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Embeddings into Orlicz Spaces for Functions from Unbounded Irregular Domains

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Abstract
We study Sobolev functions defined in unbounded irregular domains in the Euclidean n-space. We show that there exist embeddings into suitable Orlicz spaces from the space L^1_p, $1 \le p < n$. It turns out that the corresponding Orlicz function depends on the geometry of the domain. The results are sharp for L^1_1-functions.

Keywords Riesz potential · Pointwise estimate · Orlicz space · Unbounded convex domain · Non-smooth domain · Sobolev inequality · Poincaré inequality

Mathematics Subject Classification 31C15 · 42B20 · 26D10 · 46E30 · 46E35

1 Introduction

In this paper we study inequalities

$$\inf_{b \in \mathbb{R}} \|u - b\|_{L^H(D)} \le C \|\nabla u\|_{L^p(D)}, \tag{1.1}$$

in unbounded irregular domains D in $\mathbb{R}^n$. Here the target space $L^H(D)$ is an Orlicz space and it depends on the geometry of D. The function u belongs to $L^1_p(D) = \{u \in L^1_{\text{loc}}(D) : |\nabla u| \in L^p(D)\}$. Our proof is based on engulfing D by bounded domains D_i from inside. Thus we also study bounded domains and calculate

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the constants for the corresponding inequalities so that their constants do not blow up as $\operatorname{diam}(D_i) \to \infty$.

Although embeddings for functions defined in bounded irregular domains have been studied systematically, see for example [13,16], unbounded irregular domains seem to have been studied less, we refer to [10,13].

A classical example of an embedding into an Orlicz space for Sobolev functions from the Sobolev space $W^{1,n}$ is in [18]. But also, if the domain is irregular then an Orlicz space can be a natural target space for functions defined in L^1_p as in [6,8]. For papers where an Orlicz space is a target space when the functions come from another Orlicz space we refer to [3,4].

To be more precise, we assume that bounded domains D_i are φ-John domains, that is, every point can be connected to a central point of the domain by a flexible cone of the type $\{(x, x') \in \mathbb{R} \times \mathbb{R}^{n-1} : |x'| < \varphi(x)\}$. Here the function φ satisfies weak Orlicz-type conditions, we refer to Sect. 2. We showed in [7, Theorem 4.4, Theorem 3.5] that every $u \in L^1_p(D_i)$, can be estimated pointwise almost everywhere by the modified Riesz potential of its gradient

$$|u(x) - u_{D_i}| \le C \int_{D_i} \frac{|\nabla u(y)|}{\varphi(|x-y|)^{n-1}}\, dy, \tag{1.2}$$

and the modified Riesz potential can be estimated pointwise by the maximal operator

$$H\left(\int_G \frac{|f(y)|}{\varphi(|x-y|)^{n-1}}\, dy\right) \le C(Mf(x))^p, \tag{1.3}$$

where H is an N-function. This is a generalization of Hedberg's method [9, Lemma, Theorem 1]. In the present paper we modify the definition of φ-John domain so that for $t \ge 1$ the function φ grows linearly, we refer to (1.4). This definition keeps the class of uniformly bounded φ-John domains invariant but makes it possible to control the constants in (1.2) and (1.3) when $\operatorname{diam}(D_i) \to \infty$. A proper control of the constants is essential, since bounded domains should engulf the given unbounded domain and the required result for the unbounded domain is obtained as a limit of the results to the engulfing bounded domains. Then, we show that N-function H can be calculated from the geometry of the domain.

The following theorem tells which kind of N-functions we are interested in. These N-functions can encode and reveal the geometry of the domain.

Theorem 1.1 *Let $1 \le p < n$. Let the continuous, strictly increasing function $\varphi : [0,\infty) \to [0,\infty)$ be such that $\varphi(0) = \lim_{t\to 0^+} \varphi(t) = 0$ and suppose that φ satisfies the Δ_2-condition and the inequality $\frac{\varphi(t_1)}{t_1} \le \frac{\varphi(t_2)}{t_2}$ whenever $0 < t_1 \le t_2$. Assume that there exists $\alpha \in [1, n/(n-1))$ such that $t^\alpha/\varphi(t)$ is increasing for $t > 0$. If*

$$\psi(t) = \begin{cases} \varphi(t) & \text{when } \ 0 \le t \le 1; \\ \varphi(1)t & \text{when } \ t \ge 1, \end{cases} \tag{1.4}$$

then there exists an N-function H that satisfies the Δ_2*-condition, and*

$$H^{-1}(t) \approx \frac{t^{\frac{1}{p}-1}}{\psi\left(t^{-\frac{1}{n}}\right)^{n-1}} \qquad \textit{for} \quad t > 0,$$

where the implicit constant depends only on n and p.

By Theorem 1.1 we prove as an intermediate step the Sobolev-type inequality (1.1) for functions defined in bounded φ-John domains D_i, in Theorem 4.1 ($1 < p < n$) and Theorem 4.2 ($p = 1$). These results seem to be new and they recover some known results when $p = 1$. By using these bounded domains' results we obtain our main result for unbounded domains.

Theorem 1.2 *Assume that the function* φ *satisfies the conditions* (1)–(5)*, with* $C_\varphi = 1$ *in the condition* (4)*, from the beginning of Sect. 2. Assume that there exists* $\alpha \in [1, n/(n-1))$ *such that* $t^\alpha/\varphi(t)$ *is increasing for* $t > 0$*. Let the function* ψ *be defined as in* (1.4)*. Let* D *in* $\mathbb{R}^n$*,* $n \geq 2$*, be an unbounded domain that satisfies the following conditions:*

(a) $D = \cup_{i=1}^{\infty} D_i$*, where* $|D_1| > 0$*;*
(b) $\overline{D}_i \subset D_{i+1}$ *for each* i*;*
(c) *each* D_i *is a bounded* φ*-cigar John domain with a constant* c_J*.*

Let $1 \leq p < n$*. Let H be an N-function from Theorem 1.1. Then there exits a constant C such that the inequality*

$$\inf_{b\in\mathbb{R}} \|u - b\|_{L^H(D)} \leq C\|\nabla u\|_{L^p(D)},$$

holds for every $u \in L^1_p(D)$*. Here the constant C depends only on* n*,* p*,* $C_H^{\Delta_2}$*,* $C_\varphi^{\Delta_2}$*,* c_J*, and* $\operatorname{diam}(D_1)$.

We give examples in Example 4.5. Finally in Sect. 5 we show that the target space cannot be a Lebesgue space in general.

2 John Domains

Throughout the paper we let the function $\varphi : [0, \infty) \to [0, \infty)$ satisfy the following conditions

(1) φ is continuous,
(2) φ is strictly increasing,
(3) $\varphi(0) = 0$,
(4) there exists a constant $C_\varphi \geq 1$ such that

$$\frac{\varphi(t_1)}{t_1} \leq C_\varphi \frac{\varphi(t_2)}{t_2}$$

whenever $0 < t_1 \leq t_2$,

(5) φ satisfies the Δ_2-condition i.e. there exists a constant $C_\varphi^{\Delta_2} \geq 1$ such that $\varphi(2t) \leq C_\varphi^{\Delta_2}\varphi(t)$ for every $t > 0$.

We write

$$\psi(t) = \begin{cases} \varphi(t) & \text{if} \quad 0 \leq t \leq 1; \\ \varphi(1)t & \text{if} \quad t \geq 1. \end{cases} \tag{2.1}$$

Now, if φ satisfies the conditions (1)–(5), then ψ does, too, and the constant in (4) is the same for the functions φ and ψ, that is $C_\varphi = C_\psi$.

The definition of a bounded John domain goes back to John [12, Definition, p. 402] who defined an inner radius and an outer radius domain, and later this domain was renamed as a John domain in [14, 2.1].

We extend the definition of John domains following Väisälä [17, 2.1] in the classical case. Let E in $\mathbb{R}^n$, $n \geq 2$, be a closed rectifiable curve with endpoints a and b. The subcurve between x , $y \in E$ is denoted by $E[x, y]$. For $x \in E$ we write

$$q(x) = \min\left\{\ell\Big(E[a, x]\Big), \ell\Big(E[x, b]\Big)\right\},$$

where $\ell\big(E[a, x]\big)$ is the length of the subcurve $E[a, x]$.

Definition 2.1 A bounded or an unbounded domain D in $\mathbb{R}^n$ is a φ-cigar John domain if there exists a constant $c_J > 0$ such that each pair of points $a, b \in D$ can be joined by a closed rectifiable curve E in D such that

$$\operatorname{Cig} E(a, b) = \bigcup \left\{ B\left(x, \frac{\psi(q(x))}{c_J}\right) : x \in E \setminus \{a, b\} \right\} \subset D$$

where $B(x, r)$ is an open ball centered at x with a radius $r > 0$ and the function ψ is defined as in (2.1).

The set $\operatorname{Cig} E(a, b)$ is called a cigar with core E joining a and b. We point out that if D is a φ-cigar John domain with $\varphi(t) = t^p$, $p \geq 1$, then it is a φ-cigar John domain with $\varphi(t) = t^q$ for every $q \geq p$. For the case $\psi(t) = \varphi(t) = t$ for all $t \geq 0$, in Definition 2.1, we refer to [17, 2.1] and [15, 2.11 and 2.13]. Note that it is crucial that the length of the curve does not depend on the distance between the end points a and b. In bounded uniform domains the length of the cigar depends on $|a - b|$ but they are much more regular than our cigar John domains, see [15].

If D is a bounded domain then the following definition from [7, Definition 4.1] for a ψ-John domain gives an equivalent definition to a bounded φ-cigar John domain.

Definition 2.2 A bounded domain D in $\mathbb{R}^n$, $n \geq 2$, is a ψ-John domain if there exist a constants $0 < \alpha \leq \beta < \infty$ and a point $x_0 \in D$ such that each point $x \in D$ can be joined to x_0 by a rectifiable curve $\gamma : [0, \ell(\gamma)] \to D$, parametrized by its arc length, such that $\gamma(0) = x$, $\gamma(\ell(\gamma)) = x_0$, $\ell(\gamma) \leq \beta$, and

$$\psi(t) \leq \frac{\beta}{\alpha} \operatorname{dist}\big(\gamma(t), \partial D\big) \quad \text{for all} \quad t \in [0, \ell(\gamma)].$$

The point x_0 is called a John center of D and γ is called a John curve of x.

Remark 2.3 (1) If the function ψ is defined as in (2.1) with the function φ, then a bounded domain is a ψ-John domain if and only if it is a φ-John domain.
(2) If $\psi(t) = t$, then our definition for bounded ψ-John domains coincides with the definition of the classical John domains. If $\psi(t) = t^\lambda$, $\lambda \geq 1$ then our definition for bounded ψ-John domains coincides with the definition of the flexible cone condition in [2].

Theorem 2.4 *Let D be a bounded domain. If D is a ψ-John domain then D is a φ-cigar John domain. On the other hand, if D is a φ-cigar John domain with a constant c_J, then D is a ψ-John domain with constants*

$$\alpha = \frac{\psi\left(\frac{1}{4c_J}\psi\left(\frac{1}{4}\operatorname{diam}(D)\right)\right)}{c_J\varphi(1)C_\varphi(\varphi(1)+1)}, \quad \beta = \max\left\{2, \alpha, \frac{c_J \operatorname{diam}(D)}{\varphi(1)}\right\}. \tag{2.2}$$

Note that when $\operatorname{diam}(D) \to \infty$, then $\alpha \to \infty$ with the same speed as $\operatorname{diam}(D)$.

Proof Assume first that D is a ψ-John domain with a John center x_0. Let $a, b \in D$ and let the John curves γ_1 and γ_2 connect them to x_0, respectively. We may assume that $a, b \in D \setminus B(x_0, \operatorname{dist}(x_0, \partial D))$, since inside the ball the points can be connected by two straight lines going via the center of the ball $B(x_0, \operatorname{dist}(x_0, \partial D))$. Let E be a curve from a to b given by γ_1 and γ_2. Then,

$$\operatorname{Cig} E(a,b) = \bigcup_{t \in (0, \ell(\gamma_1)]} B\left(\gamma_1(t), \frac{\alpha\psi(t)}{\beta}\right) \cup \bigcup_{t \in (0, \ell(\gamma_2)]} B\left(\gamma_2(t), \frac{\alpha\psi(t)}{\beta}\right) \subset D$$

and thus D is a φ-cigar John domain.

Assume then that D is a φ-cigar John domain. Let us carefully choose a suitable John center so that the center is not too close to the boundary of D. Let $x, y \in D$ such that $|x - y| \geq \frac{1}{2}\operatorname{diam}(D)$. Let E be a core of a John cigar that connects x and y. Then the length of E is at least $\frac{1}{2}\operatorname{diam}(D)$. Let x_0 be the center of E. Then

$$\operatorname{dist}(x_0, \partial D) \geq \frac{\psi(\frac{1}{4}\operatorname{diam}(D))}{c_J}$$

and we choose

$$r = \frac{\psi\left(\frac{1}{4}\operatorname{diam}(D)\right)}{2c_J}. \tag{2.3}$$

Hence $B(x_0, 2r) \subset D$. From now on this r and the point x_0 are fixed in this proof.

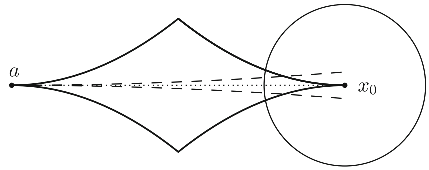

Fig. 1 The cigar from a to x_0 (the solid line), the core E (the dotted line) and a new carrot (the dashed line)

If $a \in B(x_0, 2r)$, then it can be clearly joint to x_0 by a line segment and the claim is clear.

For every $a \in D \setminus B(x_0, 2r)$ there exists a curve E such that the cigar Cig $E(a, x_0) \subset D$ (Fig. 1). Let $\ell(E)$ be the length of E, then $\ell(E) \le 2$ or by Definition 2.1 and (2.1)

$$\operatorname{diam}(D) \ge 2\frac{\psi(\ell(E)/2)}{c_J} = 2\frac{\varphi(1)\ell(E)}{2c_J},$$

i.e. $\ell(E) \le \max\left\{2, \frac{c_J \operatorname{diam}(D)}{\varphi(1)}\right\} \le \beta$.

Note that the total length of E is at least $2r$ and the length of E inside the ball $B(x_0, r)$ is at least r and thus for the points in $E \cap \partial B(x_0, r)$ the distance to the boundary is at least $\psi(r/2)/c_J$. Let us choose that

$$M = \frac{\psi(\beta)}{\psi(r/2)} = \frac{\varphi(1)\beta}{\psi(r/2)}. \tag{2.4}$$

Since $r \le \ell(E) \le \beta$ and ψ is increasing, we have $M \ge 1$.

Let $z_0 \in E$ be the first point from a that satisfies $z_0 \in \partial B(x_0, r)$. We denote by γ the function so that E is parametrized by its arc length such that $\gamma(0) = a$, $\gamma(t_0) = z_0$ and $\gamma(\ell(E)) = x_0$. We replace $E[z_0, x_0]$ by the radius of the ball $B(x_0, r)$, if needed. This new arc is written as E'. Note that $\ell(E') \le \ell(E)$.

Since $M \ge 1$ we have for $t \in (0, \frac{1}{2}\ell(E))$ that

$$\frac{\psi(t)}{M} \le \psi(t) = \psi\big(q(\gamma(t))\big). \tag{2.5}$$

By the choice of M in (2.4) we have

$$\frac{\psi(t)}{M} \le \psi\Big(\frac{r}{2}\Big) \tag{2.6}$$

for all t. On the other hand, for $t \in (\frac{1}{2}\ell(E), t_0)$ the inequality $q(\gamma(t)) \ge r/2$ holds. Hence, by (2.6)

$$\frac{\psi(t)}{M} \le \psi\big(q(\gamma(t))\big) \tag{2.7}$$

for $t \in (\frac{1}{2}\ell(E), t_0)$, too. These estimates (2.5) and (2.7) give

$$\bigcup_{t\in(0,\ell(E'))} B\left(\gamma(t), \frac{\psi(t)}{Mc_J}\right)\setminus B(x_0, r) \subset \operatorname{Cig} E(a, x_0).$$

By (2.6) we have $\psi(t) \le M\psi(r/2)$. By the definition of ψ we have $\psi(r/2) \le \varphi(1)r/2$ if $r \ge 2$, and by condition (4) the inequality $\psi(r/2) \le C_\varphi\varphi(1)r/2$ holds if $0 < r < 2$. Since $C_\varphi \ge 1$, we obtain

$$\psi(t) \le M\varphi(1)C_\varphi r/2$$

for all $t \in (0, t_0)$. Since $\varphi(1)$ might be less than one, we estimate

$$\psi(t) \le MC_\varphi(\varphi(1) + 1)r/2.$$

This inequality and the inclusion $B(x_0, 2r) \subset D$ yield that

$$\bigcup_{t\in(0,\ell(E'))} B\left(\gamma(t), \frac{\psi(t)}{MC_\varphi(\varphi(1) + 1)c_J}\right) \subset D\,.$$

Thus, by (2.4)

$$\psi(t) \le MC_\varphi(\varphi(1) + 1)c_J \operatorname{dist}(\gamma(t), \partial D) = \frac{c_J\varphi(1)C_\varphi(\varphi(1) + 1)\beta}{\psi(r/2)} \operatorname{dist}(\gamma(t), \partial D).$$

This means that we may choose $\alpha = \frac{\psi(r/2)}{c_J\varphi(1)C_\varphi(\varphi(1)+1)}$. By using (2.3) we obtain the final α. To be sure that $\alpha \le \beta$ we may choose β to be larger if it is necessary. Thus, D is a ψ-John domain with α and β given in (2.2). □

3 Pointwise Estimates

We proceed to prove pointwise estimates for domains which are not classical John domains.

We note that by the condition (4) of φ

$$\psi(t) \le C_\varphi\varphi(1)t \quad \text{for all} \quad t \ge 0. \tag{3.1}$$

We recall a covering lemma from [7, Lemma 4.3] which is valid for a bounded φ-John domain.

Lemma 3.1 [7, Lemma 4.3] *Let φ satisfy the conditions (1)–(5). Let $\psi : [0, \infty) \to [0, \infty)$ be defined as in (2.1). Let D in $\mathbb{R}^n$, $n \ge 2$, be a bounded ψ-John domain with John constants α and β. Let $x_0 \in D$ be the John center. Then for every $x \in D\setminus B(x_0, \operatorname{dist}(x_0, \partial D))$ there exists a sequence of balls $\big(B(x_i, r_i)\big)$ such that $B(x_i, 2r_i)$*

is in D for each $i = 0, 1, \dots$, and for some constants $K = K(\alpha, \operatorname{dist}(x_0, \partial D), \beta, \varphi)$, $N = N(n)$, and $M = M(n)$

- $B_0 = B\Big(x_0, \frac{1}{2}\operatorname{dist}(x_0, \partial D)\Big)$;
- $\psi(\operatorname{dist}(x, B_i)) \le K r_i$, *and* $r_i \to 0$ *as* $i \to \infty$;
- *no point of the domain D belongs to more than N balls $B(x_i, r_i)$; and*
- $|B(x_i, r_i) \cup B(x_{i+1}, r_{i+1})| \le M|B(x_i, r_i) \cap B(x_{i+1}, r_{i+1})|$.

Proof The proof is in [7, Lemma 4.3]. We recall only the proof of the inequality $\psi(\operatorname{dist}(x, B_i)) \le K r_i$, since we have to show that constant K does not blow up when $\operatorname{diam}(D) \to \infty$.

Let $x \in D \setminus B(x_0, \operatorname{dist}(x_0, \partial D))$. Let γ be a John curve joining x to x_0, its arc length written as l. We write $B_0' = B\big(x_0, \frac{1}{4}\operatorname{dist}(x_0, \partial D)\big)$ and consider the balls B_0' and

$$B\Big(\gamma(t), \frac{1}{4}\operatorname{dist}\big(\gamma(t), \partial D \cup \{x\}\big)\Big), \quad \text{where} \quad t \in (0, l).$$

By the Besicovitch covering theorem, there is a sequence of closed balls $\overline{B_1'}, \overline{B_2'}, \dots$ and $\overline{B_0'}$ that cover the set $\{\gamma(t) : t \in [0, l]\} \setminus \{x\}$ and have a uniformly bounded overlap depending on n only. We write $B(x_i, r_i) = 2B_i'$ for every $i = 0, 1, 2, \dots$, where $x_i = \gamma(t_i)$, $t_i \in (0, l)$, $r_0 = \frac{1}{2}\operatorname{dist}(x_0, \partial D)$, and $r_i = \frac{1}{2}\operatorname{dist}\big(x_i, \partial D \cup \{x\}\big)$.

By the fact that φ is an increasing function and by the definition of ψ-John domain we obtain

$$\psi(\operatorname{dist}(x, B_0)) \le \psi(l) \le \psi(\beta) \le C_\varphi \varphi(1)\beta \le \frac{c\beta r_0}{\operatorname{dist}(x_0, \partial D)}.$$

Let us suppose then that $i \ge 1$. If $r_i = \frac{1}{2}\operatorname{dist}(x_i, x)$, then by (3.1) we obtain

$$\psi(\operatorname{dist}(x, B(x_i, r_i))) \le C_\varphi \varphi(1)\operatorname{dist}(x, B(x_i, r_i)) \le 2C_\varphi \varphi(1) r_i.$$

If $r_i = \frac{1}{2}\operatorname{dist}(x_i, \partial D)$, then the fact that φ is increasing and the definition of a ψ-John domain give

$$\psi(\operatorname{dist}(x, B(x_i, r_i))) \le \psi(\operatorname{dist}(x, x_i)) \le \psi(t_i) \le \frac{\beta}{\alpha}\operatorname{dist}(\gamma(t_i), \partial D) \le \frac{2\beta}{\alpha} r_i.$$

□

Remark 3.2 (1) The constant K in the previous lemma can be chosen to be $K = \max\{\frac{c\beta}{\operatorname{dist}(x_0, \partial D)}, 2C_\varphi \varphi(1), \frac{2\beta}{\alpha}\}$.
(2) If D is a φ-cigar John domain and the John center has been chosen as in Theorem 2.4, then

$$\frac{\beta}{\operatorname{dist}(x_0, \partial D)} \le \frac{\max\left\{2, \frac{\psi\left(\frac{1}{4c_J}\psi\left(\frac{1}{4}\operatorname{diam}(D)\right)\right)}{c_J C_\varphi \varphi(1)(\varphi(1)+1)}, \frac{c_J \operatorname{diam}(D)}{\varphi(1)}\right\}}{\frac{1}{2c_J}\psi\left(\frac{1}{4}\operatorname{diam}(D)\right)} \to \max\left\{\frac{1}{2c_J C_\varphi(\varphi(1)+1)}, \frac{8c_J^2}{\varphi(1)^2}\right\}$$

and

$$\frac{\beta}{\alpha} = \frac{\max\left\{2, \frac{\psi\left(\frac{1}{4c_J}\psi\left(\frac{1}{4}\operatorname{diam}(D)\right)\right)}{c_J C_\varphi \varphi(1)(\varphi(1)+1)}, \frac{c_J \operatorname{diam}(D)}{\varphi(1)}\right\}}{\frac{\psi\left(\frac{1}{4c_J}\psi\left(\frac{1}{4}\operatorname{diam}(D)\right)\right)}{c_J C_\varphi \varphi(1)(\varphi(1)+1)}} \to \max\left\{1, \frac{16c_J^3 C_\varphi(\varphi(1)+1)}{\varphi(1)^2}\right\}$$

as $\operatorname{diam}(D) \to \infty$.

We recall the following definitions. Let G be an open set of $\mathbb{R}^n$. We denote the Lebesgue space by $L^p(G)$, $1 \le p < \infty$. By $L^1_p(G)$, $1 \le p < \infty$, we denote those locally integrable functions whose first weak distributional derivatives belongs to $L^p(G)$, that is, $L^1_p(G) = \left\{u \in L^1_{\text{loc}}(G) : |\nabla u| \in L^p(G)\right\}$. By $W^{1,p}(G)$, $1 \le p < \infty$, we denote those functions from $L^p(G)$ whose first weak distributional derivatives belongs to $L^p(G)$, that is, $W^{1,p}(G) = \{u \in L^p(G) : |\nabla u| \in L^p(G)\}$.

Theorem 2.4 and Lemma 3.1 give the following pointwise estimate which we recall from [7, Theorem 4.4].

Theorem 3.3 *Let φ satisfy the conditions (1)–(5). Let $\psi : [0,\infty) \to [0,\infty)$ be as defined in (2.1). Let D in $\mathbb{R}^n$, $n \ge 2$, be a bounded φ-cigar John domain with a John constant c_J . Then there exists a finite constant C and $x_0 \in D$ such that for every $u \in L^1_1(D)$ and for almost every $x \in D$ the inequality*

$$\left|u(x) - u_{B(x_0,\operatorname{dist}(x_0,\partial D))}\right| \le C \int_D \frac{|\nabla u(y)|}{\psi\left(|x-y|\right)^{n-1}}\, dy$$

holds. Here $C = c\left(n, c_J, C_\varphi, C_\varphi^{\Delta_2}, \varphi(1), \min\left\{\operatorname{diam}(D), 1\right\}\right)$.

We recall the definitions of N-functions and Orlicz spaces.

Definition 3.4 A function $H : [0,\infty) \to [0,\infty)$ is an N-function if

(N1) H is continuous,
(N2) H is convex,
(N3) $\lim_{t\to 0^+} \frac{H(t)}{t} = 0$ and $\lim_{t\to\infty} \frac{H(t)}{t} = \infty$.

Continuity and $\lim_{t\to 0^+}\frac{H(t)}{t}=0$ yield that $H(0)=0$.

Convexity yields that $\frac{H(t)}{t}\le\frac{H(s)}{s}$ for $0<t<s$ and thus H is a strictly increasing function.

By the notation $f\lesssim g$ we mean that there exists a constant $C>0$ such that $f(x)\le Cg(x)$ for all x. The notation $f\approx g$ means that $f\lesssim g\lesssim f$.

Two N-functions H and K are equivalent, which is written as $H\simeq K$, if there exists $m\ge 1$ such that $H(t/m)\le K(t)\le H(mt)$ for all $t>0$. Equivalent N-functions give the same space with comparable norms. We point out that $H\simeq K$ if and only if for the inverse functions $H^{-1}\approx K^{-1}$.

We assume that H satisfies the Δ_2-condition, that is, there exists a constant $C_H^{\Delta_2}$ such that

$$H(2t)\le C_H^{\Delta_2}H(t)\quad\text{for all}\quad t>0. \tag{3.2}$$

The constant $C_H^{\Delta_2}$ is called the Δ_2-constant of H.

Let G in $\mathbb{R}^n$ be an open set.

We study the Orlicz space $L^H(G)$ which means the space of all measurable functions u defined on G such that

$$\int_G H\Big(\lambda|u(x)|\Big)\,dx<\infty$$

for some $\lambda>0$.

The Orlicz space $L^H(G)$ equipped with the Luxemburg norm

$$\|u\|_{L^\Phi(G)}=\inf\left\{\lambda>0:\int_G\Phi\left(\frac{|u(x)|}{\lambda}\right)\,dx\le 1\right\}$$

is a Banach space.

Let G in $\mathbb{R}^n$ be an open set. Assume that $f\in L^1(G)$. The centered Hardy–Littlewood maximal operator is defined as

$$Mf(x)=\sup_{r>0}\fint_{B(x,r)}|f(y)\chi_G(x)|\,dx,$$

where the function $f\chi_G$ is understood to be zero in the complement of G. We recall the following theorem from [7, Theorem 3.5] which is applied to the function $f\chi_G$.

Theorem 3.5 *Let φ satisfy the conditions (1)–(5). Let $\psi:[0,\infty)\to[0,\infty)$ be defined as in (2.1). Let $1\le p<n$ be given. Suppose that there exists a continuous function $h:[0,\infty)\to[0,\infty)$ such that*

$$\sum_{k=1}^{\infty}\frac{(2^{-k}t)^n}{\psi(t2^{-k})^{n-1}}\le h(t)\quad\textit{for all}\quad t>0\,. \tag{3.3}$$

Let $\delta : (0, \infty) \to [0, \infty)$ be a continuous function and let $H : [0, \infty) \to [0, \infty)$ be an N-function satisfying the Δ_2-condition. Suppose that there exists a finite constant C_H such that the inequality

$$H\left(h(\delta(t))t + \psi(\delta(t))^{1-n}(\delta(t))^{n(1-\frac{1}{p})}\right) \le C_H t^p \tag{3.4}$$

holds for all $t > 0$. Let G in $\mathbb{R}^n$ be an open set. If $\|f\|_{L^p(G)} \le 1$, then there exists a constant C such that the inequality

$$H\left(\int_G \frac{|f(y)|}{\psi(|x-y|)^{n-1}}\,dy\right) \le C(Mf(x))^p \tag{3.5}$$

holds for every $x \in G$. Here the constant C depends on n, p, C_φ, C_H, and the Δ_2-constants of φ and H only.

Our goal is to find a formula which would give all suitable functions H. Examples of some of these functions were given in [7, Section 6].

Here we do the preparations to find H. Assume that there exists $\alpha \in [1, n/(n-1))$ such that $t^\alpha/\varphi(t)$ is increasing for $t > 0$. This yields that $t^\alpha/\psi(t)$ is increasing, too. Under this condition inequality (3.3) holds: Since

$$\begin{aligned}\frac{(2^{-k}t)^n}{\psi(t2^{-k})^{n-1}} &= \frac{(2^{-k}t)^n}{(2^{-k}t)^{\alpha(n-1)}} \cdot \frac{(2^{-k}t)^{\alpha(n-1)}}{\psi(t2^{-k})^{n-1}} \\ &\le (2^{-k}t)^{n-\alpha(n-1)} \frac{t^{\alpha(n-1)}}{\psi(t)^{n-1}} = 2^{-k(n-\alpha(n-1))} \frac{t^n}{\psi(t)^{n-1}},\end{aligned}$$

we have

$$\sum_{k=1}^{\infty} \frac{(2^{-k}t)^n}{\psi(t2^{-k})^{n-1}} \le C(n,\alpha)\frac{t^n}{\psi(t)^{n-1}}, \quad \text{where} \quad C(n,\alpha) = \frac{2^{\alpha(n-1)}}{2^n - 2^{\alpha(n-1)}}.$$

Let us define the functions h and δ such that

$$h(t) = C(n,\alpha)\frac{t^n}{\psi(t)^{n-1}} \quad \text{and} \quad \delta(t) = t^{-\frac{p}{n}} \quad \text{for all} \quad t > 0.$$

Then,

$$\begin{aligned}h(\delta(t))t+\psi(\delta(t))^{1-n}(\delta(t))^{n(1-\frac{1}{p})} &= h\left(t^{-\frac{p}{n}}\right)t + \psi\left(t^{-\frac{p}{n}}\right)^{1-n}\left(t^{-\frac{p}{n}}\right)^{n(1-\frac{1}{p})} \\ &= \frac{C(n,\alpha)t^{-p}}{\psi\left(t^{-\frac{p}{n}}\right)^{n-1}}t + \frac{t^{1-p}}{\psi\left(t^{-\frac{p}{n}}\right)^{n-1}} = \frac{(C(n,\alpha)+1)t^{1-p}}{\psi\left(t^{-\frac{p}{n}}\right)^{n-1}}.\end{aligned}$$

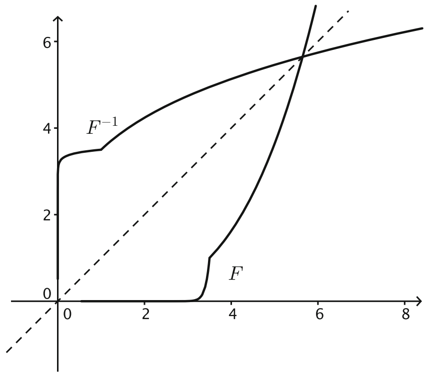

Fig. 2 The function F is not necessary convex

If we choose

$$F^{-1}(t) = \frac{(C(n,\alpha)+1)(t^{1/p})^{1-p}}{\psi\left((t^{1/p})^{-\frac{p}{n}}\right)^{n-1}} = \frac{(C(n,\alpha)+1)t^{\frac{1}{p}-1}}{\psi\left(t^{-\frac{1}{n}}\right)^{n-1}}$$

and assume that the inverse function of F^{-1} exists, that is $(F^{-1})^{-1} =: F$ exists, then

$$h(\delta(t))t + \psi(\delta(t))^{1-n}(\delta(t))^{n(1-\frac{1}{p})} = F^{-1}(t^p)$$

and thus

$$F\left(h(\delta(t))t + \psi(\delta(t))^{1-n}(\delta(t))^{n(1-\frac{1}{p})}\right) = F\left(F^{-1}(t^p)\right) = t^p.$$

Unfortunately, there is a problem with this function F to be a suitable function H; namely, the function F is not necessary convex. For example, if $n = 2$, $\varphi(t) = t^{\frac{3}{2}}$, and $p = 1.9$, then the function F is not convex, see Fig. 2. The angle at the point $(1, F^{-1}(1))$ comes from the angle of ψ at the point $(1, \psi(1))$. Our main theorem, Theorem 1.1 in Introduction, corrects this point: we show that there exists an N-function H that is equivalent with F.

Proof of Theorem 1.1 Let us write that

$$F^{-1}(t) = \frac{t^{\frac{1}{p}-1}}{\psi\left(t^{-\frac{1}{n}}\right)^{n-1}}$$

for $t > 0$ and $F^{-1}(0) = 0$. Let us first show that F^{-1} is strictly increasing. We recall that if φ satisfies condition (4), then ψ does too, and the constant is the same for both functions. We have

$$F^{-1}(t) = t^{\frac{1}{p}-1+\frac{n-1}{n}} \left(\frac{(t^{-\frac{1}{n}})}{\psi\left(t^{-\frac{1}{n}}\right)} \right)^{n-1} = t^{\frac{1}{p}-\frac{1}{n}} \left(\frac{(t^{-\frac{1}{n}})}{\psi\left(t^{-\frac{1}{n}}\right)} \right)^{n-1}.$$

Since $p < n$ the function $t \mapsto t^{\frac{1}{p}-\frac{1}{n}}$ is strictly increasing. Since the function $t \mapsto t^{-\frac{1}{n}}$ is strictly decreasing, condition (4) with $C_\varphi = 1$ yields that $t \mapsto (t^{-\frac{1}{n}})/\psi(t^{-1/n})$ is strictly increasing. These together yield that F^{-1} is strictly increasing.

This yields that the function F exists and is strictly increasing.

Let us show that $\lim_{t\to 0^+} F^{-1}(t) = 0$. Since $p < n$ we obtain

$$\lim_{t\to 0^+} F^{-1}(t) = \lim_{t\to 0^+} \frac{t^{\frac{1}{p}-1}}{\psi\left(t^{-\frac{1}{n}}\right)^{n-1}} = \lim_{t\to 0^+} \varphi(1)^{1-n} t^{\frac{n-1}{n}+\frac{1}{p}-1} = 0.$$

Let us show that $\lim_{t\to\infty} F^{-1}(t) = \infty$. Since $t/\varphi(t)$ is decreasing, by the condition (4), and by $p < n$ we obtain

$$\lim_{t\to\infty} F^{-1}(t) = \lim_{t\to\infty} \frac{t^{\frac{1}{p}-1}}{\psi\left(t^{-\frac{1}{n}}\right)^{n-1}} = \lim_{t\to\infty} t^{\frac{1}{p}-\frac{1}{n}} \left(\frac{t^{-\frac{1}{n}}}{\psi\left(t^{-\frac{1}{n}}\right)} \right)^{n-1} \geq \lim_{t\to\infty} \frac{t^{\frac{1}{p}-\frac{1}{n}}}{\varphi(1)^{n-1}} = \infty.$$

We have shown that $F^{-1} : [0, \infty) \to [0, \infty)$ is bijective.

Let us then study the condition

$$\frac{F(s)}{s} < \frac{F(t)}{t} \quad \text{for} \quad 0 < s < t. \tag{3.6}$$

Since F^{-1} is a strictly increasing bijection, inequality (3.6) is equivalent to

$$\frac{s}{F^{-1}(s)} < \frac{t}{F^{-1}(t)}.$$

Since $t^\alpha/\varphi(t)$ is increasing, then $\varphi(t)/t^\alpha$ is decreasing and $\psi(t)/t^\alpha$ is decreasing, too. We note that $1 - \frac{\alpha(n-1)}{n} > 0$, since $\alpha < \frac{n}{n-1}$. We obtain

$$\frac{s}{F^{-1}(s)} = s^{2-\frac{1}{p}} \psi\left(s^{-\frac{1}{n}}\right)^{n-1} = s^{2-\frac{1}{p}-\frac{\alpha(n-1)}{n}} \left(\frac{\psi\left(s^{-\frac{1}{n}}\right)}{\left(s^{-\frac{1}{n}}\right)^\alpha} \right)^{n-1}$$

$$= s^{1-\frac{1}{p}+1-\frac{\alpha(n-1)}{n}} \left(\frac{\psi\left(s^{-\frac{1}{n}}\right)}{\left(s^{-\frac{1}{n}}\right)^\alpha} \right)^{n-1} < t^{1-\frac{1}{p}+1-\frac{\alpha(n-1)}{n}} \left(\frac{\psi\left(t^{-\frac{1}{n}}\right)}{\left(t^{-\frac{1}{n}}\right)^\alpha} \right)^{n-1} = \frac{t}{F^{-1}(t)}$$

and thus inequality (3.6) holds.

Let us then show that $F^{-1}(cs) \geq 2F^{-1}(s)$ for all $s \geq 0$ with $c = 2^{\frac{np}{n-p}}$. The inequality $F^{-1}(cs) \geq 2F^{-1}(s)$ is equivalent to

$$2\frac{\psi\left(\left(\frac{1}{cs}\right)^{\frac{1}{n}}\right)^{n-1}}{\left(\frac{1}{cs}\right)^{1-\frac{1}{p}}} \leq \frac{\psi\left(\left(\frac{1}{s}\right)^{\frac{1}{n}}\right)^{n-1}}{\left(\frac{1}{s}\right)^{1-\frac{1}{p}}}.$$

By the condition (4) of φ and the inequality $p < n$, we obtain

$$\begin{aligned}
2\frac{\psi\left(\left(\frac{1}{cs}\right)^{\frac{1}{n}}\right)^{n-1}}{\left(\frac{1}{cs}\right)^{1-\frac{1}{p}}} &= 2\left(\frac{\psi\left(\left(\frac{1}{cs}\right)^{\frac{1}{n}}\right)}{\left(\frac{1}{cs}\right)^{\frac{1}{n}}}\right)^{n-1}\left(\frac{1}{cs}\right)^{\frac{n-1}{n}-1+\frac{1}{p}} \\
&= \left(\frac{\psi\left(\left(\frac{1}{cs}\right)^{\frac{1}{n}}\right)}{\left(\frac{1}{cs}\right)^{\frac{1}{n}}}\right)^{n-1}\left(\frac{1}{s}\right)^{\frac{n-1}{n}-1+\frac{1}{p}} \\
&\leq \left(\frac{\psi\left(\left(\frac{1}{s}\right)^{\frac{1}{n}}\right)}{\left(\frac{1}{s}\right)^{\frac{1}{n}}}\right)^{n-1}\left(\frac{1}{s}\right)^{\frac{n-1}{n}-1+\frac{1}{p}} = \frac{\psi\left(\left(\frac{1}{s}\right)^{\frac{1}{n}}\right)^{n-1}}{\left(\frac{1}{s}\right)^{1-\frac{1}{p}}}.
\end{aligned}$$

The inequality $F^{-1}(cs) \geq 2F^{-1}(s)$ yields that F satisfies the Δ_2-condition. Let us write $F(t) = s$. Then $F^{-1}(s) = t$. Since F is increasing, we have

$$F(2t) = F(2F^{-1}(s)) \leq F(F^{-1}(cs)) = cs = cF(t).$$

Since F satisfies Δ_2-condition it is finite everywhere and hence (3.6) yields that $F(0) = \lim_{s\to 0^+} F(s) = 0$ and $\lim_{s\to\infty} F(s) = \infty$. Since ψ is continuous, we find that F^{-1} is continuous on $(0, \infty)$ and hence also F is continuous on $(0, \infty)$ and moreover on $[0, \infty)$.

Hästö has shown in [11, Proposition 3.1] that if $f : [0, \infty) \to [0, \infty)$ is left-continuous, $f(0) = \lim_{s\to 0^+} f(s) = 0$, $\lim_{s\to\infty} f(s) = \infty$ and $x \mapsto f(x)/x$ is increasing, then f is equivalent to a convex function. We obtain that F is equivalent to a convex function H. Here the implicit constant depends only on the constant in the Δ_2-condition, that is, it depends only on n and p.

Using $\lim_{t\to 0^+} F^{-1}(t) = 0$ and the bijectivity, we obtain

$$\lim_{t\to 0^+}\frac{F(t)}{t} = \lim_{t\to 0^+}\frac{t}{F^{-1}(t)} = \lim_{t\to 0^+}\frac{t\,\psi\left(\left(\frac{1}{t}\right)^{\frac{1}{n}}\right)^{n-1}}{\left(\frac{1}{t}\right)^{1-\frac{1}{p}}} = \lim_{t\to 0^+}\varphi(1)^{n-1}t^{1-\frac{1}{p}+1-\frac{n-1}{n}} = 0$$

and thus also $\lim_{t\to 0^+} \frac{H(t)}{t} = 0$. This gives that H is right continuous at the origin. Since F satisfies Δ_2-condition so does H and thus it is finite everywhere. Thus by convexity the function H is continuous on $[0,\infty)$.

Since $\varphi(t)/t^\alpha$ is decreasing and $\alpha < \frac{n}{n-1}$, we obtain

$$\begin{aligned}\lim_{t\to\infty} \frac{F(t)}{t} &= \lim_{t\to\infty} \frac{t}{F^{-1}(t)} = \lim_{t\to\infty} t^{2-\frac{1}{p}} \varphi\left(t^{-\frac{1}{n}}\right)^{n-1} \\ &= \lim_{t\to\infty} t^{2-\frac{1}{p}-\frac{\alpha(n-1)}{n}} \left(\frac{\varphi\left(t^{-\frac{1}{n}}\right)}{\left(t^{-\frac{1}{n}}\right)^{\alpha}}\right)^{n-1} \geq \lim_{t\to\infty} t^{1-\frac{1}{p}+1-\frac{\alpha(n-1)}{n}} \left(\frac{\varphi(1)}{1^\alpha}\right)^{n-1} \\ &= \infty.\end{aligned}$$

Since the functions F and H are equivalent, this yields that $\lim_{t\to\infty} \frac{H(t)}{t} = \infty$. Thus we have shown that the function H satisfies the conditions (N1)–(N3). □

Remark 3.6 Later it is crucial that

$$H^{-1}(t) \approx \frac{t^{\frac{1}{p}-1}}{\psi\left(t^{-\frac{1}{n}}\right)^{n-1}} = \frac{t^{\frac{1}{p}-1}}{\varphi(1)^{n-1}\left(t^{-\frac{1}{n}}\right)^{n-1}} = \varphi(1)^{1-n} t^{\frac{n-p}{np}}$$

for $0 < t \leq 1$. Namely, for every φ the function H satisfies $H(t) \approx t^{\frac{np}{n-p}}$ whenever $0 < t \leq 1$.

Example 3.7 Functions $\varphi(t) = t^\alpha / \log^\beta(e + 1/t)$, $\alpha \in [1, \frac{n}{n-1})$ and $\beta \geq 0$, satisfy the assumptions of Theorem 1.1.

Theorems 1.1 and 3.5 yield the following result.

Theorem 3.8 *Let D be an unbounded or a bounded domain in $\mathbb{R}^n$, $n \geq 2$. Let $1 \leq p < n$. If H is the function from Theorem 1.1 and $\|f\|_{L^p(D)} \leq 1$, then there exists a constant C such that the pointwise estimate*

$$H\left(\int_D \frac{|f(y)|}{\psi(|x-y|)^{n-1}}\,dy\right) \leq C(Mf(x))^p$$

holds for every $x \in D$. Here, Mf is the Hardy–Littlewood maximal operator of f and the constant C depends on n, p, and the Δ_2-constant of H only.

As a corollary we obtain from Theorems 3.3 and 3.8:

Corollary 3.9 *Let $1 \leq p < n$. Let the function H be as in Theorem 1.1. If D is a bounded φ-cigar John domain with a constant c_J, then there exit a constant C and a point $x_0 \in D$ such that the pointwise estimate*

$$H\left(\left|u(x) - u_{B(x_0,\operatorname{dist}(x_0,\partial D))}\right|\right) \leq C(M|\nabla u|(x))^p$$

holds for all $u \in L^1_p(D)$ *with* $\|\nabla u\|_{L^p(D)} \le 1$ *and for almost every* $x \in D$*. Here the constant* C *depends on* n, p, C_H, $C_H^{\Delta_2}$, $C_\varphi^{\Delta_2}$, c_J, $\varphi(1)$ *and* $\min\Big\{\operatorname{diam}(D), 1\Big\}$ *only.*

4 On Embeddings

Corollary 3.9 is essential in the proofs of the following Theorems 4.1 and 4.2.

Theorem 4.1 (Bounded domain, $1 < p < n$) *Assume that* φ *satisfies the conditions* (1)–(5)*,* $C_\varphi = 1$ *in the condition* (4)*, and there exists* $\alpha \in [1, n/(n-1))$ *such that* $t^\alpha/\varphi(t)$ *is increasing for* $t > 0$*. Let* ψ *be defined as in* (2.1)*. Let* $D \subset \mathbb{R}^n$*,* $n \ge 2$*, be a bounded* φ*-cigar John domain with a constant* c_J*. Let* $1 < p < n$*. Then there exists an* N*-function* H*, that satisfies* Δ_2*-condition and*

$$H^{-1}(t) \approx \frac{t^{\frac{1}{p}-1}}{\psi\left(t^{-\frac{1}{n}}\right)^{n-1}} \text{ for all } t > 0\,,$$

and there exists a constant $C < \infty$ *such that the inequality*

$$\|u - u_D\|_{L^H(D)} \le C\|\nabla u\|_{L^p(D)},$$

holds for every $u \in L^1_p(D)$*. Here the constant* C *depends on* n, p, $C_H^{\Delta_2}$, $C_\varphi^{\Delta_2}$, c_J *and* $\min\{\operatorname{diam}(D), 1\}$ *only.*

Proof Theorem 2.4 implies that D is a bounded ψ-John domain. Let x_0 be a John center. Let us denote $B = B(x_0, \operatorname{dist}(x_0, \partial D))$. Assume that $\|\nabla u\|_{L^p(D)} \le 1$. Corollary 3.9 yields that $H\left(\left|u(x) - u_B\right|\right) \le C(M|\nabla u|(x))^p$, where the constant C depends on n, p, $C_H^{\Delta_2}$, $C_\varphi^{\Delta_2}$, c_J, and $\min\{1, \operatorname{diam}(D)\}$ only. By integrating over D and using the fact that the maximal operator is bounded whenever $1 < p < n$, we obtain that

$$\int_D H\left(\left|u(x) - u_B\right|\right) dx \le C\int_D (M|\nabla u|(x))^p\, dx \le C\int_D |\nabla u(x)|^p\, dx \le C.$$

This yields that the inequality $\|u - u_B\|_{L^H(D)} \le C$ holds for every $u \in L^1_p(D)$ with $\|\nabla u\|_{L^p(D)} \le 1$. If $\|\nabla u\|_{L^p(D)} = 0$ then the function is a constant function and the claim holds. Otherwise we apply this inequality to the function $u/\|\nabla u\|_{L^p(D)}$ and obtain that $\|u - u_B\|_{L^H(D)} \le C\|\nabla u\|_{L^p(D)}$.

We may assume w.l.o.g. that $\|\nabla u\|_{L^p(D)} \ne 0$. By the triangle inequality $\|u - u_D\|_{L^H(D)} \le \|u - u_B\|_{L^H(D)} + \|u_B - u_D\|_{L^H(D)}$. Here,

$$\begin{aligned}\|u_B - u_D\|_{L^H(D)} &= |u_B - u_D|\, \|1\|_{L^H(D)} \le \frac{\|1\|_{L^H(D)}}{|D|}\|u - u_B\|_{L^1(D)} \\ &\le C\frac{\|1\|_{L^H(D)}\|1\|_{L^{H^*}(D)}}{|D|}\|u - u_B\|_{L^H(D)}\end{aligned}$$

where H^* is the conjugate function of H and C is the constant in Hölder's inequality. It is well known that $\|1\|_{L^H(D)}\|1\|_{L^{H^*}(D)} \approx |D|$ see [1, Chapter 2, Theorem 5.2]. Hence, we have shown that $\|u - u_D\|_{L^H(D)} \le C\|\nabla u\|_{L^p(D)}$ for every $u \in L^1_p(D)$. □

Theorem 4.2 (Bounded domain, $p = 1$) *Assume that the function φ satisfies the conditions* (1)– (5), *$C_\varphi = 1$ in the condition* (4), *and there exists $\alpha \in [1, n/(n-1))$ such that $t^\alpha/\varphi(t)$ is increasing for $t > 0$. Let ψ be defined as in (2.1) Let $D \subset \mathbb{R}^n$, $n \ge 2$, be a bounded φ-cigar John domain with a constant c_J. Then there exists an N-function H, that satisfies Δ_2-condition and*

$$H^{-1}(t) \approx \frac{1}{\psi\left(t^{-\frac{1}{n}}\right)^{n-1}} \quad \textit{for all } t > 0\,,$$

such that the inequality

$$\|u - u_D\|_{L^H(D)} \le C\|\nabla u\|_{L^1(D)},$$

holds for some constant C and for every $u \in L^1_p(D)$. Here the constant C depends only on n, $C_H^{\Delta_2}$, $C_\varphi^{\Delta_2}$, c_J, and $\min\{1, \operatorname{diam}(D)\}$.

The term $\min\{1, \operatorname{diam}(D)\}$ means that the constant depends on the diameter only for small diameters. For large diameters the constant is independent of the diameter. □

Proof Let us consider functions $u \in L^1_1(D)$ such that $\|\nabla u\|_{L^1(D)} \le 1$. The center ball $B(x_0, \operatorname{dist}(x_0, \partial D))$ is written as B. In the proof of Theorem 2.4 we had chosen x_0 so that $\operatorname{dist}(x_0, \partial D) \ge \psi(\frac{1}{4}\operatorname{diam}(D))/c_J$. We show that there exists a constant $C < \infty$ such that the inequality

$$\int_D H(|u(x) - u_B|)\,dx \le C \tag{4.1}$$

holds whenever $\|\nabla u\|_{L^1(D)} \le 1$. This yields the claim as in the proof of Theorem 4.1.

Since H is increasing, we first estimate

$$\int_D H(|u(x) - u_B|)\,dx \le \sum_{j\in\mathbb{Z}} \int_{\{x\in D: 2^j < |u(x)-u_B| \le 2^{j+1}\}} H(2^{j+1})\,dx.$$

Let us define $v_j(x) = \max\left\{0, \min\left\{|u(x) - u_B| - 2^j, 2^j\right\}\right\}$ for all $x \in D$. If $x \in \{x \in D : 2^j < |u(x) - u_B| \le 2^{j+1}\}$, then $v_{j-1}(x) \ge 2^{j-1}$. We obtain

$$\int_D H(|u(x) - u_B|)\,dx \le \sum_{j\in\mathbb{Z}} \int_{\{x\in D: v_j(x) \ge 2^j\}} H(2^{j+2})\,dx. \tag{4.2}$$

By the triangle inequality we have

$$v_j(x) = |v_j(x) - (v_j)_B + (v_j)_B| \le |v_j(x) - (v_j)_B| + |(v_j)_B|.$$

By the $(1,1)$-Poincaré inequality in a ball B, [5, Section 7.8], there exists a constant $C(n)$ such that

$$\begin{aligned} |(v_j)_B| = (v_j)_B &= \fint_B v_j(x)\,dx \le \fint_B |u(x) - u_B|\,dx \\ &\le C(n)|B|^{\frac{1}{n}} \fint_B |\nabla u(x)|\,dx \le C(n)|B|^{\frac{1}{n}-1}. \end{aligned}$$

We continue to estimate the right hand side of inequality (4.2)

$$\begin{aligned} &\int_D H(|u(x) - u_B|)\,dx \\ &\le \sum_{j\in\mathbb{Z}} \int_{\{x\in D: |v_j(x)-(v_j)_B| + C|B|^{-1} \ge 2^j\}} H(2^{j+2})\,dx \\ &\le \sum_{j\in\mathbb{Z}} \int_{\{x\in D: |v_j(x)-(v_j)_B| \ge 2^{j-1}\}} H(2^{j+2})\,dx + \sum_{2^{j-1} \le C(n)|B|^{\frac{1}{n}-1}} \int_D H(2^{j+2})\,dx \\ &\le \sum_{j\in\mathbb{Z}} \int_{\{x\in D: |v_j(x)-(v_j)_B| \ge 2^{j-1}\}} H(2^{j+2})\,dx + \sum_{j=-\infty}^{j_0} \int_D H(2^{j+2})\,dx, \end{aligned} \tag{4.3}$$

where $j_0 = \lceil \log(C(n)|B|^{\frac{1}{n}-1}) \rceil$.

Assume first that $\operatorname{diam}(D)$ is so large that $j_0 \le -2$. When $t < 1$, then $\psi(t^{-1/n}) = \varphi(1)t^{-1/n}$ by (2.1) and thus

$$H^{-1}(t) = \frac{1}{\psi(t^{-1/n})^{n-1}} = \varphi(1)^{1-n} t^{\frac{n-1}{n}}.$$

Thus for $t < 1$ we obtain that $H(t) \approx t^{\frac{n}{n-1}}$. This yields that

$$\begin{aligned} \sum_{j=-\infty}^{j_0} \int_D H(2^{j+2})\,dx &\approx |D| \sum_{j=-\infty}^{\lceil \log(C|B|^{\frac{1}{n}-1}) \rceil} 2^{\frac{n(j+2)}{n-1}} \le C|D| 2^{\frac{n}{n-1}\cdot \lceil \log(C|B|^{\frac{1}{n}-1}) \rceil} \\ &\le C|D||B|^{\frac{n}{n-1}(\frac{1}{n}-1)} = C|D||B|^{-1} \\ &\le C \frac{\operatorname{diam}(D)^n}{(\psi(\frac{1}{4}\operatorname{diam}(D))/c_J)^n}. \end{aligned} \tag{4.4}$$

This constant does not blow up when $\operatorname{diam}(D) \to \infty$:

$$\frac{\operatorname{diam}(D)^n}{(\psi(\frac{1}{4}\operatorname{diam}(D))/c_J)^n} \to \frac{4^n c_J^n}{\varphi(1)^n} \quad \text{as} \quad \operatorname{diam}(D) \to \infty.$$

Assume then that $\operatorname{diam}(D)$ is small. This yields that for every $j_0 \in \mathbb{Z}$ the sum $\sum_{j=-2}^{j_0} H(2^{j+2})$ is finite and depends on

$$j_0 \approx \log\Big(C(n)\operatorname{dist}(x_0, \partial D)^{1-n}\Big) \le \log\Big(C(n, c_J)\psi\big(\tfrac{1}{4}\operatorname{diam}(D)\big)^{1-n}\Big).$$

We obtain

$$\sum_{j=-\infty}^{j_0} \int_D H(2^{j+2})\, dx \le \sum_{j=-\infty}^{-2} \int_D H(2^{j+2}) + \sum_{j=-2}^{j_0} H(2^{j+2}) < \infty. \tag{4.5}$$

Then, we will find an upper bound for the sum

$$\sum_{j\in\mathbb{Z}} \int_{\{x\in D: |v_j(x)-(v_j)_B| \ge 2^{j-1}\}} H(2^{j+2})\, dx\,.$$

Since $\|\nabla v_j\|_{L^1(D)} \le \|\nabla u\|_{L^1(D)} \le 1$, Corollary 3.9 yields that

$$\begin{aligned}
&\sum_{j\in\mathbb{Z}} \int_{\{x\in D: |v_j(x)-(v_j)_B| \ge 2^{j-1}\}} H(2^{j+2})\, dx \\
&\quad = \sum_{j\in\mathbb{Z}} \int_{\{x\in D: H(|v_j(x)-(v_j)_B|) \ge H(2^{j-1})\}} H(2^{j+2})\, dx \\
&\quad \le \sum_{j\in\mathbb{Z}} \int_{\{x\in D: CM|\nabla v_j|(x) \ge H(2^{j-1})\}} H(2^{j+2})\, dx.
\end{aligned}$$

We choose for every $x \in \{x \in D : CM|\nabla v_j|(x) \ge H(2^{j-2})\}$ a ball $B(x, r_x)$, centered at x and with radius r_x depending on x, such that

$$C \fint_{B(x,r_x)} |\nabla v_j(y)|\, dy \ge \frac{1}{2} H(2^{j-1})$$

with the understanding that $|\nabla v_j|$ is zero outside D. By the Besicovitch covering theorem (or the 5-covering theorem) we obtain a subcovering $\{B_k\}_{k=1}^{\infty}$ so that we may estimate by the Δ_2-conditionof H

$$\sum_{j\in\mathbb{Z}}\int_{\{x\in D:|v_j(x)-(v_j)_B|\ge 2^{j-1}\}} H(2^{j+2})\,dx \le \sum_{j\in\mathbb{Z}}\sum_{k=1}^{\infty}\int_{B_k} H(2^{j+2})\,dx$$
$$\le \sum_{j\in\mathbb{Z}}\sum_{k=1}^{\infty}|B_k|H(2^{j+2}) \le \sum_{j\in\mathbb{Z}}\sum_{k=1}^{\infty} C|B_k|\frac{H(2^{j+2})}{H(2^{j-1})}\fint_{B_k}|\nabla v_j(y)|\,dy$$
$$\le C\sum_{j\in\mathbb{Z}}\int_D|\nabla v_j(y)|\,dy.$$

Let $E_j=\{x\in D: 2^j<|u(x)-u_B|\le 2^{j+1}\}$. Since $|\nabla v_j|$ is zero almost everywhere in $D\setminus E_j$ and $|\nabla u(x)|=\sum_j|\nabla v_j(x)|\chi_{E_j}(x)$ for almost every $x\in D$, we obtain

$$\sum_{j\in\mathbb{Z}}\int_{\{x\in D:|v_j(x)-(v_j)_B|\ge 2^{j-1}\}} H(2^{j+2})\,dx \le C\int_D|\nabla u(y)|\,dy\le C. \tag{4.6}$$

Estimates (4.3), (4.4), (4.5) and (4.6) imply inequality (4.1). □

Remark 4.3 In Theorem 4.2 the N-function H is the best possible in a sense that it cannot be replaced by any N-function K that satisfies the Δ_2-condition and $\lim_{t\to\infty}\frac{K(t)}{H(t)}=\infty$.

In [7, Theorem 7.2] we have shown that the corresponding embedding in Theorem 4.2 does not hold if

$$\lim_{t\to 0^+} t^n K\left(\frac{1}{\varphi(t)^{n-1}}\right)=\infty.$$

This is valid for this function K. By the definitions of H^{-1} and ψ we obtain that

$$\lim_{t\to 0^+} t^n K\left(\frac{1}{\varphi(t)^{n-1}}\right)=\lim_{s\to\infty}\frac{1}{s}K\left(\frac{1}{\varphi\left(s^{-\frac{1}{n}}\right)^{n-1}}\right)=\lim_{s\to\infty}\frac{K\left(H^{-1}(s)\right)}{H\left(H^{-1}(s)\right)}=\infty,$$

and thus there does not exists a constant c such that $\|u-u_D\|_{L^K(D)}\le c\|\nabla u\|_{L^1(D)}$, for every $u\in L^1_p(D)$.

Remark 4.4 We refer to the detailed discussion in [6,7] for the fact that our result is optimal when $p=1$.

Next we prove our main theorem.

Proof of Theorem 1.2 The proof follows the idea of the proof of [10, Theorem 4.1]. By Theorems 4.1 and 4.2 there exists a constant C such that the inequality

$$\|u-u_{D_i}\|_{L^H(D_i)}\le C\|\nabla u\|_{L^p(D_i)} \tag{4.7}$$

holds for each D_i and all $u \in L^1_p(D)$. The constant C does not blow up when the diameter of D_i tends to infinity. In the case $1 < p < n$ this is clear. In the case $p = 1$, we refer to the discussion after (4.4) in the proof of Theorem 4.2. The constant depends on D_1 but this does not cause a problem.

When $\|\nabla u\|_{L^p(D)} \le 1$ inequality (4.7) yields that there exists a constant $C < \infty$ such that the inequality

$$\int_{D_i} H(|u(x) - u_{D_i}|)\, dx \le C,$$

holds; here the constant C is independent of i.

Let us write $u_i = u_{D_i}$. The triangle inequality yields that

$$|u_i| \le \fint_{D_1} |u(x) - u_i|\, dx + \fint_{D_1} |u(x)|\, dx.$$

Since D_i satisfies inequality (4.7), we have $u \in L^H(D_1) \subset L^1(D_1)$ and thus the second term is finite. Again, by inequality (4.7) we obtain that

$$\begin{aligned}\fint_{D_1} |u(x)-u_i|\, dx &\le \frac{C\|1\|_{L^{H^*}(D_1)}}{|D_1|}\|u-u_{D_i}\|_{L^H(D_1)} \le \frac{C\|1\|_{L^{H^*}(D_1)}}{|D_1|}\|u-u_{D_i}\|_{L^H(D_i)}\\ &\le \frac{C\|1\|_{L^{H^*}(D_1)}}{|D_1|}\|\nabla u\|_{L^p(D_i)} \le \frac{C\|1\|_{L^{H^*}(D_1)}}{|D_1|}\|\nabla u\|_{L^p(D)} < \infty.\end{aligned}$$

Thus the real number sequence (u_i) is bounded and hence there exists a convergent subsequence (u_{i_j}) and $b \in \mathbb{R}$ such that $u_{i_j} \to b$.

Since H is continuous, $\lim_{j\to\infty} \chi_{D_{i_j}} H(|u(x)-u_{i_j}|) = \chi_D H(|u(x)-b|)$. Fatou's lemma and the modular form of (4.7) yield that

$$\begin{aligned}\int_D H(|u(x) - b|)\, dx &\le \liminf_{j\to\infty} \int_D \chi_{D_{i_j}} H(|u(x) - u_{i_j}|)\, dx\\ &= \liminf_{j\to\infty} \int_{D_{i_j}} H(|u(x) - u_{i_j}|) \le \liminf_{j\to\infty} C = C\end{aligned}$$

for every $u \in L^1_{\mathrm{loc}}(D)$ with $\|\nabla u\|_{L^p(D)} \le 1$. This yields that there exists a constant C such that the inequality $\|u - b\|_{L^H(D)} \le C$ holds for every $u \in L^1_p(D)$ with $\|\nabla u\|_{L^p(D)} \le 1$. The claim follows by applying this inequality to the function $u/\|\nabla u\|_{L^p(D)}$. □

Example 4.5 Let the function φ be defined as in Theorem 1.2. The following unbounded domains satisfy the assumptions of Theorem 1.2:

(a) $\mathbb{R}^n$, $n \ge 2$.
(b) $\{(x', x_n) \in \mathbb{R}^n : x_n \ge 0 \quad\text{and}\quad |x'| < \psi(x_n)\}$.

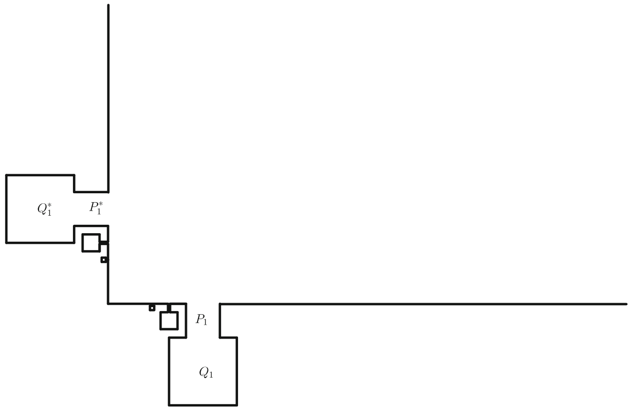

Fig. 3 Unbounded φ-cigar John domain that satisfies the assumptions of Theorem 1.2

(c) $\mathbb{R}^2 \setminus \big(\{(x, \varphi(x)) \in \mathbb{R}^2 : 0 \leq x \leq 1\} \cup \{(x, -\varphi(x)) \in \mathbb{R}^2 : 0 \leq x \leq 1\}\big)$.
(d) The undounded domain G constructed in Sect. 5, illustrated in Fig. 3.

5 Lebesgue Space Cannot be a Target Space

In this section we give an example which shows that for certain unbounded φ-cigar John domains the target space cannot be a Lebesgue space. The idea is that at near the infinity the target space should be $L^{np/(n-p)}$ but local structure of the domain may not allow so good integrability. We assume a priori that the function φ has the properties (1)–(5). Later on we give extra conditions to the function φ.

We construct a mushrooms-type domain. Let (r_m) be a decreasing sequence of positive real numbers converging to zero. Let Q_m, $m = 1, 2, \ldots$, be a closed cube in $\mathbb{R}^n$ with side length $2r_m$. Let P_m, $m = 1, 2, \ldots$, be a closed rectangle in $\mathbb{R}^n$ which has side length r_m for one side and $2\varphi(r_m)$ for the remaining $n-1$ sides. Let Q be the first quarter of the space i.e. all coordinates of the points in Q are positive. We attach Q_m and P_m together creating 'mushrooms' which we then attach, as pairwise disjoint sets, to the side $\{(0, x_2, \ldots, x_n) : x_2, \ldots, x_n > 0\}$ of Q so that the distance from the mushroom to the origin is at least 1 and at most 4, see Fig. 3. We assumed that the function φ has the properties (1)–(5), but we have to assume here also that $\varphi(r_m) \leq r_m$. We need copies of the mushrooms. By an isometric mapping we transform these mushrooms onto the side $\{(x_1, 0, \ldots, x_n) : x_1, x_3, \ldots, x_n > 0\}$ of Q and denote them by Q_m^* and P_m^*. So again the distance from the mushroom to the origin is at least 1 and at most 4. We define

$$G = \operatorname{int}\left(Q \cup \bigcup_{m=1}^{\infty}\left(Q_m \cup P_m \cup Q_m^* \cup P_m^*\right)\right). \tag{5.1}$$

See Fig. 3. We omit a short calculation which shows that G is a φ-cigar John domain.

Let us define a sequence of piecewise linear continuous functions $(u_k)_{k=1}^{\infty}$ by setting

$$u_k(x) := \begin{cases} F(r_k) & \text{in } Q_k, \\ -F(r_k) & \text{in } Q_k^*, \\ 0 & \text{in } Q, \end{cases}$$

where the function F will be given in (5.2). Then the integral average of u_k over G is zero for each k.

The gradient of u_k differs from zero in $P_m \cup P_m^*$ only and

$$|\nabla u_k(x)| = \frac{F(r_m)}{r_m}, \text{ when } x \in P_m \cup P_m^* .$$

Note that

$$\int_G |\nabla u_k(x)|^p \, dx = 2 \int_{P_m} \left(\frac{F(r_m)}{r_m} \right)^p = 2 r_m \, (\varphi(r_m))^{n-1} \, \frac{F(r_m)^p}{r_m^p} .$$

We require that $\int_G |\nabla u_k(x)|^p \, dx = 1$. Hence, we define

$$F(r_m) = \left(\frac{r_m^{p-1}}{2\varphi(r_m)^{n-1}} \right)^{1/p} . \tag{5.2}$$

Let H be an N-function. Then,

$$\begin{aligned} &\inf_{b \in \mathbb{R}} \int_G H(|u_k(x) - b|) \, dx \\ &\geq \inf_{b \in \mathbb{R}} \Big(|Q_m| \cdot H(|F(r_m) - b|) + |Q_m^*| \cdot H(|-F(r_m) - b|) \Big) \\ &\geq r_m^n H(F(r_m)) . \end{aligned}$$

Hence, we have

$$r_m^n H(F(r_m)) = r_m^n H\left(\left(\frac{r_m^{p-1}}{2\varphi(r_m^{n-1})} \right)^{1/p} \right) \geq r_m^n H\left(\frac{1}{2} \left(\frac{r_m^{p-1}}{\varphi(r_m^{n-1})} \right)^{1/p} \right).$$

Thus, there does not exist a positive constant C such that the inequality $\inf_b \|u - b\|_{L^H(G)} \leq C \|\nabla u\|_{L^p(G)}$ could hold for all u from the appropriate space if

$$\lim_{t \to 0^+} t^n H\left(\frac{1}{2} \left(\frac{t^{p-1}}{\varphi(t)^{n-1}} \right)^{1/p} \right) = \infty.$$

Assume that $\lim_{t\to 0^+} t/\varphi(t) = \infty$. If $H(t) = t^q$, then we obtain that the inequality does not hold if

$$q \geq \frac{np}{n-p}. \tag{5.3}$$

Assume then that we have a sequence (s_j) of positive numbers going to infinity. For each s_j we may choose points $x(j)$ and $y(j)$ such that the balls $B(x(j), s_j)$ and $B(y(j), s_j)$ are subsets of the first quadrant and $B(x(j), 3s_j) \cap B(y(j), 3s_j) = \emptyset$. We define a sequence of continuous functions $(v_j)_{j=1}^{\infty}$ that are radially linear on $B(x(j), 2s_j)$ and $B(y(j), 2s_j)$ by setting

$$v_j(x) := \begin{cases} s_j^{-\frac{n-p}{p}} & \text{in } B(x(j), s_j), \\ -s_j^{-\frac{n-p}{p}} & \text{in } B(y(j), s_j), \\ 0 & \text{in } G \setminus \left(B(x(j), 2s_j) \cup B(y(j), 2s_j)\right). \end{cases}$$

Now we have

$$\int_G |\nabla v_j|^p \, dx \leq C s_j^n \left| \frac{s_j^{-\frac{n-p}{p}}}{s_j} \right|^p \leq C$$

for some constant C. On the other hand, for any $b \in \mathbb{R}$

$$\begin{aligned} \int_G H(|v_j(x) - b|) \, dx &\geq C s_j^n H(|s_j^{-\frac{n-p}{p}} - b|) + C s_j^n H(|-s_j^{-\frac{n-p}{p}} - b|) \\ &\geq C s_j^n H(|s_j^{-\frac{n-p}{p}}|). \end{aligned}$$

Thus, there does not exist a positive constant C_1 such that the inequality $\inf_b \|u - b\|_{L^H(G)} \leq C_1 \|\nabla u\|_{L^p(G)}$ could hold for all u from the appropriate space if

$$\lim_{s\to\infty} s^n H(s^{-\frac{n-p}{p}}) = \lim_{s\to\infty} s^{\frac{pn}{n-p}} H\left(\frac{1}{s}\right) = \infty.$$

By choosing $H(t) = t^q$, we obtain that the inequality does not hold if

$$q < \frac{np}{n-p}. \tag{5.4}$$

If $\lim_{t\to 0^+} t/\varphi(t) = \infty$ and if there were an embedding with the Lebesgue space L^q as a target space, then by (5.3) we would have $q < \frac{np}{n-p}$ and by (5.4) we would have $q \geq \frac{np}{n-p}$. Thus the target space cannot be a Lebesgue space. The target space can be L^q only if $\lim_{t\to 0^+} t/\varphi(t) < \infty$ and in this case $q = \frac{np}{n-p}$. Note that the limit

$\lim_{t\to 0^+} t/\varphi(t)$ exists since φ is increasing and $\varphi \geq 0$. If $\lim_{t\to 0^+} t/\varphi(t) = m > 0$, then there exists $t_0 > 0$ such that $\frac{1}{2}m\varphi(t) \leq t \leq 2m\varphi(t)$.

We point out that with our assumptions the case $\lim_{t\to 0^+} t/\varphi(t) = 0$ is not possible. Namely if $\lim_{t\to 0^+} t/\varphi(t) = 0$, then $\lim_{t\to 0^+} \varphi(t)/t = \infty$, and this contradicts with condition (4).

Thus we have proved the following remarks.

Remark 5.1 Let φ satisfy (1)–(5), and assume that $\lim_{t\to 0^+} t/\varphi(t) = \infty$. Let G be the unbounded φ-cigar John domain constructed in (5.1). Let $1 \leq p < n$. Then there do not exist numbers $q \in \mathbb{R}$ and $C \in \mathbb{R}$ such that the inequality

$$\inf_{b\in\mathbb{R}} \|u - b\|_{L^q(G)} \leq C\|\nabla u\|_{L^p(G)}$$

could hold for all $u \in L^1_p(G)$.

Remark 5.2 Let the function φ satisfy conditions (1)–(5). Suppose that $\lim_{t\to 0^+} t/\varphi(t) = m \in (0,\infty)$. Then, there exists $t_0 > 0$ such that $\varphi(t) \approx t$ for all $t \in (0, t_0]$. Let G be the unbounded φ-cigar John domain constructed in (5.1). Assume that there exist numbers $q \in \mathbb{R}$ and $C \in \mathbb{R}$ such that the inequality

$$\inf_{b\in\mathbb{R}} \|u - b\|_{L^q(G)} \leq C\|\nabla u\|_{L^p(G)}$$

holds for all $u \in L^1_p(G)$. Then $q = \frac{np}{n-p}$.

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