must be imperfect. But I know and feel obliged to state (though I run the risk of seeming arrogant) that even if this work should again remain unused for another seventeen years or even longer, without entering into the actual development of science, still that time will come when it will be brought forth from the dust of oblivion and when ideas now dormant will bring forth fruit. I know that if I also fail to gather around me (as I have until now desired in vain) a circle of scholars, whom I could fructify with these ideas, and whom I could stimulate to develop and enrich them further, yet there will come a time when these ideas, perhaps in a new form, will arise anew and will enter into a living communication with contemporary developments. For truth is eternal and divine and no phase of it ... can pass without a trace; it remains in existence even if the cloth in which weak mortals dress it disintegrates into dust."

(O'Connor & Robertson 2005)

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